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Free-Space Diffraction and Interference in a Transformed Frame

John S. Briggs

Max-Planck Institute for the Physics of Complex Systems, Dresden, Germany

Correspondence: John S. Briggs (briggs@physik.uni-freiburg.de)

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ABSTRACT

For free propagation from a focus, the Hermite–Gauss wave functions of optics spread in space. In quantum mechanics, the Hermite–Gauss functions are referred to as the harmonic oscillator eigenfunctions. These functions are used here to describe the interference of wave packets. It has been shown that when transformed to a frame moving with the normal to the wave front trajectories, the Hermite–Gauss functions are constant up to a phase factor which is the Gouy phase. The Gouy phase itself assumes the role of proper space or time coordinate. Along the whole of such a trajectory, the space wave function is proportional to the wave number or momentum function. An arbitrary normalizable wave packet can be expanded using the Hermite–Gauss functions as a basis. As example, it is shown that in the transformed frame, a displaced Gaussian does not spread but rather becomes a coherent state. This allows a particularly simple representation of the Young's interference pattern from two or more slits.

1 | Introduction

This paper is concerned with the equivalence of the diffraction and interference of wave packets of light in classical optics (abbreviated as CO) and of material particles in quantum mechanics (abbreviated as QM). Of course, for light, the study of interference effects goes back at least to Young in 1801. The observation of the interference of matter waves began with electrons by Davisson and Germer in 1927 and by Stern and coworkers for atom and molecules in the early 30s, see the historical article by Friedrich and Schmidt-Böcking in ref. [1]. Today, extremely sophisticated experiments have demonstrated the interference of wave packets of molecules comprised of hundreds of atoms [2]. The sensitivity of experiments has increased such that the assembly of interference patterns by the arrival of single electrons [3] or single photons [4] can be detected.

Generically, freely moving wave packets, whether of light in CO or material particles in QM, spread out in space as they propagate. However, their Fourier transform function, in wave-number or momentum space, is invariant up to a frequency phase factor. In two previous publications ref. [5, 6], it is shown that insight into the properties of the free propagation in space or time of normalizable wave packets can be obtained by a transformation to a spatial frame defined along the locus of the normals to the instantaneous wave fronts. In QM, this appears as a space–time transition to a comoving frame which is a trajectory along the locus of the normals to the wave fronts. In QM, such trajectories are called Bohmian trajectories [7, 8]. The relevance of Bohmian trajectories to the two-slit interference pattern is discussed in detail in ref. [8].

In CO, it is not usual to think of wave packets spreading in time, principally because of the large velocity of light which implies that a stationary wave pattern is established rapidly. Hence, one refers to stream lines in space. By contrast, the much lower velocities of nonrelativistic QM make it meaningful to envisage the propagation of wave packets in time and to refer to wave front loci as trajectories, as in classical mechanics. Unfortunately, the nomenclature and terminology of CO and QM is different although referring to very similar mathematical structures.

Summary

- T In classical optics, transformation is made to a frame moving with the stream lines in which the Hermite–Gauss functions are constant. In quantum mechanics they are harmonic oscillator eigenfunctions.
- The Gouy phase forms the proper distance or time in this frame.
- Gaussian slits are non-spreading coherent states and move to coalesce and interfere in the transformed frame.

Since one aim of this paper is to unify the CO and QM phenomena, one is faced with the dilemma of which nomenclature to use. The strategy adopted here is to use the space z and time t coordinate denoting changes along the beam, largely interchangeably. This is justified by their essential equivalence which is outlined in the following section. Hence, stream lines are taken to be synonymous with trajectories and the frame defined along stream lines or trajectories will be called the transformed or comoving frame, as appropriate.

The transformation of position and time coordinates to analyze dynamics occurs throughout physics. Its particular use to transform free quantum propagation to that of a harmonic oscillator (HO) is described in ref. [9, 10]. An extensive discussion of this transformation, where it is called the quantum Arnold transformation, is given in ref. [11]. In ref. [5, 6], the transformation to the moving frame was applied to free solutions of the time-dependent Schrödinger equation (TDSE) of quantum mechanics and to the paraxial equation (PE) of classical optics.

Wave packets in CO are often described in terms of different sets of mode wave functions, among which the Hermite–Gauss (HG) modes are prominent. Similarly, the same set of functions, but usually called the HO eigenfunctions, plays a dominant role in QM.

The main results of ref. [6] in the comoving frame are:

1. The HG functions are constant along the transformed "comoving" trajectory, that is, the wave packet does not spread in space.
2. The space wave functions are proportional, along the whole trajectory, to the invariant Fourier-transformed wave number (CO) or momentum (QM) functions.
3. The Gouy phase forms the natural distance (CO) or time (QM) coordinate in the transformed frame.

With these properties, the complete set of constant HG wave functions can be used as expansion basis to describe the propagation of an arbitrary wave packet both in CO and QM.

The Gouy phase plays a significant role in the following analysis. Although discovered already in 1890 ref. [12], its meaning and interpretation has been the subject of several papers comparatively recently. For example, in optics ref. [13–15], in quantum optics ref. [16], and in matter waves ref. [17]. In ref. [5], it was shown to emerge, in the transformed frame, as a dynamic energy phase arising from mode quantization of the finite extent of the wave packet, both in QM and its equivalent in CO.

In this paper, it is demonstrated that, when represented in the basis of HG functions, a Gaussian function shifted from the origin of this basis set, transforms into a *nonspreading* propagating coherent state of quite simple form. This result is used to give an equally simple representation of the wave function describing the Young's fringes interference of two, or indeed any number, of such slits.

2 | The Equivalence of the CO PE and the TDSE of QM

The PE of CO is an approximation to the full three-dimensional Helmholtz equation for light in a vacuum. In this approximation, the variation of the wave function $\psi(x, y, z)$ in the extended z direction is considered to be slow compared to the variation in the orthogonal x and y directions. Hence, the second z derivative is neglected to give an approximate equation called paraxial. This PE is

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + 2ik_z\frac{\partial\psi}{\partial z} = 0, \tag{1}$$

where k_z is the wave number in the z direction.

In ref. [5], it is shown that the three-dimensional free *time-independent* Schrödinger equation, under the same approximation as led to the PE, reduces to the equation

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) - i\frac{\hbar p_z}{m}\frac{\partial\psi}{\partial z} = 0, \tag{2}$$

where m is the particle mass and the momentum p_z can be written as $p_z \equiv \hbar k_z$. The equivalence of this Schrödinger equation and the PE can be seen simply by multiplying Equation (1) by $-\hbar^2/(2m)$ to obtain exactly Equation (2).

The approximate Equation (2) could be used as the "paraxial" equation of QM. However, it is customary to convert to a standard TDSE. The key step in deriving the TDSE, see ref. [18], is when the variable z can be treated as a *classical variable* $z(t)$ with a corresponding *classical velocity* $\dot{z}(t) \equiv v_z$. Then, in Equation (2), one puts $p_z/m = v_z = \partial z/\partial t$, so that the PE becomes

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) - i\hbar\frac{\partial\psi}{\partial t} = 0, \tag{3}$$

which is the TDSE for motion in two dimensions.

Also, one sees that the equation in time can be obtained from the PE of Equation (1) in z, simply by putting $z = ct$, where c is the velocity of light. Then, the PE and TDSE are connected by replacing k_z/c in CO by $m/\hbar$ in QM, both constants having the same physical dimensions.

In the PE, the propagation of a two-dimensional wave packet, free along the z-axis but changing shape in the (x, y) directions, is considered. Where cartesian coordinates (x, y, z) for vacuum propagation in CO are appropriate and the HG basis is used, then the PE separates in x and y directions (see chapter 3 of ref. [19]). The same is true for free forceless propagation in QM. Since the x and y solutions are then identical, it is sufficient to consider only the x direction.

In the quantum mechanics of material particles, only expansion from a microscopic volume is observed, hence, here only the expanding half of a wave packet is considered.

3 | The Propagating HG Wavepacket in Optics

In CO, the propagation of beams as solutions of the PE in one space direction is a standard problem. In particular, the propagation in space of an initial HG wave is given in many text books. For a scalar wave in one transverse dimension, the initial HG wave packet at $z = 0$ is taken to be

$$\Psi_n(x,0) = \frac{1}{(\pi W_0^2/2)^{1/4}} \left(\frac{1}{2^n n!}\right)^{1/2} H_n\left(\frac{\sqrt{2}x}{W_0}\right) \times \exp\left(-\frac{x^2}{W_0^2}\right), \quad (4)$$

where W_0 is a constant defined below. The solution for propagation in the z direction in the laboratory frame, beginning at $z = 0$, has the form,

$$\Psi_n(x,z) = \frac{1}{[\pi W_0^2(1+\tau^2)/2]^{\frac{1}{4}}} \exp\left[-i(n+\frac{1}{2})\arctan\tau\right] \times \left(\frac{1}{2^n n!}\right)^{1/2} H_n\left(\frac{\sqrt{2}x}{W_0(1+\tau^2)^{1/2}}\right) \times \exp\left[-\frac{x^2}{W_0^2(1+\tau^2)}\right] \exp\left[i\frac{x^2\tau}{W_0^2(1+\tau^2)}\right], \quad (5)$$

where $\tau \equiv z/z_R$ is a dimensionless "distance" and z_R is the Rayleigh length. The length z_R gives the demarcation between wave front normal loci of hyperbolic form, $\tau < 1$, and the onset of the wave front normal loci becoming, for $\tau \gg 1$, almost linear. This can be seen clearly in Figure 1, where $\tau = z/z_R = 1$. The constant W_0 denotes the size of the beam waist at $z = 0$. It is related to the Rayleigh length by $W_0^2 = 2z_R/k_z$.

With the definitions $W(z) \equiv W_0(1+\tau^2)^{1/2}$ and the phase front curvature $R(z) \equiv z_R(1+\tau^2)/\tau$, the wave function can be expressed in the compact form.

$$\Psi_n(x,t) = \frac{1}{[\pi W_z^2/2]^{\frac{1}{4}}} \exp\left[-i(n+\frac{1}{2})\arctan\tau\right] \left(\frac{1}{2^n n!}\right)^{1/2} \times H_n\left(\frac{\sqrt{2}x}{W_z}\right) \exp\left[-\frac{x^2}{W_z^2}\right] \exp\left[i\frac{k_z x^2}{2R(z)}\right]. \quad (6)$$

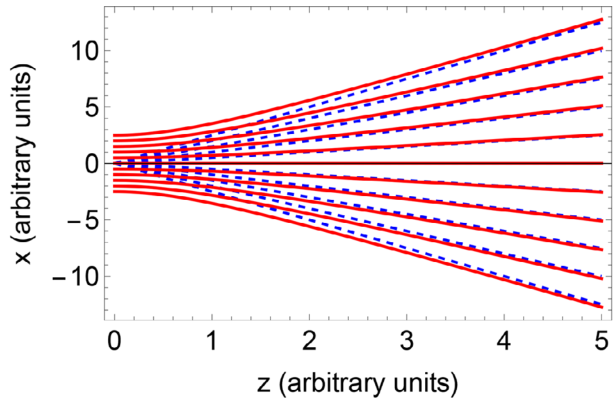

FIGURE 1 | The continuous red lines are the wave front trajectories $x(t)$ representing the spreading in one x dimension (vertical axis) of HG functions from a range of initial points x_0. The horizontal axis is the z axis. The Rayleigh distance is $z_R = 1.0$. The trajectories, beginning at x_0, are asymptotic to the straight-line $x = (k_x/k_z)\ z$ beam trajectories (blue dashed lines) which extrapolate back to $x = 0$.

This is the standard form of HG beams in optics, usually given in two transverse dimensions (x, y), see chapter 3 of Saleh and Teich [19], for example.

In ref. [6], it is shown that the Gouy phase has its origin in the "quantization" of the transverse wave to fit into the confinement of the finite space occupied by the wave at each fixed z value. Assuming a constant light velocity c in the z direction, the HG functions correspond to the mode frequency $\omega = c/z_R$. For finite distances z, an instantaneous frequency

$$\omega(z) \equiv \frac{1}{(z_R/c)(1+\tau^2)} = \frac{c}{z_R\left(1+(\frac{z}{z_R})^2\right)} \quad (7)$$

is defined.

The frequencies of all higher HG states n are given by $(n + 1/2)\omega(z)$. Then, the Gouy phase is explained as the accumulated adiabatic phase of the instantaneous eigenfrequencies

$$\exp\left[-i(n+\frac{1}{2})\arctan\tau\right] = \exp\left[-i(n+\frac{1}{2})\arctan\left(\frac{z}{z_R}\right)\right] = \exp\left[-\frac{i}{c}(n+\frac{1}{2})\int^z \omega(z')\,dz'\right]. \quad (8)$$

This result shows that the Gouy phase is due to the confinement of the harmonic transverse wave in a finite length at each fixed z value [13].

One notes that the expanding wave in the x direction of Equation (5) involves distance only in the variable $x/(1+\tau^2)^{1/2}$. If one fixes $x(0) \equiv x_0$, one can define a trajectory or stream line in the z direction given by $x(t) = x_0(1+\tau^2)^{1/2}$. Then, one can introduce a transformation of coordinates (x, τ) to a frame along this trajectory for each fixed initial value x_0. As shown in ref. [5], the "trajectories" are the loci of the normals to the wave fronts. In QM, they are known as Bohm trajectories.

The spreading direction x and the propagation direction z along the beam are scaled. That is, the space is warped by the transformations,

$$\bar{x} \equiv \frac{x}{a(z)} \qquad \bar{z} \equiv \int^{z} \frac{dz'}{a(z')^2}, \tag{9}$$

where $a(z)$ is dimensionless.

Choosing $a(z)$ to describe the stream line $a(z) = (1+\tau^2)^{1/2}$, where $\tau = z/z_R$, it is shown in ref. [6] that one transforms the free PE to the HO equation,

$$\left(-\frac{\partial^2}{\partial \bar{x}^2} + \left(\frac{2}{W_0^2}\right)^2 \bar{x}^2\right)\Phi - i2k_z\frac{\partial \Phi}{\partial \bar{z}} = 0. \tag{10}$$

From the transformation Equation (9), the new x variable $\bar{x}$ is given by $\bar{x} = x_0 = k_x W_0^2/2$. This gives the intersection point of each wave front normal locus with the x axis.

The new transformed z variable is given from Equation (9) by $\bar{z} = z_R \arctan(z/z_R) \equiv z_R \arctan(\tau)$. Then, it is consistent to define a new dimensionless variable $\bar{\tau} \equiv \bar{z}/z_R$. This gives $\bar{\tau} = \arctan(\tau)$.

Note that the full Gouy phase, from Equation (8), is an $(n+\frac{1}{2})$ multiple of $\bar{\tau}$. Since the variation of the phase is solely in $\bar{\tau}$, in the following, this function alone is often referred to as the Gouy phase. That is, up to a number, the transformed variable $\bar{\tau}$ becomes *simply the Gouy phase itself*.

The solutions to Equation (10) in the transformed frame are constant HG functions multiplied by the Gouy phase factor, that is,

$$\Phi_n(\bar{x},\bar{\tau}) = \frac{1}{(\pi W_0^2/2)^{1/4}}\left(\frac{1}{2^n n!}\right)^{1/2} H_n\left(\frac{\sqrt{2}x_0}{W_0}\right) \times \exp\left(-\frac{x_0^2}{W_0^2}\right)\exp\left(-i(n+1/2)\bar{\tau}\right). \tag{11}$$

The dimensionless parameter $\bar{\tau} \equiv \bar{z}/z_R$ can be viewed as the proper "length" in the transformed frame. The Gouy phase changes from zero at $z = 0$, to $\pi/2$ at $z = \infty$, which gives a compact mapping of the infinite z or τ axis onto the finite, zero to $\pi/2$, Gouy phase $\bar{\tau}$ axis. Hence, in the transformed frame, the wave function is constant for a given x_0. Only the Gouy phase changes linearly in $\bar{\tau} = \bar{z}/z_R$.

Using $x_0 = k_x W_0^2/2$, the wave function can be written also in terms of the wave number k_x, that is,

$$\Phi_n(\bar{x},\bar{\tau}) = \frac{1}{(\pi W_0^2/2)^{1/4}}\left(\frac{1}{2^n n!}\right)^{1/2} H_n\left(\frac{W_0 k_x}{\sqrt{2}}\right) \times \exp\left[-\frac{W_0^2 k_x^2}{4}\right]\exp\left(-i(n+1/2)\bar{\tau}\right), \tag{12}$$

which is proportional to the initial wave function in wave number space. The invariant propagation of the wave number function of a free wave packet is mirrored by the invariant propagation of the space wave function in the transformed frame.

The "trajectories" $x = x_0(1+\tau^2)^{1/2}$, where $\tau = z/z_R$, are plotted in Figure 1. The curves are hyperbolae and the beam waist $W(z)$ is just one such curve. Each curve is the locus of normals to the wave front for a particular k_x. They are asymptotic to a straight line $x = (k_x/k_z)z$, which is the beam limit of the wave description.

As is well known, the space wave function in the laboratory frame becomes proportional to the wave number function at asymptotically large z, the Fraunhofer limit. This one sees if one calculates the $z \to \infty$ limit of Equation (5) and substitutes the straight line $x = (k_x/k_z)z$. This limit is then,

$$\Psi_n(x,t) \approx \frac{W_0^{1/2}}{(2\pi)^{\frac{1}{4}}}\left[\frac{k_z}{z}\right]^{1/2}\left(\frac{1}{2^n n!}\right)^{1/2} H_n\left(\frac{W_0 k_x}{\sqrt{2}}\right) \times \exp\left[-\frac{W_0^2}{4}k_x^2\right]\exp\left[-i(n+1/2)\frac{\pi}{2}\right]\exp\left[i\frac{k_x^2}{2k_z}z\right], \tag{13}$$

which is proportional to the same wave number function of Equation (12) in the transformed frame.

The properties of the propagation of HG functions in the transformed frame are summarized as follows:

1. The loci of normals to the wave fronts define "trajectories" $x(z) = x_0(1+\tau^2)^{1/2}$ where $\tau \equiv z/z_R$.
2. The trajectories begin at $z = 0$ with $x_0 = k_x W_0^2/2$ and asymptotically become the straight lines $x = (k_x/k_z)z$.
3. The space HG wave functions are invariant along a trajectory except for a phase which is just the Gouy phase.
4. Along the complete transformed trajectory, the space wave function is proportional to the invariant wave number function. In the laboratory frame, this is only true in the *asymptotic* Fraunhofer limit.

These invariant properties are contrasted with the spreading of the space HG wave functions in the laboratory frame. In Section 5, the invariant HG functions are utilized to describe the propagation of wave packets in the transformed frame. It is shown that a shifted Gaussian wave packet assumes a particularly simple "coherent state" form, which illuminates the interference pattern of slits composed of such functions.

First, in the next section, the substitutions necessary for the equations to apply to QM are given.

4 | Transcription to Quantum Mechanics

As derived in ref. [5] and outlined in Section 2, the two-dimensional PE and TDSE, as approximations to the three-dimensional Helmholtz and Schrödinger equations, respectively, are identical mathematically. Then, all the results above for the PE apply to a quantum wave packet described by the TDSE, after suitable substitution of variables and constants. In particular, the ratio k_z/c is replaced by $m/\hbar$. In QM, the standard notation in describing Gaussian wave packets is to replace $2W_0^2$, by the notation σ^2.

In optics, the fundamental distance describing beam expansion is the Rayleigh distance z_R. The Rayleigh range $z_R = 2k_z W_0^2$ is related to the equivalent characteristic time constant $T \equiv m\sigma^2/\hbar$ of QM by the simple expression $T = z_R/c$.

In QM, the dimensionless "time" $\tau = t/T$ is introduced and the limit $\tau \gg 1$ gives the classical limit of straight-line trajectories, as explained in some detail in ref. [20].

The Gouy phase, originating from optics, is given by $\arctan(z/z_R)$. Putting $z = ct$ and $z_R = cT$, then $\tau = z/z_R = t/T$ is unchanged in the quantum case.

In the following sections, all length coordinates are taken to be scaled by $W_0/\sqrt{2}$, the width of the initial HG functions in CO or by σ, the width of initial HO function, in QM. That is, the unit of length is $W_0/\sqrt{2}$ or σ, respectively. Consistent with this, the normalization factors $(W_0/\sqrt{2})^{1/2}$ and $\sigma^{1/2}$ are put to unity. This corresponds to the square of these factors being used to scale dz or dt in the respective normalization integrals.

This introduction of dimensionless variables not only makes the equations simpler but means that *all equations apply equally to both the CO and QM cases*. At the end of the calculation, the dimensionless variables of space and time can be easily converted back to physical units.

What is to be noticed, however, although the equations are unchanged, is the enormous difference in scale between the CO and the QM cases. In optics, the typical Rayleigh range is of the order of 10^{-6} to 10^{-5} m. In atomic QM, T is typically of the order of unity in atomic units, to give an approximate z_R, dependent on longitudinal velocity, of say 10 atomic units, which is approximately 10^{-9} m.

5 | The Coherent State of Gaussian Slits

An arbitrary normalizable wave packet can be expanded in the HG basis of functions. If the wave packet in the transformed frame is denoted by $\chi(\bar{x}, \bar{\tau})$, then the space development is given by

$$\chi(\bar{x}, \bar{\tau}) = \sum_n a_n \, \Phi_n(x_0, \bar{\tau}), \tag{14}$$

since $\bar{x} = x_0$ for the transformed HG basis. For the chosen scaling of space z along the hyperbolic trajectories, only the phase factors change in $\bar{\tau}$. Then, the sum becomes simply a Fourier series.

The foregoing applies to any normalizable wave packet. As example, since it gives a simple analytical result for the space propagation, now an initial wave function of Gaussian (HO with $n = 0$) form is considered. For a grating made of slits of Gaussian shape, one needs the expansion of a Gaussian shifted w.r.t the fixed origin at $x = 0$.

For simplicity, the width of the expansion basis centered at $x = 0$ will be chosen to be the same as that of the slit. Since the width chosen for the expansion basis is arbitrary, this can always be done.

In the laboratory frame, a slit shifted from the origin of x by a length d, either in length units of $W_0/\sqrt{2}$ or in length units of σ, has a normalized wave function at $z = 0$ which is

$$\Psi_0(x-d, 0) = \frac{1}{(\pi)^{1/4}} \exp\left(-\frac{(x-d)^2}{2}\right). \tag{15}$$

Also, in the laboratory frame, the expansion basis wave functions are of the form

$$\Psi_n(x, 0) = \frac{1}{(\pi)^{1/4}} \left(\frac{1}{2^n n!}\right)^{1/2} H_n(x) \exp\left(-\frac{x^2}{2}\right). \tag{16}$$

At $z = 0$, one has $\bar{z} = \bar{\tau} = 0$, that is, the laboratory and transformed frames coincide. Then, from Equation (14), one can calculate the coefficients a_n by evaluating the laboratory-frame overlap integral

$$a_n = \int \Psi_0(x-d, 0) \Psi_n(x, 0)\, dx \tag{17}$$

to give the coefficients

$$a_n = \frac{1}{(n!)^{1/2}} \left(\frac{d}{\sqrt{2}}\right)^n \exp\left(-\frac{d^2}{4}\right). \tag{18}$$

Since, in the transformed frame, the space wave function is constant, the only $\bar{z}$ factor is the Gouy phase $\bar{\tau}$. Then, the expansion assumes the form

$$\chi(\bar{x}, \bar{\tau}) = \frac{1}{(\pi)^{1/4}} \exp\left(-\frac{x_0^2}{2} - \frac{d^2}{4}\right) e^{-i\bar{\tau}/2} \sum_n \frac{1}{n!} \left(\frac{d}{2}\right)^n \left(e^{-i\bar{\tau}}\right)^n H_n(x_0). \tag{19}$$

The summation can be carried out to give

$$\chi(\bar{x}, \bar{\tau}) = \frac{1}{(\pi)^{1/4}} \exp\left(-\frac{x_0^2}{2} - \frac{d^2}{4}\right) e^{-i\bar{\tau}/2} \exp\left(-\frac{(de^{-i\bar{\tau}})^2}{4} + x_0 d e^{-i\bar{\tau}}\right). \tag{20}$$

This result for the $\bar{\tau}$-dependent slit wave function in the transformed frame can be written as:

$$\chi(\bar{x}, \bar{\tau}) = \frac{1}{(\pi)^{1/4}} e^{-i\bar{\tau}/2} \exp\left(-\frac{d^2}{4}(1 - e^{-2i\bar{\tau}}) - \frac{(x_0 - de^{-i\bar{\tau}})^2}{2}\right). \tag{21}$$

The intensity density from this expression is particularly simple,

$$|\chi(\bar{x},\bar{\tau})|^2 = \frac{1}{(\pi)^{1/2}} \exp\left(-(x_0 - d\cos\bar{\tau})^2\right), \tag{22}$$

which is just a Gaussian of constant width. Thus, there is no spreading of the wave packet in the transformed frame. In fact, one sees that the simple Gaussian form is invariant for all the trajectory. All that happens is the center of the packet moves gradually from $x_0 = d$ for Gouy phase $\bar{\tau} = 0$ to $x_0 = 0$ for $\bar{\tau} = \pi/2$.

In QM, such a wave function as Equation (22) is called a coherent state. Usually in QM, the center of a coherent state oscillates in time. Here, there is only a single half-oscillation in $\bar{\tau}$ to change the position of the center of the Gaussian wave packet smoothly by the amount d. This is shown in the middle plot on Figure 2.

Note that, in the laboratory frame, the length x_0 is equal to $k_x W_0^2/2$. If one expresses x_0 in length units of $W_0/\sqrt{2}$, one has $x_0 = k_x/(\sqrt{2}/W_0)$. Then, it is consistent to express k_x in the inverse length units of $\sqrt{2}/W_0$, see Equation (12). In these scaled units, one has the simple assignment $x_0 = k_x$. Replacing x_0 by k_x in Equation (22) shows clearly the equivalence of space and wave number functions along a trajectory. Hence, one obtains the Fraunhofer limit of the wave number function *along the whole trajectory*.

6 | Diffraction From Two Slits or From a Grating of *N* Slits

For two slits, the Young's diffraction pattern in the laboratory frame is well-studied. Each Gaussian wave packet spreads after $\tau \approx 1$, as shown in Figure 1. As they overlap, then they interfere. The interference pattern changes as τ grows from zero to infinity. This is shown in the uppermost frame of Figure 2 with coordinates x, τ. For $\tau \gg 1$, the interference pattern assumes the asymptotic form of the wave number or momentum function, the Fraunhofer limit.

In the transformed frame, Equation (21) is the expression for a slit centered at $x = +d$. The equation for the second slit, centered at $x = -d$, is given by the same expression with d simply replaced by $-d$. If one adds the two contributions $\Psi_2(\bar{x},\bar{\tau}) = \chi_d(\bar{x},\bar{\tau}) + \chi_{-d}(\bar{x},\bar{\tau})$, one obtains the result

$$\chi_2(\bar{x},\bar{\tau}) = \frac{e^{-i\bar{\tau}/2}}{(\pi)^{1/4}} e^{-x_0^2/2} \times \exp\left(-\frac{d^2}{4}(1 + e^{-2i\bar{\tau}})\right) 2\cosh(x_0 d e^{-i\bar{\tau}}). \tag{23}$$

The square modulus of this wave function gives the detection intensity of the Young's fringes which can be put into the form

$$|\chi_2(\bar{x},\bar{\tau})|^2 = \frac{1}{(\pi)^{1/2}} \left[e^{-(x_0 - d\cos\bar{\tau})^2} + e^{-(x_0 + d\cos\bar{\tau})^2} + 2e^{-(x_0^2 + d^2\cos^2\bar{\tau})} \cos(2dx_0 \sin\bar{\tau})\right]. \tag{24}$$

This expression is valid all along the trajectory.

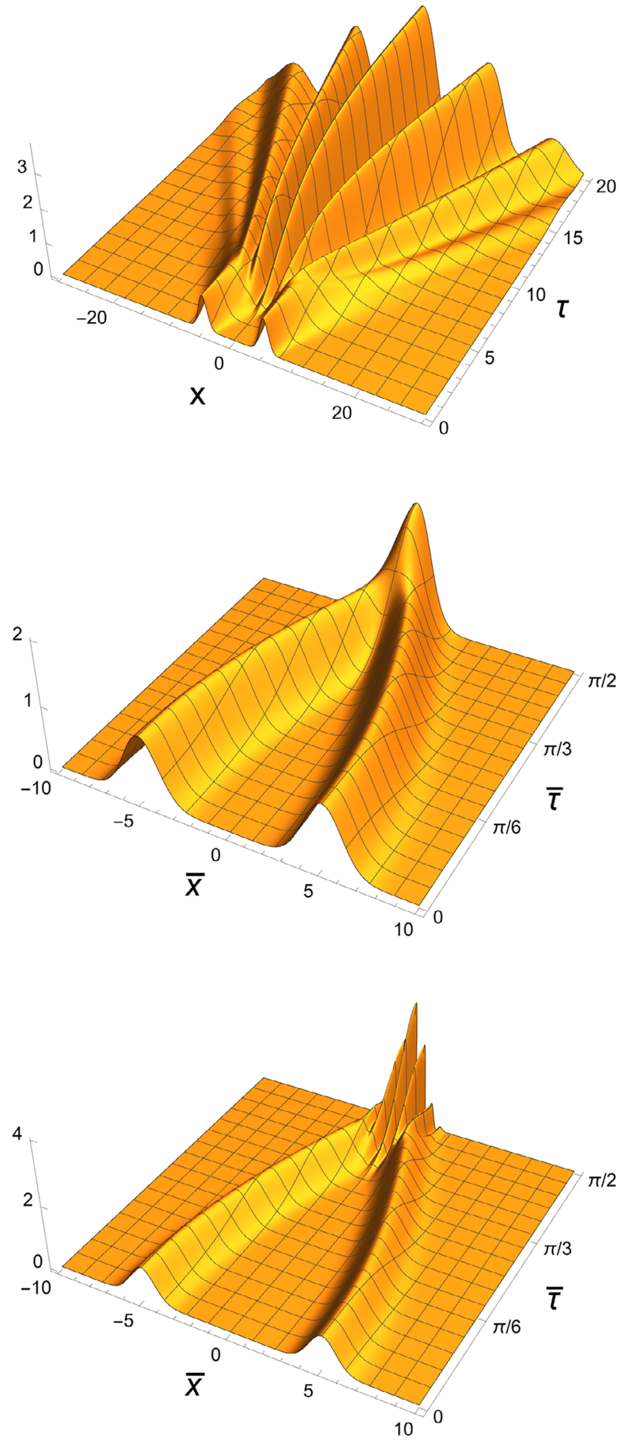

FIGURE 2 | The upper figure shows the interference of two spreading Gaussian wave packets in the laboratory frame. The abscissa is x, the ordinate is $\tau = z/z_R$ for CO or $\tau = t/T$ for QM. The middle frame shows two non-interacting coherent state Gaussians in the transformed frame. The transformed abscissa is $\bar{x}$, the ordinate is the Gouy phase $\bar{\tau} = \arctan\tau$. The Gaussians converge to the origin as $\bar{\tau}$ approaches $\pi/2$. The lower figure includes the interference between the two wave packets and shows that interference gives the asymptotic pattern only when the two wave packets overlap.

The loci of nonspreading coherent states are shown in the middle frame of Figure 2. The plot is in the transformed $\bar{x}, \bar{\tau} = \arctan\tau$ coordinates. The pattern is calculated from the r.h.s of Equation (24) simply by putting the cosine interference term to zero. The coherent states simply move from their initial position as the Gouy phase $\bar{\tau}$ develops from zero, to overlap and coalesce as the phase reaches $\pi/2$.

The interfering wave packet diffraction pattern, calculated from the full Equation (24), is shown in the lower plot of Figure 2. Since the wave packets do not spread, interference is evident only as the wave packets overlap as $\bar{\tau}$ approaches $\pi/2$. As shown in Section 3, the asymptotic interference pattern in the transformed frame (lower plot) is the same as in the laboratory coordinates (upper plot), except it is scaled down by the asymptotic condition $\bar{x} \approx x/\tau$.

At initial $z = 0$, one has $\cos(\bar{\tau}) = 1,\ \sin(\bar{\tau}) = 0$, and at final $z = \infty$, $\cos(\bar{\tau}) = 0,\ \sin(\bar{\tau}) = 1$. Hence, as can be seen from Equation (24), initially there is no interference but, as the two separated Gauss functions merge into the same function $\exp(-x_0^2)$, concomitantly, the interference factor grows to the well-known $\cos(2dx_0) = \cos(2dk_x)$ form.

That is, from Equation (24), when the Gouy phase approaches $\pi/2$, one has the asymptotic form

$$|\chi_2(\bar{x}, \bar{\tau} = \pi/2)|^2 = \frac{2}{(\pi)^{1/2}} e^{-x_0^2}[1 + \cos(2dx_0)] = \frac{4}{(\pi)^{1/2}} e^{-x_0^2} \cos^2(dx_0). \tag{25}$$

With the scaled variable relation $x_0 = k_x$, this can be written in terms of the wave number function,

$$|\chi_2(\bar{x}, \bar{\tau} = \pi/2)|^2 = \frac{4}{(\pi)^{1/2}} e^{-k_x^2} \cos^2(dk_x). \tag{26}$$

In *unscaled* length and wavenumber units, for the CO case, this becomes the equation

$$|\chi_2(\bar{x}, \bar{\tau} = \pi/2)|^2 = \frac{4}{(\pi W_0^2/2)^{1/2}} e^{-k_x^2 W_0^2/2} \cos^2(dk_x), \tag{27}$$

which has the same form as the standard two-slit interference formula in the laboratory frame.

Consider next a grating of a large number N of Gaussian slits. The total wave function for N slits at $x_0 = jd$, where j is integer, is obtained by summing Equation (21) over j to give

$$\chi_N(\bar{x}, \bar{\tau}) = \frac{1}{(\pi)^{1/4}} e^{-i\bar{\tau}/2} \sum_j \exp\left(-\frac{j^2 d^2}{4}(1 - e^{-2i\bar{\tau}}) - \frac{(x_0 - jde^{-i\bar{\tau}})^2}{2}\right). \tag{28}$$

In the asymptotic limit where the Gouy phase $\bar{\tau} = \pi/2$, corresponding in the laboratory frame to $z = \infty$, the terms in d^2 cancel to zero, to give simply

$$\chi_N(\bar{x}, \bar{\tau} = \pi/2) = \frac{e^{-i\bar{\tau}/2}}{(\pi)^{1/4}} e^{-x_0^2/2} \sum_j \exp(-ijdx_0). \tag{29}$$

With $x_0 = k_x$, the sum can be performed to obtain the standard asymptotic expression, also valid in the laboratory frame, of diffraction through a grating

$$|\chi_N(\bar{x}, \bar{\tau} = \pi/2)|^2 = \frac{1}{(\pi)^{1/2}} e^{-k_x^2} \frac{\sin^2(Ndk_x/2)}{\sin^2(dk_x/2)}. \tag{30}$$

As in the $N = 2$ case of Equation (27), in the unscaled units, the only change in this formula is to replace k_x by $W_0 k_x/\sqrt{2}$ in the exponent and change the normalization factor to $(\pi W_0^2/2)^{-1/2}$. The interference term remains the same.

The interpretation of the interference pattern in the transformed frame is then very simple, it is the extension of the two-slit patterns of Figure 2 to an array of N slits. Each Gaussian coherent state, initially centered at $\bar{x} = x = jd$, moves slowly toward the center $\bar{x} = 0$ as the Gouy phase increases from zero to $\pi/2$, to overlap completely all the other Gaussians at $\bar{x} = 0$. The different phases of the identical Gaussians then sum to give the N-slit diffraction pattern. The origin of interference is clear.

This simplicity is to be contrasted with the laboratory-frame pattern where N spreading Gaussians interfere already in the near zone. This leads to the enormous complexity of the "Talbot carpet" patterns, as can be seen in the work of Sbitnev [21], for example.

7 | Conclusions

The free expansion of HG (optics) or HO (QM) wave packets occurs along wave front streamlines (CO) or trajectories (QM). When a transformation is made to a frame coincident with the streamlines, these basis functions are invariant as the propagation ensues. They are multiplied by a phase factor which is the Gouy phase and which appears as the new variable along the trajectories.

The basis functions are defined solely by the initial x_0 position of the trajectory. This implies that the space function is always proportional to the Fourier transform function in wave number (CO) or momentum (QM) space. These properties are for basis HG functions centered on $x = 0$ in the laboratory frame.

One remembers that the frame transformation in the transverse direction is $\bar{x} = x/(1 + \tau^2)^{1/2}$, with $\tau = z/z_R$ in CO or $\tau = t/T$ in QM. The propagation of a shifted $n = 0$ Gaussian slit, initially centered on $\bar{x} = x = d$, for $\tau = 0$, is shown to be that of a nonspreading coherent state which simply moves to be centered at $\bar{x} = x/\tau = 0$ asymptotically, $\tau \gg 1$.

The same occurs for all slits, their functions move, as the Gouy phase changes from zero to $\pi/2$, to coalesce and the sum of their acquired total Gouy phases gives the interference pattern. This asymptotic pattern is the same in the scaled transformed frame as the well-known pattern in the laboratory frame. Hence, the transformed frame allows a simple interpretation of the formation of the interference pattern and the occurrence of the Fraunhofer wave-number (CO) or momentum space (QM) function in the spatial pattern.

Author Contributions

John Stuart Briggs: Conceptualization (lead), formal analysis (lead), writing–original draft (lead).

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The author confirms that he has followed the ethical policies of the journal.

Conflicts of Interest

The author declares no conflicts of interest.

Data Availability Statement

Data available on request from the author.

7.1 | Peer Review

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