Hyperbolic numbers as Einstein numbers

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Abstract. In the special theory of relativity (SR) it is usual to highlight so-called paradoxes. One of these paradoxes is the formal appearance of speed values grater then the light speed. In this paper we show that most of these paradoxes arise due to the incompleteness of relativistic calculus over velocities. Namely, operation over speeds form a group by composition. In this case, the extension to the field is usually carried out using non-relativistic operations.

1. Introduction

In the special relativity theory there are widely known paradoxes in which formally appear velocities exceeding the speed of light. The authors make the assumption that these paradoxes can be solved by introducing special operations of speed multiplication by scalar and speed multiplication along with the standard speed addition. Strictly speaking, it is necessary to extend the velocity space to algebra. In this paper, such an extension is produced by means of hyperbolic numbers by which one can represent the two-dimensional Minkowski space as well as the Lorentz transform.

2. Superluminal speed paradoxes

We list some of the well known paradoxes of superluminal motion, where the speed v of some system is greater then c.

2.1. The inclined light incidence

The simplest model of superluminal motion may serve inclined light incidence. Light pulse is formed by flat waves incident on the plane of some flat media boundary (also called screen) [1–4]. Let φ be the angle of incidence of the wave on the screen, that is, the angle between the wave vector and the normal vector of the screen. That light spot on the screen moves through the screen at a speed of:

$$v = \frac{c}{n \sin\varphi}$$

where n is the refractive index of the medium, there the light pulse propagates (the medium above the screen). As far as $\sin\varphi \leq 1 \leq c$ the speed of the light spot when decreasing the angle of incidence φ can be made greater than the speed of light c. When considering the case of wave propagation in a vacuum, this becomes most obvious:

$$v = \frac{c}{\sin\varphi}$$

Journal of Physics: Conference Series **1557** (2020) 012027 doi:10.1088/1742-6596/1557/1/012027

The velocity v can be arbitrarily large, because when the light pulse tends to a normal direction incident ($\varphi \to 0$), the velocity tends to infinity ($v \to \infty$).

2.2. The lighthouse paradox
Similarly to the incident light pulse, one can consider a rotating source [2–4]. Let us imagine a rotating spotlight. The angular velocity of the spotlight (lighthouse) is ω and the screen is at a distance r from the source. Then the light spot will move at a speed:

$$v = \omega r$$

The speed can increase unlimited, while r increasing.

2.3. The superluminal scissors
The superluminal scissors is mechanical analog of the inclined light incidence. The speed of the blades intersection point may be grater then lightspeed.

2.4. Phase speed
Phase velocity is the speed of movement of a point having a constant phase of oscillatory motion in space along a given direction. The phase velocity in the direction of the wave vector coincides with the velocity of the phase front the surface of the constant phase.

The phase velocity along the wave vector is given as follows:

$$v_p = \frac{\omega}{k} = v_p(0)$$

where ω is the angular frequency, k is the wave number. In vacuum for electromagnetic wave phase velocity value along the vector is equal to the speed of light c.

The deviation from the wave vector by an angle φ the phase velocity is equal to:

$$v_p(\varphi) = \frac{v_p(0)}{\cos\varphi}$$

so the phase velocity can be superluminal.

2.5. Hartman effect
According to Hartman's formula [5], the time of particle tunneling through the barrier, determined by the phase of the barrier transmission function, does not depend on the length of the tunneling path. With a sufficiently large path, the particle velocity can reach superluminal values [6,7].

3. Hyperbolic numbers
Hyperbolic numbers [8–10], along with elliptic and parabolic numbers, are a generalization of complex numbers. Hyperbolic numbers can be defined as follows:

$$z = x + jy, \quad j^2 = 1, \quad j \neq \pm 1.$$

The set of hyperbolic numbers is denoted as C^+. The hyperbolic numbers are also called double numbers, split complex numbers and perplex numbers.

For two hyperbolic numbers $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$ it is possible to define the following operations.

- Addition $z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$.

- Multiplication $z_1 z_2 = (x_1 x_2 + y_1 y_2) + j(x_1 y_2 + x_2 y_1)$.
- Conjunction $z^\dagger = x - jy$.
- Inverse number $z^{-1} = \frac{x}{x^2+y^2} - j\frac{y}{x^2-y^2}$.
- Division $\frac{z_1}{z_2} = \frac{x_1 x_2 - y_1 y_2}{x_2^2 - y_2^2} + j\frac{x_1 y_1 - x_1 y_2}{x_2^2 - y_2^2}$.

Hyperbolic numbers can be represented in matrix form:

$$x + jy \leftrightarrow \begin{pmatrix} x & y \\ y & x \end{pmatrix}$$

Then the addition, multiplication of numbers and finding the inverse number are reduced to the addition, multiplication of matrices and finding the inverse matrix.

For hyperbolic numbers, the analog of Euler's formula is true

$$e^{j\varphi} = \cosh\varphi + j\sinh\varphi, \quad \varphi \in R.$$

It can be proved by exponent decomposition into a series $e^{j\varphi}$:

$$e^{j\varphi} = 1 + \frac{j\varphi}{1!} + \frac{j^2\varphi^2}{2!} + \frac{j^3\varphi^3}{3!} + \ldots$$

Using $j^2 = 1$, $j^3 = j$, $j^4 = 1$, $j^5 = j$ etc. we get:

$$e^{j\varphi} = \left(1 + \frac{\varphi^2}{2!} + \frac{\varphi^4}{4!} + \ldots\right) + j\left(\frac{\varphi}{1!} + \frac{\varphi^3}{3!} + \ldots\right) = \cosh\varphi + j\sinh\varphi.$$

The Euler formula can be used to express the hyperbolic sine and cosine through an exponent:

$$\cosh\varphi = \frac{e^{j\varphi} + e^{-j\varphi}}{2}, \qquad \sinh\varphi = \frac{e^{j\varphi} - e^{-j\varphi}}{2j}.$$

These formulas are analogs to the corresponding formulas for the trigonometric cosine and sine, but instead of the imaginary unit $i^2 = -1$ the “imaginary” unit $j^2 = 1$ is used.

Hyperbolic numbers can be used to represent a point in two-dimensional Minkowski space $E^2_{1,1}$:

$$x^0 + jx^1 \leftrightarrow (x^0, x^1) \;\in\; E^2_{1,1}.$$

This is possible due to the fact that

$$|\Delta z|^2 = \Delta z \cdot \Delta z^\dagger = \Delta^2 x^0 - \Delta^2 x^1.$$

The absolute value (or modulus) r of number $z = x + jy$

$$|z| = r = \begin{cases} \sqrt{x^2 - y^2}, & |x| > |y|, \\ \sqrt{y^2 - x^2}, & |x| < |y|. \end{cases}$$

Just as complex numbers have a geometric representation using polar coordinates, hyperbolic numbers can also be represented using hyperbolic functions, which will give us a geometric representation of hyperbolic numbers on the plane (Minkowski space model $E^2_{1,1}$ on the Euclidean plane).

$$z = \begin{cases} r\left(\cosh\varphi + j\sinh\varphi\right), & |x| > |y|, \\ jr\left(\sinh\varphi + j\sinh\varphi\right), & |x| < |y|. \end{cases}$$

Journal of Physics: Conference Series **1557** (2020) 012027 doi:10.1088/1742-6596/1557/1/012027

Combining the geometric representation and Euler's formula we obtain a exponential representation

$$z = \begin{cases} re^{j\varphi}, & |x| > |y|, \\ jre^{j\varphi}, & |x| < |y|. \end{cases}$$

The value of φ is called the hyperbolic angle or argument of z ($\varphi = argz$) and is expressed in terms of the hyperbolic arctangent.

$$\varphi = \begin{cases} \tanh^{-1}\frac{y}{x}, |x| > |y|, \\ \tanh^{-1}\frac{x}{y}, |x| < |y|. \end{cases}$$

4. The hyperbolic numbers in SR

4.1. Lorentz transformations and hyperbolic numbers

With the help of the hyperbolic numbers, one can rearrange the Lorentz transformation for the two dimensional case. Let's start by writing the transformation formulas from the coordinate system (ct, x) to the dashed coordinate system (ct', x') in the classical form, and then rewrite them using hyperbolic functions.

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\frac{v}{c} \\ -\gamma\frac{v}{c} & \gamma \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} \cosh\varphi & -\sinh\varphi \\ -\sinh\varphi & \cosh\varphi \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$$

The coefficient γ is called the Lorentz factor

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

Since $\cosh\varphi = 1/\sqrt{1 - \tanh^2\varphi}$, then putting $\tanh\varphi = v/c$ get $\cosh\varphi = \gamma$. Then $\sinh\varphi = \tanh\varphi\cosh\varphi = \gamma v/c$. These relations allow us to move to the hyperbolic form of the Lorentz transformation matrix.

Since hyperbolic numbers have a matrix representation and the Lorentz transformation matrix corresponds to the matrix representing the hyperbolic number, we can replace this matrix with the corresponding hyperbolic number.

$$\begin{pmatrix} \cosh\varphi & -\sinh\varphi \\ -\sinh\varphi & \cosh\varphi \end{pmatrix} \leftrightarrow \cosh\varphi - j\sinh\varphi = e^{-j\varphi}$$

It is especially convenient to use the exponential form, since in this case the Lorentz transformation is reduced to the ratio

$$(ct', x') = e^{-j\varphi}(ct, x).$$

4.2. Velocity-addition formula

Velocity-addition formula can be derived by performing two Lorentz transformations. The composition of the two Lorentz transformations is reduced to the addition of hyperbolic angles:

$$\varphi_3 = \varphi_1 + \varphi_2.$$

Taking into account that $\tanh\varphi = v/c$ we get:

$$\tanh\varphi_3 = \tanh(\varphi_1 + \varphi_2) = \frac{\tanh\varphi_1 + \tanh\varphi_2}{1 - \tanh\varphi_1\tanh\varphi_2} = \frac{v_1/c + v_2/c}{1 - v_1 v_2/c^2}.$$

By replacing $\tanh\varphi_3 = v_3/c$ we get velocity-addition formula

$$v_3 = c\frac{v_1/c + v_2/c}{1 - v_1 v_2/c^2} = \frac{v_1 + v_2}{1 - v_1 v_2/c^2}.$$

4.3. Multiplying velocity by number

Consider the operation of multiplying the velocity vector by a scalar ($\alpha \in \mathbb{R}$), which is equivalent to multiplying the hyperbolic angle by a number:

$$\varphi_2 = \alpha\varphi_1.$$

Expressing a hyperbolic tangent through an exponent, we obtain:

$$\tanh\varphi_2 = \tanh\alpha\varphi_1 = \frac{e^{2\alpha\varphi_1}-1}{e^{2\alpha\varphi_1}+1} = \frac{\exp\left[\alpha\ln\frac{1+v/c}{1-v/c}\right]-1}{\exp\left[\alpha\ln\frac{1+v/c}{1-v/c}\right]+1} = \frac{\left(\frac{1+v/c}{1-v/c}\right)^\alpha-1}{\left(\frac{1+v/c}{1-v/c}\right)^\alpha+1}$$

Taking into account that $\tanh\varphi = v/c$, we get:

$$v_2 = c\frac{(1+v/c)^\alpha-(1-v/c)^\alpha}{(1+v/c)^\alpha+(1-v/c)^\alpha}$$

To test this formula, let us calculate $2v$ as $v+v$ and compare the results. Using multiplication formula we obtain:

$$2v = c\frac{(1+v/c)^2-(1-v/c)^2}{(1+v/c)^2+(1-v/c)^2} = c\frac{4v/c}{2\left(1+v^2/c^2\right)}.$$

Using Velocity-addition formula, we also get:

$$v+v = \frac{2v}{1+v^2/v^2}.$$

As we can see, the result are the same.

5. Conclusions

We consider the use of hyperbolic numbers to write Lorentz transformations for two-dimensional space-time and then to write the velocity-addition formula. In hyperbolic numbers, the addition of velocities was reduced to the addition of hyperbolic angles.

Since the multiplication operation by scalar $\alpha \in \mathbb{R}$ is naturally defined for the hyperbolic angle, the corresponding operation can also be defined for the relativistic velocity. This operation is not identical to the simple multiplication of the vector by a number. In the case of the natural number α it is reduced to multiple addition of velocities. It follows from the formula that multiplication by any α cannot lead to superluminal speed.

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