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Axially symmetric relativistic thin disks immersed in spheroidal matter haloes

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Abstract. Two infinite families of axially symmetric relativistic thin disks of dust immersed in spheroidal matter haloes are presented. The disks are obtained from solutions of the Einstein equations for an axially symmetric conformastatic spacetime in which the metric tensor is characterised only by one metric function. By introducing a finite discontinuity on the first derivatives of the metric tensor, solutions with a singularity of the delta function type are obtained, so describing thin disks. The nonzero components of the energy-momentum tensor, both for the disk and the halo, are obtained from the Einstein equations. In this way, the energy densities and pressures of the sources are determined. By imposing the fulfilment of all the energy conditions we obtain a constraint over the solutions, in such a way that the metric function can be properly expressed in terms of a solution of the Laplace equation. By using the solution to Laplace equation in cylindrical coordinates we find infinite disks and by using the solution to Laplace equation in oblate spheroidal coordinates we find finite disks. In both cases we obtain particular solutions with energy densities and pressures well behaved everywhere. We also show that the masses of the disks and the haloes are finite. Finally we solve the geodesic equation for circular orbits in the plane of the disk.

1. Introduction

Many isolated astrophysical objects in the universe have axial symmetry, some examples are the galaxies in thermodynamic equilibrium, stars, planets, accretion disks, black holes, etc. Since the study of these massive systems requires in some cases the use of the general relativity to give an appropriate description of the problem, it is very important to obtain as many exact solutions to the Einstein equations as possible, and to interpret them as astrophysical sources. For the case of a galactic system, the problem is not easy because those systems usually consists of many components like a thin disk, a central bulb, a black hole, a spheroidal matter halo, etc. Nevertheless, to simplify the problem, in this work we only consider the disk and the halo because they are the largest components. The problem of solving the Einstein equations with thin disks sources has been of great interest in theoretical physics due to their relevance in the description of galaxies and stars. Nevertheless, in literature there are only two references for the system consisting of a thin disk immersed in a fluid or matter halo. In the first one, Vogt and Letelier [1] apply the displacement, cut and reflect method to some known solutions to get disks with haloes. In the second work, Gutiérrez-Piñeres, González and Quevedo [2] obtain a family of disks with halo by solving the Einstein-Maxwell equations for a conformastatic spacetime.

In this work we obtain new solutions to the electro-vacuum Einstein equations for a



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conformastatic metric, and interpret them as thin disks of infinite and finite extension immersed in spheroidal matter haloes. The paper is organised as follows. First we present, in Section 2, the solutions to the Einstein equations for the halo and for the disk, the expressions to compute the mass of the system, and the rotational curves. In Section 3 we choose a particular solution to obtain an family of infinite disks immersed in spheroidal matter haloes. Finally, In Section 4 we do the same as in Section 3 but this time we find the solutions in oblate spheroidal coordinates in order to define a finite radius for the disk.

2. Axially Symmetric Disklike Solutions With Halo

In order to find the energy-momentum tensor of a system that consists of a thin disk with a halo, the disk is modeled by a thin shell Σ with equation $\phi(x^\alpha) = z$ which separates the space in two regions: M^+ (above Σ) and M^- (below Σ). By using the distributional method [3], we can write the Einstein equations as,

$$T_{\alpha\beta}^\pm = R_{\alpha\beta}^\pm - \frac{1}{2}g_{\alpha\beta}R^\pm, \quad (1)$$

$$Q_{\alpha\beta} = H_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}H, \quad (2)$$

where $Q_{\alpha\beta}$ and $H_{\alpha\beta}$ are the energy-momentum tensor and the Ricci tensor in Σ . The equations (1) are the Einstein equations for the halo, and equations (2) are the Einstein equations of the disk.

Now, we introduce the conformastatic metric,

$$ds^2 = -e^{2\psi}dt^2 + e^{-2\psi}(dr^2 + r^2d\varphi^2 + dz^2), \quad (3)$$

where we demand $\psi(r, -z) = \psi(r, z)$ in order to have axial symmetry and reflexion symmetry. With this metric and equation (1) we compute the energy density and the stress of the halo,

$$\rho = e^{2\psi}(2\nabla^2\psi - \nabla\psi \cdot \nabla\psi), \quad (4)$$

$$p = \frac{1}{3}e^{2\psi}\nabla\psi \cdot \nabla\psi, \quad (5)$$

and with equation (2) we compute the energy density of de disk

$$\sigma = 4e^\psi\psi_{,z}, \quad (6)$$

which is evaluated in $z = 0^+$. As the stress of the disk is zero, we have a fluid of dust.

In order to have physically well behaved solutions, the energy densities and the stress of the halo need to satisfy the energy conditions [4], that can be fulfilled if we consider ψ functions that are solutions to the equation

$$\nabla^2\psi = k\nabla\psi \cdot \nabla\psi, \quad (7)$$

with $k \geq 1$ and $\psi_{,z}|_{z=0^+} > 0$ (it can not be zero in order to have $\sigma \neq 0$). For the fluid in the halo, the above equation leads to

$$p = \frac{\rho}{3(2k-1)}, \quad (8)$$

the state equation of a gamma fluid. Equation (7) can be rewritten as $\nabla^2(e^{-k\psi}) = 0$, which is related to Laplace equation $\nabla^2U = 0$ through the relation

$$e^{-k\psi} = 1 - U, \quad (9)$$

where U is the solution to Laplace equation.

Now, to compute the mass of the disk-halo system, we use the Komar formulae [5],

$$M = 2 \int \left(T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta} \right) m_\alpha \xi_{(t)}^\beta \sqrt{h} d^3 y, \quad (10)$$

where m_α is the normal vector to the time-like hypersurface, $\xi_{(t)}^\beta$ is the time-like killing vector, and $\sqrt{h} d^3 y$ is the volume element. Finally, to compute the circular velocity we solve the geodesic equation for a time-like particle that moves in the plane of the disk. By constraining the four-velocity of the particle to $u^\alpha = u^0(1, 0, \omega, 0)$, the resulting expression for the square of the circular velocity yield to

$$v_c^2 = \frac{r U_{,r}}{k(1-U) - r U_{,r}}. \quad (11)$$

3. Kuzmin-Toomre Disks With Halo (Infinite Disks)

To obtain solutions describing thin disks of infinite extension immersed in a spheroidal halo, we take U as the solution to Laplace equation in spherical coordinates (R, θ, φ) ,

$$U_n(r, \theta) = - \sum_{l=0}^n \frac{A_l}{R^{l+1}} P_l(\cos \theta), \quad (12)$$

where P_l are the Legendre Polynomials and A_l are constants. Here, in order to have $\psi|_{z=0^+} \neq 0$, we need to use the displacement, cut and reflexion method [6] which is equivalent to the transformation $z \rightarrow |z| + a$, so $R^2 = r^2 + (|z| + a)^2$, and $\cos \theta = (|z| + a)/R$.

With the solution to Laplace equation presented in (12) and the relation (9) we can compute the energy density of the disk (6). For the $n = 2$ model, σ takes the form

$$\tilde{\sigma}_2 = \frac{\left[(\tilde{A}_0 - \tilde{A}_1)m^2 + 3(\tilde{A}_1 - \tilde{A}_2)m + \frac{5}{2}\tilde{A}_2(2 - \tilde{r}^2) \right] m^{\frac{5-2k}{k}}}{\left[2m^{5/2} + 2\tilde{A}_0m^2 + 2\tilde{A}_1m + \tilde{A}_2(2 - \tilde{r}^2) \right]^{\frac{1+k}{k}}}, \quad (13)$$

where $m = 1 + \tilde{r}^2$, $\tilde{r} = r/a$, $\tilde{A}_0 = A_0/a$, $\tilde{A}_1 = A_1/a^2$, $\tilde{A}_2 = A_2/a^3$, and $\tilde{\sigma}_n = (ka/4)\sigma_n$. We can also compute the energy density of the halo (4) for $n = 2$,

$$\begin{aligned} \tilde{\rho}_2 &= \frac{\tilde{M}^{(5-2k)/k} \left[\tilde{A}_0 \tilde{r} \tilde{M}^2 + (3\tilde{A}_1 \tilde{Z} + \tilde{A}_2) \tilde{r} \tilde{M} + \frac{5}{2}\tilde{A}_2(2\tilde{Z}^2 - \tilde{r}^2) \right]^2}{\left[\tilde{M}^{5/2} + \tilde{A}_0 \tilde{M}^2 + \tilde{A}_1 \tilde{M} \tilde{Z} + \frac{1}{2}\tilde{A}_2(\tilde{Z}^2 - \tilde{r}^2) \right]^{2(1+k)/k}} + \\ &\quad \frac{\tilde{M}^{(5-2k)/k} \left[(\tilde{A}_0 \tilde{Z} - \tilde{A}_1) \tilde{M}^2 + 3\tilde{Z}(\tilde{A}_1 \tilde{Z} - \tilde{A}_2) \tilde{M} + \frac{5}{2}\tilde{A}_2 \tilde{Z}(2\tilde{Z}^2 - \tilde{r}^2) \right]^2}{\left[\tilde{M}^{5/2} + \tilde{A}_0 \tilde{M}^2 + \tilde{A}_1 \tilde{M} \tilde{Z} + \frac{1}{2}\tilde{A}_2(\tilde{Z}^2 - \tilde{r}^2) \right]^{2(1+k)/k}}, \end{aligned} \quad (14)$$

where $\tilde{M} = \tilde{r}^2 + \tilde{Z}^2$, $\tilde{Z} = |\tilde{z}| + 1$, $\tilde{z} = z/a$, $\tilde{r} = r/a$, $\tilde{A}_0 = A_0/a$, $\tilde{A}_1 = A_1/a^2$, $\tilde{A}_2 = A_2/a^3$, and $\tilde{\rho}_2 = [(ka)^2/(2k-1)]\rho_2$.

To analyse the behaviour of these quantities, we plot $\tilde{\sigma}_2$ in figure 1 and $\tilde{\rho}_2$ in figure 2. The constants values were chosen to show the different behaviours that can be obtained from the solutions. We can see in both figures that the energy densities have a maximum at the center of the system and go to zero at infinity. By varying the constants we can have different energy density profiles: we can manipulate its maximum at the center, its rate of decrease, and in some cases we can obtain more than one maximum.

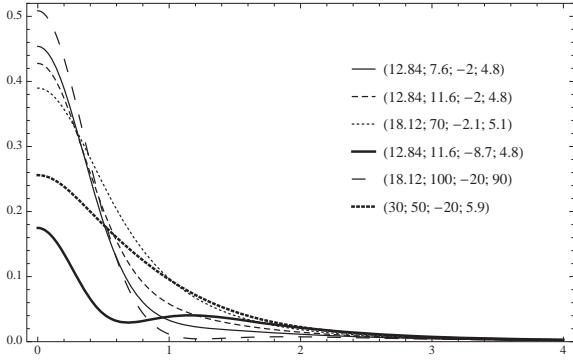


Figure 1. $\tilde{\sigma}_3$ as a function of \tilde{r} for the $n = 3$ model. Each curve is labeled by the set of numbers $(k, \tilde{A}_0, \tilde{A}_1, \tilde{A}_2)$

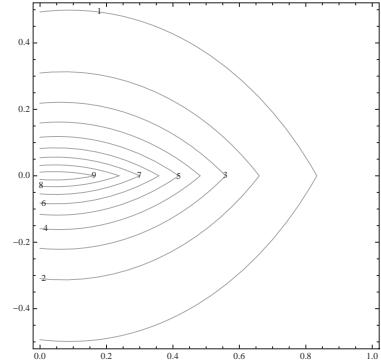


Figure 2. Contour plots of $\tilde{\rho}_2$ as a function of \tilde{r} (horizontal axis) and \tilde{z} (vertical axis) for the $n = 3$ model and for the constants $k = 18.12, \tilde{A}_0 = 100, \tilde{A}_1 = -20, \tilde{A}_2 = 90$.

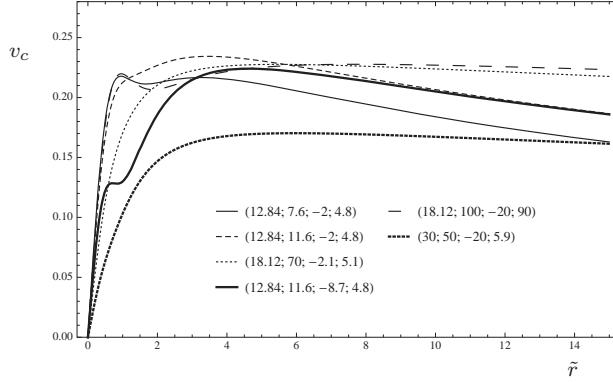


Figure 3. v_c as a function of \tilde{r} . Each curve is labeled by the set of numbers $(k, \tilde{A}_0, \tilde{A}_1, \tilde{A}_2)$

Now, we can show that the disk-halo system has finite mass, despite the fact that the matter distribution extends to infinity. To do this we use the limit comparison test to show that if the total mass of the system for the $n = 0$ model is finite, then all the family of solutions has finite masses. We compute the total mass of the system for $n = 0$ by solving the integral (10). This yields

$$M_{T_0} = M_{D_0} + M_{H_0} = \frac{8\pi A_0}{k}, \quad (15)$$

where M_{H_0} is the mass of the halo, and $M_{D_0} = \frac{8\pi a}{k} \ln(1 + A_0/k)$ is the mass of the disk. With this, it is possible to say that the masses of all the solutions are finite.

Finally, we compute v_c for the $n = 2$ model from equation (11) to obtain

$$v_{c_2}^2 = \frac{\tilde{r}^2[3\tilde{A}_2 + 3\tilde{A}_1m^2 + \tilde{A}_0m^4 + \frac{3}{2}\tilde{A}_2(3-m)m]}{k[m^2 + m\tilde{A}_1 + m^3\tilde{A}_0 + \frac{1}{2}\tilde{A}_2(3-m)]m^3 - \tilde{r}^2[3\tilde{A}_2 + 3\tilde{A}_1m^2 + \tilde{A}_0m^4 + \frac{3}{2}\tilde{A}_2(3-m)m]}. \quad (16)$$

In figure 3 we present the rotational curves associated to v_{c_2} . We can see that the circular velocity increases very fast until a maximum value which remains approximately constant. As we can see, in some cases it is possible to obtain a rotational curve with more than one maximum. This kind of behavior is qualitatively similar to those observed for spiral galaxies.

4. Kalnajs disks with halo (finite disks)

Another family of solutions can be obtained by taking U as the solution to Laplace equation in oblate spheroidal coordinates (ξ, η, ϕ) ,

$$U_m(\xi, \eta) = - \sum_{n=0}^m C_{2n} q_{2n}(\xi) P_{2n}(\eta), \quad (17)$$

where $q_{2n} = i^{2n+1} Q_{2n}(i\xi)$, with $Q_{2n}(i\xi)$ the Legendre functions of second kind of imaginary argument, and the constants C_{2n} are chosen properly to be [7]

$$C_{2n} = \frac{\tilde{M}/2}{(2n+1)q_{2n+1}(0)} \left[\frac{\pi^{1/2}(4n+1)(2m+1)!}{2^{2m}(m-n)!\Gamma(m+n+3/2)} \right] \quad (18)$$

if $n \leq m$, and zero if $n > m$ for $m \geq 1$. With U_m we can compute the energy density of the disk for the first three models of the family of solutions ($m = 1, 2, 3$) and the energy density of the halo for the $m = 3$ model. The plots of $\tilde{\sigma}_m = (ka/\tilde{M})\sigma_m$ for $m = 1, 2, 3$ are presented in figure 4, and the contours of $\tilde{\rho}_3 = [(k^2 a^2)/((2k-1)\tilde{M}^2)]\rho_3$ are presented in figure 5. We can see in these figures that the energy densities have a maximum at the center of the system. In the disk, $\tilde{\sigma}_m$ goes to zero at the radius of the disk $\tilde{r} = 1$ (finite disk), while for the halo $\tilde{\rho}_3$ goes to zero at infinity.

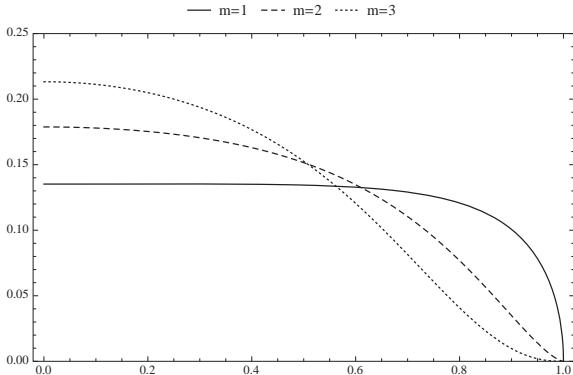


Figure 4. $\tilde{\sigma}_m$ as a function of $\tilde{r} = r/a$ for fixed values $k = 20$ and $\tilde{M} = 30$.

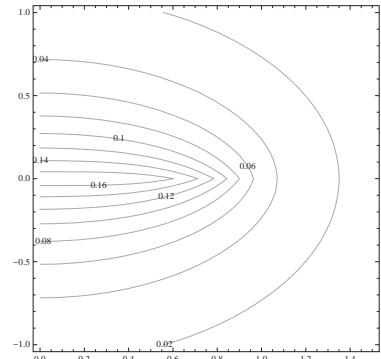


Figure 5. Contour plots of $\tilde{\rho}_3$ as a function of $\tilde{r} = r/a$ (horizontal axis) and $\tilde{z} = z/a$ (vertical axis) for the $m = 3$ model and for $k = 17.2$ and $\tilde{M} = 3.9$

Now, we can show, as in section 3, that the mass of the system is finite despite the fact that the halo matter distribution extends to infinity. To do that we use the limit comparison test and show that if the total mass is finite for $m = 1$, then all the family of Kalnajs disks with halo are finite too. By using the integral (10) we find that $M_{T_m} = M_{D_m} + M_{H_m} = 8\pi a C_0/k$, where M_{H_m} is the mass of the halo and M_{D_m} is the mass of the disk. Here we can see that the total mass is the same for every m . For $m = 1$, we have

$$M_{D_1} = \frac{64a}{k} \left[1 - \sqrt{1 + \frac{8}{3\pi C_0}} \operatorname{arcot} \sqrt{\frac{3\pi C_0}{3\pi C_0 + 8}} \right] \quad (19)$$

so the total mass of all the family of solutions are finite.

Finally, in figure 6 we present the circular velocity profiles that we find by using the functions U_1, U_2 and U_3 in the expression (11). These curves correspond to the first three solutions for v_{c_m} .

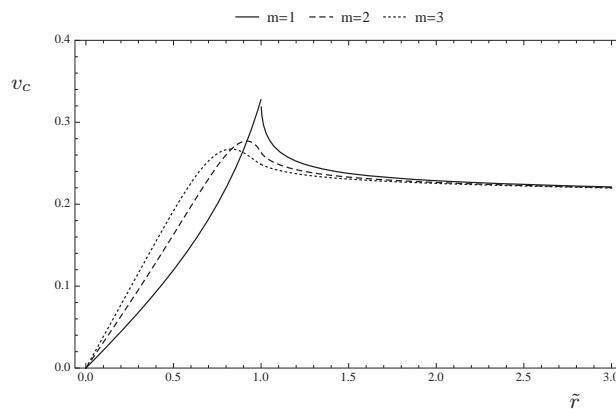


Figure 6. v_{c_m} as a function of $\tilde{r} = r/a$ for the $m = 1, 2, 3$ models, and for $k = 20$ and $\tilde{M} = 30$

In this figure we note that the rotational curve for $m = 1$ has a discontinuity in the derivative. This no longer happens when we take particular solutions with $m \geq 2$. It is clear that v_c increases until a maximum value that remains approximately constant.

5. Conclusions

We have obtained two infinite families of exact solutions to the Einstein equations that describe thin dust disks immersed in spheroidal matter haloes whose energy density and stress goes to zero at infinity. For the fluid in the halo we have obtained an equation of state of the form $p = \gamma \rho$, with $\gamma = [3(2k - 1)]^{-1}$. In the first family of solutions, the energy density of the disk-like source goes to zero at infinity, while for the second family of solutions the energy density of the disk goes to zero at the edge of the disk. All the solutions satisfy the energy conditions and yield to finite masses. Finally, we have obtained the rotational curves by solving the geodesic equation of a test particle moving in circular orbits in the plane of the disk. These curves are qualitatively similar to the observed rotational curves for spiral galaxies.

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