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Brane stability under $f(Q, \mathcal{T})$ gravity

A. R. P. Moreira[1,2,a], **Shi-Hai Dong**[1,3]

[1] Research Center for Quantum Physics, Huzhou University, Huzhou 313000, People's Republic of China
[2] Secretaria da Educação do Ceará (SEDUC), Coordenadoria Regional de Desenvolvimento da Educação (CREDE 9), Horizonte, Ceará 62880-384, Brazil
[3] Centro de 0Investigación en Computación, Instituto Politécnico Nacional, UPALM, 07700 Mexico City, CDMX, Mexico

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Abstract The focus of this paper is to investigate the behavior of a codimension-one thick brane in the modified symmetric teleparallel gravity structure $f(Q, \mathcal{T})$, where Q is the nonmetricity scalar and $\mathcal{T}$ is the trace of the energy–momentum tensor. Additionally, we examine whether braneworld scenarios are stable under small tensor perturbations and identify the associated massive and resonant modes. Our results indicate that the parameters controlling the modifications to the gravitational model-specifically those affecting nonmetricity and the energy–momentum tensor-significantly influence both the location and existence of these modes.

1 Introduction

Almost a century ago, Kaluza and Klein made a remarkable attempt to unify gravity and electrodynamics by proposing the hypothesis of extra dimensions [1,2]. Despite its elegance, the lack of experimental evidence has led to skepticism about extra dimensions within the scientific community. In 1999, the Randall–Sundrum model revived interest in extra dimensions by addressing the hierarchy problem, a longstanding issue in theoretical physics [3,4]. Over the past twenty years, higher-dimensional models, also known as braneworld models, have been extensively explored. Thick branes, in particular, emerged as an extension of the Randall–Sundrum model, introducing a real scalar field to provide thickness and a smooth deformation factor for the brane [5–12].

In thick brane models, one can consider modified gravity theories in addition to altering the dynamics of the scalar field. These include theories such as $f(R)$ and $f(R, \mathcal{T})$ [13–16], where R denotes the curvature scalar and $\mathcal{T}$ represents the trace of the energy-momentum tensor. Beyond these, curvature-free theories based on non-Riemannian geometry, such as $f(T)$, $f(T, B)$, $f(T, \mathcal{T})$, and $f(Q)$ [17–30], are also explored. Here, T is the torsion scalar, B is the boundary term, and Q is the nonmetricity scalar. Among these, $f(T)$, $f(T, B)$, and $f(T, \mathcal{T})$ are variations of the teleparallel equivalence of general relativity (TEGR) [31–43], while $f(Q)$ is a modification of the symmetric teleparallel equivalence of general relativity (STEGR) [44–50].

Although gravity $f(Q)$ proves to be very efficient in describing the evolution of the universe at both the background and perturbative levels [51–53], recent work shows that a more complete gravitational model $f(Q, \mathcal{T})$ results in a non-conserved energy-momentum tensor [54–61]. This non-conservation brings important physical consequences, including significant changes in the thermodynamics of the universe, similar to those observed in the $f(R, T)$ theory. Additionally, the non-geodesic motion of test particles results in the emergence of an additional force. Furthermore, many significant results have emerged in the study of the accelerated expansion of the universe at late times [62], baryogenesis [63], cosmological inflation [64] and cosmological perturbations [65], wormholes [66], thin-shell gravastar model [67], and the formation of primordial black holes [68]. All these results motivate us to investigate the influence of gravitational modifications $f(Q, \mathcal{T})$ in an extra-dimensional scenario. In this work, we study a five-dimensional thick brane in $f(Q, \mathcal{T})$ gravity coupled to a single scalar field. We use first-order formalism to find analytical solutions for our brane system and investigate its stability under small perturbations, as well as the localization of massless and massive modes for the graviton.

The structure of this paper is as follows: in Sect. 2, we introduce the essential concepts of STEGR and derive the

[a] e-mail: allan.moreira@fisica.ufc.br (corresponding author)

equations of motion for $f(Q, \mathcal{T})$ gravity. Section 3 examines the stability of the brane system under small perturbations and studies the localization of massive and resonant modes for the graviton. Finally, Sect. 4 presents our conclusions and discusses future perspectives.

2 Construction of the brane in $f(Q, \mathcal{T})$ gravity

To begin, we introduce the foundational concepts of symmetric teleparallel gravity and outline the equation of motion for $f(Q, \mathcal{T})$ gravity within this framework. In theories based on Riemannian geometry, the metricity condition $\nabla_M g_{NP} = 0$ must be satisfied, where g_{NP} denotes the metric and ∇_M represents the covariant derivative with respect to the Levi-Civita connection Γ^P_{MN}. The Latin index $M = 0, \ldots, D-1$ denotes coordinates in the bulk. This condition is met in General Relativity and $f(R)$ gravity, for example. However, it does not hold in theories founded on non-Riemannian geometry. In such modified gravity models, symmetric teleparallel equivalence of general relativity (STEGR) is characterized by the presence of a non-vanishing nonmetricity tensor $Q_{MNP} = \nabla_M g_{NP}$ [69]. For this tensor, we define its independent traces as $Q_M = g^{NP} Q_{MNP}$ and $\widetilde{Q}M = g^{NP} QNMP$.

Given the presence of the nonmetricity tensor, STEGR necessitates the introduction of a more generalized connection $\widetilde{\Gamma}^P_{MN}$, which is defined as

$$\widetilde{\Gamma}^P{}_{MN} = \Gamma^P{}_{MN} + L^P{}_{MN}, \tag{1}$$

where $L^P{}_{MN}$ is known as the distortion tensor, which is written in nonmetricity tensor terms as [69]

$$L^P{}_{MN} = \frac{1}{2} g^{PQ} (Q_{PMN} - Q_{MPN} - Q_{NPM}). \tag{2}$$

To formulate a gravitational action for STEGR, we introduce a comprehensive tensor that incorporates the nonmetricity tensor, its independent traces, and the distortion tensor. This tensor is known as the nonmetricity conjugate and is denoted as [69]

$$P^P{}_{MN} = -\frac{1}{2} L^P{}_{MN} + \frac{1}{4}(Q^P - \widetilde{Q}^P) g_{MN} - \frac{1}{8}(\delta^P_M Q_N + \delta^P_N Q_M), \tag{3}$$

which takes us to the nonmetricity scalar $Q = Q_{PMN} P^{PMN}$.

In this study, we focus on $f(Q, \mathcal{T})$ gravity, an extension of the Symmetric Teleparallel Equivalent of General Relativity (STEGR) [54–61,65]. The five-dimensional gravitational action in this framework is given by

$$S = \int d^5x \sqrt{-g} \left[\frac{1}{4} f(Q, \mathcal{T}) + \mathcal{L}_m \right], \tag{4}$$

where $\mathcal{L}_m$ denotes the matter Lagrangian, which will be defined in the following section. Varying the action given in Eq. (4) with respect to the metric yields the modified Einstein field equations:

$$G_{MN} = 2\Big[\mathcal{T}_{MN} - \frac{f_T}{2}(\mathcal{T}_{MN} + \theta_{MN}) \Big], \tag{5}$$

being

$$G_{MN} = \frac{2}{\sqrt{-g}} \nabla_K (\sqrt{-g} f_Q P^K{}_{MN}) - \frac{1}{2} g_{MN} f + f_Q (P_{MKL} Q_N{}^{KL} - 2 Q^L{}_{KM} P^K{}_{NL}), \tag{6}$$

where we define $f \equiv f(Q, T)$, $f_Q \equiv \partial f(Q, T)/\partial Q$ and $f_T \equiv \partial f(Q, T)/\partial T$ for simplicity. In addition, $\mathcal{T}_{MN}$ is the momentum-energy tensor given by

$$\mathcal{T}_{MN} = -2 \frac{\delta \mathcal{L}_m}{\delta g^{MN}} + g_{MN} \mathcal{L}_m, \tag{7}$$

while the tensor θ_{MN} is defined as

$$\theta_{MN} = g^{AB} \frac{\delta T_{AB}}{\delta g^{MN}}. \tag{8}$$

We can still vary the action (4) with respect to connection to obtain

$$\nabla_M \nabla_N (\sqrt{-g} f_Q P_K{}^{MN}) = 0. \tag{9}$$

After examining STEGR, we are now prepared to explore brane-world scenarios. To start, we will consider a single scalar field that interacts exclusively with an extra dimension, which contributes to the thickness of the brane:

$$\mathcal{L}_m = -\partial_M \phi \partial^M \phi / 2 - V(\phi). \tag{10}$$

To derive the brane equations, one adopts a flat metric akin to Randall-Sundrum, expressed as

$$ds^2 = e^{2A} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2, \tag{11}$$

In this metric, $\eta_{\mu\nu}$ denotes the Minkowski metric, e^{2A} represents the warp factor, and y corresponds to the extra dimension. The Greek indices μ and ν range from 0 to 3. Additionally, we impose the coincident gauge condition $\widetilde{\Gamma}^P{}_{MN} = 0$. Under these conditions, the equations governing the scalar field and gravity are given by

$$\Big(1 + \frac{3}{4} f_{\mathcal{T}}\Big)\phi'' + \Big[(4 + 3 f_{\mathcal{T}}) A' + \frac{3}{4} f'_{\mathcal{T}}\Big]\phi' = \Big(1 + \frac{5}{4} f_{\mathcal{T}}\Big) V_\phi, \tag{12}$$

$$12 f_Q A'^2 - \frac{1}{2} f = \Big(1 + \frac{3}{2} f_{\mathcal{T}}\Big)\phi'^2 - 2V, \tag{13}$$

$$3(f_Q A'' + f'_Q A') = -\Big(2 + \frac{3 f_{\mathcal{T}}}{2}\Big)\phi'^2. \tag{14}$$

In the given equations, the prime (′) denotes the derivative with respect to the extra dimension. Here, we provide the units adopted by the international system of units. In

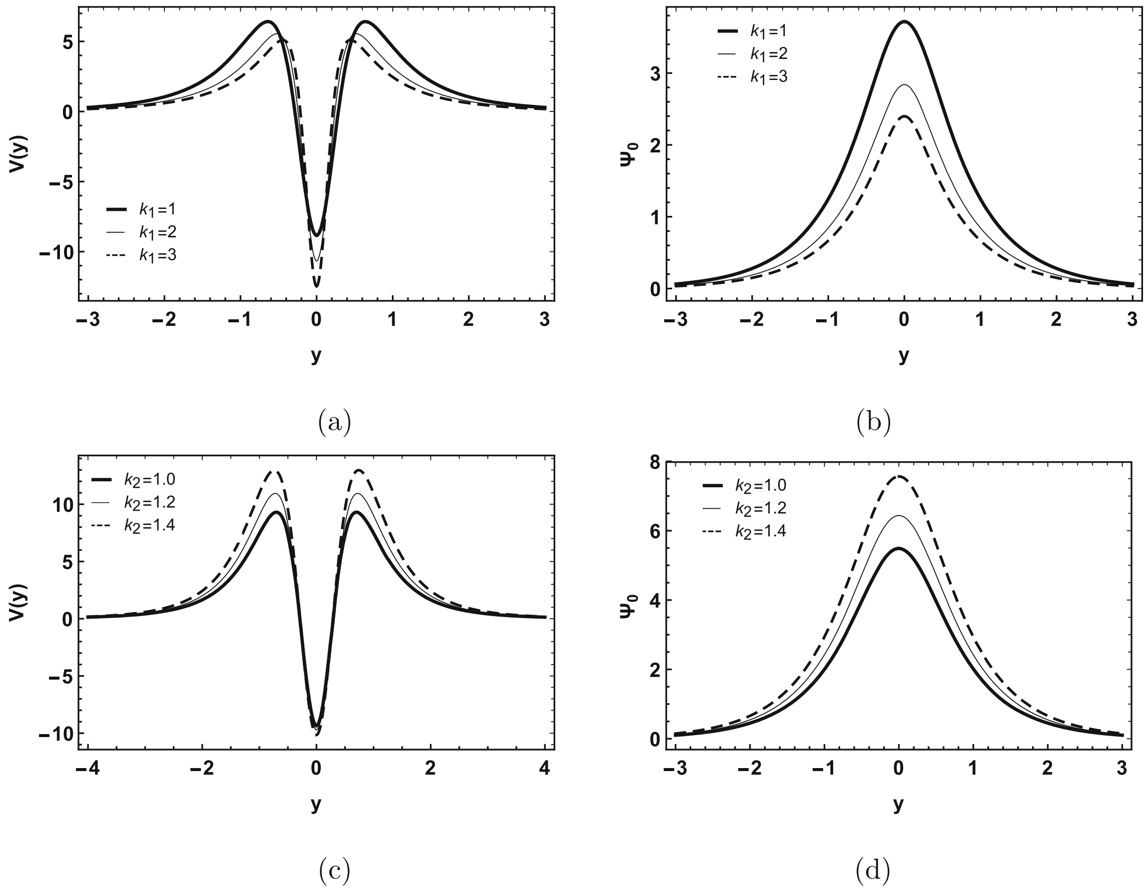

Fig. 1 For the sine-Gordon superpotential with $a = \beta = 1$ and $\alpha = 0, 25$ ($n = 1$). Being $k_2 = 0.5$ **a** effective potential $V(y)$ and **b** zero-mode $\psi(y)$. Being $k_1 = 1$, **c** effective potential $V(y)$ and **d** zero-mode $\psi(y)$

dimensional analysis, the mass dimensions for the fields and the coupling constants are $[g_{MN}] = 1$, $[\phi] = M^{3/2}$, $[V(\phi)] = M^5$ and $[Q] = M^2$. Furthermore, it is worth noting that the units of measurement for physical quantities can be related to the dimension of mass in the system. For example, in Ref. [70] the authors observed that if $M \sim 1$ TeV is considered, the brane width is $\gtrsim 10^{-18}$ cm, and if $M < 1$ TeV a better lower bound for the brane width is obtained. To solve these equations, we employ a first-order formalism in which the derivative of the warp factor with respect to the extra dimension is expressed in terms of the scalar field. Specifically, we have $A' = -aW(\phi)$ [5,18,27].

The next step is to select a specific form for the superpotential $W(\phi)$, which is derived from the first-order formalism. We consider three distinct superpotentials. For $n = 1$, we use the sine-Gordon superpotential $W(\phi) = \beta^2 \sin\left(\frac{\phi}{\beta}\right)$ and a polynomial superpotential $W(\phi) = \beta\phi - \frac{1}{3\beta}\phi^3$. For $n = 2$, we select a linear superpotential $W(\phi) = \beta\phi$. Additionally, we define the function $f(Q, \mathcal{T})$ as $f(Q, \mathcal{T}) = Q + k_1 Q^n + k_2 \mathcal{T}$, where k_1 governs the effect of nonmetricity and k_2 governs the effect of the trace of the energy-momentum tensor.

3 Brane stability and gravity localization

In this section, we will examine the stability of the braneworld by analyzing small tensor perturbations of the form $\eta_{\mu\nu} \to \eta_{\mu\nu} + h_{\mu\nu}(x^\mu, y)$, where $h_{\mu\nu}$ represents the graviton, and small scalar field perturbations of the form $\phi \to \phi(y)+\epsilon(x^\mu, y)$, with x^μ denoting the four-dimensional position vector. For simplicity, we will assume that the tensor perturbation $h_{\mu\nu}$ satisfies the transverse and traceless conditions: $\partial^\mu h_{\mu\nu} = 0$ (transverse) and $h = \eta^{\mu\nu} h_{\mu\nu} = 0$ (traceless). Under these perturbations, the metric takes the form [16,19,24,27,29]

$$ds^2 = e^{2A}[\eta_{\mu\nu} + h_{\mu\nu}(x^\mu, y)]dx^\mu dx^\nu + dy^2. \tag{15}$$

For this perturbed metric, we write the non vanishing non-metricity tensor as being [29]

$$Q^{(1)\rho}{}_{\mu\nu} = \partial^\rho_{\mu\nu}, \tag{16}$$

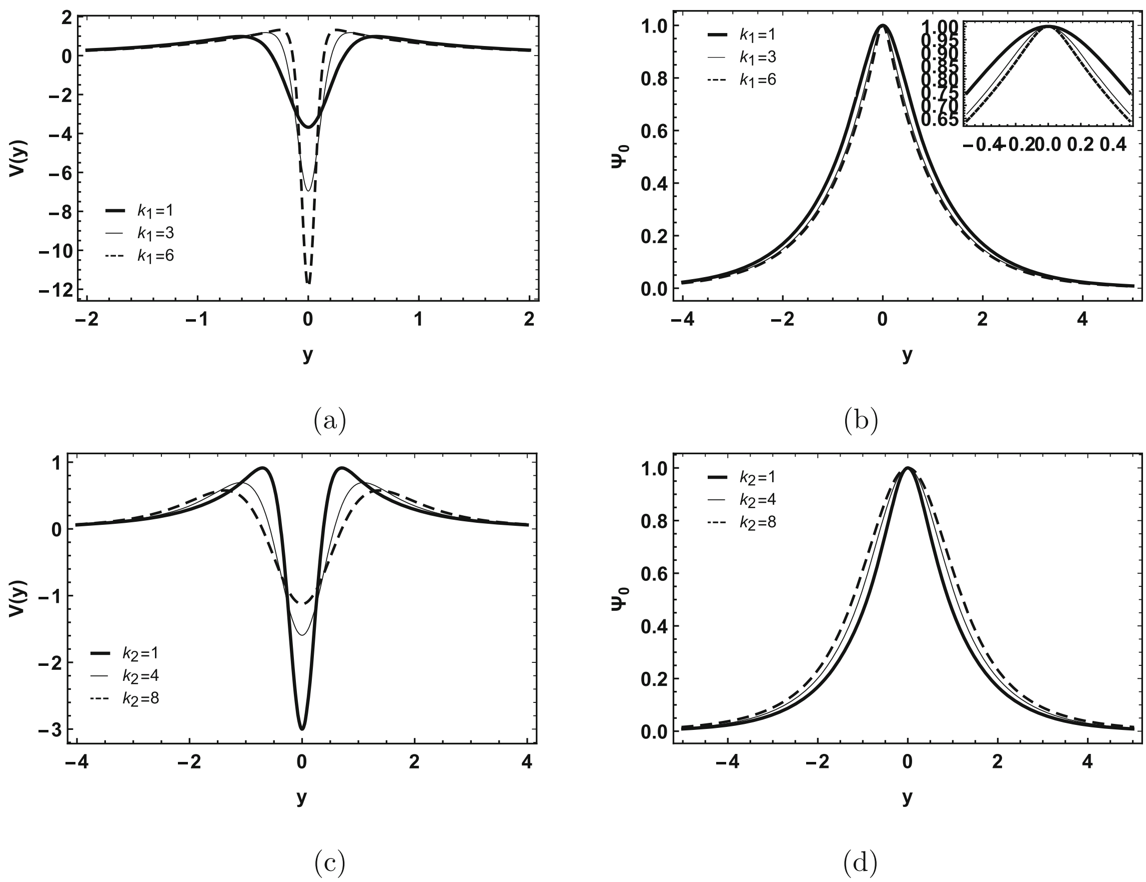

Fig. 2 For the polynomial superpotential with $a = \beta = 1$ and $\alpha = 0, 25$ ($n = 1$). Being $k_2 = 0.5$ **a** effective potential $V(y)$ and **b** zero-mode $\psi(y)$. Being $k_1 = 1$ **c** effective potential $V(y)$ and **d** zero-mode $\psi(y)$

$$Q^{(1)4}{}_{\mu\nu} = e^{2A}(2A' h_{\mu\nu} + h'_{\mu\nu}) \tag{17}$$

and the pertburbed nonmetricity conjugate is derived as [29]

$$P^{(1)\rho}{}_{\mu\nu} = -\frac{1}{4}\Big[\partial^\rho h_{\mu\nu} - (\partial_\mu h^\rho_\nu + \partial_\nu h^\rho_\mu)\Big], \tag{18}$$

$$P^{(1)4}{}_{\mu\nu} = \frac{1}{4}e^{2A}(6A' h_{\mu\nu} - h'_{\mu\nu}), \tag{19}$$

$$P^{(1)\rho}{}_{4\nu} = P^{(1)\rho}{}_{\nu 4} = \frac{1}{4}h'^{\rho}_{\nu}. \tag{20}$$

On the other hand, we write the perturbation of the momentum-energy tensor as

$$\mathcal{T}^{(1)\rho}{}_{\mu\nu} = -e^{2A}\Big(\frac{1}{2}\overline{\phi}'^2 h_{\mu\nu} + \overline{\phi}'\epsilon'\eta_{\mu\nu} + V h_{\mu\nu} + V_\phi \epsilon \eta_{\mu\nu}\Big). \tag{21}$$

With these expressions, the perturbed version of equation leads to equation of motion for $h_{\mu\nu}$

$$h''_{\mu\nu} + \left(4A' + \frac{f'_Q}{f_Q}\right) h'_{\mu\nu} = e^{-2A}\Box h_{\mu\nu}. \tag{22}$$

We write the Kaluza–Klein (KK) decomposition for graviton as follows

$$h_{\mu\nu}(x^\lambda, y) = \sum \widehat{h}_{\mu\nu}(x^\lambda)\chi(y), \tag{23}$$

where the 4D tensor satisfies $\Box\widehat{h}_{\mu\nu} = m^2\widehat{h}_{\mu\nu}$. Substituting (23) into (22), we obtain the following equation for $\chi(y)$

$$\chi'' + \left(4A' + \frac{f'_Q}{f_Q}\right)\chi' = -m^2 e^{-2A}\chi. \tag{24}$$

Solving the above equation is a pretty hard task. Then it is useful to introduce the conformal coordinate $dz = e^{-A}dy$ writing the equation as

$$\ddot{\chi}(z) + 2H\dot{\chi}(z) = -m^2\chi(z), \tag{25}$$

where

$$H = \frac{1}{2}\Big(3\dot{A} + \frac{\dot{f}_Q}{f_Q}\Big). \tag{26}$$

Here, the dot ($\dot{}$) denotes the derivative with respect to the conformal coordinate z. The next step is to convert equation (25) into a Schrödinger-like form by applying the transformation $\chi(z) = e^{K(z)}\psi(z)$, where $K(z) = -\int H, dz$. With

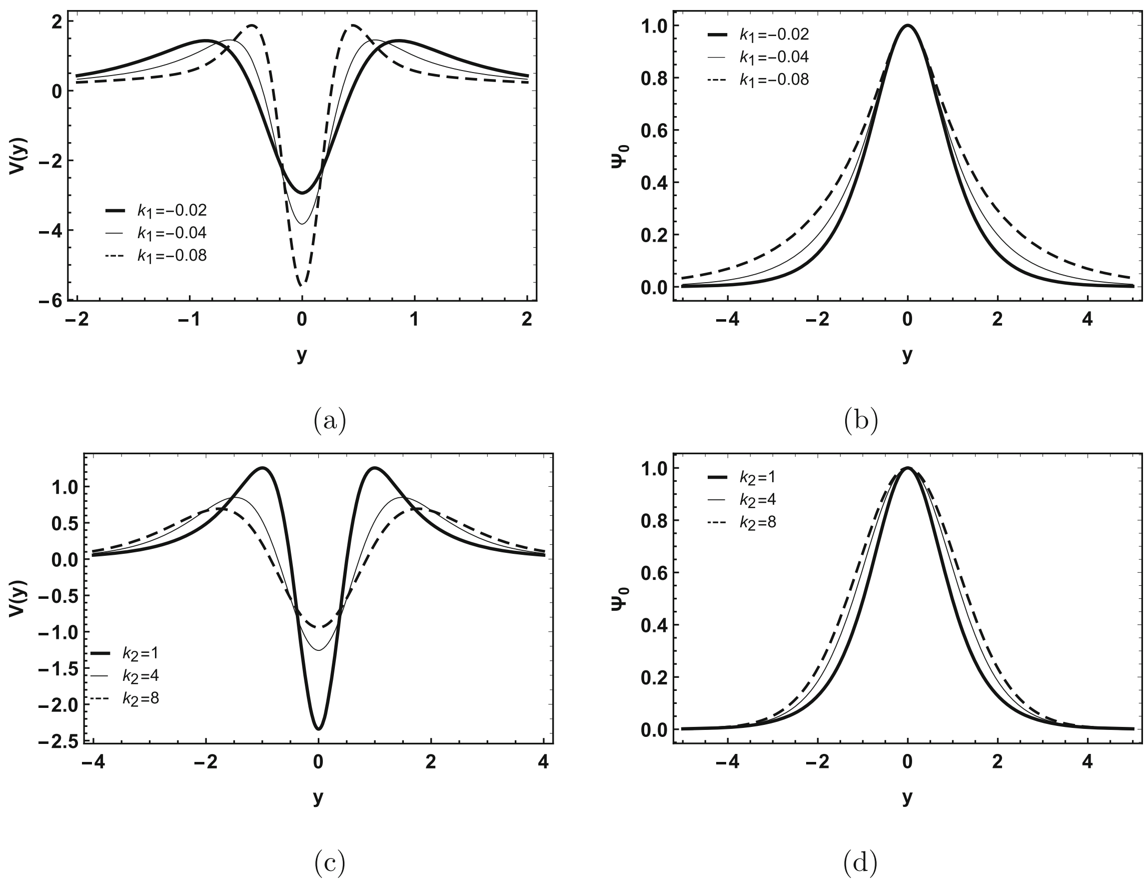

Fig. 3 For the linear superpotential with with $a = \beta = 1$ and $\alpha = 0, 25$ $(n = 2)$. Being $k_2 = 0.5$ **a** effective potential $V(y)$ and **b** zero-mode $\psi(y)$. Being $k_1 = -0.02$ **c** effective potential $V(y)$ and **d** zero-mode $\psi(y)$

this transformation, we obtain the following Schrödinger-like equation:

$$-\ddot{\psi} + V\psi = m^2\psi, \tag{27}$$

where we have defined the effective potential

$$V = \dot{H} + H^2. \tag{28}$$

Equation (27) represents a Schrödinger-like equation derived from supersymmetric quantum mechanics. Expressing Eq. (27) in this form is crucial because it ensures the stability of the graviton spectrum and allows for the existence of a massless mode (zero-mode), which is given by

$$\psi_0(z) = N_0 e^{\frac{3}{2}A + \frac{1}{2}\int \frac{\dot{f}_Q}{f_Q} dz}, \tag{29}$$

where N_0 is a normalization constant. The next step is to analyze the localization of gravity for the three superpotentials discussed in the previous section.

3.1 Massless modes

The first superpotential considered is the sine-Gordon type $(n = 1)$. The effective potential in this case features two barriers around the origin and a finite well at the origin. As the nonmetricity parameter k_1 increases, the potential barriers decrease while the well at the origin becomes deeper (see Fig. 1a). Consequently, the zero-mode responds to these changes by decreasing its amplitude at the origin (see Fig. 1b). Conversely, increasing the momentum-energy parameter k_2 enhances the potential barriers (see Fig. 1c), leading to an increase in the zero-mode amplitude at the origin (see Fig. 1d). It is important to note that the parameters k_1 and k_2 can affect the localization of massless modes, making them either more or less localized.

The second superpotential is of the polynomial type $(n = 1)$. In this case, the effective potential features two barriers around the origin and a well at the origin. Increasing the k_1 parameter intensifies the potential well (see Fig. 2a), which results in a more localized zero-mode (see Fig. 2b). Conversely, increasing the k_2 parameter reduces the depth of the

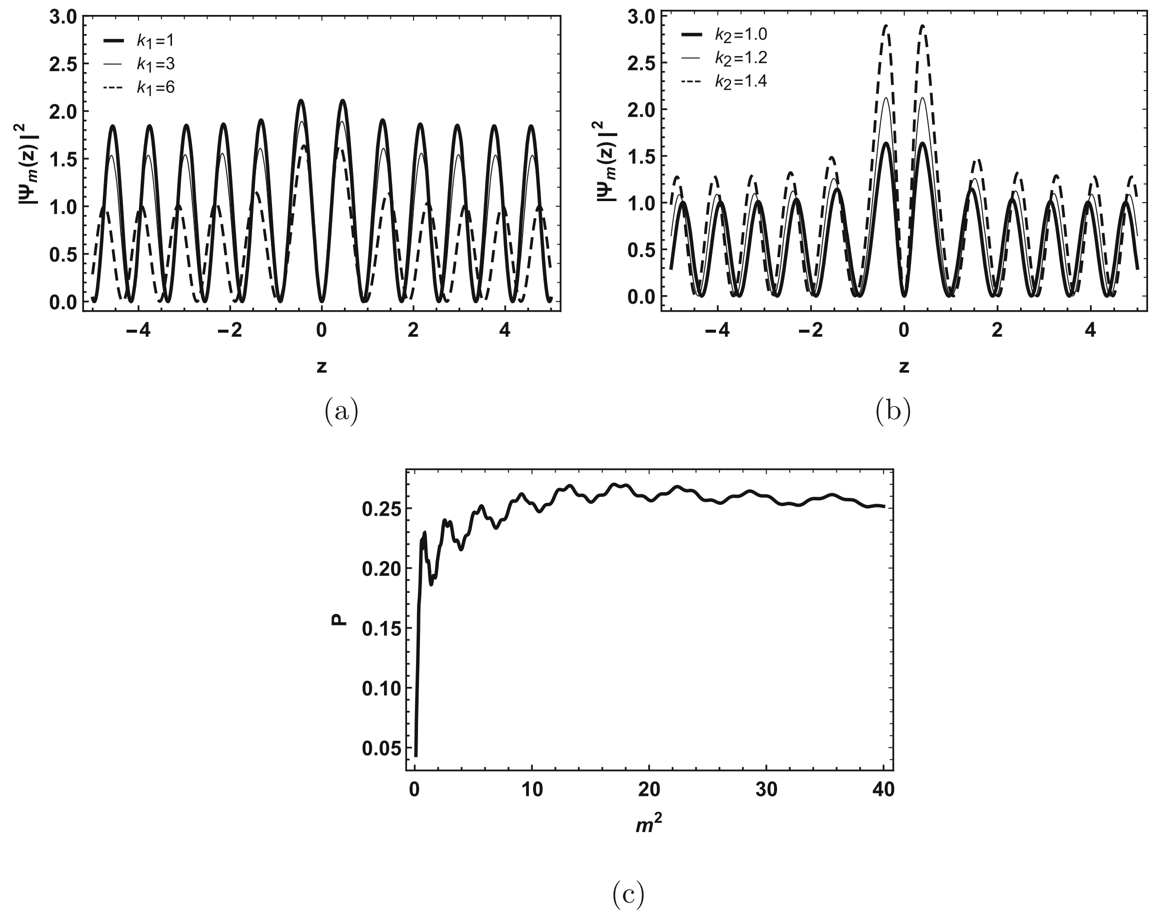

Fig. 4 For the sine-Gordon superpotential with $a = \beta = 1$ and $\alpha = 0, 25$ ($n = 1$). Massive modes with $m^2 = 2.866$. **a** $k_2 = 1$. **b** $k_1 = 1$. **c** Relative probability

potential well (see Fig. 2c), leading to a less localized zero-mode (see Fig. 2d).

Finally, for $n = 2$, the chosen superpotential is linear. Decreasing the value of the parameter k_1 increases the intensity of both the barriers around the origin and the well at the origin of the effective potential (see Fig. 3a). This change in the potential is detected by the zero-mode, which adjusts accordingly (see Fig. 3b). Conversely, increasing the parameter k_2 reduces the depth of the potential well (see Fig. 3c), which directly affects the zero-mode (see Fig. 3d). It is important to note that the parameters k_1 and k_2 have a direct impact on the localization of massless modes, potentially making them more localized.

3.2 Massive modes

We obtain the massive modes by solving Eq. (27). These massive solutions are determined numerically. To do this, we employ the interpolation method and apply the standard boundary conditions $\dot{\psi}(-\infty) = \dot{\psi}(\infty) = 0$.

Additionally, it is important to analyze the massive resonant modes. Although the massive modes are generally not localized, we can identify modes with significant amplitudes near the origin, indicating a higher probability of finding the particle close to the core. These modes are referred to as resonant modes. We identify resonant modes by examining the relative probability $P(m)$, which represents the likelihood of finding a particle with mass m within a narrow band $2z_b$. [26,28,71]

$$P(m) = \frac{\int_{-z_b}^{z_b} |\psi_m(z)|^2 dz}{\int_{-z_{max}}^{z_{max}} |\psi_m(z)|^2 dz}, \tag{30}$$

where z_{max} is representing the limit of the domain.

For the sine-Gordon superpotential, we identify a resonant mode at $m^2 = 2.866$. It is important to note that the parameters $k_{1,2}$ influence the amplitude of the massive mode oscillations. Increasing the parameter k_1 tends to dampen these oscillations and reduce their amplitudes (see Fig. 4a). Conversely, increasing the parameter k_2 enhances the amplitude of the oscillations, making the resonant mode more pronounced (see Fig. 4b). Thus, the parameters k_1 and k_2 play a

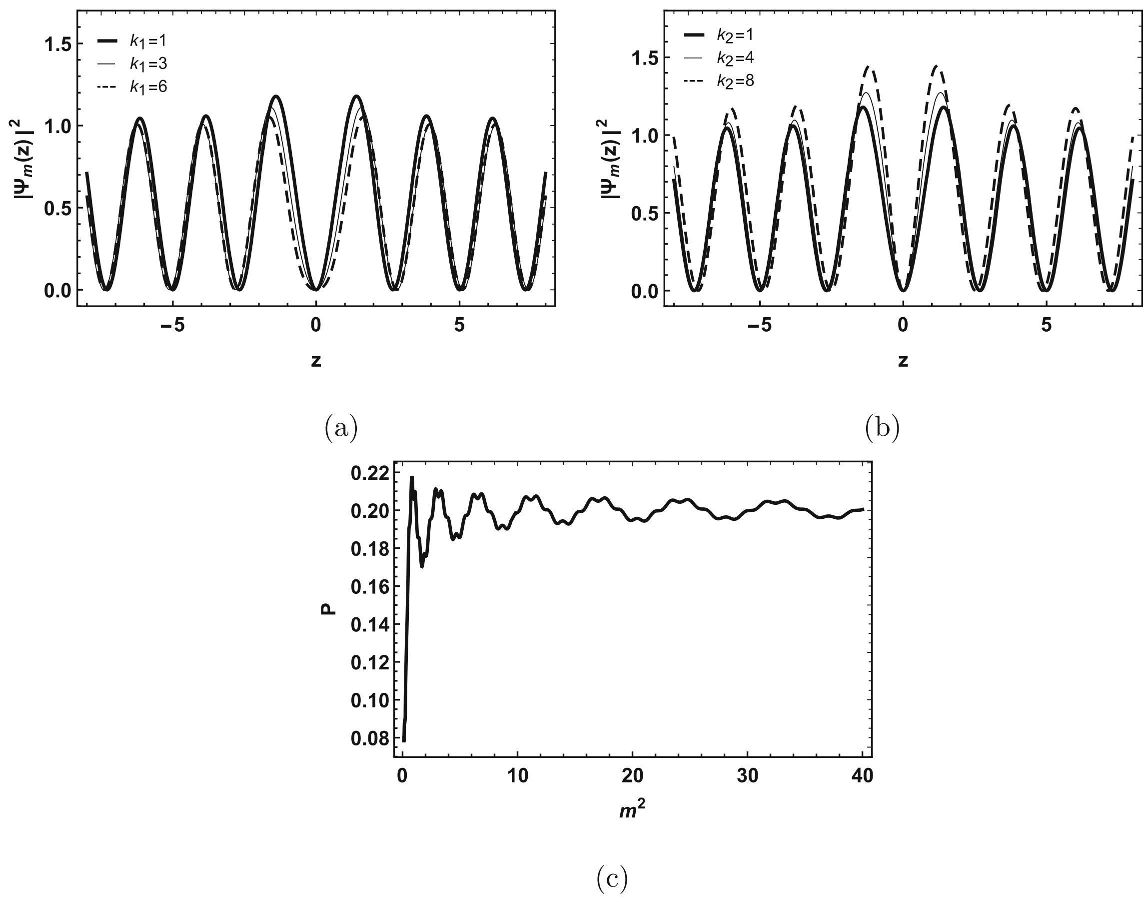

Fig. 5 For the polynomial superpotential with $a = \beta = 1$ and $\alpha = 0, 25$ ($n = 1$). Massive modes with $m^2 = 0.893$. **a** $k_2 = 1$. **b** $k_1 = 1$. **c** Relative probability

crucial role in determining the existence and localization of resonant modes.

For the polynomial superpotential, we find a potential resonant mode at $m^2 = 0.893$. The behavior of the massive modes is influenced by the parameters $k_{1,2}$. Increasing k_1 results in a reduction of the oscillation amplitudes (see Fig. 5a). In contrast, increasing k_2 makes the resonant mode more prominent (see Fig. 5b).

For the linear superpotential, decreasing the value of k_1 results in smaller amplitudes for the massive mode oscillations (see Fig. 6a). Similarly, increasing k_2 also leads to a reduction in the amplitude of the oscillations (see Fig. 6b). The relative probability analysis reveals a possible resonant mode at $m^2 = 1.125$ (see Fig. 6c).

4 Final remarks

In this paper, we have investigated a five-dimensional braneworld scenario governed by a single scalar field within the framework of nonmetricity-based modified gravity. We employed the first-order formalism and explored three well-known superpotentials to fully describe the braneworld model, characterized by the function $f(Q, \mathcal{T}) = Q + k_1 Q^n + k_2 \mathcal{T}$, where the parameters k_1 and k_2 control the effects of nonmetricity and the trace of the energy-momentum tensor, respectively.

Our analysis of gravitational perturbations revealed the stability of the model and led to a Schrödinger-like equation, which is akin to a supersymmetric quantum mechanics equation and permits a normalizable massless mode. Both the effective potentials and the massless modes are influenced by the parameters k_1 and k_2, with these parameters affecting the localization of the zero-mode. Additionally, we examined the massive modes, which also depend on k_1 and k_2. To identify potential resonant modes, we used the relative probability $P(m)$. Notably, variations in k_1 and k_2 significantly impact the existence and localization of these resonant modes.

It is important to highlight that for $n = 1$, that is, for the model $f(Q, \mathcal{T}) = Q + k_1 Q + k_2 \mathcal{T}$, the term k_2 that controls The influence of the trace of the energy-momentum tensor has a greater influence on the location of the zero-modes and massive modes, and in turn has a more radical influence on the stability results. However, for $n = 2$, that

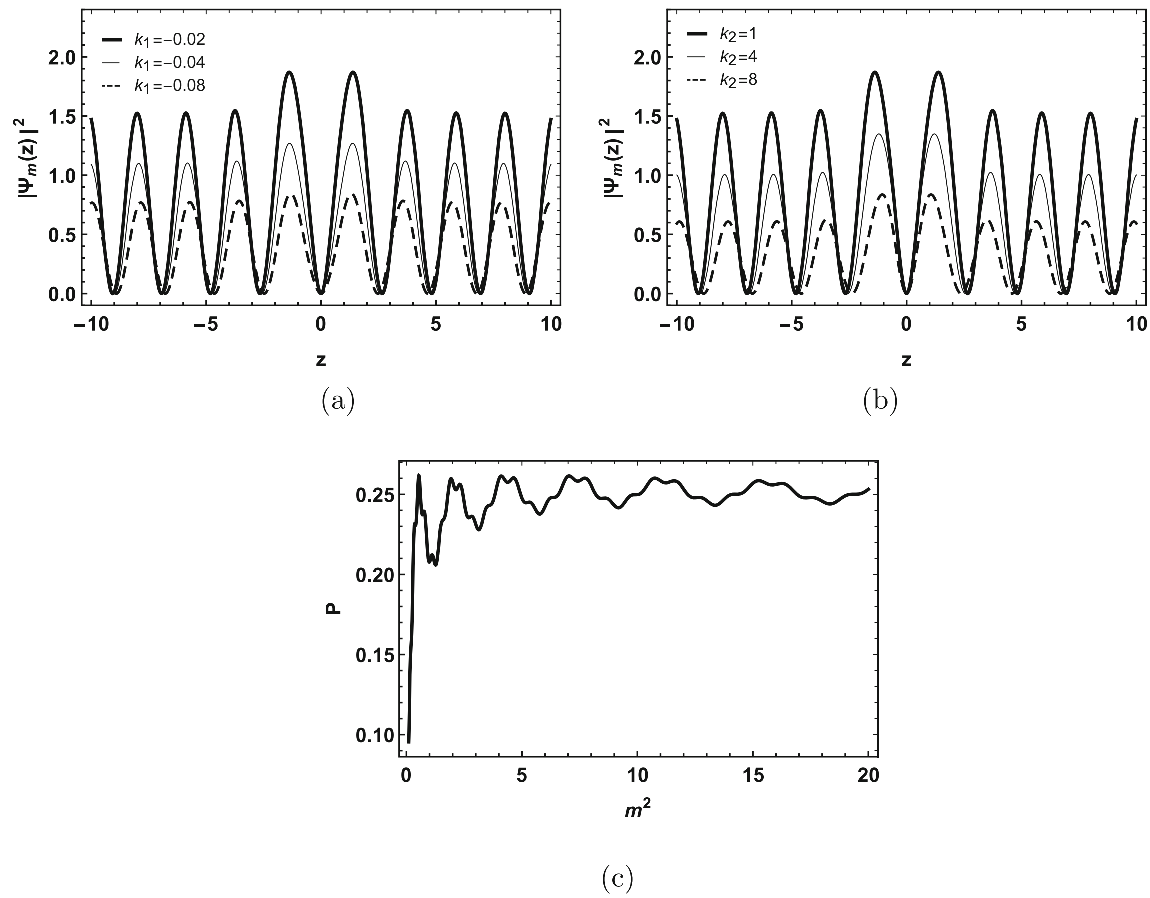

Fig. 6 For the linear superpotential with $a = \beta = 1$ and $\alpha = 0, 25$ ($n = 2$). Massive modes with $m^2 = 1.125$. **a** $k_2 = 1$. **b** $k_1 = -0.02$. **c** Relative probability

is, $f(Q, \mathcal{T}) = Q + k_1 Q^2 + k_2 \mathcal{T}$, the term k_1 that controls the nonmetricity is the one that has the most influences the location of these modes and the stability of the model. In other words, depending on the choice of n, we observe a greater influence of Q or $\mathcal{T}$ on the stability of the model.

Based on the study carried out in the gravitational model $f(T, B)$ [72], we leave as a future perspective the study of the location of fermions in $f(Q, \mathcal{T})$-brane scenario. Furthermore, for a detailed analysis of the influence of gravitational modification on the location of the fermion, we intend to use quantum information theories, such as Shannon entropy and Fisher Information. The study of brane information theories can potentially serve as probes for the detection of new particles and can open new avenues for the exploration of holography and brane cosmography.

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