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Particle acceleration near a magnetized Ernst black hole

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We investigate the collision of particles around a magnetized Ernst black hole. We derive general expression of center-of-mass energy of the colliding particles near the magnetized Ernst black hole. We focus on a specific collision scenario. That is, the collision of particles between charged particle revolving at the innermost stable circular orbit (ISCO) and neutral particle freely falling from infinity is considered. Similarly, the collision of two particles with angular momenta near the black hole is also taken into account. We show that arbitrarily high energy can be extracted by the collision process when the ISCO radius becomes very close to the black hole horizon.

I. INTRODUCTION

The effect of infinite energy can be produced near a rotating black hole horizon due to the collision of particles. It has been shown by Banados, Silk and West (BSW) [1] that a rotating spherical black hole acts as a particle accelerator to arbitrary high energies in the center of mass frame of the collision of a pair of particles near an extremal Kerr black hole. This means that the extremal rotating black hole could be regarded as a Planck-energy-scale collider, which might bring us possible visible signals from ultra high energy collisions, such as, dark matter particles. Nowadays, the BSW mechanism has been attracted much attention in the recent years [see, e.g. 2–17]. Such process by a spinning black hole [3], charged black hole [5], weakly magnetized black hole [9], Kerr naked singularity [10, 18], and a magnetized non-rotating black hole [19] has been analysed on the possibility of the production of the particles with unlimited energies. Authors of the paper [12] studied the collision of two particles with the different rest masses moving in the equatorial plane in a Kerr-Taub-NUT spacetime and found that the center of mass energy depends not only on the rotation parameter a but also on the gravitomagnetic monopole charge of the central black hole. Furthermore authors of the paper [13] have studied motion of particles and particle acceleration under the strong gravitational field of a rotating black hole in Randall-Sundrum brane acting as a particle accelerator. The mechanism of formation of black holes from collisions of particles in the vicinity of the strong gravitational supermassive black hole acting as a particle accelerator trough BSW effect and investigation of BH-BH collision, reaching large values of the center-of-mass energy have been also studied in the paper [14]. Such a process regarding high energy radiation connected with the Penrose effect in the spacetime of rotating black hole with nonvanishing gravitomagnetic charge has been studied by the authors of the paper [20].

Although the possibility of reaching the arbitrary high energy near the event horizon of the different kinds of central objects by colliding particles has been elegantly described in the several papers, in the present paper we aim to show that a such effect of particle collision with center of mass energy is also possible when one include of the Ernst spacetime and briefly reproduce and try to approach this effect by investigating in the case of different collisions

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of particles in the vicinity of a black hole immersed in strong magnetic field having already caused spacetime to be significantly distorted.

In this work we aim to study particle collisions near the magnetized Ernst black hole by considering several collision cases. In Sec. II we study the particle motion and consider trajectory of particles around the black hole. In Sec. III we briefly describe the center of mass energy and study the collision of two particles in the case i) when both are freely falling from infinity and ii) when one is charged and orbiting at the ISCO and another one is neutral and radially falling from infinity. We summarize our concluding remarks of the obtained results in the Sec. IV.

In this work we use a system of units in which $G = c = 1$. Greek indices are taken to run from 0 to 3, Latin indices from 1 to 3.

II. CHARGED PARTICLE DYNAMICS AROUND A MAGNETIZED ERNST BLACK HOLE

In this section, we consider a charged particle with the rest mass m and charge q around a magnetized Ernst black hole. We discuss how the magnetic field affects on the motion of charged particles.

The spacetime metric is given by [21]

$$ds^2 = \Lambda^2 \left(-f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\theta^2 \right) + \frac{r^2 \sin\theta^2}{\Lambda^2} d\phi^2 \,, \tag{1}$$

where

$$f(r) = 1 - \frac{2M}{r} \,, \tag{2}$$

$$\Lambda = 1 + B^2 r^2 \sin^2\theta \,. \tag{3}$$

Here the parameter B is described as the magnetic field parameter. In the case when $B \to 0$ it reduces to the Schwarzschild solution. The event horizon is also $r_h = 2M$ as in the Schwarzschild black hole. The vector potential for the magnetic field has the form as

$$A_\mu dx^\mu = -\frac{Br^2 \sin^2\theta}{2\Lambda} d\phi \,. \tag{4}$$

In Boyer-Lindquist coordinates system, the timelike and axial killing vectors are given by $\xi^a = (\partial/\partial t)^a$ and $\psi^a = (\partial/\partial\phi)^a$, respectively. With help of the Killing vectors ξ_t and ψ_φ, we have the following conserved quantities along a nongeodesics as

$$E \;=\; -(\xi_t)^\mu \pi_\mu = \Lambda^2 \left(1 - \frac{2M}{r}\right) \frac{dt}{d\tau}, \tag{5}$$

$$L \;=\; (\psi_\varphi)^\mu \pi_\mu = r^2 \sin^2\theta \left(\frac{1}{\Lambda^2} \frac{d\varphi}{d\tau} + \frac{qB}{2\Lambda} \right), \tag{6}$$

where $\pi_\mu = mp_\mu + qA_\mu$ is the four momentum of the particle with mass m and charge q and defined by $g_{\mu\nu}\frac{dx^\nu}{d\tau}$, τ is the proper time for timelike geodesics.

Continuing our assumption that the deflection in θ direction is reasonably small and orbits of the particles are in the quasi-equatorial plane $\theta = \pi/2 + \delta\theta(t)$, we restrict our attention to particle motion on the equatorial plane ($\theta = \pi/2$) so that $\pi^\theta = 0$. Then, the equations of motion for a charged particle around a black hole described by Ernst metric (1) immersed in a strong magnetic field take forms

$$\frac{dt}{d\tau} \;=\; \frac{Er}{\Lambda^2 (r - 2M)} \,, \tag{7}$$

$$\frac{d\varphi}{d\tau} \;=\; \frac{1}{r^2} \left(L - \frac{qBr^2}{2\Lambda} \right) \Lambda^2, \tag{8}$$

$$\frac{dr}{d\tau} \;=\; \left(\frac{E^2}{\Lambda^4} - \left(1 - \frac{2M}{r}\right) \left[\frac{1}{\Lambda^2} + \frac{1}{r^2} \left(L - \frac{qBr^2}{2\Lambda} \right)^2 \right] \right)^{1/2} . \tag{9}$$

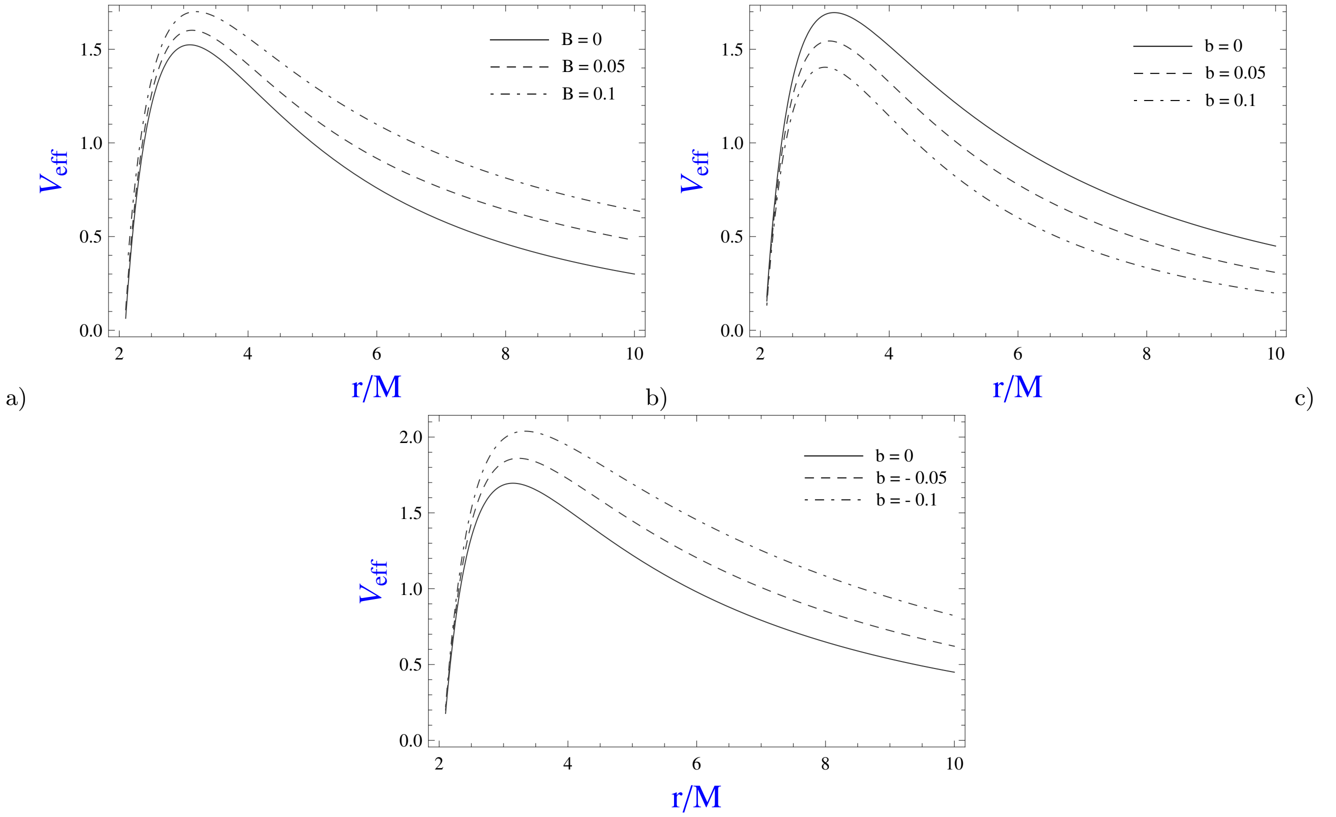

FIG. 1: Radial dependence of the effective potential on radial motion of particles near the magnetized Ernst black hole for a) a neutral particle with the magnetic parameter $b = 0$ and b) a charged particle with the magnetic field of strength $B = 0.1$.

The radial equation for a charged particle moving along non-geodesics is described by

$$\frac{1}{2}\dot{r}^2 + V_{eff}(r) = 0, \tag{10}$$

where the effective potential for the equatorial plane $\theta = \pi/2$ has a following form

$$V_{eff}(r) = -\frac{1}{2}\left(\frac{\mathcal{E}^2}{\Lambda^4} - \left(1 - \frac{2M}{r}\right)\left[\frac{1}{\Lambda^2} + \frac{1}{r^2}\left(\mathcal{L} - \frac{\mathcal{B}r^2}{2\Lambda}\right)^2\right]\right). \tag{11}$$

Here, we denote parameters $\mathcal{E} = E/m$, $\mathcal{L} = L/m$, and the parameter $b = eBM/m$ measures influence of the magnetic field on charged particle motion.

1. *Effective potential*

Here we analyse and study the properties of the effective potential described by expression (11) so as to understand more deeply nature of the magnetized Ernst black hole and its electromagnetic field. Fig. 1(a) shows the radial dependence of the effective potential on the radial motion of the neutral particle at the equatorial plane of the Ernst black hole. As can be seen from Fig. 1(a), it shows that the shape of the effective potential is shifted upward in the wake of an increase in the value of magnetic field B. It means that with increasing of the magnetic field the

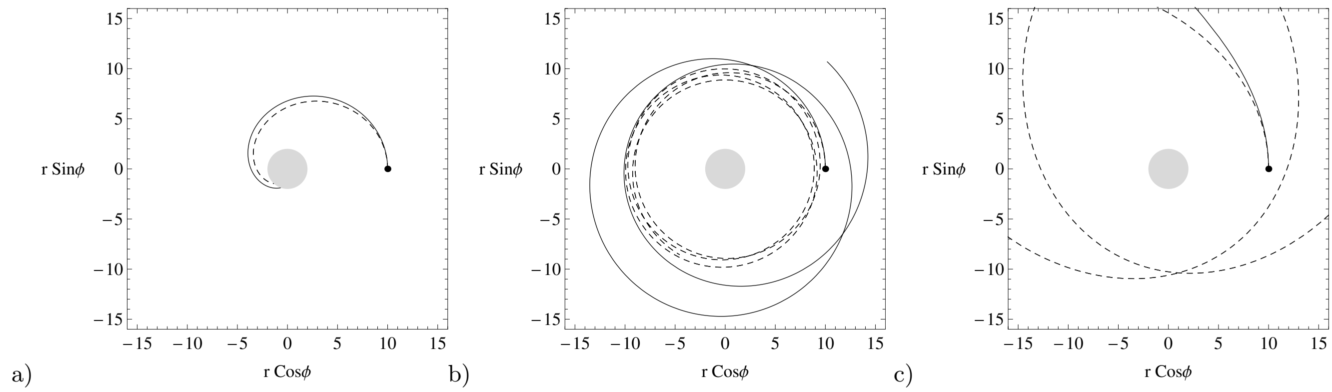

FIG. 2: Trajectories of the particles at the equatorial plane around the magnetized Ernst black hole in the case when the magnetic field $B = 0$ (solid curve) and $B = 0.01$ (dashed curve) for the specific angular momentum a) $\mathcal{L} = 3$, b) $\mathcal{L} = 4$, and c) $\mathcal{L} = 6$ for all possible orbits. In each case particle starts moving from $r_0 = 10$ towards the black hole.

gravitational force of the black hole becomes more and more stronger, and so the particles can be forced to approach to the central object. On the other hand, the magnetic field plays an important role in changing the geometry of the black hole spacetime. Due to this fact the orbit of the freely falling particles can be placed near the horizon of the black hole. Fig. 1(b) also illustrates the same behaviour as the in Fig. 1(a), but it describes the radial dependence of the effective potential on the radial motion of the charged particles. From Fig. 1(b) the behaviour of the effective potential acts the reverse and the presence of the charged particle slightly shifts the shape of the effective potential downward and makes circular orbits more and more stable. That is, the particles can be able to orbit at the ISCO, whereas, in Fig. 1(c), one can easily notice that a charged particle's orbit shifts toward right to larger r as we increase the value of magnetic parameter b.

2. Trajectory of the particles around the magnetized Ernst black hole

Let us the turn to the discussion of particle motion in the black hole exterior. Here, we consider non-geodesic motion of test particles and their trajectories around the magnetized Ernst black hole. We can assume that the trajectory of particles lies on the equatorial plane. Fig. 2 illustrates the behavior of the particle's trajectory around the black hole. It is sufficient to find out nature of orbits and turning points, allowing one to define possible orbits. From Fig. 2(a) it is apparent that the presence of magnetic field B gives rise to particle orbits to become more unstable and particles to fall into the black hole immediately than that of particles in the case when $B = 0$ for the given value of angular momentum $\mathcal{L}$. In this case the final point of trajectory of particles is singularity. Therefore there are available such unstable orbits, which correspond to terminating orbit (TO) for particles. From Fig. 2(b) shows the bound orbits (BO), for which the centrifugal force is balanced with gravitational force, and thus particles move on the bound orbits (BO) for both vanishing and nonvanishing magnetic field cases for the given value of the angular momentum. In Fig. 2(c), one can easily notice that there exist no bound orbits but exist the escape orbits (EO), according to which the trajectories of particles are always open to infinity due to the fact that the repulsive centrifugal force acting on particles becomes larger than the attractive gravitational force. Based on the results one may say that when the magnetic field is larger the bound orbits (BO) then approach the black hole's horizon in comparison with that of bound orbits induced by the small magnetic field despite the fact that the particles have larger angular momentum in the vicinity of the magnetized Ernst black hole.

3. The innermost stable circular orbits (ISCO)

According to the general description,the ISCO can be defined by the equations in the case when the first and second derivatives of the effective potential $V_{eff}(r)$ are equal to zero, in doing so we make the relevant numerical calculations to study the properties of the ISCO around the Ernst black hole. In Table I the values of the ISCO of the neutral particles moving around the magnetized Ernst black hole for the different values of B are tabulated. The results listed in Table I explain the behavior of the ISCO for the different values of B near the black hole. From the results one can easily notice that with increasing the magnetic field parameter the ISCO becomes arbitrarily close to the black hole horizon, and for the special value of the magnetic field $B_{cr} = 0.5$ the ISCO is almost equal to the black hole

TABLE I: The values of the ISCO of the neutral particles moving around the magnetized Ernst black hole for the different values of magnetic field parameter B.

B	0	0.01	0.03	0.05	0.07	0.09	0.1	0.2	0.3	0.4
$r_{\rm ISCO}$	6.0	5.73399	4.99513	4.56610	4.31245	4.14983	4.08776	3.47001	2.78998	2.31971
B	0.41	0.43	0.47	0.49	0.499	0.49999	$0.4(5\times 9)$	$0.4(7\times 9)$	$0.4(8\times 9)$	0.5
$r_{\rm ISCO}$	2.28218	2.21127	2.08421	2.02712	2.00267	$2.\ (3\cdot 10^{-5})$	$2.\ (26\cdot 10^{-6})$	$2.\ (26\cdot 10^{-8})$	$2.\ (2\cdot 10^{-9})$	2.0

TABLE II: The value of the ISCO of the charged particles moving around the magnetized Ernst black hole for the given value of the magnetic field parameter B.

				$B=0.5$				
b	3	30	70	120	200	300	500	700
$r_{\rm ISCO}$	2.72554	2.03967	2.01670	2.00969	2.00580	2.00386	2.00231	2.00165

horizon $r_h = 2$. In the Table II the value of the ISCO radius of the charged particles moving around the magnetized Ernst black hole for the different values of the magnetic parameter b are tabulated. As can be seen from Table II, the ISCO radius becomes close to the black hoe horizon with increasing b. From the results, one can notice that for the ISCO radius it is not possible to reach the horizon even though the electromagnetic force is much stronger than the gravitational force.

III. CENTER OF MASS ENERGY OF TWO PARTICLES AROUND THE MAGNETIZED ERNST BLACK HOLE

1. *Collision of freely falling particles*

In this section, we investigate the center of mass energy for the collision of two particles at the equatorial plane of a black hole immersed in a strong magnetic field. Consider two colliding particles with rest masses m_1 and m_2 at rest at infinity ($E = m_{1,2}$). The four-momentum and the total momenta of the two colliding particles $i (i = 1, 2)$ are given by

$$p_i^\mu = m_0 u_i^\mu , \tag{12}$$
$$p_t^\mu = p_1^\mu + p_2^\mu \,. \tag{13}$$

where u_i^μ is the four velocity of particle i. The center of mass energy $E_{c.m.}$ of collision between two particles is then given by [1]

$$\frac{E_{c.m.}^2}{2m_1m_2} = \frac{m_1^2+m_2^2}{2m_1m_2} - g_{\mu\nu}u_1^\mu u_2^\nu \,. \tag{14}$$

Here, we consider two particles coming from infinity with $E_1/m_1 = E_2/m_2 = 1$ for simplicity. Using the equation of particle motion around black hole (7)–(9), one can easily obtain the following relation for the center-of-mass energy of collision of two freely falling particles in the equatorial plane of the Ernst spacetime:

$$\begin{aligned}\frac{E_{\rm c.m.}^2}{2m_1m_2} &= \frac{m_1^2+m_2^2}{2m_1m_2} + \frac{r}{(r-2)\Lambda^2} - \frac{(l_1\Lambda-br^2)(l_2\Lambda-br^2)}{r^2} - \frac{r\Lambda^2}{r-2}\sqrt{\frac{1}{\Lambda^4}-\frac{r-2}{r}\left(\frac{1}{\Lambda^2}+\left(l_1-\frac{br^2}{\Lambda}\right)^2\frac{1}{r^2}\right)} \\ &\times\sqrt{\frac{1}{\Lambda^4}-\frac{r-2}{r}\left(\frac{1}{\Lambda^2}+\left(l_2-\frac{br^2}{\Lambda}\right)^2\frac{1}{r^2}\right)}\,, \end{aligned} \tag{15}$$

where the parameter $\Lambda = 1 + B^2r^2$ and the mass of the black hole is taken to be $M = 1$.

We consider limiting cases of the center of mass energy in order to find out such energy that whether it can be arbitrary high. Considering that one of the colliding particles has the maximum angular momentum l_{max} and other one has the minimum angular momentum l_{min}, we obtain the center of mass energy when freely falling charged particles collide at the equatorial plane ($\theta = \pi/2$) and near the horizon of the black hole in the case of strong magnetic field. In case of similar masses $m_1 = m_2 = m$ the center of mass energy has the following form

$$E^2_{\rm c.m.}(r \to 2) \;=\; \frac{m}{2}\Big(16 + (1+4B^2)^2(l_2 - l_1)^2 - 8b\big(2b - (1+4B^2)(l_1+l_2)\big)\Big)^{1/2} . \tag{16}$$

In the case of freely falling neutral particles the center of mass energy of collision Eq. (17) reduces to

$$E^2_{\rm c.m.}(r \to 2) \;=\; \frac{m}{2}\Big(16 + (1+4B^2)^2(l_2 - l_1)^2\Big)^{1/2} . \tag{17}$$

2. Collision of a freely falling neutral particle with a charged particle at ISCO

We consider a collision of two particles, each of mass m and conserved energy of per unit mass E. One of the particles is taken to be freely falling neutral particle with zero angular momentum and the other one is charged and revolving at the ISCO orbit of the black hole in the case of strong magnetic field. Taking into account a circular motion around the black hole one can easily write the momentum of a charged particle at the orbit of radius r on the equatorial plane as following

$$p^\mu \;=\; m\gamma\left(\frac{1}{\Lambda}\left(\frac{r}{r-2}\right)^{1/2}\delta^\mu_t + \frac{\Lambda}{r}\upsilon\delta^\mu_\varphi\right) , \tag{18}$$

where υ (which can be both positive and negative) is a velocity of the particle with respect to a rest frame, and γ is the Lorentz gamma factor which is defined by the following form

$$\gamma = \frac{1}{\sqrt{1-\upsilon^2}} . \tag{19}$$

On the basis of relation $d\varphi/d\tau = \upsilon\gamma/r$ and (8) one can find

$$\upsilon\gamma = \left(\frac{\mathcal{L}}{r^2} - \frac{b}{\Lambda}\right)\Lambda^2 , \tag{20}$$

which allows one to obtain expression of the velocity

$$\upsilon = \frac{r\chi(r)}{\sqrt{1+r^2\chi^2(r)}} , \tag{21}$$

where the parameter $\chi(r)$ is defined as

$$\chi(r) = \left(\frac{\mathcal{L}}{r^2} - \frac{b}{\Lambda}\right)\Lambda^2 . \tag{22}$$

From the equations above one can find the Lorentz gamma factor as following

$$\gamma^2 = 1 + r^2\left(\frac{\mathcal{L}}{r^2} - \frac{b}{\Lambda}\right)^2\Lambda^4 . \tag{23}$$

We represent by m_n and m_q the mass of freely falling particle and the mass of charged particle orbiting along the ISCO of radius r, and by p_n and p_q their four momenta of neutral and charged particles, respectively. The total momentum of two colliding particles is

$$P = p_n + p_q \,, \tag{24}$$

and the corresponding center of mass energy of collision of neutral and charged particles has the form

$$E^2_{\rm c.m.} = m_n^2 + m_q^2 - 2g_{\mu\nu}p^\mu p^\nu \,. \tag{25}$$

By direct replacement of (7), (8), and (18) into (25) one can find the collision energy of two particles

$$E^2_{\rm c.m.} = m_n^2 + m_q^2 + 2m_q\gamma_q\left(\frac{\mathcal{E}_n}{\Lambda}\left(\frac{r}{r-2}\right)^{1/2} - \frac{\Lambda\upsilon}{r}\mathcal{L}_n\right) , \tag{26}$$

Since second term in the bracket of expression (26) is finite in any case, the corresponding the center of mass energy has the form when the collision occurs close to the black hole horizon ($r \to 2$)

$$E^2_{\rm c.m.} \sim \frac{2m_q\gamma_q\mathcal{E}_n\sqrt{r}}{(1+B^2r^2)\sqrt{r-2}} \,. \tag{27}$$

Based on the results regarding the ISCO listed in Tables I and II we showed that the center of mass energy of colliding particles can be arbitrary high in the case when one is neutral at the ISCO and another one is also neutral and radially falling toward the black hole horizon. The presence of magnetic field pushes the ISCO toward the horizon, thus leading to collision at the horizon where the black hole can, in principle, accelerate the particles to arbitrary high energies. And so high energy collisions can be produced by the collision of neutral particles in the vicinity of the magnetized Ernst black hole. Also we considered the collision between the neutral particle radially falling and the charged particle orbiting at the ISCO, so we showed that the center of mass energy can be large but not produced arbitrarily high energy in their collision. From the results listed in Table II one can easily see that the value of the magnetic parameter b corresponds to the different ISCO. That is, due to large Lorentz force the ISCO gets decreased but never reaches the horizon radius. For this specific collision scenario collision energy is not arbitrary high energy despite the fact that the particles can be essentially accelerated near the horizon due to the the magnetic field. In conclusion, for the charged particle case, the energy of colliding particles would be high but not infinite energy in their center of mass frame as compared to the one for neutral particles which have arbitrarily high energy as a result of their collision at the ISCO radius.

IV. CONCLUSIONS

In this paper, we have studied the collision of two particles coming from infinity with the different rest masses moving in the equatorial plane of the black hole and the collision of two particles between charged particle revolving at the ISCO and neutral particle freely falling. We have derived the general formula for the center of mass energy of the colliding two particles for specific collision scenarios.

We have shown that the center of mass energy can be approached to arbitrary high energy in the collision between neutral particle freely falling from infinity and charged particle at the ISCO in vicinity of the magnetized Ernst black hole. We have shown that this energy infinitely grows when the ISCO radius becomes arbitrarily close to the horizon. Also, for the charged particle case the ISCO would not reach the black hole's horizon, and thus the center of mass energy can not be arbitrary high because the rate of acceleration of two particles near the horizon becomes slower than that of colliding neutral particles. Furthermore, we have studied the properties of particle trajectories around the magnetized Ernst black hole. The dependence of the trajectory of the particles from the magnetic field parameter B shows that the bound orbits (BO) approach the black hole horizon and become more stable with increasing the magnetic field parameter.

Acknowledgments

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