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Parton helicities at arbitrary x and Q^2 in double-logarithmic approximation

B. I. Ermolaev[a]

Ioffe Institute, 194021 St. Petersburg, Russia

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Abstract Description of spin-dependent hadronic processes at high energies in terms of parton helicities is a both effective and technically convenient means. In the present paper, we obtain explicit expressions for the parton helicities when either collinear or KT forms of QCD factorization are used. Starting our studies with calculation of the helicities in the double-logarithmic approximation (DLA) in the region of small x and large Q^2, we generalize the results in order to obtain formulae valid at arbitrary x and Q^2. We argue against using collinear factorization, when the parton orbital angular momenta are accounted for, and prove that KT factorization should be used instead. We also consider in detail the small-x asymptotics of the parton helicities, compare them with the DGLAP-asymptotics in LO, NLO, etc. and prove that the DGLAP asymptotics are less singular at small x than the Regge asymptotics

1 Introduction

Parton helicities correspond to longitudinal components of the parton spins when hadron masses are neglected and by this reason the helicities have often been used for description of spin-dependent hadronic reactions at high energies. A new wave of interest to parton helicities was initiated recently by Yu.V. Kovchegov and his colleagues who investigated the proton spin at high energies. They suggested in Refs. [1–9] a new method, namely KPSCTT evolution equations, to calculate the quark and gluon helicities. Their calculations were done mostly with double-logarithmic (DL) accuracy though some of the papers included single-logarithmic contributions as well. The obtained results were represented in the form of the small-x asymptotics. To make sure that the KPSCTT was correct, the small-x asymptotics of the parton helicities were compared with the asymptotics of the spin-dependent structure function g_1 calculated in Refs. [10,11] by other means. The comparison led to the full agreement between those asymptotics. After that, the KPSCTT method was applied to calculations of the proton spin at high energies in Refs. [12–17] and to scattering of polarized protons in Ref. [18].

Recognizing that description of the nucleon spin at high energies in terms of parton helicities was the optimal means for studying the proton spin problem, we investigated this problem in Ref. [19] in terms of the parton helicities with DL accuracy. Both the quark and gluon helicities (which we denote $h_{q,g}$ throughout the present paper) depend on $x = Q^2/(2pq)$ and Q^2 (with q and p being momenta of the parton and the proton respectively). We skipped tracing the Q^2-dependence in Ref. [19] as we followed the pattern of the RHIC experiments[20,21] where Q^2 was fixed at 10 GeV^2. This approximation allowed us to apply the results on $h_{q,g}$ obtained in Ref. [22] (see also the overview Ref. [23]).

However, accounting for the Q^2-dependence is mandatory in order to get a complete of the parton helicities. Moreover, the formulae for $h_{q,g}$ of Refs. [22,23] can be used in combination with collinear factorization only, they are absolutely incompatible with KT Factorization. Nevertheless, there are situations where the use of KT Factorization can be beneficiary and even unavoidable. It makes necessary deriving expressions for the parton helicities in the form compatible with KT Factorization. Obtaining such expressions for the parton helicities that account for both x- and Q^2-dependence and also are compatible with both collinear and KT factorizations is the main subject of the present paper. First we calculate them in the DL approximation (DLA) at small x and large Q^2, and then, engaging the DGLAP formulae, construct interpolation formulae valid at arbitrary x and Q^2.

[a] e-mail: boris.ermolaev@cern.ch (corresponding author)

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Our calculations are done with composing and solving appropriate infra-red evolution equations (IREE).[1] The key point here is that DL contributions to amplitudes (i.e. scattering amplitudes, structure functions, form factors, etc.) of QCD processes arrive from the integration regions where some of the virtual partons are almost on-shell. In this case, the amplitudes become IR-singular and they need to be regulated by IR cut-offs μ which are primarily free parameters and specified a posteriori. Studying evolution of the amplitudes with respect to μ is the essence of the IREE approach. Constructing IREEs exploits factorization of DL contributions of virtual particles with minimal transverse momentum $k_\perp$ even if their energies are not small, which was first noticed in the QED context by Gribov in Ref. [24] and then extended to non-Abelian theories by Lipatov [25–28]. IREEs in the QCD context for $2 \to 2$ processes were suggested in Ref. [27,28] whilst IREEs for inelastic $2 \to 2 + n$ processes were suggested in Refs. [29,30]. The IREE approach proved to be much simpler than preceding tools even in the relatively simple case of elastic processes (cf. for example Refs. [27,28,31]), not to mention inelastic reactions.

In contrast to Ref. [19], we consider in more detail technique of constructing IREEs for the helicities and dwell upon the role of the t-channel color octet amplitudes. As a result, we obtain the expressions for $h_{q,g}(x, Q^2)$ which can be used in the framework of Collinear Factorization and also more complex expressions compatible with KT factorization. We argue that it is KT Factorization that is supposed to be used when parton orbital angular momenta (OAM) are accounted for.

After obtaining explicit expressions for $h_{q,g}(x, Q^2)$, we consider in detail calculating their small-x asymptotics, applying the Saddle-Point method to their parent expressions. By doing so, we demonstrate that despite $h_q(x, Q^2)$, $h_g(x, Q^2)$ and $g_1(x, Q^2)$ in DLA are represented by different formulae, their asymptotics are identical. Comparing these asymptotics with the ones obtained in the framework of N^{n}LO DGLAP [32–35], we prove that the DGLAP asymptotics in LO, NLO, NNLO, etc do not acquire the Regge form. Then we examine the impact of the involved fits on the small-x behaviour of the helicities and discuss a problem of "false intercept" i.e. of the intercept supposedly dependent on Q^2.

Our paper is organized as follows: in Sect. 2 we briefly remind basic features of the IREE method and introduce our basic definitions. We present there a brief prescription for constructing IREEs in general. In order to account for both perturbative and non-perturbative contributions to the parton helicities, we introduce in Sect. 3 their representation throughout QCD factorization and discuss involved mass scales. The kinematic regions where formulae for $h_{q,g}$ are different are introduced in Sect. 4. IREEs for perturbative contributions to the perturbative components of $h_{q,g}$, where all external partons are off-shell, are derived in Sect. 5. A simpler case involving both off-shell and on-shell external partons is considered in Sect. 6 whereas the case of only on-shell partons is for Sect. 7. In contrast to the IREEs for the DIS structure function g_1, IREEs for the parton helicities involve the t-channel color octet contributions from the starting point. We discuss the situation with the color octets in Sect. 8. Explicit DL expressions for $h_{q,g}$ in the context of both collinear and KT factorization are presented in Sect. 9. Besides, we discuss there using orbital angular momenta for description of nucleon spins. Section 10 is for extending these expressions to arbitrary x and Q^2 so as to construct interpolation formulae for $h_{q,g}(x, Q^2)$ valid at arbitrary x and Q^2. Sect. 11 contains detailed consideration of small-x asymptotics of $h_{q,g}$. Finally, Sect. 12 is for our concluding remarks.

2 Preliminary notes

2.1 List of our basic notations

To make reading easier, we list below basic notations that we use throughout the paper:

$h_{q,g}\left(x, Q^2\right)$ denote the parton helicities, they are defined in Eqs. (7, 8).

$\Phi_{q,g}$ are initial parton distributions in collinear factorization, introduced in Eq. (7).

$\varphi_{q,g}$ are related to $\Phi_{q,g}$ by Eq. (1).

$\Phi^{KT}_{q,g}$ are initial parton distributions in KT Factorization, introduced in Eq. 8).

$\varphi^{KT}_{q,g}$ are related to $\Phi^{KT}_{q,g}$ by Eq. (1).

$A_{ij}\left(w, Q^2, k^2_\perp\right)$ denotes totally off-shell parton-parton amplitudes, introduced in Eq. (1).

$M_{ij}\left(w, Q^2, k^2_\perp\right)$ denotes imaginary parts of amplitudes $A_{ij}\left(w, Q^2, k^2_\perp\right)$, introduced in Eq. (5).

$F_{ij}\left(\omega, Q^2, k^2_\perp\right)$ are related to amplitudes $A_{ij}\left(w, Q^2, k^2_\perp\right)$ by Eq. (1).

$A'_{ij}\left(w, Q^2\right)$ denotes partly off-shell parton-parton amplitudes, introduced in Eq. (9).

$M'_{ij}\left(w, Q^2\right)$ denotes imaginary parts of amplitudes $A_{ij}\left(w, Q^2, k^2_\perp\right)$, introduced in Eq. (9).

$F'_{ij}\left(\omega, Q^2\right)$ are related to amplitudes $A_{ij}\left(w, Q^2\right)$ by Eq. (1).

$A''_{ij}\left(w\right)$ denotes on-shell parton-parton amplitudes, introduced in Eq. (10).

$M''_{ij}\left(w\right)$ denotes imaginary parts of amplitudes $A''_{ij}\left(w\right)$, introduced in Eq. (10).

$f_{ij}\left(\omega\right)$ are related to amplitudes $A''_{ij}\left(w\right)$ by Eq. (1).

[1] Detailed explanations on constructing and solving IREEs in the DIS context can be found in the overview[23] whilst a brief description of this method is given in Sect. 2.

2.2 Mellin transform

Throughout the paper we will use the Mellin transform which is defined as follows:

$$A_{ij}\left(w, Q^2, k_\perp^2\right) = \int_{-\imath\infty}^{\imath\infty} \frac{d\omega}{2\pi\imath}\left(w/\mu^2\right)^{\omega} F_{ij}\left(\omega, Q^2, k_\perp^2\right), \tag{1}$$

where $w = 2pq$ and μ is a mass scale. It is often associated with the factorization scale. We consider its value in the next section. The integration line in Eq. (1) runs to the right of the rightmost singularity of $F_{ij}(\omega, y)$. It is convenient to consider $A_{ij}\left(x, Q^2, k_\perp^2\right)$, using the logarithmic notations:

$$A_{ij}\left(w, Q^2, k_\perp^2\right) = \int_{-\imath\infty}^{\imath\infty} \frac{d\omega}{2\pi\imath} e^{\omega\rho} F_{ij}(\omega, y_1, y_2), \tag{2}$$

where the logarithmic variables ρ, ξ, y defined as follows:

$$\rho = \ln\left(w/\mu^2\right), \quad y_1 = \ln\left(Q^2/\mu^2\right), \; y_2 = \ln\left(k_\perp^2/\mu^2\right)$$
$$\xi = \rho - y_1 = \ln(1/x). \tag{3}$$

Definition of ρ in Eq. (3) does not respect analytical properties of w-dependence of an amplitude $A_{ij}(w)$. Indeed, $A_{ij}(w)$ should have a cut at positive $s = (p+q)^2 \approx w$. It means that $\ln w$ should be replaced by $\ln\left(s\, e^{-\imath\pi}\right) = \ln w - \imath\pi$ and therefore

$$A_{ij}(\ln w - \imath\pi) \approx A_{ij}(\rho) - \imath\pi \frac{dA_{ij}}{d\rho}, \tag{4}$$

i.e.

$$M_{ij} \approx -\pi \frac{dA_{ij}}{d\rho}. \tag{5}$$

Combining Eqs. (2) and (5), obtain

$$M_{ij}(w, y_1, y_2) = -\pi \frac{d}{d\rho}\int_{-\imath\infty}^{\imath\infty} \frac{d\omega}{2\pi\imath} e^{\omega\rho}$$
$$F_{ij}(\omega) = -\pi \int_{-\imath\infty}^{\imath\infty} \frac{d\omega}{2\pi\imath} e^{\omega\rho}\, \omega\, F_{ij}(\omega, y_1, y_2). \tag{6}$$

In what follows we will drop the factor π, assuming that it can be attributed to the parton distributions $\Phi_{q,g}$. We will use the same transform to relate M'_{ij} to F'_{ij} and M''_{ij} to $f_{ij}(\omega)$.

2.3 General remark on the IREE technology

Technology of constructing IREEs is universal for all QCD reactions. We briefly remind it below, using as an example a relatively simple case of $2 \to 2$ scattering of partons. We denote A the scattering amplitude of this process. IREE for amplitude A is depicted in the diagram form in Fig. 1. DL contributions to A arrive from the kinematics where transverse momenta $k_{j\perp}$, $j = 1, ..$ of virtual partons are strongly ordered, so there always can be found a virtual parton with minimal $k_\perp$ (softest parton) although its energy can be large. DL contribution of such parton can be factorized.

There are two options to factorize the softest parton:

Option (i) The softest parton is a non-ladder gluon.[2] Its factorizing means that its propagator is attached to pairs of the external lines in every possible way as shown on graphs (a, b, c) in Fig. 1 (mirror graphs are implied) whereas $k_\perp$ acts as a new IR cut-offs for momenta of all other virtual partons in the blobs. Note that graph (a) does not yield DL contribution at $t = 0$, so we will drop it in what follows. The blobs on graphs (b,c) correspond to the t-channel colour octets. Indeed, the blob in the l.h.s. is the color singlet but after emission of the softest gluon, the blob corresponds to the sum of non-singlet $SU(3)$-representations in the t-channel. The simplest non-singlet representation is the octet. It can be described in terms of gluon wave functions while the higher representations require introducing other, exotic fields. By this reason we consider the octet contributions only. Introducing the octet and singlet projection operators and applying them in turn both to graphs (a, b, c) in Fig. 1 and to the blobs on these graphs demonstrates that the contributions of the singlet blobs are zeros which leaves us with the octet blobs only, see details e.g. in Refs. [27,28].

Option (ii) The softest parton (quark or gluon) is a ladder parton. Result of its factorizing is that the primary amplitude is represented by convolutions (d) and (e) of two $2 \to 2$-amplitudes whereas $k_\perp$ acts again as a new IR cut-off for all other virtual partons. All blobs in these convolutions are color singlets. Note that the intermediate states, with the number of t-channel partons greater than 2, do not bring DL contributions. Integration over $k_\perp$ involves μ as a lowest limit. Value of μ is primarily arbitrary save the obvious requirement $\mu > \Lambda_{QCD}$. Specifying μ is discussed in the next section.

Applying the standard Feynman rules to the graphs in Fig. 1 leads to converting this IREE to analytic form. The most important difference between IREEs and other equations of the Bethe–Salpeter type is that the blobs/kernels in Fig. 1 do not depend on longitudinal components of k though they depend on $k_\perp$ in the sense that $k_\perp$ acts a new IR cut-off for momenta of other virtual partons. Finally, notice that the convolutions (d, e) in Fig. 1 look much simpler in the ω-space after the Mellin transform has been used. This technology was used to describe many various QED, QCD and Electroweak processes in DLA. We apply it below to calculate the parton helicities.

[2] We imply the Feynman gauge.

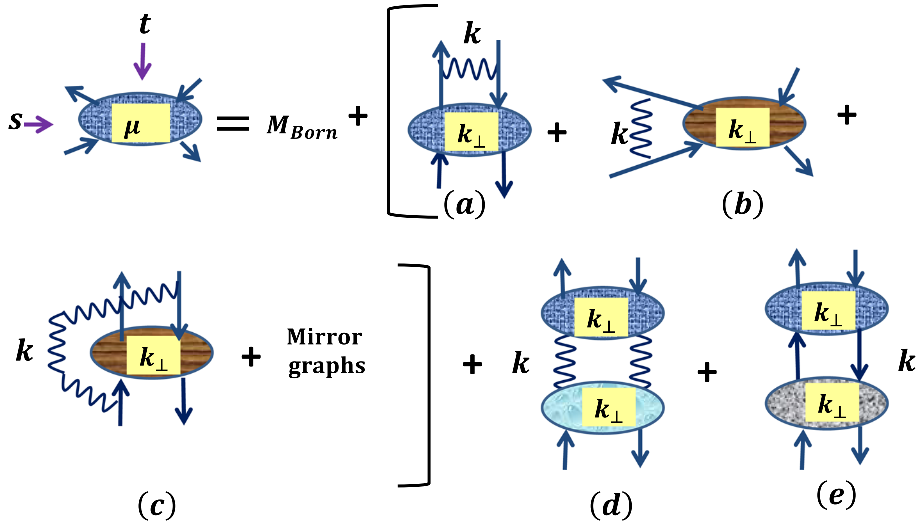

Fig. 1 Constructing IREE for amplitude A of $2 \to 2$-parton scattering, using factorization of the softest partons with momentum k. Letters s, t in the l.h.s. denote the standard Mandelstam variables. The first term in the r.h.s. stands for the Born contribution. The blobs denote that radiative corrections in DLA are accounted for. The blobs on graphs (**b**, **c**) (marked with horizontal strips) are t-channel color octets while the other blobs are color singlets. The letters on the blobs stand for IR cut-offs. Graphs (**a**, **b**, **c**) correspond to factorization of non-ladder partons (Option (i)) whilst factorization of ladder ones (Option (ii)) is shown on graphs (**d**) and (**e**)

3 Parton helicities within the framework of QCD factorization

According to the QCD factorization concept, the parton helicities $h_i\left(x, Q^2\right)$, can be represented as convolutions of the perturbative components M_{ij} (i,j = q,g) and polarized parton distributions Φ_j. In the framework of collinear factorization depend on two arguments: $M_{ij} = M_{ij}\left(x, Q^2\right) M_{ij}$ while virtualities of the initial partons are $\approx \mu^2$, where μ is the factorization scale. It also plays the role of mass shell for the partons, so we will address as partly off-shell amplitudes and denote $M'_{ij}\left(x, Q^2\right)$. Therefore

$$
\begin{aligned}
h_q\left(x, Q^2\right) &= M'_{qq}\left(x, Q^2\right) \otimes \Phi_q(x) \\
&\quad + M'_{qg}\left(x, Q^2\right) \otimes \Phi_g(x), \\
h_g\left(x, Q^2\right) &= M'_{gq}\left(x, Q^2\right) \otimes \Phi_q(x) \\
&\quad + M'_{gg}\left(x, Q^2\right) \otimes \Phi_g(x).
\end{aligned} \tag{7}
$$

In contrast, K_T factorization involves dependence of M_{ij} on one more argument: $M_{ij} = M_{ij}\left(x, Q^2, k_\perp^2\right)$, so the initial parton distributions $\Phi^{KT}_{q,g}$ depend on x and $k_\perp^2$:

$$
\begin{aligned}
h_q\left(x, Q^2\right) &= M^{KT}_{qq}\left(x, Q^2, k_\perp^2\right) \otimes \Phi^{KT}_q(x, k_\perp^2) \\
&\quad + M^{KT}_{qg}\left(x, Q^2, k_\perp^2\right) \otimes \Phi^{KT}_g(x, k_\perp^2), \\
h_g\left(x, Q^2\right) &= M^{KT}_{gq}\left(x, Q^2, k_\perp^2\right) \otimes \Phi^{KT}_q(x, k_\perp^2) \\
&\quad + M^{KT}_{gg}\left(x, Q^2, k_\perp^2\right) \otimes \Phi^{KT}_g(x, k_\perp^2).
\end{aligned} \tag{8}
$$

In what follows we will name amplitudes depending on three arguments, i.e. the ones like A^{KT}_{ij}, M^{KT}_{ij}, totally off-shell amplitudes. The partly off-shell amplitudes A'_{ij} and M'_{ij} are related to the totally off-shell amplitudes A_{ij}, M_{ij} as follows:

$$
\begin{aligned}
A_{ij}\left(x, Q^2, k_\perp^2\right)|_{k_\perp^2=\mu^2} &= A'_{ij}\left(x, Q^2\right), \\
M_{ij}\left(x, Q^2, k_\perp^2\right)|_{k_\perp^2=\mu^2} &= M'_{ij}\left(x, Q^2\right).
\end{aligned} \tag{9}
$$

On-shell parton-parton amplitudes $A''_{ij}(x)$ and their imaginary parts $M''_{ij}(x)$ are defined similarly:

$$
\begin{aligned}
A'_{ij}\left(x, Q^2\right)|_{Q^2=\mu^2} &= A''_{ij}(x), \\
M'_{ij}\left(x, Q^2\right)|_{Q^2=\mu^2} &= M''_{ij}(x).
\end{aligned} \tag{10}
$$

The convolution symbol $\otimes$ in Eqs. (7, 8) implies integrations over momenta of intermediate t-channel partons. The number of involved integrations depends on the specific form of factorization. In collinear factorization $\otimes$ means the integrations over the longitudinal components of the intermediate parton momenta and K_T factorization involves in addition integration over $k_\perp$. Equations (7, 8) are illustrated in Fig. 2, where the s-cuts of each graph are implied.

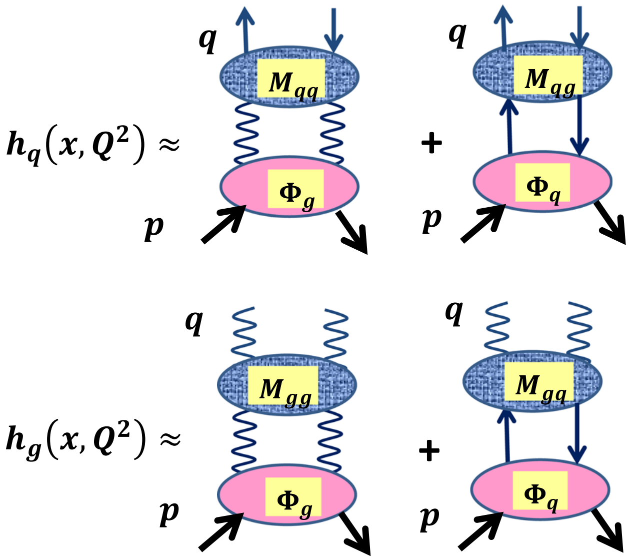

Fig. 2 Representation of the parton helicities through convolutions of perturbative (upper blobs) and non-perturbative components in any of the forms of QCD factorization. Waved lines denote gluons with momenta q, the thick straight lines attached to the lowest blobs correspond to the hadron with momentum p, the other straight lines attached to the upper blobs denote quarks with momenta q. The cuts with respect to $s = (p + q)^2$ on each graph are implied

In general, the number of the intermediate t-channel partons in convolutions is not fixed, it depends on the type of the parton-parton collisions under consideration. We account for the single-parton collisions, which corresponds to two-parton intermediate states, as depicted in Fig. 2. Fortunately, the single-parton collision scenario is perfectly consistent with applying DLA.

3.1 Remark on mass scales

Strictly speaking, theory of polarized DIS involves several mass scales which sometimes are used in the literature:

(i) Factorization scale μ_F;
(ii) IR cut-off μ_{IR} for regulating IR singularities related to DL contributions to perturbative components of the parton helicities;
(iii) Starting point Q_0^2 of the Q^2-evolution;
(iv) The mass scale μ in the Mellin transform (1).

The IREE approach involves the IR cut-off in the transverse space, so μ_{IR} is the minimal transferred momentum and because of that it plays the role of the mass shell for the partons. Its value was estimated in Ref. [23] on basis of principle of minimal sensitivity[44]: $\mu_{IR} \approx 1$ GeV. Throughout the paper we will denote it μ. For the sake of simplicity, we suggest that values of the factorization scale and Q_0 are also $\approx \mu$ although this simplification is not principal for both our reasoning and our results.

4 Kinematic regions for integration of the parton helicities

Both M_{ij} and the parent amplitudes A_{ij} are given by different formulae, depending on x and Q^2. Below we present these regions. The whole kinematic region D_{tot} where the variables x and Q^2 run is

$$D_{tot} : 0 < x < 1, \quad 0 \le Q^2 < w. \tag{11}$$

We divide D_{tot} into four regions:

$$D_{tot} = D_A \oplus D_B \oplus D_C \oplus D_D. \tag{12}$$

First of all, there is the “hard” region D_A:

$$D_A : \quad x \sim 1, \quad Q^2 \gg \mu^2. \tag{13}$$

As is well-known, DGLAP was especially constructed for operating in this region. In what follows, we will address D_A as the DGLAP kinematic region. The next is region D_B, where DL contributions are leading:

$$D_B : \quad x \ll 1, \quad Q^2 \gg \mu^2. \tag{14}$$

In what follows, we will address D_B as the DL kinematic region. Both D_A and D_B are the large-Q^2 regions where $Q^2 > \mu^2$. In contrast to them, there are also the small-Q^2 regions D_C and D_D in D_{tot}, where $Q^2 \le \mu^2$:

$$D_C : \quad x \ll 1, \quad Q^2 \le \mu^2, \tag{15}$$
$$D_D : \quad x \sim 1, \quad Q^2 \le \mu^2. \tag{16}$$

We start with calculating the totally off-shell and partly off-shell amplitudes in the region D_B and then move to considering the other regions. The technical means tat we will use in region D_B is constructing and solving IREEs. However, there are two subregions of D_B, where IREEs for the totally off-shell amplitudes $A_{ij}(w, Q^2, k_\perp^2)$ are different. These subregions are:

(i) Moderate-virtual (MV) kinematics, where

$$Q^2 k_\perp^2 < w\mu^2. \tag{17}$$

Equation (17) can also be written as follows:

$$k_\perp^2 < \frac{\mu^2}{x}. \tag{18}$$

In what follows we will denote D^{MV} the subregion (17) of D_B and keep the notations $A_{ij}(w, Q^2, k_\perp^2)$ for the off-shell amplitudes in D^{MV}.

(ii) Deeply-virtual (DV) kinematics, where

$$Q^2 k_\perp^2 > w\mu^2, \tag{19}$$

i.e.

$$k_\perp^2 > \frac{\mu^2}{x}. \tag{20}$$

Throughout the paper we will keep the notation D^{DV} for this subregion and denote $A_{ij}^{DV}(w, Q^2, k_\perp^2)$ the off-shell amplitudes in the region (19).

5 Totally off-shell amplitudes in region D_B

In the present section, we first demonstrate how to apply the general prescription for constructing IREEs presented in Sect. 2.3 to constructing IREEs for amplitudes A_{ij} and solve the obtained equations. Then, we move to the totally off-shell amplitudes in DV kinematics.

5.1 Totally off-shell amplitudes in region D^{MV}

We demonstrate below how to apply the general prescription depicted in Fig. 1 to constructing IREEs for amplitudes $A_{ij}(w, Q^2, k_\perp^2)$. It is convenient to write IREEs in the differential form, applying operator $-\mu^2 d/d\mu^2$ to all graphs in Fig. 1. Also, it is convenient to write the IREEs in the Mellin representation because the convolutions corresponding to graphs (d, e) in Fig. 1 look much simple in the ω-space. We denote $F_{ij}(\omega, y_1, y_2)$ the Mellin amplitudes corresponding to $A_{ij}(w, Q^2, k_\perp^2)$. As a result, we obtain the following IREE:

$$\left[\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2} + \omega\right] F_{ij}(\omega, y_1, y_2) = \frac{1}{8\pi^2} F'_{in}(\omega, y_1) F'_{nj}(\omega, y_2) + \frac{1}{8\pi^2} \widetilde{T}_{ij}(\omega, y_1, y_2), \tag{21}$$

where $i, j, n = q, g$ and summation over n is implied. The l.h.s. of Eq. (21) corresponds to differentiation of l.h.s. of the IREE in Fig. 1. It contains two derivatives while the last term corresponds to differentiation of the factor $e^{\omega\rho}$ in Eq. (2). The first term in the r.h.s. of Eq. (21) corresponds to graphs (d,e) in Fig. 1. It relates $F_{ij}(\omega, y_1, y_2)$ to partly off-shell amplitudes F'_{ij}. The last term in the r.h.s. of Eq. (21) corresponds to the sum of graphs (b) and (c) in Fig. 1 as well as the mirror graphs. It involves the color octet amplitude (the blobs on graphs (b, c) and mirror graphs), so we provide it with the superscript (8). Neither Born term nor graph (a) in Fig. 1 contribute to Eq. (21): the Born term does not depend on μ when the external partons are virtual, so it vanishes when differentiated over μ; graph (a) does not yield DL contribution at $t = 0$. So, Eq. (21) makes it possible to express $F_{ij}(\omega, y_1, y_2)$ through amplitudes F'_{ij} which are simpler than $F_{ij}(\omega, y_1, y_2)$. We compose and solve IREEs for them in the next section. However, it involves also the octet contributions $\widetilde{T}_{ij}(\omega, y_1, y_2)$ which are quite complicated. IREEs for them involve partly off-shell octet amplitudes, so these IREEs should be constructed and solved before working on Eq. (21) so that their solutions could be substituted in Eq. (21). This problem has not been solved up to now, although our work on it is underway, so we show in Sect. 8 how to approximately account for $\widetilde{T}_{ij}$.

The main difference between Eq. (21) and IREEs for the perturbative components of the structure function g_1 is that the latter do not involve $\widetilde{T}_{ij}(\omega, y_1, y_2)$ because in this case the upper partons in Fig. 1 are replaced by virtual photons which cannot be coupled to the lower partons. It is shown in Sect. 8 that this difference does not affect the small-x behavior of the parton helicities.

Equation (21) demonstrates the general strategy of the IREE method: IR equations with n arguments involve simpler amplitudes depending on $n - 1$ arguments; then, IREEs for them involve amplitudes depending on $n - 2$ arguments and so on until arriving at amplitudes depending on one variable only. Now we proceed to solving Eq. (21). It looks simpler in terms of variables $z_{1,2}$ defined as follows:

$$z_1 = y_1 + y_2,$$
$$z_2 = y_1 - y_2. \tag{22}$$

The inverse relations are

$$y_1 = (z_1 + z_2)/2,$$
$$y_2 = (z_1 - z_2)/2. \tag{23}$$

In terms of $z_{1,2}$, Eq. (21) is

$$\left[\frac{\partial}{\partial z_1} + \frac{\omega}{2}\right] F_{ij}(\omega, y_1, y_2) = \frac{1}{16\pi^2} F'_{ir}(\omega, y_1) F'_{rj}(\omega, y_2) + \frac{1}{16\pi^2} \widetilde{T}_{ij}(\omega, y_1, y_2). \tag{24}$$

In order to simplify Eq. (24), introduce auxiliary amplitudes Ψ_{ij}, ψ_{ij} and $V_{ij}^{(8)}$:

$$\Psi_{ij} = \frac{1}{8\pi^2} F_{ij},$$
$$\psi_{rj} = \frac{1}{8\pi^2} F'_{rj},$$
$$\widetilde{V}_{ij} = \frac{1}{8\pi^2} \widetilde{T}_{ij}. \tag{25}$$

Then obtain from (24)

$$\left[\frac{\partial}{\partial z_1}+\frac{\omega}{2}\right]\Psi_{ij}(\omega,y_1,y_2)$$
$$=\frac{1}{2}\left[\psi_{ir}(\omega,y_1)\psi_{rj}(\omega,y_2)+\widetilde{V}_{ij}(\omega,y_1,y_2)\right]. \quad (26)$$

The general solution to it is:

$$\Psi_{ij}(\omega,y_1,y_2)=e^{-\omega z_1/2}\,\widetilde{C}_{ij}(\omega,z_1,z_2), \quad (27)$$

with $\widetilde{C}_{ij}$ obeying

$$\frac{\partial\widetilde{C}_{ij}}{\partial z_1}=\frac{e^{\omega z_1/2}}{2}\times\left[\psi_{ir}(\omega,y_1)\psi_{rj}(\omega,y_2)+\widetilde{V}_{ij}(\omega,y_1,y_2)\right]. \quad (28)$$

General solution to (28) is

$$\widetilde{C}_{ij}=\frac{1}{2}\int_0^{z_1}dz_1'e^{\omega z_1'/2}\left[\psi_{ir}(\omega,y_1')\psi_{rj}(\omega,y_2')+\widetilde{V}_{ij}(\omega,y_1',y_2')\right]+C(\omega,z_2), \quad (29)$$

where $C(\omega,z_2)$ is arbitrary while $y'_{1,2}$ are defined as follows:

$$y_1'=(z_1'+z_2)/2,$$
$$y_2'=(z_1'-z_2)/2. \quad (30)$$

In order to specify the general solution we first assume that

$$y_1>y_2 \quad (31)$$

and then use matching:

$$F_{ij}(\omega,y_1,y_2)|_{y_2=0}=F'_{ij}(\omega,y_1), \quad (32)$$

i.e.

$$\Psi_{ij}(\omega,y_1,y_2)|_{y_2=0}=\psi_{ij}(\omega,y_1), \quad (33)$$

where $F'_{ij}(\omega,y_1)$ denotes partly off-hell amplitudes and $\psi_{ij}(\omega,y_1)$ is defined in Eq. (25). Obtain

$$\Psi_{ij}(\omega,y_1,y_2)=e^{-\omega y_2}\psi_{ij}(\omega,z_2)+\frac{e^{-\omega z_1/2}}{2}\int_{z_2}^{z_1}dz_1'e^{\omega z_1'/2}\times\left[\psi_{ir}(\omega,y_1')\psi_{rj}(\omega,y_2')+\widetilde{V}_{ij}(\omega,y_1',y_2')\right] \quad (34)$$

and therefore the amplitude $A_{ij}\left(w,Q^2,k_\perp^2\right)$ in the MV kinematics is

$$A_{ij}\left(w,Q^2,k_\perp^2\right)=\frac{1}{8\pi^2}\int_{-\imath\infty}^{\imath\infty}\times\frac{d\omega}{2\pi\imath}\left(w/\mu^2\right)^{\omega}\Psi_{ij}\left(\omega,Q^2,k_\perp^2\right). \quad (35)$$

It is easy to check that Eq. (34) satisfies both matching (33) and (26). It expresses the totally off-shell amplitudes through less complicated partly off-shell amplitudes. The ordering (31) can easily be lifted with replacement y_2 by $|y_2|$.

5.2 Totally off-shell amplitudes in deeply-virtual kinematics

In contrast to $A_{ij}(w,Q^2,k_\perp^2)$, amplitudes $A_{ij}^{DV}(w,Q^2,k_\perp^2)$ do not depend on μ, i.e. they are infrared-stable, so IREEs for them are much simpler:

$$\left[\frac{\partial}{\partial\rho}+\frac{\partial}{\partial y_1}+\frac{\partial}{\partial y_2}\right]A_{ij}^{DV}(w,Q^2,k_\perp^2)=0 \quad (36)$$

and the general solution to Eq. (36) is also simple:

$$A_{ij}^{DV}(\rho,y_1,y_2)=C_{ij}^{DV}(\rho-y_1,\rho-y_2), \quad (37)$$

where C_{ij}^{DV} are arbitrary analytic functions. We specify it by matching A_{ij}^{DV} with amplitude A_{ij} of Eq. (35) on the border between MV and DV kinematics, where

$$\rho=y_1+y_2. \quad (38)$$

The matching yields

$$C_{ij}^{DV}(\rho-y_1,\rho_-y_2)|_{\rho=y_1+y_2}=A_{ij}(\rho,y_1,y_2)|_{\rho=y_1+y_2} \quad (39)$$

and therefore

$$C_{ij}^{DV}(y_2,y_1)=A_{ij}(y_1+y_2,y_1,y_2), \quad (40)$$

which fixes C_{ij}^{DV} on the border (38). In the region of DV kinematics (19, 20), i.e. off the border, $C_{ij}^{DV}(\rho-y_1,\rho_-y_2)$ is obtained out of $A_{ij}(y_1+y_2,y_1,y_2)$ with replacements $y_1\to\rho-y_2$, $y_2\to\rho-y_1$, i.e. $z_1\to 2\rho-y_1-y_2=\rho-z_1$ and $z_2\to y_2-y_1=-z_2$:

$$A_{ij}^{DV}(\rho,y_1,y_2)=A_{ij}(2\rho-y_1-y_2,\rho-y_2,\rho-y_1). \quad (41)$$

and

$$M_{ij}^{DV}(\rho,y_1,y_2)=M_{ij}(2\rho-y_1-y_2,\rho-y_2,\rho-y_1). \quad (42)$$

The specific form of dependence of the r.h.s. of Eqs. (41, 42) on ρ, $y_{1,2}$ clearly demonstrates that A_{ij}^{DV} and M_{ij}^{DV} do not depend on the IR cut-off, i.e. these expression are IR stable.

6 Partly off-shell amplitudes

Being guided by the general prescription (see Sect. 2.3) for composing IREEs and remembering how Eq. (21) was constructed, one can easily compose IREEs for $F'_{ij}(\omega, y)$ or related to them amplitudes ψ_{ij} (see Eq. (25)). Actually, the system of IREEs for ψ_{ij} splits into two subsystems:

$$\begin{aligned}\frac{\partial \psi_{qq}(\omega, y)}{\partial y} &= \psi_{qq}(\omega, y)\, h_{qq}(\omega) \\ &\quad + \psi_{qg}(\omega, y)\, h_{gq}(\omega) + V'_{qq}(\omega, y)\,, \\ \frac{\partial \psi_{qg}(\omega, y)}{\partial y} &= \psi_{qq}(\omega, y)\, h_{qg}(\omega) \\ &\quad + \psi_{qg}(\omega, y)\, h_{gg}(\omega) + V'_{gq}(\omega, y)\end{aligned} \tag{43}$$

and

$$\begin{aligned}\frac{\partial \psi_{gq}(\omega, y)}{\partial y} &= \psi_{gq}(\omega, y)\, h_{qq}(\omega) + \psi_{gg}(\omega, y)\, h_{gq}(\omega) \\ &\quad + V'_{qg}(\omega, y)\,, \\ \frac{\partial \psi_{gg}(\omega, y)}{\partial y} &= \psi_{gq}(\omega, y)\, h_{qg}(\omega) + \psi_{gg}(\omega, y)\, h_{gg}(\omega) \\ &\quad + V'_{gg}(\omega, y)\,,\end{aligned} \tag{44}$$

where $V'_{ij}(\omega, y)$ stand for color octets. They correspond to the sum of convolutions (a, c) and mirror in Fig. 1. Contribution of the Born term vanishes when differentiated over μ. Amplitudes h_{ij} do not depend on y, so we refer to them as on-shell amplitudes. Let us note once more that Eq. (21) differs from IREEs for the DIS structure function g_1 in that the latter do not involve octet contributions.

First step to solve Eqs. (43, 44) is to find general solutions to homogeneous equations, where $V'_{ij} = 0$:

$$\begin{aligned}\psi_{qq} &= C_1(\omega) e^{y\Omega_+} + C_2(\omega) e^{y\Omega_-}\,, \\ \psi_{qg} &= C_1(\omega) r_1(\omega) e^{y\Omega_{(+)}} + C_2(\omega) r_2(\omega) e^{y\Omega_{(-)}}\,,\end{aligned} \tag{45}$$

and

$$\begin{aligned}\psi_{gq} &= C_3(\omega) r_3(\omega) e^{y\Omega_{(+)}} + C_4(\omega) r_4(\omega) e^{y\Omega_{(-)}}\,, \\ \psi_{gg} &= C_3(\omega) e^{y\Omega_+} + C_4(\omega) e^{y\Omega_-}\,,\end{aligned} \tag{46}$$

where

$$\Omega_{(\pm)} = \frac{1}{2}\left[h_{qq} + h_{gg} \pm \sqrt{R}\right], \tag{47}$$

with

$$R = (h_{gg} - h_{qq})^2 + 4 h_{qg} h_{gq} \tag{48}$$

and

$$\begin{aligned} r_1 &= \frac{1}{2h_{gq}}\left[h_{gg} - h_{qq} + \sqrt{R}\right], \\ r_2 &= \frac{1}{2h_{gq}}\left[h_{gg} - h_{qq} - \sqrt{R}\right], \\ r_3 &= \frac{1}{2h_{qg}}\left[h_{qq} - h_{gg} - \sqrt{R}\right], \\ r_4 &= \frac{1}{2h_{qg}}\left[h_{qg} - h_{gg} + \sqrt{R}\right].\end{aligned} \tag{49}$$

The factors $C_{1,2,3,4}(\omega)$ in Eqs. (45, 46) are arbitrary. Let us focus on Eq. (45). It can be solved with applying standard mathematical means. Put $C_{1,2}$ be dependent on y and substitute it in (43):

$$\begin{aligned} dC_1/dy\, e^{y\Omega_+} + dC_2/dy\, e^{y\Omega_-} &= V'_{qq}(\omega, y)\,, \\ dC_1/dy\, r_1\, e^{y\Omega_+} + dC_2/dy\, r_2\, e^{y\Omega_-} &= V'_{qg}(\omega, y)\,.\end{aligned} \tag{50}$$

Solution to (50) is:

$$\begin{aligned}\frac{dC_1(\omega, y)}{dy} &= e^{-y\Omega_+}\, \frac{V'_{qq}(\omega, y) - r_2\, V'_{qg}(\omega, y)}{r_2 - r_1}, \\ \frac{dC_2(\omega, y)}{dy} &= e^{-y\Omega_-}\, \frac{-r_1\, V'_{qq}(\omega, y) + V'_{qg}(\omega, y)}{r_2 - r_1}.\end{aligned} \tag{51}$$

Integrating (51) yields:

$$\begin{aligned} C_1(\omega, y) &= \widetilde{C}_1(\omega) + \frac{1}{r_2 - r_1}\int_0^y d' e^{-y'\Omega_+} \\ &\quad \times \left[V'_{qq}(\omega, y') - r_2\, V'_{qg}(\omega, y')\right], \\ C_2(\omega, y) &= \widetilde{C}_2(\omega) + \frac{1}{r_2 - r_1}\int_0^y d' e^{-y'\Omega_-} \\ &\quad \times \left[-r_1\, V'_{qq}(\omega, y) + V'_{qg}(\omega, y)\right],\end{aligned} \tag{52}$$

where $\widetilde{C}_{1,2}$ do not depend on y. Substituting it in (45), obtain

$$\begin{aligned}\psi_{qq} &= e^{y\Omega_+}\left[\widetilde{C}_1(\omega) + \frac{1}{r_2 - r_1}\int_0^y d' e^{-y'\Omega_+}\right. \\ &\quad \left.\times \left[V'_{qq}(\omega, y') - r_2\, V'_{qg}(\omega, y')\right]\right. \\ &\quad + e^{y\Omega_-}\left[\widetilde{C}_2(\omega) + \frac{1}{r_2 - r_1}\int_0^y d' e^{-y'\Omega_-}\right. \\ &\quad \left.\times \left[-r_1\, V'_{qq}(\omega, y) + V'_{qg}(\omega, y)\right]\right], \\ \psi_{qg} &= r_1\, e^{y\Omega_+}\left[\widetilde{C}_1(\omega) + \frac{1}{r_2 - r_1}\int_0^y d' e^{-y'\Omega_+}\right. \\ &\quad \left.\times \left[V'_{qq}(\omega, y') - r_2\, V'_{qg}(\omega, y')\right]\right. \\ &\quad + r_2\, e^{y\Omega_-}\left[\widetilde{C}_2(\omega) + \frac{1}{r_2 - r_1}\int_0^y d' e^{-y'\Omega_-}\right. \\ &\quad \left.\times \left[-r_1\, V'_{qq}(\omega, y) + V'_{qg}(\omega, y)\right]\right].\end{aligned} \tag{53}$$

In order to fix $\widetilde{C}_{1,2}(\omega)$, we use the matching with the on-shell amplitudes $h_{ij}(\omega)$:

$$\psi_{qq}(\omega, y)|_{y=0} = h_{qq}(\omega), \;\; \psi_{qg}(\omega, y)|_{y=0} = h_{qg}(\omega). \quad (54)$$

It yields

$$\widetilde{C}_1 = \frac{r_2 h_{qq} - h_{qg}}{r_2 - r_1}, \quad \widetilde{C}_2 = \frac{-r_1 h_{qq} + h_{qg}}{r_2 - r_1} \quad (55)$$

and therefore (53) takes the following form:

$$\begin{aligned}
\psi_{qq} &= \frac{e^{y\Omega_+}}{r_2 - r_1}\Big[\left(r_2 h_{qq} - h_{qg}\right) + \int_0^y d' e^{-y'\Omega_+} \\
&\times \left[V'_{qq}\left(\omega, y'\right) - r_2\, V'_{qg}\left(\omega, y'\right)\right] \\
&+ \frac{e^{y\Omega_-}}{r_2 - r_1}\Big[\left(-r_1 h_{qq} + h_{qg}\right) + \int_0^y d' e^{-y'\Omega_-} \\
&\times \left[-r_1\, V'_{qq}\left(\omega, y\right) + V'_{qg}\left(\omega, y\right)\right], \\
\psi_{qg} &= \frac{r_1\, e^{y\Omega_+}}{r_2 - r_1}\Big[\left(r_2 h_{qq} - h_{qg}\right) + \int_0^y d' e^{-y'\Omega_+} \\
&\times \left[V'_{qq}\left(\omega, y'\right) - r_2\, V'_{qg}\left(\omega, y'\right)\right] \\
&+ \frac{r_2\, e^{y\Omega_-}}{r_2 - r_1}\Big[\left(-r_1 h_{qq} + h_{qg}\right) + \int_0^y d' e^{-y'\Omega_-} \\
&\times \left[-r_1\, V'_{qq}\left(\omega, y\right) + V'_{qg}\left(\omega, y\right)\right]. \quad (56)
\end{aligned}$$

Expressions (44) for ψ_{gq} and ψ_{gg} can be solved in the same way, however with using the other matching:

$$\psi_{gq}(\omega, y)|_{y=0} = h_{gq}(\omega), \;\; \psi_{gg}(\omega, y)|_{y=0} = h_{gg}(\omega). \quad (57)$$

As a result, obtain

$$\begin{aligned}
\psi_{gg} &= \frac{e^{y\Omega_+}}{r_4 - r_3}\Big[\left(r_4 h_{gg} - h_{gq}\right) + \int_0^y d' e^{-y'\Omega_+} \\
&\times \left[V'_{gg}\left(\omega, y'\right) - r_4\, V'_{gq}\left(\omega, y'\right)\right] \\
&+ \frac{e^{y\Omega_-}}{r_4 - r_3}\Big[\left(-r_3 h_{gg} + h_{gq}\right) + \int_0^y d' e^{-y'\Omega_-} \\
&\times \left[-r_1\, V'_{gg}\left(\omega, y\right) + V'_{gq}\left(\omega, y\right)\right], \\
\psi_{gq} &= \frac{r_1\, e^{y\Omega_+}}{r_4 - r_3}\Big[\left(r_4 h_{gg} - h_{gq}\right) + \int_0^y d' e^{-y'\Omega_+} \\
&\times \left[V'_{gg}\left(\omega, y'\right) - r_4\, V'_{gq}\left(\omega, y'\right)\right] \\
&+ \frac{r_4\, e^{y\Omega_-}}{r_4 - r_3}\Big[\left(-r_4 h_{gg} + h_{gq}\right) + \int_0^y d' e^{-y'\Omega_-} \\
&\times \left[-r_3\, V'_{qq}\left(\omega, y\right) + V'_{gq}\left(\omega, y\right)\right]. \quad (58)
\end{aligned}$$

Equations (56, 58) represent the partly off-shell amplitudes in terms of on-sell amplitudes $h_{ij}(\omega)$ and the color octet contributions V'_{ij}. First, we proceed to calculating the on-shell amplitudes.

7 On-shell amplitudes

IREEs for the on-shell amplitudes $f_{ij}(\omega)$ can also be obtained from the general pattern presented in Sect. 2.3 and depicted in Fig. 1 in the same way as the IREEs for the off-shell amplitudes considered above. However, they differ from off-shell equations. First, the on-shell l.h.ss. do not involve derivatives because $f_{ij}(\omega)$ do not depend on y, which leads to algebraic equations. Second, the on-shell Born terms depend on μ, so, in contrast to Eqs. (21, 43, 44), they contribute to on-shell IREEs.

7.1 IREEs for on-shell amplitudes

Accounting for all terms in Fig. 1 leads to the following IREEs:

$$\omega f_{ij}(\omega) = a^B_{ij}(\omega) + \frac{1}{8\pi^2}\, f_{in}(\omega) f_{nj}(\omega) + \frac{1}{8\pi^2}\, t^{(8)}_{ij}(\omega), \quad (59)$$

where a^B_{ij} stand for the Born terms, $n = q, g$ and $t^{(8)}(\omega)$ corresponds to the sum of convolutions (b, c) in Fig. 1 when all external partons are on-shell. The contribution of graph (a) does not yield double-logs because $t = 0$, so this graph is dropped. As in the previous cases, we introduce amplitudes h_{ij} instead of f_{ij}:

$$h_{ij}(\omega) = \frac{1}{8\pi^2}\, f_{ij}(\omega). \quad (60)$$

In terms of h_{ij}, IREEs for the on-shell amplitudes are as follows:

$$\begin{aligned}
\omega h_{qq}(\omega) &= b_{qq} + h_{qq}(\omega) h_{qq}(\omega) + h_{qg}(\omega) h_{gq}(\omega), \\
\omega h_{qg}(\omega) &= b_{qg} + h_{qq}(\omega) h_{qg}(\omega) + h_{qg}(\omega) h_{gg}(\omega), \\
\omega h_{gq}(\omega) &= b_{gq} + b_{gq}(\omega) h_{qq}(\omega) + h_{gg}(\omega) h_{gq}(\omega), \\
\omega h_{gg}(\omega) &= b_{gg} + h_{gq}(\omega) h_{qg}(\omega) + h_{gg}(\omega) h_{gg}(\omega), \quad (61)
\end{aligned}$$

where inhomogeneous terms b_{ij} contain the Born contributions and contributions of the color octets, see Ref. [22]:

$$b_{ij}(\omega) = a_{ij}(\omega) + V_{ij}(\omega), \quad (62)$$

with

$$a_{ij}(\omega) = \frac{1}{8\pi^2}\, a^B_{ij}(\omega), \;\; V_{ij}(\omega) = \frac{1}{8\pi^2}\, t^{(8)}_{ij}(\omega). \quad (63)$$

The Born contributions a_{ij} can be found Ref. [22,23] and discussion on the color octets is in Sect. 8.

7.2 Solutions to the equations for the on-shell-amplitudes

Equations in Eq. (61) for the on-shell amplitudes h_{ij} are algebraic but non-linear. Solution to them can be found analytically. We remind that the DIS structure function g_1 in DLA is also made out of amplitudes h_{ij}. Approximate solution to Eq. (61) in the analytic form was presented in Refs. [22,23]. Below we represent this solution in a more convenient form. First, introduce auxiliary amplitudes

$$h_{\pm} = h_{qq} \pm h_{gg} \tag{64}$$

and express h_{qg} and h_{gg} in terms of h_+:

$$\omega h_{qg} = \frac{b_{qg}}{\omega - h_+}, \quad h_{gq} = \frac{b_{gq}}{\omega - h_+}, \tag{65}$$

Then rewrite the remaining equations in (61) in terms of $h_{\pm}$:

$$\begin{aligned} h_- &= \frac{b_-}{\omega - h_+}, \\ \omega h_+ &= b_+ + h_{qq}^2 + h_{gg}^2 + 2b_{qg}b_{gq}/(\omega - h_+)^2, \end{aligned} \tag{66}$$

where

$$b_{\pm} = b_{qq} \pm b_{gg}. \tag{67}$$

Replace $h_{qq}^2 + h_{gg}^2$ by $(1/2)\left[h_+^2 + h_-^2\right]$ and arrive at the following equation for h_+:

$$\begin{aligned} \omega h_+ &= b_+ + \frac{1}{2}\left[h_+^2 + \frac{(b_{qq} - b_{gg})^2}{(\omega - h_+)^2}\right] + \frac{2b_{qg}b_{gq}}{(\omega - h_+)^2} \\ &= b_+ + \frac{1}{2}\left[(\omega - h_+)^2 + 2\omega h_+ - \omega^2 + \frac{(b_{qq} - b_{gg})^2}{(\omega - h_+)^2}\right] \\ &\quad + \frac{2b_{qg}b_{gq}}{(\omega - h_+)^2}. \end{aligned} \tag{68}$$

Denoting $(\omega - h_+)^2 = u$, obtain an algebraic quadratic equation for u:

$$u^2 - [\omega^2 - 2(b_{qq} + b_{gg})]\, u + [(b_{qq} - b_{gg})^2 + 4b_{qg}b_{gq}] = 0. \tag{69}$$

This equation has four roots but only one of them satisfies matching with the Born amplitudes

$$h_{ik} = \frac{b_{ik}}{\omega} \tag{70}$$

at $\omega \gg 1$. As a result we obtain explicit expressions for h_{ik}:

$$\begin{aligned} h_{qq} &= \frac{1}{2}\left[\omega - Z - \frac{b_{gg} - b_{qq}}{Z}\right], \qquad h_{qg} = \frac{b_{qg}}{Z}, \\ h_{gg} &= \frac{1}{2}\left[\omega - Z + \frac{b_{gg} - b_{qq}}{Z}\right], \qquad h_{gq} = \frac{b_{gq}}{Z}, \end{aligned} \tag{71}$$

where

$$Z = \sqrt{\frac{1}{2}\left[U + \sqrt{W}\right]}, \tag{72}$$

with

$$U = \omega^2 - 2b_+ \tag{73}$$

and

$$W = U^2 - 4(b_-^2 + 4b_{qg}b_{gq}) = (\omega^2 - 2b_+)^2 - 4(b_-^2 + 4b_{qg}b_{gq}). \tag{74}$$

Making use of Eqs. (72, 74) brings (71) to the more convenient form:

$$\begin{aligned} h_{qq} &= \frac{1}{2}\left[\omega - \frac{\sqrt{U + \sqrt{W}}}{\sqrt{2}} + \frac{b_-}{\sqrt{b_-^2 + 4b_{qg}b_{gq}}}\frac{\sqrt{U - \sqrt{W}}}{\sqrt{2}}\right], \\ h_{gg} &= \frac{1}{2}\left[\omega - \frac{\sqrt{U + \sqrt{W}}}{\sqrt{2}} - \frac{b_-}{\sqrt{b_-^2 + 4b_{qg}b_{gq}}}\frac{\sqrt{U - \sqrt{W}}}{\sqrt{2}}\right], \\ h_{qg} &= \frac{b_{qg}}{\sqrt{b_-^2 + 4b_{qg}b_{gq}}}\frac{\sqrt{U - \sqrt{W}}}{\sqrt{2}}, \\ h_{gq} &= \frac{b_{gq}}{\sqrt{b_-^2 + 4b_{qg}b_{gq}}}\frac{\sqrt{U - \sqrt{W}}}{\sqrt{2}}. \end{aligned} \tag{75}$$

8 Remark on octet contributions

Terms $\widetilde{V}_{ij}$, V'_{ij} and V_{ij} in Eqs. (26, 43, 44) and (61) involve color octet amplitudes in the following way:

$$\hat{V}_{ij} = \kappa_{ij}\frac{\alpha_s}{\pi}\frac{\hat{F}^{(8)}}{\omega^2}, \tag{76}$$

where κ_{ij} are the color factors while $\hat{V}_{ij}$ and $\hat{F}^{(8)}$ are generic notations of color octet amplitudes and their contributions to IREEs in the kinematics considered above (totally off-shell, partly off-shell and on-shell kinematics). IREEs for $\hat{F}^{(8)}$ are more involved than ones for the color singlets. Strictly speaking, only one particular case has been studied so far. Namely, there was obtained and solved the IREE for the on-shell amplitude $h_{qq}^{(8)}(\omega)$ whereas all other $h_{ij}^{(8)}(\omega)$

were neglected (flavour non-singlet case). This equation was obtained in Ref. [27,28]:

$$\omega h_{qq}^{(8)}(\omega) = a_{qq}^{(8)} + \kappa_{qq} \frac{\alpha_s}{\pi} \frac{d}{d\omega} h_{qq}^{(8)}(\omega) + h_{qq}^{(8)}(\omega) h_{qq}^{(8)}(\omega), \quad (77)$$

with $a_{qq}^{(8)} = -\alpha_s/(4\pi N)$ and $\kappa_{qq} = 2C_F$. Solution to this equation is

$$h_{qq}^{(8)}(\omega) = \frac{\alpha_s N}{2\pi} \frac{d}{d\omega} \ln\left[e^{z^2/4} D_p(z)\right], \quad (78)$$

where $D_p(z)$ denotes the parabolic cylinder function, with $z = \omega/\sqrt{\alpha_s N/(2\pi)}$ and $p = -1/(2N^2)$. Straightforward substitution of Eq. (78) in

$$h_{qq} = \frac{1}{2}\sqrt{\omega^2 - 4b_{qq}(\omega)}, \quad (79)$$

where b_{qq} is given by (62), with $a_{qq} = \alpha_s C_F/(2\pi)$ and $\kappa_{qq} = N/2$, leads to the very complicated expression which can be treated with numerical calculations only. However, $h_{qq}^{(8)}(\omega)$ quickly decreases at large ω. This makes it possible to approximate $h_{qq}^{(8)}(\omega)$ by its Born value a_{qq}/ω. Such an approximation was made in Refs. [10,11,27,28] for color non-singlets and then generalized in Refs. [22,23] to the case of the singlet g_1. Approximation expressions for b_{ij} can be found in Refs. [22,23]. Explicit expressions for the octet amplitudes contributing to Eqs. (21, 43, 44) are unknown yet although our work on them is underway. So, in the present paper we account for them only approximately, through V_{ij} in the Born approximation.

It is worth mentioning in this regard that the impact of the octets on the intercepts of h_{ij} and g_1 is known to be not so large. For instance, the intercept ω_0 of the small-x asymptotics of h_{qq} is

$$\omega_0 = \omega_0' \sqrt{\frac{1 + \sqrt{1 + 2/(NC_F)}}{2}} \approx \omega_0' \left[1 + 1/2N^2\right], \quad (80)$$

where $\omega_0' = \sqrt{2\alpha_s C_F/\pi}$ accommodates contributions of the ladder graphs only.[3] The octet contribution in Eq. (80) is presented by the term $2/(NC_F)$ which stems from accounting for non-ladder graphs.

9 Expressions for helicities in the region D_B

Expressions for the parton helicities are given in Eqs. (7, 8) in the symbolic form. In the present section. we obtain explicit expressions for them in both collinear and KT forms of QCD factorization.

[3] Throughout the paper we imply the Feynman gauge for virtual gluons.

9.1 Helicities in collinear factorization

Collinear factorization operates with partly on-shell amplitudes M'_{ij}, where virtualities of the initial partons are fixed by the factorization scale μ^2, so

$$\begin{aligned} h_q\left(x, Q^2\right) &= \int_x^1 d\beta \left[M'_{qq}\left(x/\beta, Q^2\right) \otimes \Phi_q(\beta) \right. \\ &\left. \quad + M'_{qg}\left(x/\beta, Q^2\right) \otimes \Phi_g(\beta)\right], \\ h_g\left(x, Q^2\right) &= \int_x^1 d\beta \left[M'_{gq}\left(x, Q^2\right) \otimes \Phi_q(x) \right. \\ &\left. \quad + M'_{gg}\left(x, Q^2\right) \otimes \Phi_g(x)\right]. \end{aligned} \quad (81)$$

Combining Eqs. (25) and (81), we obtain expressions for the parton helicities $h_{q,g}$ in the framework of collinear factorization:

$$\begin{aligned} h_q(x, y_1) &= \int_{-\imath\infty}^{\imath\infty} \frac{d\omega}{2\pi\imath} x^{-\omega}\, \omega \\ &\quad \times \left[\psi_{qq}(\omega, y_1)\, \varphi_q(\omega) + \psi_{qg}(\omega, y_1)\, \varphi_g(\omega)\right], \\ h_g(x, y_1) &= \int_{-\imath\infty}^{\imath\infty} \frac{d\omega}{2\pi\imath} x^{-\omega}\, \omega \\ &\quad \times \left[\psi_{gq}(\omega, y_1)\, \varphi_q(\omega) + \psi_{gg}(\omega, y_1)\, \varphi_g(\omega)\right], \end{aligned} \quad (82)$$

where ψ_{ij} are given by Eqs. (56, 58). The parton distributions $\varphi_{q,g}$ will be specified in Sect. 11.

9.2 Helicities in KT factorization

Expressions for $h_{q,g}$ in the framework of KT Factorization in a generic form are given by obtained by Eq. (8). We convert it in the constructive form, replacing the symbolic sign $\otimes$ by explicit integrations over β and $k_\perp$. First step to do it is to fix the integration region which denote it D_{KT}. The region D_{KT} shown in Fig. 2. It is restricted by the condition $1 \geq \beta \geq x + k_\perp^2/w$ and consists of subregions D_{MV} and D_{DV}. Each of them includes the condition $1 \geq \beta \geq x + k_\perp^2/w$ and the restrictions for $k_\perp^2$ corresponding to Eqs. (17, 19):

$$D_{MV}: \quad \mu^2/x \geq k_\perp^2 \geq \mu^2, \; 1 \geq \beta \geq x + k_\perp^2/w, \quad (83)$$

$$D_{DV}: \quad w(1-x) \geq k_\perp^2 > \mu^2/x, \; 1 \geq \beta \geq x + k_\perp^2/w. \quad (84)$$

So, we obtain

$$\begin{aligned} h_q^{(KT)}\left(x, Q^2\right) &= \int_{\mu^2}^{\mu^2/x} \frac{dk_\perp^2}{k_\perp^2} \int_{x+k_\perp^2/w}^1 \frac{d\beta}{\beta} \\ &\quad \times \left[M_{qq}\left(x/\beta, Q^2, k_\perp^2\right)\, \Phi_q^{KT}(\beta, k_\perp^2) \right. \\ &\quad \left. + M_{qg}\left(x/\beta, Q^2, k_\perp^2\right)\, \Phi_g^{KT}(\beta, k_\perp^2)\right] \end{aligned}$$

$$
\begin{aligned}
&+\int_{\mu^2/x}^{w(1-x)} \frac{dk_\perp^2}{k_\perp^2} \int_{x+k_\perp^2/w}^{1} \frac{d\beta}{\beta} \\
&\times \Big[M_{qq}^{DV}\left(x/\beta, Q^2, k_\perp^2\right)\ \Phi_q^{KT}(\beta, k_\perp^2) \\
&+ M_{qg}^{DV}\left(x/\beta, Q^2, k_\perp^2\right)\ \Phi_g^{KT}(\beta, k_\perp^2)\Big], \\
h_g^{(KT)}\left(x, Q^2\right) = &\int_{\mu^2}^{\mu^2/x} \frac{dk_\perp^2}{k_\perp^2} \int_{x+k_\perp^2/w}^{1} \frac{d\beta}{\beta} \\
&\times \Big[M_{gq}\left(x/\beta, Q^2, k_\perp^2\right)\ \Phi_q^{KT}(\beta, k_\perp^2) \\
&+ M_{gg}\left(x/\beta, Q^2, k_\perp^2\right)\ \Phi_g^{KT}(\beta, k_\perp^2)\Big] \\
&+\int_{\mu^2/x}^{w(1-x)} \frac{dk_\perp^2}{k_\perp^2} \int_{x+k_\perp^2/w}^{1} \frac{d\beta}{\beta} \\
&\times \Big[M_{gq}^{DV}\left(x/\beta, Q^2, k_\perp^2\right)\ \Phi_q^{KT}(\beta, k_\perp^2) \\
&+ M_{gg}^{DV}\left(x/\beta, Q^2, k_\perp^2\right)\ \Phi_g^{KT}(\beta, k_\perp^2)\Big],
\end{aligned}
\tag{85}
$$

with M_{ij} defined in Eqs. (25, 34) and M_{ij}^{DV} defined in Eq. (42). Expressions for $h_{q,g}^{(KT)}$ in Eq. (85) are more involved than the ones in Collinear Factorization but sometimes involving KT factorization can be unavoidable. We consider one such situation below.

9.3 KT factorization and orbital angular momenta

KT factorization is more general form of QCD factorization than collinear one and involves more complicated formulae than collinear factorization. It is the reason why collinear factorization is used more often than KT factorisation. However, there are situations when using KT factorization is mandatory. In particular, it takes place when impact of orbital angular momenta (OAM) $L_{q,g}$ of the partons on the nucleon spin is investigated. Indeed, the conventional scenario of description of the nucleon spin S_N is that S_N is formed by spins of the partons (quarks and gluons) complemented by their OAMs $L_{q,g}$:

$$
S_N = \frac{1}{2}\Delta\Sigma + \Delta G + L_q + L_g, \tag{86}
$$

where $(1/2)\Delta\Sigma$ and ΔG are conventional notations for contributions of the quark and gluon spins respectively. Aim of adding $L_{q,g}$ to them in Refs. [36–38] was to increase the r.h.s. of Eq. (86). Extensive overview of the literature on OAM can be found in Ref. [39]. Let us choose the frame where a high-energy nucleon moves along the z-axis, i.e. its momentum $\vec{P}$ has the only component P_z. Then the nucleon spin/helicity is also directed along the z-axis. Obviously, $L_{q,g}$ can increase the total angular momentum if they have z-components $(L_{q,g})_z$. However, $(L_{q,g})_z \neq 0$ only when the parton momenta have transverse components. Indeed, consider a very simple example: the z-component of OAM L_z of a parton with momentum k is

$$
L_z = y k_x - x k_y, \tag{87}
$$

i.e. $\vec{k}$ has both longitudinal component k_z and transverse component(s). The latter excludes involvement of collinear factorization for evolving $L_{q,g}$ but leaves possible applying KT factorization. The above arguments hold when $L_{q,g}$ is applied to other hadronic reactions, see e.g. the recent paper [40]. In order to avoid misunderstanding, we stress that we are not against using OAM contributions for description of longitudinal nucleon spin but we argue against using collinear factorization for this reason. We remind also that involvement of collinear factorization is not considered in Refs. [36–39].

10 Helicities in collinear factorization at arbitrary x and Q^2

DGLAP evolution equations [32–35] can be used for calculation of the parton helicities $h_{q,g}^{DGLAP}$ at large x and Q^2, i.e. in region D_A. The DGLAP coefficient functions and anomalous dimensions for $h_{q,g}^{DGLAP}$ are calculated in several orders in α_s. They contain contributions essential in region D_A but do not contain total resummations at all. In contrast, $h_{q,g}$ calculated in DLA involve all-order DL contributions which are leading in region D_B but are inessential in D_A. It seems scarcely possible to modify the IREE technology so as to include non-leading contributions. It leaves us with the only possibility to describe the parton helicities in region $D_A \oplus D_B$: to obtain an interpolation formulae which should coincide with the DGLAP ones at large x, Q^2 and at the same time it should account for the total resummation of DL contributions.

10.1 Helicities in the region $D_A \oplus D_B$

Expression for $h_{q,g}$ in the region D_B are given in Eq. (82). In order to adapt them to the region $D_A \oplus D_B$, they should be modified so as to combine both the resummation of DL terms and DGLAP non-DL terms in the coefficient functions and anomalous dimensions. In order to make such formulae, the following steps should be done:

(i) subtract DL terms from C_{ij}^{DGLAP} and $\Omega_{(\pm)}^{DGLAP}$. We denote the result $\bar{C}_{ij}^{DGLAP}$ and $\bar{\Omega}_{(\pm)}^{DGLAP}$.
(ii) Add $\bar{C}_{ij}^{DGLAP}$ and $\bar{\Omega}_{(\pm)}^{DGLAP}$ to the expressions for $C_{ij}^{(\pm)}$ and $\Omega_{(\pm)}$ defined in Eqs. (52, 57, 47) respectively:

$$
\begin{aligned}
\widetilde{\Omega}_{(\pm)} &= \Omega_{(\pm)} + \widetilde{\Omega}_{(\pm)}^{DGLAP}, \\
\widetilde{C}_{ij}^{(\pm)} &= C_{ij}^{(\pm)} + \widetilde{C}_{ij}^{DGLAP}.
\end{aligned}
\tag{88}
$$

Replacing $C_{ij}^{(\pm)}$ and $\Omega_{(\pm)}$ in Eqs. (52, 55, 47) by $\widetilde{C}_{ij}^{(\pm)}$ and $\widetilde{\Omega}_{(\pm)}$, we obtain interpolating expressions, which we denote ψ_{ij}^{AB} for the helicities, which is valid in the large-Q^2 region $A \oplus B$.

10.2 Helicities in the region $D_A \oplus D_B \oplus D_C \oplus D_D$

It was shown in Ref. [41,42] that extension of the large-Q^2 expressions ψ_{ij}^{AB} to the small-Q^2 region $D_C \oplus D_D$ can be done with the replacement of Q^2 by $\bar{Q}^2$:

$$\widetilde{Q}^2 = Q^2 + \mu^2 \tag{89}$$

and therefore x is replaced by $\bar{x}$:

$$\widetilde{x} = x + \mu^2/2pq, \tag{90}$$

where μ is the IR cut-off. It is worth reminding that shift of Q^2 was suggested in Refs. [43,45–47] where the parameter μ of the shift was introduced on phenomenological basis whereas it was proved in Refs. [41,42] that μ should be the IR cut-off. As a result, replacements of $C_j, \Omega_{(\pm)}$ by $\widetilde{C}, \widetilde{\Omega}_{(\pm)}$ followed by replacement of x, Q^2 by $\widetilde{x}$, $\widetilde{Q}^2$ convert ψ_{ik} into $\widetilde{\psi}_{ik}$ which can be used in the whole region $D_A \oplus D_B \oplus D_C \oplus D_D$. Convoluting $\widetilde{\psi}_{ik}$ with the initial parton distributions, we arrive at the expressions for the parton helicities $\widetilde{\psi}_{q,g}$ valid at arbitrary Q^2 and arbitrary x:

$$\begin{aligned}
\widetilde{\psi}_q(\omega, y) &= \widetilde{\psi}_{qq}(\omega, y)\,\varphi_q(\omega) + \widetilde{\psi}_{qg}(\omega, y)\,\varphi_g(\omega),\\
\widetilde{\psi}_g(\omega, y) &= \widetilde{\psi}_{gq}(\omega, y)\,\varphi_q(\omega) + \widetilde{\psi}_{gg}(\omega, y)\,\varphi_g(\omega).
\end{aligned} \tag{91}$$

In the next section, we will show that the inputs $\phi_{q,g}$ can be approximated by constants:

$$\phi_q \approx N_q, \quad \phi_g \approx N_g \tag{92}$$

and therefore $h_{q,g}$ in the region D_{tot} can be represented as follows:

$$\begin{aligned}
h_q &= \int_{-\imath\infty}^{\imath\infty} \frac{d\omega}{2\pi\imath} \widetilde{x}^{-\omega} \left[\widetilde{\psi}_{qq}(\omega, \widetilde{y})\, N_q + \widetilde{\psi}_{qg}(\omega, \widetilde{y})\, N_g\right],\\
h_g &= \int_{-\imath\infty}^{\imath\infty} \frac{d\omega}{2\pi\imath} \widetilde{x}^{-\omega} \left[\widetilde{\psi}_{gq}(\omega, \widetilde{y})\, N_q + \widetilde{\psi}_{gg}(\omega, \widetilde{y})\, N_g\right],
\end{aligned} \tag{93}$$

where $\widetilde{x}$ is defined in Eq. (90) while $\widetilde{\psi}_{ij}(\omega, \widetilde{y})$ correspond to the expressions in Eqs. (56, 58) with replacements of Eqs. (88, 89). The non-perturbative factors $N_{q,g}$ cannot be calculated in the QCD framework and should be specified by using other ways, for example by fitting experimental data.

11 Small-x asymptotics of the parton helicities

Small-x asymptotics of h_q and h_g are the same save pre-exponential factors both in collinear and KT forms of Factorizations, so we consider below h_q in collinear factorization. First we consider the small-x asymptotics of h_q in DLA, then in the DGLAP approach and compare them. Quite often in the literature, the high-energy asymptotics are simply represented by the factor s^{ω_0} (where ω_0 is the rightmost/leading singularity) without any derivation. However, we find it useful to derive the asymptotics from the parent expressions for the helicities, applying the Saddle-Point method, see also the recent paper Ref. [48].

11.1 Small-x asymptotics of $h_{q,g}$ in DLA

Expressions of Eq. (47) for $\Omega_{(\pm)}$ read that $\Omega_{(+)} > \Omega_{(-)}$, so we neglect the terms $\sim \Omega_{(-)}$ in (47). Then

$$\begin{aligned}
h_q(x, y_1) &\approx \int_{-\imath\infty}^{\imath\infty} \frac{d\omega}{2\pi\imath} x^{-\omega}\,\omega\\
&\times \left[C_{qq}^{(+)}(\omega)\varphi_q(\omega) + C_{qg}^{(+)}(\omega)\varphi_g(\omega)\right] e^{y_1\Omega_+(\omega)},\\
h_g(x, y_1) &\approx \int_{-\imath\infty}^{\imath\infty} \frac{d\omega}{2\pi\imath} x^{-\omega}\,\omega\\
&\times \left[C_{gq}^{(+)}(\omega)\varphi_q(\omega) + C_{gg}^{(+)}(\omega)\varphi_g(\omega)\right] e^{y_1\Omega_+(\omega)}.
\end{aligned} \tag{94}$$

It is convenient to write Eq. (94) in such a way:

$$\begin{aligned}
h_q(x, y_1) &\approx \int_{-\imath\infty}^{\imath\infty} \frac{d\omega}{2\pi\imath} e^{\omega\xi + E_q(\omega) + y_1\Omega_{(+)}(\omega)},\\
h_g(x, y_1) &\approx \int_{-\imath\infty}^{\imath\infty} \frac{d\omega}{2\pi\imath} e^{\omega\xi + E_g(\omega) + y_1\Omega_{(+)}(\omega)},
\end{aligned} \tag{95}$$

with $\xi = \ln(1/x)$ and

$$\begin{aligned}
E_q(\omega) &= \ln\left[C_{qq}^{(+)}(\omega)\varphi_q(\omega) + C_{qg}^{(+)}(\omega)\varphi_g(\omega)\right],\\
E_g(\omega) &= \ln\left[C_{gq}^{(+)}(\omega)\varphi_q(\omega) + C_{gg}^{(+)}(\omega)\varphi_g(\omega)\right]
\end{aligned} \tag{96}$$

and apply the Saddle Point method to Eq. (94). Dealing with the both integrals in Eq. (94) is identical, so we focus on h_q. The first step is to expand the exponent in the first equation in Eq. (94) in the power series, retaining three terms:

$$\begin{aligned}
\omega\rho + E_q(\omega) + y_1\Omega_{(+)}(\omega) &\approx \left[\omega_0\rho + E_q(\omega_0) + y_1\Omega_{(+)}(\omega_0)\right]\\
&+ \left[\xi + E'_q(\omega_0) + y_1\Omega'_{(+)}(\omega)\right](\omega - \omega_0)\\
&+ \frac{1}{2}\left[E''_q(\omega_0) + y_1\Omega''_{(+)}(\omega)\right](\omega - \omega_0)^2,
\end{aligned} \tag{97}$$

with the prime signs denoting derivatives $d/d\omega$ and ω_0 being the stationary point which is found from the stationary equation:

$$\xi + E'_q(\omega_0) + y_1 \Omega'_{(+)}(\omega) = 0 \tag{98}$$

at $\xi \to \infty$. Combining Eqs. (95, 97, 98), obtain

$$\begin{aligned} h_q(x, y_1) &\approx e^{-\omega_0 \xi + E_q(\omega_0) + y_1 \Omega_{(+)}(\omega_0)} \\ &\times \int_{-\imath\infty}^{\imath\infty} \frac{d\omega}{2\pi\imath} e^{\left[(1/2)\left(E''_q(\omega_0) + y_1 \Omega''_{(+)}(\omega_0)\right)(\omega-\omega_0)^2\right]} \\ &= \Pi_q(\omega_0)\, \xi^{-3/2}\, e^{-\omega_0 \xi + y_1 \Omega_{(+)}(\omega_0)}, \\ h_g(x, y_1) &\approx \Pi_g(\omega_0)\, \xi^{-3/2}\, e^{-\omega_0 \xi + y_1 \Omega_{(+)}(\omega_0)}, \end{aligned} \tag{99}$$

with

$$\begin{aligned} \Pi_q(\omega_0) &= \left[C^{(+)}_{qq}(\omega_0)\varphi_q(\omega_0) + C^{(+)}_{qg}(\omega_0)\varphi_g(\omega_0)\right] \\ &\times \sqrt{\frac{2\pi}{E''_q(\omega_0) + y_1 \Omega''_{(+)}(\omega_0)}}, \\ \Pi_g(\omega_0) &= \left[C^{(+)}_{gq}(\omega_0)\varphi_q(\omega_0) + C^{(+)}_{gg}(\omega_0)\varphi_g(\omega_0)\right] \\ &\times \sqrt{\frac{2\pi}{E''_q(\omega_0) + y_1 \Omega''_{(+)}(\omega_0)}}. \end{aligned} \tag{100}$$

Now consider Eq. (98) in more detail. Obviously, it holds at any Q^2 when

$$E'_{q,g}(\omega) \to -\infty \tag{101}$$

at $\omega \to \omega_0$. So, $E'_g(\omega_0)$ should be singular at $\omega \to \omega_0$. Basically, there can be many singularities in expressions in Eq. (98) but we need to find out the leading one. We assume that the fits $\varphi_{q,g}(\omega)$ are regular in ω and, remembering that the factors $C^{(+)}_{q,g}$ are made out of h_{ij}, represent the derivatives $E'_q(\omega)$ as follows:

$$E'_q(\omega) \approx \frac{\partial E_q}{\partial h_{ij}} \frac{\partial h_{ij}}{\partial Z} \left[\frac{\partial Z}{\partial U} \frac{1}{2\sqrt{U - \sqrt{V}}} \frac{dU}{d\omega} - \frac{\partial Z}{\partial V} \frac{1}{2\sqrt{V}} \frac{dV}{d\omega}\right], \tag{102}$$

where we have retained only the terms which can be singular. Equation (102) demonstrates that the leading singularity ω_0 can come either from $\sqrt{U - \sqrt{V}} = 0$ or from $\sqrt{V} = 0$, both of them are square-root branching points. However, $1/\sqrt{U - \sqrt{V}} \to +\infty$ and therefore it cannot equate ρ, so the only option for the leading singularity corresponds to $V = 0$, in the term $-1/\sqrt{V}$ which $\to -\infty$ at $\omega \to \omega_0$. It means that the last term in brackets in Eq. (102) is the most important and therefore

$$\begin{aligned} E''_q(\omega) &\approx -\frac{\partial E_q}{\partial h_{ij}} \frac{\partial h_{ij}}{\partial Z} \frac{dV}{d\omega} \frac{d}{d\omega}\left(\frac{1}{2\sqrt{V}}\right) \\ &= \frac{\partial E_q}{\partial h_{ij}} \frac{\partial h_{ij}}{\partial Z} \frac{1}{4V^{3/2}} \left(\frac{dV}{d\omega}\right)^2. \end{aligned} \tag{103}$$

It explains appearance of the factors $\xi^{-3/2}$ in Eq. (99). Then, we remind that the structure function g_1 is also made out of amplitudes h_{ik}, so the equation for the leading singularity for g_1 also includes the expression in brackets of Eq. (102) and leads to the requirement $V = 0$. It proves that the asymptotics of the helicities and g_1 are identical save the factors Π_i:

$$h_q \sim h_g \sim g_1 \sim \left[\ln^{-3/2}\left(Q^2/\mu^2\right)\right] x^{-\omega_0} \left(Q^2/\mu^2\right)^{\omega_0/2}, \tag{104}$$

with ω_0 being the intercept. Obviously, ω_0 does not depend on Q^2. When the running coupling effects are accounted for, the intercept $\omega_0 = 0.86$ at $\mu = 1$ GeV, see Ref. [23] for more detail. The identity of the asymptotics of $h_{q,g}$ and g_1 in Eq. (104) without using the Saddle-Point method was obtained in Refs. [1–9].

The square-root singularity $1/\sqrt{V} = 0$ is the result of total resummation of DL contributions The perturbation series of DL contributions to a QCD amplitude M can generically be written in the ω-space as follows (we skip here the y-dependence for simplicity):

$$M = c_0 + c_1 \frac{\alpha_s}{\omega^2} + c_2 \frac{\alpha_s^2}{\omega^4} + c_3 \frac{\alpha_s^3}{\omega^6} + \cdots, \tag{105}$$

where c_i, with $i = 1, 2, ..$, are numerical factors. Although each term in Eq. (105) contains a pole singularity at $\omega = 0$, the total sum of them forms a more complicated singularity (e.g. the square-root one) and as a result leads to the Regge asymptotics. In contrast, the truncated sum of DL terms in (105) never leads to asymptotics of the Regge type. We prove it, discussing the DGLAP small-x asymptotics. Besides, one more difference between the asymptotics obtained in our approach and DGLAP is that the intercept in our approach is equally generated by the coefficient functions and the anomalous dimensions while the coefficient function impact is negligibly small in DGLAP.

11.2 DGLAP small-x asymptotics

In this section, we discuss the small-x asymptotics generated by the perturbative components of DGLAP [32–35], postponing consideration of an influence of the fits for parton distributions until the next section. Consider some structure function/helicity F in the framework DGLAP, assuming that the fits for initial parton distributions do not have singularities in x, so their impact can be neglected. Represent F

generically as follows (cf. Eq. (1)):

$$F = \int_{-\imath\infty}^{\imath\infty} \frac{d\omega}{2\pi\imath} e^{\omega\xi + \ln C(\omega) + yh(\omega)}, \tag{106}$$

where $C(\omega)$ $(h(\omega))$ corresponds to the coefficient functions (anomalous dimensions). Expand the exponent in the series:

$$\begin{aligned} &\omega\xi + \ln C(\omega) + yh(\omega) \approx \omega_n\xi + \ln C(\omega_n) + yh(\omega_n) \\ &\quad + \left[\xi + (\ln C(\omega_n))' + yh'(\omega_n)\right](\omega - \omega_n) \\ &\quad + \frac{1}{2}\left[(\ln C(\omega_n))'' + yh''(\omega_n)\right](\omega - \omega_n)^2. \end{aligned} \tag{107}$$

Equation for ω_n:

$$\xi + \ln C(\omega_n))' + yh'(\omega_n) = 0. \tag{108}$$

DL contributions to C and h in LO, NLO, NNLO, etc. can be represented as follows:

$$\begin{aligned} C(\omega) &= 1 + c_1\frac{\alpha_s}{\omega^2} + c_2\frac{\alpha_s^2}{\omega^4} + c_3\frac{\alpha_s^3}{\omega^6} + \cdots + c_n\frac{\alpha_s^n}{\omega^{2n}} \\ &= \left[\omega^{2n} + c_1\alpha_s\omega^{2n-2} + c_2\alpha_s^2\omega^{2n-4}\cdots + c_n\alpha_s^n\right]\omega^{-2n}, \\ h(\omega) &= c_1'\frac{\alpha_s}{\omega} + c_2'\frac{\alpha_s^2}{\omega^3} + \cdots + c_n'\frac{\alpha_s^n}{\omega^{2n-1}}. \end{aligned} \tag{109}$$

Therefore,

$$\ln C = \ln\left[\omega^{2n} + c_1\alpha_s\omega^{2n-2} + c_2\alpha_s^2\omega^{2n-4}\cdots + c_n\alpha_s^n\right] - 2n\ln\omega \tag{110}$$

and

$$(\ln C(\omega))' = \frac{\left[2n\omega^{2n} + c_1(2n-2)\alpha_s\omega^{2n-2} + \cdots + c_{n-1}2\alpha_s^{n-1}\omega\right]}{\omega\left[\omega^{2n} + c_1\alpha_s\omega^{2n-2}\cdots + c_n\alpha_s^n\right]} - \frac{2n}{\omega}. \tag{111}$$

When $\omega \to n$, the singular contribution to $(\ln C(\omega))'$ is

$$(\ln C(\omega))'_{sing} = -\frac{2n}{\omega}, \tag{112}$$

while the singular contribution to h is

$$h'_{sing} = -(2n-1)c_n'\frac{\alpha_s^n}{\omega^{2n}}. \tag{113}$$

Substituting Eqs. (112, 113) in Eq. (108), obtain

$$\xi - \frac{2n}{\omega_n} - y\ (2n-1)c_n'\frac{\alpha_s^n}{\omega_n^{2n}} \approx \xi - y\ (2n-1)c_n'\frac{\alpha_s^n}{\omega_n^{2n}} = 0, \tag{114}$$

therefore

$$\omega_n = \left(y\ (2n-1)c_n'\alpha_s^n/\xi\right)^{1/2n}, \tag{115}$$

which leads to the following asymptotics of F:

$$F \sim e^{\xi\omega_n} = \exp\left[\xi^{(1-1/2n)} y^{1/2n}\left((2n-1)\alpha_s^n c_n'\right)^{1/2n}\right]. \tag{116}$$

In particular, putting $n = 1$ leads to the well-known formula for asymptotics of LO DGLAP: $F_{LO} \sim \exp\left[c_{LO}\sqrt{\alpha_s\xi y}\right]$, whereas the asymptotics of NLO DGLAP corresponds to $n = 2$: $F_{NNLO} \sim \exp\left[c_{NLO}\xi^{3/4}y^{1/4}\sqrt{\alpha}\right]$.

Remark on the false DGLAP intercept

Sometimes,[4] when DGLAP applies to experimental data processing, the DGLAP asymptotics (116) are interpreted in the Regge-like way: Neglecting the term $1/2n$ in the exponential $\xi^{1-1/2n}$, they approximate Eq. (116) by the Regge-like expression

$$F \sim \xi^{\Delta}, \tag{117}$$

where the false "intercept"

$$\Delta = \sqrt{\alpha_s}\ C\ y^{1/2n} \tag{118}$$

depends on Q^2 and n corresponds to N^{n}LO DGLAP while C is a numerical factor. Obviously, Δ contradicts both the phenomenological Regge theory and the asymptotics (104). The crucial difference between them is Q^2-dependence of Δ whereas the Q^2-dependence of the small-x asymptotics in Eq. (104) comes from the factor $(Q^2/\mu^2)^{\omega_0/2}$. Obviously, the Q^2-dependence predicted by Eq. (104) is much sharper than the one in Eq. (118). Which is those predictions is true, can be checked with analysis of experimental data.

11.3 Asymptotics induced by DGLAP fits for initial parton densities

Adding to Eq. (106) the fit for initial parton densities converts Eq. (106) into

$$F = \int_{-\imath\infty}^{\imath\infty} \frac{d\omega}{2\pi\imath} e^{\omega\xi + \ln C(\omega) + yh(\omega)}\varphi(\omega), \tag{119}$$

where $\varphi(\omega)$ stands for the fit. Typical DGLAP fits for the parton densities have the following generic structure:

$$\Phi = Nx^{-a}(1-x)^b(1+cx^d), \tag{120}$$

where parameters N, a, b, c, d are positive. Expand Φ into the power series:

$$\Phi = N\left[\sum_{k=0}^{\infty}\lambda_k^b\ x^{-a+k} + c\sum_{k=0}^{\infty}\lambda_k^b\ x^{-a+d+k}\right], \tag{121}$$

[4] For instance, when DGLAP is combined with BFKL.

where λ_k^b are binomial factors. From the point of the Regge theory, the expansion in Eq. (121) is the series of Reggeons. In the ω-space it converts into the series of the simple poles:

$$\varphi(\omega) = N\left[\sum_{k=0}^{\infty}\frac{\lambda_k^b}{\omega - a + k} + c\sum_{k=0}^{\infty}\frac{\lambda_k^b}{\omega - a + d + k}\right], \quad (122)$$

with the pole

$$\varphi_0(\omega) = \frac{N}{\omega - a} \quad (123)$$

being leading. Substituting φ_0 in Eq. (119), obtain the small-x asymptotics of F of the Regge kind:

$$F \sim [NC(a)]\ x^{-a}\ \left(Q^2/\mu^2\right)^{h(a)}, \quad (124)$$

whereas the other poles in Eq. (122) bring sub-leading corrections. It is interesting to notice that the asymptotics (124) and (104), in contrast to Eq. (118), predict the Regge kind of x- dependence with intercepts independent of Q^2 .

12 Conclusions and outlook

In this work, we have obtained explicit expressions for the parton helicities $h_{q,g}(x, Q^2)$ at high energies, which can be used in the framework of both Collinear and KT forms of QCD factorization. It was done in three steps:

Firstly, we calculated the perturbative components M_{ij} of the helicities in the DLA by constructing and solving appropriate IREEs. This allowed us to obtain $h_{q,g}(x, Q^2)$ in the kinematic region of small x and large Q^2. We demonstrated both similarity and difference between IREEs for the parton helicities and for the DIS structure function g_1. Also, we considered the color octets impact on the helicities.

Secondly, we obtained interpolation expressions for $h_{q,g}(x, Q^2)$ which hold at arbitrary x and large Q^2. It was done by combining the appropriate DGLAP formulae with the formulae obtained in DLA. By doing so, we obtained the universal expressions for $h_{q,g}(x, Q^2)$, which on one hand coincide with the DGLAP expressions at large and medium x and on the other hand contain total resummations of DL contributions important at small x.

Thirdly, we extended our formalism so as to include in consideration the region of small Q^2. It was done with the shift of Q^2 introduced in Eq. (89). As a result, we arrived at the expressions for $h_{q,g}(x, Q^2)$ valid at arbitrary x and Q^2. They can be used in both collinear and KT factorization context.

After obtaining the expressions for $h_{q,g}(x, Q^2)$, we considered in detail the topic usually discussed too briefly in the literature, namely the small-x asymptotics of the helicities. As the asymptotics in collinear and KT forms of QCD factorization are quite similar, we focused on collinear factorization. We demonstrated that though $h_q(x, Q^2)$, $h_g(x, Q^2)$, and the structure function $g_1(x, Q^2)$ are represented in DLA by different formulae, their small-x asymptotics are identical. Then we compared the small-x asymptotics of $h_{q,g}(x, Q^2)$ in DLA and DGLAP. We showed that neither LO DGLAP nor N^{n}LO DGLAP can provide the helicities with the Regge asymptotics whereas the factors x^{-a} in the DGLAP fits for initial parton densities can make it.

Discussing KT factorization, we argued that this form of QCD factorization is the only self-consisting means for including in consideration the parton orbital angular momenta (OAM) when the longitudinal nucleon spin is discussed whereas applying collinear factorization is inconsistent from the theory considerations. As a consequence, results of the papers combining OAM and collinear factorization (e.g. through using DGLAP) should be revised. It is worth mentioning that Refs. [36–38], where the OAM contribution to the proton spin was suggested, as well as the overview [39] do not relate OAM to collinear factorization. On the other hand, we found out that involving OAM to description of the transverse spin of nucleons is compatible with collinear factorization. To conclude this subject we stress once more that we agree that accounting for OAM is the interesting and important subject but we argue against combining collinear factorization and OAM.

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