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The Cosmological Constant and Trapped Surfaces

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Abstract. The cosmological constant is a model of dark energy that is one explanation for the accelerated expansion of the universe. The concept of a trapped surface provides a precise characterization of gravitational collapse that has proceeded beyond the point of no return. Therefore, because of its importance, we examined spherically-symmetric spacetimes with a cosmological constant to determine its impact on the characteristic development of trapped surfaces. The principal result of this study is a relationship between the mass and the cosmological constant that is a necessary and sufficient condition for trapped surfaces to develop to the future of a branch of a marginally trapped hypersurface.

1. Introduction

A trapped surface is a compact, spacelike two-surface having the property that all null geodesics meeting it orthogonally converge locally to the future. Penrose introduced this concept as a characterization of gravitational collapse that has proceeded beyond the point of no return [Penrose 1965], resulting in a black hole. Under rather general conditions, if an object collapses sufficiently far that trapped surfaces develop, then the spacetime containing the object must be singular [Penrose 1965, Hawking and Penrose 1970].

Several studies addressed the problem of determining conditions under which trapped surfaces develop [Pajerski and Newman 1971, Demmie and Janis 1973, and Demmie 1975]. These studies exploited the property of the Schwarzschild space-time [Schwarzschild 1917] that a region containing trapped surfaces is separated from a region that does not contain trapped surfaces by a marginally trapped hypersurface (MTH). A MTH is a null hypersurface whose normal has vanishing divergence. They determined the conditions for which trapped surfaces develop to its future. However, these studies did not include a cosmological constant in the analyses. With the importance of the cosmological constant as a model of dark energy, we performed the present study to understand the role of the cosmological constant on the development of trapped surfaces and the concomitant impact on black holes.

2. Formalism for Spherically-Symmetric Spacetimes

The Newman-Penrose (NP) formalism [Newman and Penrose 1962] is useful for investigating the characteristic development of trapped surfaces. Spherical symmetry leads to considerable simplification in the NP equations [Demmie 1971]. [Demmie 1971] showed that

- In a spherically-symmetric spacetime, there exists a class of coordinate systems and associated null tetrad systems such that in any one of these coordinate systems, (u, r, θ, ϕ), the associated null tetrad system is $\{D,\ \Delta,\ \delta,\ \overline{\delta}\}$ with

$$D = \frac{\partial}{\partial r}, \Delta = \frac{\partial}{\partial u} + U\frac{\partial}{\partial r}, \text{and } \delta = \frac{1}{\sqrt{2}}\, N \left(\frac{\partial}{\partial \theta} + i\, csc\theta\, \frac{\partial}{\partial \phi}\right). \qquad 1.$$

- The spin coefficients ρ, and μ are real functions of u and r given by[1]

$$\rho = -l_{\mu;\nu} m^{\mu} \bar{m}^{\nu}, \ \mu = n_{\mu;\nu} \bar{m}^{\mu} m^{\nu}, \tag{2.}$$

  where the l^{μ}, n^{μ}, m^{μ} are the respective components of $\{D,\ \Delta,\ \delta,\ \bar{\delta}\}$.
- The metric is

$$ds^2 = 2U\, du^2 - 2\, du\, dr + N^{-2}\, (d\theta^2 + \sin^2\theta\, d\phi^2)\,. \tag{3.}$$

- The spacelike two-surface, $S(u,r) = \{(u, r, \theta, \phi)\colon u \text{ and } r \text{ are constant}\}$, is a trapped surface if and only if it is compact and ρ and μ satisfy

$$\rho > 0 \ \ \textit{and} \ \ \mu < 0. \tag{4.}$$

3. Solution Procedure and Results

For any one of these spacetimes with a particular null coordinate system (u, r, θ, ϕ) and associated null tetrad system $\{D,\ \Delta,\ \delta,\ \bar{\delta}\}$, the main conditions that we adopted are that the MTH is given by $u = 0$ and the variables are analytic functions of u and r.

The NP equations in [Demmie, 1971] and $u = 0$ being a MTH imply

$$\rho(0,r) = 0, \textit{and } N(0,r) = N_0 \tag{5.}$$

where N_0 is an arbitrary constant. Comparison of N_0 with the Schwarzschild solution in suitable coordinates [Israel 1968 and Demmie 1971] establish that $N_0 = 1/2m$ where m is the mass.

Solving the NP equations with the MTH given by $u = 0$, yields

$$\rho(u,r) = \rho_1 u + \{\rho_1^2 r\}u^2 + \cdots \text{, where } \rho_1 = \tfrac{1}{2}(N_0^2 - \Lambda), \tag{6.}$$

[2]

$$\mu(u,r) = -\rho_1 r - \frac{1}{2}\{\rho_1(2N_0^2 - \Lambda)r^2\}u - \frac{1}{6}\{\rho_1^2(9N_0^2 - 4\Lambda)r^3\}u^2 + \cdots \tag{7.}$$

4. Characteristic Development of Trapped Surfaces

4.1 Case 1: $N_0^2 > \Lambda$ $(\rho_1 > 0)$

In this case, we observe that for any positive value of r, r_0, there exists a sufficiently small value of u, u_0, such that $\rho(u,r) > 0$ and $\mu(u,r) < 0$ for $0 < u \le u_0$ and $0 < r \le r_0$. Therefore, since r_0 can be arbitrarily large, we have

- *In a spherically-symmetric spacetime with mass (m) and cosmological constant (Λ), trapped surfaces develop to the future (u > 0) of a MTH (u = 0) on the r > 0 branch of this hypersurface if and only if*

$$N_0^2 > \Lambda\,, \textit{which is } 1/(2m)^2 > \Lambda \tag{8.}$$

For a negative cosmological constant, (8) implies that trapped surfaces exist to the future of a branch of a MTH for all masses.

4.2 Case 2: $N_0^2 = \Lambda$ $(\rho_1 = 0)$

In this case, we have the following exact solution to the NP equations:

$$U(u,r) = \tfrac{1}{2}\Lambda\, r^2\,, \ N(u,r) = \sqrt{\Lambda}, \ \rho(u,r) = \mu(u,r) = 0 \tag{9.}$$

Since ρ and μ are both zero, there is no development of trapped surfaces.

4.3 Case 3: $N_0^2 < \Lambda$ $(\rho_1 < 0)$

We observe that u and r must both be negative to have $\rho > 0$ and $\mu < 0$ near $u = 0$. Therefore,

- *In a spherically-symmetric spacetime with mass (m) and cosmological constant*

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[1] We employ the Einstein summation convention. A bar over a quantity denotes its complex conjugate.

[2] The units of the cosmological constant (Λ) are meters^{-2}. Hence, the unit of mass (m) in (6) is meters. The conversion from kg to meters is G/c^2 where G is the gravitational constant and c is the speed of light.

(Λ), trapped surfaces exist in the past (u < 0) of a MTH (u = 0) on the r < 0 branch of this hypersurface if and only if

$$N_0^2 < \Lambda\,, which\ is\ 1/(2m)^2 < \Lambda \qquad 10$$

For a positive cosmological constant, we can only assert the existence of trapped surfaces near this hypersurface in the region $u < 0$, and $r < 0$.

5. Further Discussion of Results

For a given positive cosmological constant (Λ), we define a critical mass ($m_{critical}$) by

$$\Lambda = 1/4m_{critical}^2\,. \qquad 11$$

The curve (11) defines the boundary in the m-Λ plane between the region where trapped surfaces exist to the future of a MTH and where they do not exist.

5.1 Cosmological Constant Obtained from the Planck-Satellite Data and Critical Mass

The cosmological constant is determined from the ratio (Ω_Λ) of the energy density attributed to the cosmological constant (ρ_Λ) to the critical density of the universe (ρ_C), which is the value of density necessary to keep a spatially-flat universe from expanding forever. Now $\Omega_\Lambda = \frac{\rho_\Lambda}{\rho_C}$, where $\rho_C = \frac{3H^2}{8\pi G}$ and $\rho_\Lambda = \frac{\Lambda c^2}{8\pi G}$, H is Hubble's parameter, G is the gravitational constant, and c is the speed of light. Therefore,

$$\Lambda = 3\Omega_\Lambda (H/c)^2\,. \qquad 12$$

The Planck-satellite data estimate Ω_Λ to be 0.6889 ± 0.0056 and H to be 67.66 ± 0.42 km/s/Mpc = (2.1927664 ± 0.0136) $\times 10^{-18}$ s^{-1} [Planck Collaboration 2018]. Therefore,

$$\Lambda = 1.1057 \times 10^{-52}\ m^{-2}\,. \qquad 13$$

This value for Λ in (11) implies that the critical mass is

$$m_{critical} = \frac{1}{2\sqrt{\Lambda}}\frac{c^2}{G} = \frac{c}{2H\sqrt{3\Omega_\Lambda}}\frac{c^2}{G} = 6.40\times10^{52}\ kg = 3.22x10^{22}\,M_\odot\ , \qquad 14$$

where $M_\odot$ is the mass of the sun. The maximum mass for a black hole must be strictly less than the critical mass given the current value of the cosmological constant. Estimates of supermassive black holes as large as $6.6x10^{10}\ M_\odot$ (Ton 618) were made [Shemmer, et al. 2004]. The black hole at the center of the Milky Way galaxy (Sagittarius A*) is about $4.2x10^6\ M_\odot$ [Ghez, et al. 2008].

5.2 A Cosmological Implication

A closed homogeneous, isotropic universe with a cosmological constant given by (13) must be spherically symmetric with a closed, spherical two-surface bounding the spatial section. Since light cannot escape from this surface, it must be a trapped surface. Hence, the maximum mass for this universe must be less than the critical mass in (14). [Mercier 2019] estimated the mass of the observable universe to be $1.73x10^{53}$ kg. This mass exceeds the mass of $6.40x10^{52}$ kg in (14). Therefore, we conclude that this universe cannot be closed and bounded.

5.3 Cosmological Constant for Critical Mass Equal to the Planck Mass

Consider an object whose mass (m_P) equals the Planck mass [Tomilin 1999]

$$m_P = \sqrt{\hbar c/G} \approx 2.176434(24) \times 10^{-8}\ kg\,, \qquad 15$$

where $\hbar$ is Planck's constant divided by 2π. The value for the cosmological constant corresponding to this mass is $\Lambda_P = 9.570263 \times 10^{68}\ m^{-2}$ Therefore, the ratio of Λ_P to Λ in (13) is

$${}^{\Lambda_P}/_{\Lambda} = 8.656503 \times 10^{120}\,. \qquad 16$$

Predictions for the cosmological constant using quantum field theory exceed observations by 120 orders of magnitude in [Weinberg 1989]. We note 54 orders were found in [Martin 2012].

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