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# A Concrete Construction of a Topological Operator in Factorization Algebras

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Factorization algebras play a central role in the formulation of quantum field theories given by Kevin Costello and Owen Gwilliam. In this paper, we propose a concrete construction of a topological operator using their formulation. We focus on a shift symmetry of a 1D massless scalar theory and discuss its $\mathbb{Z}$-gauging, i.e. compact scalar theory and its $\theta$-vacuum.

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## 1. Introduction

Quantum field theory is a central topic in theoretical physics. It is valid for describing various kinds of phenomena and also useful in the generation of some mathematical conjectures. In spite of such effectiveness, we still do not understand the complete mathematical formulation of quantum field theories.

In the last 10 years, Kevin Costello and Owen Gwilliam have developed a new formulation based on factorization algebras [1,2]. This formulation works well in perturbative field theories, conformal field theories, and topological quantum field theories. These days, some people are working on formulations of nonperturbative aspects of quantum field theories [3,4].

In such a context, nonperturbative Noether theorem is also proposed in Ref. [5], motivated by recent developments of generalized symmetries [6,7,8]. The dogma of generalized symmetries is

$$\text{Symmetry} = \text{Topological operator}. \tag{1}$$

A topological operator with $\mathcal{U}(U)$ some support $U$ is *invariant* under homotopic transformations of $U$.

A question arises: what does this *invariance* mean? We will show that $\mathcal{U}(U_1)$ and $\mathcal{U}(U_2)$ are in the same equivalence class of Batalin–Vilkovisky cohomology when $U_1$ and $U_2$ are homotopic. We will propose that this is the definition of the topological operators.

In this paper, we focus only on the case of 1D free massless scalar theory (i.e. quantum mechanics without potential terms). This theory has a shift symmetry $\Phi \mapsto \Phi + \alpha$ where $\Phi$ is a field and $\alpha$ is a real number. We give a concrete construction of the topological operator representing this shift $\mathbb{R}$-symmetry. One virtue of the topological operator is that we can perform gauging of the subsymmetry $\mathbb{Z} \subset \mathbb{R}$. By this $\mathbb{Z}$-gauging, we can obtain a compact scalar theory. This is an interesting theory because it has instanton effects. Work has been done on compact $U(1)$ $p$-form gauge theory [4] motivated by S-duality. The case of $p = 0$ is essentially the same


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as this paper in terms of using $\mathbb{Z}$-gauging. However, our construction is more concrete, and we will also discuss the relation to generalized symmetry notions and the $\theta$-vacuum.

This paper is organized as follows. In Section 2, we will review the formulation by Costello and Gwilliam. In Section 3, we will propose a concrete construction of the topological operator. In Section 4, we will see the $\theta$-degree and the formulation. Section 5 is devoted to the conclusion and discussion.

## 2. Review of factorization algebra of free real scalar theory

### 2.1. *Observable space*

In this section, we will define an observable $\mathcal{O}$. Let $M$ be a $d$-dimensional manifold and a real scalar field $\Phi$ be in $C^\infty(M)$. $\mathcal{O}$ must have two properties:

- $\mathcal{O}$ is a functional; i.e. $\mathcal{O}$ is a map from a field $\Phi$ to a number $\mathcal{O}(\Phi)$.
- $\mathcal{O}$ has a concept of *locality*. In other words, $\mathcal{O}$ has support $U \subset M$.

Here $U$ is an open subset of $M$. If the shape of $U$ is like a $d$-dimensional open ball $B^d$, $\mathcal{O}$ is a point operator with a UV cutoff. In the case of $U$ being $B^{d-1} \times S^1$, it is a loop operator.

Motivated by the above two properties, we will find an example of observables: a *linear observable*,

$$\mathcal{O}_{\text{linear}} : \Phi \mapsto \int_M f\Phi, \quad f \in C_{\text{c}}^\infty(U, \mathbb{C}), \tag{2}$$

where $f$ is a $\mathbb{C}$-function with compact support in $U$. The integral $\int_M f\Phi$ refers to only the $\Phi$ on $U$; hence in this sense $\mathcal{O}_{\text{linear}}$ has locality. We identify $\mathcal{O}_{\text{linear}}$ with $f$, then define *linear observable space on* $U$ as

$$\text{Obs}_{\text{linear}}(U) := C_{\text{c}}^\infty(U, \mathbb{C}). \tag{3}$$

A more general observable can be regarded as a polynomial of functions with compact supports: $\mathcal{O} = c + f + f_1 * f_2 + \cdots$, where $*$ is a formal symmetric product, $c \in \mathbb{C}$, and $f, f_1, f_2, \cdots \in C_{\text{c}}^\infty(U, \mathbb{C})$. $\mathcal{O}$ acts $\Phi$ as

$$\mathcal{O} : \Phi \mapsto c + \int_M f\Phi + \int_M f_1\Phi \int_M f_2\Phi + \cdots . \tag{4}$$

**Definition 1.** Observable space $\text{Obs}(U)$ is defined as

$$\text{Obs}(U) := \text{Sym}\left(C_{\text{c}}^\infty(U, \mathbb{C})\right) \tag{5}$$

and the elements are called observables on $U$.

### 2.2. *Classical derived observable space*

Roughly speaking, *derived* means that we will add a concept of *degree* to the observable space $\text{Obs}(U)$. This formulation was originally given by Batalin and Vilkovisky [9,10]:

$$C_{\text{c}}^\infty(U)^0 := \left(C_{\text{c}}^\infty(U, \mathbb{C}), 0\right), \tag{6}$$

$$C_{\text{c}}^\infty(U)^{-1} := \left(C_{\text{c}}^\infty(U, \mathbb{C}), -1\right). \tag{7}$$

The first one is a *degree-0 linear observable*, and the second one is a *degree-*$(-1)$ *linear observable*, which corresponds to an *antifield* in the physics literature. For simplicity, we denote $(f, 0) \in C_{\text{c}}^\infty(U)^0$ and $(f, -1) \in C_{\text{c}}^\infty(U)^{-1}$ as $f$ and $f^\star$. The symmetric product $*$ is defined for them as

$$a * b = (-1)^{|a||b|} b * a. \tag{8}$$

**Definition 2.** Classical derived observable space $\mathrm{Obs}^{\mathrm{cl}}(U)$ is defined as

$$\mathrm{Obs}^{\mathrm{cl}}(U) := \left( \mathrm{Sym}\left( C_{\mathrm{c}}^{\infty}(U)^{-1} \oplus C_{\mathrm{c}}^{\infty}(U)^{0} \right), \Delta_{\mathrm{BV}}^{\mathrm{cl}} \right) \tag{9}$$

where $\Delta_{\mathrm{BV}}^{\mathrm{cl}}$ is a classical Batalin–Vilkovisky operator defined in Definition 3.

By Eq. (8), we can rewrite classical derived observable space $\mathrm{Obs}^{\mathrm{cl}}(U)$:[1]

$$\begin{aligned} \mathrm{Obs}^{\mathrm{cl}}(U) = \Bigg( \cdots \oplus & \left( \bigwedge^{2} C_{\mathrm{c}}^{\infty}(U)^{-1} * \mathrm{Sym}\left( C_{\mathrm{c}}^{\infty}(U)^{0} \right) \right) \\ \oplus & \left( C_{\mathrm{c}}^{\infty}(U)^{-1} * \mathrm{Sym}\left( C_{\mathrm{c}}^{\infty}(U)^{0} \right) \right) \\ \oplus & \mathrm{Sym}\left( C_{\mathrm{c}}^{\infty}(U)^{0} \right), \Delta_{\mathrm{BV}}^{\mathrm{cl}} \Bigg). \end{aligned} \tag{10}$$

This is a decomposition about the degree. The first line is degree $(-2)$, second one is degree $(-1)$, and third one is degree 0. Actually $\Delta_{\mathrm{BV}}^{\mathrm{cl}}$ is defined as a map from degree $(-n)$ to degree $(-n+1)$:

$$\Delta_{\mathrm{BV}}^{\mathrm{cl}} : \bigwedge^{n} C_{\mathrm{c}}^{\infty}(U)^{-1} * \mathrm{Sym}\left( C_{\mathrm{c}}^{\infty}(U)^{0} \right) \to \bigwedge^{n-1} C_{\mathrm{c}}^{\infty}(U)^{-1} * \mathrm{Sym}\left( C_{\mathrm{c}}^{\infty}(U)^{0} \right). \tag{11}$$

The concrete form of $\Delta_{\mathrm{BV}}^{\mathrm{cl}}$ depends on what theory we want. In this paper, we are interested in free theory; thus we will define it as follows.

**Definition 3.** The classical Batalin–Vilkovisky operator $\Delta_{\mathrm{BV}}^{\mathrm{cl}}$ is a map $C_{\mathrm{c}}^{\infty}(U)^{-1} \ni f^{\star} \mapsto -(-\Delta + m^2) f \in C_{\mathrm{c}}^{\infty}(U)^{0}$. Here $\Delta$ is a Laplacian of $M$. We define it with the Leibniz rule; then we have

$$\begin{aligned} \Delta_{\mathrm{BV}}^{\mathrm{cl}} \left( f_1^{\star} * \cdots * f_n^{\star} * P \right) &= \sum_{i=1}^{n} f_1^{\star} * \cdots * f_{i-1}^{\star} * (-1)^{i-1} \left( \Delta_{\mathrm{BV}}^{\mathrm{cl}} f_i^{\star} \right) * f_{i+1}^{\star} * \cdots * f_n^{\star} * P \\ &= \sum_{i=1}^{n} f_1^{\star} * \cdots * \widehat{f_i^{\star}} * \cdots * f_n^{\star} * (-1)^{i-1} \left( \Delta_{\mathrm{BV}}^{\mathrm{cl}} f_i^{\star} \right) * P \end{aligned} \tag{12}$$

where $f_i^{\star} \in C_{\mathrm{c}}^{\infty}(U)^{-1}$, $P \in \mathrm{Sym}\left( C_{\mathrm{c}}^{\infty}(U)^{0} \right)$. $(-1)^{i-1}$ comes from the rule that $\Delta_{\mathrm{BV}}^{\mathrm{cl}}$ and $f_i^{\star}$ are anticommutative, since $\Delta_{\mathrm{BV}}^{\mathrm{cl}}$ has degree $+1$.

By some calculations, we have

- *Leibniz rule*: $\Delta_{\mathrm{BV}}^{\mathrm{cl}}(A * B) = \Delta_{\mathrm{BV}}^{\mathrm{cl}}(A) * B + (-1)^{|A|} A * \Delta_{\mathrm{BV}}^{\mathrm{cl}}(B)$,
- *Nilpotency*: $\left( \Delta_{\mathrm{BV}}^{\mathrm{cl}} \right)^2 = 0$.

[1] To be precise, we need to take a completion in order to make $\mathrm{Obs}^{\mathrm{cl}}(U)$ a topological vector space. However, we omit the discussion. The interested reader can see Ref. [1] for details.

*Nilpotency* means that

$$\begin{aligned}\mathrm{Obs}^{\mathrm{cl}}(U) = \Bigg( \cdots \xrightarrow{\Delta^{\mathrm{cl}}_{\mathrm{BV}}} \bigwedge^{2} C^{\infty}_{\mathrm{c}}(U)^{-1} * \mathrm{Sym}\left(C^{\infty}_{\mathrm{c}}(U)^{0}\right) \\ \xrightarrow{\Delta^{\mathrm{cl}}_{\mathrm{BV}}} C^{\infty}_{\mathrm{c}}(U)^{-1} * \ \mathrm{Sym}\left(C^{\infty}_{\mathrm{c}}(U)^{0}\right) \\ \xrightarrow{\Delta^{\mathrm{cl}}_{\mathrm{BV}}} \mathrm{Sym}\left(C^{\infty}_{\mathrm{c}}(U)^{0}\right) \Bigg)\end{aligned} \tag{13}$$

is a chain complex. This is called the *classical Batalin–Vilkovisky complex*. The cohomology of the classical Batalin–Vilkovisky complex is called *classical Batalin–Vilkovisky cohomology*:

$$H^{*}\left(\mathrm{Obs}^{\mathrm{cl}}(U)\right). \tag{14}$$

The *Leibniz rule* means that $H^{*}\left(\mathrm{Obs}^{\mathrm{cl}}(U)\right)$ has a product $*$, because the product of $A + \Delta^{\mathrm{cl}}_{\mathrm{BV}}(\cdots)$ and $B + \Delta^{\mathrm{cl}}_{\mathrm{BV}}(\cdots)$ can be rewritten as $A * B + \Delta^{\mathrm{cl}}_{\mathrm{BV}}(\cdots)$.[2]

**Remark 1.** On the other hand, the quantum Batalin–Vilkovisky operator $\Delta^{\mathrm{q}}_{\mathrm{BV}}$ does NOT have a Leibniz rule, as we discuss later, so quantum cohomology $H^{*}\left(\mathrm{Obs}^{\mathrm{q}}(U)\right)$ does NOT have the product $*$.

Let us consider the physical meanings of the cohomology $H^{0}\left(\mathrm{Obs}^{\mathrm{cl}}(U)\right)$. We take 0-degree observables $\mathcal{O}_1$, $\mathcal{O}_2$ and assume that these are the same in the cohomology, i.e. $\exists X$ s.t.:

$$\mathcal{O}_2 - \mathcal{O}_1 = \Delta^{\mathrm{cl}}_{\mathrm{BV}} X. \tag{16}$$

Let $\Phi_{\mathrm{cl}}$ be a solution of the equation of motion $(-\Delta + m^2)\Phi = 0$.[3] We have

$$\begin{aligned}\mathcal{O}_2(\Phi_{\mathrm{cl}}) - \mathcal{O}_1(\Phi_{\mathrm{cl}}) &= \Delta^{\mathrm{cl}}_{\mathrm{BV}} X(\Phi_{\mathrm{cl}}) \\ &= 0.\end{aligned} \tag{17}$$

Hence we can see $H^{0}\left(\mathrm{Obs}^{\mathrm{cl}}(U)\right)$ as the on-shell evaluation of observables. However, $[\mathcal{O}_1](= [\mathcal{O}_2])$ is not a number. Then we define a map called a *state* $\langle - \rangle$.

**Definition 4.** A state $\langle - \rangle$ is a smooth map:

$$\langle - \rangle : H^{0}\left(\mathrm{Obs}^{\mathrm{cl}}(M)\right) \to \mathbb{C}. \tag{18}$$

## 2.3. *Concrete calculations of classical Batalin–Vilkovisky cohomology*

In some situations, we can calculate $H^{*}\left(\mathrm{Obs}^{\mathrm{cl}}(M)\right)$ explicitly.

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[2]To check this, let us think about $(A + \Delta^{\mathrm{cl}}_{\mathrm{BV}}\tilde{A}) * (B + \Delta^{\mathrm{cl}}_{\mathrm{BV}}\tilde{B})$ where $A$, $B$ satisfy $\Delta^{\mathrm{cl}}_{\mathrm{BV}} A = 0,\ \Delta^{\mathrm{cl}}_{\mathrm{BV}} B = 0$:

$$\begin{aligned}\left(A + \Delta^{\mathrm{cl}}_{\mathrm{BV}}\tilde{A}\right) * \left(B + \Delta^{\mathrm{cl}}_{\mathrm{BV}}\tilde{B}\right) &= A * B + \left(\Delta^{\mathrm{cl}}_{\mathrm{BV}}\tilde{A}\right) * B + A * \left(\Delta^{\mathrm{cl}}_{\mathrm{BV}}\tilde{B}\right) + \left(\Delta^{\mathrm{cl}}_{\mathrm{BV}}\tilde{A}\right) * \left(\Delta^{\mathrm{cl}}_{\mathrm{BV}}\tilde{B}\right) \\ &= A * B + \left(\Delta^{\mathrm{cl}}_{\mathrm{BV}}\tilde{A}\right) * B + (-1)^{|\tilde{A}|}\, \tilde{A} * \left(\Delta^{\mathrm{cl}}_{\mathrm{BV}} B\right) + A * \left(\Delta^{\mathrm{cl}}_{\mathrm{BV}}\tilde{B}\right) \\ &\quad + (-1)^{|A|}\left(\Delta^{\mathrm{cl}}_{\mathrm{BV}} A\right) * \tilde{B} + \left(\Delta^{\mathrm{cl}}_{\mathrm{BV}}\tilde{A}\right) * \left(\Delta^{\mathrm{cl}}_{\mathrm{BV}}\tilde{B}\right) + (-1)^{|\tilde{A}|}\, \tilde{A} * \left(\Delta^{\mathrm{cl}}_{\mathrm{BV}}\right)^{2} \tilde{B} \\ &= A * B + \Delta^{\mathrm{cl}}_{\mathrm{BV}}\left(\tilde{A} * B\right) + (-1)^{|A|}\, \Delta^{\mathrm{cl}}_{\mathrm{BV}}\left(A * \tilde{B}\right) + \Delta^{\mathrm{cl}}_{\mathrm{BV}}\left(\tilde{A} * \left(\Delta^{\mathrm{cl}}_{\mathrm{BV}}\tilde{B}\right)\right) \\ &= A * B + \Delta^{\mathrm{cl}}_{\mathrm{BV}}\left(\cdots\right).\end{aligned} \tag{15}$$

[3]To obtain a theory with some interactions, we need to add some terms to $\Delta^{\mathrm{cl}}_{\mathrm{BV}}$.

**Theorem 1.** *If $M$ is compact and $m^2 > 0$, then*

$$H^n\left(\mathrm{Obs}^{\mathrm{cl}}(M)\right) = \begin{cases} \mathbb{C} & (n = 0) \\ 0 & (\text{otherwise}) \end{cases}. \tag{19}$$

*Proof.*

$$A := \left(C_{\mathrm{c}}^{\infty}(M)^{-1} \xrightarrow{\Delta_{\mathrm{BV}}^{\mathrm{cl}}} C_{\mathrm{c}}^{\infty}(M)^{0}\right). \tag{20}$$

This is an isomorphism. Thus $H^*(A) = 0$, then $H^0(\mathrm{Sym}(A)) = \mathbb{C}$ and $H^{n\le -1}(\mathrm{Sym}(A)) = 0$. □

**Theorem 2.** *If $M = \mathbb{R}$ and $I \subset M$ is an interval,*

$$H^n\left(\mathrm{Obs}^{\mathrm{cl}}(I)\right) = \begin{cases} \mathbb{C}[q, p] & (n = 0) \\ 0 & (\text{otherwise}) \end{cases}. \tag{21}$$

*Here $q, p$ has degree 0.*

*Proof.* We show a quasi-isomorphism:

$$A \sim B \tag{22}$$

where

$$A = \left(C_{\mathrm{c}}^{\infty}(I)^{-1} \xrightarrow{\Delta_{\mathrm{BV}}^{\mathrm{cl}}} C_{\mathrm{c}}^{\infty}(I)^{0}\right),\ B = (0 \to \mathbb{C}^2). \tag{23}$$

$\mathbb{C}^2$ sits in degree 0 and we denote the basis of $\mathbb{C}^2$ as $q, p$. The cohomology of $\mathrm{Sym}(A)$ and $\mathrm{Sym}(B)$ are $H^*(\mathrm{Obs}^{\mathrm{cl}}(I))$ and $\mathbb{C}[q, p]$ respectively.

First of all, we will see the following commutative diagram:

$$\begin{array}{ccc} C_{\mathrm{c}}^{\infty}(I)^{-1} & \longrightarrow & C_{\mathrm{c}}^{\infty}(I)^{0} \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ 0 & \longrightarrow & \mathbb{C}^2 \end{array} \tag{24}$$

where $\pi_2$ is defined as

$$\pi_2(g) := q\int_I dx\, g(x)\phi_q(x) + p\int_I dx\, g(x)\phi_p(x) \tag{25}$$

for $g \in C_{\mathrm{c}}^{\infty}(I)^0$. The definition of $\phi_q, \phi_p \in C^{\infty}(\mathbb{R})$ is as follows. In the case that $m > 0$, we define $\phi_q, \phi_p \in C^{\infty}(\mathbb{R})$ as

$$\phi_q(x) = \frac{1}{2}(e^{mx} + e^{-mx}),\ \phi_p(x) = \frac{1}{2m}(e^{mx} - e^{-mx}). \tag{26}$$

They form the kernel of $-\Delta + m^2$. If $m = 0$, we define

$$\phi_q(x) = 1,\ \phi_p(x) = x. \tag{27}$$

We note that $\phi_p'(x) = \phi_q(x)$. We can easily check that $\pi_2(g) = 0$ holds if $g = \Delta_{\mathrm{BV}}^{\mathrm{cl}} f^{\star}$.

Next, we will see $H^0(A) = H^0(B)$. By definition,

$$H^0(A) = \frac{C_{\mathrm{c}}^{\infty}(I)^0}{\mathrm{im}\left(\Delta_{\mathrm{BV}}^{\mathrm{cl}}\right)}. \tag{28}$$

We will show that $\mathrm{im}(\Delta_{\mathrm{BV}}^{\mathrm{cl}}) = \ker(\pi_2)$. If it holds, $H^0(A) = H^0(B)$. Obviously,

$$\mathrm{im}\left(\Delta_{\mathrm{BV}}^{\mathrm{cl}}\right) \subset \ker(\pi_2). \tag{29}$$

Then we will check $\mathrm{im}(\Delta^{\mathrm{cl}}_{\mathrm{BV}}) \supset \ker(\pi_2)$. Take $f \in \ker(\pi_2)$. $f$ satisfies

$$\int_I dx\, f(x)e^{mx} = 0 \text{ and } \int_I dx\, f(x)e^{-mx} = 0 \quad \text{(for the massive case)},$$
$$\int_I dx\, f(x)x = 0 \text{ and } \int_I dx\, f(x) = 0 \quad \text{(for the massless case)}, \tag{30}$$

because $\int_I dx f(x)\phi_p(x) = 0$ and $\int_I dx f(x)\phi_q(x) = 0$. Let $G \in C^0(\mathbb{R})$ be the Green's function:

$$G(x) = \frac{1}{2m}e^{-m|x|} \quad \text{(for the massive case)},$$
$$G(x) = -\frac{1}{2}|x| \quad \text{(for the massless case)}. \tag{31}$$

Then the convolution of $f$ and $G$ is

$$(G \cdot f)(x) := \int_I dy\, G(x-y)f(y). \tag{32}$$

This is in $C^\infty_{\mathrm{c}}(I)^0$ because of Eq. (30). For $(G \cdot f)^\star \in C^\infty_{\mathrm{c}}(I)^{-1}$,

$$f = \Delta^{\mathrm{cl}}_{\mathrm{BV}}(G \cdot f)^\star. \tag{33}$$

Thus $f \in \mathrm{im}(\Delta^{\mathrm{cl}}_{\mathrm{BV}})$, and $\mathrm{im}(\Delta^{\mathrm{cl}}_{\mathrm{BV}}) \supset \ker(\pi_2)$.

Finally, we will see $H^{-1}(A) = H^{-1}(B)$. Clearly $H^{-1}(B) = 0$. Then let us think of

$$H^{-1}(A) = \ker\left(\Delta^{\mathrm{cl}}_{\mathrm{BV}}\right). \tag{34}$$

This is trivial since there are no nontrivial solutions of

$$(-\Delta + m^2)f = 0 \tag{35}$$

for $f \in C^\infty_{\mathrm{c}}(I)$. Then $H^{-1}(A) = H^{-1}(B) = 0$. □

**Theorem 3.** *If $M = \mathbb{R}$, then for any value of $m$*

$$H^n\left(\mathrm{Obs}^{\mathrm{cl}}(\mathbb{R})\right) = \begin{cases} \mathbb{C}[q, p] & (n = 0) \\ 0 & (\text{otherwise}) \end{cases}. \tag{36}$$

*Here $q, p$ has degree 0.*

*Proof.* The proof of this is essentially the same as Theorem 2. □

Theorem 1 tells us that the state $\langle - \rangle$ can be given as an isomorphism; i.e. the state $\langle - \rangle$ is canonically determined. On the other hand, for the cases of Theorems 2 and 3, there is room to choose the state $\langle - \rangle$. Why is there such a difference? In the massive case, we have a reasonable interpretation for this difference.

In the case of the noncompact manifold $M_{\mathrm{non}}$, the boundary condition is necessary to compute the path integral. If not, the scalar field $\Phi$ might diverge on $\partial \overline{M_{\mathrm{non}}}$. Then the path integral calculation fails. In order to compute the value of the path integral, we must choose a boundary condition. Reflecting this, we must choose a map $\langle - \rangle : H^0\left(\mathrm{Obs}^{\mathrm{cl}}(M_{\mathrm{non}})\right) \to \mathbb{C}$.

On the other hand, in the case of the compact manifold $M_{\mathrm{com}}$, the scalar field $\Phi$ never diverges on $\partial M_{\mathrm{com}}$. Thus we can decide the value of the path integral. Similarly, we find the state $\langle - \rangle$ canonically.

The above discussion is valid for the massive case. However, in the massless case, even for a compact manifold, $H^0\left(\mathrm{Obs}^{\mathrm{cl}}(M_{\mathrm{com}})\right)$ is not isomorphic to $\mathbb{C}$. In other words, we cannot decide the expectation value naturally, though there is no $\Phi$ divergence on the boundary $\partial M_{\mathrm{com}}$. In order to understand this, we have to see another factor to interrupt the path integral computation: *IR divergence*. We will propose a treatment of IR divergence in a forthcoming paper (M. Kawahira, manuscript in preparation).

### 2.4. *Definition of position observable Q and momentum observable P*

We define a position observable $Q$ as a function in $C_c^\infty(I)$ satisying

$$\int_I Q(x)\phi_q(x)\,dx = 1 \text{ and } Q(-x) = Q(x). \tag{37}$$

By the second condition, we obtain $\int_I Q(x)\phi_p(x)\,dx = 0$. We define a momentum observable as

$$P := -Q' \in C_c^\infty(I)^0. \tag{38}$$

Now we see why we call $Q$ and $P$ the position and momentum. If $Q$, $P$ act on field $\Phi \in C^\infty(\mathbb{R})$, we get

$$Q(\Phi) = \int_{I_t} Q(x)\Phi(x)\,dx,\;\; P(\Phi) = \int_{I_t} Q(x)\Phi'(x)\,dx. \tag{39}$$

Especially in the massless case, $Q(\cdot)$ is a "smeared $\delta$-function" because we have

$$\int_I Q(x)\,dx = 1 \text{ and } Q(-x) = Q(x) \tag{40}$$

by using $\phi_q = 1$. Hence, if we narrow the width of the interval, $Q$ gets close to the $\delta$-function, and $Q(\Phi)$ and $P(\Phi)$ approach $\Phi(t)$ and $\Phi'(t)$.

In the massive case, $Q$ is also a kind of smeared $\delta$-function. To see this, we remind ourselves that

$$\phi_q(x) = \frac{1}{2}(e^{mx} + e^{-mx}). \tag{41}$$

Then we have

$$\int_I \delta(x)\phi_q(x) = 1 \text{ and } \delta(-x) = \delta(x). \tag{42}$$

Therefore, we also expect $Q$ to get close to the $\delta$-function in the massive case.

$Q$ and $P$ have the following important property.

**Theorem 4.** $[Q]$ *and* $[P]$ *are the generators of* $H^0(\mathrm{Obs}^{\mathrm{cl}}(\mathbb{R}))$.

*Proof.* We have already seen $H^0(\mathrm{Obs}^{\mathrm{cl}}(\mathbb{R})) = \mathbb{C}[q, p]$ in Theorem 3. Take

$$Q + \Delta_{\mathrm{BV}}^{\mathrm{cl}} X,\, P + \Delta_{\mathrm{BV}}^{\mathrm{cl}} Y, \tag{43}$$

where $X, Y \in \mathrm{Sym}(C_c^\infty(\mathbb{R})) * C_c^\infty(\mathbb{R}))^{-1}$. By $\pi_2$ action,

$$\pi_2\left(Q + \Delta_{\mathrm{BV}}^{\mathrm{cl}} X\right) = \pi_2(Q) = q, \tag{44}$$

$$\pi_2\left(P + \Delta_{\mathrm{BV}}^{\mathrm{cl}} Y\right) = \pi_2(P) = p. \tag{45}$$

We have used $\phi_p'(x) = \phi_q(x)$. □

### 2.5. *Quantum derived observable space*

**Definition 5.** Quantum derived observable space $\mathrm{Obs}^{\mathrm{q}}(U)$ is defined as

$$\mathrm{Obs}^{\mathrm{q}}(U) := \left(\mathrm{Sym}\left(C_c^\infty(U)^{-1} \oplus C_c^\infty(U)^0 \oplus \mathbb{C}\hbar\right), \Delta_{\mathrm{BV}}^{\mathrm{q}} = \Delta_{\mathrm{BV}}^{\mathrm{cl}} + \hbar\delta\right) \tag{46}$$

where $\Delta^{\mathrm{q}}_{\mathrm{BV}} = \Delta^{\mathrm{cl}}_{\mathrm{BV}} + \hbar\delta$ is a quantum Batalin–Vilkovisky operator and $\delta$ is defined in Definition 6.

Note that $C^\infty_{\mathrm{c}}(U) * \mathbb{C} = C^\infty_{\mathrm{c}}(U)$; we can rewrite this as a formal series of $\hbar$:

$$\begin{aligned}\mathrm{Obs}^{\mathrm{q}}(U) = \Bigg( \cdots \oplus & \left( \bigwedge^2 C^\infty_{\mathrm{c}}(U)^{-1} * \mathrm{Sym}\left(C^\infty_{\mathrm{c}}(U)^0\right)[\hbar] \right) \\ & \oplus \left( C^\infty_{\mathrm{c}}(U)^{-1} * \mathrm{Sym}\left(C^\infty_{\mathrm{c}}(U)^0\right)[\hbar] \right) \\ & \oplus \mathrm{Sym}\left(C^\infty_{\mathrm{c}}(U)^0\right)[\hbar], \Delta^{\mathrm{q}}_{\mathrm{BV}} \Bigg). \end{aligned} \tag{47}$$

$\delta$ is a quantum correction and will be defined as a map from $-n$ degree to $-n+1$ degree:

$$\delta : \bigwedge^n C^\infty_{\mathrm{c}}(U)^{-1} * \mathrm{Sym}\left(C^\infty_{\mathrm{c}}(U)^0\right) \to \bigwedge^{n-1} C^\infty_{\mathrm{c}}(U)^{-1} * \mathrm{Sym}\left(C^\infty_{\mathrm{c}}(U)^0\right). \tag{48}$$

The definition of $\delta$ is slightly complicated. We thus take three steps.

**Definition 6.** (1) For $f^\star * g \in C^\infty_{\mathrm{c}}(U)^{-1} * C^\infty_{\mathrm{c}}(U)^0$, the quantum correction $\delta$ is defined as

$$\begin{array}{ccc} C^\infty_{\mathrm{c}}(U)^{-1} * C^\infty_{\mathrm{c}}(U)^0 & \xrightarrow{\delta} & \mathbb{R} \subset \mathrm{Sym}(C^\infty_{\mathrm{c}}(U)^0) \\ \Cup & & \Cup \\ f^\star * g & \longmapsto & \int_U f(x)g(x)dx \end{array} \tag{49}$$

and thus $\delta$ has degree $+1$.

(2) For $f^\star * P \in C^\infty_{\mathrm{c}}(U)^{-1} * \mathrm{Sym}\left(C^\infty_{\mathrm{c}}(U)^0\right)$, it is sufficient to consider the $n$th term $g_1 * \cdots * g_n$ of $P$:

$$\begin{array}{ccc} C^\infty_{\mathrm{c}}(U)^{-1} * \mathrm{Sym}\left(C^\infty_{\mathrm{c}}(U)^0\right) & \xrightarrow{\delta} & \mathrm{Sym}\left(C^\infty_{\mathrm{c}}(U)^0\right) \\ \Cup & & \Cup \\ f^\star * g_1 * \cdots * g_n & \longmapsto & \sum_{i=1}^n g_1 * \cdots * \widehat{g_i} * \cdots * g_n * \delta(f^\star * g_i). \end{array} \tag{50}$$

(3) For $f_1^\star * \cdots * f_m^\star * P \in \bigwedge^m C^\infty_{\mathrm{c}}(U)^{-1} * \mathrm{Sym}\left(C^\infty_{\mathrm{c}}(U)^0\right)$,

$$\begin{array}{ccc} \bigwedge^m C^\infty_{\mathrm{c}}(U)^{-1} * \mathrm{Sym}\left(C^\infty_{\mathrm{c}}(U)^0\right) & \xrightarrow{\delta} & \bigwedge^{m-1} C^\infty_{\mathrm{c}}(U)^{-1} * \mathrm{Sym}\left(C^\infty_{\mathrm{c}}(U)^0\right) \\ \Cup & & \Cup \\ f_1^\star * \cdots * f_m^\star * P & \longmapsto & \sum_{i=1}^m f_1^\star * \cdots * \widehat{f_i^\star} * \cdots * f_m^\star * (-1)^{i-1}\delta\left(f_i^\star * P\right). \end{array} \tag{51}$$

$\delta$ is nilpotent[4] and anticommutes with $\Delta^{\mathrm{cl}}_{\mathrm{BV}}$:[5] $\delta^2 = 0,\ \Delta^{\mathrm{cl}}_{\mathrm{BV}}\delta = -\delta\Delta^{\mathrm{cl}}_{\mathrm{BV}}$. Then we have

$$\begin{aligned}\left(\Delta^{\mathrm{q}}_{\mathrm{BV}}\right)^2 &= \left(\Delta^{\mathrm{cl}}_{\mathrm{BV}} + \hbar\delta\right)\left(\delta^{\mathrm{cl}}_{\mathrm{BV}} + \hbar\delta\right) \\ &= 0. \end{aligned} \tag{52}$$

[4]To check this, we use $\delta(f^\star * \delta(\tilde{f}^\star * P)) = \delta(\tilde{f}^\star * \delta(f^\star * P))$.
[5]We can understand that this is because $\Delta^{\mathrm{cl}}_{\mathrm{BV}}$ and $\delta$ have degrees $+1$.

This means that the following one is a chain complex. This is called the *quantum Batalin–Vilkovisky complex*:

$$\mathrm{Obs}^{\mathrm{q}}(U) = \Bigg( \cdots \xrightarrow{\Delta^{\mathrm{q}}_{\mathrm{BV}}} \bigwedge^{2} C^{\infty}_{\mathrm{c}}(U)^{-1} * \mathrm{Sym}\left(C^{\infty}_{\mathrm{c}}(U)^{0}\right)[\hbar] \xrightarrow{\Delta^{\mathrm{q}}_{\mathrm{BV}}} C^{\infty}_{\mathrm{c}}(U)^{-1} * \mathrm{Sym}\left(C^{\infty}_{\mathrm{c}}(U)^{0}\right)[\hbar] \xrightarrow{\Delta^{\mathrm{q}}_{\mathrm{BV}}} \mathrm{Sym}\left(C^{\infty}_{\mathrm{c}}(U)^{0}\right)[\hbar] \Bigg). \tag{53}$$

$\mathrm{Obs}^{\mathrm{q}}(U)$ does not have a Leibniz rule. Instead, we have the following theorem.

**Theorem 5.** *For* $A, B \in \mathrm{Obs}^{\mathrm{q}}(U)$ *with some degree* $|A|, |B|$,

$$\Delta^{\mathrm{q}}_{\mathrm{BV}}(A * B) = \left(\Delta^{\mathrm{q}}_{\mathrm{BV}} A\right) * B + (-1)^{|A|} A * \left(\Delta^{\mathrm{q}}_{\mathrm{BV}} B\right) + (-1)^{|A|}\hbar\{A, B\}. \tag{54}$$

Here $\{-, -\}$ is the antibracket, which is defined below.[6] By this theorem, $H^*\left(\mathrm{Obs}^{\mathrm{q}}(U)\right)$ does NOT have a product $*$ in contrast to $H^*\left(\mathrm{Obs}^{\mathrm{cl}}(U)\right)$.

**Definition 7.** For $f^\star \in C^{\infty}_{\mathrm{c}}(U)^{-1},\ g \in C^{\infty}_{\mathrm{c}}(U)^{0}$, we define an antibracket:

$$\{f^\star, g\} := \int_U f(x)g(x)dx \in \mathbb{R} \subset \mathrm{Sym}\left(C^{\infty}_{\mathrm{c}}(U)^{0}\right). \tag{55}$$

For any $A, B \in \mathrm{Obs}^{\mathrm{cl,q}}(U)$ with some definite degrees, we define an antibracket with the following properties:

- $\{A, B\} = 0$ if $A, B \in C^{\infty}_{\mathrm{c}}(U)^{0}$ or $A, B \in C^{\infty}_{\mathrm{c}}(U)^{-1}$
- $\{A, B\} = -(-1)^{(|A|+1)(|B|+1)}\{B, A\}$
- $\{A, B * C\} = \{A, B\} * C + (-1)^{(|A|+1)|B|} B * \{A, C\}$.

However, in special situations we can define a product $*$ for quantum cohomology! Let $A$ and $B$ have compact support in $U_1$ and $U_2$ respectively and assume that $U_1$ and $U_2$ are disjoint open subsets of $U$. Then

$$\{A, B\}_U = 0. \tag{56}$$

The subscript $U$ denotes that this is computed in $U$. This is because the computation of $\{A, B\}_U$ reduces to the integrals

$$\int_U (U_1\text{-supported function}) \times (U_2\text{-supported function}) = 0. \tag{57}$$

Equation (56) means that we have a Leibniz rule for such $A$, $B$; thus we can define a product $*$ for them.

[6] This definition is motivated by the usual antibracket in the physics literature:

$$\{a, b\} := \sum_{i=1}^{N} \left[ \left( a \frac{\overleftarrow{\partial}}{\partial x^i} \right) \left( \frac{\overrightarrow{\partial}}{\partial x^{\star i}} b \right) - \left( a \frac{\overleftarrow{\partial}}{\partial x^{\star i}} \right) \left( \frac{\overrightarrow{\partial}}{\partial x^i} b \right) \right]$$

where $x^i$ is bosonic and $x^{\star i}$ is fermionic.

**Theorem 6.** *Let $U_1$ and $U_2$ be disjoint open subsets of $U$. The quantum Batalin–Vilkovisky cohomology has a product:*

$$\begin{array}{ccc} H^*\left(\mathrm{Obs}^{\mathrm{q}}(U_1)\right) \times H^*\left(\mathrm{Obs}^{\mathrm{q}}(U_2)\right) & \longrightarrow & H^*\left(\mathrm{Obs}^{\mathrm{q}}(U)\right) \\ \Downarrow & & \Downarrow \\ ([A]_{U_1}, [B]_{U_2}) & \longmapsto & [A]_{U_1} * [B]_{U_2} = [A * B]_U \,. \end{array} \tag{58}$$

*Here the $[\,\cdot\,]_-$ subscript denotes that it is in $H^*\left(\mathrm{Obs}^{\mathrm{q}}(-)\right)$. This product is the same as the "operator product" in the physics literature.*

The cohomology is a formal series of $\hbar$; hence the definition of a state changes slightly.

**Definition 8.** A state $\langle - \rangle$ is a smooth map:

$$\langle - \rangle : H^0\left(\mathrm{Obs}^{\mathrm{q}}(M)\right) \to \mathbb{C}[\hbar]. \tag{59}$$

### 2.6. *Weyl algebra in a 1D system*

In the case of $M = \mathbb{R}$, the quantum cohomology forms a Weyl algebra (the canonical commutation relation).

**Theorem 7.** *If $M = \mathbb{R}$, then for any value of $m$*

$$H^n\left(\mathrm{Obs}^{\mathrm{q}}(\mathbb{R})\right) = \begin{cases} \text{Weyl algebra} & (n = 0) \\ 0 & (\text{otherwise}) \end{cases}. \tag{60}$$

The aim of this section is to explain what the generators of Weyl algebra are. Roughly speaking, the generators of Weyl algebra are $\hbar$, $[Q]$, $[P]$.[7] However, this seems strange, because $Q$, $P$ have the same support $I = (-1/2, 1/2)$; thus we cannot define the products of them.

In order to take the products, we will define the *modified position observable* $\mathcal{Q}_t$ and *modified momentum observable* $\mathcal{P}_t$.

**Definition 9.** The modified position and momentum observables $\mathcal{Q}_t$ and $\mathcal{P}_t$ are in $C_c^\infty(I_t)$, $I_t = (-1/2 + t, 1/2 + t)$. The definition of them is

$$\begin{aligned} \mathcal{Q}_t(x) &:= \phi_p(t) P_t(x) - \dot{\phi}_p(t) Q_t(x), \\ \mathcal{P}_t(x) &:= \phi_q(t) P_t(x) - \dot{\phi}_q(t) Q_t(x) \end{aligned} \tag{61}$$

where $\phi_q$ and $\phi_p$ are defined in Eq. (26) and Eq. (27), and $Q_t$ and $P_t$ are defined as

$$Q_t(x) := Q(x - t), \;\; P_t(x) := P(x - t). \tag{62}$$

Modified observables have the following special property:

**Theorem 8.**

$$\frac{\partial}{\partial t} \mathcal{Q}_t(\cdot) = \Delta^{\mathrm{q}}_{\mathrm{BV}} \left(\phi_p(t) Q_t^\star(\cdot)\right), \tag{63}$$

$$\frac{\partial}{\partial t} \mathcal{P}_t(\cdot) = \Delta^{\mathrm{q}}_{\mathrm{BV}} \left(\phi_q(t) Q_t^\star(\cdot)\right). \tag{64}$$

*We remind ourselves that $Q = \mathcal{Q}_{t=0}$ and $P = \mathcal{P}_{t=0}$, then for any $t$*

$$[Q] = [\mathcal{Q}_t], \;\; [P] = [\mathcal{P}_t]. \tag{65}$$

[7] The definition of $Q$ and $P$ is given in Section 2.4.

*Proof.* We compute $\frac{\partial}{\partial t}\mathcal{Q}_t(x)$.

$$
\begin{aligned}
\frac{\partial}{\partial t}\mathcal{Q}_t(x) &= \dot{\phi}_p(t)P_t(x) + \phi_p(t)\frac{\partial}{\partial t}P_t(x) - \ddot{\phi}_p(t)Q_t(x) - \dot{\phi}_p(t)\frac{\partial}{\partial t}Q_t(x) \\
&= -\dot{\phi}_p(t)\frac{\partial}{\partial x}Q(x-t) - \phi_p(t)\frac{\partial}{\partial t}\frac{\partial}{\partial x}Q(x-t) - \ddot{\phi}_p(t)Q(x-t) - \dot{\phi}_p(t)\frac{\partial}{\partial t}Q(x-t) \\
&= -\phi_p(t)\frac{\partial}{\partial t}\frac{\partial}{\partial x}Q(x-t) - \ddot{\phi}_p(t)Q(x-t) \\
&= -\phi_p(t)\frac{\partial}{\partial t}\frac{\partial}{\partial x}Q(x-t) - m^2\phi_p(t)Q(x-t) \\
&= \phi_p(t)\left(\frac{\partial}{\partial x}\right)^2 Q(x-t) - m^2\phi_p(t)Q(x-t) \\
&= \left[\left(\frac{\partial}{\partial x}\right)^2 - m^2\right]\phi_p(t)Q(x-t) \\
&= \Delta^{\mathrm{q}}_{\mathrm{BV}}\left(\phi_p(t)Q^\star_t(x)\right).
\end{aligned} \tag{66}
$$

By similar calculations, we also obtain

$$\frac{\partial}{\partial t}\mathcal{P}_t(x) = \Delta^{\mathrm{q}}_{\mathrm{BV}}\left(\phi_q(t)Q^\star_t(x)\right). \tag{67}$$

□

The support of $\mathcal{Q}_t$ and $\mathcal{P}_t$ is $I_t = (-1/2 + t, 1/2 + t)$. Therefore we can define the products of them like

$$[\mathcal{Q}_t]_{I_t} * [\mathcal{P}_s]_{I_s} \ \text{ or } \ [\mathcal{Q}_t * \mathcal{P}_s] \ \ (|t-s| > 1) \tag{68}$$

where $I_t$ and $I_s$ denote being in $H^*(\mathrm{Obs}^{\mathrm{q}}(I_t))$ and $H^*(\mathrm{Obs}^{\mathrm{q}}(I_s))$. Note that this product is independent for the choice $t$ and $s$ because of Theorem 8.

**Theorem 9.** *We have the canonical commutation relation*

$$[\mathcal{P}_t * \mathcal{Q}_s] - [\mathcal{Q}_t * \mathcal{P}_s] = \hbar[1] \ \ (s - t > 1). \tag{69}$$

$[\mathcal{Q}_\bullet]$ *and* $[\mathcal{P}_\bullet]$ *are the generators. Then* $H^0(\mathrm{Obs}^{\mathrm{q}}(\mathbb{R}))$ *is a Weyl algebra.*

*Proof.* It is enough to show that for some $t > 1$

$$[\mathcal{Q}_0 * (\mathcal{P}_{-t} - \mathcal{P}_t)] = \hbar[1]. \tag{70}$$

Reminding ourselves that

$$\frac{\partial}{\partial t}\mathcal{P}_t(x) = \Delta^{\mathrm{q}}_{\mathrm{BV}}\left(\phi_q(t)Q^\star_t(x)\right) \tag{71}$$

and integrating over $[t_1, t_2]$, we have

$$\mathcal{P}_{t_2} - \mathcal{P}_{t_1} = \Delta^{\mathrm{q}}_{\mathrm{BV}} h^\star_{t_1,t_2}, \tag{72}$$

$$h^\star_{t_1,t_2}(x) := \int_{t_1}^{t_2} dt\ \phi_q(t)Q^\star_t(x) \in C^\infty_{\mathrm{c}}(\mathbb{R})^{-1}. \tag{73}$$

We have defined $h_{t_1,t_2}(x) := \int_{t_1}^{t_2} dt\ \phi_q(t)Q_t(x) \in C^\infty_{\mathrm{c}}(\mathbb{R})^0$.

Let us think of

$$S_t := \mathcal{Q}_0 * h^\star_{-t,t} \in C^\infty_{\mathrm{c}}(\mathbb{R})^0 * C^\infty_{\mathrm{c}}(\mathbb{R})^{-1}. \tag{74}$$

$\Delta^{\mathrm{q}}_{\mathrm{BV}}$ act on it:

$$\begin{aligned}\Delta^{\mathrm{q}}_{\mathrm{BV}} S_t &= Q_0 * \left(\Delta^{\mathrm{q}}_{\mathrm{BV}} h^{\star}_{-t,t}\right) + \hbar \int_{\mathbb{R}} dx\, Q_0(x) h_{-t,t}(x) \\ &= Q_0 * (\mathcal{P}_t - \mathcal{P}_{-t}) + \hbar \int_{\mathbb{R}} dx \int_{-t}^{t} du\, Q_0(x)\phi_q(u)\mathcal{Q}_u(x) \\ &= Q_0 * (\mathcal{P}_t - \mathcal{P}_{-t}) + \hbar \int_{\mathbb{R}} dx \int_{-t}^{t} du\, Q(x)\phi_q(u)Q(x-u). \end{aligned} \tag{75}$$

Therefore we will show that

$$\hbar \int_{\mathbb{R}} dx \int_{-t}^{t} du\, Q(x)\phi_q(u)Q(x-u) = \hbar \tag{76}$$

for some $t > 1$.

Now we take a limit $t \to \infty$:[8]

$$\Delta^{\mathrm{q}}_{\mathrm{BV}} S_\infty = Q_0 * (\mathcal{P}_\infty - \mathcal{P}_{-\infty}) + \hbar \int_{\mathbb{R}} dx \int_{\mathbb{R}} du\, Q(x)\phi_q(u)Q(x-u) \tag{77}$$

Since $Q$ is an even function,

$$\Delta^{\mathrm{q}}_{\mathrm{BV}} S_\infty = Q_0 * (\mathcal{P}_\infty - \mathcal{P}_{-\infty}) + \hbar \int_{\mathbb{R}} dx \int_{\mathbb{R}} du\, Q(x)\phi_q(u)Q(u-x). \tag{78}$$

By a change of variable, $u \to u + x$,

$$\Delta^{\mathrm{q}}_{\mathrm{BV}} S_\infty = Q_0 * (\mathcal{P}_\infty - \mathcal{P}_{-\infty}) + \hbar \int_{\mathbb{R}} dx \int_{\mathbb{R}} du\, Q(x)\phi_q(u+x)Q(u). \tag{79}$$

By the assumption, for $m \neq 0$,

$$\int_{\mathbb{R}} dx\, Q(x)\phi_q(x) = \frac{1}{2}\int_{\mathbb{R}} dx\, Q(x)(e^{mx} + e^{-mx}) = 1 \tag{80}$$

and $Q$ is even; hence

$$\int_{\mathbb{R}} dx\, Q(x)e^{mx} = \int_{\mathbb{R}} dx\, Q(x)e^{-mx} = 1. \tag{81}$$

The second term on the right-hand side of Eq. (79) is

$$\begin{aligned}\hbar \int_{\mathbb{R}} dx \int_{\mathbb{R}} du\, Q(x)\phi_q(u+x)Q(u) &= \frac{\hbar}{2}\int_{\mathbb{R}} dx \int_{\mathbb{R}} du\, Q(x)(e^{m(x+u)} + e^{-m(x+u)})Q(u) \\ &= \hbar. \end{aligned} \tag{82}$$

We have used Eq. (81) for the last line.

In the case of $m = 0$, since $\phi_q = 1$, we can easily check that the second term on the right-hand side of Eq. (79) is $\hbar$. □

## 3. Construction of the topological operator

### 3.1. *Shift symmetry*

In the case of $m = 0$, we have a shift symmetry

$$\Phi \mapsto \Phi + \alpha \quad (\alpha \in \mathbb{R}) \tag{83}$$

for the equation of motion $\Delta\Phi = 0$. Reflecting this, $P_t$ must be conserved. This is true because $\mathcal{P}_t$ is conserved and in the massless case

$$\mathcal{P}_t(x) = \phi_q(t)P_t(x) - \dot{\phi}_q(t)Q_t(x) = P_t(x). \tag{84}$$

In other words, $P_t$ is the Noether charge.

[8] When $u$ is sufficiently large, $Q(x)Q(x-u) = 0$, then the integrand goes to zero. Therefore we can take the limit $t \to \infty$.

### 3.2. *Construction of the topological operator*

First of all, we will review the usual construction of the topological operators. Let $\hat{q}$ and $\hat{p}$ be the generators of Weyl algebra; i.e. they satisfy $[\hat{q}, \hat{p}] = \hbar$. In the physics literature, the topological operator of the shift symmetry is[9]

$$\hat{V}_\alpha := \exp(\alpha \hat{p}) = \sum_{n=0}^{\infty} \frac{1}{n!}(\alpha \hat{p})^n. \tag{85}$$

Obviously we have

$$\hat{q}\hat{V}_\alpha = \hat{V}_\alpha(\hat{q} + \alpha\hbar) \because \ \hat{q}\hat{p}^n = \hat{p}^n\hat{q} + n\hbar\hat{p}^{n-1}. \tag{86}$$

Then the naive construction of the topological operator in the Batalin–Vilkovisky formalism is

$$[\mathcal{V}_{\alpha,t}] := \sum_{n=0}^{\infty} \frac{1}{n!}\, \alpha^n [P_{s_0+t} * P_{s_1+t} * \cdots * P_{s_n+t}],$$
$$(s_1 - s_0 > 1,\ s_2 - s_1 > 1, \cdots, s_n - s_{n-1} > 1). \tag{87}$$

The supports of $I_{s_0+t}, I_{s_0+t}, \cdots$ are disjoint. Hence each term of $[\mathcal{V}_{\alpha,t}]$ works well in terms of "topologicalness" and "action for $Q_0$".

- Topologicalness
  By using Theorem 8 and $\mathcal{P}_t = P_t$,
  $$\frac{d}{dt}[P_{s_0+t} * P_{s_1+t} * \cdots * P_{s_n+t}] = 0. \tag{88}$$
- Action for $Q_0$
  Similar to Eq. (86), we have
  $$\left[Q_0 * P_{s_0+t_+} * P_{s_1+t_+} * \cdots * P_{s_n+t_+}\right] = \left[P_{s_0+t_-} * P_{s_1+t_-} * \cdots * P_{s_n+t_-} * Q_0\right]$$
  $$+ n\hbar \left[P_{s_0+t_-} * P_{s_1+t_-} * \cdots * P_{s_{n-1}+t_-}\right] \tag{89}$$
  where $t_+$ is sufficiently large and $t_-$ is sufficiently small.

However, there is a problem. The support of $\mathcal{V}_\alpha$ is infinitely wide. This is not good for the topological operator because we cannot define the action.

Therefore we define the topological operator in another way.[10]

**Definition 10.**

$$[\mathcal{U}_{\alpha,t}] := \sum_{n=0}^{\infty} \frac{1}{n!}\, \alpha^n [\underbrace{P_t * P_t * \cdots * P_t}_{n}] = \sum_{n=0}^{\infty} \frac{1}{n!}\, \alpha^n \left[P_t^{*n}\right]. \tag{90}$$

$\mathcal{U}_{\alpha,t}$ obviously has the finite support $I_t$.

Instead of the finite support, we lost the clarity of the following properties:

- Topologicalness
  $$\left[\frac{d}{dt}\mathcal{U}_{\alpha,t}\right]_{I_t} = 0 \tag{91}$$

[9] We have no $i$ in front of $\alpha\hat{p}$ because the Weyl algebra is $[\hat{q}, \hat{p}] = \hbar$.

[10] $\mathrm{Obs}^{\mathrm{q}}(U)$ is made of $\mathrm{Sym}(C_c^\infty(U))$. If we take "Sym" literally, $\mathrm{Obs}^{\mathrm{q}}(U)$ has only polynomials. Hence $\mathcal{U}_{\alpha,t}$ is technically outside of the definition of $\mathrm{Obs}^{\mathrm{q}}(U)$. However, we take a completion implicitly as in footnote 1. It might make $\mathcal{U}_{\alpha,t}$ being in $\mathrm{Obs}^{\mathrm{q}}(U)$.

- Action for $Q_0$

$$[Q_0 * \mathcal{U}_{\alpha,t}] = [\mathcal{U}_{\alpha,s} * (Q_0 + \alpha\hbar)], \quad (t > 1, -1 > s). \tag{92}$$

In the following sections, we will prove these properties.

### 3.3. *Proof of topologicalness*

**Theorem 10.** *We have a topological operator $\mathcal{U}_{\alpha,t}$. Thus*

$$\left[\frac{d}{dt}\mathcal{U}_{\alpha,t}\right]_{I_t} = 0. \tag{93}$$

*Proof.* It is enough to show that

$$\frac{d}{dt}\left[P_t^{*n}\right]_{I_t} = 0. \tag{94}$$

We have

$$\frac{d}{dt}P_t^{*n} = nP_t^{*(n-1)} * \frac{d}{dt}P_t, \tag{95}$$

$$\frac{d}{dt}P_t = \Delta_{\mathrm{BV}}^{\mathrm{q}}\left(\phi_q(t)Q_t^{\star}\right) = \Delta_{\mathrm{BV}}^{\mathrm{q}}Q_t^{\star} \tag{96}$$

since $\phi_q = 1$ for the case of $m = 0$. Then we obtain

$$\frac{d}{dt}P_t^{*n} = nP_t^{*(n-1)} * \Delta_{\mathrm{BV}}^{\mathrm{q}}Q_t^{\star}. \tag{97}$$

It is enough to show that

$$\Delta_{\mathrm{BV}}^{\mathrm{q}}\left(P_t^{*(n-1)} * Q_t^{\star}\right) = P_t^{*(n-1)} * \Delta_{\mathrm{BV}}^{\mathrm{q}}Q_t^{\star}. \tag{98}$$

However, in fact, $P_t^{*(n-1)} \in \mathrm{Sym}(C_{\mathrm{c}}^{\infty}(I_t)^0)$, $Q_t^{\star} \in C_{\mathrm{c}}^{\infty}(I_t)^{-1}$, and then

$$\Delta_{\mathrm{BV}}^{\mathrm{q}}\left(P_t^{*(n-1)} * Q_t^{\star}\right) = P_t^{*(n-1)} * \Delta_{\mathrm{BV}}^{\mathrm{q}}Q_t^{\star} + \hbar\delta\left(P_t^{*(n-1)} * Q_t^{\star}\right). \tag{99}$$

Hence we need to see that $\delta(P_t^{*(n-1)} * Q_t^{\star})$ vanishes:

$$\delta\left(P_t^{*(n-1)} * Q_t^{\star}\right) = (n-1)P_t^{*(n-2)}\int_{\mathbb{R}} P_t(x)Q_t(x)dx. \tag{100}$$

We will see that $\int_{\mathbb{R}} P_t(x)Q_t(x)dx$ vanishes:

$$\begin{aligned}\int_{\mathbb{R}} P_t(x)Q_t(x)dx &= -\int_{\mathbb{R}} Q'(x-t)Q(x-t)dx \\ &= -\int_{\mathbb{R}} Q'(x)Q(x)dx \\ &= -\frac{1}{2}\int_{\mathbb{R}}\frac{\partial}{\partial x}(Q(x)Q(x))dx.\end{aligned} \tag{101}$$

$Q(x)$ has compact support in $I$, so $\int_{\mathbb{R}} P_t(x)Q_t(x)dx = 0$. □

### 3.4. *Proof of action for operators*

**Theorem 11.** *$Q_0$ is a charged operator against $\mathcal{U}_{\alpha,t}$:*

$$[Q_0 * \mathcal{U}_{\alpha,t}] = [\mathcal{U}_{\alpha,s} * (Q_0 + \alpha\hbar)], \quad (t > 1, -1 > s). \tag{102}$$

*Proof.* It is enough to show that

$$\left[Q_0 * \left(P_t^{*n} - P_{-t}^{*n}\right)\right] = n\hbar\left[P_0^{*(n-1)}\right] \tag{103}$$

for some $t > 1$. We have obtained

$$\frac{d}{dt}P_t^{*n} = n\Delta_{\mathrm{BV}}^{\mathrm{q}}\left(P_t^{*(n-1)} * Q_t^{\star}\right) \tag{104}$$

in the proof of Theorem 10. Hence we have

$$P_{t_2}^{*n} - P_{t_1}^{*n} = \Delta_{\mathrm{BV}}^{\mathrm{q}} H_{t_1,t_2}(n),$$
$$H_{t_1,t_2}(n) := n\int_{t_1}^{t_2} du\, P_u^{*(n-1)} * Q_u^{\star}. \tag{105}$$

By multiplication of $Q_0$,

$$Q_0 * \left(P_t^{*n} - P_{-t}^{*n}\right) = Q_0 * \Delta_{\mathrm{BV}}^{\mathrm{q}} H_{-t,t}(n). \tag{106}$$

We will show that the right-hand side is $n\hbar P_0^{*(n-1)}$ up to the cohomology.

We know $\delta(P_u^{*(n-1)} * Q_u^{\star}) = 0$ from the proof of Theorem 11; then we have

$$\Delta_{\mathrm{BV}}^{\mathrm{q}}(Q_0 * H_{-t,t}(n)) = Q_0 * \Delta_{\mathrm{BV}}^{\mathrm{q}}(H_{-t,t}(n)) + \hbar\delta(Q_0 * H_{-t,t}(n)); \tag{107}$$

thus it is enough to show that $\hbar\delta(Q_0 * H_{-t,t})$ is equivalent to $n\hbar P_0^{*(n-1)}$ up to the cohomology.

We compute $\hbar\delta(Q_0 * H_{-t,t}(n))$:

$$\begin{aligned}
\hbar\delta(Q_0 * H_{-t,t}(n)) &= n\hbar\int_{-t}^{t} du\,\delta\left(Q_0 * P_u^{*(n-1)} * Q_u^{\star}\right)\\
&= n\hbar\int_{-t}^{t} du\,\left(\delta\left(Q_0 * Q_u^{\star}\right)P_u^{*(n-1)} + Q_0 * \delta\left(P_u^{*(n-1)} * Q_u^{\star}\right)\right)\\
&= n\hbar\int_{-t}^{t} du\,\delta\left(Q_0 * Q_u^{\star}\right)P_u^{*(n-1)}\\
&= n\hbar\int_{-t}^{t} du\,P_u^{*(n-1)}\int_{\mathbb{R}} dx Q_0(x)Q_u(x)\\
&= n\hbar\int_{-t}^{t} du\,P_u^{*(n-1)}\int_{\mathbb{R}} dx Q(x)Q(x-u).
\end{aligned} \tag{108}$$

Here we have used $\delta(P_u^{*(n-1)} * Q_u^{\star}) = 0$. Substituting $P_u^{*(n-1)} - P_c^{*(n-1)} = \Delta_{\mathrm{BV}}^{\mathrm{q}}\tilde{H}_{c,u}(n-1)$,

$$\begin{aligned}
\hbar\delta(Q_0 * H_{-t,t}(n)) &= n\hbar P_c^{*(n-1)}\int_{-t}^{t} du\int_{\mathbb{R}} dx Q(x)Q(x-u)\\
&\quad + n\hbar\int_{-t}^{t} du\,\Delta_{\mathrm{BV}}^{\mathrm{q}}\tilde{H}_{c,u}(n-1)\int_{\mathbb{R}} dx Q(x)Q(x-u).
\end{aligned} \tag{109}$$

In the second term on the right-hand side, $Q(x)Q(x-u)$ is just a number;[11] thus this term is trivial in cohomology. It remains to be shown that

$$\int_{-t}^{t} du\int_{\mathbb{R}} dx Q(x)Q(x-u) = 1 \tag{110}$$

for some $t > 1$. When $u$ is sufficiently large, $Q(x)Q(x-u) \to 0$. Therefore we can take a limit $t \to \infty$:

$$\int_{\mathbb{R}} du\int_{\mathbb{R}} dx Q(x)Q(x-u) = \int_{\mathbb{R}} du\int_{\mathbb{R}} dx Q(x)Q(u-x) = \int_{\mathbb{R}} du\int_{\mathbb{R}} dx Q(x)Q(u) = 1. \tag{111}$$

□

[11] These trivially act for the scalar field $\Phi \in C^{\infty}(\mathbb{R})$.

**Theorem 12.** *$P_0$ has no charge against $\mathcal{U}_{\alpha,t}$:*

$$[P_0 * \mathcal{U}_{\alpha,t}] = [\mathcal{U}_{\alpha,s} * P_0], \quad (t > 1, -1 > s). \tag{112}$$

*Proof.* It is enough to show that

$$\left[P_0 * \left(P_t^{*n} - P_{-t}^{*n}\right)\right] = 0 \tag{113}$$

for some $t > 1$. We already have

$$P_{t_2}^{*n} - P_{t_1}^{*n} = \Delta_{\mathrm{BV}}^{\mathrm{q}} H_{t_1,t_2}, \tag{114}$$

$$H_{t_1,t_2} := n \int_{t_1}^{t_2} du\, P_u^{*(n-1)} * Q_u^{\star}. \tag{115}$$

By multiplication of $P_0$,

$$P_0 * \left(P_{t_2}^{*n} - P_{t_1}^{*n}\right) = P_0 * \Delta_{\mathrm{BV}}^{\mathrm{q}} H_{t_1,t_2}. \tag{116}$$

We will show that the right-hand side is zero up to the cohomology.

Since

$$\Delta_{\mathrm{BV}}^{\mathrm{q}}(P_0 * H_{t_1,t_2}) = P_0 * \Delta_{\mathrm{BV}}^{\mathrm{q}}(H_{t_1,t_2}) + \hbar\delta(P_0 * H_{t_1,t_2}), \tag{117}$$

it is enough to show that $\hbar\delta(P_0 * H_{t_1,t_2}) = 0$ up to the cohomology.

We compute $\hbar\delta(P_0 * H_{t_1,t_2})$:

$$\begin{aligned}
\hbar\delta(P_0 * H_{t_1,t_2}) &= \hbar \int_{-t}^{t} du\, \delta\left(P_0 * P_u^{*(n-1)} * Q_u^{\star}\right) \\
&= \hbar \int_{-t}^{t} du\, \left(\delta\left(P_0 * Q_u^{\star}\right) P_u^{*(n-1)} + P_0 * \delta\left(P_u^{*(n-1)} * Q_u^{\star}\right)\right) \\
&= \hbar \int_{-t}^{t} du\, \delta\left(P_0 * Q_u^{\star}\right) P_u^{*(n-1)} \\
&= \hbar \int_{-t}^{t} du\, P_u^{*(n-1)} \int_{\mathbb{R}} dx\, P_0(x) Q_u(x) \\
&= \hbar \int_{-t}^{t} du\, P_u^{*(n-1)} \int_{\mathbb{R}} dx\, Q'(x) Q(x-u).
\end{aligned} \tag{118}$$

Here we have used $\delta(P_u^{*(n-1)} * Q_u^{\star}) = 0$. Substituting $P_u^{*(n-1)} - P_c^{*(n-1)} = \Delta_{\mathrm{BV}}^{\mathrm{q}} H_{c,u}$,

$$\begin{aligned}
\hbar\delta(P_0 * H_{t_1,t_2}) = P_c^{*(n-1)} \hbar \int_{-t}^{t} du\, \int_{\mathbb{R}} dx\, Q'(x) Q(x-u) \\
+ \hbar \int_{-t}^{t} du\, \left(\Delta_{\mathrm{BV}}^{\mathrm{q}} H_{c,u}\right) \int_{\mathbb{R}} dx\, Q'(x) Q(x-u).
\end{aligned} \tag{119}$$

In the second term on the right-hand side, $Q'(x)Q(x-u)$ is just a number; thus this term is trivial in cohomology. It remains to be shown that

$$\int_{-t}^{t} du\, \int_{\mathbb{R}} dx Q'(x) Q(x-u) = 0 \tag{120}$$

for some $t > 1$. When $u$ is sufficiently large, $Q'(x)Q(x-u) \to 0$. Hence we can take a limit $t \to \infty$:

$$\int_{\mathbb{R}} du \int_{\mathbb{R}} dx Q'(x)Q(x-u) = \int_{\mathbb{R}} du \int_{\mathbb{R}} dx Q'(x)Q(u-x) = \int_{\mathbb{R}} du \int_{\mathbb{R}} dx Q'(x)Q(u) = 0. \tag{121}$$

□

## 4. Gauging and the compact scalar

In this section, we will give an application of the topological operator. Some discussion is not established rigorously, but is physically natural.

### 4.1. $\mathbb{Z}$-*gauging*

Topological operators have the advantage over Noether charge; thus we can define *discrete gauging*.[12] This is because topological operators describe finite group transformations, while Noether charges only describe infinitesimal transformations. Hence now we have a group action like

$$\mathbb{R} \curvearrowright H^*(\mathrm{Obs}^{\mathrm{q}}(\mathbb{R})). \tag{122}$$

The subgroup $\mathbb{Z} \subset \mathbb{R}$ also acts as

$$\mathbb{Z} \curvearrowright H^*(\mathrm{Obs}^{\mathrm{q}}(\mathbb{R})). \tag{123}$$

Hence we can define[13]

$$\frac{H^*(\mathrm{Obs}^{\mathrm{q}}(\mathbb{R}))}{\mathbb{Z}}. \tag{124}$$

The generators are

$$\hbar, \quad [\mathrm{Exp}(\pm iQ)], \quad [P] \tag{125}$$

where $\mathrm{Exp}(\bullet) := \sum_n \frac{1}{n!} \bullet^{*n}$. They satisfy

$$[\mathrm{Exp}(\pm iQ) * P_{-t}] - [\mathrm{Exp}(\pm iQ) * P_t] = [\mathrm{Exp}(\pm iQ)] \quad (t > 1). \tag{126}$$

This is the same as $[\hat{p}, e^{\pm i\hat{q}}] = \hbar e^{\pm i\hat{q}}$ in the usual notation.

**Definition 11.** We saw the algebra generated by

$$\hat{p}, e^{i\hat{q}}, e^{-i\hat{q}}, \hbar 1 \tag{127}$$

and satisfying

$$[\hat{p}, e^{\pm i\hat{q}}] = \hbar e^{\pm i\hat{q}}. \tag{128}$$

We call this a periodic Weyl algebra.

We can say

$$\frac{H^0(\mathrm{Obs}^{\mathrm{q}}(\mathbb{R}))}{\mathbb{Z}} \cong \text{periodic Weyl algebra} \tag{129}$$

similarly to Theorem 7.

[12] More precisely, we need to show the fusion rule, $\mathcal{U}_\alpha \mathcal{U}_\beta = \mathcal{U}_{\alpha+\beta}$, to make the discussion of this section complete.
[13] In (pre)factorization algebra, it is better to consider a finite interval: $I \rightsquigarrow H^*(\mathrm{Obs}^{\mathrm{q}}(I))/\mathbb{Z}$.

### 4.2. *Compact scalar and $\theta$-vacuum*

After $\mathbb{Z}$-gauging, massless scalar theory becomes compact scalar theory, as explained in Refs. [11,12]. In compact scalar theory, we have $\theta$-degree. We can express $\theta$ as a $\theta$-term or $\theta$-vacuum. However, implementation of the $\theta$-term is difficult in this formalism, because it is based on an equation of motion and the equation of motion does not have a $\theta$-term. On the other hand, implementing a $\theta$-vacuum is relatively easy since a vacuum is just a map from the cohomology to $\mathbb{C}[\hbar]$:

$$\frac{H^*(\mathrm{Obs}^{\mathrm{q}}(\mathbb{R}))}{\mathbb{Z}} \to \mathbb{C}[\hbar]. \tag{130}$$

One way to give a state is to think of a representation of $H^*(\mathrm{Obs}^{\mathrm{q}}(\mathbb{R}))/\mathbb{Z}$. For simplicity, we use $e^{\pm i\hat{q}}$ and $\hat{p}$ rather than $[\mathrm{Exp}(\pm iQ)]$ and $[P]$. Take the eigenvector $v_p$:

$$\hat{p}v_p = pv_p. \tag{131}$$

Then we find that $p$ must be quantized. To see this, consider a vector $e^{i\hat{q}}v_p$. This satisfies

$$\hat{p}(e^{i\hat{q}}v_p) = (p+\hbar)(e^{i\hat{q}}v_p). \tag{132}$$

Therefore $e^{\pm i\hat{q}}$ are ladder operators. The spectrum of $\hat{p}$ is

$$\lambda_r(\hat{p}) = \{\hbar(n+r) \mid n \in \mathbb{Z},\ r \in \mathbb{R}\}, \tag{133}$$

which depends on $r$. Obviously the $r+1$-spectrum is equivalent to the $r$-spectrum. Physicists denote $r$ as $\theta/2\pi$. At $\theta = 0$ and $\theta = \pi$, we can see the $\mathbb{Z}_2$-symmetry of the spectrum: $p \mapsto -p$. Hence the map (130) is given by a trace:

$$\langle - \rangle_{r=\frac{\theta}{2\pi}} : X \mapsto \sum_{p\in\lambda_r(\hat{p})} (v_p, Xv_p). \tag{134}$$

We can see the $\theta$-dependence in the state.

## 5. Conclusion and discussion

We have seen the construction of a topological operator $\mathcal{U}_{\alpha,t}$. The point of the construction is the order: first taking products $*$, then taking a cohomology $[\cdot]$ (we call this order A). If we take reversed order (we call this order B), we have

$$[\mathcal{V}_\alpha] := \sum_{n=0}^{\infty} \frac{1}{n!}(i\alpha)^n [P]_{I_{s_0}} * [P]_{I_{s_1}} * \cdots * [P]_{I_{s_n}},$$
$$(s_1 - s_0 > 1,\ s_2 - s_1 > 1, \cdots, s_n - s_{n-1} > 1), \tag{135}$$

and it has infinitely wide support. Order A versus order B is similar to path integral formalism versus operator formalism. In path integral formalism, we can formally think of multiplications of operators like $\mathcal{O}_1(\Phi)\mathcal{O}_2(\Phi)\cdots\mathcal{O}_n(\Phi)$ because these are just c-numbers. On the other hand, in operator formalism, in order to give multiplications of operators, we need to get rid of UV divergences like $:\mathcal{O}_1\mathcal{O}_2\cdots\mathcal{O}_n:$ since they are q-numbers. In order A, observables $\mathcal{O}$ are just c-numbers; thus we can freely take the products $\mathcal{O}_1 * \mathcal{O}_2 * \cdots * \mathcal{O}_n$. This is the same as path integral formalism. However, in order B, products of $[\mathcal{O}]$ cannot be taken freely; therefore this corresponds to operator formalism. In order to take products, the support of $[\mathcal{O}_1] * [\mathcal{O}_2] * \cdots * [\mathcal{O}_n]$ need to be wider than the original support. This is a kind of UV divergence.

In addition, we have seen the $\theta$-dependence of the state $\langle - \rangle$ by representation theory. It is interesting to construct the map in another way. For example, the state of the massive scalar theory on $M = \mathbb{R}$ can be given by the embedding

$$f : C_c^\infty(\mathbb{R}) \hookrightarrow \mathcal{S}(\mathbb{R}) \tag{136}$$

where $\mathcal{S}(\mathbb{R})$ is a set of Schwartz functions on $\mathbb{R}$. By considering $\mathrm{Obs}_\mathcal{S} := \mathrm{Sym}(\mathcal{S}(\mathbb{R})^{-1} \to \mathcal{S}(\mathbb{R})^0)$, we obtain $H^0(\mathrm{Obs}^q_\mathcal{S}(\mathbb{R}))) \cong \mathbb{C}[\hbar]$. Then $f$ induces the state

$$H^0(\mathrm{Obs}^q(\mathbb{R})) \to H^0\left(\mathrm{Obs}^q_\mathcal{S}(\mathbb{R})\right) \cong \mathbb{C}[\hbar]. \tag{137}$$

It is attractive to give a state in a similar way in the case of the compact scalar. However, in order to achieve this, we need a technique to remove the IR divergence because the compact scalar is massless. Such a technique is discussed in our forthcoming paper (M. Kawahira, manuscript in preparation).

There is another way to consider the compact scalar theory. It is to focus on factorization homology. The factorization algebra of the compact scalar is essentially a functor that assigns a periodic Weyl algebra to each open subset. Hence, if we consider this system in $S^1$, the factorization homology is the same as the Hochschild homology of the periodic Weyl algebra. To extract the $\theta$-information from the Hochschild homology is fascinating work.

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