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Review

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Review

Applications of Symmetries to Nonlinear Partial Differential Equations

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Abstract: This review begins with the standard Lie symmetry theory for nonlinear PDEs and explores extensions of symmetry analysis. First, it introduces three key symmetry reduction methods: the classical symmetry method, conditional symmetries, and the CK direct method. Next, it presents two finite symmetry transformation group methods—one related to Lax pairs and one independent of them. The fourth section reviews four nonlocal symmetry methods based on conserved forms, conformal invariants, Darboux transformations, and Lax pairs. The final section covers supersymmetry theory and supersymmetric dark equations. Each method is illustrated with examples and references.

Keywords: classical Lie symmetry approach; partial differential equation; finite symmetry transformation group; supersymmetric equation; supersymmetric dark equation

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1. Introduction

Symmetries of nonlinear partial differential equations (PDEs) and ordinary differential equations (ODEs) refer to transformations that map certain solutions to new solutions. The fundamental theory of the symmetries of nonlinear PDEs was constructed by Marius Sophus Lie and the basic idea is to use infinitesimal generators to describe the continuous symmetry group [1]. Noether extended infinitesimals to any finite-order dependent variables [2]. Noether's theorem demonstrated the relation between Lie symmetries and conservation law.

Nonlinear PDEs are widely used to model nonlinear scientific phenomena, representing smooth variations in space and time. Symmetry groups, along with their associated similarity solutions and reductions, have several key applications in PDEs:

(1). Reducing the number of independent and dependent variables, potentially transforming a PDE with two variables into a one-variable ODE.
(2). Linking conservation laws to symmetry properties.
(3). Classifying PDEs into equivalence classes and identifying simpler representations.
(4). Generating new solutions from existing ones.

This paper reviews various methods in the symmetry group theory of nonlinear PDEs and highlights recent advancements. The structure is as follows: Section 2 introduces three fundamental methods, while Section 4 focuses on four nonlocal symmetry methods. Section 3 explores finite symmetry transformation groups. Supersymmetric and supersymmetric dark equations are covered in Section 5, with a summary and discussion provided in Section 6.

2. Basic Methods of Symmetry Analysis

A Lie symmetry group analysis of a PDE includes the following steps [3–5]:

(1): Identify classical Lie point symmetries for the suggested model.

(2): Construct an algebra based on the identified symmetries.
(3): Determine similarity variables corresponding to each symmetry.
(4): Utilize the obtained symmetries to reduce the PDE to a lower-order PDE or ODE.
(5): Obtain solutions from the ODE.

In this section, we will review three basic methods of symmetry reductions.

2.1. Classical Lie Group Method

Many studies in the literature have commented on classical methods [3,4,6]. Consider w as a dependent variable, t and x are two dependent variables, and $w^{(l)}$ means partial derivatives of order l of w. One PDE with the function $w(t,x)$ as an independent variable reads

$$S = S(t,x,w,w^{(l)}(t,x),\ldots,w^{(n)}(t,x)) = 0. \tag{1}$$

Now, assume that system (1) admits the transformations including an infinitesimal parameter ϵ in the form of

$$\bar{t} = t + \epsilon T(t,x,w) + O(\epsilon^2)\,, \tag{2}$$
$$\bar{x} = x + \epsilon X(t,x,w) + O(\epsilon^2)\,, \tag{3}$$
$$\bar{w} = w + \epsilon W(t,x,w) + O(\epsilon^2)\,. \tag{4}$$

Under these transformations for Equation (1), a set of overdetermined equations for T, X, and W can be obtained. The corresponding vector fields of Lie algebra reads

$$\widetilde{\mathbf{V}} = T(t,x,w)\frac{\partial}{\partial t} + X(t,x,w)\frac{\partial}{\partial x} + W(t,x,w)\frac{\partial}{\partial w}. \tag{5}$$

We will apply the famous Boussinesq equation as an instance to demonstrate this traditional group method in this subsection.

Example 1. *Classical Lie group for the Boussinesq equation [6–8].*

The Boussinesq equation reads

$$S \equiv w_{tt} + ww_{xx} + w_x^2 + w_{xxxx} = 0. \tag{6}$$

The solution of (6) should be unchanged under the traditional Lie group transformation (4). Then, the fourth prolongation of (5) $\mathrm{pr}^{(4)}\widetilde{\mathbf{V}}$ should satisfy

$$\mathrm{pr}^{(4)}\widetilde{\mathbf{V}}(S)|_{S=0} = 0, \tag{7}$$

where $\mathrm{pr}^{(4)}\widetilde{\mathbf{V}}$ is given by

$$\mathrm{pr}^{(4)}\widetilde{\mathbf{V}} = \widetilde{\mathbf{V}} + W^x\frac{\partial}{\partial w_x} + W^{xx}\frac{\partial}{\partial w_{xx}} + W^{tt}\frac{\partial}{\partial w_{tt}} + W^{xxxx}\frac{\partial}{\partial w_{xxxx}}\,, \tag{8}$$

where W^x, W^{xx}, W^{tt} and W^{xxxx} are represented explicitly by X, T, and W [3,4].

Then, one obtains twelve determining equations [6]. Solving these determining equations, we then obtain the following solution for X, T, and W:

$$T = 2\,\delta\,t + \psi,\;\; X = \delta\,x + \theta,\;\; W = -2\,\delta\,w, \tag{9}$$

where δ, θ, and ψ are arbitrary constants. Correspondingly, $\widetilde{\mathbf{V}}$ turns into

$$\begin{aligned}\widetilde{\mathbf{V}} &= (2\,\delta\,t + \psi)\frac{\partial}{\partial t} + (\delta\,x + \theta)\frac{\partial}{\partial x} - 2\,\delta w\frac{\partial}{\partial w}\\ &= \delta\left(2\,t\frac{\partial}{\partial t} + x\,\frac{\partial}{\partial x} - 2\,w\,\frac{\partial}{\partial w}\right) + \psi\,\frac{\partial}{\partial t} + \theta\,\frac{\partial}{\partial x}\,.\end{aligned} \tag{10}$$

The generators $\widetilde{\mathbf{V}}_1 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 2w\frac{\partial}{\partial w}$, $\widetilde{\mathbf{V}}_2 = \frac{\partial}{\partial t}$, and $\widetilde{\mathbf{V}}_3 = \frac{\partial}{\partial x}$, which correspond to the invariance of scale transformation and spatiotemporal translation.

The following characteristic equations can derive the similarity variables:

$$\frac{\mathrm{d}w}{W} = \frac{\mathrm{d}t}{T} = \frac{\mathrm{d}x}{X} \,. \tag{11}$$

Based on whether δ is equal to zero or not, one can obtain two different types of similarity variables: similarity solutions and reduction equations [6–8]. The final results read

$$\begin{aligned} &w = \frac{W(\zeta)}{2\delta t + \psi}, \quad \zeta = \frac{\delta x + \theta}{\sqrt{2\delta t + \psi}}, \\ &\delta^2 W_{\zeta\zeta\zeta\zeta} + (\zeta^2 + W)W_{\zeta\zeta} + W_\zeta^2 + 7\zeta W_\zeta + 8W = 0\,, \end{aligned} \tag{12}$$

for $\delta \neq 0$ and

$$\begin{aligned} &w = W(\eta), \quad \eta = \psi x - \theta t, \\ &\psi^4 W_{\eta\eta\eta\eta} + \theta^2 W_{\eta\eta} + \psi^2(WW_{\eta\eta} + W_\eta^2) = 0, \end{aligned} \tag{13}$$

for $\delta = 0$.

The classical Lie group method is widely applied in various PDEs [9–12].

2.2. *Conditional Symmetry Method*

The concept of partially invariant solutions was developed in 1962 [13,14]. In 1969, the nonclassical Lie group method of group-invariant solutions was proposed [15]. The nonclassical Lie group symmetries are also known as conditional symmetries [16–18] and partial symmetries [19,20].

In order to obtain symmetries using the nonclassical Lie group method, we should add an auxiliary first-order equation to Equation (1), namely [18]

$$\Phi \equiv T(t,x,w)w_t + X(t,x,w)w_x - W(t,x,w) = 0. \tag{14}$$

The auxiliary equation is related to Formula (5).

In the nonclassical Lie group method, both (1) and (14) should be invariant under the transformation (4). Then, the system for $T(t,x,w)$, $X(t,x,w)$, and $W(t,x,w)$ can be obtained. Next, we can use the classical method to seek the symmetry group for Formulas (1) and (14). The prolongation for (5) should satisfy

$$\mathrm{pr}^{(1)}\widetilde{\mathbf{V}}(\Phi)|_{S=0,\Phi=0} = 0, \tag{15}$$

$$\mathrm{pr}^{(\mathrm{n})}\widetilde{\mathbf{V}}(\mathrm{S})|_{S=0,\Phi=0} = 0. \tag{16}$$

Formula (16) is trivial and no restriction on T, X, and W is imposed. The same as for the classical method, we will also apply the Boussinesq equation to demonstrate the conditional symmetry method.

Example 2. *Conditional symmetry group of the Boussinesq Equation* (6) *[6,18].*

For the (2+1)-dimensional Boussinesq Equation (6), two nontrival conditions should be considered. The first condition is $T \neq 0$ and the second condition is $\{T = 0, X \neq 0\}$.

Codition 1 is $T \neq 0$. T can be set as $T = 1$. The auxiliary first-order Equation (14) shows that

$$w_t = W - Xw_x, \tag{17}$$

$$w_{xt} = W_x + W_w w_x - (X_x w_x + X_w w_x^2 + Xw_{xx}), \tag{18}$$

$$w_{tt} = W_t + W_w w_t - X_t w_x - X_w w_x w_t - Xw_{xt}. \tag{19}$$

Then, eliminating u_{tt} in the Boussinesq Equation (6) yields

$$\begin{aligned} &W_t + W_w(W - Xw_x) - X_t w_x - X_w w_x(W - Xw_x) \\ &-X[W_x + W_w w_x - (X_x w_x + X_w w_x^2 + X w_{xx})] + w w_{xx} + w_x^2 + w_{xxxx} = 0. \end{aligned} \tag{20}$$

Equation (20) is an ODE, with $w(x)$ as the variable. Applying the classical Lie group method to Equation (20), and eliminating w_{xxxx} with the help of (20), one then obtains a system of determining equations. Solving the seven determining equations, the following symmetry operator can be obtained:

$$\begin{aligned} \widetilde{\mathbf{V}} &= [xf(t) + g(t)]\partial_x + \partial_t + W\partial_w, \\ W &= -\big[2f(t)w + 2(f(t)\dot{g}(t) + g(t)\dot{f}(t) + 4g(t)f(t)^2)x \\ &\quad +2f(t)x^2(\dot{f}(t) + 2f(t)^2) + 2g(t)(\dot{g}(t) + 2f(t)g(t))\big], \end{aligned} \tag{21}$$

where the constraint conditions are

$$\ddot{g}(t) + 2f(t)\dot{g}(t) - 4f(t)^2 g(t) = 0, \quad \ddot{f}(t) + 2f(t)\dot{f}(t) - 4f(t)^3 = 0, \tag{22}$$

with dots denoting time derivatives.

We introduce $J(t)$ and $K(t)$, governed by

$$\dot{J} - Kg = 0, \;\; \dot{K} + fK = 0, \;\; \ddot{K} + AK^5 = 0, \;\; \ddot{J} + AK^4(J - X_0)\,, \tag{23}$$

where the constants X_0 and A are free. Then, the nonclassical similarity solution related to (21) turns into

$$\begin{aligned} w &= W(z)K^2 + (AK^4 - K^{-2}\dot{K}^2)x^2 - 2[AK^3(J - X_0) - K^{-2}\dot{K}\dot{J}]x \\ &\quad +K^2[A(J - X_0)^2 + B] - K^{-2}\dot{J}^2, \end{aligned} \tag{24}$$

with the arbitrary constant B, while the function $W = W(z) = W(Kx - J)$ satisfies the reduction ODE

$$W_{zzzz} + [W + B + A(z + X_0)^2]W_{zz} + W_z^2 + 3A(z + X_0)W_z = 0\,. \tag{25}$$

This equation amounts to the Painlevé IV Formula [6,18].

Codition 2 is $\{T = 0, X \neq 0\}$. One can suppose that $X = 1$, and Formula (14) is simplified as

$$w_x = W(t, x, w). \tag{26}$$

Substituting Formula (26) into the Boussinesq Equation (6) yields

$$\begin{aligned} &w_{tt} + w(W_x + WW_w) + W^2 + W_{xxx} + W_w W_{xx} + 3WW_x W_{ww} + 3W_x W_{xw} \\ &+W_x W_w^2 + W^3 W_{www} + 3W^2 W_{xww} + 4W^2 W_w W_{ww} + 3WW_{xxw} + 5WW_w W_{xw} + WW_w^3 = 0, \end{aligned} \tag{27}$$

This equation is equivalent to an ODE with $w(t)$ being the variable.

Applying the classical Lie group method to Equation (27), one obtains three determining equations. Solving the three determining equations, one obtains two solutions of W and their corresponding similarity solutions of w in the forms of

$$\begin{cases} w(t,x) = -\dfrac{12}{x^2} + w_1(t)x^2\,, \\ W = \dfrac{2w}{x + x_0} + \dfrac{48}{(x + x_0)^3}\,, \\ \ddot{w}_1(t) + 6w_1(t)^2 = 0\,, \end{cases} \tag{28}$$

and

$$\begin{cases} w(t,x) = w_2(t) + x\varphi_1(t) + x^2\varphi_2(t), \\ W = 2x\varphi_2(t) + \varphi_1(t), \\ \ddot{\varphi}_1(t) + 6\varphi_1(t)\varphi_2(t) = 0,\ \ddot{w}_2(t) + 2w_2(t)\varphi_2(t) + \varphi_1(t)^2 = 0,\ \ddot{\varphi}_2(t) + 6\varphi_2(t)^2 = 0, \end{cases} \tag{29}$$

where the functions $\{w_1(t), w_2(t), \varphi_1(t), \varphi_2(t)\}$ are determined by the reduction equations.

Generally speaking, the results obtained from the conditional symmetry method are more abundant than the results from the classical method. However, for some PDEs, such as the KdV equation, the symmetries obtained by the conditional symmetry method are same as those obtained by the classical Lie group method [6].

2.3. CK Direct Method

A common characteristic of the above two methods for symmetries for PDEs lies in the application of group theory. In Ref. [21], Clarkson and Kruskal (CK) developed the direct method of deriving similarity reductions for PDEs. The CK direct method can obtain symmetry reduction without using symmetry groups. Ref. [18] provided the group theoretical explanation for this method.

For a PDE in the form of

$$S(y_1, y_2, \cdots, y_p, w(y_1, y_2, \cdots, y_p)) = 0, \tag{30}$$

its similarity reduction is generally written as [22]

$$\begin{aligned} w(y_1, y_2, \cdots, y_p) = W(y_1, y_2, \cdots, y_p, V(Y_1, Y_2, \cdots, Y_q)), \\ (Y_i = Y_i(y_1, y_2, \cdots, y_p), i = 1, 2, \cdots, q < p). \end{aligned} \tag{31}$$

The basic idea for the CK direct method is to replace (31) with a simpler form [21]:

$$w(y_1,\ y_2, \cdots,\ y_p) = \phi(y_1,\ y_2, \cdots,\ y_p) + \psi(y_1,\ y_2, \cdots,\ y_p)V(Y_1,\ Y_2, \cdots,\ Y_q). \tag{32}$$

The combination of (30) and (31) will cause the result to be a PDE with a lower dimension or an ODE, and then one can solve W [23]. For some PDEs, it is enough to replace (31) with (32).

Example 3. *Application of the CK direct method to the Boussinesq equation [6,21,24].*

The Boussinesq equation is in the form of (6). Its general similarity reduction is

$$w(t,x) = W(t,x,V(Y(t,x))\ . \tag{33}$$

Plugging this formula into the Boussinesq Equation (6), we then demand the updated equation to be an ODE with $V(Y)$ as the dependent variable. For the Boussinesq Equation (6), ref. [21] proved that (33) can be replaced by the simpler form

$$w(t,x) = \phi(t,x) + \psi(t,x)V(Y(t,x)). \tag{34}$$

Substituting Formula (34) into the Boussinesq equation and classifying coefficients of of V and its derivatives, one obtains an equation concluding ϕ, ψ, V, and Y. The obtained equation is required to be an ODE for $V(Y)$ and the coefficients are supposed to related to Y only. Solving the proportional relationship between these coefficients, one can obtain the symmetry reductions. For the Boussinesq Equation (6), whether Y_x is equal to 0 or not should be considered. Ref. [21] discussed the case of $Y_x \neq 0$, and ref. [24] proposed the discussions on $Y_x = 0$.

Condition 1 $Y_x \neq 0$**.** Substituting (34) into Formula (6) yields a system on $V(Y)$ in the form of

$$\psi Y_x^4 V'''' + \psi^2 Y_x^2 V V'' + (\psi_x^2 + \psi\psi_{xx})V^2 + (\phi_{tt} + \phi\phi_{xx} + \phi_x^2 + \phi_{xxxx}) + G(V) = 0, \tag{35}$$

where $G(V)$ includes the terms of $\{V''', V'', VV', (V')^2, V', V\}$ with the superscript $'$ indicating a Y derivative.

The coefficients are supposed to be related to Y only, and the normalizing coefficient can be arbitrarily selected. In ref. [21], the coefficient of V'''' has been selected, and the proportional coefficient is expressed as $\Gamma(Y)$. Then, the following eight formulas are obtained:

$$\begin{cases} \psi Y_x^4 \Gamma_1(z) = \psi^2 Y_x^2, \\ \psi Y_x^4 \Gamma_2(z) = \psi_x^2 + \psi\psi_{xx}, \\ \psi Y_x^4 \Gamma_3(z) = \phi_{tt} + \phi\phi_{xx} + \phi_x^2 + \phi_{xxxx}, \\ \qquad\qquad \vdots \end{cases} \tag{36}$$

Solving (36), one can obtain the similarity solutions for the Boussinesq Formula (6):

$$w = K^2[W + AY^2 + 2AX_0Y + B] - \frac{1}{K^2}\left(x\frac{\mathrm{d}K}{\mathrm{d}t} - \frac{\mathrm{d}J}{\mathrm{d}t}\right)^2, \qquad Y(t,x) = -J + Kx, \tag{37}$$

with the functions $K(t)$ and $J(t)$ satisfying the conditions

$$\ddot{K} + AK^5 = 0, \; \ddot{J} + AK^4(J - X_0) = 0. \tag{38}$$

It is not difficult to verify that (37) is the same as the result of the nonclassical symmetry reduction solutions (24), when $W(Y)$ and Y are rewritten as $W(z)$ and z, respectively. Thus, $W = W(z)$ in (37) satisfies the reduction Equation (25).

Condition 2. $Y_x = 0$**.** When $Y_x = 0$, Formula (34) degenerates to

$$w(t,x) = \phi(t,x) + \psi(t,x)V(Y(t)). \tag{39}$$

Plugging Formula (39) with $z = t$ into the Boussinesq Equation (6) yields

$$\begin{aligned} &\psi Y'' + 2\psi_t Y' + (\psi_{tt} + \psi_{xxxx} + 2\phi_x\psi_x + \phi\psi_{xx} + \phi_{xx}\psi)Y \\ &\quad + (\psi_x^2 + \psi\psi_{xx})Y^2 + (\phi_{xxxx} + \phi_{tt} + \phi\phi_{xx} + \phi_x^2) = 0, \end{aligned} \tag{40}$$

where the superscripts indicate t derivatives.

In ref. [24], ψ relating to Y'' was selected to be the comparison coefficient, and the proportional coefficients are $\{\Gamma_i(t), i = 4,5,6,7\}$. So,

$$\begin{cases} \psi\Gamma_4(t) = 2\psi_t\,, \\ \psi\Gamma_5(t) = \psi_{tt} + \psi_{xxxx} + 2\phi_x\psi_x + \phi\psi_{xx} + \phi_{xx}\psi\,, \\ \psi\Gamma_6(t) = \phi_{xxxx} + \phi_{tt} + \phi\phi_{xx} + \phi_x^2\,, \\ \psi\Gamma_7(t) = \psi_x^2 + \psi\psi_{xx}\,. \end{cases} \tag{41}$$

There are two canonical types of reduction as follows:

$$w(t,x) = x^2 w_1(t) - \frac{12}{x^2} \tag{42}$$

and

$$w(t,x) = w_2(t) + x\phi_1(t) + x^2\phi_2(t), \tag{43}$$

which are the same as the solution of w in (28) and (29), respectively.

3. Finite Symmetry Transformation Groups

Generally speaking, one should first research symmetry algebras and the corresponding symmetry groups to study symmetries of PDEs. The CK direct method shows that one can directly obtain symmetry reductions of PDEs without using symmetry groups. Inspired by the method, two finite symmetry transformation group methods were brought up. One method is based on Lax pairs of the PDEs, which can only be applied to Lax integrable models. The other method is independent of a Lax pair of the PDEs, which is known as the modified CK (MCK) direct method. In this section, we will introduce the progress of the finite symmetry transformation group methods.

3.1. MCK Direct Method

In ref. [23], the main idea of the MCK method was proposed, and the calculation process was displayed by using the Kadomtsev–Petviashvili (KP) equation as an example. The results include the traditional Lie symmetry groups and some non-Lie symmetry groups.

The basic idea of the MCK method is that the PDE (30) is required to be invariant under the transformation $\{w,\ y_1, y_2, \cdots, y_n\} \rightarrow \{V,\ Y_1, Y_2, \cdots, Y_p\}$, which means that U satisfies the PDE when

$$\begin{aligned} w(y_1, y_2, \cdots, y_p) = W(y_1, y_2, \cdots, y_p, V(Y_1, Y_2, \cdots, Y_p)),\\ (Y_i = Y_i(y_1, y_2, \cdots, y_p), i = 1, 2, \cdots, p) \end{aligned} \tag{44}$$

is substituted into Equation (30).

The same as the CK direct method, the group transformation (44) is assumed that it can be replaced by a simpler form, as follows:

$$w(y_1,\ y_2, \cdots, y_p) = \phi(y_1,\ y_2, \cdots, y_p) + \psi(y_1,\ y_2, \cdots, y_p)V(Y_1,\ Y_2, \cdots, Y_p) \tag{45}$$

for some PDEs.

The difference between the CK and MCK methods lie in the fact that the number of variables Y_i is different. The number of variables Y_i in (31) and (32) is q, which is less than p. However, the number of the variables X_i in (44) and (45) is p. So, the symmetry group can be obtained while the variables of the reduced equation remain the same as those of the original PDE using the MCK method. We will use the KP system as an example to demonstrate this method.

Example 4. *Transformation group for the KP equation by the MCK method [23].*

The KP equation reads

$$w_{xt} + 3w_{yy} + w_{xxxx} - 6(ww_x)_x = 0. \tag{46}$$

Firstly, we should prove that the assumption of the group transformation

$$w = \phi(t,y,x) + \psi(t,y,x)V(T,Y,X) \tag{47}$$

is enough to replace

$$w = W(t,y,x,V(T,Y,X)) \tag{48}$$

for Equation (46), where $V(T,Y,X)$ also satisfies Equation (46).

Plugging Formula (48) into Equation (46) and classifying coefficients for $\{V_{Y^4}, V_{X^{10}}, V_{TT}\}$, one can obtain

$$Y_x = 0,\ T_y = 0,\ T_x = 0, \tag{49}$$

where $V_{X^n} \equiv \frac{\partial^n V}{\partial X^n}$. Then,

$$Y \equiv Y(t, y), \tag{50}$$
$$T \equiv T(t). \tag{51}$$

The combinations of (46), (48), (50) and (51) leads to

$$W_{VV}X_x^4 V_{XX}^2 + H_1(t, y, x, V, V_X)V_{XX} + H_2(t, y, x, V, V_X, V_{X^3}, V_{X^4}, V_Y, V_{YY}, V_T, V_{XY}) = 0, \tag{52}$$

with $\{H_1,\ H_2\}$ being two specific functions. The coefficient of the the first term V_{XX}^2 is $W_{VV}X_x^4$. When $X_x = 0$, only a trivial transformation can be obtained, then one concludes that $X_x \neq 0$. Furthermore, the formula $W_{VV} = 0$ is naturally obtained, which shows the effectiveness of Formual (47) in replacing Formula (48).

The combination of (46), (47), (50) and (51) leads to

$$X = C_1 T_t^{1/3} x - \frac{1}{18} C_1 T_t^{-2/3} T_{tt} y^2 - \frac{\gamma Y_{0t} y}{6 C_1 T_t^{1/3}} + X_0, \tag{53}$$

$$Y = \gamma C_1^2 T_t^{2/3} y + Y_0\,, \tag{54}$$

$$\phi = \frac{1}{18}(\ln T_t)_t x - \frac{(3T_t T_{ttt} - 4\tau_{tt}^2)y^2}{324\tau_t^2} - \frac{C_1 \gamma y T_t^{1/3}}{36}\left(\frac{Y_{0t}}{T_t}\right)_t + \frac{C_1^2 (Y_{0t}^2 + 12 X_{0t} T_t)}{72 T_t^{4/3}}, \tag{55}$$

$$\psi = C_1^2 T_t^{2/3}\,, \tag{56}$$

where γ and C_1 are governed by

$$\gamma = \pm 1, \quad C_1^3 = 1, \tag{57}$$

and $\{X_0 = X_0(t), Y_0 = Y_0(t), T = T(t)\}$ are arbitrary functions. Then, the following theorem can be concluded [23].

Theorem 1. Provided that $V(t, y, x)$ is a solution for Equation (46), then

$$\begin{aligned} w = {} & \frac{1}{18}(\ln T_t)_t x - \frac{(3T_t T_{ttt} - 4\tau_{tt}^2)y^2}{324\tau_t^2} - \frac{C_1 \gamma y T_t^{1/3}}{36}\left(\frac{Y_{0t}}{T_t}\right)_t \\ & + \frac{C_1^2 (Y_{0t}^2 + 12 X_{0t} T_t)}{72 T_t^{4/3}} + C_1^2 T_t^{2/3} V(T, Y, X), \end{aligned} \tag{58}$$

is also a solution for Equation (46), where the parameters satisfy (51), (53), (54), and (57).

To obtain the equivalent form by traditional methods from Theorem 1, one sets

$$\gamma = C_1 = 1, X_0 = \epsilon g, Y_0 = \epsilon h, T = \epsilon f + t, \tag{59}$$

where the parameter ϵ is infinitesimal, and $\{f, g, h\}$ are functions of t. Formula (58) turns into

$$
\begin{aligned}
w &= V + \epsilon\sigma(V) + O(\epsilon^2), \\
\sigma(V) &= \left(h - \frac{y}{6}g_t + \frac{x}{3}f_t - \frac{y^2}{18}f_{tt}\right)V_x + \left(g + \frac{2}{3}f_t y\right)V_y + \frac{2}{3}f_t V \\
&\quad + fV_t + \frac{y}{36}g_{tt}(t) - \frac{x}{18}f_{tt} - \frac{1}{6}h_t + \frac{y^2}{108}f_{ttt}.
\end{aligned} \tag{60}
$$

The corresponding symmetry vector reads

$$
\begin{aligned}
\widetilde{\mathbf{V}} = &\left(h - \frac{y}{6}g_t + \frac{x}{3}f_t - \frac{y^2}{18}f_{tt}\right)\frac{\partial}{\partial x} + \left(g + \frac{2}{3}f_t y\right)\frac{\partial}{\partial y} + f\frac{\partial}{\partial t} \\
&+ \left(\frac{x}{18}f_{tt} - \frac{y^2}{108}f_{ttt} - \frac{2}{3}f_t V - \frac{y}{36}g_{tt} + \frac{1}{6}h_t\right)\frac{\partial}{\partial V},
\end{aligned} \tag{61}
$$

which is equivalent to the traditional result [25].

3.2. Lax Pair-Assisted Finite Symmetry Transformation Group

Lax pair is one very valuable concept for integrable PDEs, and Lax integrable is one of the most important integrals for integrable models. When we research infinite symmetries, Lax pairs have been proven to be very helpful. In Refs. [26,27], Lax pairs are also proven to be useful in deriving finite symmetries. We will still take the KP equation as an example to display the method.

Example 5. *Lax pair-assisted finite symmetry transformation groups of the KP equation [26,27].*

The KP equation reads

$$
\left(w_t + \frac{1}{4}w_{xxx} + \frac{3}{2}ww_x\right)_x + \frac{3}{4}\sigma w_{yy} = 0,\ (\sigma = \pm 1), \tag{62}
$$

which can map into (46) by some simple scale transformations. Its Lax pair is in the form of

$$
\Phi_{xx} + w'\Phi + \sqrt{\sigma}\Phi_y = 0, \tag{63}
$$

$$
\Phi_t + \Phi_{xxx} + \frac{3}{2}w'\Phi_x - \frac{3}{4}\left(w'_x - \sqrt{\sigma}\int w'_y \mathrm{d}x\right)\Phi = 0, \tag{64}
$$

with w' being a solution of Equation (62).

Set

$$
\Phi = G\Psi(T, Y, X), \tag{65}
$$

where Ψ also satisfies Formulas (63) and (64), and $\{T, Y, X, G\}$ are functions related to $\{t, y, x\}$. Then,

$$
\Psi_{XX} + w(T, Y, X)\Psi + \sqrt{\sigma}\Psi_Y = 0, \tag{66}
$$

$$
\Psi_T + \Psi_{XXX} + \frac{3}{2}w(T, Y, X)\Psi_X - \frac{3}{4}\left[w_X(T, Y, X) - \sqrt{\sigma}\int w_Y(T, Y, X)\mathrm{d}X\right]\Psi = 0. \tag{67}
$$

Substituting (65), (66) and (67) into (63) and (64), and classifying the coefficients for Ψ and its derivatives, we obtain

$$
X = T_t^{\frac{1}{3}}x - \frac{2}{9}\frac{y^2 T_{tt}}{\sigma T_t^{\frac{2}{3}}} - \frac{2}{3}\frac{\theta_t y}{\sigma T_t^{\frac{1}{3}}} + \gamma, \quad Y = T_t^{\frac{2}{3}}y + \theta, \quad T = T(t), \tag{68}
$$

$$
w' = -\frac{2}{9}\frac{xT_{tt}}{T_t} + T_t^{\frac{2}{3}}w - \frac{4\tau_{tt}{}^2 y^2}{81\sigma T_t^2} - \frac{4\tau_{tt}\theta_t y}{27\sigma T_t^{5/3}} - \frac{\theta_t{}^2}{9\sigma T^{4/3}} - \frac{\sqrt{\sigma}G_{0y}}{G_0}. \tag{69}
$$

where θ, γ, and T are arbitrary functions of t. Then, the following theorem can be concluded [26].

Theorem 2. Provided that $w = w(t, y, x)$ is a solution for Equation (62), then

$$\begin{aligned} w' &= T_t^{\frac{2}{3}} w(T, Y, X) - \frac{16}{81} \frac{(T_{tt} y)^2}{\sigma T_t^2} - \frac{4}{9} \frac{T_{tt} y \theta_t}{\sigma T_t^{\frac{5}{3}}} - \frac{2}{9} \frac{\theta_t^2}{\sigma T_t^{\frac{4}{3}}} \\ &\quad - \frac{2}{9} \frac{x T_{tt}}{T_t} + \frac{4}{27} \frac{y^2 T_{ttt}}{\sigma T_t} + \frac{4}{9} \frac{y \theta_{tt}}{\sigma T_t^{\frac{2}{3}}} - \frac{2}{3} \frac{\gamma_t}{T_t^{\frac{1}{3}}} \end{aligned} \tag{70}$$

is also a solution of Equation (62), where T, Y, and X are governed by (68).

To re-derive the traditional symmetries from Formula (70), the variables in Formula (68) can be set as follows:

$$T = t + \epsilon f, \ \theta = \epsilon g, \ \gamma = \epsilon h(t), \tag{71}$$

where the parameter ϵ is infinitesimal, and $\{f, g, h\}$ are functions of t. Furthermore, Formual (70) turns into

$$w' = w + \epsilon \sigma(w) + O(\epsilon^2), \tag{72}$$

$$\begin{aligned} \sigma(w) &= f w_t + \left(h - \frac{2y}{3} \sigma g_t + \frac{x}{3} f_t - \frac{2y^2}{9} \sigma f_{tt} \right) w_x + \left(\frac{2y}{3} f_t + g \right) w_y \\ &\quad + \frac{2w}{3} f_t - \frac{2x}{9} f_{tt} + \frac{4y^2}{27} \sigma f_{ttt} + \frac{4y}{9} g_{tt} + \frac{2}{3} h_t. \end{aligned} \tag{73}$$

It is obvious that (73) is equivalent to the result by the standard Lie approach [25].

The method of deriving finite symmetry transformation groups from its Lax pair has been extended to study other Lax integrable models, such as the sine-Gordon equation, the Csmassa–Holm equation, and some other PDEs [26].

4. Nonlocal Symmetries

Initially, the symmetry analysis of PDEs showed local symmetries. A infinitesimal of local symmetry is related to local variables only. Nonlocal symmetries were first studied by Krasil'shchik and Vinogradov [28–30]. Ref. [28] extensively discussed the definition of nonlocal symmetries and pointed out that a generating function for nonlocal symmetry is a differential-integro-type operator [28]. Since then, many nonlocal symmetry methods have been proposed. In this section, we select four typical nonlocal symmetry methods from the literature to introduce the progress of the nonlocal symmetry theory.

4.1. Nonlocal Symmetries Derived from Conserved Form

For a given PDE S, Bluman et al. presented many methods to systematically find nonlocal symmetries [31–34]. They introduce auxiliary potential variables to research nonlocal symmetries, where the auxiliary variables are related to conservation laws [35]. They demonstrate that $2q - 1$ nonlocal equations can be derived from an equation possessing q local conservation laws.

There are many methods to transform PDEs that are not in conservative forms into conservative forms. A q-order conservative form of PDE S reads [31]

$$\sum_{i=1}^{p} \frac{\partial}{\partial y_i} F_i(y_1, y_2, \cdots, y_p, w, w^{(l)}(y), \cdots, w^{(q-1)}(y)) = 0, \tag{74}$$

where $w^{(l)}$ denotes partial derivatives of order l of w, with one dependent variable w, and $p \geq 2$ independent variables $y = (y_1, y_2, \cdots, y_p)$. For Equation (74), there is an determined system T

$$G_1 = \frac{\partial \psi_1}{\partial y_2}, \quad (75)$$

$$G_j = (-1)^{(j-1)} \left(\frac{\partial \psi_{j-1}}{\partial y_{j-1}} + \frac{\partial \psi_j}{\partial y_{j+1}} \right), \quad (76)$$

$$G_p = (-1)^{p+1} \frac{\partial \psi_{p-1}}{\partial y_{p-1}}, \quad (77)$$

where $1 < j < p$.

When $p = 4$, let $G_1, G_2, G_3, G_4, y_1, y_2, y_3$, and y_4 equal to A, B, C, E, x, y, z, and t, respectively. Then, Equation (74) turns into

$$\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z} + \frac{\partial E}{\partial t} = 0. \quad (78)$$

Its determined system T is

$$\frac{\partial \psi_1}{\partial y} = A(x, y, z, t, w, w^{(1)}, \cdots, w^{(q-1)}), \quad (79)$$

$$-\left(\frac{\partial \psi_2}{\partial z} + \frac{\partial \psi_1}{\partial x} \right) = B(x, y, z, t, w, w^{(1)}, \cdots, w^{(q-1)}), \quad (80)$$

$$\frac{\partial \psi_3}{\partial t} + \frac{\partial \psi_2}{\partial y} = C(x, y, z, t, w, w^{(1)}, \cdots, w^{(q-1)}), \quad (81)$$

$$-\frac{\partial \psi_3}{\partial z} = E(x, y, z, t, w, w^{(1)}, \cdots, w^{(q-1)}). \quad (82)$$

Assume that there exists the following symmetry transformations for the system T:

$$y^* = y + \lambda\, \xi_T(y, w, \psi) + O(\lambda^2), \quad (83)$$

$$w^* = w + \lambda\, \eta_T(y, w, \psi) + O(\lambda^2), \quad (84)$$

$$\psi^* = \psi + \lambda\, \zeta_T(y, w, \psi) + O(\lambda^2). \quad (85)$$

with λ as the infinitesimal parameter for symmetries.

When ξ_T and η_T are actually related to ψ, Formulas (83) and (84) are new symmetry transformations. Due to the existence of ψ, Formulas (83)–(85) are neither Lie–Bäcklund transformations nor Lie point symmetry transformations. In Formulas (75)–(77), ψ is defined as the derivative form, which leads symmetry transformations (83)–(85) to be nonlocal symmetries. Next, we will demonstrate the potential nonlocal symmetry method with the help of the nonlinear diffusion equation.

Example 6. *Potential nonlocal symmetries for the diffusion equation [31].*

The diffusion system S is in the form of

$$\frac{\partial}{\partial x}\left[K(w) \frac{\partial w}{\partial x} \right] - \frac{\partial w}{\partial t} = 0. \quad (86)$$

Equation (86) is already in a conserved form:

$$\frac{\partial A}{\partial x} - \frac{\partial B}{\partial t} = 0, \quad (87)$$

where

$$A = K(w)\frac{\partial w}{\partial x}\,, \tag{88}$$

$$B = w\,. \tag{89}$$

The corresponding system T is

$$\frac{\partial \psi}{\partial t} = K(w)\frac{\partial w}{\partial x}\,, \tag{90}$$

$$\frac{\partial \psi}{\partial x} = w\,. \tag{91}$$

Solving the symmetries of System (86) and System (90) and (91) by the classical Lie algorithm, one finds that their symmetry groups depend on the form of $K(w)$. The group G_s of (86) was listed in Refs. [14,22]. The group G_T of (90) and (91) was firstly presented in ref. [31]. The most special condition is

$$K(w) = \frac{\mathrm{e}^{\left(c_3 \int \frac{\mathrm{d}w}{c_2+w^2+c_1 w}\right)}}{c_2 + w^2 + c_1 w}\,, \tag{92}$$

where the parameters $\{c_1, c_2, c_3\}$ do not satisfy the following relationships:

$$(1)\; c_3 = \pm 2,\; {c_1}^2 - 4c_2 > 0, \tag{93}$$

$$(2)\; c_3 = 0,\; {c_1}^2 - 4c_2 = 0. \tag{94}$$

In this case, the symmetry group G_T of System (90) and (91) is

$$L_0 = \frac{\partial}{\partial \psi}\,,\quad L_1 = \frac{\partial}{\partial x}\,,\quad L_2 = \frac{\partial}{\partial t}\,,\quad L_3 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} + \psi\frac{\partial}{\partial \psi}\,,$$
$$L_4 = \psi\frac{\partial}{\partial x} + (c_3 - c_1)t\frac{\partial}{\partial t} - (w^2 + c_1 u + c_2)\frac{\partial}{\partial w} - (c_2 x + c_1 \phi)\frac{\partial}{\partial \psi}\,. \tag{95}$$

Comparing G_s and G_T, expressed by (95), we can see that new symmetries for Equation (86) can be derived from G_T. The new symmetries are related to nonlocal symmetries.

Due to the fact that a given PDE can have different conserved forms, a PDE could have multiple nonlocal symmetries [31].

4.2. Nonlocal Symmetries Derived from the Conformal Invariant Form and Residual Symmetry

For a PDE with one independent variable ψ, its Möbious transformation is

$$\psi \longrightarrow \frac{a_1 + a_2\psi}{a_3 + a_4\psi}\,, \qquad (a_1 a_4 \neq a_2 a_3), \tag{96}$$

which is also called conformal transformation. A PDE in its Schwartzian form (conformal invariant form) can remain unchanged under Transformation (96). With the help of the Painlevé expansion or other transformations, most integrable PDEs can be transformed into their Schwartzian forms.

Ref. [36] pointed out that one can obtain infinitely many nonlocal symmetries. The well-known KP and the KdV equation were selected as examples to demonstrate the effectiveness of nonlocal symmetries derived from their Schwartzian forms [36,37]. In this review paper, the KdV equation is selected to demonstrate this nonlocal symmetry method.

Example 7. *Nonlocal symmetry of the KdV equation deriving from its Schwartzian form [36].*

The KdV equation is written as

$$w_t - w_{xxx} - 6ww_x = 0. \quad (97)$$

Under the transformation

$$w = \gamma - \frac{1}{2}\left(\frac{\psi_{xx}}{\psi_x}\right)_x - \frac{1}{4}\left(\frac{\psi_{xx}}{\psi_x}\right)^2, \quad (98)$$

Equation (97) transforms into the Schwartzian KdV equation

$$\frac{\psi_t}{\psi_x} = 6\gamma + \{\psi; x\}, \quad (99)$$

where

$$\{\psi; x\} = \left(\frac{\psi_{xxx}}{\psi_x}\right) - \frac{3}{2}\left(\frac{\psi_{xx}}{\psi_x}\right)^2 \quad (100)$$

denotes a Schwartz derivative. Formula (99) can remain unchanged when it is imposed by Transformation (96). A type of parameter combination is $\{a_1 = 0, a_2 = 1, a_3 = 1, a_4 = \epsilon\}$, where ϵ is a infinitesimal parameter. In this condition, Transformation (96) turns into a symmetry form:

$$\psi \longrightarrow \psi - \epsilon\psi^2. \quad (101)$$

So $-\psi^2$ is one symmetry σ^ψ for Schwartzian Formula (99).

Formula (98) leads to the relationship of symmetry σ^w and symmetry σ^ψ is governed by

$$\sigma^w = -\frac{1}{2}\partial_x\left(\frac{\partial_x^2}{\psi_x} - \frac{\psi_{xx}}{\psi_x^2}\partial_x\right)\sigma^\psi - \frac{1}{2}\left(\frac{\psi_{xx}}{\psi_x}\right)\left(\frac{\partial_x^2}{\psi_x} - \frac{\psi_{xx}}{\psi_x^2}\partial_x\right)\sigma^\psi. \quad (102)$$

The substitution of $\sigma^\psi = -\psi^2$ into Formula (102) leads to a nonlocal symmetry σ^w in the form of

$$\sigma^w = 2\psi_{xx}, \quad (103)$$

where ψ satisfies the Schwartzian KdV Equation (99).

It should be noted that the method of deriving a nonlocal symmetry from the conformal invariant form was developed into a residual symmetry in 2013 [38].The residue in a truncated Painlevé expansion for a Painlevé integrable model is found to be a nonlocal symmetry, so the nonlocal symmetry is called the residual symmetry. For ease of comparison, we still take the KdV Equation (97) as an example to demonstrate residual symmetry [38].

Example 8. *Residual symmetry of the KdV equation [38].*

Equation (97) could pass the Painlevé integrable test with the help of the Painlevé expansion:

$$w = \frac{w_0}{\psi^2} + \frac{w_1}{\psi} + w_2 + w_3\psi + w_4\psi^2 + \cdots, \quad (104)$$

where

$$w_0 = -2\psi_x^2, \quad w_1 = 2\psi_{xx}, \quad w_2 = \frac{1}{6}\psi_x^{-2}(3\psi_{xx}^2 + \psi_x\psi_t - 4\psi_x\psi_{xxx}). \quad (105)$$

In particular, $w_1 = 2\psi_{xx}$ is the residue in the Laurent-like series (104). According to the application of the standard truncated Painlevé expansion, a type of exact solution for Equation (97) can be written as

$$w = \frac{w_0}{\psi^2} + \frac{w_1}{\psi} + w_2\,, \tag{106}$$

where the functions w_0, w_1, and w_2 are expressed by (105).

Substituting (105) and (106) into the KdV Equation (97) and collecting the coefficient of ψ^{-2}, we can obtain the Schwartzian KdV Equation (99). Formula (103) indicates that $2\psi_{xx}$ is one symmetry for Equation (97), and Formula (105) indicates that $2\psi_{xx}$ is the residue in the truncated Painlevé expansion for Equation (97). Thus, this type of symmetry is called the residual symmetry [38].

The localization of a nonlocal symmetry is presented in detail in ref. [38]. Regarding why residual symmetry is nonlocal, ref. [39] provides a proof with the help of the mKdV equation. As a result of the simplicity of the truncated Painlevé expansion, nonlocal symmetries on the (2+1)-dimensional Chaffee–Infante equation [40], a new (3+1)-dimensional generalized KP equation [41], the Whitham–Broer–Kaup equation [42], the (3+1)-dimensional shallow water wave equation [43], the Davey–Stewartson III equation [44], and some other equations are researched using the residual symmetry method.

4.3. Nonlocal Symmetries Derived from Darboux Transformations

Using Darboux transformations, we can usually derive new solutions from an old solution. To find nonlocal symmetries, ref. [45] provides a method to find nonlocal symmetries by Darboux transformations. The invariance of PDEs caused by Darboux transformations can be used to derive nonlocal symmetries of PDEs, such as the KP equation, the KdV equation, and some other equations [37,45]. We will apply the KdV equation as an instance to demonstrate this method.

Example 9. *Nonlocal symmetry of the KdV equation deriving from Darboux transformations [45].*

The KdV equation is written as [45]

$$w_t + w_{xxx} - 6ww_x = 0. \tag{107}$$

Transformations $t \to -t$ and $w \to -w$ can transform Equation (107) into Equation (97). The Lax pair of Equation (107) reads

$$\psi_t = -w_x\,\psi + (2w + 4\gamma)\psi_x\,, \tag{108}$$

$$\psi_{xx} = (w - \gamma)\psi\,. \tag{109}$$

where the spectral function ψ is an arbitrary function of $\{t, x, \gamma\}$, and γ is an arbitrary constant.

Setting $\gamma = \gamma_0$, then $\psi(t, x, \gamma) = \psi(t, x, \gamma_0)$, where γ_0 is an arbitrary constant. $\psi(t, x, \gamma_0)$ can be set as $\psi(t, x, \gamma_0) = f(t, x)$. For the KdV Equation (107), there exists a Darboux transformation [46,47]

$$\widetilde{w}(t, x) = w - 2(\ln f)_{xx}, \tag{110}$$

$$\Phi(t, x, \gamma) = \psi_x(t, x, \gamma) - \frac{f_x}{f}\psi(t, x, \gamma) \tag{111}$$

with $\widetilde{w}(t, x)$ as the solution for Equation (107), and $\Phi(t, x, \gamma)$ as the spectral function corresponding to $\widetilde{w}(t, x)$.

We should notice that both of γ and γ_0 are arbitrary constants; thus, (110) indicates that

$$\widetilde{w}(t, x, \gamma) = w(t, x) - 2\ln\psi_{xx}(t, x, \gamma) \tag{112}$$

is also a solution for Equation (107). When $\gamma = 0$, (112) degenerates to

$$\widetilde{w}(t,x,0) = w(t,x) - 2\ln\psi_{xx}(t,x,0). \tag{113}$$

We may set

$$W(t,x) = \widetilde{w}(t,x,0) = w(t,x) - 2\ln\psi_{xx}(t,x,0). \tag{114}$$

Then, $W(t,x)$ is also a solution for Equation (107). We can then expand Formula (112) to

$$\begin{aligned} \widetilde{w}(t,x,\gamma) &= w(t,x) - 2\ln\psi_{xx}(t,x,0) + \gamma\left[\left(-2\frac{\partial^2}{\partial x^2}\ln\psi\right)_\gamma\Big|_{\gamma=0}\right] + O(\gamma^2) \\ &= W(t,x) + \gamma\left[\left(-2\frac{\partial^2}{\partial x^2}\ln\psi\right)_\gamma\Big|_{\gamma=0}\right] + O(\gamma^2). \end{aligned} \tag{115}$$

So, $\left(-2\frac{\partial^2}{\partial x^2}\ln\psi\right)_\gamma\Big|_{\gamma=0}$ is a symmetry for Equation (107) corresponding to W. We set

$$\varphi(t,x) = \psi(t,x,0), \tag{116}$$
$$\widetilde{\varphi}(t,x) = \psi_\gamma(t,x,0). \tag{117}$$

Then, Formula (114) turns into

$$w = W(t,x) + 2\ln\psi_{xx}(t,x,0) = W(t,x) + 2\ln\varphi_{xx}\,. \tag{118}$$

When γ tends to zero, $\psi(t,x,\gamma)$ can be expanded into

$$\begin{aligned} \psi(t,x,\gamma) &= \psi(t,x,0) + \gamma\psi_\gamma(t,x,0) + O(\gamma^2) \\ &= \varphi(t,x) + \gamma\widetilde{\varphi}(t,x) + O(\gamma^2) \\ &= \varphi(t,x) + \gamma\widetilde{\varphi}(t,x). \end{aligned} \tag{119}$$

The symmetry $\left(-2\frac{\partial^2}{\partial x^2}\ln\psi\right)_\gamma\Big|_{\gamma=0}$ turns into

$$\sigma^w = -2[(\ln\psi)_{xx}]_\gamma\Big|_{\gamma=0} = -2\left\{\frac{\partial^2}{\partial x^2}\ln[\varphi(t,x) + \gamma\widetilde{\varphi}(t,x)]\right\}_\gamma\Bigg|_{\gamma=0} = -2\left(\frac{\widetilde{\varphi}}{\varphi}\right)_{xx}. \tag{120}$$

Let us discuss the constraints on φ and $\widetilde{\varphi}$. When $\gamma = 0$, substituting (116)–(118) into the Lax pair (108) and (109) and replacing W by w, we obtain

$$\varphi_t + \varphi(2(\ln\varphi)_{xxx} + w_x) - \varphi_x(4(\ln\varphi)_{xx} + 2w) = 0\,, \tag{121}$$
$$\varphi_{xx} - \varphi(2(\ln\varphi)_{xx} + w) = 0\,. \tag{122}$$

When λ tends to zero, substituting (118), (119), (121) and (122) into the the Lax pair (108) and (109) and replacing W by w, one can obtain

$$\widetilde{\varphi}_t + \widetilde{\varphi}(2(\ln\varphi)_{xxx} + w_x) - \widetilde{\varphi}_x(4(\ln\varphi)_{xx} + 2w) = 4\varphi_x\,, \tag{123}$$
$$\widetilde{\varphi}_{xx} - \widetilde{\varphi}(2(\ln\widetilde{\varphi})_{xx} + w) = -\varphi\,. \tag{124}$$

In summary, the following theorem holds [45].

Theorem 3. *When $\varphi(t,x)$ and $\widetilde{\varphi}(t,x)$ are constrained by (121)–(124), $\left(\frac{\widetilde{\varphi}}{\varphi}\right)_{xx}$ is a symmetry for Equation (107).*

Ref. [45] provides further explanation on the relationship between $\varphi(t,x)$ and $\widetilde{\varphi}(t,x)$, and proves the nonlocality of the symmetry derived from Darboux transformations.

4.4. Nonlocal Symmetries Derived from Lax Pair

Taking the mKdV equation as an example, ref. [48] proposes a method of finding a nonlocal symmetry by the Lax pair. This subsection will present some important details from ref. [48].

Example 10. *Nonlocal symmetry of the mKdV equation from a Lax pair [48].*

The mKdV equation reads

$$w_t = w_{xxx} - 6w^2 w_x \,. \tag{125}$$

The Lax pair of Equation (125) is known as [49]

$$\phi_{xx} = -2w\phi_x, \quad \phi_t = -2(w_x + w^2)\phi_x \,, \tag{126}$$

where $\phi = \phi(x,t)$ is the spectral parameter.

The linearized equation for Equation (125) is

$$\sigma_t - \sigma_{xxx} + 6w^2\sigma_x + 12w\sigma w_x = 0, \tag{127}$$

where σ is a symmetry of Equation (125) and it is supposed to be written as

$$\sigma = \xi(t,x,w,\phi,\phi_x)w_x + \tau(t,x,w,\phi,\phi_x)w_t - W(t,x,w,\phi,\phi_x). \tag{128}$$

Due to the existence of auxiliary functions ϕ and ϕ_x, the symmetry σ differs from traditional Lie symmetries and Lie–Bäcklund symmetries.

Plugging Formula (128) into Formula (127), removing w_t, ϕ_{xx}, and ϕ_t with the help of Formulas (125) and (126), one obtains the determining equations of the functions ξ, τ, and W. The calculation of ξ, τ, and W lead to the solution of σ in the following form:

$$\sigma = \left(c_6 - 6\,c_5\,t + \frac{c_1 x}{3}\right)w_x + (c_1 t + c_2)w_t - 3c_3\,\phi_x + \frac{c_1 w}{3} - \frac{c_5\phi + c_4}{\phi_x}. \tag{129}$$

where σ contains two parts:

$$\sigma_1 = -6\,c_5\,t\,w_x - 3c_3\,\phi_x - \frac{c_5\phi + c_4}{\phi_x}\,, \tag{130}$$

$$\sigma_2 = \left(c_6 + \frac{c_1 x}{3}\right)w_x + (c_1 t + c_2)w_t + \frac{c_1 w}{3}\,. \tag{131}$$

σ_1 is related to a nonlocal symmetry and σ_2 corresponds to the traditional Lie group symmetry.

In order to obtain a symmetry reduction corresponding to the nonlocal symmetry, the authors of ref. [48] assumed that the symmetry is in the form of

$$\sigma^w = \xi(t,x,w,\phi,\phi_1)w_x + \tau(t,x,w,\phi,\phi_1)w_t - W(t,x,w,\phi,\phi_1), \tag{132}$$

$$\sigma^\phi = \xi(t,x,w,\phi,\phi_1)\phi_x + \tau(t,x,w,\phi,\phi_1)\phi_t - \Phi(t,x,w,\phi,\phi_1), \tag{133}$$

$$\sigma^{\phi_1} = \xi(t,x,w,\phi,\phi_1)\phi_{1x} + \tau(t,x,w,\phi,\phi_1)\phi_{1t} - \Phi_1(t,x,w,\phi,\phi_1), \tag{134}$$

where

$$\phi_1 = \phi_x. \tag{135}$$

Applying symmetry conponents (132)–(134) to Equations (125), (126), and (135), one can obtain the solution of ξ, τ, W, Φ, and Φ_1, which are govened by

$$\tau = c_2 + c_1 t, \qquad \xi = c_4 + \frac{c_1 x}{3}, \qquad W = c_3\phi_1 - \frac{c_1 u}{3},$$
$$\Phi = -c_3\Phi^2 + c_5\Phi + c_6,\ \Phi_1 = -2c_3\,\Phi\,\Phi_1 + c_5\Phi_1 - \frac{c_1\Phi_1}{3}. \tag{136}$$

From Formula (136), the following six operators can be obtained:

$$V_1 = \frac{x}{3}\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \frac{w}{3}\frac{\partial}{\partial w} - \frac{\Phi_1}{3}\frac{\partial}{\partial \Phi_1},\ V_2 = \frac{\partial}{\partial t},\ V_3 = \Phi_1\frac{\partial}{\partial w} - \Phi^2\frac{\partial}{\partial \Phi} - 2\Phi\Phi_1\frac{\partial}{\partial \Phi_1},$$
$$V_4 = \frac{\partial}{\partial x}, \qquad V_5 = \Phi\frac{\partial}{\partial \Phi} + \Phi_1\frac{\partial}{\partial \Phi_1}, \qquad V_6 = \frac{\partial}{\partial \Phi}. \tag{137}$$

Applying commutator operators $[V_k, V_j] = V_kV_j - V_jV_k$, one can list the corresponding Lie bracket presented in Table 1, where the (k, j)-th entry denotes $[V_k, V_j]$.

Table 1. Lie bracket [48].

Lie	V_1	V_2	V_3	V_4	V_5	V_6
V_1	0	$-V_2$	0	$-\frac{1}{3}V_4$	0	0
V_2	V_2	0	0	0	0	0
V_3	0	0	0	0	$-V_3$	$2V_5$
V_4	$\frac{1}{3}V_4$	0	0	0	0	0
V_5	0	0	V_3	0	0	$-V_6$
V_6	0	0	$-2V_5$	0	V_6	0

Applying Table 1 and the Lie series in the form of

$$Ad(\exp(\varepsilon V_k))Vj = V_j - \varepsilon[V_k, V_j] + \frac{1}{2}\varepsilon^2[V_k, [V_k, V_j]] - ,\cdots \tag{138}$$

one can obtain the adjoint representation present in Table 2, where the (k, j)-th entry denotes $Ad(\exp(\varepsilon V_k))V_j$.

Table 2. Adjoint representation [48].

Lie	V_1	V_2	V_3	V_4	V_5	V_6
V_1	V_1	$e^{\varepsilon}V_2$	V_3	$e^{\frac{1}{3}\varepsilon}V_4$	V_5	V_6
V_2	$V_1 - \varepsilon V_2$	V_2	V_3	V_4	V_5	V_6
V_3	V_1	V_2	V_3	V_4	$V_5 + \varepsilon V_3$	$V_6 - 2\varepsilon V_5 - \varepsilon^2 V_3$
V_4	$V_1 - \frac{1}{3}\varepsilon V_4$	V_2	V_3	V_4	V_5	V_6
V_5	V_1	V_2	$e^{-\varepsilon}V_3$	V_4	V_5	εV_6
V_6	V_1	V_2	$V_3 + 2\varepsilon V_5 - \varepsilon^2 V_6$	V_4	$V_5 - \varepsilon V_6$	V_6

4.5. *Localization of Nonlocal Symmetries*

In this subsection, we will take the MKdV Equation (125) as the instance to demonstrate the localization method of nonlocal symmetries. The content of this subsection is firstly reported in this review.

Example 11. *Localization method of nonlocal symmetries for the MKdV Equation* (125)*.*

From Equation (126), ϕ can be solved with the final result:

$$\phi = a_1\int \exp(2v)\mathrm{d}x + a_0,\ v_x = w,\ v_t = w_{xx} - 2w^3 = v_{xxx} - 2v_x^3. \tag{139}$$

Using the relation (139), the symmetry (129) can be rewritten as

$$\sigma = a_1(3tw_t + xw_x + w) + a_2w_x + a_3w_t + a_4v_{1x} + a_5v_{2x} + a_6(v_1v_{2x} - 6tw_x), \tag{140}$$

with arbitrary constants $a_1, a_2, \ldots, a_6$ while v_1 and v_2 are related to v by $\{v_1; x\} \equiv \frac{v_{1xxx}}{v_{1x}} - \frac{3}{2}\frac{v_{1xx}^2}{v_{1x}^2}$,

$$v_{2x} = \exp(-2v),\ v_{2t} = -2\exp(-2v)\big(v_{2xx} + v_{2x}^2\big) = \{v_2; x\}v_{2x}, \tag{141}$$

$$v_{1x} = \exp(2v),\ v_{1t} = 2\exp(2v)\big(v_{1xx} - v_{1x}^2\big) = \{v_1; x\}v_{1x}. \tag{142}$$

In the symmetry expression (140), the a_1 part is related to the scaling invariance. It is interesting that to find exact solutions via nonlocal symmetries, the a_2 and a_3 parts are related to the space and time translation invariance, the a_4 and a_5 parts are the Darboux transformation-related nonlocal symmetries, while the a_6 part corresponds to the nonlocal Galileo transformation invariance.

In order to find some invariant solutions related to the nonlocal symmetries, one has to use the localization method [38]. In this subsection, we take $a_5 = a_6 = 0$ in (140) for simplicity. In this special case, the nonlocal symmetry of w can be rewritten as

$$\sigma = a_1(3tw_t + xw_x + w) + a_2w_x + a_3w_t + a_4\exp(2v), \tag{143}$$

with

$$v_x = w,\ v_t = w_{xx} - 2w^3 = v_{xxx} - 2v_x^3\,. \tag{144}$$

The nonlocal symmetry (143) cannot be directly used to find invariant solutions. The first step in the nonlocalization procedure is to find a related symmetry transformation of v, $v \to v + \epsilon\sigma_1$, from (144). The result reads

$$\sigma_1 = a_1(3tv_t + xv_x) + a_2v_x + a_3v_t + a_4v_1\,, \tag{145}$$

with

$$v_{1x} = \exp(2v),\ v_{1t} = 2\exp(2v)\big(v_{1xx} - v_{1x}^2\big). \tag{146}$$

The next step is to find the related symmetry transformation of v_1, $v_1 \to v_1 + \epsilon\sigma_2$, from (146), with the result

$$\sigma_2 = a_1(3tv_{1t} + xv_{1x} - v_1) + a_2v_{1x} + a_3v_{1t} + a_4v_1^2\,, \tag{147}$$

even though the symmetry (140) is nonlocal for the modified KdV equation (125). However, the symmetries (140), (145), and (147), i.e.,

$$\Sigma = \begin{pmatrix} \sigma \\ \sigma_1 \\ \sigma_2 \end{pmatrix} = \begin{pmatrix} a_1(3tu_t + xu_x + u) + a_2u_x + a_3u_t + a_4\exp(2v) \\ a_1(3tv_t + xv_x) + a_2v_x + a_3v_t + a_4v_1 \\ a_1(3tv_{1t} + xv_{1x} - v_1) + a_2v_{1x} + a_3v_{1t} + a_4v_1^2 \end{pmatrix}, \tag{148}$$

are local symmetries for the prolonged system of {(125), (144), and (146) }. Now, it is standard to find invariant solutions of {(125), (144), and (146) } by solving $\Sigma = 0$ from (148) with two important special cases:

Case 1. $a_1 = 1,\ a_2 = a_3 = 0,\ a_4 = c$. In this case, the symmetry invariant solution reads

$$\begin{cases} v_1 = \dfrac{1}{c + t^{-1/3} V(xt^{-1/3})}, \\ v = \dfrac{1}{2} \ln(v_{1x}), \\ w = \dfrac{1}{2} [\ln(v_{1x})]_x, \end{cases} \tag{149}$$

while $V(xt^{-1/3}) \equiv V(\xi)$ is determined by the reduction equation

$$\{V,\ \xi\} + \frac{1}{3}(\xi V)_\xi = 0. \tag{150}$$

Case 2. $a_1 = 0,\ a_2 = b,\ a_3 = 1,\ a_4 = c$. In this situation, the symmetry invariant solution possesses the form

$$\begin{cases} v_1 = \dfrac{1}{ct + W(\eta)},\ \eta = x - bt, \\ v = \dfrac{1}{2} \ln[-W_\eta (ct + W)^{-2}], \\ w = \dfrac{W_{\eta\eta}}{2W_\eta} - \dfrac{W_\eta}{W + ct}\ . \end{cases} \tag{151}$$

The invariant function $W \equiv W(\eta) = W(x - bt)$ is a solution of the ODE

$$\{W,\ \eta\} + bW_\eta - c = 0, \tag{152}$$

which can be solved by elliptic integration:

$$U \equiv W_\eta,\ \int^U \frac{\mathrm{d}z}{\sqrt{Cz^3 + 2bz^2 - cz}} = \eta - \eta_0, \tag{153}$$

with two arbitrary constants C and η_0.

In addition to the four methods of finding nonlocal symmetries and the above localization method, many other methods are proposed in the literature. Due to the space limitations of this paper, we will not review them one by one.

5. Supersymmetric Equation and Supersymmetric Dark Equation

To unify bosons and fermions in physics, the idea of supersymmetry was introduced into physics [50,51]. In mathematics, on the basis of the standard commuting and bosonic variables, the anticommuting and fermionic variables were increased to describe supersymmetries [52]. By means of extending variables, many integrable PDEs were extended to supersymmetric systems [53]. A supersymmetric KdV equation and the corresponding Lax pair were proposed in ref. [54]. Unfortunately, the supersymmetric KdV equation did not satisfy the supersymmetry invariant [55] even though it was a super-integrable model. The KP hierarchy was extended to supersymmetric fields and the integrablility was proposed in ref. [56]. Many branches of mathematics have been extended and their super analogues have been constructed and studied; examples include super Lie algebra and supermanifold [57,58]. From the supersymmetric equations, the concepts of dark equations and supersymmetric dark equations have been proposed in the literature. In this section, we will review the basic properties of supersymmetric equations and dark equations, and select some typical methods from the literature to introduce the construction of supersymmetric equations and supersymmetric dark equations, and introduce the bosonization method of the supersymmetric integrable system.

5.1. Supersymmetric Equations

Time dependence is implicit everywhere, so only a space supersymmetric invariance is considered. With the help of a Grassman variable θ, the classical spacetimes (t, x) are generally extended into super-spacetimes (t, x, θ). The Grassman variable θ is fermionic and anticommuting, and it satisfies $\theta^2 = 0$. Simultaneously, a super dependent variable $\Phi(t, x, \theta)$ was introduced to replace the dependent variable $w(t, x)$ of a PDE. The super dependent variable can be a fermionic variable or a bosonic variable.

Applying the Taylor expansion and $\theta^2 = 0$, the super dependent variable $\Phi(t, x, \theta)$ can be expanded to a simpler form. When $\Phi(t, x, \theta)$ is a fermionic variable, it turns into

$$\Phi(t, x, \theta) = \theta w(t, x) + \xi(t, x), \tag{154}$$

with $\xi(t, x)$ being a anticommuting variable, and $w(t, x)$ being a commuting variable. When $\Phi(t, x, \theta)$ is a bosonic variable, it expands into

$$\Phi(t, x, \theta) = \theta \lambda(t, x) + w(t, x), \tag{155}$$

where $\lambda(t, x)$ is an anticommuting variable.

The spatial covariant derivative $\mathcal{D}$ is an indispensable concept in supersymmetry theory, which is governed by $\mathcal{D} \equiv \theta\partial_x + \partial_\theta$. It satisfies

$$\begin{aligned} \mathcal{D}^2\Phi(t, x, \theta) &= (\partial_\theta + \theta\partial_x)(\partial_\theta + \theta\partial_x)(\theta w + \xi) \\ &= (\partial_\theta + \theta\partial_x)(w + \theta\xi_x) \\ &= \theta w_x + \xi_x \\ &= \partial_x\Phi(t, x, \theta), \end{aligned} \tag{156}$$

where $w = w(t, x)$ and $\xi = xi(t, x)$. So, $\mathcal{D}^2 = \partial_x$. The spatial covariant derivative rules for function products are

$$\mathcal{D}(hk) = \begin{cases} (\mathcal{D}h)\,k - h\,(\mathcal{D}k), & \text{if } h \text{ is a fermionic field,} \\ (\mathcal{D}h)\,k + h(\mathcal{D}\,k), & \text{if } h \text{ is a bosonic field.} \end{cases} \tag{157}$$

There have been many methods proposed for constructing supersymmetric systems, among which the simplest one is the direct construction method. Now, we will use the KdV equation to demonstrate the construction of supersymmetric equations using the direct construction method.

Example 12. *The construction of a supersymmetric KdV equation via the direct construction method [59].*

The KdV equation is written as

$$w_t + w_{xxx} - 6ww_x = 0, \tag{158}$$

where w is a commuting field. Firstly, we expand the potential function $w(t, x)$ into a superfield. There are two possible types of extension. One is fermionic extension:

$$\Phi(t, x, \theta) = \xi(t, x) + \theta w(t, x), \tag{159}$$

and the other is bosonic extension in the form of

$$W(t, x, \theta) = w(t, x) + \theta\xi(t, x), \tag{160}$$

where ξ and θ are the anticommuting variable and Grassman variable, respectively.

For the sake of constructing a supersymmetric KdV equation, every term in Equation (158) is multiplied by the parameter θ and can be rewritten with the help of the symbol $\mathcal{D}$ and a superfield. A superfield is usually represented by characters such as Φ, Ψ, etc. In this review paper, we choose Ψ. In this way, the fermionic part of every term is equal to the original term. The corresponding transformations are

$$\theta w_t \longrightarrow \Psi_t \,, \tag{161}$$

$$\theta w w_x \longrightarrow (\mathcal{D}\Psi)\Psi_x \text{ or } (\mathcal{D}\Psi_x)\Psi \,, \tag{162}$$

$$\theta w_{xxx} \longrightarrow \Psi_{xxx} \,, \tag{163}$$

where $\Psi(t,x,\theta) = \xi(t,x) + \theta w(t,x)$.

The second term $\theta(ww_x)$ corresponds to the fermionic parts of both $(\mathcal{D}\Psi)\Psi_x$ and $(\mathcal{D}\Psi_x)\Psi$, so they are linearly combined together. Then, the supersymmetric extension of Equation (158) can be written as

$$\Psi_t - (6-a)(\mathcal{D}\Psi)\Psi_x - a(\mathcal{D}\Psi_x)\Psi + \Psi_{xxx} = 0. \tag{164}$$

This equation is called the sKdV-a equation. Its component forms read

$$\begin{cases} w_t - 6ww_x + a\xi\xi_{xx} + w_{xxx} = 0\,, \\ \xi_t - (6-a)\xi_x w - a\xi w_x + \xi_{xxx} = 0\,. \end{cases} \tag{165}$$

Among the various super symmetric extensions of Equation (158), Formula (164) is the most general nontrivial case.

When $a = 3$, Formula (164) degenerates to the sKdV-3 equation

$$\Psi_t - 3(\mathcal{D}\Psi)\Psi_x - 3(\mathcal{D}\Psi_x)\Psi + \Psi_{xxx} = 0. \tag{166}$$

Ref. [59] proved that the sKdV-3 equation is completely integrable. When $a = 0$, Formula (164) turns into a trivial case.

In addition to a fermionic superfield, there is also a bosonic superfield governed by

$$W(t,x,\theta) = w(t,x) + \theta\xi(t,x). \tag{167}$$

Equation (158) is correspondingly extended to

$$W_t + W_{xxx} - 6WW_x = 0. \tag{168}$$

Its component form is

$$w_t + w_{xxx} - 6ww_x = 0\,, \tag{169}$$

$$\xi_t + \xi_{xxx} - 6(\xi w)_x = 0\,. \tag{170}$$

This type of superfield extension is also trivial. The triviality in sKdV-0 and the bosonic superfield extension lies in that one component equation is just Equation (158).

5.2. Supersymmetric Dark Equation

In modern astrophysics, dark matter is crucial. Gravitational effects of dark matter in cosmological environments provide evidence for dark matter [60,61]. There is a similar concept of "dark energy". The universe is accelerating its expansion, and the reason is believed to be related to dark energy [62–64]. Dark energy has a repulsive force. In ref. [65], the concept of "dark equations" was proposed and some dark equations in homogeneous linear forms were demonstrated. In ref. [66], dark equations were promoted to nonlinear forms and nonhomogeneous linear forms.

A PDE S reads

$$w_t = A(t, x, w, w_x, w_{xx}, \ldots, w_{x^q}) \equiv A(w), \quad w = (w_1, w_2, \ldots, w_q)^T. \tag{171}$$

Here, T represents the transpose of the matrix, and q is an integer. Its higher-order symmetry flow can be written as

$$w_\tau = B(t, x, w, w_x, w_{xx}, \ldots, w_{x^p}) \equiv B(w), \tag{172}$$

where p is an arbitrary integer and satisfies $p > q$. If Formulas (171) and (172) satisfy the commutator relationship

$$w_{t\tau} - w_{\tau t} = [A, B] = A'B - B'A = 0, \tag{173}$$

with the superscript $'$ denoting linearized operators and p being a suitable integer, then the PDE S is called symmetry integrable [66,67].

Ref. [65] provides the definition of a dark system in a homogeneous linear form. Homogeneous linear extensions of Formulas (171) and (172) can be written as

$$W_t = \begin{pmatrix} w \\ v \end{pmatrix}_t = \tilde{A}(W) = \begin{pmatrix} A(w) \\ C(w)v \end{pmatrix}, \tag{174}$$

and

$$W_\tau = \begin{pmatrix} w \\ v \end{pmatrix}_\tau = \tilde{B}(W) = \begin{pmatrix} B(w) \\ E(w)v \end{pmatrix}, \tag{175}$$

where $W \equiv (w, v)^T$, $v = (v_1, v_2, \cdots, v_p)^T$, and $C(w)$ and $E(w)$ are $p \times p$ matrix operators independent of v. If Formulas (174) and (175) satisfy the commutator relationship

$$W_{t\tau} - W_{\tau t} = 0\,, \tag{176}$$

then Formulas (174) and (175) are referred to as homogeneous linear dark equations for Equations (171) and (172), where $W_{t\tau} - W_{\tau t} = [\tilde{A}, \tilde{B}] = \tilde{A}'\tilde{B} - \tilde{B}'\tilde{A}$.

Ref. [66] extends dark systems from homogeneous linear forms to nonhomogenous linear forms. Nonhomogenous linear extensions of Formulas (171) and (172) can be written as

$$W_t = \begin{pmatrix} w \\ v \end{pmatrix}_t = \hat{A}(W) = \begin{pmatrix} A(w) \\ C(w)v + C_0(w) \end{pmatrix}, \tag{177}$$

and

$$W_\tau = \begin{pmatrix} w \\ v \end{pmatrix}_\tau = \hat{B}(W) = \begin{pmatrix} B(w) \\ E(w)v + E_0(w) \end{pmatrix}, \tag{178}$$

where $C(w), E(w), C_0(w)$ and $E_0(w)$ depend only on w . If Formulas (177) and (178) satisfy the commutator relationship (176), then Formulas (177) and (178) are referred to as nonhomogenous linear dark equations for Equations (171) and (172).

In ref. [66], not only are more forms of dark equations proposed, but dark equations are also promoted to supersymmetric dark equations. We will demonstrate the construction of supersymmetric dark equations with the help of the KdV equation.

Example 13. *The construction of supersymmetric dark KdV systems [65,66].*

The KdV equation is in the form of (158). For a symmetry integrable equation, there should exist a high-order equation that satisfies the consistent commuting condition. For the KdV Equation (158), its higher-order symmetry flow can be chosen as the fifth-order KdV equation:

$$w_\tau = 10ww_{xxx} + 20w_xw_{xx} - 30w^2w_x - w_{xxxxx}\,. \tag{179}$$

Through direct calculation, we can find that Equations (158) and (179) satisfy the compatibility condition (173); therefore, Equation (179) is a higher symmetry flow of Equation (158).

Supersymmetric dark equations are based on supersymmetric equations. The supersymmetric KdV equations and the supersymmetric fifth-order KdV equations should be constructed first.

From the previous subsection, we know that the most general nontrivial supersymmetric extension of the KdV equation is in the form of (164). The sKdV-3 Equation (166) is the most special one because of its complete integrability. By applying the direct construction method, one can obtain the general nontrivial supersymmetric extension for Equation (179):

$$\begin{aligned}\Psi_\tau \;=\;& b\,\Psi_x(\mathcal{D}\Psi_{xx}) + (20-b)\,(\mathcal{D}\Psi_x)\,\Psi_{xx} + c\,(\mathcal{D}\Psi)\,\Psi_{xxx} + (10-c)\,\Psi\,(\mathcal{D}\Psi_{xxx}) \\ & -2\,d\,\Psi\,(\mathcal{D}\Psi)\,(\mathcal{D}\Psi_x) - (30-2d)\,(\mathcal{D}\Psi)^2\,\Psi_x - \Psi_{xxxxx}\,. \end{aligned} \tag{180}$$

Among them, the most special one is

$$\begin{aligned}\Psi_\tau \;=\;& 10\,\Psi_x(\mathcal{D}\Psi_{xx}) + 10\,(\mathcal{D}\Psi_x)\,\Psi_{xx} + 5\,(\mathcal{D}\Psi)\,\Psi_{xxx} + 5\,\Psi\,(\mathcal{D}\Psi_{xxx}) \\ & -20\,\Psi\,(\mathcal{D}\Psi)\,(\mathcal{D}\Psi_x) - 10(\mathcal{D}\Psi)^2\,\Psi_x - \Psi_{xxxxx}\,, \end{aligned} \tag{181}$$

which belongs to the supersymmetric KdV hierarchy and is integrable [55,68].

The above calculation shows that sKdV-3 Equation (166) and the supersymmetric fifth-order system (181) actually satisfy the commutativity condition (173). Then, we focus on constructing dark systems on Formulas (166) and (181).

The supersymmetric dark equations related to sKdV-3 Equation (166) are assumed to be

$$\begin{cases} \Psi_t = 3(\mathcal{D}\Psi_x)\Psi + 3(\mathcal{D}\Psi)\Psi_x - \Psi_{xxx}\,, \\ \Phi_t = -a_1\Phi_{xxx} + a_2\Psi(\mathcal{D}\Phi_x) + a_3(\mathcal{D}\Psi_x)\Phi + a_4\Psi_x(\mathcal{D}\Phi) + a_5(\mathcal{D}\Psi)\Phi_x\,, \end{cases} \tag{182}$$

where Φ means the dark field, and a_1, a_2, a_3, a_4, a_5 are undetermined coefficients. In Formula (182), all relevant terms are linearly combined together. By doing so, a general dark extension for sKdV-3 Equation (166) can be constructed. Similarly, the supersymmetric dark fifth-order KdV equation also has a general form with some undetermined coefficients.

Different parameter combinations of $\{a_1, a_2, a_3, a_4, a_5\}$ will lead to different dark extensions. These different dark extensions correspond to different extensions for the supersymmetric fifth-order KdV equations because they need to satisfy compatibility condition (176). Sixteen types of nontrivial dark extensions of sKdV-3 Equation (166) are listed in ref. [66]. We only demonstrate the first type of the dark extensions here, written as

$$\begin{cases} \Psi_t = 3(\mathcal{D}\Psi_x)\Psi + 3(\mathcal{D}\Psi)\Psi_x - \Psi_{xxx}\,, \\ \Phi_t = 3\Psi\mathcal{D}\Phi_x - \Phi_{xxx}\,. \end{cases} \tag{183}$$

Its corresponding higher-order symmetry flow system is

$$\begin{cases} \Psi_\tau = 10\,\Psi_x(\mathcal{D}\Psi_{xx}) + 10\,(\mathcal{D}\Psi_x)\,\Psi_{xx} + 5\,(\mathcal{D}\Psi)\,\Psi_{xxx} + 5\,\Psi\,(\mathcal{D}\Psi_{xxx}) \\ \qquad -20\,\Psi\,(\mathcal{D}\Psi)\,(\mathcal{D}\Psi_x) - 10(\mathcal{D}\Psi)^2\,\Psi_x - \Psi_{xxxxx}\,, \\ \Phi_\tau = 5\Psi_x(\mathcal{D}\Phi_{xx}) + 5\Psi(\mathcal{D}\Phi_{xxx}) + 5\Psi_{xx}(\mathcal{D}\Phi_x) - 10\Psi(\mathcal{D}\Psi)(\mathcal{D}\Phi_x) - \Phi_{xxxxx}\,. \end{cases} \tag{184}$$

Formulas (183) and (184) meet compatibility condition (176); so, the supersymmetric dark KdV system (183) is symmetry integrable. Conservation laws and the Lax integrability of these supersymmetric dark KdV equations are discussed in ref. [66].

5.3. Bosonization of Supersymmetric Integrable System

Because of the noncommutative property of the supersymmetric systems, to solve supersymmetric models is very difficult. In ref. [38], a powerful method, the bosonization approach, is proposed, and all the known solutions of the classical integrable systems are extended to the supersymmetric ones. In this subsection, we apply the bosonization approach to the KdV-a system (165).

Example 14. *Bosonization of supersymmetric integrable KdV-a system* (165).

If a special solution of the KdV-a system (165) possesses two Grassmann parameters, ζ_1 and ζ_2, then the solution can be written as

$$\xi = v_1\zeta_1 + v_2\zeta_2,\ w = w_0 + w_{12}\zeta_1\zeta_2, \tag{185}$$

where v_1, v_2, w_0, and w_{12} are four boson fields. These boson fields satisfy a special type of dark equation:

$$\begin{cases} w_{0t} + w_{0xxx} - 6w_0w_{0x} = 0, \\ v_{1t} + v_{1xxx} - (6-a)w_0v_{1x} - aw_{0x}v_1 = 0, \\ v_{2t} + v_{2xxx} - (6-a)w_0v_{2x} - aw_{0x}v_2 = 0, \\ w_{12t} + w_{12xxx} - 6w_0w_{12x} - 6w_{0x}w_{12} + a(v_1v_{2xx} - v_2v_{1xx}) = 0. \end{cases} \tag{186}$$

The first equation of (186) is just the classical KdV equation with various known exact solutions. For a fixed solution w_0, the other three equations related to the boson fields v_1, v_2, and w_{12} are only three graded linear equations. Some of the explicit examples are given in ref. [69].

If n arbitrary Grassmann constants are included in a special solution of the supersymmetric KdV system (165), there are 2^n bosonic fields which satisfy some types of graded dark equations. For instance, the related dark equation for $n = 3$ reads

$$\begin{cases} w_{0t} + w_{0xxx} - 6w_0w_{0x} = 0, \\ v_{it} + v_{ixxx} - (6-a)w_0v_{ix} - aw_{0x}v_i = 0,\ (i = 1,\ 2,\ 3), \\ v_{123t} + v_{123xxx} - (6-a)(w_0v_{v123x} + w_{12}v_{3x} - w_{13}v_{2x} + w_{23}v_{1x}) \\ \qquad -a(w_{0x}v_{123} + v_1w_{23x} - v_2w_{13x} + v_3w_{12x}) = 0, \\ w_{ijt} + w_{ijxxx} - 6(w_0w_{ij})_x + a(v_iv_{jxx} - v_jv_{ixx}) = 0,\ (i < j = 1,\ 2,\ 3). \end{cases} \tag{187}$$

6. Conclusions and Discussion

Several symmetry group methods of PDEs, including basic symmetry group methods, finite symmetry transformation group methods, nonlocal symmetry methods, and supersymmetric theory, are introduced in this review paper. We provide an example for each method.

Three basic group methods are proposed, which are all illustrated using the Boussinesq equation as an example. Compared to the conditional symmetry method, the standard Lie symmetry method generates a greater number of determining equations. Therefore, the conditional symmetry method for some PDEs can produce more results. For some other PDEs, the symmetries deriving from the two methods are the same, such as the KdV equation. We need to notice that the relevant vector fields related to the conditional symmetry method do not form Lie algebras or vector spaces [6]. One should treat the conditional symmetry method and the classical method equally in terms of symmetry reduction [18]. Compared to the above two symmetry methods, not using group theory is an important characteristic of the CK direct method. For the Boussinesq equation, the symmetry reduction results derived by means of the CK direct method can all be derived by means of the conditional symmetry method [6]. For many PDEs, the conditional symmetry method and the CK direct method are equivalent [24,70,71].

Two finite symmetry transformation group methods are introduced. One is the MCK direct method, which is independent of Lax pairs. The other method is based on Lax pairs, which can only be applied to Lax integrable models. By applying either of the two methods, one can obtain finite transformation groups, and the corresponding reduced equations have the same dimensions as the original PDEs. When the parameters chosen have some special value, the finite symmetry transformation groups degenerate to the result by the standard Lie approach.

Four methods for finding nonlocal symmetries and one localization method of nonlocal symmetries are introduced. Nonlocal symmetries can be derived from conservative forms, conformal transformations, Darboux transformations, and Lax pairs. The commonality among these methods is that they all introduce auxiliary functions or auxiliary equations on the basis of traditional symmetry. Auxiliary functions or functions in auxiliary equations produce a nonlocality.

As an important branch of the symmetry group theory, supersymmetries are briefly introduced. We briefly introduce some basic properties of supersymmetry and introduce the direct construction method of supersymmetric equations by taking the KdV equation as the instance. Supersymmetric dark equations were only proposed earlier this year, and we report on them in this review paper. One bosonization method of supersymmetric integrable system is also reported.

The symmetry methods can be applied not only to integer dimensional equations, but also to fractional dimensional equations [72–74]. In recent years, the fractional equations have attracted great attention. In 1998, E.Buckwar and Y. Luchko proposed the symmetry groups of scaling transformations for some linear fractional PDEs [75]. In ref. [76,77], R.K. Gazizov and A.A. Kasatkin et al. discussed the Lie point symmetries of some nonlinear fractional PDEs. The symmetry methods can be used for both time fractional PDEs and space fractional PDEs, and can be used for both single PDEs and coupled PDEs [78–83].

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