Extended DeWitt-Schwinger subtraction scheme, heavy fields and decoupling*

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Keywords: Renormalization techniques; quantum field theories.

1. Introduction

A standard result of quantum field theory is the existence of divergences when computing observables. This requires the construction of techniques, called regularization and renormalization schemes, to overcome these infinities and produce well defined physical observables. A standard example is the computation of vacuum polarization effects in perturbative Quantum Electrodynamics (QED) which results in the well-known Lamb Shift an the anomalous magnetic momentum of the electron. As a byproduct of these methods, an arbitrary mass scale parameter μ present in the calculations implies a dependence under this scale of the different coupling constants of the theory.[1] Instead of becoming an obstacle it offers a very useful tool, the renormalization group equation,[1,2] to understand the behaviour of the theories at some particular ranges, without knowing the exact details of the theory. The relevant magnitudes that encode the information of this equation are the beta functions, $\beta_O = \mu\frac{dO_R}{d\mu}$, where O_R is any possible renormalized coupling constant of the theory.

At the level of quantum field theory in curved spacetime, similar divergences arise. These appear even when computing observables in the vacuum state since the standard method of regularizing the vacuum divergences in Minkowski, known as *normal ordering*, is not available anymore since it strictly depends on the definition of the vacuum state, which is not unique in this context. Nevertheless, many regularization methods, including Dimensional Regularization and Pauli-Villars have been inherit from the Minkowski spacetime case, and have obtained finite renormalized

*Talk given by A. Ferreiro at the Sixteenth Marcel Grossmann Meeting (2021).

quantities, e.g. the Feynman propagator $G(x,x')$ and the expectation value of the stress energy tensor T_{ab} for scalar, Dirac or gauge fields in curved spacetime.[3–5]

Most methods rely on the convenient expression of the propagator in terms of an integral in the DeWitt-Schwinger proper time

$$G(x,x') = \frac{\Delta^{1/2}(x,x')}{(4\pi)^2} \int_0^\infty \frac{ds}{(is)^2} e^{-im^2 s + \frac{\sigma(x,x')}{2is}} F(x,x',is) \ . \tag{1}$$

The advantage of expression (1) is that it admits an expansion

$$F(x,x',is) = \sum_{j=0}^{\infty} a_j(x,x')(is)^j. \tag{2}$$

Note that the effective action and consequently the stress-energy tensor can be derived from $G(x,x')$ in the limit $x' \to x$ where this expression becomes divergent. In the case of four dimensions $d = 4$, the divergences are encoded and isolated in the first two terms of the expansion (the first three terms in case of the effective action). There are several ways to regulate expression (1). For instance, it can be analytically extended to higher dimension $d > 4$, via dimensional regularization or it can be expanded in terms of point splitting parameter σ. The infinities are recovered in the case of $d \to 4$ and $\sigma \to 0$ respectively. It can be shown[3] that all the divergent pieces of the one loop corrections of a free field can be consistently reabsorbed in the original action

$$S_G = \int d^4x\sqrt{-g}\left(-\Lambda + \frac{1}{2\kappa}R + \alpha_1 C^2 + \alpha_2 R^2 + \alpha_3 E + \alpha_4 \Box R\right), \tag{3}$$

where $\kappa = 8\pi G$. It is in principle possible to use the DeWitt-Schwinger decomposition (1) to compute the exact one-loop contribution.[3,5] In this case, as a byproduct of the DeWitt-Schwinger decomposition we can construct a natural subtraction scheme without invoking any artificial regulator

$$G_{\mathrm{ren}}(x,x) = \lim_{x'\to x} G(x,x') - \frac{1}{(4\pi)^2}\sum_{j=0}^{1} a_j(x) \int_0^\infty \frac{ds}{(is)^{(2-j)}} e^{-im^2 s} \ . \tag{4}$$

We can also construct subtraction schemes from dimensional regularization or point splitting. In any case the different regularization and subtraction schemes differ in terms that can always be reabsorbed in the action (3).[4]

Note that the subtracted contribution of (4) needs to include an small imaginary $m^2 - i\epsilon$ to avoid an infrared singularity. In the mass-less case this is an obstacle since the infrared divergence is present in the limit $s \to 0$.[4] In order to overcome this problem we can insert an artificial parameter μ^2 as a mass-scale instead of m^2 in (4) and then take the limit $\mu^2 \to 0$. This has been done for instance to obtain the conformal anomaly.[3] However, another interesting approach is to maintain in all the potential calculations μ^2 different to zero and to modify consistently the coefficients $a_j(x)$ of (4) in order to ensure that no extra divergences appear in the case of $\mu^2 \neq 0$. In the most general case, without assuming $m^2 = 0$, the extended DeWitt-Schwinger subtraction scheme would be[6]

$$\bar{G}_{\rm ren}(x,x) = \lim_{x'\to x} G(x,x') - \frac{1}{(4\pi)^2}\sum_{j=0}^{1}\bar{a}_j(x)\int_0^\infty \frac{ds}{(is)^{(2-j)}}e^{-i(m^2+\mu^2)s} \,. \quad (5)$$

Here the the new coefficients are $\bar{a}_0 = 1$, $\bar{a}_1 = a_1 + \mu^2$ and $\bar{a}_2 = a_1\mu^2 + \frac{1}{2}\mu^4$ [a]. The introduction of this extra parameter is required for the massless limit. However, it also offers a possible running coupling interpretation analogue to the Minimal subtraction scheme in dimensional regularization. We will analyze this in the next section.

An important property of the regularization and renormalization program is that it should not spoil the decoupling of heavy massive fields in the low energy limit. For instance, we should not have to compute the vacuum polarization of the quark top mass when performing a low energy (below the top mass) scattering process in QED. The decoupling of heavy field in renormalizable theories has been well stated by the Appelquist-Carazzone theorem.[8] In the case of QED, Minimal Subtraction is not compatible with decoupling[9] and a different subtraction scheme has to be taken in order to have a physical interpretation of the μ parameter at low energy regime. A possible scheme is known as *Momentum subtraction scheme*,[9] which not only decouples the heavy fields, i.e., $\beta_O \to 0$ when $m^2 \to \infty$ but also recovers the Minimal Subtraction beta functions in the limit of $\mu^2 \to \infty$ or $m^2 \to 0$.

In the case of quantum fields in curved spacetime an analog to the Momentum subtraction scheme has been elusive. A notable result of the subtraction scheme of (5) is that it is consistent with the decoupling of heavy massive fields,[6] as we will show in the next section.

2. Running of the couplings and decoupling of massive fields

In order to compute the running of the couplings with the parameter μ it is useful to use the extended DeWitt-Schwinger subtraction scheme for the effective action, which can be constructed in a similar manner to (5).[7] It is also useful for pedagogical purposes to consider a complex scalar field interacting with an electromagnetic field obeying the Klein-Gordon equation

$$\left(g^{\mu\nu}D^x_\mu D^x_\nu + m^2 + \xi R\right)G(x,x') = -|g(x)|^{-1/2}\delta(x'-x), \quad (6)$$

where $D_\mu = \nabla_\mu + iqA_\mu$. In this case, the extended DeWitt-Schwinger subtraction scheme takes the form[7]

$$W^{(1)}_{\rm ren} = W^{(1)} - \frac{2i}{2(4\pi)^2}\int d^4x\sqrt{-g}\sum_{j=0}^{2}\bar{a}_j(x)\int_0^\infty \frac{ds}{(is)^{(3-j)}}e^{-i(m^2+\mu^2)s} \quad (7)$$

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[a]Note here that in terms of adiabatic order, the μ^2 in the exponential of (5) is of adiabatic order zero while the μ^2 of the coefficients are of adiabatic order two. This is equivalent to the methodology taken in[7] for Parker-Fulling adiabatic regularization.

where $W^{(1)}$ is the divergent one-loop effective action constructed from (1) and the coefficients are

$$\begin{aligned}
&\bar{a}_0(x) = 1 \ , \qquad \bar{a}_1(x) = \left(\frac{1}{6} - \xi\right) R + \mu^2 \\
&\bar{a}_2(x) = \frac{1}{180} R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} - \frac{1}{180} R^{\alpha\beta} R_{\alpha\beta} - \frac{1}{6}\left(\frac{1}{5} - \xi\right) \Box R \\
&+ \frac{1}{12}\left(\xi - \frac{1}{6}\right)^2 R^2 - \frac{q^2}{12} F_{\mu\nu} F^{\mu\nu} + \left(\frac{1}{6} - \xi\right) R\mu^2 + \frac{1}{2}\mu^4 \ .
\end{aligned} \tag{8}$$

The subtraction terms of (7) can be absorbed into the original action

$$S^c = S_G + \int d^4x \sqrt{-g} \left(-\frac{1}{4} Z_A \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} + (\tilde{D}_\mu \phi)^\dagger \tilde{D}^\mu \phi + m^2 |\phi|^2 + \xi R |\phi|^2 \right) \tag{9}$$

with $\tilde{D}_\mu = \nabla_\mu + iqZ_A^{1/2} \tilde{A}_\mu$, where we have re-scaled the potential to introduce an extra coupling constant Z_A. The complete effective action would be

$$S^{(1)}_{\rm ren} = W^{(1)}(m, \xi, qZ_A^{1/2}) + S^c(m, \xi, q, Z_A, \Lambda, \kappa, \alpha_i), \tag{10}$$

where we have made visible the explicit dependence of the bare couplings and fields. The complete action in terms of the physical renormalized couplings O_R is

$$S^{(1)}_{\rm ren} = W^{(1)}_{\rm ren}(m_R, \xi_R, q_R Z_R^{1/2}) + S^c_{\rm ren}(Z_R, \Lambda_R, \kappa_R, \alpha_{iR}) \tag{11}$$

where $S^c_{\rm ren}$ is the original S^c upgrading the bare divergent coupling constants to the corresponding renormalized finite ones.In the particular case of a free complex scalar field, it is not difficult to see that expression (7) does not involve any divergent piece that need to be reabsorbed neither in m^2, nor ξ and the combination $qZ^{1/2}$ and therefore we can already write

$$\xi = \xi_R \qquad m = m_R \qquad qZ_A^{1/2} = q_R Z_{AR}^{1/2}. \tag{12}$$

This is of course not a general result and indeed this couplings constants could be divergent in theories including other interactions.[10] We will from now on drop the subscript $_R$ from these terms.

In order to compute the beta functions $\beta_O = \mu \frac{dO_R}{d\mu}$ we will impose the invariance of the renormalized action with respect to the arbitrary parameter μ

$$\mu \frac{d}{d\mu} S^{(1)}_{\rm ren} = 0. \tag{13}$$

From (7), it is easy to check

$$\mu \frac{d}{d\mu} W^{(1)}_{\rm ren} = \int d^4x \sqrt{-g} \frac{1}{16\pi^2 (m^2 + \mu^2)} \left(2\mu^2 a_2(x) - 2\mu^4 a_1(x) - \mu^6\right). \tag{14}$$

Using (12), (13) and (14), we finally obtain the following beta functions

$$\beta_{\alpha 1} = -\frac{1}{960\pi^2}\frac{\mu^2}{m^2+\mu^2} \qquad \beta_{\alpha 2} = -\frac{\left(\xi-\frac{1}{6}\right)^2}{16\pi^2}\frac{\mu^2}{m^2+\mu^2}$$
$$\beta_\Lambda = \frac{1}{16\pi^2}\frac{\mu^6}{m^2+\mu^2} \qquad \beta_{\kappa^{-1}} = \frac{\xi-\frac{1}{6}}{4\pi^2}\frac{\mu^4}{m^2+\mu^2} \qquad \beta_q = \frac{q_R^3}{48\pi^2}\frac{\mu^2}{m^2+\mu^2} \quad . \tag{15}$$

It can be easily checked that the running of dimensionless coupling constants coincides with the Minimal Subtraction beta functions,[11] but with the advantage that the structure $\frac{\mu^2}{m^2+\mu^2}$, similar to the cutoff Wilsonian scheme[12] makes a vanishing contribution when $m^2 \to \infty$ thus making this renormalization scheme compatible with decoupling of heavy massive fields. The remarkable result is that also the running of both the Newton constant and the cosmological constant also decouple in this limit, thus enforcing the fact that higher massive fields do not contribute to the low energy regime, even when the gravitational field is present.

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