

The Einstein-Infeld-Hoffmann legacy in mathematical relativity I: The classical motion of charged point particles

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Einstein, Infeld, and Hoffmann (EIH) claimed that the field equations of general relativity theory alone imply the equations of motion of neutral matter particles, viewed as point singularities in space-like slices of spacetime; they also claimed that they had generalized their results to charged point singularities. While their analysis falls apart upon closer scrutiny, the key idea merits our attention. This rapport identifies necessary conditions for a well-defined general-relativistic joint initial value problem of N classical point charges and their electromagnetic and gravitational fields. Among them, in particular, is the requirement that the electromagnetic vacuum law guarantees a finite field energy-momentum of a point charge. This disqualifies the Maxwell(-Lorentz) law used by EIH. On the positive side, if the electromagnetic vacuum law of Bopp, Landé-Thomas, and Podolsky (BLTP) is used, and the singularities equipped with a non-zero bare rest mass, then a joint initial value problem can be formulated in the spirit of the EIH proposal, and shown to be locally well-posed — *in the special-relativistic zero- G limit*. With gravitational coupling (i.e. $G > 0$), though, changing Maxwell's into the BLTP law and assigning a bare rest mass to the singularities is by itself not sufficient to obtain even a merely well-defined joint initial value problem: the gravitational coupling also needs to be changed, conceivably in the manner of Jordan and Brans-Dicke.

1. Brief History and State of Affairs

“I am plaguing myself with the derivation of the equations of motion of material points, conceived of as singularities [in the gravitational field], from the equations of general relativity.”

Albert Einstein, in a letter to Max Born on Dec. 4, 1926.

We don't know when **Einstein** first conceived of the notion of point particles as singularities in relativistic fields,^a but his letter to Max Born makes it plain that by the end of 1926 his ideas had matured to the point where he pursued a dynamical theory for such point singularities, expecting that their law of motion could be extracted from his gravitational field equations. Already a month later **Einstein & Grommer** announced that “the law of motion is completely determined by the field equations, though shown in this work only for the case of equilibrium.” In that paper⁷ the case of a static, spherically symmetric spacetime with a single time-like singularity was studied. The truly dynamical many-body problem was treated a decade later by **Einstein, Infeld**, and **Hoffmann** in their famous paper Ref. 8, with follow-ups in Refs. 9, 10. They argued explicitly that the field equations of general relativity theory alone determine the equations of motion of neutral matter

^aIn 1909 he remarked that “light quanta” might be point singularities in “a field,” their motion being guided by the electromagnetic field. See part II of our rapport.

particles, viewed as point singularities in space-like slices of spacetime. They also claimed that they had generalized their results to charged point-singularities, with the details written up in a set of notes deposited with the secretary of the IAS. In 1941 the motion of charged point-singularities was revisited by Infeld's student **P. R. Wallace**, who presented the details of the calculations in Ref. 28.

Here is the gist of the **Einstein-Infeld-Hoffmann** argument (modern terminology):

- Suppose you have a four-dimensional, time-oriented, asymptotically flat electromagnetic **Lorentz** spacetime $\mathcal{M}^{1,3}$ with N charged, time-like singularities of infinite extent, presentable as a graph over $\mathbb{R}^{1,3} \setminus \{N$ time-like world-lines $\}$.
- Away from the singularities the spacetime structure obeys **Einstein**'s equations

$$\mathbf{R} - \frac{1}{2}R\mathbf{g} = \frac{8\pi G}{c^4}\mathbf{T}[\mathbf{F}, \mathbf{g}], \quad (1)$$

where $\mathbf{T}[\mathbf{F}, \mathbf{g}]$ is the energy-momentum-stress tensor of the electromagnetic vacuum field \mathbf{F} , satisfying **Maxwell**'s field equations in vacuum,

$$d\mathbf{F} = \mathbf{0} \quad \& \quad d^* \mathbf{F} = \mathbf{0}. \quad (2)$$

- The twice contracted second **Bianchi** identity implies energy-momentum conservation:

$$\nabla \cdot (\mathbf{R} - \frac{1}{2}R\mathbf{g}) = \mathbf{0} \implies \nabla \cdot \mathbf{T}[\mathbf{F}, \mathbf{g}] = \mathbf{0}. \quad (3)$$

- “Massive, charged” singularities are associated with field “fluxes,” and thus

$\nabla \cdot \mathbf{T}[\mathbf{F}, \mathbf{g}] = \mathbf{0}$ & flux conditions

\implies law of the time-like singularities.

(4)

The main bullet point is of course the last one. Here are **EIH** in their own words (p.66): “It is shown that for two-dimensional [closed] spatial surfaces containing singularities , certain surface integral conditions are valid which determine the motion .” [Emphasis ours.]

Unfortunately, despite its publication in the Annals of Mathematics, the 1938 **EIH** paper is not only not rigorous, it contains questionable technical assumptions and serious blunders. Some were addressed in Ref. 10, yet *their main conclusions turn out to be false*. All the same, the core idea merits a deeper inquiry.

The issue is how to correctly handle singularities. **EIH** state (p.66): “By means of a new method of approximation, specially suited to the treatment of quasi-stationary fields, the gravitational field due to moving particles is determined.” This approximation method assumes that the particles are moving *slowly* and the field strengths (as seen in a **Lorentz** frame of **Minkowski** space, in which the particles move slowly) are *weak*. As a consequence, one has to choose the radii of the closed surfaces sufficiently large to satisfy the weak-field assumption needed for the

convergence of the expansion (which was not shown). Yet, on p. 92 one reads: “It is most convenient to take definite, infinitesimally small spheres whose centers are at the singularities, ...” which clearly violates their weak-field condition. Indeed, **EIH** realize that: “... in this case infinities of the types

$$\lim \text{const.}/r^n, \quad n \text{ a positive integer,} \quad r \rightarrow 0 \quad (5)$$

can occur in the values of the partial integrals,” but then commit a major blunder by stating (**EIH**, p. 92): “Since these must cancel, however, in the final result, we may merely ignore them throughout the calculation of the surface integrals.”

Alas, the infinities do not cancel!

In the same year **P. A. M. Dirac** invented *negative infinite bare mass renormalization* to handle those infinities, in the simpler special-relativistic purely electrodynamical setting.⁶ For an electron with positive “observable mass” m_{obs} and charge $-e$ he obtained the equation of motion (in **Misner–Thorne–Wheeler** notation)

$$m_{\text{obs}} \frac{d^2}{d\tau^2} \mathbf{q} = \mathbf{f}^{\text{ext}} + \mathbf{f}^{\text{LAUE}}, \quad (6)$$

where

$$\mathbf{f}^{\text{ext}} = -\frac{e}{c} \mathbf{F}^{\text{ext}}(\mathbf{q}) \cdot \frac{d}{d\tau} \mathbf{q} \quad (7)$$

is a **Lorentz Minkowski**-force due to “external sources,”

$$\mathbf{f}^{\text{LAUE}} = \frac{2e^2}{3c^3} \left(\mathbf{g} + \frac{1}{c^2} \frac{d}{d\tau} \mathbf{q} \otimes \frac{d}{d\tau} \mathbf{q} \right) \cdot \frac{d^3}{d\tau^3} \mathbf{q} \quad (8)$$

is **von Laue**’s radiation-reaction **Minkowski**-force of the electron, and

$$m_{\text{obs}} = \lim_{r \downarrow 0} \left(m_b(r) + \frac{e^2}{2c^2} \frac{1}{r} \right) \quad (9)$$

defines $m_b(r)$. [N.B.: $m_b(r) \downarrow -\infty$ as $r \downarrow 0$]. Here, r is the radius of a sphere in the instantaneous rest-frame of the electron, centered at the electron, which plays the role of the surfaces containing singularities invoked by **EIH**.

As is well-known, such mass-renormalization computations have become the template for the much more elaborate — and quite successful — renormalization group computations in quantum electrodynamics (and more generally, quantum field theory). Be that as it may, **Dirac** himself later in life became very dissatisfied with this approach, and so are we.

First of all, supposing a point electron has a bare mass, then how could it possibly depend on the radius r of a sphere over which a theoretical physicist averages the fields?

Second, the third proper time derivative featuring in the **von Laue Minkowski**-force means that (6) is a third-order ODE for the position of the particle as a function of proper time, requiring vector initial data for position, velocity, and acceleration. Yet a classical theory of point particle motion should only involve initial data for position and velocity.

In 1951 **Lev Landau & Eugenii Lifshitz** addressed the $\ddot{\mathbf{q}}$ problem as follows:

- Test particle theory works well for many practical purposes.
- In such situations **von Laue**'s $\ddot{\mathbf{q}}$ force term must be a small perturbation of \mathbf{f}^{ext} .
- Compute $\ddot{\mathbf{q}}$ perturbatively: take the proper time derivative of the test-particle law,

$$\frac{d^3}{d\tau^3}\mathbf{q} \approx -\frac{e}{m_{\text{obs}}c}\frac{d}{d\tau}(\mathbf{F}^{\text{ext}}(\mathbf{q}) \cdot \frac{d}{d\tau}\mathbf{q}). \quad (10)$$

- The right-hand side depends only on \mathbf{q} , $\dot{\mathbf{q}}$, $\ddot{\mathbf{q}}$. Substitute it for $\frac{d^3}{d\tau^3}\mathbf{q}$ in (8).

R.h.s.(8) with r.h.s.(10) substituted for $\frac{d^3}{d\tau^3}\mathbf{q}$ will be called the **Landau–Lifshitz–Minkowski**-force of radiation-reaction, denoted \mathbf{f}^{LL} . Equation (6) with \mathbf{f}^{LAUE} replaced by \mathbf{f}^{LL} is known as the **Landau–Lifshitz** equation of motion for the electron. It seems to work quite well for practical purposes in which \mathbf{F}^{ext} can be approximated by some smooth field tensor, on time scales beyond the one where test particle theory works well, but not arbitrarily far beyond.²⁵ However⁴, as soon as \mathbf{F}^{ext} is taken to be the field generated by all other particles the **Landau–Lifshitz** equation of motion is typically well-posed only until the moment that a point charge meets the forward initial light cone of another point charge, a ludicrously short time span!

Dirac's idea of infinite negative bare mass renormalization and **Landau–Lifshitz**'s perturbative treatment of the **von Laue** radiation-reaction **Minkowski**-force have become standard ingredients also in general-relativistic treatments of charged point particle motion. Thus, for a point electron moving in a given curved background, **Eric Poisson, Adam Pound, & Ian Vega** in their review Ref. 23 present the following equations of motion:

$$m_{\text{obs}}\frac{D}{d\tau}\mathbf{u} = \mathbf{f}^{\text{ext}} + \mathbf{f}^{\text{LAUE}} + \mathbf{f}^{\text{tail}}, \quad (11)$$

where $\mathbf{u} := \frac{d}{d\tau}\mathbf{q}$ and $\frac{D}{d\tau}\mathbf{u} := \frac{d}{d\tau}\mathbf{u} + \mathbf{\Gamma}^{\text{ext}}(\mathbf{u}, \mathbf{u})$, and $\mathbf{f}^{\text{ext}} = -\frac{e}{c}\mathbf{F}^{\text{ext}}(\mathbf{q}) \cdot \mathbf{u}$ as before, but now

$$\mathbf{f}^{\text{LAUE}} = \frac{2}{3}e^2 \left(\mathbf{g} + \frac{1}{c^2}\mathbf{u} \otimes \mathbf{u} \right) \cdot \left(\frac{1}{6}\mathbf{R}^{\text{ext}} \cdot \frac{1}{c}\mathbf{u} + \frac{1}{c^3}\frac{D^2}{d\tau^2}\mathbf{u} \right) \quad (12)$$

with

$$\frac{D^2}{d\tau^2}\mathbf{u} \approx -\frac{e}{m_{\text{obs}}c}\frac{D}{d\tau}(\mathbf{F}^{\text{ext}}(\mathbf{q}) \cdot \mathbf{u}), \quad (13)$$

and

$$\mathbf{f}^{\text{tail}} = 2e^2 \int_{-\infty}^{\tau} \mathbf{H}^{\text{ret}}(\mathbf{q}(\tau), \mathbf{q}(\tau')) \cdot \mathbf{u}(\tau') d\tau' \cdot \mathbf{u}(\tau), \quad (14)$$

where $\mathbf{H}^{\text{ret}}(\mathbf{q}(\tau), \mathbf{q}(\tau'))$ is a retarded type of Green function for the electromagnetic field tensor in curved spacetime. Equation (11) does not yet include gravitational radiation-reaction, which **Poisson, Pound, & Vega** discuss also, in particular the approaches of **Quinn & Wald** and **Detweiler & Whiting** (see Ref. 23), but we don't need to go there because (11) already displays a major problem due to the so-called tail force term:

Equation (11), even with r.h.s.(13) substituted for $\frac{D^2}{d\tau^2}\mathbf{u}$ at r.h.s.(12), does not pose a second-order initial value problem for the position of the point electron but instead requires the input of the entire past history of the motion!

One can try to extricate oneself from this dilemma by once again having recourse to a **Landau–Lifshitz**-type perturbation argument: the tail force, also a radiation-reaction term, must be small in situations where test particle theory works well. In this case, backward from the initial instant (say at $\tau = 0$) one can approximately replace $\mathbf{q}(\tau')$ and $\mathbf{u}(\tau')$ in the integrand by the pertinent expressions computed from test particle theory, with particle data for position and velocity given, and then treat this so-modified equation as a second-order integro-differential equation from the initial instant on forward, with $\mathbf{q}(\tau')$ and $\mathbf{u}(\tau')$ in the integrand for $\tau' > 0$ no longer test-particle expressions. This set of “effective equations of motion” may work well in many practical situations.

Yet from a mathematical relativity point of view this state of affairs is very unsatisfactory, both technically (being non-rigorous) and conceptually (involving heuristic but arbitrary arguments).

2. Rigorous Approach

In the following we report on recent rigorous advances in formulating a joint initial value problem for classical charged point particles and the electromagnetic and gravitational fields they generate, with the key idea of the 1938 **EIH** paper, as outlined on p.1 of this rapport, serving as our point of departure. To avoid the mistakes made by **EIH**, we inquire into necessary conditions on the energy-momentum-stress tensor which allow one to extract a law of motion associated with the time-like singularities from an equation like (4) without invoking infinite mass renormalization, nor arbitrary averaging over some neighborhood of an a-priori ill-defined force field, as done in Ref. 23. For the simpler special-relativistic zero-gravity limit (cf. Ref. 16, 17) we even state a well-posedness theorem, so this case is treated first.

2.1. The zero- G Limit

2.1.1. Time-like particle world-lines in Minkowski spacetime

In the limit $G \downarrow 0$, (1) is solved by $\mathcal{M}^{1,3} = \mathbb{R}^{1,3} \setminus \{N \text{ time-like world-lines}\}$, with metric $\mathbf{g} = \boldsymbol{\eta}$ away from the world-lines. The question is which conditions on \mathbf{T} lead, in a mathematically clean way, to the **EIH**-type conclusion

$$\boxed{\nabla \cdot \mathbf{T}[\mathbf{F}, \boldsymbol{\eta}] = \mathbf{0} \quad \& \quad \text{flux conditions}} \implies \text{law of the time-like world-lines.} \quad (1)$$

To answer this question, we extend $\mathcal{M}^{1,3}$ continuously to $\mathbb{R}^{1,3}$ (by adding the time-like world-lines), and switch to a distributional formulation. We can formulate everything for N charged, massive, time-like world-lines, but for simplicity we set $N = 1$ in the following.

Since the issue is the formulation of a well-posed initial value problem, we also choose an arbitrary Lorentz frame, with space vector $\mathbf{s} \in \mathbb{R}^3$ and time $t \in \mathbb{R}$. Then the space part of l.h.s.(1) becomes the *local conservation law for the total momentum*,

$$\frac{\partial}{\partial t} \mathbf{\Pi}(t, \mathbf{s}) + \nabla \cdot \mathbf{T}(t, \mathbf{s}) = \mathbf{0}, \quad (2)$$

where it is *postulated* (equivalent to “minimal coupling”) that

$$\mathbf{\Pi}(t, \mathbf{s}) := \mathbf{\Pi}^{\text{field}}(t, \mathbf{s}) + \mathbf{\Pi}^{\text{charge}}(t, \mathbf{s}) \quad (3)$$

is the total momentum vector-density, with $\mathbf{\Pi}^{\text{field}}(t, \mathbf{s})$ the contribution from the field and $\mathbf{\Pi}^{\text{charge}}(t, \mathbf{s})$ (a distribution) the usual contribution from the point charge, which must be assigned a non-vanishing bare rest mass m_b . Similarly, it is *postulated* that

$$\mathbf{T}(t, \mathbf{s}) := \mathbf{T}^{\text{field}}(t, \mathbf{s}) + \mathbf{T}^{\text{charge}}(t, \mathbf{s}) \quad (4)$$

is the symmetric total stress tensor, with $\mathbf{T}^{\text{charge}}(t, \mathbf{s})$ the usual stress tensor of the point particle, and $\mathbf{T}^{\text{field}}(t, \mathbf{s})$ that of the field (except for our unconventional choice of sign!)

Incidentally, energy conservation follows as a corollary from momentum conservation.

Next, we integrate (2) over all $\mathbf{s} \in \mathbb{R}^3$, at $t \in \mathbb{R}$. This yields total momentum conservation

$$\frac{d}{dt} \mathbf{p}(t) = -\frac{d}{dt} \int_{\mathbb{R}^3} \mathbf{\Pi}^{\text{field}}(t, \mathbf{s}) d^3s. \quad (5)$$

Clearly, for (5) to make sense, the field momentum vector-density $\mathbf{\Pi}^{\text{field}}(t, \mathbf{s})$ has to be integrable over \mathbb{R}^3 , and this integral differentiable in time. This rules out the **Maxwell–Lorentz** field equations, but leaves other options, notably the **Maxwell–Born–Infeld** (**MBI**) and **Maxwell–Bopp–Landé–Thomas–Podolsky** (**MBLTP**) field equations (see below).

Comparing equation (5) with **Newton**’s law for the rate of change of momentum,

$$\frac{d}{dt} \mathbf{p}(t) = \mathbf{f}(t), \quad (6)$$

where $\mathbf{f}(t)$ is the force acting on the particle at time t , it is clear that the force $\mathbf{f}(t)$ needs to be extracted from r.h.s.(5). Since the particle’s bare momentum $\mathbf{p}(t)$ is given in terms of its bare mass m_b and velocity $\mathbf{v}(t) = \frac{d}{dt} \mathbf{q}(t)$ by the **Einstein–Lorentz–Poincaré** law

$$\mathbf{p}(t) := m_b \frac{\mathbf{v}(t)}{\sqrt{1 - \frac{1}{c^2} |\mathbf{v}(t)|^2}}, \quad (7)$$

the expression for the force also has to be compatible with the requirements of a second-order initial value problem for the position of the point particle! Thus, beside the existence of the time-derivative of the space integral over the field momentum vector-density, it is important that the result involves, initially, only the initial

electromagnetic fields and the initial data for position and velocity of the point particle and, at later times $t > 0$, at most the history of position, velocity, and acceleration of the particle, and of the fields, from the initial instant on, yet not beyond t . Whenever this is possible we obtain a well-defined joint initial value problem for field and particle, which may or may not be well-posed.

We were able to explicitly extract a well-defined force on the point charge from the **MBLTP** field equations, and to prove that the resulting joint initial value problem for charge and field is well-posed. We expect this to be feasible also for the **MBI** field equations, but their formidable nonlinearity makes rigorous progress a slow process.

Common to all these classical systems of electromagnetic field equations are the *pre-metric Maxwell–Lorentz field equations*, viz. the evolution equations

$$\partial_t \mathbf{B}(t, \mathbf{s}) = -c \nabla \times \mathbf{E}(t, \mathbf{s}) \quad (8)$$

$$\partial_t \mathbf{D}(t, \mathbf{s}) = +c \nabla \times \mathbf{H}(t, \mathbf{s}) + 4\pi e \dot{\mathbf{q}}(t) \delta_{\mathbf{q}(t)}(\mathbf{s}) \quad (9)$$

and the constraint equations

$$\nabla \cdot \mathbf{B}(t, \mathbf{s}) = 0 \quad (10)$$

$$\nabla \cdot \mathbf{D}(t, \mathbf{s}) = -4\pi e \delta_{\mathbf{q}(t)}(\mathbf{s}) \quad (11)$$

for the \mathbf{B} , \mathbf{D} fields. They differ in the *Electromagnetic Vacuum Law*: $(\mathbf{B}, \mathbf{D}) \leftrightarrow (\mathbf{H}, \mathbf{E})$. The **Born–Infeld** law² reads

$$\mathbf{H} = \frac{\mathbf{B} - \frac{1}{b^2} \mathbf{D} \times (\mathbf{D} \times \mathbf{B})}{\sqrt{1 + \frac{1}{b^2} (|\mathbf{B}|^2 + |\mathbf{D}|^2) + \frac{1}{b^4} |\mathbf{B} \times \mathbf{D}|^2}} \quad (12)$$

$$\mathbf{E} = \frac{\mathbf{D} - \frac{1}{b^2} \mathbf{B} \times (\mathbf{B} \times \mathbf{D})}{\sqrt{1 + \frac{1}{b^2} (|\mathbf{B}|^2 + |\mathbf{D}|^2) + \frac{1}{b^4} |\mathbf{B} \times \mathbf{D}|^2}} \quad (13)$$

The **Bopp–Landé–Thomas–Podolsky** law^{1,19,20,22} reads

$$\mathbf{H}(t, \mathbf{s}) = (1 + \varkappa^{-2} \square) \mathbf{B}(t, \mathbf{s}) \quad (14)$$

$$\mathbf{D}(t, \mathbf{s}) = (1 + \varkappa^{-2} \square) \mathbf{E}(t, \mathbf{s}). \quad (15)$$

(N.B.: $\square := c^{-2} \partial_t^2 - \Delta$.) When $b \rightarrow \infty$, respectively when $\varkappa \rightarrow \infty$, both these vacuum laws reduce to the **Maxwell–Lorentz** law $\mathbf{H} = \mathbf{B}$ & $\mathbf{E} = \mathbf{D}$.

For given particle motions with subluminal velocity $|\dot{\mathbf{q}}(t)| < c$, the field **Cauchy** problems are globally well-posed in the sense of distributions for both the **ML** and **MBLTP** field equations. The **MBI** field Cauchy problem, unfortunately, has not yet been conquered to the extent needed. Global well-posedness of the classical initial value problem has only been shown with small data (no charges!) in Ref. 24; **F. Pasqualotto** presented an extension of **Speck**’s result to **MBI** field evolutions on the **Schwarzschild** background.²¹ A local well-posedness result for **MBI** field evolutions with subluminal point sources (and inevitably large data) should be possible, but so far only the special case of electrostatic solutions with N point

charge sources has been conquered in Ref. 15. There it was shown that a unique finite-energy electrostatic weak solution of the **MBI** field equations with N point charges placed anywhere in \mathbb{R}^3 exists, and that the solution is real analytic away from the point charges for any choices of their signs and magnitudes.

For the field momentum densities $\mathbf{\Pi}$ (dropping the superscript “field”) one has the following expressions. For the **ML** and for **MBI** field equations,

$$4\pi c\mathbf{\Pi} = \mathbf{D} \times \mathbf{B}, \quad (16)$$

whereas for the **MBLTP** field equations,

$$4\pi c\mathbf{\Pi} = \mathbf{D} \times \mathbf{B} + \mathbf{E} \times \mathbf{H} - \mathbf{E} \times \mathbf{B} - \varkappa^{-2}(\nabla \cdot \mathbf{E})(\nabla \times \mathbf{B} - \varkappa \dot{\mathbf{E}}). \quad (17)$$

For **MBLTP** field evolutions with point sources we showed that $\mathbf{\Pi}(t, \mathbf{s})$ is in $L^1_{loc}(\mathbb{R}^3)$ about each $\mathbf{q}(t)$, see Ref. 17. We expect such a result also for **MBI** fields. It is surely false for **ML** fields!

With appropriate decay rates at spatial infinity imposed on the field initial data, the **MBLTP** field momentum $\int_{\mathbb{R}^3} \mathbf{\Pi}(t, \mathbf{s}) d^3s$ exists for all t . Moreover, given **Lipschitz** maps $t \mapsto \mathbf{q}(t)$, $t \mapsto \mathbf{v}(t)$ and bounded $t \mapsto \mathbf{a}(t)$, we showed that $\frac{d}{dt} \int_{\mathbb{R}^3} \mathbf{\Pi}(t, \mathbf{s}) d^3s$ exists for all t .

The crucial step in showing that (5), with (7), yields an equation of motion is now the following. The fields $\mathbf{B}, \mathbf{D}, \mathbf{E}, \dot{\mathbf{E}}$ (and \mathbf{H}) at (t, \mathbf{s}) are given by explicit functionals of the vector functions $\mathbf{q}(\cdot)$ and $\mathbf{v}(\cdot)$, and \mathbf{D} & \mathbf{H} also involve $\mathbf{a}(\cdot)$; their dependence on $\mathbf{a}(\cdot)$ is linear. For $t < 0$ we set $\mathbf{q}(t) = \mathbf{q}(0) + \mathbf{v}(0)t$ and $\mathbf{v}(t) = \mathbf{v}(0)$, and $\mathbf{a}(t) = \mathbf{0}$. Treating $\mathbf{q}(\cdot)$ and $\mathbf{v}(\cdot)$ as given, and $\mathbf{a}(\cdot)$ as independent vector function variable for $t > 0$, (5) together with (7) is equivalent to a **Volterra** integral equation for $\mathbf{a} = \mathbf{a}[\mathbf{q}, \mathbf{p}]$, viz.

$$\mathbf{a} = W[\mathbf{p}] \cdot \left(\mathbf{f}^{\text{vac}}[\mathbf{q}, \mathbf{v}] + \mathbf{f}^{\text{source}}[\mathbf{q}, \mathbf{v}; \mathbf{a}] \right) \quad (18)$$

where

$$\mathbf{v} = \frac{1}{m_b} \frac{\mathbf{p}}{\sqrt{1 + \frac{|\mathbf{p}|^2}{m_b^2 c^2}}}; \quad m_b \neq 0 \quad (19)$$

and

$$W[\mathbf{p}] := \text{sgn}(m_b) \frac{c}{\sqrt{m_b^2 c^2 + |\mathbf{p}|^2}} \left[\mathbf{I}_{3 \times 3} - \frac{\mathbf{p} \otimes \mathbf{p}}{m_b^2 c^2 + |\mathbf{p}|^2} \right], \quad (20)$$

and where we have written the field as a sum of a source-free (vacuum) field and a field having the point charge as source, resulting in a **Lorentz** force due to that vacuum field,

$$\mathbf{f}^{\text{vac}}[\mathbf{q}, \mathbf{v}](t) \equiv -e \left[\mathbf{E}^{\text{vac}}(t, \mathbf{q}(t)) + \frac{1}{c} \mathbf{v}(t) \times \mathbf{B}^{\text{vac}}(t, \mathbf{q}(t)) \right], \quad (21)$$

and a “self”-type force $\mathbf{f}^{\text{source}}[\mathbf{q}, \mathbf{v}; \mathbf{a}]$, in **BLTP electrodynamics**^b given by

$$\mathbf{f}^{\text{source}}[\mathbf{q}, \mathbf{v}; \mathbf{a}](t) = -\frac{d}{dt} \int_{\mathbb{R}^3} \mathbf{\Pi}^{\text{source}}(t, \mathbf{s}) d^3s \quad (22)$$

$$= -\frac{d}{dt} \int_{B_{ct}(\mathbf{q}_0)} (\mathbf{\Pi}^{\text{source}}(t, \mathbf{s}) - \mathbf{\Pi}^{\text{source}}(0, \mathbf{s} - \mathbf{q}_0 - \mathbf{v}_0 t)) d^3s \quad (23)$$

$$= \frac{e^2}{4\pi} \left[-\mathbf{Z}_{\xi}^{[2]}(t, t) + \mathbf{Z}_{\xi^{\circ}}^{[2]}(t, t) \right] \quad (24)$$

$$- \sum_{0 \leq k \leq 1} c^{2-k} (2-k) \int_0^t \left[\mathbf{Z}_{\xi}^{[k]}(t, t^r) - \mathbf{Z}_{\xi^{\circ}}^{[k]}(t, t^r) \right] (t - t^r)^{1-k} dt^r \quad (25)$$

$$- \sum_{0 \leq k \leq 2} c^{2-k} \int_0^t \left[\frac{\partial}{\partial t} \mathbf{Z}_{\xi}^{[k]}(t, t^r) - \frac{\partial}{\partial t} \mathbf{Z}_{\xi^{\circ}}^{[k]}(t, t^r) \right] (t - t^r)^{2-k} dt^r \quad (26)$$

where $\xi(t) \equiv (\mathbf{q}, \mathbf{v}, \mathbf{a})(t)$, and $\xi^{\circ}(t) \equiv (\mathbf{q}_0 + \mathbf{v}_0 t, \mathbf{v}_0, \mathbf{0})$, and where

$$\mathbf{Z}_{\xi}^{[k]}(t, t^r) = \int_0^{2\pi} \int_0^{\pi} (1 - \frac{1}{c} |\mathbf{v}(t^r)| \cos \vartheta) \mathbf{\pi}_{\xi}^{[k]}(t, \mathbf{q}(t^r) + c(t - t^r) \mathbf{n}) \sin \vartheta d\vartheta d\varphi, \quad (27)$$

with $\mathbf{n} = (\sin \vartheta \cos \varphi, \sin \vartheta \sin \varphi, \cos \vartheta)$, and where, with the kernels

$$\mathbf{K}_{\xi}(t', t, \mathbf{s}) := \frac{J_1(\varkappa \sqrt{c^2(t-t')^2 - |\mathbf{s} - \mathbf{q}(t')|^2})}{\sqrt{c^2(t-t')^2 - |\mathbf{s} - \mathbf{q}(t')|^2}}, \quad (28)$$

$$\mathbf{K}_{\xi}(t', t, \mathbf{s}) := \frac{J_2(\varkappa \sqrt{c^2(t-t')^2 - |\mathbf{s} - \mathbf{q}(t')|^2})}{c^2(t-t')^2 - |\mathbf{s} - \mathbf{q}(t')|^2} (\mathbf{s} - \mathbf{q}(t') - \mathbf{v}(t')(t - t')), \quad (29)$$

we have

$$\mathbf{\pi}_{\xi}^{[0]}(t, \mathbf{s}) = -\varkappa^4 \frac{1}{4} \left[\frac{(\mathbf{n}(\mathbf{q}, \mathbf{s}) - \frac{1}{c} \mathbf{v}) \times (\mathbf{n}(\mathbf{q}, \mathbf{s}) \times \frac{1}{c} \mathbf{v})}{\left(1 - \frac{1}{c} \mathbf{v} \cdot \mathbf{n}(\mathbf{q}, \mathbf{s})\right)^2} \right]_{\text{ret}} \quad (30)$$

$$+ \varkappa^4 \frac{1}{2} \left[\frac{\mathbf{n}(\mathbf{q}, \mathbf{s}) - \frac{1}{c} \mathbf{v}}{1 - \frac{1}{c} \mathbf{v} \cdot \mathbf{n}(\mathbf{q}, \mathbf{s})} \right]_{\text{ret}} \times \int_{-\infty}^{t_{\xi}^{\text{ret}}(t, \mathbf{s})} \mathbf{v}(t') \times \mathbf{K}_{\xi}(t', t, \mathbf{s}) dt' \quad (31)$$

$$- \varkappa^4 \frac{1}{2} \left[\frac{\mathbf{n}(\mathbf{q}, \mathbf{s}) \times \frac{1}{c} \mathbf{v}}{1 - \frac{1}{c} \mathbf{v} \cdot \mathbf{n}(\mathbf{q}, \mathbf{s})} \right]_{\text{ret}} \times \int_{-\infty}^{t_{\xi}^{\text{ret}}(t, \mathbf{s})} c \mathbf{K}_{\xi}(t', t, \mathbf{s}) dt' \quad (32)$$

$$- \varkappa^4 \int_{-\infty}^{t_{\xi}^{\text{ret}}(t, \mathbf{s})} c \mathbf{K}_{\xi}(t', t, \mathbf{s}) dt' \times \int_{-\infty}^{t_{\xi}^{\text{ret}}(t, \mathbf{s})} \mathbf{v}(t') \times \mathbf{K}_{\xi}(t', t, \mathbf{s}) dt' \quad (33)$$

$$- \varkappa^4 c \int_{-\infty}^{t_{\xi}^{\text{ret}}(t, \mathbf{s})} \mathbf{K}_{\xi}(t', t, \mathbf{s}) dt' \int_{-\infty}^{t_{\xi}^{\text{ret}}(t, \mathbf{s})} \mathbf{K}_{\xi}(t', t, \mathbf{s}) \mathbf{v}(t') dt', \quad (34)$$

^bNone of the four original contributors formulated a well-defined expression for the force, yet we believe that our formulation accomplishes what they had intended; hence the name of the theory.

$$\pi_{\xi}^{[1]}(t, s) = -\varkappa^2 \frac{1}{2} \left[\mathbf{n}(\mathbf{q}, s) \times \frac{(\mathbf{n}(\mathbf{q}, s) - \frac{1}{c}\mathbf{v}) \times \mathbf{a}}{c^2 (1 - \frac{1}{c}\mathbf{v} \cdot \mathbf{n}(\mathbf{q}, s))^3} \right]_{\text{ret}} \quad (35)$$

$$- \varkappa^2 \left[\mathbf{n}(\mathbf{q}, s) \times \frac{(\mathbf{n}(\mathbf{q}, s) - \frac{1}{c}\mathbf{v}) \times \mathbf{a}}{c^2 (1 - \frac{1}{c}\mathbf{v} \cdot \mathbf{n}(\mathbf{q}, s))^3} \right]_{\text{ret}} \times \int_{-\infty}^{t_{\xi}^{\text{ret}}(t, s)} \mathbf{v}(t') \times \mathbf{K}_{\xi}(t', t, s) dt' \quad (36)$$

$$+ \varkappa^2 \left[\mathbf{n}(\mathbf{q}, s) \times \left[\mathbf{n}(\mathbf{q}, s) \times \frac{(\mathbf{n}(\mathbf{q}, s) - \frac{1}{c}\mathbf{v}) \times \mathbf{a}}{c^2 (1 - \frac{1}{c}\mathbf{v} \cdot \mathbf{n}(\mathbf{q}, s))^3} \right]_{\text{ret}} \right] \times \int_{-\infty}^{t_{\xi}^{\text{ret}}(t, s)} c \mathbf{K}_{\xi}(t', t, s) dt' \quad (37)$$

$$- \varkappa^3 \left[\frac{1}{1 - \frac{1}{c}\mathbf{v} \cdot \mathbf{n}(\mathbf{q}, s)} \right]_{\text{ret}} \int_{-\infty}^{t_{\xi}^{\text{ret}}(t, s)} \mathbf{K}_{\xi}(t', t, s) [\mathbf{v}(t_{\xi}^{\text{ret}}(t, s)) + \mathbf{v}(t')] dt', \quad (38)$$

$$\pi_{\xi}^{[2]}(t, s) = -\varkappa^2 \left[\frac{1}{(1 - \frac{1}{c}\mathbf{v} \cdot \mathbf{n}(\mathbf{q}, s))^2} \frac{1}{c}\mathbf{v} \right]_{\text{ret}} \quad (39)$$

$$+ \varkappa^2 \left[\left[1 - \frac{1}{c^2} |\mathbf{v}|^2 \right] \mathbf{n}(\mathbf{q}, s) \times \frac{\mathbf{n}(\mathbf{q}, s) - \frac{1}{c}\mathbf{v}}{(1 - \frac{1}{c}\mathbf{v} \cdot \mathbf{n}(\mathbf{q}, s))^3} \right]_{\text{ret}} \times \int_{-\infty}^{t_{\xi}^{\text{ret}}(t, s)} c \mathbf{K}_{\xi}(t', t, s) dt' \quad (40)$$

$$- \varkappa^2 \left[\left[1 - \frac{1}{c^2} |\mathbf{v}|^2 \right] \frac{\mathbf{n}(\mathbf{q}, s) - \frac{1}{c}\mathbf{v}}{(1 - \frac{1}{c}\mathbf{v} \cdot \mathbf{n}(\mathbf{q}, s))^3} \right]_{\text{ret}} \times \int_{-\infty}^{t_{\xi}^{\text{ret}}(t, s)} \mathbf{v}(t') \times \mathbf{K}_{\xi}(t', t, s) dt', \quad (41)$$

and $|_{\text{ret}}$ means that $\mathbf{q}(\tilde{t})$, $\mathbf{v}(\tilde{t})$, $\mathbf{a}(\tilde{t})$ are evaluated at $\tilde{t} = t_{\xi}^{\text{ret}}(t, s)$.

Remark 2.1. The decomposition of the electromagnetic fields into a sum of two types of fields, one with the point charge as source, the other source-free, is to some extent arbitrary. For this reason it is futile to try to identify the self-field force of a point charge. At best one can speak of a “self”-field force, the scare quotes referring to the ambiguity in identifying how much of the field is “self”-generated by the charge and how much is not.

The following *key proposition* about the Volterra equation is proved in Ref. 17. Its proof takes several dozen pages of careful estimates.

Proposition 2.2. *Given $C^{0,1}$ maps $t \mapsto \mathbf{q}(t)$ and $t \mapsto \mathbf{p}(t)$, with $\text{Lip}(\mathbf{q}) = v$, $\text{Lip}(\mathbf{v}) = a$ big enough, and $|\mathbf{v}(t)| \leq v < c$, the Volterra equation (18) as a fixed point map has a unique C^0 solution $t \mapsto \mathbf{a}(t) = \alpha[\mathbf{q}(\cdot), \mathbf{p}(\cdot)](t)$ for $t \geq 0$. Moreover, the solution depends **Lipschitz** continuously on the maps $t \mapsto \mathbf{q}(t)$ and $t \mapsto \mathbf{p}(t)$ (treated as independent).*

The well-posedness result of the joint initial value problem for **MBLTP** fields and their point charge sources is a corollary of Proposition 2.2. Namely, now setting $\mathbf{a}(t) := \frac{d}{dt}\mathbf{v}(t)$ and $\mathbf{v}(t) := \frac{d}{dt}\mathbf{q}(t)$, and recalling (7), the solution to the **Volterra** integral equation for the acceleration poses a **Newton**-type second-order initial value problem for the position of the point charge with a complicated yet **Lipschitz**-continuous force. Once the motion is computed, inserting the vector functions $\mathbf{q}(\cdot)$,

$\mathbf{v}(\cdot)$, and $\mathbf{a}(\cdot)$ of the solution into the functionals of the fields yields $\mathbf{B}, \mathbf{D}, \mathbf{E}, \dot{\mathbf{E}}$ at (t, \mathbf{s}) for $t > 0$, too. This is a Theorem in Ref. 17, summarized informally as follows.

Theorem 2.3. *As a consequence of Proposition 2.2, the joint initial value problem for **MBLTP** fields and their point charge source is equivalent to the fixed point equations*

$$\mathbf{q}(t) = \mathbf{q}(0) + \frac{1}{m_b} \int_0^t \frac{\mathbf{p}}{\sqrt{1 + |\mathbf{p}|^2/m_b^2}}(\tilde{t}) d\tilde{t} =: Q_t(\mathbf{q}(\cdot), \mathbf{p}(\cdot)) \quad (42)$$

$$\mathbf{p}(t) = \mathbf{p}(0) - \int_{\mathbb{R}^3} (\mathbf{\Pi}(t, \mathbf{s}) - \mathbf{\Pi}(0, \mathbf{s})) d^3s =: P_t(\mathbf{q}(\cdot), \mathbf{p}(\cdot)), \quad (43)$$

where Q_t and P_t are **Lipschitz** maps. Thus, **BLTP** electrodynamics is locally well-posed.

In fact, in Ref. 17 the **Cauchy** problem for the **MBLTP** field with N point charges is treated. *Local well-posedness* is proved for *admissible initial data* (see below) & $m_b \neq 0$, and *global well-posedness* shown to hold if in finite time: (a) no particle reaches the speed of light; (b) no particle is infinitely accelerated; (c) no two particles reach the same location.

By “admissible” initial data we mean the following: the initial particle velocities are subluminal ($|\mathbf{v}(0)| < c$) and no two particles occupy the same location; the electromagnetic initial fields are the sum of a spatially sufficiently rapidly decaying vacuum field plus N fields each with a single point charge source — the sourced fields are boosted electrostatic fields with boost velocity equal to the initial velocity of the source.

Although our result seems to be the first formulation of a well-posed joint initial value problem for classical electromagnetic fields and their point charge sources, and this endows **BLTP** electrodynamics with a mathematically superior status compared to the ill-defined **Lorentz** electrodynamics, we do not claim that **BLTP** electrodynamics is the correct classical limit of the illusive mathematically well-formulated quantum theory of electromagnetism. In particular, the **MBLTP** field equations feature “physical” oddities: (a) a field energy functional which is unbounded below; (b) subluminal transversal electromagnetic wave modes; (c) longitudinal electrical wave modes.

Moreover, since the **MBLTP** field equations demand initial data for $\mathbf{B}, \mathbf{D}, \mathbf{E}, \dot{\mathbf{E}}$ at $t = 0$ while physically we only can prescribe \mathbf{B}, \mathbf{D} (N.B.: in **Lorentz** electrodynamics, $\mathbf{D} = \mathbf{E}$; not so in **BLTP** electrodynamics), one needs to find a prescription which expresses $\mathbf{E}, \dot{\mathbf{E}}$ at $t = 0$ in terms of \mathbf{B}, \mathbf{D} at $t = 0$. In Ref. 17 we show that a reasonable choice is the map $(\mathbf{B}, \mathbf{D})_0 \mapsto (\mathbf{E}, \dot{\mathbf{E}})_0$ obtained by *maximizing the field energy* w.r.t. \mathbf{E}_0 and $\dot{\mathbf{E}}_0$ (treated independently). This can be made co-variant by stipulating that the maximization is carried out in the **Lorentz** frame in which total momentum vanishes initially.

Back to the mathematically superior status of **BLTP** electrodynamics, one can now apply rigorous analysis, and controlled numerical techniques, to study

the theory. For instance, a rigorous comparison of our expression (22) for the “self”-force with a differently defined “self”-field force which was studied in Ref. 12 was carried out recently by Hoang & Radosz and their students, see Ref. 5, 13. One of our next projects is to rigorously extract effective equations of motion with more user-friendly expressions for the “self”-force. In particular, whether or to which extent the **Landau–Lifshitz** equation approximately governs the motion is an interesting question.

2.1.2. *Topologically non-trivial flat spacetimes with time-like singularities*

The *zero-gravity limit* of singular spacetimes does not automatically yield **Minkowski** spacetime minus a number of world-lines. In Ref. 27 the zero- G limit of **Carter**’s maximal analytical extension of the electromagnetic **Kerr–Newmann** spacetime was analyzed. The limiting spacetime is axially symmetric and static. It is locally isometric to **Minkowski** spacetime, but is topologically non-trivial, featuring **Zipoy** topology. Its constant-time slices are double-sheeted, and have the topology of \mathbb{R}^3 branched over the un-knot. The spacetime is singular on a 2-dimensional time-like cylinder $\mathbb{S}^1 \times \mathbb{R}$, which is the world-tube of a space-like ring-type singularity. The most intriguing aspect of this solution is that at any instant of time the ring singularity, when viewed from one of the two sheets of space, appears to be positively charged, and from the other sheet, negatively charged, as already noticed by **Carter**. The electromagnetic fields it supports were discovered in the 19th century by **P. Appell** as “multi-valued electromagnetic fields,” while **A. Sommerfeld** soon realized that they represent single-valued electromagnetic fields on a topologically non-trivial multi-sheeted space.

A natural next step would be to formulate the corresponding zero-gravity two-body problem for two space-like ring-type singularities of zGKN-type, evolving in time jointly with the electromagnetic fields they generate. We know for example that the space should be four-sheeted (in general, 2^N sheets are needed for N ring-type singularities). However, as the zGKN fields solve the **Maxwell–Lorentz** vacuum field equations away from the singularities, it is not surprising that one again encounters the infinite field energy-momentum problems which plague **Lorentz** electrodynamics. Thus one would first need to find either **MBI** or **MBLTP** analogues of the **Appell–Sommerfeld** fields. The nonlinearity of the **MBI** field equations makes this a daunting task, but we are confident that the feat can be accomplished with the **MBLTP** field equations. In that case the zero- G **EIH**-type approach explained in the previous subsection should allow the formulation of a well-posed initial value problem for ring-type singularities and their electromagnetic **MBLTP** fields. Note that the law of motion would most likely be a system of partial differential equations, as the ring-type singularity has to be allowed to bend, twist, warp, stretch, and such.

2.2. Turning on gravity: $G > 0$

2.2.1. The neighborhood of Minkowski spacetime

By rigorously establishing well-posedness of **BLTP** electrodynamics with point charges as a consequence of postulating the conservation law (2) for the total momentum vector-density (3), with the expressions for the particles given be the usual special-relativistic ones and those for the fields determined by the field equations, Ref. 17 demonstrates that a key idea of the 1938 **EIH** paper is viable in the zero- G limit when applied with a suitable set of electromagnetic field equations, and with non-zero bare rest mass assigned to the point charges. We now address the question whether this result extends continuously to a $G > 0$ neighborhood of special relativity, free of black holes. By a result of **Geroch–Traschen**¹¹ a no-Black-Holes spacetime with a one-dimensional time-like singularity cannot exist if the singularity has positive bare mass, different from the special-relativistic case, where the bare mass merely had to be non-zero.

One of the main ingredients of the **EIH** argument is of course, that the conservation law of energy-momentum is implied by the twice contracted 2nd **Bianchi** identity in concert with **Einstein**'s general-relativistic field equations. However, for the spacetimes with time-like singularities envisaged by **EIH** (and **Weyl**) this is not automatically true. An important step, therefore, is to determine the **Lorentz** spacetimes with time-like singularities on which the twice contracted 2nd **Bianchi** identity holds in a weak form.

Together with **A. Y. Burtscher** we have begun a systematic study of the favorable conditions. In Ref. 3 we study the simplest non-trivial case: static spherically symmetric spacetimes with a single time-like singularity. We obtain some necessary and, for certain electromagnetic vacuum laws, also sufficient conditions that the twice contracted 2nd **Bianchi** identity holds in a weak form. Interestingly, in the naked singularity regime (no Black Hole!), the answer is *negative* for the electromagnetic **Reissner–Weyl–Nordström** spacetime, but *positive* for the **Hoffmann** spacetime^c in which electromagnetic **MBI** fields are coupled with **Einstein**'s gravity; see Ref. 26 for a rigorous discussion. Our goal is to generalize our study, one step at a time, to dynamical spacetimes without symmetry.

The crucial question, then, is whether the weak twice contracted 2nd **Bianchi** identity implies the law of the electromagnetic spacetime's time-like singularities with bare energy-momentum assigned to them. Interestingly, it seems that merely changing **Maxwell**'s into the **BLTP** or **MBI** vacuum law is by itself not sufficient to allow the formulation of a well-defined joint initial value problem for the massive point charges and the electromagnetic and gravitational fields they generate: the gravitational coupling also needs to be changed, for instance in the manner of

^cIt is curious that although **Hoffmann**¹⁴ had worked out his spacetime solution by 1935, the fact that his spacetime with non-positive bare mass is less singular than the **RWN** spacetime did not in 1938 compell **EIH** to use the **MBI** instead of **Maxwell–Lorentz** field theory.

Jordan, Brans–Dicke, or $f(\mathbf{R})$ gravity, to obtain a well-defined joint **Cauchy** problem. Put differently, the gravitational coupling of spacetime structure with bare matter and the electromagnetic fields needs to be “mediated” by a certain type of scalar field which moderates the strength of the spacetime singularities enough so that the strategy explained in the previous subsection can be applied. This scalar field in the classical theory would play a role vaguely reminiscent of the role played by the scalar Higgs field in the quantum field-theoretical standard model of elementary particle physics.

2.2.2. The neighborhood of $zGKN$ -type spacetimes

Everything stated in the previous subsection about the neighborhood of the **Minkowski** spacetime has an analogue problem for the gravitational neighborhood of topologically non-trivial flat spacetimes of (generalized) $zGKN$ type.

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