

# Refractive Bending of Light due to Thermal Gradients in Air.

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ATLAS Muon Note  
October 24, 2001

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## Abstract

The principal feature of an alignment system is a straight line reference. Ideally, light rays can be considered as straight lines. This is however not true in air, where the refractive index is sensitive to both temperature and pressure variations. The required precision of the various alignment devices used for the alignment system of the ATLAS muon spectrometer is at a level where refractive bending becomes non-negligible. A simple setup to measure these refractive effects is presented, and the results are compared to calculations.

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## 1 Refraction of Light

The deviation of a light ray from the straight line path can be calculated from the following considerations. The refractive index  $n$  varies in the vertical direction  $y$ , where  $x$  is the direction of propagation of the light ray.

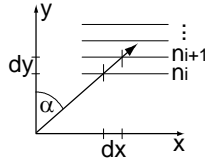


Figure 1: light ray

The propagation in the  $x$  and  $y$  directions expressed in infinitesimal steps  $dx$  can be written as:

$$x_n = x_{n-1} + dx \quad (1)$$

$$y_n = y_{n-1} + \cot \alpha \, dx . \quad (2)$$

Using  $G_y(x)$  as the gradient of the refractive index in the vertical direction which changes along the axis of propagation  $x$ , the refractive index  $n$  can be expressed in discrete steps  $dx$  like:

$$n(x_{n+1}) = n(x_n) + G_y(x_n) \cot \alpha(x_n) \, dx . \quad (3)$$

To calculate the angle of propagation  $\alpha(x)$

$$\alpha(x_{n+1}) = \alpha(x_n) + d\alpha(x_n)$$

as a function of the gradient  $G_y(x)$  one can start with Snellius law:

$$n(x_n) \sin \alpha(x_n) = n(x_{n+1}) \sin \alpha(x_{n+1}) .$$

Then,

$$\begin{aligned} \frac{\sin \alpha(x_n)}{\sin \alpha(x_{n+1})} &= 1 + \frac{G_y(x_n)}{n(x_n)} \cot \alpha(x_n) dx \\ \frac{\sin \alpha(x_n)}{\sin (\alpha(x_n) + d\alpha(x_n))} &= 1 + \frac{G_y(x_n)}{n(x_n)} \cot \alpha(x_n) dx \\ 1 - \cot \alpha(x_n) d\alpha(x_n) + O(d\alpha)^2 &= 1 + \frac{G_y(x_n)}{n(x_n)} \cot \alpha(x_n) dx , \\ d\alpha(x_n) &= -\frac{G_y(x_n)}{n(x_n)} dx , \end{aligned} \quad 1$$

and thus the recursion for  $\alpha(x_n)$  can be written as:

$$\alpha(x_{n+1}) = \alpha(x_n) - \frac{G_y(x_n)}{n(x_n)} dx . \quad (4)$$

Using the recursion relations (1) - (4) to calculate the refractive bending of a light ray for different distributions of the refractive index gradient  $G_y(x)$ , one can show that the overall effect only depends on integral changes in the refractive index,  $\int_0^l G_y(x) dx$ .

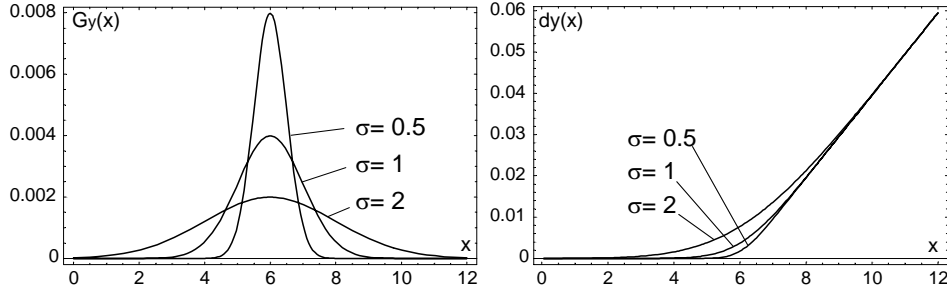


Figure 2: left: different distributions of  $G_y(x)$ , right: resulting refractive bending.

Knowing that the overall effect is independent of the distribution of the gradient  $G_y(x)$ , changes of the refractive index can be reduced to a function that is constant along the light path:

$$n(y) = n_0(1 - \varepsilon y) , \quad (5)$$

where  $n_0\varepsilon$  is the gradient in the vertical direction is.

Since we are interested in the overall refractive bending, the vector form of a light ray can also be written using the following differential equation:

$$\frac{d}{ds} \left( n \frac{d\vec{r}}{ds} \right) = G_y(s) , \quad (6)$$

where  $\vec{r} = \vec{e}_x x + \vec{e}_y y$ ;  $\vec{e}_x$  and  $\vec{e}_y$  are the unit vectors in the  $x$  and  $y$  directions, and  $s$  is the distance along the ray. To a very good approximation,  $s = x$ ; thus Eq. (6) can be reduced to

$$\frac{d}{dx} \left( n \frac{dy}{dx} \right) = -n_0\varepsilon . \quad (7)$$

For a ray starting at  $x = 0$  parallel to  $\vec{e}_x$ , where  $n = n_0$ , the solution to Eq. (7) is,

$$y = -\varepsilon x^2/2 . \quad (8)$$

This shows that for a constant gradient in the refractive index the bending of the light ray is a quadratic function in  $x$ , whereas the angular deviation of the light ray from the straight line path depends linearly on the distance  $x$ .

## 2 Setup

The idea to measure refractive bending effects due to temperature gradients in air is to compare two light rays, one of which passes the area of varying refractive index ("bending region") once (fig. 3:  $\overline{LED1, BCAM}$ ), while the other one passes it twice (fig. 3:  $\overline{LED2, BCAM}$ ).

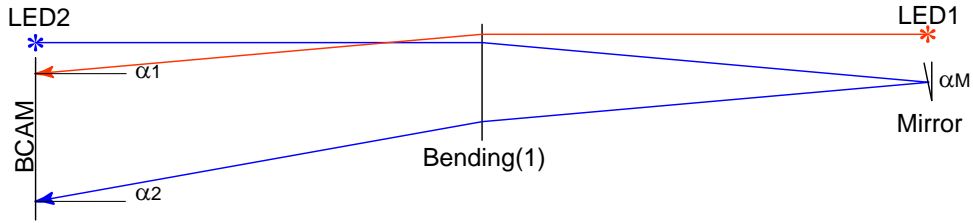


Figure 3: comparison of two light rays.

As illustrated in fig. (3) the two different light rays enter the angular monitor (BCAM [1]) at different angles  $\alpha_1$  and  $\alpha_2$ , which depend on the gradient of the refractive index,  $G_y(x)$ . Assuming that the gradients applied to the two different rays are the same, the following relation holds:

$$\alpha_1(G_y(x)) = \frac{\alpha_2(G_y(x)) - 2\alpha_M}{2} . \quad (9)$$

Here  $2\alpha_M$  is the angle enclosed by the straight lines  $\overline{LED2, Mirror}$  and  $\overline{Mirror, BCAM}$ . The angle  $\alpha_M$  is nonzero because LED1 and the mirror have to be at different positions. Looking at the rays entering the BCAM shows that moving the bending region along the axis of propagation (fig. 4) causes a shift of the "single passing" ray, whereas the "double passing" ray is independent of that.

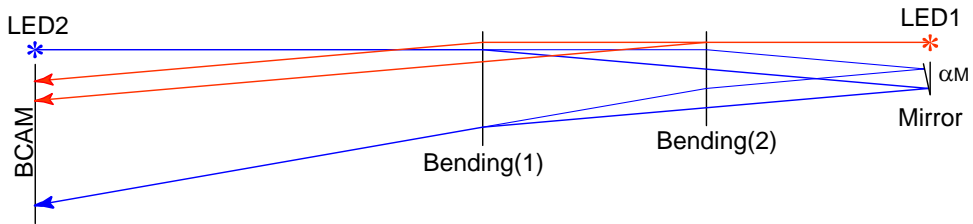


Figure 4: shifting the "bending region" causes a shift of the single passing ray

The angles  $\alpha_1$  and  $\alpha_2$  are, however, not affected by such a movement. They are only a function of the total amount of refractive bending  $\int G_y(x) dx$ .

To measure this effect the light rays were passing a tube ( $l = 1600$  mm,  $d = 74$  mm) which was heated from below with 2 times 900 W heating power. In such a way a thermal gradient in  $y$  direction, and thus a gradient of the refractive index in  $y$  direction, was obtained. Fig. (5) illustrates the setup.

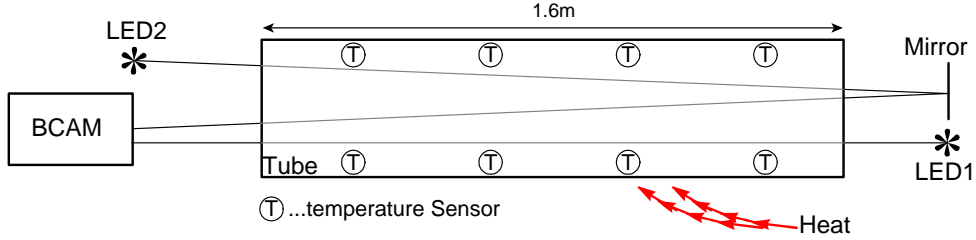


Figure 5: experimental setup

Eight temperature sensors have been positioned such, that an estimate about the temperature distribution inside the tube is possible.

For the measurement the tube was heated for 30 minutes, fig. (6, 7). Figure 6, left shows the average temperature over all 8 sensors. The average temperature gradient over the vertical distance of 5 cm is shown in fig. (6, right). The gradient is measured by subtracting the readings of the 4 lower temperature sensors from the readings of the 4 upper temperature sensors.

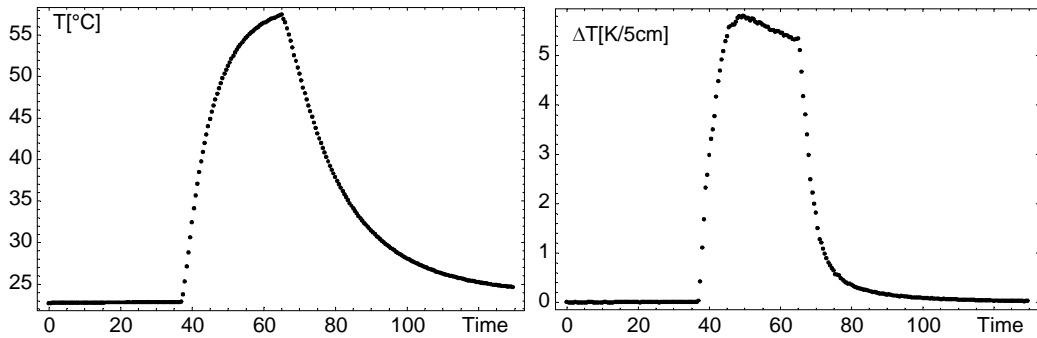


Figure 6: left: average temperature inside the tube, right: average temperature difference over a distance of 5 cm

The corresponding angle measurements using a BCAM are showing the expected effects for the two different light rays in fig. (7).

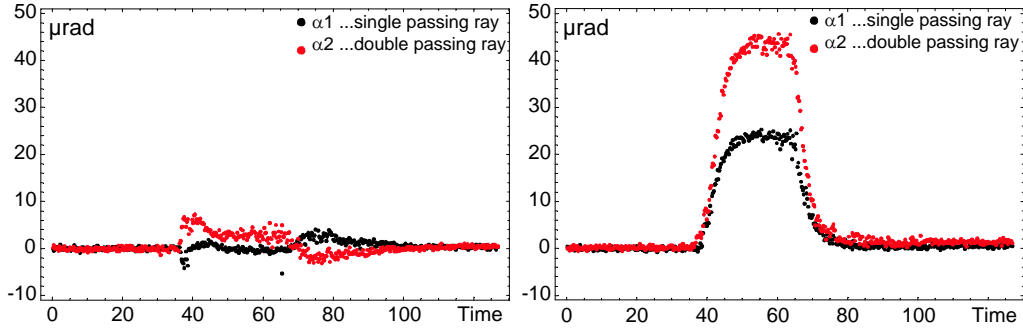


Figure 7: relative angles  $\alpha_1$  and  $\alpha_2$  measured by the BCAM. left: in  $x$ -direction, right: in  $y$ -direction.

The left picture in fig. (7) represents the measurement of the angle deviation in horizontal direction. The deviations from zero can be explained with temperature variations in horizontal direction while heating and therefore bending the rays also in  $x$  direction. They can be used to estimate the systematic error. Looking at the angular deviations in  $y$  direction, the angle of the "double passing" ray changes about twice as much as the angle of the "single passing" ray, as expected. According to eq. (9), plotting  $\Delta\alpha_2$  vs.  $\Delta\alpha_1$  should give a straight line with slope two and zero intercept. Figure (8) shows this linear correlation between the two angles  $\Delta\alpha_1$  and  $\Delta\alpha_2$  due to the generated temperature variations.

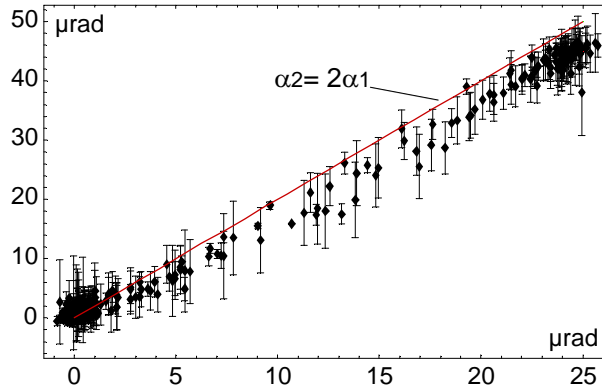


Figure 8: measured angles in  $y$ -direction:  $\alpha_2$  vs.  $\alpha_1$

The spread of the data and the deviation from the expected slope can be explained by local variations of the temperature gradient inside the tube, therefore the bending of the two light rays is slightly different. Since their actual distance in  $y$  was comparable to the inner diameter of the tube, a difference in the thermal gradient of the order of 10% between bottom and top of the tube is possible. Deviations of the refractive index in  $x$  direction fig. (7) are indicating this systematic error.

### 3 Refractive index as a function of temperature

The following discussion is taken from Ref.[2] and reproduced here for the convenience of the reader.

The most dominant contribution to changes of the refractive index in air is due to temperature gradients. Starting with the index of refraction  $n(y)$  as a function of the vertical position  $y$ :

$$n(y) = n_0(1 - \varepsilon y) \quad , n_0 \varepsilon \dots \text{gradient in vertical direction}$$

$\varepsilon$  can be derived from the Lorentz-Lorentz formula:

$$A = \frac{1}{3} \frac{\mu}{p} RT (n^2 - 1) \quad (10)$$

with:  $A \dots$  molar refractivity,  $R \dots$  gas constant,  $T \dots$  temperature and  $p \dots$  pressure.

Equation (10) can be rearranged as

$$n - 1 = \frac{3Ap}{RT(n + 1)}$$

with  $n + 1 \cong 2$ :

$$n - 1 \cong \frac{3A}{2R} \frac{p}{T}$$

At standard temperature and pressure (298 K and 760 Torr)  $n - 1 = 3 \cdot 10^{-4}$ . Thus  $\frac{3A}{2R}$  is equal to  $1.2 \cdot 10^{-4}$  K/Torr.

Pressure and Temperature can be expressed as a function of  $y$ , to the first order as

$$\begin{aligned} p(y) &= p_0 + \frac{\partial p}{\partial y} y = p_0 + p' y , \\ T(y) &= T_0 + \frac{\partial T}{\partial y} y = T_0 + T' y , \end{aligned}$$

If the ratio  $p(y)/T(y)$  is expanded in a Taylor series about  $y_0 = 0$  then,

$$\frac{p(y)}{T(y)} = \frac{p_0}{T_0} + \frac{p' y}{T_0} - \frac{p_0 T' y}{T_0^2} + O^3$$

where  $p_0$  and  $T_0$  are the nominal pressure and temperature in the line of sight at  $y_0$ .

Now substituting for  $n$ , yields

$$n_0 - n_0 \varepsilon y - 1 = \frac{3A}{2R} \frac{p_0}{T_0} + \frac{p' y}{T_0} - \frac{p_0 T' y}{T_0^2} .$$

At  $y = 0$ ,

$$n_0 - 1 = \frac{3A}{2R} \frac{p_0}{T_0} = 1.2 \cdot 10^{-4} \frac{p_0}{T_0}$$

thus

$$\varepsilon = \frac{1.2 \cdot 10^{-4}}{n_0} \frac{p_0 T'}{T_0^2} - \frac{p'}{T_0} \quad \frac{\text{K}}{\text{Torr}} \quad (11)$$

The value of  $n_0$  is 1.0003, the pressure gradient  $p'$  is  $-1.15 \cdot 10^{-4} \text{ p}_0/\text{m}$  (for air) and is hence in general negligible. The main contribution for changes of the refractive index is due to temperature variations.

Now, knowing the relation between temperature and the refractive index, one can try to predict the measurement using the equations (11) and (8). Figure (9) shows the predicted and the measured angles  $\Delta\alpha_1$  for the temperature distribution displayed in fig. (6).

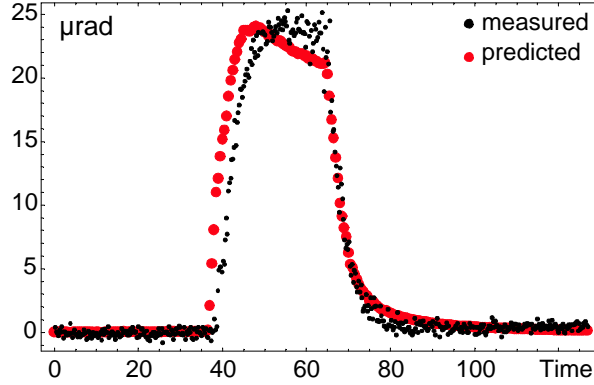


Figure 9: measured and predicted angles  $\alpha_1$  using the measured temperatures inside the tube

The prediction of the angle deviation is mostly dependent on the quality of the temperature measurement and the derived temperature gradient. The way of predicting the deviation due to a measured gradient can be used to estimate the effect of refractive bending.

## 4 Possible implementation in ATLAS

The ATLAS end-cap alignment uses BCAMs for the polar lines which are connecting the end-cap wheels to each other. These polar lines are long-distance lines and mostly horizontal. As thermal gradients are expected to be the largest in vertical direction, these polar lines are most affected by thermal diffraction. The following setup explains the possible implementation to correct for the thermal effects.

Each BCAM used in the ATLAS alignment consists of two light sources and one CCD. In the polar alignment always two BCAMs are looking at each other. By additionally mounting one mirror to one of the two BCAMs, one can generate three alignment rays like shown in fig. (10)

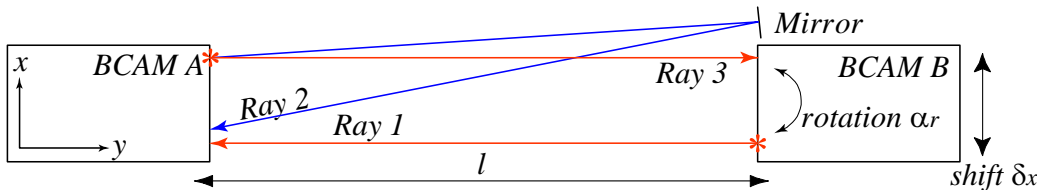


Figure 10: sketch of the setup of one polar alignment ray using two BCAMs looking at each other plus one additional mirror fixed to one BCAM

The two BCAMS are intended to measure relative shifts of their positions as well as relative angular changes. As a simple approach, where the light source is at the same position as the pivot point of the optical system of the BCAM, the following relations hold:

$$\begin{aligned}
\delta\alpha_1 &= \frac{\delta x}{l} + \delta\alpha_t \\
\delta\alpha_2 &= 2\delta\alpha_r + 2\delta\alpha_t \\
\delta\alpha_3 &= \delta\alpha_r + \frac{\delta x}{l} + \delta\alpha_t,
\end{aligned} \tag{12}$$

where  $\delta\alpha_i$  is the angle of the ray  $i$  measured by the BCAM,  $\delta\alpha_r$  is the rotation of the BCAM B relative to BCAM A,  $\frac{\delta x}{l}$  is the angle caused by the shift of BCAM B relative to the viewing direction of BCAM A. The angle  $\delta\alpha_t$  represents the effects due to thermal diffraction. Rearranging the eqs. (12) shows that in addition to the conventional "two ray" setup where the relative shift  $\delta x$  and the relative rotation  $\delta\alpha_r$  is measured, information about diffraction effects  $\delta\alpha_t$  can be obtained:

$$\begin{aligned}
\delta\alpha_r &= \delta\alpha_3 - \delta\alpha_1 \\
\frac{\delta x}{l} &= \delta\alpha_s = \delta\alpha_3 - \frac{\delta\alpha_1}{2} \\
\delta\alpha_t &= \delta\alpha_1 - \delta\alpha_3 + \frac{\delta\alpha_2}{2}.
\end{aligned}$$

Referring to eq. (9) the determination of the absolute influence due to thermal gradients on the angles  $\alpha_1$  and  $\alpha_2$  requires the knowledge of  $\alpha_M$ , which would mean performing a measurement where no thermal gradients exist, or calibrating the system once and measuring relative to this calibration.

One drawback of this setup is, that the angle  $\alpha_M$  of the mirror has to be adjusted very precisely and the angular acceptance of the reflected ray reduces with the distance  $l$  between the two BCAMs.

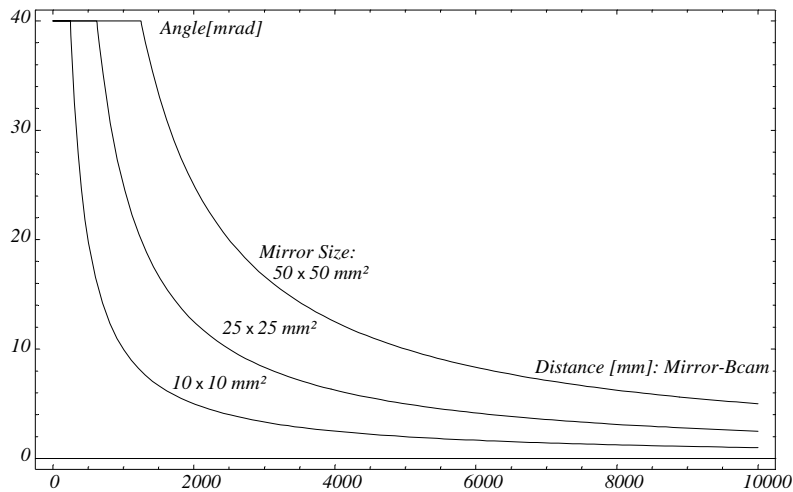


Figure 11: Angular acceptance as a function of distance and mirror size

## 5 Summary

It has been shown how correcting for refractive bending due to thermal gradients in air is in principle possible. The measurements have been performed knowing that the angle  $\alpha_M$  of the mirror does not significantly change during the short period. Using the described method for long term and distance measurements makes the determination and adjustment of  $\alpha_M$  difficult. Still, the most reasonable implementation would be for large optical distances like the polar lines used in the ATLAS muon end-cap alignment.

## References

- [1] The BCAM Camera: J. Bensinger, K. Hashemi, ATL-MUON-2000-024 . (ATL-COM-MUON-2000-011 )
- [2] The Poisson Line as a Straight Line Reference: Lee V. Griffith, Lawrence Livermore National Laboratory.