



Faculty of Technology and Science
Physics

Jonas Björnsson

Strings, Branes and Non-trivial Space-times

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Abstract

This thesis deals with different aspects of string and p -brane theories. One of the motivations for string theory is to unify the forces in nature and produce a quantum theory of gravity. p -branes and related objects arise in string theory and are related to a non-perturbative definition of the theory. The results of this thesis might help in understanding string theory better. The first part of the thesis introduces and discusses relevant topics for the second part of the thesis which consists of five papers.

In the three first papers we develop and treat a perturbative approach to relativistic p -branes around stretched geometries. The unperturbed theory is described by a string- or particle-like theory. The theory is solved, within perturbation theory, by constructing successive canonical transformations which map the theory to the unperturbed one order by order. The result is used to define a quantum theory which requires for consistency $d = 25+p$ dimensions for the bosonic p -branes and $d = 11$ for the supermembrane. This is one of the first quantum results for extended objects beyond string theory and is a confirmation of the expectation of an eleven-dimensional quantum membrane.

The two last papers deal with a gauged WZNW-approach to strings moving on non-trivial space-times. The groups used in the formulation of these models are connected to Hermitian symmetric spaces of non-compact type. We have found that the GKO-construction does not yield a unitary spectrum. We will show that there exists, however, a different approach, the BRST approach, which gives unitarity under certain conditions. This is the first example of a difference between the GKO- and BRST construction. This is one of the first proofs of unitarity of a string theory in a non-trivial non-compact space-time. Furthermore, new critical string theories in dimensions less than 26 or 10 are found for the bosonic and supersymmetric string, respectively.

List of publications

- I J. Björnsson and S. Hwang,
“Stretched quantum membranes”
Nucl. Phys. B **727** (2005) 77 [[hep-th/0505269](#)].
- II J. Björnsson and S. Hwang,
“The BRST treatment of stretched membranes”
Nucl. Phys. B **736** (2006) 156 [[hep-th/0511217](#)].
- III J. Björnsson and S. Hwang,
“On small tension p-branes,”
Phys. Lett. B **662** (2008) 270 [arXiv:0801.1748](#) [[hep-th](#)].
- IV J. Björnsson and S. Hwang,
“On the unitarity of gauged non-compact WZNW strings”
Nucl. Phys. B **797** (2008) 464 [arXiv:0710.1050](#) [[hep-th](#)].
- V J. Björnsson and S. Hwang,
“On the unitarity of gauged non-compact world-sheet
supersymmetric WZNW models,”
Submitted to Nucl. Phys. B [arXiv:0802.3578](#) [[hep-th](#)].

These papers will henceforth be referred to as paper I–V, respectively. In these articles I have made, more or less, all computations. The ideas, techniques and solutions of the articles have grown out of a series of discussions with my co-author. In addition, one more paper not included in the thesis is

1. J. Björnsson and S. Hwang,
“The membrane as a perturbation around string-like configurations”
Nucl. Phys. B **689** (2004) 37 [[hep-th/0403092](#)].

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Jonas Björnsson
Karlstad University
April, 2008.

Introduction

Most of the phenomena discovered so far in our universe can be explained by two fundamental theories; the theory of general relativity and the standard model of particle physics. General relativity was more or less the work of one man, Albert Einstein, and describes gravitation [1]. The standard model is a theory for the smallest particles observed in our universe, the elementary particles from which everything is built¹. It was a collective work of many scientists and was, more or less, completed in the seventies and is formulated as so-called gauge theories. The foundation of the standard model is a combination of quantum principles and special relativity into one theory called quantum field theory, see [2–4] for an introduction of quantum field theory and the standard model. Quantum theory conflicts with our everyday view of our world. In our everyday world waves, e.g. water waves and sound waves are distinct from particles, which are thought of and behave as billiard balls. In quantum theory the two concepts are unified into one. Thus, particles have properties which we think only waves have and the other way around. This has far-reaching consequences, for instance, one cannot measure the momentum and position of particles at an arbitrary level of accuracy and particles do not necessary take the shortest path between points, it may take any possible path.

In the theory of gravity, one views space-time as something dynamical. The dynamics of space-time is governed by the distribution of energy. General relativity is not a quantum theory, it is a classical theory. If one tries to quantize gravity

¹It should be mentioned, that it is now known that this corresponds to only about 4% of the total energy in the universe.

one will encounter problems. If one computes graviton exchange beyond leading order, one finds an infinite answer. This also occurs in the theories that describe the standard model, but there is a difference, in the theory of gravity one cannot find a rigorous way to extract finite values for physical measurements. A reason why such theory should be formulated is that there are situations where the energy density is large, thus gravitational effects are not negligible. For example, in the vicinity of black holes and in the early times of our universe it is believed that quantum effects of gravity are essential.

A candidate for a theory of quantum gravity is string theory. It was proposed as a theory of the strong force [5], but one found that it predicts a massless particle with spin two. It was later proposed that this particle is the graviton, thus, string theory involves gravity. One interesting property of the model is that it predicts the number of space-time dimensions. The problem is that it is not four, but 10 or 26 in a flat background. These conditions arise in two different models. They are the bosonic string, which only has bosonic particles in its spectrum, and the supersymmetric string, which has both bosonic and fermionic particles in its spectrum. The latter model also has a symmetry which relates bosonic and fermionic particles to each other, called supersymmetry. For the bosonic model there is a problem because there is a tachyon in the spectrum, which indicates an instability of the vacuum. For the latter model, the supersymmetric string, one does not have this problem. But the supersymmetric string theory is not unique in 10 dimensions, there are five different ones. These are linked to the different possible ways of realizing the supersymmetry in 10 dimensions. Studying perturbations around the free string each model looks they are different, but non-perturbative aspects of the theories imply that they are related. Furthermore, the non-perturbative properties of string theory yield other dynamical objects of the theory, the Dirichlet branes (D-branes for short). That the different theories seem to be related, hints that one is studying different aspects of one and the same theory. The unifying theory is usually called M-theory, which is a theory assumed to live in 11 dimensions. The membrane, which might be a part of

its formulation, is one of the topics of this thesis.

As briefly mentioned above, string theory predicts the number of dimensions of space-time. The problem is that it does not predict the observed number, which is four. One can solve this problem by assuming that space-time consists of two pieces. One piece is a four-dimensional non-compact part which is the space-time we observe. The other part is a compact space which is too small for us to have been detected so far. One model which seems to yield consistent string theories on a number of different spaces is the Wess-Zumino-Novikov-Witten (WZNW) model and the connected model, the gauged WZNW-model. These can also be used to formulate models for the string on non-trivial non-compact space-times as well. This is the other topic of this thesis. One motivation for studying such models is the *AdS/CFT*-conjecture by Maldacena [6] which predicts a duality between gravity and gauge theory. Another is general properties of string theory formulated on non-trivial non-compact space-time.

The thesis consists of two parts. The first part is a general introduction to string theory and focuses on different topics connected to the papers. This part begins with a discussion about properties of constrained theories; such theories are relevant for all fundamental models of our universe. The second chapter is an introduction to strings, membranes and a brief discussion about the conjectured M-theory. In the third chapter we give a short discussion about different limits of membranes. A few properties and conjectures about M-theory are discussed in chapter four. In the fifth chapter a different route is set out, it concerns non-trivial backgrounds in string theory and will be the main topic in the chapters to follow. In this chapter, the WZNW-model is introduced and a few properties and connections to so-called affine Lie algebras are discussed. The chapter after this discusses generalizations of the WZNW-model to so-called gauged WZNW-models. In these models, one introduces gauge symmetry in the model to reduce the number of degrees of freedom. In the last chapter of the first part, models with a non-trivial time direction are discussed.

The second part consists of five articles. Paper I-III discusses a perturbative

approach p -branes, which we call stretched configurations. In paper I we deal with stretched membranes and formulate them as a free string-like theory with a non-trivial perturbation. In the lightcone gauge we show that one can, with successive canonical transformations, map the theory to the unperturbed one order by order in perturbation theory. Paper II deals with a more covariant approach to stretched membranes in which we show that one can generalize the results to hold also for the BRST charge of the stretched membrane. In paper III we generalize the results to arbitrary p -branes and also consider two different limits, the limits where the unperturbed theory is a string-like theory and when it is a particle-like theory. Papers IV and V discuss string theories formulated on non-trivial space-times, which are connected to Hermitian symmetric spaces of non-compact type. In these papers we prove the unitarity of one sector of these models using a BRST approach. Furthermore, we discover that the GKO-construction of these models yields a non-unitary spectrum. This is the first example where the GKO-construction and the BRST construction differ. Furthermore, this is the first example of a unitarity proof of strings in a non-trivial background apart from the $SU(1,1)$ WZNW-model. Paper IV treats the bosonic string and in paper V we generalize to the world-sheet supersymmetric case. Here, we also find new critical string theories in dimensions less than 26 or 10 for the bosonic and supersymmetric string, respectively.

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Chapter 1

Treatment of theories with constraints

This chapter will introduce techniques to how to treat systems which are interesting for physics, systems which have constraints. Constraints are relations between different degrees of freedom. Examples of systems which have constraints are the theories which describe the fundamental forces. The first who discussed constraints in detail was Dirac which, in a series of papers and lecture notes, gave a general formulation of constrained systems [7,8]. I will here give a brief review of how one treat theories with constraints. This will in general be limited to theories with finite degrees of freedom and using the Hamiltonian approach. In this chapter, a few of the tools which are used in the papers will be discussed, the representation theory of the BRST charge and the Lifschetz trace formula. some references about constraint theories are the book by Henneaux and Teitelboim [9], which the later part of this chapter is based on, and lecture notes by Marnelius [10,11].

1.1 The Dirac procedure

Consider a general theory with finite degrees of freedom described by an action

$$S = \int dt L(q^i, \dot{q}^i), \quad (1.1)$$

where q^i are the coordinates of the theory. The equations of motion of this action can be written as

$$\ddot{q}^i M_{ij} = \frac{\delta L}{\delta q^j} - \dot{q}^i N_{ij} \quad (1.2)$$

where

$$\begin{aligned} M_{ij} &= \frac{\delta^2 L}{\delta \dot{q}^i \delta \dot{q}^j} \\ N_{ij} &= \frac{\delta^2 L}{\delta q^i \delta \dot{q}^j}. \end{aligned} \quad (1.3)$$

For eq. (1.2) to yield solutions for all variables, the matrix M_{ij} has to be invertible. If we define the canonical momentum of the theory,

$$p_i = \frac{\delta L}{\delta \dot{q}^i}, \quad (1.4)$$

we find

$$\frac{\delta p_i}{\delta \dot{q}^j} = M_{ij}. \quad (1.5)$$

Thus, if the determinant the matrix M_{ij} is zero, there will exist relations between the momenta and coordinates. This implies that not all coordinates and momenta are independent. If this is the case, one can construct non-trivial phase-space functions which are weakly zero¹

$$\phi_{a'}(q^i, p_i) \approx 0. \quad a' = 1, \dots \quad (1.6)$$

¹The symbol \approx is used to show that the constraints can have non-zero Poisson brackets with phase-space functions.

The Hamiltonian of the theory is defined as a Legendre transformation of the Lagrangian

$$H_0(q^i, p_i) = \dot{q}^i p_i - L(q^i, \dot{q}^i). \quad (1.7)$$

Because there are constraints, there is a redundancy in the definition. One can use the constraints to define a total Hamiltonian as

$$H_{tot} = H_0 + \lambda^{a'} \phi_{a'}. \quad (1.8)$$

For the theory to be consistent, the time evolution of the constraints has to be zero. This will lead to the following consistency conditions²

$$\dot{\phi}_{a'} = \{\phi_{a'}, H_0\} + \lambda^{b'} \{\phi_{a'}, \phi_{b'}\} \approx 0. \quad (1.9)$$

There are four different cases: (a) this is trivially satisfied, (b) will yield secondary constraints, (c) determine the unknown functions $\lambda^{b'}$ or (d) yield that the theory is inconsistent. Henceforth, we assume that the theory is consistent. In the end, one will get a set of constraints all satisfying eq. (1.9). One can classify the constraints into two groups. Constraints, ψ_a , which satisfy

$$\{\psi_a, \phi_{b'}\} = U_{ab'}{}^{c'} \phi_{c'}, \quad (1.10)$$

and constraints that do not satisfy this. The constraints that satisfy eq. (1.10) are called first class constraints and the other kind is called second class constraints. As a consequence, the second-class constraints, Φ_u , satisfy

$$\det [\{\Phi_u, \Phi_v\}]_M \neq 0, \quad (1.11)$$

where M denote the restriction to the physical phase-space. This subspace is defined as the space where all constraints are strongly set to zero. A way to eliminate the second-class constraints is to put them to zero and use them to reduce the number of

²We have here assumed that $\phi_{a'}$ does not have any explicit time dependence.

independent degrees of freedom. Another way is to introduce additional degrees of freedom to make the second class constraints first class. A systematic construction of the later was first discussed in [12]. This is used in papers IV and V, which is discussed in chapter six of this thesis. Following the first approach one needs to introduce a new bracket since the constraints cannot be set to zero within the usual Poisson brackets. The bracket that fulfils this is the Dirac bracket defined as

$$\{A, B\}^* = \{A, B\}|_{M'} - \{A, \Psi_u\}|_{M'} [\{\Psi_u, \Psi_v\}|_{M'}]^{-1} \{\Psi_v, B\}|_{M'}, \quad (1.12)$$

where M' is the subspace where all second-class constraints have been put to zero. This bracket satisfies $\{A, \Psi_u\}^* = 0$, thus, projects out the second-class constraints. Assume now that all second-class constraints have been eliminated. Due to eq. (1.10), first-class constraints satisfy a closed algebra

$$\{\psi_a, \psi_b\} = U_{ab}{}^c \psi_c. \quad (1.13)$$

To treat theories with first-class constraints there are two different approaches, one can either reduce the constraints or use the more powerful BRST formalism.

Let me first, briefly, discuss the former. In order to reduce the constraints one has to specify gauge conditions, $\chi^a \approx 0$, which should satisfy that the matrix of Poisson brackets of all constraints and gauge conditions is invertible,

$$\det [\{\chi^a, \psi_b\}|_M] \neq 0. \quad (1.14)$$

The problem is now the same as for the reduction of second class constraints, one has to define a new bracket which projects out the constraints and gauge fixing functions

$$\begin{aligned} \{A, B\}^* &= \{A, B\}|_M - \{A, \psi_a\}|_M [\{\psi_a, \chi^b\}|_M]^{-1} \{\chi^b, B\}|_M \\ &\quad - \{A, \chi^a\}|_M [\{\chi^a, \psi_b\}|_M]^{-1} \{\psi_b, B\}|_M \\ &\quad + \{A, \psi_a\}|_M [\{\psi_a, \chi^c\}|_M]^{-1} \{\chi^c, \chi^d\}|_M [\{\chi^d, \psi_b\}|_M]^{-1} \{\psi_b, B\}|_M. \end{aligned} \quad (1.15)$$

Let me end this section by briefly mentioning quantization of theories with first class constraints using the above approach. One can either reduce the degrees of freedom

before quantization or impose the constraints as conditions on the state space. The later is called “old covariant quantization” (OCQ). In many cases, one has to impose the constraints not as strong conditions, but as conditions on the scalar products between physical states

$$\langle \phi | \psi_a | \phi' \rangle = 0, \quad (1.16)$$

where $|\phi\rangle$ and $|\phi'\rangle$ are physical states. Since this is not the most powerful way of treating theories with first-class constraints, especially when quantizing theory. I will in the next section discuss the more general treatment using the BRST symmetry. Using this approach, where one does not fix a gauge for the theory.

1.2 BRST formalism

In this section a more general and powerful formalism of treating first class constraints will be discussed. This is the BRST formalism. Here, one introduces extra degrees of freedom with the opposite Grassman parity to the constraints. These extra degrees of freedom are needed to construct a theory invariant under the BRST symmetry. This symmetry will project out the unphysical degrees of freedom.

The first who realized that one could add extra degrees of freedom to cancel the effects of the unphysical degrees of freedom was Feynman [13] and DeWitt [14]. The more general treatment was given by Faddeev and Popov [15], where they used it for the path integral quantization of Yang-Mills theories. They found that the determinant which arose in the path integral could be described by an action involving ghost fields. Later it was found that the resulting action possessed, in certain gauges, a rigid fermionic symmetry. This was first discovered by Becchi, Rouet and Stora [16] and, independently, by Tyutin [17]. It was further developed so that the BRST formalism worked even where one could not use the Fadeev-Popov formalism [18]. The symmetry is nilpotent, so that performing two such transformations yields zero.

This is reflected in the BRST charge by³

$$\{Q, Q\} = 0, \quad (1.17)$$

which is a nontrivial property, as the charge is fermionic. The existence and form of the BRST charge was elaborated upon in a series of papers by Fradkin and Vilkovisky [19], Batalin and Vilkovisky [20] and Fradkin and Fradkina [21]. The form of it is

$$Q = \psi_a c^a + \sum_{r=1}^N C^{a_1, \dots, a_r} b_{a_1} \cdot \dots \cdot b_{a_r}, \quad (1.18)$$

where c^a are the additional coordinates of the extended phase-space called ghosts. The fields b_a are the corresponding momenta for these ghosts. These fields can be chosen in such a way that

$$\{c^a, b_b\} = \delta_b^a. \quad (1.19)$$

The coefficients C^{a_1, \dots, a_i} are determined by the nilpotency condition and the algebra of the constraints. These coefficients involve $i+1$ ghost fields. The ghosts have always the opposite Grassman parity to the constraints, so that e.g., if a constraint has even Grassman parity, then the ghost has odd. This yields that the corresponding BRST charge has odd Grassmann parity, as expected. A simple example is the charge when the constraints are bosonic and satisfy an algebra,

$$\{\psi_a, \psi_b\} = U_{ab}^c \psi_c, \quad (1.20)$$

where U_{ab}^c are constants, for which the charge is

$$Q = \psi_a c^a - \frac{1}{2} U_{ab}^c c^a c^b b_c. \quad (1.21)$$

The BRST charge admits a conserved charge, the ghost number charge, which has the form

$$N = \frac{1}{2} \sum_a (c^a b_a + (-)^{\epsilon_a} b_a c^a), \quad (1.22)$$

³Here, and all Poisson brackets and commutators which follow, has been generalized to the larger phase space which also involve the ghosts and their corresponding momenta.

where ϵ_a is the Grassman parity of c^a , thus equals zero or one if it is even or odd, respectively. The fundamental fields in the extended phase-space and the BRST charge has the following values w.r.t. the charge

$$\begin{aligned}
 \{N, q^i\} &= 0 \\
 \{N, p_i\} &= 0 \\
 \{N, c^a\} &= c^a \\
 \{N, b_a\} &= -b_a \\
 \{N, Q\} &= Q.
 \end{aligned} \tag{1.23}$$

The ghost number charge will induce a grading of the phase-space functions

$$\mathcal{F} = \bigoplus_i \mathcal{F}^i. \tag{1.24}$$

The BRST charge now acts as

$$Q : \mathcal{F}^i \rightarrow \mathcal{F}^{i+1}. \tag{1.25}$$

The nilpotency condition on Q implies that it is analogous to the differential form, d , in differential geometry and the grading corresponds to the form degree⁴. The physical phase-space functions are now defined as the non-trivial functions invariant under Q . They are, therefore, classified by the cohomology group defined by

$$H^r = \frac{Z^r}{B^r} \tag{1.26}$$

where the subspaces Z^r and B^r are defined as

$$\begin{aligned}
 Z^r &= \{\forall X^r \in \mathcal{F}^r : \{Q, X^r\} = 0\} \\
 B^r &= \{\forall X^r \in \mathcal{F}^r \exists Y^{r-1} \in \mathcal{F}^{r-1} : X^r = \{Q, Y^{r-1}\}\}
 \end{aligned} \tag{1.27}$$

Here I have only applied the BRST formalism to the classical theory. Its true advantages arise when one quantizes the theory. In the rest of this chapter, the parts which

⁴ c^a corresponds to a basis of the cotangent space and b_a to a basis of the tangent space.

are interesting for paper IV and V will be discussed. The discussion will mainly be systems with finite degrees of freedom, or systems which have enumerable number of degrees of freedom. Furthermore, one usually redefines the ghost momenta such that the commutator is the same as the Poisson bracket defined in eq. (1.19). The quantum BRST charge satisfies

$$\begin{aligned} [Q, Q] &= 2Q^2 = 0 \\ Q^\dagger &= Q \end{aligned} \tag{1.28}$$

The ghost number operator defined in eq. (1.22) is anti-Hermitian and introduces a grading of the state space as well as on the operators. The state space, which henceforth will be denoted by W , splits as a sum of eigenspaces of N

$$\begin{aligned} W &= \sum_p W_p \\ N |\phi_p\rangle &= p |\phi_p\rangle \quad |\phi_p\rangle \in W_p. \end{aligned} \tag{1.29}$$

As the ghost number operator is anti-Hermitian, the non-zero scalar products are between states of ghost number k and $-k$, respectively. From this property, one can prove that $p \in \mathbb{Z}$ or $p \in \mathbb{Z} + 1/2$ if the number of independent bosonic constraints are even or odd, respectively. The physical state condition is

$$Q |\phi\rangle = 0. \tag{1.30}$$

States which satisfy this condition are in the kernel of the mapping by Q and are called BRST closed. There is a subspace of these states of the form $|\phi\rangle = Q |\chi\rangle$, i.e. in the image of the mapping by Q . Such states are called BRST exact. Of these spaces the true physical states are identified as

$$H_{st}^*(Q) = \frac{\text{Ker}(Q)}{\text{Im}(Q)}. \tag{1.31}$$

Thus, the physical states are identified as $|\phi\rangle \sim |\phi\rangle + Q |\chi\rangle$, which is a generalization of a gauge transformation of classical fields to the space of states. Therefore, one

is interested in the cohomology of the BRST charge. The BRST exact states do not couple to the physical states as can be seen from computing the scalar product between a physical state and a BRST exact term

$$\begin{aligned}
 \langle \phi | (|\phi'\rangle + Q|\chi\rangle) &= \langle \phi | \phi'\rangle + \langle \phi | Q|\chi\rangle \\
 &= \langle \phi | \phi'\rangle + \langle Q\phi | \chi\rangle \\
 &= \langle \phi | \phi'\rangle .
 \end{aligned}
 \tag{1.32}$$

Let us now discuss in more detail the results needed for the papers IV and V. One implicit result needed is the representation theory of the BRST charge. The state space at ghost number k can be decomposed into three parts

$$W_k = E_k \oplus G_k \oplus F_k, \tag{1.33}$$

where G_k is the image of Q (the BRST exact states), E_k are the states which are BRST invariant but not exact, therefore, $(E_k \oplus G_k)$ is the kernel of Q (the BRST closed states) and F_k is the completion. The physical states are the states in E_k . These spaces satisfy the properties

$$\begin{aligned}
 G_k &= QF_{k-1} \\
 \dim F_k &= \dim G_{-k} \\
 \dim E_k &= \dim E_{-k}.
 \end{aligned}
 \tag{1.34}$$

We can now state a theorem about the most general representation of the BRST charge [22], [23] and [24]

Theorem 1 *The most general representation of the BRST charge*

$$\begin{aligned}
 Q^2 &= 0 & [Q, N] &= Q \\
 Q^\dagger &= Q & N^\dagger &= -N
 \end{aligned}
 \tag{1.35}$$

decomposes into the following irreducible representations

1. *Singlet:*

$$\begin{aligned} Qe &= 0 \\ Ne &= 0 \\ \langle e|e \rangle &= \pm 1, \end{aligned} \tag{1.36}$$

where $e \in E_0$.

2. *Non-null doublet:*

$$\begin{aligned} Qe = Qe' &= 0 \\ Ne &= ke \\ Ne' &= -ke' \\ \langle e|e' \rangle &= 1, \end{aligned} \tag{1.37}$$

where $k \neq 0$, $e \in E_k$ and $e' \in E_{-k}$.

3. *Null doublet at ghost number $\pm\frac{1}{2}$:*

$$\begin{aligned} Qf &= g \\ Nf &= -\frac{1}{2}f \\ \langle f|g \rangle &= \pm 1, \end{aligned} \tag{1.38}$$

where $g \in G_{1/2}$ and $f \in F_{-1/2}$.

4. *Quartet:*

$$\begin{aligned} Qf &= g \\ Qf' &= g' \\ Nf &= (k-1)f \\ Nf' &= -kf' \\ \langle f|g' \rangle &= 1 \\ \langle f'|g \rangle &= 1 \end{aligned} \tag{1.39}$$

The proof of this theorem follows from the choice of basis for E_k , G_k and F_k . Let us state, and prove, a central theorem of this thesis. Define the Lefschitz trace as

$$\mathrm{Tr} [(-1)^{\Delta N} A] = \sum_{k-\nu \in \mathbb{Z}} (-1)^k \mathrm{Tr}_k [A], \quad (1.40)$$

where A is a BRST invariant operator, ΔN is the ghost number operator minus its value on the vacuum, ν is the ghost number eigenvalue of the vacuum. Tr_k is the trace of the states over the subspace W_k and its dual W_{-k} .

Theorem 2 *Let A be a BRST invariant operator then*

$$\mathrm{Tr} [(-1)^{\Delta N} A] = \mathrm{Tr}_E [(-1)^{\Delta N} A], \quad (1.41)$$

where Tr_E denotes the trace over the subspaces E_k .

A connection to Theorem 2 in the framework of BRST was first discussed in [25], where also a simple proof of the no-ghost theorem for the bosonic string was given.

Proof: Let me here give a simple proof of the theorem by using the results of the representation theory above. We consider the two last cases of the irreducible representations of the BRST charge.

Case 3;

$$\begin{aligned} \mathrm{Tr} [(-1)^{\Delta N} A]_3 &= \langle f | (-1)^{1/2-\nu} A | g \rangle + \langle g | (-1)^{-1/2-\nu} A | f \rangle \\ &= (-1)^{1/2-\nu} (\langle f | A | g \rangle - \langle f | A | g \rangle) \\ &= 0, \end{aligned} \quad (1.42)$$

where $\mathrm{Tr}[\dots]_3$ denotes the trace over states in case 3.

Case 4;

$$\begin{aligned} \mathrm{Tr} [(-1)^{\Delta N} A]_4 &= \langle f | (-1)^{-k+1-\nu} A | g' \rangle + \langle g' | (-1)^{k-1-\nu} A | f \rangle \\ &+ \langle f' | (-1)^{k-\nu} A | g \rangle + \langle g | (-1)^{-k-\nu} A | f' \rangle \\ &= (-1)^{k-\nu} (\langle f' | A | g \rangle - \langle g' | A | f \rangle) \\ &+ (-1)^{-k-\nu} (\langle g | A | f' \rangle - \langle f | A | g' \rangle) \\ &= 0. \end{aligned} \quad (1.43)$$

Thus, the trace does not get any contribution from the spaces G_k and F_k . Therefore, the trace over the non-trivial states in the cohomology is equal to the trace over all states. \square

This theorem has a connection to the Euler-Poincaré theorem, by taking the trace over the identity operator, or some other diagonalizable operator. For example, the zero mode for the Virasoro algebra and/or some momentum operators. In the next chapter, the proof of the no-ghost theorem of [25] will be reproduced and generalized to the supersymmetric string.

Let me end this chapter by discussing anomalies at the quantum level in the BRST formalism. In going from a classical theory to a quantum theory ordering problems often arise. Thus, one could face a problem when the algebra of the quantum constraints gets an anomaly such that the constraints are not first class, i.e. that the algebra does not close. This will in general imply that the corresponding BRST charge is not nilpotent, depending on whether or not there is a compensating term arises from the ghosts. In some cases, e.g. in string theory the cancellation occurs if certain conditions are satisfied. The BRST charge is nilpotent only in 26 or 10 dimensions for the bosonic string and supersymmetric string in flat space, respectively.

Chapter 2

String theory, an introduction

This chapter will give an introduction to string theory and other theories related to the first papers, namely membrane theory and the conjectured M-theory. The chapter begins with the simplest one, the string model. After this, membrane theory and some of its difficulties is discussed. Furthermore, generalizations to general p -branes will be made to connect to paper III. In the last section the conjecture about the existence of M-theory, as proposed by Witten [26], will be briefly reviewed.

2.1 Strings

This section is in part based on a few of the books on the subject; Polchinski [27,28] and Green, Schwarz and Witten [29,30]. New books on the subject is Becker, Becker and Schwarz [31] and Kiritsis [32]

The action for a propagating string is proportional to the area that the string traces out in space-time

$$S_1 = -T_s \int_{\Sigma} d^2\xi \sqrt{-\det(\partial_i X^\mu \partial_j X_\mu)}, \quad (2.1)$$

where Σ is the world-sheet traced out by the string, T_s is the string tension, $\mu = 0, \dots, D-1$; $i, j = 0, 1$; $\partial_i \equiv \frac{\partial}{\partial \xi^i}$ and where ξ^0 is the time-like parameter on the world-sheet. For historical reasons, one defines the Regge slope α' , by $T_s^{-1} = 2\pi\alpha'$. An

action of this kind was first proposed by Dirac [33] when he introduced an extended model for the electron (this was a charged membrane). This action was reinvented independently by Nambu [34] and Goto [35] to give an action for the string. A classically equivalent action was later proposed by Brink, De Vecchia and Howe [36] and, independently, by Deser and Zumino [37]

$$S_p = -\frac{T_s}{2} \int_{\Sigma} d^2\xi \sqrt{-\gamma} \gamma^{ij} \partial_i X^\mu \partial_j X_\mu, \quad (2.2)$$

where a metric on the world-sheet, γ_{ij} with Lorentzian signature, has been introduced. Due to the important work of Polyakov [38, 39], where he discovered the importance of γ_{ij} in the perturbative formulation of interacting strings, this action is commonly referred to as the Polyakov action. The fact that the two actions are classically equivalent can be seen by studying the equations of motion for γ_{ij} . The actions in eqs. (2.1) and (2.2) both possess global space-time Poincaré invariance and local reparametrization invariance of the world-sheet. The action in eq. (2.2) is also invariant under local Weyl rescaling of the world-sheet metric, $\gamma^{ij} \rightarrow e^{2\sigma} \gamma^{ij}$.

The local invariance of the action in eq. (2.2) make it possible to choose the metric of the world-sheet to be conformally flat, $\gamma^{ij} = e^{2\sigma} \eta^{ij}$, where $\eta^{ij} = \text{diag}\{-1, 1\}$. If one Wick rotates time, $\xi^2 = i\xi^0$, such that the world-sheet is Euclidean and introduces $z = \exp[i\xi^1 + \xi^2]$ the action can be written as

$$S = T_s \int_{\Sigma} d^2z \partial X_\mu \bar{\partial} X^\mu, \quad (2.3)$$

where $d^2z = dzd\bar{z}$, $\partial = \partial_z$ and $\bar{\partial} = \partial_{\bar{z}}$. In this action one can, in a simple way, introduce world-sheet supersymmetry. Under the assumption that the fermions are Majorana, thus can be represented by real degrees of freedom, the supersymmetric action is

$$S = \frac{1}{4\pi} \int_{\Sigma} d^2z \left[\frac{2}{\alpha'} \partial X_\mu \bar{\partial} X^\mu + \psi \bar{\partial} \psi + \tilde{\psi} \partial \tilde{\psi} \right]. \quad (2.4)$$

This is also called the “spinning string”. The equations of motion for this action are

$$\begin{aligned}\partial(\bar{\partial}X^\mu) &= \bar{\partial}(\partial X^\mu) = 0 \\ \bar{\partial}\psi^\mu &= 0 \\ \partial\tilde{\psi}^\mu &= 0.\end{aligned}\tag{2.5}$$

For the bosonic fields, the solution can be expanded as¹

$$X^\mu(z, \bar{z}) = q^\mu - i\frac{\alpha'}{2}p^\mu \ln(|z|^2) + i\left(\frac{\alpha'}{2}\right)^{1/2} \sum_{m \neq 0} \frac{1}{m} \left(\frac{\alpha_m^\mu}{z^m} + \frac{\tilde{\alpha}_m^\mu}{\bar{z}^m} \right), \tag{2.6}$$

where p^μ is the center of mass momentum, which is conserved. For the fermions, on the other hand, one has two different possibilities for the boundary conditions. If we classify them on the cylinder, the Ramond sector (R) has periodic boundary conditions and the Neveu-Schwarz sector (NS) has anti-periodic. Transforming this to the annulus, and Laurent expanding the solution, we find

$$\psi^\mu(z) = \sum_{n \in \mathbb{Z}} \frac{d_n^\mu}{z^{n+1/2}} \quad \text{R-sector} \tag{2.7}$$

$$\psi^\mu(z) = \sum_{n \in 1/2 + \mathbb{Z}} \frac{b_n^\mu}{z^{n+1/2}}, \quad \text{NS-sector} \tag{2.8}$$

and similar expressions for the left-moving field. Thus, the R-sector is anti-periodic on the annulus and the NS-sector is periodic.

To construct the physical states of the theory we need to know the constraints of the theory. These follow from the vanishing of the energy-momentum tensor and the supercurrent,

$$T(z) = -\frac{1}{\alpha'} \partial X^\mu \partial X_\mu - \frac{1}{2} \psi^\mu \partial \psi_\mu \tag{2.9}$$

$$G(z) = i \left(\frac{2}{\alpha'} \right)^{1/2} \psi^\mu \partial X_\mu, \tag{2.10}$$

¹This is for the closed case; for the open one will get relations between the two different families of modes.

etc. for the left-moving sector. These constraints generate a superconformal algebra. To get the expressions for the bosonic string, one puts the fermions to zero, and the algebra is then a conformal algebra. As we are dealing with a quantum theory, these operators have to be normalized such that they have finite eigenvalue on the physical states.

We have now a theory with constraints, which we discussed in the previous chapter. As was described in the previous chapter, one can define a quantum theory in different ways: Either one gauge-fixes the action completely using, for example, the lightcone gauge, where only physical fields are left, or one imposes the constraints on the state space. The third way, and the most general one, is the BRST approach. We begin by using the second approach, namely the “old covariant quantization” (OCQ) approach, to quantize the string. Later in the section, the BRST approach will be used to show the no-ghost theorem, which is a simplified version of the proof used in the papers IV and V to show unitarity of certain gauged non-compact WZNW-models.

The Fourier modes constituting the physical fields are treated as operators when quantizing the theory. The non-zero commutators are

$$\begin{aligned}
 [\alpha_m^\mu, \alpha_n^\nu] &= m\delta_{m+n,0}\eta^{\mu\nu} \\
 [d_r^\mu, d_s^\nu] &= \delta_{r+s,0}\eta^{\mu\nu} \\
 [b_r^\mu, b_s^\nu] &= \delta_{r+s,0}\eta^{\mu\nu}.
 \end{aligned}
 \tag{2.11}$$

Define a vacuum for the string by

$$\begin{aligned}
 \alpha_m^\mu |0\rangle &= 0 && \text{for } m > 0 \\
 \alpha_m^\mu |0, \beta, R\rangle &= 0 && \text{for } m > 0 \\
 d_m^\mu |0, \beta, R\rangle &= 0 && \text{for } m > 0 \\
 \alpha_m^\mu |0, NS\rangle &= 0 && \text{for } m > 0 \\
 b_m^\mu |0, NS\rangle &= 0 && \text{for } m > 0,
 \end{aligned}
 \tag{2.12}$$

where $|0\rangle$, $|0, \beta, R\rangle$ and $|0, NS\rangle$ is the vacuum for the bosonic string, Ramond sector and the Neveu-Schwarz sector for the supersymmetric string, respectively. If we also expand the constraints,

$$\begin{aligned} T(z) &= \sum_{m \in \mathbb{Z}} \frac{L_m}{z^{m+2}} \\ G(z) &= \sum_{m \in \mathbb{Z}} \frac{G_m^R}{z^{m+3/2}} && \text{R-sector} \\ G(z) &= \sum_{m \in 1/2 + \mathbb{Z}} \frac{G_m^{NS}}{z^{m+3/2}} && \text{NS-sector,} \end{aligned} \quad (2.13)$$

one can determine the zero-mode of the Virasoro algebra

$$L_0 = \frac{\alpha'}{4} p^2 + \sum_{m=1}^{\infty} \alpha_{-m}^{\mu} \alpha_{m,\mu} \quad \text{bosonic string} \quad (2.14)$$

$$L_0 = \frac{\alpha'}{4} p^2 + \sum_{m=1}^{\infty} \alpha_{-m}^{\mu} \alpha_{m,\mu} + \sum_{m=1}^{\infty} m d_{-m}^{\mu} d_{m,\mu} \quad \text{R-sector} \quad (2.15)$$

$$L_0 = \frac{\alpha'}{4} p^2 + \sum_{m=1}^{\infty} \alpha_{-m}^{\mu} \alpha_{m,\mu} + \sum_{r=1}^{\infty} \left(r - \frac{1}{2} \right) b_{-r+1/2}^{\mu} b_{r-1/2,\mu} \quad \text{NS-sector.} \quad (2.16)$$

The OCQ procedure prescribes that the physical states satisfies

$$\begin{aligned} (L_0 - a) |\phi\rangle &= 0 \\ L_m |\phi\rangle &= G_m |\phi\rangle = 0 \quad \text{for } m > 0. \end{aligned} \quad (2.17)$$

One will find, using [27], $a = \frac{D-2}{24}$, $a = 0$ and $a = \frac{D-2}{16}$ for the bosonic string, R-sector and NS-sector, respectively. Here one uses the fact that each on-shell bosonic field contributes to a with $-1/24$, R-fermions with $1/24$ and NS-fermions with $-1/48$.

The conditions on the physical states yield that the ground-state for the bosonic string is tachyonic, that the first excited state is a gauge boson for the open string, and that there exists a graviton in the massless part of the spectrum for the closed string. Furthermore, one finds a constraint on the number of space-time dimensions, namely that it is 26.

For the superstring, on the other hand, the constraints on the NS-sector imply that the critical dimension is $D = 10$. Also, all states in the NS-sector are bosonic. The ground-state is tachyonic and the first excited level is massless and describes, for the open case, a gauge boson. This state is created by $b_{-1/2}^\mu$.

Let us now discuss the R-sector. The ground-state is massless because $a = 0$ in this sector. However, it is not a singlet, but rather is representation of the ten-dimensional Clifford algebra as d_0^μ generates this algebra. This is a spinor representation of $\text{Spin}(1,9)$. Thus, all states in this sector are fermionic because the ground-state is fermionic. As the dimension is even, one can decompose the spinor representation into two Weyl representations which have different values of the Γ^{10} matrix. The representation of the ground-state is $16 \oplus 16'$. Using the constraints, one will get a Dirac equation of the state, which reduces the number of degrees of freedom by a factor of one half. Thus, the physical ground-state reduces to $8 \oplus 8'$ on-shell degrees of freedom. Therefore, we have a difference between the NS-sector and the R-sector. One can get rid of this by introducing the GSO-projection [40] on each of the modes. This will project out, in the NS-sector, the states with an even number of b_{-r}^μ excitations and, in the R-sector, one of the chiralities (thus, one of the Weyl representations). This produces an equal number of bosonic and fermionic states at all mass levels. Also, it projects out the tachyon in the bosonic sector. Thus, the theory has the possibility to be space-time supersymmetric.

We have only discussed the spectrum for the open superstring, or to be more precise, one of the sectors of the closed superstring. The open string is a part of the type I string, which consists of unoriented closed strings and open strings with $\text{SO}(32)$ gauge freedom at the ends of the string. There also exist two other string theories which are constructed in the same way: The type IIA and type IIB string theories. These theories consist only of closed strings² where one projects out fermions such that the ones left have the opposite chirality (type IIA) or the same chirality (type IIB).

²From a perturbative perspective.

Let us in the end discuss the BRST approach to the theories above, and limit us to the class of bosonic and type II string. The algebra of the constraints is

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{r+s,0} \\
[L_m, G_n] &= \frac{m-2r}{2}G_{m+r} \\
[G_m, G_n] &= 2L_{m+n} + \frac{c}{12}(4r^2-1)\delta_{r+s,0},
\end{aligned} \tag{2.18}$$

In these equations we have defined the conformal anomaly, c , which for the bosonic string in flat background is equal to the number of dimensions, D . For the supersymmetric string it is $3D/2$ in a flat background. From these equations one can see that the quantized constraints are anomalous. But as described in the previous chapter, one can define a BRST charge of the theory as

$$\begin{aligned}
Q &= \sum_n :c_{-n}L_n + \sum_r : \gamma_{-r}G_r : - \sum_{m,n} (m-n) : c_{-m}c_{-n}b_{m+n} : \\
&+ \sum_{m,r} \left(\frac{3}{2}m+r \right) : c_{-m}\beta_{-r}\gamma_{m+r} : - : \sum_{r,s} \gamma_{-r}\gamma_{-s}\beta_{r+s} - ac_0,
\end{aligned} \tag{2.19}$$

where $:\dots:$ indicate normal ordering, putting creation operators to the left and annihilation operators to the right. a is a normal ordering constant for the BRST charge. Furthermore, the sum over r and s is over integers or half-integers for the Ramond or Neveu-Schwarz sector, respectively. This charge is nilpotent for $d = 26$ and $a = 1$ for the bosonic string and $d = 10$ for the superstring where $a = 1/2$ and $a = 0$ for the Neveu-Schwarz and Ramond sector, respectively. We can now prove the no-ghost theorem for the bosonic and supersymmetric string. We will do this by computing the character and signature function. This follows a method first presented in [25] and uses the trace formula in Theorem 2. To use this we first restrict to the relative cohomology

$$\begin{aligned}
Q|\phi\rangle &= 0 \\
b_0|\phi\rangle &= 0,
\end{aligned} \tag{2.20}$$

so that the states we need to consider do not have any excitation of c_0 . This will get rid of the degeneration of the ghost vacuum. One can construct a relative BRST charge and ghost number operator by extracting the dependence of the zero modes of the ghosts and ghost momenta. The BRST charge is nilpotent on states as long as the charge

$$L_0^{\text{tot.}} = L_0 + L_0^{\text{gh.}} - a, \quad (2.21)$$

where

$$L_0^{\text{gh.}} = nb_{-n}c_n + r\beta_{-r}\gamma_r, \quad (2.22)$$

has zero eigenvalue. This one can satisfy by introducing a delta function in the trace as

$$\delta(L_0^{\text{tot.}}) \equiv \int d\tau \exp [2\pi i\tau L_0^{\text{tot.}}], \quad (2.23)$$

where $L_0^{\text{tot.}}$ in these equations should be thought of as the eigenvalue of the operator. We define the character to be

$$\begin{aligned} \chi(\theta^\mu) &= \int d\tau \text{Tr} [(-1)^{\Delta N_{gh}} \exp [2\pi i\tau L_0^{\text{tot.}}] \exp [p_\mu \theta^\mu]] \\ &= \exp [p_\mu \theta^\mu] \int d\tau \chi(\tau)_\alpha \chi(\tau)_{d/b} \chi(\tau)_{gh.}, \end{aligned} \quad (2.24)$$

where we in the last equality have decomposed the character into different sectors. $\chi(\tau)_\alpha$ is the character involving the bosonic fields, $\chi(\tau)_{d/b}$ is the character for the world-sheet fermions d_{-r}^μ or b_{-r}^μ and $\chi(\tau)_{gh.}$ is the character corresponding to the ghosts. Let us go through the computation of these parts. As all the α_{-m}^μ are independent, one can compute the character for each operator and multiply together each individual part. A generic state is of the form

$$|\phi\rangle = (\alpha_{-m}^\mu)^n |0\rangle \quad (2.25)$$

which has the eigenvalue mn of $L_0^{\text{tot.}}$. Therefore, the part of the character corresponding to this operator is

$$q^{-a} \sum_{n=0}^{\infty} q^{mn} = q^{-a} \frac{1}{1 - q^m}, \quad (2.26)$$

where $\tau \equiv \exp[2\pi i\tau]$. Thus, the character for the bosonic part is

$$\chi(q)_\alpha = q^{-a} \prod_{m=1}^{\infty} \frac{1}{(1-q^m)^D}. \quad (2.27)$$

One can now proceed to compute the character for the fermionic ghosts and the fermions and bosonic ghosts in the Neveu-Schwarz sector in a similar way. The result is

$$\begin{aligned} \chi(q)_b &= \prod_{m=1}^{\infty} (1+q^{m-1/2})^D \\ \chi(q)_{ferm. gh.} &= \prod_{m=1}^{\infty} (1-q^m)^2 \\ \chi(q)_{bos. gh.} &= \prod_{m=1}^{\infty} \frac{1}{(1-q^{m-1/2})^2}. \end{aligned} \quad (2.28)$$

Putting things together, we get the characters for the bosonic string and Neveu-Schwarz sector of the supersymmetric string are

$$\begin{aligned} \chi(\theta^\mu)_{bos.} &= \exp[p_\mu \theta^\mu] \oint dq q^{\frac{\alpha'}{4} p^2 - 1 - 1} \prod_{m=1}^{\infty} \frac{1}{(1-q^m)^{24}} \\ \chi(\theta^\mu)_{bos.} &= \exp[p_\mu \theta^\mu] \oint dq q^{\frac{\alpha'}{4} p^2 - 1/2 - 1} \prod_{m=1}^{\infty} \left(\frac{1+q^{m-1/2}}{1-q^m} \right)^8. \end{aligned} \quad (2.29)$$

Here, the integration over q can be excluded because p^2 can take any value. One can also define a signature function, in which we sum over states so that positive/negative normed states contribute with a positive/negative sign.

$$\begin{aligned} \Sigma(q, \theta^\mu) &= \text{Tr}' \left[(-1)^{\Delta N_{gh}} q^{L_0^{\text{tot.}}} \exp[p_\mu \theta^\mu] \right] \\ &= \exp[p_\mu \theta^\mu] \Sigma(q)_\alpha \Sigma(q)_{d/b} \Sigma(q)_{gh.}, \end{aligned} \quad (2.30)$$

here the prime denote the trace where the sign of the state has been accounted for.

One can in the same way as for the characters determine the individual parts

$$\begin{aligned}
\Sigma(q)_\alpha &= q^{-a} \prod_{m=1}^{\infty} \frac{1}{(1-q^m)^{D-1} (1+q^m)} \\
\Sigma(q)_b &= \prod_{r=1}^{\infty} (1+q^{r-1/2})^{D-1} (1-q^{r-1/2}) \\
\Sigma(q)_{ferm. gh.} &= \prod_{m=1}^{\infty} (1-q^m) (1+q^m) \\
\Sigma(q)_{bos. gh.} &= \prod_{r=1}^{\infty} \frac{1}{(1-q^{r-1/2}) (1+q^{r-1/2})}. \tag{2.31}
\end{aligned}$$

The signature functions for the bosonic string and the Neveu-Schwarz sector of the supersymmetric string are

$$\begin{aligned}
\Sigma(q, \theta^\mu)_{bos.} &= \exp[p_\mu \theta^\mu] q^{\frac{\alpha'}{4} p^2 - 1} \prod_{m=1}^{\infty} \frac{1}{(1-q^m)^{24}} \\
\Sigma(q, \theta^\mu)_{bos.} &= \exp[p_\mu \theta^\mu] q^{\frac{\alpha'}{4} p^2 - 1/2} \prod_{m=1}^{\infty} \left(\frac{1+q^{m-1/2}}{1-q^m} \right)^8. \tag{2.32}
\end{aligned}$$

These expressions are precisely the same expressions as in eq. (2.29), when the integration $\oint \frac{dq}{q}$ has been removed. This proves the no-ghost theorem for the bosonic and the Neveu-Schwarz sector of the supersymmetric string. This follows since the character determines the number of states at a certain level and the signature function determines the difference of the number of states with positive and negative norms. If the character and signature functions are equal, the number of states with negative norm is zero.

Let us here also discuss the Ramond-sector. The major problem in computing the character and signature functions in this case is the degeneracy of the vacuum due to d_0^μ . This implies that the character is of the form of an alternating and divergent sum and the signature function is equal to zero times infinity. This problem we solved in paper V by introducing an operator, $\exp[\phi N']$, where $N' = \sum_{i=0}^4 (d_0^{-i} d_0^{+i} - \gamma_0 \beta_0)$ and in the end letting $\phi \rightarrow 0$ to recover the BRST invariant piece. Here we have

defined

$$\begin{aligned} d_0^{\pm,0} &= \frac{1}{\sqrt{2}} [d_0^1 \pm d_0^0] \\ d_0^{\pm,i} &= \frac{1}{\sqrt{2}} [d_0^{2i+1} \pm i d_0^{2i}] \quad i = 1, 2, 3, 4 \end{aligned} \quad (2.33)$$

Such that $d_0^{i,+}$, $i = 0, \dots, 4$, annihilates the R-vacuum. Assuming now that the representation is the Majorana representation of Spin(1,9) one finds the characters for the fermions and bosonic ghosts

$$\begin{aligned} \chi(q, \phi)_d &= (1 + \exp[\phi])^5 \prod_{m=1}^{\infty} (1 + q^m)^{10} \\ \chi(q, \phi)_{bos. gh.} &= \frac{1}{1 + \exp[\phi]} \prod_{m=1}^{\infty} \frac{1}{(1 - q^{m-1/2})^2}. \end{aligned} \quad (2.34)$$

The signature functions are

$$\begin{aligned} \Sigma(q, \phi)_d &= (1 + \exp[\phi])^4 (1 - \exp[\phi]) \prod_{m=1}^{\infty} (1 + q^m)^9 (1 - q^m) \\ \Sigma(q, \phi)_{bos. gh.} &= \frac{1}{1 - \exp[\phi]} \prod_{m=1}^{\infty} \frac{1}{(1 - q^m)(1 + q^m)}. \end{aligned} \quad (2.35)$$

Combining this with the known bosonic part, which is the same as for the bosonic string, and taking the limit $\phi \rightarrow 0$, yields the character

$$\chi(q, \theta^\mu) = 2^4 \exp[p_\mu \theta^\mu] q^{\frac{\alpha'}{4} p^2} \prod_{m=1}^{\infty} \left(\frac{1 + q^m}{1 - q^m} \right)^8, \quad (2.36)$$

which is exactly the same as the signature function, thus shows unitary. One can also show that the supersymmetric string has a possibility to be space-time supersymmetric by applying the GSO-projection to the characters. The GSO-projection acting on the Neveu-Schwarz sector projects out all even numbers of excitations, which can easily be achieved for the character. For the Ramond sector, the projection to one

Weyl representation halves the number of degrees of freedom

$$\begin{aligned}
 \chi(q, \theta^\mu)_{Bos.} &= \exp[p_\mu \theta^\mu] q^{\frac{a'}{4} p^2} \left\{ \frac{1}{\sqrt{q}} \left(\prod_{m=1}^{\infty} \left(\frac{1+q^{m-1/2}}{1-q^m} \right)^8 \right. \right. \\
 &\quad \left. \left. - \prod_{m=1}^{\infty} \left(\frac{1-q^{m-1/2}}{1-q^m} \right)^8 \right) \right\} \\
 \chi(q, \theta^\mu)_{Ferm.} &= \exp[p_\mu \theta^\mu] q^{\frac{a'}{4} p^2} \left\{ 8 \prod_{m=1}^{\infty} \left(\frac{1+q^m}{1-q^m} \right)^8 \right\}. \tag{2.37}
 \end{aligned}$$

These two expressions are equal because of an identity proved by Jacobi. This shows that one has an equal number of fermionic and bosonic states, which is required by space-time supersymmetry.

In this section we have discussed three consistent string theories with world-sheet supersymmetry as well as space-time supersymmetry. In addition to these, there also exist two heterotic string theories, the $\mathfrak{so}(32)$ and $E_8 \oplus E_8$, which are constructed by taking bosonic left-moving modes and world-sheet supersymmetric right-moving modes (or the other way around). These are the known consistent string theories in ten-dimensions. But, if any of them would describe our universe, which one is it? A conjecture which, if true, would answer this question was presented in the mid 1990's when Witten [26], see also [41], argued that there exists a theory in eleven dimensions called M-theory. This theory would, in different limits, yield the different string theories. This conjecture was based on many results of non-perturbative string theory. As this theory may involve the supermembrane, it is important to study membranes in more detail. In the next section I will introduce this theory.

2.2 Membranes

This section is in part based on reviews by Duff [42], Nicolai and Helling [43], Taylor [44] and de Wit [45].

Membrane theory is formulated in the same way as string theory. It is a geometric theory and, therefore, its dynamics is encoded in an action proportional to the world-volume that the membrane traces out in space-time. But, there exist differences as well; the membrane is a self-interacting theory from the start and there does not exist a parameter that can work as a perturbation parameter. This is in contrast to the case of the string, where the vacuum expectation value of the dilaton field acts as a perturbation parameter. One may, however, introduce different types of perturbation parameters. One example is our work in paper I, where the tension of the membrane acts as a perturbation parameter. Another example is the work of Witten [26], where he argues for the existence of an M-theory. There one introduces a perturbation parameter through compactification, where the size of the compact dimension acts as a perturbation parameter. This, we will see, is connected to the vacuum expectation value of the dilaton field of type IIA string theory.

One additional difference between the string and the membrane, is that one cannot introduce world-sheet supersymmetry in a simple way [46–49]. The action that exist [50, 51] does not have an obvious connection to the space-time supersymmetric formulation of the membrane action [52, 53]. One can also formulate a string action which is space-time supersymmetric, the Green-Schwarz superstring [54–56] (GS-superstring).

Let us begin by introducing the Dirac action [33] for the bosonic membrane

$$S_2 = -T_2 \int_{\Sigma} d^3\xi \sqrt{-\det h_{ij}}, \quad (2.38)$$

where Σ is the world-volume which is traced out by the membrane, T_2 is the tension of the membrane, $i, j = 0, 1, 2$ and $h_{ij} = \partial_i X^\mu \partial_j X_\mu$ is the induced metric of the world-volume. This action is of the same kind as the free string action. It has rigid Poincaré invariance in target space and local reparametrization invariance of the

world-volume. The constraints corresponding to local reparametrization invariance are

$$\begin{aligned}\phi_0 &= \frac{1}{2} \{ \mathcal{P}^2 + T_2^2 \det [h_{ab}] \} \\ \phi_a &= \mathcal{P}_\mu \partial_a X^\mu,\end{aligned}\tag{2.39}$$

where $a, b = 1, 2$. If we Legendre transform to the Hamiltonian formulation of the theory, we will find that the Hamiltonian is weakly zero, thus, verifying that the theory is reparametrization invariant. The constraints satisfy a closed Poisson bracket algebra, but the structure functions depend on the phase-space coordinates.

One can also introduce a metric on the world-volume to get the action

$$S'_2 = -\frac{T_2}{2} \int d^3\xi \sqrt{-\gamma} [\gamma^{ij} \partial_i X^\mu \partial_j X_\mu - 1].\tag{2.40}$$

Here we see a difference between the string and the membrane. One has an extra term corresponding to a cosmological constant, which is not there for the string. A consequence of this is that the action does not possess Weyl invariance³. Since we only have three constraints, and the metric has five independent components, we cannot fix the metric to be conformally flat.

The equations of motion are those of an interacting theory, as can be shown as follows. Choose the Hamiltonian to be proportional to ϕ_0 . The equations of motion which follow from this Hamiltonian are

$$\begin{aligned}\ddot{X}^\mu &= \partial_1(\partial_1 X^\mu (\partial_2 X)^2) - \partial_2(\partial_1 X^\mu (\partial_1 X \partial_2 X)) \\ &+ \partial_2(\partial_2 X^\mu (\partial_1 X)^2) - \partial_1(\partial_2 X^\mu (\partial_1 X \partial_2 X)),\end{aligned}\tag{2.41}$$

which are non-linear and, therefore, not equations of motion for a free theory. Thus, the three-dimensional theory on the world-volume, contrary to the string world-sheet theory, is an interacting theory.

³One can formulate an action which possesses Weyl invariance [57]. One of the solutions of the equations, which arises from the variation of γ^{ij} , can be used to yield the action in eq. (2.38). For this action, one can introduce linearized world-volume supersymmetry [50, 51].

To introduce world-sheet supersymmetry in this action is, as stated above, difficult. Therefore, one usually introduces space-time supersymmetry instead. Assume that we can introduce $N = 1$ supersymmetry in space-time and that the fermions can be taken to be Majorana. The basic fields in this action are X^μ , which form a vector representation of $SO(1, D)$, and the fermionic fields θ^α , which form the spinor representation of $\text{Spin}(1, D)$ which is the universal covering group of $SO(1, D)$. Under the assumption that the fermions are Majorana, the fermionic field will have $2^{\lfloor D/2 \rfloor}$ real degrees of freedom. The ways the supersymmetry transformations act on the fields are

$$\begin{aligned}\delta_Q \theta^\alpha &= \epsilon^\alpha \\ \delta_Q \bar{\theta}^\alpha &= \bar{\epsilon}^\alpha \\ \delta_Q X^\mu &= i\bar{\epsilon}\Gamma^\mu\theta,\end{aligned}\tag{2.42}$$

where δ_Q is a supersymmetry transformation and ϵ^α is a spinor. One can construct a field, $\Pi_i^\mu \equiv \partial_i X^\mu - i\bar{\theta}\Gamma^\mu\partial_i\theta$, which is invariant under these space-time transformations. This can be used to construct an action for the supermembrane based on the Dirac action by exchanging $\partial_i X^\mu$ by Π_i^μ . But, if one does this one discovers a problem. This is because the number of degrees of freedom does not match. One has $D - 3$ on-shell bosonic and $2^{\lfloor D/2 \rfloor - 1}$ on-shell fermionic degrees of freedom. Because M-theory is a theory in eleven dimensions and supermembranes would be a part of it, the interesting number of space-time dimensions to formulate this theory in is eleven⁴. In this number of dimension we find twice as many fermions as bosons. The way out of this is to postulate that the action also has to be invariant under kappa symmetry. This symmetry is a link between the space-time and world-sheet supersymmetry. This acts on the fermions as

$$\delta\theta^\alpha = \kappa(1 + \Gamma)^\alpha\tag{2.43}$$

⁴It is not possible, even at the classical level, to formulate an action for supersymmetric membranes in an arbitrary dimension, only $D = 4, 5, 7$ and 11 is possible for scalar world-sheet fields.

where⁵

$$\Gamma \equiv \frac{1}{3! \sqrt{-\det h_{ij}}} \epsilon^{ijk} \Pi_i^\mu \Pi_j^\nu \Pi_k^\lambda \Gamma_{\mu\nu\lambda}. \quad (2.44)$$

In this equation ϵ^{ijk} is a totally anti-symmetric pseudo-tensor defined by $\epsilon^{012} = -1$ and

$$\Gamma_{\mu\nu\lambda} = \Gamma_{[\mu} \Gamma_{\nu} \Gamma_{\lambda]} \quad (2.45)$$

where square bracket indicates antisymmetrization of the indices with unit norm. As $\text{Tr}[\Gamma] = 0$ and $\Gamma^2 = 1$, eq. (2.43) reduces the number of fermions by a factor of two. Thus, in eleven dimensions, there exist 8 bosonic and 8 fermionic on-shell degrees of freedom, so that, the degrees of freedom match and the theory has the possibility to have unbroken supersymmetry. The action which possesses this symmetry⁶ is the Dirac action formulated as a pullback of the space-time metric plus a three-form [52, 53]

$$S_2^s = - \int d^3 \xi \sqrt{-h} + \int C_3. \quad (2.46)$$

Here I have put $T_2 = 1$ and $h \equiv \det [h_{ij}]$. This action is of the same principal form as the covariant action for the Green-Schwarz string [58]. It was thought that one could not formulate such actions for other models than particles and strings. This was proved wrong by the construction in [59], where an action was presented for a three-brane in six dimensions. In flat space the three-form is

$$\int C_3 = \frac{i}{2} \int d^3 \xi \epsilon^{ijk} \Gamma_{\mu\nu} \partial_i \theta \left[\Pi_j^\mu \partial_k X^\nu - \frac{1}{3} \bar{\theta} \Gamma^\mu \partial_j \theta \bar{\theta} \Gamma^\nu \partial_k \theta \right], \quad (2.47)$$

where $\Gamma_{\mu\nu} \equiv \Gamma_{[\mu} \Gamma_{\nu]}$. To analyze this action one has to choose a gauge. A suitable gauge is to fix it partially to the lightcone gauge⁷ [60]. To do this, we first choose

⁵Here I have written down the expression for a membrane in flat background.

⁶Provided that the background fields satisfy the on-shell constraints of eleven-dimensional supergravity.

⁷This is not exactly the same kind of lightcone gauge that we use in paper I.

lightcone coordinates

$$\begin{aligned} A^+ &= \frac{1}{\sqrt{2}} (A^{D-1} + A^0) \\ A^- &= \frac{1}{\sqrt{2}} (A^{D-1} - A^0). \end{aligned} \quad (2.48)$$

Then we fix the kappa-invariance by imposing $\Gamma^+\theta = 0$ where

$$\Gamma^+ = 1_{16} \otimes \begin{pmatrix} 0 & 0 \\ \sqrt{2}i & 0 \end{pmatrix}. \quad (2.49)$$

1_{16} is the 16×16 unit matrix. Fixing two out of the three reparametrization invariance by⁸

$$\begin{aligned} \chi_1 &= X^+ - \xi^1 \approx 0 \\ \chi_{2,a} &= \partial_a X^- + i\psi \partial_a \psi + \partial_a X_I \partial_0 X_I \approx 0, \end{aligned} \quad (2.50)$$

for $I = 1, \dots, D-2$. Here we have redefined the non-zero fermions by $\psi^\alpha = (2)^{1/4} \theta^{2\alpha}$.

In this gauge one can choose $\mathcal{P}^+ = P^+ \sqrt{w(\xi^1, \xi^2)}$ where P^+ is a constant and

$$\int d^2\xi \sqrt{w} = 1. \quad (2.51)$$

Thus, P^+ is the center of momentum. This will, in the end, yield the Hamiltonian [60]

$$H = \frac{1}{P^+} \int d^2\xi \left[\frac{\mathcal{P}^2 + \det(\partial_a X \partial_a X)}{2\sqrt{w}} + P^+ \epsilon^{ab} \psi \gamma_I \partial_a \psi \partial_b X^I \right], \quad (2.52)$$

where ϵ^{ab} is a totally antisymmetric pseudo-tensor defined by $\epsilon^{12} = 1$ and γ_I is defined by

$$\Gamma_I = \gamma_I \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2.53)$$

The constraints that are still left are

$$\phi = \epsilon^{ab} \left[\partial_a \left(\frac{1}{\sqrt{w}} P \right) \partial_b X + \partial_a \left(\frac{1}{\sqrt{w}} S \right) \partial_b \psi \right] \approx 0 \quad (2.54)$$

$$G^\alpha = S^\alpha - i\sqrt{w} P^+ \psi^\alpha \approx 0, \quad (2.55)$$

⁸The second of these equations is only one gauge fixing condition.

where the first constraint is due to the single-valuedness of the X^- ⁹ and the second is the Dirac constraint for the fermions. Regularization of this Hamiltonian and other limits of the membrane action will be discussed in the next chapter. Let us briefly mention the generalization to general p . An action of a p -brane is a generalization of the action of the string

$$S_p = -T_p \int_{\Sigma} d^{p+1}\xi \sqrt{-h_{ij}}, \quad (2.56)$$

where Σ is the world-hypervolume traced out by the p -brane, T_p is the tension of the p -brane, $i, j = 0, \dots, p$ and $h_{ij} = \partial_i X_{\mu} \partial_j X^{\mu}$. The constraints of this action follow straightforward from the definition of the canonical momentum, \mathcal{P}_{μ} , and looking at the string and membrane constraints

$$\begin{aligned} \phi_0 &= \frac{1}{2} [\mathcal{P}^2 + T_p^2 \det[h_{ab}]] \\ \phi_a &= \mathcal{P}_{\mu} \partial_a X^{\mu}, \end{aligned} \quad (2.57)$$

where $a = 1, \dots, p$. The Hamiltonian is weakly zero. One can also here define an action where one has introduced a metric on the world-hypervolume

$$S'_p = -\frac{T_p}{2} \int_{\Sigma} d^{p+1}\xi \sqrt{-\gamma} [\gamma^{ij} \partial_i X^{\mu} \partial_j X_{\mu} - (p-1)]. \quad (2.58)$$

For the regular p -branes even less is known than for the membranes. There are branes which has connections to the p -branes called D-branes [61] which actions for type II string theory was formulated in [62,63]. Let us, present the arguments by Witten [26] for why there should exist a unifying theory called M-theory where D-branes play an essential role.

2.3 M-theory

In this section I will briefly discuss one of the arguments presented by Witten [26] for the existence of a theory in eleven dimensions which, in different limits, should

⁹The first of these constraints is a local condition on the field X^- , but there also exist global constraints on the field, namely, that the line integral over a closed curve of $P\partial_a X + S\partial_a \phi$ should vanish.

describe the five known consistent superstring theories.

Let us first begin with the type IIA supergravity limit, for which the bosonic part of the action is $I_{IIA} = I_{NS} + I_R$, where I_{NS} is the NS-NS part of the massless bosonic fields and I_R is the part which is quadratic in the R-R-sector fields. The massless fields in NS-NS-sector is the metric $g_{\hat{\mu}\hat{\nu}}$, the anti-symmetric field $b_{\hat{\mu}\hat{\nu}}$, which we write as a two-form $B_2 = \frac{1}{2}b_{\hat{\mu}\hat{\nu}}dx^{\hat{\mu}} \wedge dx^{\hat{\nu}}$, and the dilaton ϕ . The massless fields in the R-R sector are a one-form A_1 , and a three-form A_3 . From these, one can construct the corresponding field strengths $H_3 = dB_2$, $F_2 = dA_1$ and $F_4 = dA_3$. We also need $F'_4 = dA_3 + A_1 \wedge H_3$. After defining these fields the bosonic part of the supergravity action is

$$\begin{aligned} I_{NS} &= \frac{1}{2} \int d^{10}x \sqrt{g} e^{-2\gamma} \left\{ R + 4(\nabla\phi)^2 - \frac{1}{12} |H_3|^2 \right\} \\ I_R &= -\frac{1}{2} \int d^{10}x \left\{ \frac{1}{2!} |F_2|^2 + \frac{1}{4!} |F'_4|^2 \right\} - \frac{1}{4} \int F_4 \wedge F_4 \wedge B_2, \end{aligned} \quad (2.59)$$

where $|F_p|^2 = F_{\mu_1, \dots, \mu_p} F^{\mu_1, \dots, \mu_p}$. This action was derived from the N=1 supergravity theory in $D = 11$ [64] by dimensional reduction. The N=1 supergravity theory in $D = 11$ [64] is simpler and the bosonic part is

$$I_{11} = \int d^{11}x \sqrt{G} \{ R + |dA_3|^2 \} + \int A_3 \wedge dA_3 \wedge dA_3, \quad (2.60)$$

where $G_{\mu\nu}$ is the metric in 11 dimensions.

For the type IIA theory we have two chiralities. Thus, we will have a separation between two different supersymmetry charges, Q_α and $Q'_{\dot{\alpha}}$. From the point of view of fundamental string theory, these are independent. Let us now focus on the gauge field A_1 . There is a possibility to add a central term in the anti-commutator between the two different supersymmetry charges, which has a non-zero charge Z , with respect of the gauge field A_1 :

$$\{Q_\alpha, Q'_{\dot{\alpha}}\} \sim \delta_{\alpha, \dot{\alpha}} Z. \quad (2.61)$$

There exists a Bogomol'nyi bound [65] for states which relates the mass to the charge. If we assume that the allowed values of Z are discrete, what mass would such a states

have? First of all, they should decouple when the string coupling $\lambda = e^{\phi_0}$, where ϕ_0 is the vacuum expectation value of the dilaton field, goes to zero. Thus, $M \sim \lambda^{-m}$ where m is a positive number. One can also argue, under the assumption that the kinetic energy is independent of ϕ , that $m = 1$. Thus, the Bogomol'nyi bound for such a state is

$$M \geq c_1 \frac{|n|}{\lambda}, \quad n \in \mathbf{Z} \quad (2.62)$$

where c_1 is a constant. Using arguments by Hull and Townsend [66] and Townsend [41] one can show that these states are quantized black holes. The BPS black holes are the states where one has equality as in eq. (2.62). The BPS states are completely determined by the charges carried by the states and remain BPS as the coupling constants vary. These states, therefore, survive in the non-perturbative sector of the theory and decouple when $\lambda \rightarrow 0$. These states will not appear as elementary string states and, therefore, will be new sectors of the theory. Furthermore, these states will be parts of a non-perturbative formulation of string theory. The existence, and what these states corresponds to, will be discussed in the next chapter.

We will now argue that these states arise because an extra dimension in the type IIA string theory appears when the coupling constant differs from zero. Consider the N=1 supergravity in $D = 11$ and make a Kaluza-Klein ansatz, see for instance [67], by

$$\begin{aligned} ds^2 &= e^{-\gamma} g_{\hat{\mu}\hat{\nu}} dx^{\hat{\mu}} dx^{\hat{\nu}} + e^{2\gamma} (dx^{11} - A_{\hat{\mu}} dx^{\hat{\mu}})^2 \\ B_{\hat{\mu}\hat{\nu}} &= A_{\hat{\mu}\hat{\nu}10}. \end{aligned} \quad (2.63)$$

This yields that the action in eq. (2.60) has a similar form as I_{IIA} if one makes the identification $\gamma = 2\phi_0/3$. Thus, the radius, $r(\lambda) = e^\gamma$, is related to the perturbation parameter of the type IIA theory; in the strong coupling limit, $\lambda \rightarrow \infty$, of type IIA an extra dimension appears, in which the field theory is weakly coupled¹⁰. The

¹⁰Consider the simplest interacting model, $\int d^5x \{(\partial\Phi)^2 + \Phi^3\}$. Compactifying $\phi = R^{1/2}\Phi$, yields $\int d^4x \{(\partial\phi)^2 + R^{-1/2}\phi^3\}$ and the theory is weakly coupled when R gets large.

masses of the Kaluza-Klein modes are proportional to

$$e^{-\gamma/2}/r(\lambda) = c' \frac{|n|}{\lambda}, \quad (2.64)$$

this is of the same form as eq. (2.62). Thus, the central charges of the supersymmetry algebra appear as Kaluza-Klein modes of the compactified dimension.

What I have described here is one of the dualities in a web of dualities which relate different string theories in different dimensions.

Chapter 3

Limits of Membrane theory

In this chapter a few different limits of membrane theories will be discussed. One of the most interesting is the matrix model approximation, which can be used for membranes in the lightcone gauge [60, 68, 69]. This will, in the supersymmetric case, yield a quantum-mechanical model of a maximally supersymmetric $\mathfrak{su}(N)$ matrix model [70–72]. These models are quantizable and it is argued that in the limit $N \rightarrow \infty$, they will describe the quantum membrane. This limit is, as will be discussed in the next chapter, not thought to be a simple one. For instance, if one chooses different $N \rightarrow \infty$ limits, one can prove that the different algebras one obtains are in pairs non-isomorphic [73], so that one has not a unique algebra of $\mathfrak{su}(\infty)$.

There exist limits of the membrane where string theory arises naturally. One example is the double-dimensional reduction [74, 75]. One can find other limits, where one keeps some of the dependence of the ξ^2 -direction. In the limit, when the radius of the compact dimension goes to zero, one will get the rigid string [76].

3.1 Matrix approximation

This section is in part based on [45], which is a review of supermembranes and supermatrix models. Let us consider the bosonic part of the Hamiltonian in eq. (2.52)

$$H = \frac{1}{P^+} \int d^2\xi \left\{ \frac{\mathcal{P}^2 + \det(\partial_a X \partial_a X)}{2\sqrt{w}} \right\}. \quad (3.1)$$

Let us define a bracket

$$\{A, B\} \equiv \frac{\epsilon^{ab}}{\sqrt{w}} \partial_a A \partial_b B. \quad (3.2)$$

This bracket satisfies the Jacobi identity,

$$\{A, \{B, C\}\} + \{B, \{C, A\}\} + \{C, \{A, B\}\} = 0. \quad (3.3)$$

Therefore, if one has a complete set of functions which are closed under the bracket; the functions satisfy an algebra.

Using the bracket, the Hamiltonian and the corresponding constraint can be written as

$$\begin{aligned} H &= \frac{1}{2P^+} \int d^2\xi \sqrt{w} \left[\frac{\mathcal{P}^2}{w} + \frac{1}{2} \{X^I, X^J\}^2 \right] \\ \phi &= \{w^{-1/2} P_I, X^I\} \approx 0. \end{aligned} \quad (3.4)$$

This action is invariant under area-preserving diffeomorphisms

$$\xi^a \rightarrow \xi^a + \varphi^a, \quad (3.5)$$

with

$$\partial_a (\sqrt{w} \varphi^a) = 0, \quad (3.6)$$

such that the bracket is invariant in eq. (3.2). This condition can be simplified by defining

$$\varphi_a \equiv \epsilon_{ab} \sqrt{w} \varphi^b. \quad (3.7)$$

Thus, eq. (3.6) shows that φ_a may be used to construct a closed one-form. Under the assumption that we have a simply connected membrane, closed one-forms are also exact,

$$\varphi_a = \partial_a \varphi. \quad (3.8)$$

The infinitesimal transformations act as

$$\delta_\varphi X^I = \frac{\epsilon^{ab}}{\sqrt{w}} \varphi_a \partial_b X^I. \quad (3.9)$$

If we consider the commutator of two such transformations, parameterized by $\varphi^{(1)}$ and $\varphi^{(2)}$, one will get a resulting transformation parameterized by

$$\varphi_a^{(3)} = \partial_a \left(\frac{\epsilon^{bc}}{\sqrt{w}} \varphi_b^{(2)} \varphi_c^{(1)} \right), \quad (3.10)$$

which shows that $\varphi^{(3)}$ is exact even if $\varphi^{(1)}$ and $\varphi^{(2)}$ are not exact. If we use the bracket defined in eq. (3.2) one can see that

$$\{\varphi^{(2)}, \varphi^{(1)}\} = \varphi^{(3)}. \quad (3.11)$$

We can expand the fields in terms of a complete set of functions $\{Y^A\}$ of the two-dimensional surface

$$\begin{aligned} \partial_a X^I &= \sum_A X_A^I \partial_a Y^A(\xi) \\ P^I &= \sum_A \sqrt{w} P_A^I Y^A(\xi), \end{aligned} \quad (3.12)$$

The closed one-forms are then dY^A . As they constitute a complete set, they satisfy

$$\{Y^A, Y^B\} = f^{AB}{}_C Y^C. \quad (3.13)$$

From eq. (3.11) we find that $f^{AB}{}_C$ are the structure constants of the area-preserving diffeomorphisms in the basis $\{Y^A\}$. Since these functions constitute a complete set, one can define an invariant metric

$$\int d^2\xi \sqrt{w} Y^A Y^B = \eta^{AB}, \quad (3.14)$$

which can be used to raise indices. In the compact case, η^{AB} is changed to δ^{AB} . One can show that the diffeomorphism group of the membrane can be approximated by a finite dimensional algebra [68, 69, 77–80], (for closed membranes it is $\mathfrak{su}(N)$ if we subtract the center of momentum) which in a particular limit describes the membranes. Let us consider the simplest example, namely the toroidal membrane, to see how this works.

A basis for these membranes is

$$Y_{\vec{m}} = \frac{1}{2\pi} e^{i\vec{m}\cdot\vec{\xi}}, \quad (3.15)$$

where $\vec{m} = [m_1, m_2]$, $\vec{\xi} = [\xi^1, \xi^2]$ and $0 < \xi^1, \xi^2 < 2\pi$. If one computes the bracket one will get

$$\{Y_{\vec{m}}, Y_{\vec{n}}\} = -\frac{1}{2\pi} (\vec{m} \times \vec{n}) Y_{\vec{m}+\vec{n}}, \quad (3.16)$$

where $(\vec{m} \times \vec{n})$ denotes $m_1 n_2 - m_2 n_1$. To see that this indeed can be reconstructed by approximating to $\mathfrak{u}(N)$ matrices¹ and taking $N \rightarrow \infty$, we use the t'Hooft clock and shift matrices

$$U = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ 1 & 0 & \cdots & \cdots & 0 \end{pmatrix} \quad V = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \omega & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \omega^{N+1} \end{pmatrix}, \quad (3.17)$$

where $\omega = e^{2\pi i k/N}$. Products of these matrices form a basis of Hermitian $N \times N$ matrices and satisfy the matrix commutation relations

$$[V^{m_1} U^{m_2}, V^{n_1} U^{n_2}] = (\omega^{m_2 n_1} - \omega^{m_1 n_2}) U^{m_1+n_1} V^{m_1+n_2}, \quad (3.18)$$

as can be shown by using $UV = \omega VU$. If we define the generators of $\mathfrak{u}(N)$ as

$$T_{\vec{m}} = -\frac{iN}{(2\pi)^2 k} V^{m_1} U^{m_2} \quad (3.19)$$

¹ $\mathfrak{u}(N) \cong \mathfrak{su}(N) \oplus \mathfrak{u}(1)$ and the $\mathfrak{u}(1)$ corresponds to the center of momentum.

we see that in the limit

$$\lim_{N \rightarrow \infty} [T_{\vec{m}}, T_{\vec{n}}] = -\frac{1}{2\pi} (\vec{m} \times \vec{n}) T_{\vec{m}+\vec{n}}. \quad (3.20)$$

This has the same form as eq. (3.16).

The Hamiltonian for this matrix theory is

$$H = \frac{1}{P^+} \text{Tr} \left[\frac{1}{2} P^2 + \frac{1}{4} [X^I, X^J]^2 \right], \quad (3.21)$$

where I have made the following identifications

$$\begin{aligned} \int d^2\xi(\cdot) &\iff \text{Tr}(\cdot) \\ \int d^2\xi \sqrt{w} Y^A = 0 &\iff \text{Tr}[T^A] = 0 \\ \{Y^A, Y^B\} = f^{AB}{}_C Y^C &\iff [T^A, T^B] = f^{AB}{}_C T^C. \end{aligned} \quad (3.22)$$

T^A are the generators of the $\mathfrak{su}(N)$ algebra and $[\cdot, \cdot]$ is the usual commutator of matrices. One can also do the same for the supersymmetric matrix model to get [60]

$$H = \frac{1}{P^+} \text{Tr} \left[\frac{1}{2} P^2 + \frac{1}{4} [X^I, X^J]^2 + P^+ \theta^T \gamma_I [\theta, X^I] \right]. \quad (3.23)$$

Area preserving diffeomorphisms is also connected to $W(\infty)$ -algebras [81–83]. The W_N -algebra is an extension of the Virasoro algebra to also consist of fields with integer spin less than equal to N introduced by Zamolodchikov [84].

3.1.1 The spectrum of the matrix model

In this subsection I will briefly discuss the results for the spectrum of the matrix Hamiltonians in eqs. (3.21) and (3.23). It was thought that the latter of these models possesses discrete mass spectrum due to the work of [85, 86]. But it was found by de Wit, Lüscher and Nicolai [87] that this was not the case. To describe this proof in detail is beyond the scope of this thesis. Instead, I will describe a simpler toy model which captures the essential features of the matrix model for the supermembrane.

Consider the two-dimensional quantum mechanical model defined by the self-adjoint supercharge

$$Q = \begin{pmatrix} -xy & i\partial_x + \partial_y \\ i\partial_x - \partial_y & xy \end{pmatrix}. \quad (3.24)$$

This charge can be used to construct a Hamiltonian

$$\begin{aligned} H &= \frac{1}{2} \{Q, Q^\dagger\} \\ &= \begin{pmatrix} -\Delta + x^2y^2 & x + iy \\ x - iy & -\Delta + x^2y^2 \end{pmatrix}, \end{aligned} \quad (3.25)$$

where $\Delta = \partial_x^2 + \partial_y^2$. The associated bosonic operator to this Hamiltonian is the diagonal part of the matrix. One can see that the potential $V(x, y) = x^2y^2$ has flat directions. The particle is, therefore, not confined in the potential. This non-confinement does not survive quantization. This can be seen as a zero-point effect for the harmonic oscillator at fixed value of x (or y). The zero-point effect is proportional to $|x|$ and the Hamiltonian is, therefore, bounded from below by the potential $V_q = |x| + |y|$. For the supersymmetric model, this is not the case. Because of supersymmetry, the fermionic degrees of freedom cancel the zero-point effects of the bosonic degrees of freedom. This implies that there is no potential barrier which confines the particle to the center of the potential. Consequently, all energies of the particle are possible except negative ones since the Hamiltonian is bounded from below.

The membrane has the same features as this toy model. Thus, the mass spectrum for the supermembrane is continuous and the allowed masses in the spectrum are

$$M \in [0, \infty). \quad (3.26)$$

This result does not rule out the possibility that there may exist a discrete subspectrum. Furthermore, this result is for finite N , which one assumes to hold also in the limit $N \rightarrow \infty$. The continuous spectrum is interpreted as the membranes not having a fixed topology. Membranes can form spikes with zero area and, therefore,

of zero energy. These spikes can reattach to another part of the surface to create a membrane with another genus. Furthermore, the number of membranes is not fixed because the membranes can attach themselves to each other by spikes.

Let us now go back to discuss the BPS-states that were mentioned in the previous chapter. These states exist, at least for $N = 1$, as they correspond to a single D0-brane of type IIA string theory [61]. The argument for the existence of a unique BPS-state for $N \neq 1$, which corresponds to N bounded D0-branes, is different. To analyze this one can use a more indirect method by computing the Witten index [88] for the corresponding group $SU(N)$. The number one should find is one for all N . Furthermore, one also needs to show that there are no fermionic ground states. This has been computed and proven for $N = 2$ in ref. [89]. For $N > 2$ one has shown that there exist bounded states by a computation made in ref. [90] combined with the results in ref. [91]. But one only gets that there exist bound states not any of their properties. It is very difficult to determine the states for $N \geq 2$ and only the asymptotic form of the ground state wave function has been computed, for $N = 2$ in ref. [92–95] and for $N = 3$ in ref. [96]. In ref. [94] a summary of the known properties of $SU(N)$ -models was made. One quite interesting feature is that there seems to not exist any normalizable ground state for the other cases, $D = 2, 3$ and 5 , in which one can formulate a supersymmetric matrix model. This could indicate that there exists a critical dimension for these matrix models and, therefore, for the supermembranes.

3.2 Type IIA string from the supermembrane

This section is a review of the arguments in ref. [74]. Assume that $D = 11$. As we will see, this is related to the arguments of M-theory presented in the previous chapter. We start from the bosonic part of the action in eq. (2.46),

$$S_2 = - \int d^3\xi \left\{ \sqrt{-\det \partial_i X^\mu \partial_j X^\nu g_{\mu\nu}} - \frac{1}{3!} \epsilon^{ijk} \partial_i X^\mu \partial_j X^\nu \partial_k X^\lambda C_{\mu\nu\lambda} \right\}. \quad (3.27)$$

Let us now perform a double-dimensional reduction by gauging

$$X^{D-1} = \xi^2, \quad (3.28)$$

and choosing the other fields to be independent of the ξ^2 -variable. Furthermore, compactify the theory on a circle by the Kaluza-Klein ansatz:

$$g_{\mu\nu} = \Phi^{-2/3} \begin{pmatrix} \hat{g}_{\hat{\mu}\hat{\nu}} + \Phi^2 A_{\hat{\mu}} A_{\hat{\nu}} & \Phi^2 A_{\hat{\mu}} \\ \Phi^2 A_{\hat{\nu}} & \Phi^2 \end{pmatrix}, \quad (3.29)$$

where the hatted indices are in $D - 1$. Then, split the antisymmetric field as

$$\begin{aligned} C_{\mu\nu\lambda} &= (\hat{C}_{\hat{\mu}\hat{\nu}\hat{\lambda}}, \hat{C}_{\hat{\mu}\hat{\nu}10}) \\ &= (\hat{C}_{\hat{\mu}\hat{\nu}\hat{\lambda}}, b_{\hat{\mu}\hat{\nu}}). \end{aligned} \quad (3.30)$$

Inserting this into the action in eq. (3.27) and integrating out the ξ^2 -dependence yields

$$S' = - \int d^2\xi \left\{ \sqrt{-\det \partial_i X^{\hat{\mu}} \partial_j X^{\hat{\nu}} \hat{g}_{\hat{\mu}\hat{\nu}}} - \frac{1}{2} \epsilon^{\hat{i}\hat{j}} \partial_i X^{\hat{\mu}} \partial_j X^{\hat{\nu}} b_{\hat{\mu}\hat{\nu}} \right\}, \quad (3.31)$$

where $\hat{i}, \hat{j} = 0, 1$. This is the Dirac-Nambu-Goto action for the string in curved background with a Wess-Zumino interaction term corresponding to the antisymmetric field $b_{\hat{\mu}\hat{\nu}}$. What we have done is to wrap the membrane around the compact dimension, choosing the radius of this tenth dimension to be small, or to be more precise, zero. This procedure truncates the fields to the zero modes of the Kaluza-Klein excitations. One can generalize the results to hold for the supermembrane as well by introducing supervielbeins, E_M^A , and the pullback of them to the world-volume, $E_i^A = \partial_i Z^M E_M^A$, where $Z^M = (X^m, \theta^{\bar{\alpha}})$. We gauge partially by setting $E_i^{D-1} = \delta_{i,2}$ and letting the other fields be independent of ξ^2 . We also make the Kaluza-Klein reduction

$$\begin{aligned} E_M^A &= \begin{pmatrix} E_{\hat{M}}^{\hat{\mu}} & E_{\hat{M}}^{10} & E_{\hat{M}}^{\alpha} \\ E_{10}^{\hat{\mu}} & E_{10}^{10} & E_{10}^{\alpha} \end{pmatrix} \\ &= \Phi^{-1/3} \begin{pmatrix} \hat{E}_{\hat{M}}^{\hat{\mu}} & \Phi A_{\hat{M}} & \hat{E}_{\hat{M}}^{\alpha} + A_{\hat{M}} \chi^{\alpha} \\ 0 & \Phi & \chi^{\alpha} \end{pmatrix}, \end{aligned} \quad (3.32)$$

where A_M is the $U(1)$ charged superfield, χ^α and Φ is the superfields whose leading components are the dilatino and dilation, respectively. Moreover, we split the Wess-Zumino term as

$$\begin{aligned} C_{ABC} &= (C_{\hat{A}\hat{B}\hat{C}}, C_{\hat{A}\hat{B}10}) \\ &= (\hat{C}_{\hat{A}\hat{B}\hat{C}}, B_{\hat{A}\hat{B}}). \end{aligned} \quad (3.33)$$

Inserting this ansatz into the action eq. (2.46), written in terms of supervielbeins gives

$$S_2 = - \int d^3\xi \left\{ \sqrt{-\det E_i^A E_j^B \delta_{AB}} - \frac{1}{3!} \epsilon^{ijk} E_i^A E_j^B E_k^C C_{ABC} \right\}, \quad (3.34)$$

and integrating out the ξ^2 -dependence yields

$$S'_1 = - \int d^2\xi \left\{ \sqrt{-\det(\hat{E}_i^{\hat{A}} \hat{E}_j^{\hat{B}} \delta_{\hat{A}\hat{B}})} - \frac{1}{2!} \epsilon^{\hat{i}\hat{j}} \hat{E}_i^{\hat{A}} \hat{E}_j^{\hat{B}} B_{\hat{A}\hat{B}} \right\}, \quad (3.35)$$

This is the GS-action of the superstring in a curved background [97]. If one studies the kappa-invariance of the reduced action, one finds that it is invariant if the background fields satisfy the on-shell field constraints of type IIA supergravity in $D = 10$. Thus, the theory we have found is the type IIA string theory.

Let us connect these considerations to the perturbation theory treated in paper I using the lightcone gauge, and generalized to hold also for covariant² approach using BRST charge for the bosonic membrane in paper II. In these articles we considered the opposite limit, where the ξ^2 -direction of the membrane covers a large part of the X^{D-1} -direction of space-time. In this limit we also need the membrane tension to be small which, if one analyzes the arguments above, it is the opposite limit.

Let me also make a comment about another limit in the double-dimensional reduction of the bosonic membrane in flat background. We choose the parameterization of the compact dimension as $X^{D-1} = \xi^2 \in [0, 2\pi R]$. Expand up to linear order in the fields

$$X^{\hat{\mu}}(\xi^2) = x^{\hat{\mu}} + y^{\hat{\mu}} R \xi^2, \quad (3.36)$$

²This is covariant in $D - 1$ directions because one has partially reduced the system.

where $x^{\hat{\mu}}$ and $y^{\hat{\mu}}$ are independent of ξ^2 . If we expand the determinant of the induced metric to order two in R and linear in ξ^2 , insert the ansatz into eq. (2.38) and integrate out the ξ^2 -dependence, one finds

$$S_{rs} = T_1 \int d^2\xi \sqrt{-\det h_{ij}} \left\{ 1 + h^{ij} \partial_i x^{\hat{\mu}} \partial_j B_{\hat{\mu}} + \frac{1}{2} \alpha T_1 [B^{\hat{\mu}} B_{\hat{\mu}} - B_{\hat{\mu}} \partial_i x^{\hat{\mu}} h^{ij} B_{\hat{\nu}} \partial_j x^{\hat{\nu}}] \right\} \quad (3.37)$$

where $h_{ij} \equiv \partial_i x^{\hat{\mu}} \partial_j x^{\hat{\mu}}$, $B^{\hat{\mu}} \equiv \pi R^2 y^{\hat{\mu}}$, $T_1 \equiv 2\pi R T_2$ and $\alpha \equiv (2\pi^3 R^3 T_2)^{-1}$. This is the rigid string proposed by Polyakov [76], written in the setting of [98]. This limit was discovered in [99].

3.3 Arguments for a critical dimension

I will here discuss the results of [100, 101], which, prior to our work, claimed critical dimensions for the bosonic membrane and the supermembrane. We will here only consider the bosonic membrane and mention the general features of the computations. First one gauge fixes the membrane action in eq. (2.38) completely using the lightcone gauge and a static gauge for ξ^2 ,

$$X^{D-2} = \xi^2. \quad (3.38)$$

When quantizing the theory, phase-space functions turn into operators. One then needs a prescription of how to order the operators. The prescription which was outlined in [100, 101] is to make a Weyl ordering between the phase-space operators

$$W(X^I P_J) = \frac{1}{2} [X^I P_J + P_J X^I]. \quad (3.39)$$

For the other products of fields one uses the concept of point splitting

$$A(\xi)B(\xi) \rightarrow A(\xi + 1/2\epsilon)B(\xi - 1/2\epsilon). \quad (3.40)$$

The most general expansion of the $X^I X^J$ and $\mathcal{P}_I \mathcal{P}_J$ operators, which is $SO(D-3)$ invariant, is

$$\begin{aligned} X^I X^J(\xi, \epsilon) &= Z_x(\epsilon)[X^I X^J]_R(\xi, \epsilon) + \delta^{IJ} s_x(\epsilon) + R_x(\epsilon, \xi) \\ \mathcal{P}_I \mathcal{P}_J(\xi, \epsilon) &= Z_p(\epsilon)[\mathcal{P}^I \mathcal{P}^J]_R(\xi, \epsilon) + \delta_{IJ} s_p(\epsilon) + R_p(\epsilon, \xi), \end{aligned} \quad (3.41)$$

where the regularization functions are symmetric in ϵ . In [100] one required the operators to satisfy

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \frac{s(\epsilon)}{Z(\epsilon)} &\rightarrow \infty \\ \lim_{\epsilon \rightarrow 0} Z(\epsilon) &\neq 0 \\ \lim_{\epsilon \rightarrow 0} R(\epsilon) &= 0. \end{aligned} \quad (3.42)$$

By these expansions one can define normalized operators, e.g.

$$[X^I X^J]_R(\xi) = \lim_{\epsilon \rightarrow 0} \frac{1}{Z_x(\epsilon)} (X^I X^J(\xi, \epsilon) - \delta^{IJ} s_x(\epsilon)). \quad (3.43)$$

The function Z_x can be seen as a field renormalization of X^I

$$X_R^I = Z_x^{-1/2} X^I, \quad (3.44)$$

and the same way for \mathcal{P}^I . Since one has reduced the system, one of the problems to have a consistent quantum theory is the closure of the Lorentz algebra. In [100] one studied a reduced part of the Lorentz algebra. Thus, one only checked the closure of the operators which were truncated to products of only three fields. The interesting one is

$$\begin{aligned} \tilde{M}^{-I} &= -\frac{1}{P^+} \int d^2 \xi \left\{ \frac{1}{2} [X^I \mathcal{P}_J \mathcal{P}^J]_R + \frac{1}{2} [X^I \partial_1 X_J \partial_1 X^J]_R \right. \\ &\quad \left. + [K^I \mathcal{P}_J \partial_1 X^J]_R \right\}. \end{aligned} \quad (3.45)$$

where \tilde{M}^{-I} is the truncated Lorentz generator and K^I is defined by $P^I = \partial_1 K^I$. Computing the commutator between \tilde{M}^{-I} and \tilde{M}^{-J} yields that it is zero only if

$D = 27$. This is only a necessary condition for an anomaly free Lorentz algebra for the quantum membrane.

Let us make some general comments on these calculations. First, and foremost, the ordering and regularization of operators might not be consistent with a choice of a physical ground-state having finite energy. Only if one can show that such a ground-state exists does the ordering prescribed here make sense. The calculations of the critical dimension in paper I and paper II, yielding $D = 27$ for the bosonic membrane, prescribes an ordering for which there exist physical ground-states with finite energy. The particular ordering is defined perturbatively and is highly non-trivial in terms of the original fields. Furthermore, is not the same ordering as in [100, 101].

Chapter 4

M(atr ix)-theory

Let us discuss some of the known properties of M-theory. One knows that M-theory in the field theory limit is the N=1 supergravity action in D=11 dimensions. The massless particles in this supergravity limit are the graviton $g_{\mu\nu}$, an antisymmetric three form $C_{\mu\nu\lambda}$, and a Majorana gravitino ψ_α . Thus, these have to exist for the microscopic theory as well. All in all, there are 44+84 bosonic and 128 fermionic degrees of freedom. Furthermore, one knows some of objects that exist in M-theory, there is a supermembrane and an M5-brane which couples electrically and magnetically to a three-form potential. In addition to these there also exists a KK11 monopole, which is a six-dimensional brane with an extra isometry in one space direction.

M-theory also has to yield the perturbatively formulated type IIA string theory when compactified on a circle with small radius. Furthermore, one needs to match the different objects which exist in the type IIA theory. A few of these one can find by looking at wrapped or unwrapped objects in M-theory. The wrapped supermembrane yields the fundamental string, while the unwrapped supermembrane is the D2-brane. The wrapped M5-brane is the D4-brane and the unwrapped M5-brane is the NS5-brane of the type IIA theory. Other objects in the theory, for instance the D0-brane and D6-brane and the A_μ -field, are constructed in a different way. First, the D0-brane is the supergraviton with a momentum $1/R$ in the compact direction, R is the

radius of the compactified dimension. This state couples to the A_μ -field, which is a part of the Kaluza-Klein reduction of the metric, see eqs. (3.29) and (3.32). The D6-brane arises from the KK11 monopole solution of the $D = 11$ supergravity action. The D6-brane is the brane where the extra isometry direction of the Kaluza-Klein monopole is compactified. This brane couples magnetically to the A_μ Kaluza-Klein field. For a description of the relation between different M-branes in M-theory and D-branes in type IIA string theory, see for example [28]. It should also yield the heterotic $E_8 \oplus E_8$ -theory when compactified on S^1/\mathbb{Z}_2 [102].

4.1 BFSS-conjecture

In 1996 Banks, Fischler, Shenker and Susskind presented a conjecture [103], later called the BFSS-conjecture or simply the matrix conjecture. This conjecture is based on the following facts about M-theory compactified on a space-like circle, $X^{10} = X^{10} + 2\pi R$. If we let the momentum around the compactified direction, $p_{10} = N/R$, be non-zero then the states are N bounded D0-branes in the type IIA picture. If we go to the infinite momentum frame [104, 105] such that $p_{10}, R, N \rightarrow \infty$, the conjecture states that this will capture all the degrees of freedom of M-theory in this frame. As one can, by Lorentz invariance, transform any state in M-theory to this frame the conjecture should capture all degrees of freedom of M-theory. In this limit fundamental string excitations decouple except the open massless strings which end on these D0-branes. Thus, M-theory would be described by sets of infinite stacks of D0-branes which interact by open strings ending on them. In the limit when p_{10} goes to infinity the D0-brane dynamics would be non-relativistic. A matrix model for N D0-branes for low energies, thus non-relativistic, is [106]

$$H = \frac{R}{2} \text{Tr} \left[P^I P^I + \frac{1}{2} [X^I, X^J] [X^I, X^J] + 2\theta^T \gamma_I [\theta, X^I] \right], \quad (4.1)$$

which is the same matrix model previously studied for the membrane action [60]. Here R is the radius of the extra dimension and, in addition, this Hamiltonian is

written in string units, $l_s \equiv \lambda^{-1/3} l_p$, where l_p is the ten-dimensional Planck length and $\lambda = e^{\phi_0}$. The Lagrangian corresponding to this Hamiltonian is

$$L = \frac{1}{2} \text{Tr} \left[\frac{1}{R} D_t X^I D_t X^I - \frac{1}{2} [X^I, X^J] [X^I, X^J] - 2\theta^T D_t \theta - 2R\theta^T \gamma_I [\theta, X^I] \right], \quad (4.2)$$

where $D_t = \partial_t + iA$. This action one can use to compute interactions for D0-branes. The claim that this action will capture the microscopic degrees of freedom of uncompactified M-theory is based on the following facts. First, it captures the known long-range interactions between two supergravitons [107]. Also, one finds that there exist classical membrane states in the spectrum.

This conjecture also puts the results of the continuous spectrum of supermembranes on a different footing. Since M-theory should be described as a multi-particle and an interacting theory, it is a second quantized theory from the outset. Therefore, the continuous mass-spectrum is a consequence of the masses of the intermediate states of the theory. Furthermore, the discrete spectrum found for the bosonic membrane shows that this membrane is not consistent with this picture.

This conjecture also has a few difficulties. One of them is to show that one has 11-dimensional Lorentz invariance in the limit $N \rightarrow \infty$. This has been shown to be true for the classical theory in [79], where the deviation from the closure of the algebra, at finite N , is shown to vanish in the limit.

4.2 DLCQ-conjecture, finite N -conjecture

The discrete lightcone quantization conjecture (DLCQ-conjecture) arose shortly after the BFSS-conjecture. Susskind [108] initiated the work that the conjecture, described in the previous section, would by reformulating it, also describe M-theory for finite N . This work was continued, and strengthened, by Sen [109] and Seiberg [110].

This conjecture is based on a matrix model compactified on a light-like circle

instead of a space-like one

$$X^+ = X^+ + 2\pi R, \quad (4.3)$$

which quantized yields that the allowed values of P^- are discrete, $P^- = N/R$. That this model would describe M-theory even at finite N is argued in [108] using the following arguments. The first is that if one compactify the theory on a small space-like circle, with radius R' , one will get the type IIA string theory compactified on a time-like circle, the DLCQ of the string. This argument is supported by the identification of the supersymmetric Yang-Mills theory at infinite coupling to the superconformal fixed point theory [111, 112]. The latter yields a free theory which can be identified as the DLCQ-string. Another argument is that if one compactify the theory on T^n , $n \leq 6$, T- and S-dualities exist even for finite N [109, 110].

4.3 Consistency checks of the conjecture

In this last section I will discuss the problems one has found in the matrix conjecture. One of the tests of the conjecture is to compute the interactions between D0-branes separated by a large distance and compare the results which one has for supergravity in $D = 11$. In doing these computations one finds an interesting feature of the matrix model. In computing the leading, non-trivial, order interactions between two D0-branes one has to make a one-loop quantum mechanical computation. This is quite surprising because the leading order supergravity computation is a classical computation for the linearized theory.

What one has found is that these computations match at one-loop, for an early computation see [107, 113] and for a review see [44]. They also match at two-loops, [114, 115]. To these orders one has proved non-renormalisation theorems which protect the results [116, 117]. But, one has also found disparities between the supergravity computations and matrix model computations. These are the computations made by Dine *et al* [118] which, for a three-loop computation for $N = 4$, yield differences between the two. This suggests that the matrix conjecture might not be

correct, or at least, that the $N \rightarrow \infty$ limit might be more problematic than expected. Another critical issue is the Lorentz invariance of the theory. The theory is only invariant in this limit and checking if this is true or not is difficult. One has, at least classically, shown that one recovers the full Lorentz algebra [79], but doing this quantum mechanically seems to be a difficult task. The results of [118] may also indicate that one has to add terms to this matrix model for it to be Lorentz invariant. This would prove the conjecture wrong.

These considerations indicate that one, in the end, needs to formulate M-theory in the continuum. To do this one needs a non-perturbative formulation of string theory. This is a difficult task and only a few ingredients are known for such a formulation. One can also attack the problem in a different way. Starting from the eleven-dimensional supergravity [64] and compute higher order derivative corrections to the action using known ingredients from string theory. Another approach is to use symmetries of the action and Bianchi identities to get relations for the higher derivative terms of eleven-dimensional supergravity, see [119] and references therein concerning higher order derivative terms of supergravity theories.

Chapter 5

Strings on curved backgrounds

So far in the thesis, only string theory in flat background and results from it has been treated. The rest of the thesis will discuss backgrounds which are non-trivial, but where the background is restrictive enough such that the quantized theory is well-defined and even almost determined fully. Before discussing concrete models, I will motivate them by going through some more general properties of string theory on non-trivial backgrounds and see what restrictions string theory has on them. This chapter will deal with bosonic strings, and will only briefly discuss generalizations to world-sheet supersymmetric string theories. The first section of this chapter is based on Polchinski [27] and lecture notes by D'Hoker published in [120]. The section on Lie algebras is based on the book by Fuchs and Schweigert [121], but is quite standard and can be found in many different books. The part where WZNW-models and affine Lie algebras are discussed is based on Fuchs [122], Fuchs and Schweigert [121], Di Francesco, Mathieu and Sénéchal [123] and Kac [124], apart from original references.

5.1 Non-trivial backgrounds in string theory

One interesting feature of string theory is that it yields non-trivial constraints on the background. The first example has already been discussed, that the string in flat

background sets constraints on the number of space-time dimensions. Let us briefly discuss other conditions that the string model imposes on the background. Consider here the bosonic closed string and, henceforth, disregard the problems of the model connected to the existence of a tachyon in the spectrum. One straightforward generalization of the action in eq. (2.2) is to introduce a non-flat background, exchanging $\eta_{\mu\nu}$ with $g_{\mu\nu}$. One can introduce further non-trivial background fields, namely the antisymmetric field, $B_{\mu\nu}$, and the dilaton, ϕ . The action will be of the form

$$S_{p,general} = \frac{1}{4\pi\alpha'} \int_{\Sigma} d^2\xi \sqrt{\gamma} (\gamma^{ij} g_{\mu\nu}(X) + \epsilon^{ij} B_{\mu\nu}(X)) \partial_i X^\mu \partial_j X^\nu + 2\alpha' R\phi(X), \quad (5.1)$$

where R is the Gaussian curvature of the world-sheet, which when integrated, is connected to the Euler characteristic of the world-sheet. One can see that these fields all correspond to massless excitations of the closed string by assuming that the background deviates only a little from the flat background. This shows that the theory described by eq. (5.1) is the same theory as the string formulated in flat background, as it is connected by some non-trivial excitation of closed string states. The action above is invariant under general coordinate transformations in space-time and on the world-sheet. One further invariance is the $U(1)$ -invariance of $B_{\mu\nu}$, $B_{\mu\nu} \rightarrow B_{\mu\nu} + \partial_\mu \lambda_\nu - \partial_\nu \lambda_\mu$. Because the model should be conformal, the action should be invariant under Weyl rescalings. But the action in eq. (5.1) is not since the part involving the dilaton is not. One can make the action invariant by combining a Weyl transformation and a local coordinate transformation of space-time. This yields that the background has to satisfy the equations

$$\begin{aligned} \beta_{\mu\nu}^g &= \nabla_\mu \nabla_\nu \phi = 0 \\ \beta_{\mu\nu}^B &= \frac{1}{2} \nabla^\kappa \phi H_{\kappa\mu\nu} = 0 \\ \beta_{\mu\nu}^\phi &= 2\nabla^2 \phi = 0, \end{aligned} \quad (5.2)$$

where ∇_μ is the covariant derivative and $H = dB$, which in components is $H_{\mu\nu\kappa} = \partial_\mu B_{\nu\kappa} + \partial_\nu B_{\kappa\mu} + \partial_\kappa B_{\mu\nu}$. This is the classical contribution to the Weyl transformations.

To get the quantum contributions of a Weyl rescaling one has to treat the action in eq. (5.1) differently. This is a two-dimensional non-linear sigma model which is a renormalizable theory. To get the equations which the background has to satisfy, without making any expansions, is in general an impossible task. Therefore, one usually makes a low energy expansion of the background field and treats the problem order by order. The low energy expansion is an expansion of α' as the limit $\alpha' \rightarrow 0$ corresponds to a limit where the all internal degrees of freedom, except for massless states, of the string are frozen. Expanding the background fields and treating the corresponding theory as a two-dimensional quantum field theory one gets equations which the background has to satisfy to be invariant under Weyl rescalings. The result to first order in $\beta_{\mu\nu}^g$ and $\beta_{\mu\nu}^B$ and first and second order in $\beta_{\mu\nu}^\phi$ is¹

$$\begin{aligned}\beta_{\mu\nu}^g &= \frac{1}{2}R_{\mu\nu}^g - \frac{1}{8}H_{\mu\kappa\sigma}H_{\nu}{}^{\kappa\sigma} + \nabla_\mu\nabla_\nu\phi = 0 \\ \beta_{\mu\nu}^B &= -\frac{1}{4}\nabla^\kappa H_{\kappa\mu\nu} + \frac{1}{2}\nabla^\kappa\phi H_{\kappa\mu\nu} = 0 \\ \beta_{\mu\nu}^\phi &= \frac{D-26}{6} + \alpha' \left(2\nabla_\mu\phi\nabla^\mu\phi - 2\nabla_\mu\nabla^\mu\phi - \frac{1}{2}R^g + \frac{1}{24}H^2 \right) = 0,\end{aligned}\quad (5.3)$$

where the contribution from the ghost system, $-13/3$, has been added. Assume now that the background is a generic background, then the only possibility to satisfy the equations above is to set $D = 26$ as α' is an expansion parameter. The equations (5.3) can be viewed as equations of motion for some 26-dimensional theory. The action which yields these equations of motion is

$$I_{26} = \frac{1}{2\kappa^2} \int d^{26}x \sqrt{-g} e^{-2\phi} \left(R^g + 4\nabla_\mu\phi\nabla^\mu\phi - \frac{1}{12}H^2 \right), \quad (5.4)$$

where $g = \det(g_{\mu\nu})$. This is the particle approximation of the string, called the gravity approximation of the closed bosonic string theory. This is the NS-NS part of the supergravity approximations of string theory, eq. (2.59). This was discussed in detail in papers [125–128].

¹The derivation of these coefficients is out of scope of the thesis, but can be found in D'Hoker's lecture notes [120].

As we now have studied backgrounds in general, let us discuss the issue that string theory needs more space-time dimensions than four. One way of interpreting the unobserved extra dimensions is that they are too small to have been discovered. This assumes a split in the metric² as

$$g_{\mu\nu} = \begin{pmatrix} \hat{g}_{\hat{\mu}\hat{\nu}} & 0 \\ 0 & g_{mn} \end{pmatrix}. \quad (5.5)$$

The non-compact part of this split will be discussed in chapter 7. In this and the next chapter the focus will be on the compact part of the theory. One can exchange the compactified part of the theory by any conformal field theory with $c = 22$ or $c = 9$ for the bosonic and supersymmetric string theory, respectively. For compactifications of the superstring, which will not be discussed, one usually want to partially keep the supersymmetries unbroken such that the theory in four dimensions has $N = 1$ supersymmetry. This leads to the condition that the compact manifold must be Kähler. Another condition one can impose is that $H = 0$ which fixes the manifold to be Ricci flat. This yields that the six-dimensional manifold is Calabi-Yau. These manifolds are highly restrictive and results from differential geometry give the massless fields and interactions in the four-dimensional non-compactified space. This is a subject of its own and will not be treated in the thesis.

Let us, henceforth, focus on the bosonic string and make the restriction that the compact space is a group manifold, or connected to a group manifold. The model we consider will, from the start, give consistent backgrounds in which the string can propagate. Before discussing the model, we give a few basic properties of finite dimensional Lie algebras.

²We have here made a simplification; compare with eq. (3.29), one general reference on Kaluza-Klein compactifications is [67].

5.2 Some basic facts of Lie algebras

Before discussing strings on group manifolds, consider Lie algebras and the case of (semi)simple Lie algebras which are essential to the models we treat in the rest of the chapters. A Lie algebra, denote it by \mathfrak{g} , is an algebra with a Lie bracket, $[\cdot, \cdot]$, which is antisymmetric and satisfies the Jacobi identity. A simple Lie algebra is a Lie algebra which has no proper ideals and is not abelian³. Semisimple Lie algebras are direct sums of simple algebras. Define the killing form by

$$\kappa(x, y) \equiv \text{Tr}(\text{ad}_x \circ \text{ad}_y), \quad (5.6)$$

where ad is the adjoint action defined as $\text{ad}_x(y) = [x, y]$ and \circ is the composition of maps. The killing form is bilinear, symmetric and satisfies the invariance property, $\kappa(x, [y, z]) = \kappa([x, y], z)$.

For a semisimple Lie algebra, the killing form is invertible, which is used to classify the simple Lie algebras. To classify the simple Lie algebras one usually uses a special basis, the Cartan-Weyl basis. In this basis one first chooses the largest set of linearly independent generators which span an abelian subalgebra. Denote the abelian subalgebra by \mathfrak{g}_0 and the generators by H^i , and the dimension of this subalgebra by r . This number is independent of the choice of abelian subalgebra and is called the rank of the algebra. The other elements can be chosen to be eigenoperators of H^i

$$\begin{aligned} [H^i, E^\alpha] &= \alpha(H^i) E^\alpha \\ &= \alpha^i E^\alpha. \end{aligned} \quad (5.7)$$

Where α are non-zero r -dimensional vectors called roots. Denote the set of all roots by $\Delta^{\mathfrak{g}}$; or Δ in short. The properties of the roots are that they are nondegenerate and the only multiples of roots in Δ are $\pm\alpha$. Using the algebra and the killing form,

³For an abelian Lie algebra all brackets vanish. An ideal \mathfrak{J} is defined as $[\mathfrak{g}, \mathfrak{J}] \subseteq \mathfrak{J}$

one can define an inner product on the root space

$$\begin{aligned} (\alpha, \beta) &\equiv \frac{\alpha(H^\beta)}{\kappa(E^\beta, E^{-\beta})} \\ &= \frac{\kappa(H^\alpha, H^\beta)}{\kappa(E^\alpha, E^{-\alpha})\kappa(E^\beta, E^{-\beta})}. \end{aligned} \quad (5.8)$$

The space is Euclidian as $\kappa^{ij} = \frac{1}{I_{ad}} \sum_{\alpha \in \Phi} \alpha^i \alpha^j$. One can here also fix the normalization of E^α by letting $\kappa(E^\alpha, E^{-\alpha}) = 2/(\alpha, \alpha)$. As the root space is Euclidian, one can define positive and negative roots by introducing a hyperplane in the root space and define the roots on one side as positive, denote this set of roots by Δ_+ and the corresponding algebra by \mathfrak{g}_+ . The rest of the roots are negative, denoted this set by Δ_- and the corresponding algebra by \mathfrak{g}_- ⁴. One can also define the Weyl vector as half the sum of the positive roots, $\rho \equiv \frac{1}{2} \sum_{\alpha > 0} \alpha$. Among the positive roots one can define simple roots as $\Delta_s = \{\alpha^{(i)} \in \Delta_+ | \alpha^{(i)} - \alpha \neq \Delta_+ \forall \alpha \in \Delta_+\}$. There are r simple roots and they span the root space. These are also the roots which are closest to the hyperplane. One can choose the elements in the Cartan subalgebra such that $\kappa(H^i, H^j) = (\alpha^{(i)\vee}, \alpha^{(j)\vee}) \equiv G^{ij}$ where the coroot is defined by

$$\alpha^\vee \equiv \frac{2\alpha}{(\alpha, \alpha)}. \quad (5.9)$$

The fundamental weights $\Lambda_{(i)}$ are defined as $(\Lambda_{(i)}, \alpha^{(j)\vee}) = \delta_i^j$. As the simple roots span the root space, all roots can be written as $\beta = \sum_i b_i \alpha^{(i)}$, which can be used to introduce the height, $h = \sum_i b_i$, of a root. For simple Lie algebras there is a unique root of maximal height called the highest root. Denote this root by θ . This root satisfies

$$(\theta, \theta) \geq (\alpha, \alpha) \quad \forall \alpha \in \Delta \quad (5.10)$$

One can use this root to fix the normalization of the roots, one standard choice is $(\theta, \theta) = 2$, which simplifies expressions. We now have

$$[H^i, E_\pm^j] = \pm A^{ji} E_\pm^j \quad (5.11)$$

⁴In short, one can use $\alpha > 0$ and $\alpha < 0$ to denote positive and negative roots, respectively.

where $E_{\pm}^j \equiv E^{\pm\alpha^{(j)}}$ and the Cartan matrix $A^{ij} \equiv (\alpha^{(i)}, \alpha^{(j)\vee})$, which satisfies

$$\begin{aligned} \det(A^{ij}) &> 0 \\ A^{ii} &= 2 \\ A^{ij} &\in \mathbb{Z} \\ A^{ij} &\leq 0 \quad i \neq j \\ A^{ij} = 0 &\Leftrightarrow A^{ji} = 0. \end{aligned} \tag{5.12}$$

One can use these properties, together with the property that the matrix A^{ij} is not block diagonal, to classify all simple Lie algebras. There are four large classes, $A_r \cong \mathfrak{sl}(r+1)$ ($r \geq 1$), $B_r \cong \mathfrak{so}(2r+1)$ ($r \geq 2$), $C_r \cong \mathfrak{sp}(2r)$ ($r \geq 3$) and $D_r \cong \mathfrak{so}(2r)$ ($r \geq 3$) and five isolated E_6, E_7, E_8, G_2 and F_4 . The Cartan matrix of the different cases is summarized in the Dynkin diagrams in Figure 5.1. Here the node corresponds to the Cartan subalgebra (and simple roots). The number of lines between each node is determined by $\max\{|A^{ij}|, |A^{ji}|\}$ and the arrow is pointing from the largest absolute value to the smallest. Let us summarize and give the non-zero Lie brackets in the Cartan-Weyl basis

$$\begin{aligned} [H^i, E^\alpha] &= \alpha^i E^\alpha \\ [E^\alpha, E^\beta] &= e_{\alpha,\beta} E^{\alpha+\beta} + \delta_{\alpha+\beta,0} \alpha^\vee_i H^i, \end{aligned} \tag{5.13}$$

where $e_{\alpha,\beta} \neq 0$ if $\alpha + \beta \in \Delta_+$.

In Physics the abstract algebras are usually not interesting; instead it is how they act on suitable spaces, usually physical states. A restriction of the possible representations of the Lie algebra is that the space which they act on is a vector space. This makes the representation of a specific element a linear map. Let us be less abstract and focus on one specific type of representation, so-called highest weight representations. These representations are constructed from a highest weight state, denote it by $|\mu\rangle$. This state satisfies

$$\begin{aligned} R(H^i) |\mu\rangle &= \mu^i |\mu\rangle \quad i = 1, \dots, r \\ R(E^\alpha) |\mu\rangle &= 0 \quad \alpha \in \Delta_+, \end{aligned} \tag{5.14}$$

where $R(\)$ denote a representation of the Lie algebra element; this will henceforth be suppressed. From the algebra one can see that $|\mu - \alpha\rangle = E^{-\alpha} |\mu\rangle$. By applying all creation operators, operators in \mathfrak{g}_- , one will get a representation called the Verma module. This is a representation of the universal enveloping algebra⁵. This algebra has one interesting element, which can be used to characterize the representations, the quadratic Casimir

$$C_2 = G_{ij} H^i H^j + \sum_{\alpha \in \Delta} \frac{(\alpha, \alpha)}{2} E^{-\alpha} E^{\alpha} \quad (5.15)$$

where G_{ij} is the inverse of G^{ij} . C_2 commute with all elements in \mathfrak{g} and will, therefore, have the same eigenvalue for all states in the representation of the Verma module. The eigenvalue is $(\mu, \mu + 2\rho)$. The representations here need not to be irreducible, as there might exist states, $|\phi\rangle$, such that $E^{\alpha} |\phi\rangle = 0 \ \forall \alpha \in \Delta_+$. These states are highest weight states as well, with the same value of the quadratic Casimir. As states in different highest weight representation decouple, these highest weight representation states should be excluded from the irreducible part of the representation. If the representation is to be finite dimensional, all $\mathfrak{sl}(2)$ representations should be finite dimensional. One can show that a necessary and sufficient condition is $(\alpha^{(i)\vee}, \mu) \in \mathbb{Z}_+$, therefore, the highest weight of these representations can be written as $\mu = \sum_i b^i \Lambda_{(i)}$ where $b^i \in \mathbb{Z}_+$. In Physics one is usually not interested in representations of the complex Lie algebra, which we have treated so far, but representations of a real form of the algebra. A way to get a few of the representation spaces of a real form is to start from the compact real form where all eigenvalues of the killing form are negative and write the complex Lie algebra as

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{R}} \oplus i\mathfrak{g}_{\mathbb{R}} \quad (5.16)$$

where $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{g}_{\mathbb{R}}$ denote the complex and compact real form, respectively. To get the relevant real form one can introduce Hermitian conjugation rules which are consistent

⁵In short, this algebra consists of arbitrary number of tensor products of the elements of the Lie algebra modulo elements of the form $t^a \otimes t^b - t^b \otimes t^a - [t^a, t^b]$ and can be written as $U(\mathfrak{g}_-) \times U(\mathfrak{g}_0) \times U(\mathfrak{g}_+)$

with the algebra. A choice which yields the correct representation spaces for the compact real form is

$$\begin{aligned} H^{i\dagger} &= H^i \\ E_{\pm}^{i\dagger} &= E_{\mp}^i. \end{aligned} \quad (5.17)$$

This implies that the finite dimensional representations are also unitary representations of the compact real form. This also gives unitary representations of the Lie group by exponentiation of the representation of the Lie algebra by

$$\exp[\xi_a R(t^a)], \quad (5.18)$$

where ξ_a are real parameters.

An interesting function, which one can define, is the character of a representation

$$\chi_{\mu}(\phi) = \sum_{\lambda} \text{mult}_{\mu}(\lambda) \exp[i(\lambda, \phi)], \quad (5.19)$$

where $\phi = \sum \phi_i \alpha^{(i)\vee}$, λ is a weight in the representation and $\text{mult}_{\mu}(\lambda)$ is the multiplicity of the weight λ in the representation. For dominant highest weight representations one can prove that this expression is equal to [129, 130]

$$\chi(\phi) = \exp[i(\mu, \phi)] \frac{\sum_{w \in W} \text{sign}(w) \exp[i(w(\mu + \rho) - \mu - \rho, \phi)]}{\prod_{\alpha > 0} (1 - \exp[-i(\alpha, \phi)]). \quad (5.20)$$

Here W is the Weyl group and w is an element in this group. Furthermore, we have the denominator identity

$$\sum_{w \in W} \text{sign}(w) \exp[i(w(\rho) - \rho, \phi)] = \prod_{\alpha > 0} (1 - \exp[-i(\alpha, \phi)]), \quad (5.21)$$

using the character of the trivial representation. A proof of eq. (5.20) is based on the Weyl invariance of the dominant highest weight representations and that the quadratic Casimir has the same eigenvalue for the embedded Verma modules, [131]. The Weyl group is generated by reflections in the root space. A reflection is defined as

$$w_{\alpha}(\lambda) \equiv \lambda - (\lambda, \alpha^{\vee}) \alpha. \quad (5.22)$$

The Weyl group is generated by the elements connected to the simple roots, called letters, and all elements in the group are written as products of these elements, called words. $\text{sign}(w) = 1$ or -1 for even or odd number of letters in the word⁶. Let us end by presenting another function which is interesting, the Shapovalov determinant [132]

$$\det [F_\lambda] = C \prod_{\alpha > 0} \prod_{n=1}^{\infty} \left((\mu + \rho, \alpha) - n \frac{(\alpha, \alpha)}{2} \right)^{\text{mult}(\lambda + n\alpha)} \quad (5.23)$$

where F_λ is a matrix consisting of the scalar product between all states with weight $\lambda = \mu - \sum_i n_i \alpha^{(i)}$ in the representation. This expression shows that another class of interesting representations, called antidominant representations [133], where $\mu^i + 1 < 0$, have an irreducible Verma module⁷.

5.3 WZNW-models

Let us now consider strings on group manifolds. Here, mainly bosonic strings are considered and only in the end a short discussion about a world-sheet supersymmetric extension of this model will be discussed. Strings on group manifolds are described by an action which is a sum of two parts, the first term is an action which describes a so-called principal sigma model associated to the group and the second one is a Wess-Zumino term [134]

$$\begin{aligned} S_\sigma &= \frac{1}{a^2} \int d^2 z \kappa (\partial g, \bar{\partial} g^{-1}) \\ S_{WZ} &= \frac{1}{24\pi} \int_{\mathcal{B}} d^3 y \epsilon^{abc} \kappa (\dot{g}^{-1} \partial_a \dot{g}, [\dot{g}^{-1} \partial_b \dot{g}, \dot{g}^{-1} \partial_c \dot{g}]), \end{aligned} \quad (5.24)$$

where g and \dot{g} take values in some Lie group with the difference that \dot{g} depends on three variables and satisfies $\dot{g}|_\Sigma = g$. Σ is the world-sheet and \mathcal{B} is the generalized ball with boundary being the world-sheet, $\partial \mathcal{B} = \Sigma$. There is an ambiguity in defining

⁶One other way to define the sign is by saying that if the Weyl element induces a rotation it is an even element, or a rotation and one reflection it is an odd element.

⁷There are also other representations which yield an irreducible Verma module.

\mathcal{B} , [135]. As Physics should not depend on this, the ambiguity has to vanish in the path integral. The ambiguity is equal to the WZ-term integrated over S^3 which gives 2π ⁸. Therefore, S_{WZ} can arise in the action only as kS_{WZ} where $k \in \mathbb{Z}$. Varying these two actions yields

$$\begin{aligned}
\delta S_\sigma &= \frac{1}{a^2} \int d^2 z \kappa (g^{-1} \delta g, \partial (g^{-1} \bar{\partial} g) + \bar{\partial} (g^{-1} \partial g)) \\
&= \frac{1}{a^2} \int d^2 z \kappa (\delta g g^{-1}, \partial (\bar{\partial} g g^{-1}) + \bar{\partial} (\partial g g^{-1})) \\
\delta S_{WZ} &= -\frac{k}{4\pi} \int d^2 z \kappa (g^{-1} \delta g, \partial (g^{-1} \bar{\partial} g) - \bar{\partial} (g^{-1} \partial g)) \\
&= -\frac{k}{4\pi} \int d^2 z \kappa (\delta g g^{-1}, \bar{\partial} (\partial g g^{-1}) - \partial (\bar{\partial} g g^{-1})), \tag{5.25}
\end{aligned}$$

where

$$\begin{aligned}
\int d^3 y \epsilon^{abc} \partial_c (\dots) &= \int d^2 \xi \epsilon^{ab} (\dots)|_\Sigma \\
d^2 \xi &= \frac{d^2 z}{2i z \bar{z}} \\
\epsilon^{z\bar{z}} &= 2i z \bar{z}, \tag{5.26}
\end{aligned}$$

has been used. If we require $\delta S = 0$ and put the equations in eq. (5.25) together, we find

$$\begin{aligned}
\left(\frac{1}{a^2} - \frac{k}{4\pi}\right) \partial (g^{-1} \bar{\partial} g) + \left(\frac{1}{a^2} + \frac{k}{4\pi}\right) \bar{\partial} (g^{-1} \partial g) &= 0 \\
\left(\frac{1}{a^2} - \frac{k}{4\pi}\right) \bar{\partial} (\partial g g^{-1}) + \left(\frac{1}{a^2} + \frac{k}{4\pi}\right) \partial (\bar{\partial} g g^{-1}) &= 0. \tag{5.27}
\end{aligned}$$

Thus, if $a^2 = 4\pi/k$ we get that

$$\bar{\partial} (g^{-1} \partial g) = \partial (\bar{\partial} g g^{-1}) = 0. \tag{5.28}$$

These equations show that the solutions to the equations of motion holomorphically factorize

$$g(z, \bar{z}) = \tilde{f}(\bar{z}) f(z). \tag{5.29}$$

⁸Except for the case of $SO(3)$, using the normalization for $SU(2)$.

This enlarges the global $G \times G$ invariance to a local $G(z) \times G(\bar{z})$ invariance

$$g(z, \bar{z}) \rightarrow \tilde{\Omega}(\bar{z})g(z, \bar{z})\Omega^{-1}(z). \quad (5.30)$$

The action which has this holomorphic factorization is called the Wess-Zumino-Novikov-Witten (WZNW) action⁹ [136]

$$S_{WZNW} = \frac{k}{4\pi} \int_{\Sigma} d^2z \kappa (\partial g, \bar{\partial} g^{-1}) + \frac{k}{24\pi} \int_{\mathcal{B}} d^3y \epsilon^{abc} \kappa (\dot{g}^{-1} \partial_a \dot{g}, [\dot{g}^{-1} \partial_b \dot{g}, \dot{g}^{-1} \partial_c \dot{g}]). \quad (5.31)$$

Let us now move to the quantum properties of this action. Consider the infinitesimal transformation

$$\delta_w g = -g(z, \bar{z})w(z), \quad (5.32)$$

The WZNW-action then transforms as

$$\begin{aligned} \delta_w S &= -\frac{k}{2\pi} \int d^2z \kappa (w(z), \bar{\partial} (g^{-1} \partial g)) \\ &= -\frac{1}{2\pi} \oint dz \kappa (w(z), J(z)) \end{aligned} \quad (5.33)$$

where $J(z) = k(g^{-1} \partial g)$. We can now use this to derive Ward identities using

$$\begin{aligned} 0 &= \delta \langle X \rangle \\ &= \int \mathcal{D}\Phi (X + \delta X) \exp[i(S + \delta S)] - \int \mathcal{D}\Phi (X) \exp[iS] \\ &= \langle \delta X \rangle + i \langle \delta S X \rangle, \end{aligned} \quad (5.34)$$

where X is a generic field insertion and we have assumed that δ is a symmetry transformation of S . Expanding $J = \sum_a t_a J^a$ and $w = \sum_a t_a w^a$ and using the above equation yields

$$\langle \delta_w J^b(z) \rangle = -\frac{1}{2\pi i} \oint dz' w_a(z') \langle J^a(z') J^b(z) \rangle. \quad (5.35)$$

⁹The action is also referred to as the Wess-Zumino-Witten (WZW) action.

Using

$$\delta_w J^a = f_{bc}{}^a w^b(z) J^c(z) - k \partial w^a(z), \quad (5.36)$$

gives

$$-\frac{1}{2\pi i} \oint dz' w_a(z') \langle J^a(z') J^b(z) \rangle = -f^{ab}{}_c w_a \langle J^c(z) \rangle - k \kappa_{ab} \partial w_a(z) \quad (5.37)$$

which has the solution

$$J^a(z') J^b(z) \sim \frac{f^{ab}{}_c J^c(z)}{z' - z} + \frac{k \kappa^{ab}}{(z' - z)^2}, \quad (5.38)$$

where \sim denotes equality up to regular terms. This is a representation of an affine Lie algebra at level k , which is shown by making a Laurent expansion of the current $J^a(z)$,

$$J^a(z) = \sum_{n \in \mathbb{Z}} J_n^a z^{-n-1}, \quad (5.39)$$

to get

$$[J_m^a, J_n^b] = f^{ab}{}_c J_{m+n}^c + m k \kappa^{ab} \delta_{m+n,0}. \quad (5.40)$$

For the other current, $\bar{J} = -k \bar{\partial} g g^{-1}$, the end result is the same and the two different current modes commute.

Let us focus now on the affine Lie algebra, which is a generalization of the Lie algebra discussed above. The Lie brackets of the algebra, in a Cartan-Weyl basis is

$$\begin{aligned} [H_m^i, H_n^j] &= m K G^{ij} \delta_{m+n,0} \\ [H_m^i, E_n^\alpha] &= \alpha^i E_{m+n}^\alpha \\ [E_m^\alpha, E_n^\beta] &= E_{m+n}^{\alpha+\beta} + \delta_{\alpha+\beta,0} \left(\alpha^\vee_i H_{m+n}^i + m \frac{2}{(\alpha, \alpha)} K \delta_{m+n,0} \right) \\ [D, H_m^i] &= m H_m^i \\ [D, E_m^\alpha] &= m E_m^\alpha. \end{aligned} \quad (5.41)$$

The rest of the brackets are zero. Most of the definitions and results for the finite dimensional Lie algebra can be generalized to the affine Lie algebra. One is that the Cartan subalgebra can be chosen as (H_0^i, K, D) . The definition of the killing form for the semisimple Lie algebra in eq. (5.6) is not well-defined for the affine Lie algebra, but can be generalized using its properties and the results for the zero mode to hold for the affine lie algebra. The non-zero parts is

$$\begin{aligned}\kappa(J_m^a, J_n^b) &= \kappa(J_0^a, J_0^b)\delta_{m+n,0} \\ \kappa(K, D) &= 1,\end{aligned}\tag{5.42}$$

where $J_m^a = \{H_m^i, E_m^\alpha\}$. The roots are defined in the same way as in the finite dimensional case, as the (non-zero) eigenvalues of the elements in the Cartan subalgebra. Therefore, the roots are all of the form

$$\hat{\alpha} = \begin{cases} (\alpha, 0, n) & \alpha \in \Delta \\ (0, 0, n) & n \in \mathbb{Z}, n \neq 0 \end{cases} .\tag{5.43}$$

Denote these roots by $\hat{\Delta}$. From the killing form one gets that the roots $(0, 0, n)$ are light like. One can also separate the roots into positive and negative roots,

$$\hat{\Delta}_+ = \left\{ (\alpha, 0, n) \in \hat{\Delta} : n > 0 \text{ or } n = 0, \alpha \in \Delta_+ \right\} .\tag{5.44}$$

Among the roots one can define simple roots $\alpha^{(i)}$ using the requirement that if $\hat{\alpha} > 0$ then $b_i \geq 0$ in a decomposition $\hat{\alpha} = \sum_{i=0}^r b_i \alpha^{(i)}$. Thus, the simple roots are of the form

$$\begin{aligned}\alpha^{(0)} &= (-\theta, 0, 1) \\ \alpha^{(i)} &= (\alpha^{(i)}, 0, 0) \quad \text{for } i = 1, \dots, r.\end{aligned}\tag{5.45}$$

One can define the Cartan matrix $A^{ij} \equiv (\alpha^{(i)}, \alpha^{(j)\vee})$. Which shows that one in the above prescription gets all non-twisted affine Lie algebras, see Figure 5.2. The twisted Lie algebras will not be used in this thesis. One cannot for the affine Lie algebra use the same definition for the Weyl vector, but instead use the property $(\alpha^{(i)\vee}, \hat{\rho}) = 1$

to define it. The result is $\hat{\rho} = (\rho, g^\vee, 0)$, where g^\vee is the dual coxeter number defined as half the eigenvalue of the Casimir operator in the adjoint representation.

The highest weight representations of the affine Lie algebra are a straightforward generalization of the finite dimensional case. The highest weight state is defined as

$$\begin{aligned} E^{\hat{\alpha}} |\hat{\mu}\rangle &= 0 & \hat{\alpha} \in \hat{\Delta}_+ \\ H^i |\hat{\mu}\rangle &= \mu^i |\hat{\mu}\rangle & i = 1, \dots, r \\ K |\hat{\mu}\rangle &= k |\hat{\mu}\rangle \\ D |\hat{\mu}\rangle &= 0. \end{aligned} \tag{5.46}$$

A subset of all representations are the integrable highest weight representations where all $\mathfrak{sl}(2)$ -representations corresponding to real roots are finite dimensional. Thus, for the highest weight one will get the condition that $(\hat{\alpha}^\vee, \hat{\mu}) \in \mathbb{Z}_+$. This yields that the conditions which the weight has to satisfy is $\mu^i \in \mathbb{Z}_+$ for $i = 1, \dots, r$ and $k - (\theta, \mu) \in \mathbb{Z}_+$. One can show that these representations are unitary representations of the compact real form of the affine Lie algebra, see for instance [124]. One can generalize the Shapovalov determinant to hold also for the affine case, the generalization is called the Shapovalov-Kac-Kazhdan determinant [137]. Using this one also finds that for antidominant representations, $(\hat{\mu} + \hat{\rho}, \alpha^{(i)}) < 0$, the Verma module is irreducible.

To discuss the characters, one needs to discuss the Weyl group of the affine Lie algebra. For the finite dimensional Lie algebras, the Weyl group is finite dimensional, but for the affine algebra the group is infinite dimensional. The letters of the Weyl group is defined from the simple roots; the reflections are defined in the same way as for the simple Lie algebras. The words and the sign of it, is defined as before. The Weyl group can be split into two parts, one associated to the finite dimensional algebra and a translation

$$w_{\hat{\alpha}} = w_\alpha (t_{\alpha^\vee})^n, \tag{5.47}$$

where $\hat{\alpha} = (\alpha, 0, n)$. The translation is defined as

$$t_{\alpha^\vee} = w_{(-\alpha, 0, 1)} w_\alpha \tag{5.48}$$

and satisfies

$$\begin{aligned} t_{\alpha^\vee} t_{\beta^\vee} &= t_{\alpha^\vee + \beta^\vee} \\ w_\alpha t_{\beta^\vee} w_\alpha^{-1} &= t_{w_\alpha(\beta^\vee)} \end{aligned} \quad (5.49)$$

This makes it possible to write a generic element in the affine Weyl group as an element in the Weyl group associated to the finite dimensional Lie algebra times a translation, t_{β^\vee} , where the β^\vee is an element in the coroot lattice, L^\vee , spanned by the coroots.

The character is defined in the same way as for the finite dimensional Lie algebra. One can prove, using the properties of the affine Lie algebra, which one can define a quadratic Casimir and Verma modules, to get

$$\chi_{\hat{\mu}}(\hat{\phi}) = \frac{\sum_{\hat{w} \in \hat{W}} \text{sign}(\hat{w}) \exp \left[i \left(\hat{\phi}, \hat{w}(\hat{\mu} + \hat{\rho}) \right) \right]}{\sum_{\hat{w} \in \hat{W}} \text{sign}(\hat{w}) \exp \left[i \left(\hat{\phi}, \hat{w}(\hat{\rho}) \right) \right]}, \quad (5.50)$$

where \hat{W} is the affine Weyl group. This is the Kac-Weyl character formula for affine Lie algebras which was proven in [131]. As discussed above, the affine Weyl group can be split into two pieces

$$\sum_{\hat{w} \in \hat{W}} \text{sign}(\hat{w}) \exp \left[i \left(\hat{\phi}, \hat{w}(\hat{\mu}) \right) \right] = \sum_{w \in W} \text{sign}(w) \sum_{\alpha^\vee \in L^\vee} \exp \left[i \left(\hat{\phi}, t_{\alpha^\vee}(w(\hat{\mu})) \right) \right]. \quad (5.51)$$

Defining the theta function as

$$\Theta_{w(\hat{\mu})}(\hat{\phi}) = \exp \left[-i(\hat{\mu}, \hat{\mu}) \frac{(\hat{\phi}, \delta)}{(\hat{\mu}, \delta)} \right] \prod_{\alpha^\vee \in L^\vee} \exp \left[i \left(t_{\alpha^\vee}(w(\hat{\mu})), \hat{\phi} \right) \right], \quad (5.52)$$

where $\delta = (0, 0, 1)$. Using this, the character can be written as

$$\chi_{\hat{\mu}}(\hat{\phi}) = \exp \left[i s_{\hat{\mu}}(\delta, \hat{\phi}) \right] \frac{\sum_{w \in W} \Theta_{w(\hat{\mu} + \hat{\rho})}(\hat{\phi})}{\sum_{w \in W} \Theta_{w(\hat{\rho})}(\hat{\phi})}, \quad (5.53)$$

where $s_{\bar{\mu}}$ is the modular anomaly of the representation. It has the form

$$s_{\bar{\mu}} = \frac{(\mu, \mu + 2\rho)}{2(k + g^\vee)} - \frac{1}{24}c \quad (5.54)$$

where c is defined as

$$c = \frac{kd}{k + g^\vee}, \quad (5.55)$$

and $d = \dim(g)$. In string theory one usually leaves out the contribution corresponding to $\exp[ic/24]$ in the character.

So, we now have all the ingredients needed to discuss further properties of WZNW-model. One interesting feature is that the Virasoro generators can be constructed from the currents, thus, are also elements in the universal enveloping algebra of the affine Lie algebra,

$$L_n = \frac{1}{2(k + g^\vee)} \left[\sum_m G_{ij} : H_{-m}^i H_{m+n}^j + \sum_m \frac{(\alpha, \alpha)}{2} : E_{-m}^{-\alpha} E_{m+n}^\alpha \right]. \quad (5.56)$$

This is the Sugawara construction of the energy-momentum tensor [138, 139] for which the correct prefactor was given in [140–143]. The commutators between the Virasoro generators and the generators of the affine algebra are

$$\begin{aligned} [L_m, J_n^a] &= -nJ_{m+n}^a \\ [L_m, L_n] &= (m - n)L_{m+n} + \frac{1}{12}cm(m^2 - 1)\delta_{m+n,0}, \end{aligned} \quad (5.57)$$

where the value of the conformal anomaly is given by eq. (5.55).

Let me end this chapter by discussing a few properties of the WZNW-model. One property of the compact real form is that one has, for all finite values of the level k , a finite number of primary fields. There is a correspondence between primary fields and states in the integrable highest weight representations of the horizontal part of the affine Lie algebra. For the WZNW-model, where one has holomorphic factorization, one can discuss each chirality separately, and apply the level matching condition $(L_0 - \bar{L}_0)|\phi\rangle$. Then the states corresponding to primary fields satisfy

$L_n |\phi\rangle = J_n^a |\phi\rangle = 0$ for $n > 0$, $L_0 |\phi\rangle = \Delta |\phi\rangle$, and $H_0^i |\phi\rangle = \mu^i |\phi\rangle$. Furthermore, $\Delta = \frac{(\mu_i \mu^i + 2\rho)}{2(k+g^\vee)}$. The finiteness of the number of primary fields is a property of a subclass of all conformal field theories called rational conformal field theories. As a comment, we have here implicitly assumed that the model is right-left invariant, thus the representations come in pairs, this is the *diagonal* WZNW-model or, *the* WZNW-model. This will implicitly be assumed henceforth as well.

One can also make the WZNW-model world-sheet supersymmetric by adding fermions to the model [144, 145]. The action can be written as

$$\begin{aligned} S_{SWZ\text{NW}} &= \frac{k}{4\pi} \int_{\Sigma} d^2z d^2\theta \kappa (DG, \bar{D}G^{-1}) \\ &+ \frac{k}{4\pi} \int_{\text{B}} d^2z dt d^2\theta \kappa \left(\dot{G} \partial_t \dot{G}^{-1}, D\dot{G} \bar{D}\dot{G}^{-1} + \bar{D}\dot{G} D\dot{G}^{-1} \right), \end{aligned} \quad (5.58)$$

where $G = g + i\theta\psi + i\bar{\theta}\tilde{\psi} + \theta\bar{\theta}F$, $D = \partial_{\theta}^L + \theta\partial$, $\bar{D} = \partial_{\bar{\theta}}^L + \bar{\theta}\bar{\partial}$ and G^{-1} is defined as $G^{-1}G = 1$. Furthermore $\dot{G}\Big|_{\Sigma} = G$. The equations of motion which arise from this action is $\bar{D}(G^{-1}DG) = D(\bar{D}GG^{-1}) = 0$. Solving the equations for the F -term, $F = -\tilde{\psi}g^{-1}\psi$, yields

$$\begin{aligned} \bar{\partial}(g^{-1}\psi) &= 0 \\ \partial(\tilde{\psi}g^{-1}) &= 0 \\ \partial(\bar{\partial}gg^{-1} - \tilde{\psi}g^{-1}\tilde{\psi}g^{-1}) &= 0 \\ \bar{\partial}(g^{-1}\partial g + g^{-1}\psi g^{-1}\psi) &= 0 \end{aligned} \quad (5.59)$$

Using the equation for the F -term and integrating out θ and $\bar{\theta}$ yields a simplified expression

$$S_{SWZ\text{NW}} = S_{WZ\text{NW}} + \frac{k}{4\pi} \int_{\Sigma} d^2z \left[\kappa (g^{-1}\psi, \partial(g^{-1}\psi)) + \kappa (\tilde{\psi}g^{-1}, \bar{\partial}(\tilde{\psi}g^{-1})) \right]. \quad (5.60)$$

Defining the supercurrents as $\mathcal{J} \equiv kG^{-1}DG \equiv i\lambda(z) + \theta J(z)$, and using a similar

analysis as above for the bosonic model, yields the algebra

$$\begin{aligned}
 [\lambda_r^a, \lambda_s^b] &= k\kappa^{ab}\delta_{r+s,0} \\
 [J_m^a, \lambda_r^b] &= f^ab{}_c \lambda_{m+r}^c \\
 [J_m^a, J_n^b] &= f^ab{}_c J_{m+n}^c + mk\kappa^{ab}\delta_{m+n,0},
 \end{aligned} \tag{5.61}$$

where $r, s \in \mathbb{Z} + 1/2$ or $r, s \in \mathbb{Z}$ if the fermions are Neveu-Schwarz or Ramond. Furthermore, the superconformal algebra is generated by

$$\begin{aligned}
 L_m &= \frac{1}{2k}\kappa_{ab} \sum_{n \in \mathbb{Z}} : J_{-n}^a J_{m+n}^a + \frac{1}{4k} \sum_{r \in \mathbb{Z} + \nu} (m+2r)\kappa_{ab} : \lambda_{-r}^a \lambda_{m+r}^b : \\
 G_r &= \frac{1}{k}\kappa_{ab} \sum_{n \in \mathbb{Z}} J_{-m}^a \lambda_{r+m}^b - \frac{2}{3k^2} f_{abc} \sum_{r,s \in \mathbb{Z} + \nu} : \lambda_{-s}^a \lambda_{-q}^b \lambda_{r+s+q}^c :,
 \end{aligned} \tag{5.62}$$

where $\nu = 0, 1/2$ for the Neveu-Schwarz and Ramond sector, respectively, and f_{abc} is defined from $f^ab{}_c$ by using the killing form. One can also simplify the algebra in eq. (5.61) by defining $\hat{J}^a \equiv J^a + \frac{1}{2k} f_{ab}{}^c \lambda^b \lambda^c$ which yield a representation of an affine algebra at level $k - g^\vee$ times a free fermion representation. This makes the spectrum of the models simple, but the world-sheet supersymmetry less explicit. The conformal anomaly for this model is

$$c = \frac{k - g^\vee}{k} d + \frac{1}{2} d \tag{5.63}$$

What has been discussed in this chapter are the WZNW-model at level $k \neq -g^\vee$. Aspects of the WZNW-model at the critical level has been analyzed in [146–148] in the setting of non-compact groups. WZNW-models of non-compact type will be discussed in the last chapter. This limit is thought of as a tensionless limit of the string moving on the group manifold.

Figure 5.1: Tables of Dynkin diagrams of simple Lie algebras

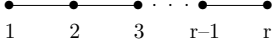
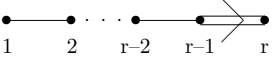
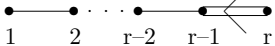
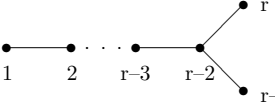
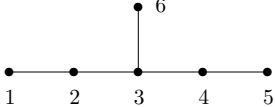
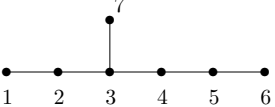
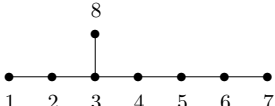
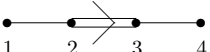


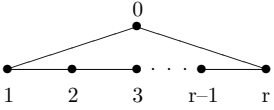
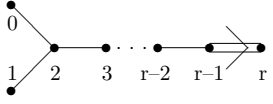
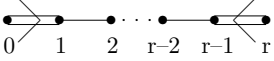
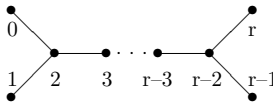
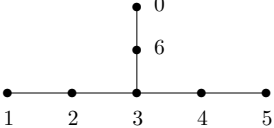
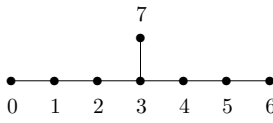
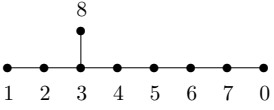
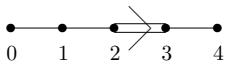
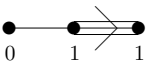
<p>Dynkin diagram of A_r:</p> 	<p>Dynkin diagram of B_r:</p> 
<p>Dynkin diagram for C_r:</p> 	<p>Dynkin diagram of D_r:</p> 
<p>Dynkin diagram of E_6:</p> 	<p>Dynkin diagram of E_7:</p> 
<p>Dynkin diagram of E_8:</p> 	<p>Dynkin diagram for F_4:</p> 
<p>Dynkin diagram for G_2:</p> 	

Figure 5.2: Tables of Dynkin diagrams of untwisted affine Lie algebras

<p>Dynkin diagram for $A_1^{(1)}$:</p> 	<p>Dynkin diagram of $A_r^{(1)}$:</p> 
<p>Dynkin diagram of $B_r^{(1)}$:</p> 	<p>Dynkin diagram for $C_r^{(1)}$:</p> 
<p>Dynkin diagram of $D_r^{(1)}$:</p> 	<p>Dynkin diagram of $E_6^{(1)}$:</p> 
<p>Dynkin diagram of $E_7^{(1)}$:</p> 	<p>Dynkin diagram of $E_8^{(1)}$:</p> 
<p>Dynkin diagram for $F_4^{(1)}$:</p> 	<p>Dynkin diagram for $G_2^{(1)}$:</p> 

Chapter 6

Gauged WZNW-models

The models treated in the last chapter have one restriction. They all have conformal charge larger than the rank of the algebra, $r \leq c < d$. Therefore, one cannot describe models which have conformal charge less than one. But there is a way out of this; one can subtract contributions to the energy-momentum tensor which makes it possible to get more conformal field theories. This was first developed by Goddard, Kent and Olive [149], the GKO construction, but has its roots in early history of string theory [150, 151]. This construction can also be generalized to world-sheet supersymmetric models by the Kazama-Suzuki model [152, 153]. The GKO construction will, in the compact case, be equivalent to a gauged WZNW-model, [154–164]. As the gauged WZNW-model is a theory with gauge invariance one can introduce ghosts [155] and find that the theory is invariant under a BRST symmetry transformation [161]. This also generalizes to the world-sheet supersymmetric model [165].

6.1 The GKO construction

One can in a simple way add contributions to the energy-momentum tensor by assuming that the individual contributions are independent. But let us see if one can subtract contributions to the energy-momentum tensor in a simple way. This one

can do by considering theories where one gauge a subalgebra, denote this by $\hat{\mathfrak{h}}$, of the model. In an operator formulation, the GKO construction, this makes the physical states $|\phi\rangle$ satisfy

$$\begin{aligned} J_m^a |\phi\rangle &= 0 & m > 0 \\ E_0^\alpha |\phi\rangle &= 0, \end{aligned} \quad (6.1)$$

where $J_m^a = \{E_m^\beta, H_m^i\}$ denotes generators of the subalgebra and α a positive root in \mathfrak{h} . The Virasoro generators have to commute with the subalgebra and can be found from the Sugawara construction of each algebra. For the algebra $\hat{\mathfrak{g}}$ the Virasoro generators are

$$L_m^{\mathfrak{g}} = \frac{1}{2(k + g^\vee)} \left[\sum_m G_{ij}^{\mathfrak{g}} : H_{-m}^i H_{m+n}^j : + \sum_{m, \alpha \in \Delta^{\mathfrak{g}}} \frac{(\alpha, \alpha)}{2} : E_{-m}^{-\alpha} E_{m+n}^\alpha : \right]. \quad (6.2)$$

For the subalgebra, the Virasoro generators have the form

$$L_m^{\mathfrak{h}} = \frac{1}{2(\kappa + g^\vee)} \left[\sum_m G_{ij}^{\mathfrak{h}} : H_{-m}^i H_{m+n}^j : + \sum_{m, \alpha \in \Delta^{\mathfrak{h}}} \frac{(\alpha, \alpha)}{2} : E_{-m}^{-\alpha} E_{m+n}^\alpha : \right]. \quad (6.3)$$

where κ is the level times the Dynkin index of the embedding

$$I_{\mathfrak{h} \subset \mathfrak{g}} = \frac{(\theta(\mathfrak{g}), \theta(\mathfrak{g}))}{(\theta(\mathfrak{h}), \theta(\mathfrak{h}))}, \quad (6.4)$$

where $\theta(\mathfrak{h})$ is the highest root of \mathfrak{h} expressed as a root in \mathfrak{g} . If we take the linear combination

$$L_m^{(\mathfrak{g}, \mathfrak{h})} = L_m^{\mathfrak{g}} - L_m^{\mathfrak{h}}, \quad (6.5)$$

one finds that

$$[L_m^{(\mathfrak{g}, \mathfrak{h})}, J_n^a] = 0. \quad J_n^a \in \mathfrak{h} \quad (6.6)$$

Thus, $L_m^{\mathfrak{h}}$ commutes with $L_m^{(\mathfrak{g}, \mathfrak{h})}$. The conformal charge for this model is

$$c = \frac{k d_{\mathfrak{g}}}{k + g_{\mathfrak{g}}^\vee} - \frac{\kappa d_{\mathfrak{h}}}{\kappa + g_{\mathfrak{h}}^\vee}. \quad (6.7)$$

For the dominant highest weight representations, the state space of $\hat{\mathfrak{g}}$ decomposes completely into dominant highest weight representations of the subalgebra. Thus, the character of the algebra $\hat{\mathfrak{g}}$ decomposes into a sums of characters of dominant highest weight representations of the subalgebra

$$\chi_{\hat{\mu}}^{\hat{\mathfrak{g}}}(q, \theta_i, \phi_I) = \sum_{\hat{\nu}} b_{\hat{\mu}, \hat{\nu}}^{(\hat{\mathfrak{g}}, \hat{\mathfrak{h}})}(q, \phi_I) \chi_{\hat{\nu}}^{\hat{\mathfrak{h}}}(q, \theta_i), \quad (6.8)$$

where $b_{\hat{\mu}, \hat{\nu}}^{(\hat{\mathfrak{g}}, \hat{\mathfrak{h}})}(q, \phi_I)$ are called branching functions. Here the sum is over highest weights $\hat{\nu}$, at level κ .

One interesting feature, and also one of the motivations in studying these models, is that one can get minimal models where $c < 1$. As discussed before, for WZNW-models one only gets $c \geq 1$. One interesting model is the diagonal coset $(\widehat{\mathfrak{su}}_k(2) \oplus \widehat{\mathfrak{su}}_1(2), \widehat{\mathfrak{su}}_{k+1}(2))$ which yields the series of unitary minimal models [149]

$$c = 1 - \frac{6}{(k+2)(k+3)}. \quad (6.9)$$

6.2 Gauged WZNW-model

The models treated above, can alternatively be defined within a path integral formulation using gauged WZNW-models. Let us here restrict our consideration to the adjoint action of the subalgebra where the action is invariant under $g(z, \bar{z}) \rightarrow h(z, \bar{z})g(z, \bar{z})h^{-1}(z, \bar{z})$. The gauged WZNW-model is described by the action

$$\begin{aligned} S_{WZNW}(g, A, \bar{A}) &= S_{WZNW}(g) + \frac{1}{2\pi k} \int_{\Sigma} d^2z [\kappa (kg^{-1}\partial g, \bar{A}) - \kappa (A, k\bar{\partial}gg^{-1}) \\ &+ \kappa (A, g\bar{A}g^{-1}) - \kappa (A, \bar{A})], \end{aligned} \quad (6.10)$$

where A is a \mathfrak{h} -valued connection. The action is invariant under

$$\begin{aligned} g(z, \bar{z}) &\rightarrow h(z, \bar{z})g(z, \bar{z})h^{-1}(z, \bar{z}) \\ A(z, \bar{z}) &\rightarrow h(z, \bar{z})A(z, \bar{z})h^{-1}(z, \bar{z}) - k\partial h(z, \bar{z})h^{-1}(z, \bar{z}) \\ \bar{A}(z, \bar{z}) &\rightarrow h(z, \bar{z})\bar{A}(z, \bar{z})h^{-1}(z, \bar{z}) - k\bar{\partial} h(z, \bar{z})h^{-1}(z, \bar{z}). \end{aligned} \quad (6.11)$$

This can be checked using the Polyakov-Wiegmann identity [166, 167]

$$\begin{aligned} S_{WZ\bar{N}W}(g_1 g_2) &= S_{WZ\bar{N}W}(g_1) + S_{WZ\bar{N}W}(g_2) \\ &\quad - \frac{k}{2\pi} \int_{\Sigma} d^2 z \kappa (g_1^{-1} \partial g_1 \bar{\partial} g_2 g_2^{-1}). \end{aligned} \quad (6.12)$$

The connection is flat, $[\bar{\partial} + \bar{A}, \partial + A] = 0$, which makes it possible to choose the gauge

$$\begin{aligned} A &= -k \partial h h^{-1} \\ \bar{A} &= -k \bar{\partial} \tilde{h} \tilde{h}^{-1}. \end{aligned} \quad (6.13)$$

This implies that

$$S_{WZ\bar{N}W}(g, A, \bar{A}) = S_{WZ\bar{N}W}(h^{-1} g \tilde{h}) - S_{WZ\bar{N}W}(h^{-1} \tilde{h}). \quad (6.14)$$

Therefore, the action is classically equivalent to the GKO construction. The gauge choice in eq. (6.13) gives rise to a determinant in the path integral. This can be rewritten as a path integral over ghost fields [155]. In the end, one finds that this action possesses a BRST symmetry [161]. One interesting feature is that one has to add an auxiliary H WZNW-model at a level $-\kappa - 2g_6^{\vee}$ to the theory.

Let us see how the BRST symmetry can be imposed within an operator formulation. One simple way of understanding why one needs to add an additional sector is that the classical constraints corresponding to the coset construction is second class at the classical level [168]. Add a sector, denote the elements by $\{\tilde{E}_m^{\alpha}, \tilde{H}_m^i\}$, with level $\tilde{\kappa}$; let us keep the level arbitrary for the moment. Define the highest weight state in the ghost sector to satisfy

$$\begin{aligned} c_m^{\alpha} |0\rangle_{bc} &= c_m^{\alpha} |0\rangle_{bc} = 0 \\ c_{n,i} |0\rangle_{bc} &= b_p^i |0\rangle_{bc} = 0, \end{aligned} \quad (6.15)$$

for $m \geq 0$ when $\alpha \in \Delta_+$, $m > 0$ when $\alpha \in \Delta_-$, $n > 0$ and $p \geq 0$. The BRST charge

is constructed as

$$\begin{aligned}
Q &= \sum_{m \in \mathbb{Z}} c_{-m,i} \left(H_m^i + \tilde{H}_m^i \right) + \sum_{m \in \mathbb{Z}} c_{-m}^{-\alpha} \left(E_m^\alpha + \tilde{E}_m^\alpha \right) + \sum_{m,n \in \mathbb{Z}} \alpha^i : c_{-m,i} c_{-n}^\alpha b_{m+n}^{-\alpha} : \\
&- \frac{1}{2} \sum_{m,n \in \mathbb{Z}} e_{\alpha,\beta} c_{-m}^{-\alpha} c_{-n}^{-\beta} b_{m+n}^{\alpha+\beta} - \frac{1}{2} \sum_{m,n \in \mathbb{Z}} \alpha_i^\vee c_{-m}^{-\alpha} c_{-n}^\alpha b_{m+n}^i,
\end{aligned} \tag{6.16}$$

where one sum over all roots α and β in the subalgebra and all i such that H^i is in the Cartan subalgebra of \mathfrak{h} . The BRST charge is nilpotent only when the additional sector has the level $\tilde{\kappa} = -\kappa - 2g_{\mathfrak{h}}^\vee$. The ghost number operator is

$$\begin{aligned}
N &= \frac{1}{2} (c_{0,i} b_0^i - b_0^i c_{0,i}) + \frac{1}{2} \sum_{\alpha > 0} (c_0^{-\alpha} b_0^\alpha - b_0^{-\alpha} c_0^\alpha) \\
&+ \sum_{n > 0} (c_{-n,i} b_n^i - b_{-n}^i c_{n,i} + c_{-n}^{-\alpha} b_n^\alpha - b_{-n}^{-\alpha} c_n^\alpha)
\end{aligned} \tag{6.17}$$

The Virasoro generators are of the form

$$L_n^{tot} = L_n^{\mathfrak{g}} + L_n^{\mathfrak{h}} + L_n^{gh}, \tag{6.18}$$

where $L_n^{\mathfrak{h}}$ are the Virasoro generators of the auxiliary sector and

$$L_n^{gh} = \sum_{i,m \in \mathbb{Z}} m : b_{n-m}^i c_{m,i} + \sum_{\alpha \in \Delta_{\mathfrak{h}}, m \in \mathbb{Z}} m : b_{n-m}^{-\alpha} c_m^\alpha, \tag{6.19}$$

the Virasoro generators of the ghost system. One can show that the Virasoro algebra is equal to the one arising from the GKO construction up to a BRST trivial term

$$\begin{aligned}
L_n^{tot} &= L_n^{GKO} + \frac{1}{2(\kappa + g_{\mathfrak{h}}^\vee)} \left[Q, \sum_{m \in \mathbb{Z}} b_{-m}^i G_{ij}^{\mathfrak{h}} \left(H_{m+n}^j - \tilde{H}_{m+n}^j \right) \right. \\
&\left. + \sum_{m \in \mathbb{Z}, \alpha \in \Delta^{\mathfrak{h}}} b_{-m}^\alpha \left(E_{m+n}^{-\alpha} - \tilde{E}_{m+n}^{-\alpha} \right) \right],
\end{aligned} \tag{6.20}$$

where $L_n^{GKO} = L_n^{\mathfrak{g}} - L_n^{\mathfrak{h}}$. From the BRST charge one can define the elements of an

affine Lie algebra with vanishing central charge

$$\begin{aligned}
H_m^{\text{tot},i} &\equiv [Q, b_m^i] \\
&= H_m^i + \tilde{H}_m^i - \sum_{m \in \mathbb{Z}, \alpha \in \Delta^{\mathfrak{h}}} \alpha^i : b_{m-n}^{-\alpha} c_n^\alpha : . \\
E_m^{\text{tot},\alpha} &\equiv [Q, b_m^\alpha] \\
&= E_m^\alpha + \tilde{E}_m^\alpha + \alpha^i c_{-n,i} b_{m+n}^\alpha - e_{\alpha,\beta} c_{-n}^{-\beta} b_{m+n}^{\alpha+\beta} - \beta_i^\vee c_{-n}^\beta b_{m+n}^i . \quad (6.21)
\end{aligned}$$

Let us now focus on the spectrum of the model. The BRST states are non-trivial states which satisfy

$$Q |\phi\rangle = 0, \quad (6.22)$$

but one has a redundancy in the ghost part of the spectrum due to the $c_{i,0}$ excitations¹. One can get rid of this by fixing

$$b_0^i |\phi\rangle = 0. \quad (6.23)$$

This will fix the spectrum of physical ket-states to be at one fixed ghost number. One consequence of this is that the charge $H_0^{\text{tot},i}$ has zero eigenvalue on all physical states satisfying eq. (6.23). One can write the a BRST charge as $\hat{Q} = Q - c_{i,0} H_0^{\text{tot},i} - \hat{H}_0^{\text{tot},i} b_0^i$, where \hat{Q} is independent of $c_{i,0}$ and b_0^i and construct a ghost number operator, \hat{N} , where one has excluded the part involving $c_{i,0}$ and b_0^i . The BRST charge is nilpotent on states $|\phi\rangle$ which satisfy $H_0^{\text{tot},i} |\phi\rangle = 0$. One can now prove that the spectrum in the BRST and GKO constructions is equal for $h = \hat{u}_k(1)$ [161] and equal if one assumes that the additional sector has only antidominant highest weight representations in the generic case, [164].

One can define the character for the coset by

$$\chi_{\hat{\mu}, \tilde{\mu}}^{(\hat{\mathfrak{a}}, \hat{\mathfrak{h}})}(q, \theta_i, \phi_I) = \text{Tr} \left[(-1)^{\Delta N_{gh}} q^{L_0^{\mathfrak{g}} + L_0^{\tilde{\mathfrak{g}}} + L_0^{gh} + c/24} \exp \left[i \left(\theta_i H_0^{\text{tot},i} + \phi_I H_0^I \right) \right] \right], \quad (6.24)$$

¹The argument is the same as for the bosonic string in flat background.

where ΔN_{gh} is the ghost number relative to the highest weight state. H^I are chosen such that they commute with all elements in the subalgebra, therefore, commute with the BRST charge. One here faces a problem since this character does not project down to the relative space. The way out of this problem is to project out to the states $|\phi\rangle$ with $H_0^{\text{tot},i}|\phi\rangle \neq 0$, [164]. Formally this can be written as an integration over θ^i . Thus, the character is defined as

$$\chi_{\hat{\mu}, \hat{\nu}}^{(\hat{\mathfrak{g}}, \hat{\mathfrak{h}})}(q, \phi_I) = \int \prod_i \frac{d\theta_i}{2\pi} \text{Tr} \left[(-1)^{\Delta N_{gh}} q^{L_0^{\mathfrak{g}} + L_0^{\mathfrak{h}} + L_0^{gh} + c/24} \exp \left[i \left(\theta_i H_0^{\text{tot},i} + \phi_I H_0^I \right) \right] \right]. \quad (6.25)$$

One can prove that the character above is equal to the branching functions in eq. (6.8) [169]. The equality reads $b_{\hat{\mu}, \hat{\nu}}^{(\hat{\mathfrak{g}}, \hat{\mathfrak{h}})}(q, \phi_I) = \chi_{\hat{\mu}, -\hat{\nu}-2\hat{\rho}}^{(\hat{\mathfrak{g}}, \hat{\mathfrak{h}})}(q, \phi_I)$

6.3 Perturbed gauged WZNW-models

One interesting feature of the gauged WZNW-model is that one can use it to describe sine-Gordon theory, and generalizations called ‘‘symmetric space sine-Gordon’’ theories, by perturbing the gauged WZNW-model [170–172]. Sine-Gordon theories are 1 + 1 dimensional integrable theories which has solitonic solutions. It is classically equivalent to the sigma model on S^2 [173]. Furthermore, the relation between the perturbed gauged WZNW-models to right invariant sigma models will be discussed. This section is based on papers [171, 174]. Assume that we have a Hermitian symmetric space of the form F/G . Choose a realization of the corresponding algebra as $\mathfrak{f} = \mathfrak{k} \oplus \mathfrak{g}$ such that

$$\begin{aligned} [\mathfrak{g}, \mathfrak{g}] &\subset \mathfrak{g} \\ [\mathfrak{k}, \mathfrak{g}] &\subset \mathfrak{k} \\ [\mathfrak{k}, \mathfrak{k}] &\subset \mathfrak{g} \end{aligned} \quad (6.26)$$

and let T and \bar{T} be two element in \mathfrak{k} . Determine a subalgebra of \mathfrak{g} by

$$\mathfrak{h} = \{t \in \mathfrak{g} : [t, T] = [t, \bar{T}] = 0\}, \quad (6.27)$$

Then the gauged WZNW-model² with the addition of the term $-\frac{\mu^2}{2\pi} \int d^2z \kappa (gT, g^{-1}\bar{T})$ describes the “symmetric space sine-Gordon” theory. Let me briefly show a few of the steps. First of all, one has the identities

$$\begin{aligned} [\bar{\partial} + \bar{A}, T] &= 0 \\ [g^{-1}\bar{T}g, \partial + g^{-1}\partial g + g^{-1}Ag] &= 0 \end{aligned} \quad (6.28)$$

which follow from that T and \bar{T} commute with \mathfrak{h} . The equations of motion for the action is

$$[\bar{\partial} + \bar{A}, \partial + g^{-1}\partial g + g^{-1}Ag] - \mu^2 [T, g^{-1}\bar{T}g] = 0. \quad (6.29)$$

Using eqs. (6.28) and (6.29) one can write the equations of motion in Lax form using two Lax pairs³

$$\left[\bar{\partial} + \bar{A} + \lambda \mu^2 g^{-1}\bar{T}g, \partial + g^{-1}\partial g + g^{-1}Ag + \frac{1}{\lambda}T \right] = 0 \quad (6.30)$$

where λ is a non-zero parameter. This equation can be viewed as an integrability condition for equations of motion for a field in the larger space F of the form

$$\begin{aligned} (\bar{\partial} + \bar{A} + \mu^2 \lambda g^{-1}\bar{T}g) \Psi &= 0 \\ \left(\partial + g^{-1}\partial g + g^{-1}Ag + \frac{1}{\lambda}T \right) \Psi &= 0. \end{aligned} \quad (6.31)$$

In addition to the equations of motion in eq. (6.30), one has to consider the constraint equations arising from the variation of the connections A and \bar{A}

$$\begin{aligned} (-\bar{\partial}g g^{-1} - \bar{A} + g\bar{A}g^{-1})|_{\mathfrak{h}} &= 0 \\ (g^{-1}\partial g - A + g^{-1}Ag)|_{\mathfrak{h}} &= 0, \end{aligned} \quad (6.32)$$

where $|_{\mathfrak{h}}$ denotes to the \mathfrak{h} part of the equation. Using this and the equations of motion in eq. (6.30) yields that one has a flat connection $[\partial + A, \bar{\partial} + \bar{A}] = 0$. This

²We here put $k = 1$ and consider the algebras such that the embedding index is one.

³This shows that the theory is integrable.

makes it possible to choose a gauge of the form $A = \bar{A} = 0$. The equations motion and constraints give

$$\begin{aligned}\bar{\partial}(g^{-1}\partial g) - \mu^2 [T, g^{-1}\bar{T}g] &= 0 \\ g^{-1}\partial g|_{\mathfrak{g}} = \bar{\partial}gg^{-1}|_{\mathfrak{g}} &= 0.\end{aligned}\tag{6.33}$$

One can now make some realizations of different symmetric spaces. For instance, the $F = SO(n)$, $G = SO(n-1)$ and $H = SO(n-2)$ triplet yields equations of motion of the form [171]

$$\bar{\partial} \left[\frac{\partial V_i}{\sqrt{1-V_k^2}} \right] = \mu V_i,\tag{6.34}$$

which are the equations of motion of the ‘‘symmetric space sine-Gordon’’ model [175–179]. Here V_i are the non-zero linear independent elements of $g^{-1}\bar{T}g$.

Let us show that these equations also follow from a right invariant sigma model. The action for the right invariant sigma model of the coset F/G is

$$S_g = \int d^2z \kappa(P, \bar{P})\tag{6.35}$$

where $P = f^{-1}\partial f|_{\mathfrak{k}}$ and $\bar{P} = f^{-1}\bar{\partial}f|_{\mathfrak{k}}$ where $f \in F$. The equations of motion which follow from this action are

$$\bar{D}P + D\bar{P} = 0,\tag{6.36}$$

where D and \bar{D} are the covariant derivatives defined as $DX = [\partial + A, X]$ where $A = g^{-1}\partial g|_{\mathfrak{g}}$ and the same for \bar{D} . The currents $J = f^{-1}\partial f$ and $\bar{J} = f^{-1}\bar{\partial}f$ corresponds to a \mathfrak{f} -valued connection which is flat

$$\begin{aligned}\bar{\partial}P - \partial\bar{P} &= 0 \\ \bar{\partial}A - \partial\bar{A} + [\bar{P}, P] + [\bar{A}, A] &= 0.\end{aligned}\tag{6.37}$$

Using the equations of motion and the flatness of the currents J and \bar{J} one finds that the equations of motion can be written in Lax form

$$\left[\partial + A + \lambda P, \bar{\partial} + \bar{A} + \frac{1}{\lambda} \bar{P} \right] = 0.\tag{6.38}$$

The energy momentum tensor components are

$$\begin{aligned} T &= -\frac{1}{2}\kappa(P, P) \\ \bar{T} &= -\frac{1}{2}\kappa(\bar{P}, \bar{P}) \end{aligned} \quad (6.39)$$

which are conserved, $\partial\bar{T} = \bar{\partial}T = 0$. They can be set to $T = \bar{T} = \mu^2$, where μ is a constant, by conformal transformations⁴. Let us make some further restrictions on the algebra we consider. Let us assume that the algebra satisfies

$$\begin{aligned} \mathfrak{k} &= \mathfrak{n} \oplus \mathfrak{a} & \mathfrak{g} &= \mathfrak{m} \oplus \mathfrak{h} \\ [\mathfrak{a}, \mathfrak{a}] &= 0 & [\mathfrak{a}, \mathfrak{h}] &= 0 \\ [\mathfrak{m}, \mathfrak{m}] &\subset \mathfrak{h} & [\mathfrak{m}, \mathfrak{h}] &\subset \mathfrak{m} \\ [\mathfrak{m}, \mathfrak{a}] &\subset \mathfrak{n} & [\mathfrak{a}, \mathfrak{n}] &\subset \mathfrak{m} \end{aligned} \quad (6.40)$$

Assume also that the abelian algebra \mathfrak{a} is one-dimensional. One can here use the global invariance to gauge P , or \bar{P} , to be \mathfrak{a} -valued. Then, the equations of motion for P read

$$\begin{aligned} \bar{\partial}P &= 0 \\ [\bar{B}, P] &= 0, \end{aligned} \quad (6.41)$$

where $\bar{B} = f^{-1}\bar{\partial}f|_{\mathfrak{m}}$. The solution to these equations is

$$\begin{aligned} P &= \mu T \\ \bar{B} &= 0, \end{aligned} \quad (6.42)$$

where T is normalized as $\kappa(T, T) = -2$. The energy momentum tensor yields that μ is a constant. From the same tensor one finds $\bar{P} = \mu g^{-1}Tg$. The equations of motion for \bar{P} then become

$$\partial(g^{-1}Tg) + [A, g^{-1}Tg] = 0, \quad (6.43)$$

⁴The theory is classically conformal invariant.

which implies that A can be written as

$$A = g^{-1}\partial g + g^{-1}A'g, \quad (6.44)$$

where A' is a \mathfrak{h} -valued function. The remaining equation of motion is

$$[\bar{\partial} + \bar{A}, \partial + g^{-1}\partial g + g^{-1}A'g] = \mu^2 [T, g^{-1}Tg]. \quad (6.45)$$

This shows that the action in eq. (6.35) yields the same equations of motion at the classical level as the perturbed gauged WZNW-model. This action will be briefly discussed in the next chapter as the bosonic part of the action for type IIB strings on $AdS_5 \times S^5$, which can be written as a right invariant action [180].

Chapter 7

String theories on non-compact groups

In the beginning of chapter four we discussed some general properties of strings propagating on curved backgrounds. Furthermore, in chapter four and five, consistent theories of non-trivial compact backgrounds were discussed. But one would also like to study, and construct models, of strings propagating on manifolds with a non-trivial time direction as well, thus, strings on non-compact space-times. There are many different reasons in studying these models; one obvious is that strings moving in backgrounds can contain black holes. Here the metric is far from flat. Assuming that the backgrounds arise from non-compact groups, or cosets of groups, one can list all possible groups by requiring that these should only have one time direction [181]. One example of a background constructed from non-compact groups is the two-dimensional black hole which is formulated as a gauge WZNW-model of the coset $SL(2, \mathbb{R})/U(1)$ [182].

Another reason for studying strings in non-trivial backgrounds is the *AdS/CFT* conjecture [6], which is thought to be a more general correspondence between gravity and gauge theory. The only possible groups, or cosets, which have one time direction is the above mentioned *AdS* spaces and spaces connected to Hermitian symmetric

spaces of non-compact type, see Table 7.1. In paper IV and paper V, the latter models are of interest, where a string model is formulated as a gauged WZNW-model. The finite dimensional Lie algebras corresponding to these Hermitian symmetric spaces will be discussed here. In addition, a simple string model corresponding to $SU(1, 1)$ by using the methods in paper IV. This is the only case which is both an AdS -space and connected to a Hermitian symmetric space of non-compact type.

7.1 AdS/CFT correspondence

The AdS/CFT correspondence makes a few of the old discovered connections of gauge theory and string theory more precise. For instance, the ideas that one could describe QCD by the flux tubes, which behave as non-critical strings. Furthermore, the work by Polyakov [38] which shows that the string quantized in a non-critical dimension has one additional degree of freedom, the Liouville field. Effectively this means that the string moves in one more dimension. Connecting this to QCD, the string model describing QCD would be formulated in five dimensions, but the extra dimension would be highly non-trivial. One further indication that Yang-Mills theories are connected to string theories arises from the t'Hooft limit of asymptotically free pure Yang-Mills or Yang-Mills with matter fields in the adjoint representation [183]. The t'Hooft limit is a limit where the number of colors, N , is taken to infinity such that the combination $\lambda = g_{YM}^2 N$ is finite. Here g_{YM} is the coupling constant of Yang-Mills.

Let me here briefly review the arguments behind the AdS/CFT -conjecture which link string theory on $AdS_5 \times S^5$ to the conformal invariant $\mathcal{N} = 4$ supersymmetric $SU(N)$ Yang-Mills theory in four dimensions. The basic ingredients are two different limits of the problem with a three-dimensional singularity at the BPS-bound. One is connected to the supergravity approach to type IIB string theory in the presence of D3-branes and the other to the low energy effective Lagrangian of the massless open strings ending on the D3-branes [184].

Consider the supergravity approach first. The supergravity solution of a stack of N D3-branes describes an extended black hole. The stack of D3-branes carry N units of charge w.r.t. the R-R gauge field A_4 . The metric for the supergravity solution can be written as [185], see also [186, 187] for connected solutions,

$$\begin{aligned} ds^2 &= \frac{1}{\sqrt{f(r)}} (-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + \sqrt{f(r)} (dr^2 + r^2 d\Omega_5^2) \\ f &= 1 + \frac{R^4}{r^4} \\ R^4 &\equiv 4\pi g_s \alpha'^4 N, \end{aligned} \tag{7.1}$$

where $d\Omega_5^2$ is the metric of the five sphere. One can consider the low energy limit of this theory. In this limit the low energy excitations consist of two separate pieces, one bulk supergravity piece and the other near horizon fluctuations. The metric of the near horizon region can be approximated to

$$ds^2 = \frac{r^2}{R^2} (-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + \frac{R^2}{r^2} dr^2 + R^2 d\Omega_5^2, \tag{7.2}$$

which is the metric of $AdS_5 \times S^5$. The conditions for the supergravity approximation to be valid is that the curvature of AdS is small, this transforms into $g_s N \gg 1$.

In the other perspective, the low energy limit consists of the same bulk part as in the supergravity approximation, but with the difference that strings, which end on the D3-brane, are described by a $U(N)$ $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in four dimensions [184]. This limit one can trust when $g_{YM}^2 N \ll 1$. Thus, combining these two descriptions one can see that the physics close to the D3-brane is described by two different theories in two different limits. One can then conjecture that the two different theories are dual to each other. Thus, one get the conjecture that string theory on $AdS_5 \times S^5$ and the $SU(2, 2|4)^1$ gauge theory $D = 4$ is dual to each other. One can argue in the same way for other near horizon regions for black p -brane solutions in string theory to find dualities between different theories.

¹This is the superconformal group in $D = 4$ with $\mathcal{N} = 4$ supersymmetry

7.2 Pohlmeyer reduction of the $AdS_5 \times S^5$ supersymmetric string

Let us briefly go back to the model discussed in the end of chapter six, the gauged WZNW-model with a perturbation. What was found there was that a non-linear sigma model of a coset F/G can be described by a gauged WZNW-model of G/H . One can make this reduction for the supersymmetric string of the non-compact supergroup $SU(2, 2|4)$ as well to discover that one can describe this theory by a gauged WZNW-model of the bosonic subgroup $SO(4, 1) \times SO(5)$ gauged by $SU(2) \times SU(2) \times SU(2) \times SU(2)$ ² plus a perturbation of the same form as for the bosonic case and a fermionic term. In [174] it was argued that the relation between the theories also holds for the quantized theories, as well, due to the fact that the background yields a superconformal field theory.

7.3 Coset models with one time direction

Let us list the possible groups which can be used to get space-times with one time direction. Here, we restrict to the case where the time direction is non-trivially embedded into the group. Let us go through a simple example: The group $G = SU(p, q)$ has $(p + q)^2 - 1$ linearly independent elements where $p^2 + q^2 - 1$ and $2pq$ are compact and non-compact, respectively. The maximal compact subgroup of this coset is $S(U(p) \times U(q))$, which has $p^2 + q^2 - 1$ linearly independent elements. But one can split the group into $SU(p) \times SU(q) \times U(1)$. Therefore, the coset space G/H , where $H = SU(p) \times SU(q)$, has one compact direction, which plays the role of time, and $2pq$ non-compact directions. In Table 7.1 all possible groups are listed which satisfy the assumptions. This is reproduced from Quevedo and Ginsparg [181]. The last case in the table corresponds to the bosonic part of AdS_{p+1} space. The other spaces

²Here the first two $SU(2)$'s are subgroups of $SO(4, 1)$ and the other two subgroups of $SO(4, 1) \times SO(5)$

Table 7.1: Relevant coset models where G is simple.

G	H	Signature
$SU(p, q)$	$SU(p) \times SU(q)$	$(1, 2mn)$
$SO(p, 2)$	$SO(p)$	$(1, 2p)$
$SP(2p, \mathbb{R})$	$SU(p)$	$(1, p(p+1))$
$SO^*(2p)$	$SU(p)$	$(1, p(p-1))$
$E_{6 -14}$	$SO(10)$	$(1, 32)$
$E_{7 -25}$	$E_{6 -78}$	$(1, 54)$
$SO(p, 2)$	$SO(p, 1)$	$(1, p)$

are also interesting. These are the groups connected to the Hermitian symmetric spaces of non-compact type. These spaces correspond to cases where the maximal compact subgroup of the non-compact real form has a one-dimensional center as is e.g. the case in the simple example above. If one considers the coset space where one divides out all but the center of the compact real form one arrives at a coset space which has one time direction. These are the spaces which are of interest in paper IV and paper V. In these papers we formulate the theories as gauged WZNW-models. In the next section, we will discuss the finite dimensional Lie algebras.

A short comment before we proceed, the gauged WZNW-model which arises in the Pohlmeyer reduction of the type IIB string on $AdS_5 \times S^5$ is not part of the cosets in the table above, as one has no time direction left, when gauging the subgroup.

7.4 Non-compact real forms of Lie algebras

For the algebras which are connected to a Hermitian symmetric spaces of non-compact type, one can choose that the Cartan subalgebra to be compact. Furthermore, one can get the representations spaces of these algebras by choosing different

Hermitian conjugation rules for the generators corresponding to simple roots

$$(E_-^i)^\dagger = \pm E_+^i, \quad (7.3)$$

where the plus and minus signs are for compact and non-compact roots, respectively. Furthermore, one can choose a realization such that the maximal compact real form has a regular embedded and that the highest root is non-compact. A choice which satisfies these properties are shown as Dynkin diagrams in [188] and reproduced in the appendix of paper IV. Before treating the general case, consider the simplest non-trivial non-compact algebra, $\mathfrak{su}(1, 1)$. The Hermitian conjugation for this algebra is that the roots are non-compact. The condition on the highest weight for the representation to be unitary, follows from the first excited state, $E_-^1 |\mu\rangle$, which yields $\mu < 0$ ³.

Consider now the unitarity of the general case, which was proven in [188, 189]. One of the properties of the realization considered is that roots can only consist of at most one non-compact simple root. Thus, all compact roots can be written as linear combinations of compact simple roots. Consider first only excitations of the compact subalgebra. A condition that the state space of the representation is unitary is that the highest weight is dominant and integer. Thus, the components of the highest weight corresponding to the compact subalgebra have to be dominant and integer for unitarity. Construct the following representation space

$$N_{\mathfrak{g}}(\mu) = U(\mathfrak{p}_-) \otimes V_{\mathfrak{b}'}(\mu), \quad (7.4)$$

where $V_{\mathfrak{b}'}(\mu)$ is the irreducible state space corresponding to the subalgebra of the semisimple part of the compact real form. \mathfrak{p}_- is the set of all negative non-compact roots and $U(\mathfrak{p}_-)$ (a representation of) the universal enveloping algebra of \mathfrak{p}_- . If $(\mu + \rho, \theta) < 0$, where θ is the highest root, then the representation in eq. (7.4) is irreducible [190]. Thus, one can take a limit $\mu^I \rightarrow -\infty$, where μ^I is the component of

³It is simple to see that this is indeed enough as all states can be written as $(E_-^1)^n |\mu\rangle$ which has norm squared $n! \frac{(-\mu+n)!}{(-\mu)!} > 0$

the highest weight which is connected to the one-dimensional center of the maximal compact subalgebra. This is possible since the Shapovalov-Kac-Kazhdan determinant has no zeros corresponding to the non-compact roots. Therefore, the signature of the representation is unchanged in the limit. In this limit the representation simplifies such that the matrix of inner products diagonalizes. We have

$$[E^\beta, E^{-\beta}]|\phi\rangle = (\mu, \beta^\vee)|\phi\rangle, \quad (7.5)$$

where $\beta \in \mathfrak{p}_-$. Thus, the representation is unitary if $(\mu, \alpha^{(i)\vee}) \in \mathbb{Z}_+$ and $(\mu + \rho, \theta) < 0$, where $\alpha^{(i)}$ is a simple compact root. One can also determine the characters of these representations in a simple way. Assume that one has diagonalized the state space $V_{\mathfrak{h}'}(\mu)$. Then the character can be thought of as sums of characters corresponding to Verma modules spanned by the non-compact roots. These characters are simple to determine. Assume that the weight of the state corresponding to the finite dimensional algebra is $\mu - \sum_i n_i \alpha^{(i)}$ where $n_i \in \mathbb{Z}_+$ and $\alpha^{(i)}$ is a simple compact root. Then this part of the character is

$$\chi_{\mu - \sum_i n_i \alpha^{(i)}}^{part.}(\phi) = \exp \left[i \left(\phi, \mu - \sum_i n_i \alpha^{(i)} \right) \right] \prod_{\alpha \in \Delta_+^n} \frac{1}{(1 - \exp[-i(\phi, \alpha)])}. \quad (7.6)$$

We can now sum all terms to get

$$\begin{aligned} \chi_\mu(\phi) &= \prod_{\alpha \in \Delta_+^n} \frac{1}{(1 - \exp[-i(\phi, \alpha)])} \\ &\times \sum_{C(n_i, \mu)} \text{mult}_\mu \left(\mu - \sum_i n_i \alpha^{(i)} \right) \exp \left[i \left(\phi, \mu - \sum_i n_i \alpha^{(i)} \right) \right], \quad (7.7) \end{aligned}$$

where $C(n_i, \mu) = \{n : |\mu - \sum_i n_i \alpha^{(i)}\rangle \in V_{\mathfrak{h}'}(\mu)\}$ and $\text{mult}_\mu(\mu - \sum_i n_i \alpha^{(i)})$ is the multiplicity of weight $\mu - \sum_i n_i \alpha^{(i)}$. Comparing eqs. (7.7) and (5.19), the second part of the character is the same as the character of the compact subalgebra. Thus, we can write eq. (7.7) as

$$\chi_\mu(\phi) = e^{i(\mu, \phi)} \frac{\sum_{w \in W_c} \text{sign}(w) \exp[i(\phi, w(\mu + \rho) - \mu - \rho)]}{\prod_{\alpha \in \Delta_+} (1 - \exp[-i(\phi, \alpha)])}, \quad (7.8)$$

where W_c is the Weyl group generated by the simple compact roots. These are not the representations considered in paper IV and paper V. The main reason is that the character for the affine algebra is not known for these representations. There might be possible to determine them, but for the signature function, this would be difficult task as one needs to keep track of the sign of each state.

7.5 AdS_3 and the $SU(1, 1)$ string

Apart from the example of AdS_5/CFT_4 -duality, there is also another well studied example, the duality between strings on AdS_3 and a two-dimensional conformal field theory. The studied examples are of the form $AdS_3 \times S^3 \times M_4$ where M_4 is a compact manifold. On the string theory side, one gets a theory in an R-R background. But due to the $SL(2, \mathbb{Z})$ -symmetry of the IIB string theory this theory should be related to a theory with NS-NS background. Thus, the string theory can be formulated as a $SU(1, 1)$ WZNW-model. This model is part of the Hermitian symmetric spaces treated above, as well as being a AdS space.

The $SU(1, 1)$ string has been treated in many different papers, a few of them are [191–201]. Let us first discuss a simple proof of unitarity of one of the sectors of the theory. For unitary representations of the finite dimensional algebra at grade zero, the highest weight has to satisfy $\mu < 0$, as discussed in the previous sector. There is also a state of the form $E_{-1}^+ |\mu\rangle$, where $E_m^{\pm\alpha(1)} = E_m^\pm$. This state is unitary if $k < \mu$, but there is also a state of the form $H_{-1} |\mu\rangle$, which yields $k > 0$. Thus, the representations are not unitary, which is not a surprise as the signature is $(1, 2)$. However, the physical states should also be highest weights of the Virasoro algebra with weight 1. Construct the BRST charge corresponding to the Virasoro algebra of the $SU(1, 1)$ string. Assume that one has, in addition to the contribution corresponding to a representation of the $\hat{\mathfrak{su}}(1, 1)$ affine Lie algebra, a unitary conformal field theory such that the conformal charge is 26. As the state $E_{-1}^+ |\mu\rangle$ is a highest weight w.r.t. the Virasoro algebra, we have $k < \mu$. Thus, the representations which

are relevant have $\mu < 0$ and $k < \mu$. Let us make a restriction on the representations, that they are antidominant. Therefore, $\mu + 1 < 0$ and $k + 1 + \mu < 0$. One can now compute the character and signature function

$$\begin{aligned}\chi_\mu(q, \theta) &= q^{\frac{(\mu, \mu + 2\rho)}{2(k+2)} + N - 1} \frac{e^{i(\theta, \mu)}}{(1 - e^{-2\theta})} \prod_{n=1}^{\infty} \frac{1}{(1 - q^n)(1 - q^n e^{-2\theta})(1 - q^n e^{2\theta})} \\ \Sigma_\mu(q, \theta) &= q^{\frac{(\mu, \mu + 2\rho)}{2(k+2)} + N - 1} \frac{e^{i(\theta, \mu)}}{(1 - e^{-2\theta})} \prod_{n=1}^{\infty} \frac{1}{(1 + q^n)(1 - q^n e^{-2\theta})(1 - q^n e^{2\theta})},\end{aligned}\quad (7.9)$$

where $N \geq 0$ is the contribution from the unitary conformal field theory. The ghost contribution is

$$\begin{aligned}\chi^{gh} &= \prod_{n=1}^{\infty} (1 - q^n)^2 \\ \Sigma^{gh} &= \prod_{n=1}^{\infty} (1 - q^n)(1 + q^n),\end{aligned}\quad (7.10)$$

which when multiplied together yields that the signature function and the character is equal. Thus, the representation is unitary. But there are additional sectors. A first indication of this is that the L_0 value is bounded from above, thus, there is a highest excitation level. This seems not to be correct as it, for instance, would not be consistent in a theory which involves interactions of strings. This would imply that the joining of two physical string states yields a state which is not physical. Another indication is that the partition function is not modular invariant. This problem was the main motivation to add new sectors to the theory [198]. The states which arise are connected to the representations above by spectral flows. These are constructed by applying elements of the loop group of $SU(1, 1)$ to the currents. The loop group of $SU(1, 1)$ is non-trivial and isomorphic to \mathbb{Z} . This will produce new representations which solves the problem with a largest excitation level. These states were discussed in detail in [199–201]. In short, the spectral flow acts on g as $\bar{z}^{-wH/2} g z^{wH/2}$ where

$g \in G$ and $w \in \mathbb{Z}$. This makes the operators in a Laurent expansion to transform as

$$\begin{aligned}\tilde{H}_m &= H_m - wk\delta_{m,0} \\ \tilde{E}_m^\pm &= E_{m\mp w}^\pm \\ \tilde{L}_n &= L_n - \frac{1}{2}wkH_n + \frac{1}{4}w^2\delta_{n,0}.\end{aligned}\tag{7.11}$$

The spectrally flowed states in the horizontal part are required to satisfy

$$\begin{aligned}E_{m\mp w}^\pm \left| \tilde{\mu}, \tilde{\lambda} \right\rangle &= 0 \\ H_m \left| \tilde{\mu}, \tilde{\lambda} \right\rangle &= 0\end{aligned}\tag{7.12}$$

for $m > 0$. The mass-shell condition on a state is

$$(L_0 - 1) |\phi\rangle = \left(\frac{1}{2}\tilde{\mu}(\tilde{\mu} + 2) + \frac{1}{2}w\tilde{\lambda} + \frac{1}{4}kw^2 + N + h - 1 \right) |\phi\rangle,\tag{7.13}$$

where $|\phi\rangle = \left| \tilde{\mu}, \tilde{\lambda}; h \right\rangle \otimes |\varphi\rangle_N$. Here, $|\varphi\rangle_N$ is a state in the unitary conformal field theory at grade N . From eq. (7.13) one can see that there is no limit on N and h as $k < 0$ and $\tilde{\lambda} < 0$ if $w \neq 0$. This is one of the sectors of the theory; one has also a sector where the horizontal part is a unitary continuous representation and spectrally flowed sectors of it.

Modular invariance has not been discussed before in the thesis, but is connected to global properties of the reparametrization invariance of string theory. Obstructions of this invariance can arise in the loop amplitudes of the string theory. It is sufficient to determine that the partition function, which is the 1-loop amplitude with no insertions, is invariant under $\tau \rightarrow \tau + 1$ and $\tau \rightarrow -1/\tau$. These transformations generate the modular group $PSL(2, \mathbb{Z})^4$. As the WZNW-model discussed in paper IV and V has the same properties as the $SU(1, 1)$ string, we believe that one should add more sector connected to the loop group of the coset. One can argue that the loop group of the background discussed in paper IV and V is the same as for $SU(1, 1)$

⁴ P is because the modular transformation is isomorphic to $SL(2, \mathbb{Z})$ up to an over all sign.

manifold. The argument is based on that the Hermitian symmetric space of non-compact type is simply connected and that the center of the subgroup is isomorphic to S^1 . Thus, spaces considered in paper IV and V are topologically equivalent to $\mathbb{R}^{2n} \times S^1$. For this space the loop group is isomorphic to \mathbb{Z} .

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