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Quantization of two- and three-player cooperative games based on QRA

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Abstract

In this paper, a novel quantization scheme for cooperative games is proposed. The circuit is inspired by the Eisert–Wilkens–Lewenstein protocol, which was modified to represent cooperation between players and extended to 3–qubit states. The framework of Clifford algebra is used to perform necessary computations. In particular, we use a direct analogy between Dirac formalism and Quantum Register Algebra (QRA) to represent circuits. This analogy enables us to perform automated proofs of the circuit equivalence in a simple fashion. The expected value of the Shapley value concerning quantum probabilities is employed to distribute players' payoffs after the measurement. We study how entanglement, representing the level of pre-agreement between players, affects the final utility distribution. The paper also demonstrates how the QRA and GAALOP software can automate all necessary calculations.

Keywords: quantum games, cooperative games, quantum computing, shapley value, quantum register algebra

1. Introduction

Game theory is a branch of applied mathematics that studies optimal decisions of entities between which conflict of interests has occurred [33]. The conflict is formalized into a game, where the entities take the role of players with clearly defined strategies (set of possible decisions) and payoff functions, describing utilities the players can obtain [25]. Cooperative

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game theory enables the study of situations where players can form coalitions to cooperate and improve their utilities [26]. In classical (superadditive) cooperative games, the main interest dwells in finding a fair allocation of the utility produced by the grand coalition concerning players' contributions [29].

Recently, game theory has been enriched with a new class of quantum games [15]. Originally, quantum game theory studied the quantization of non-cooperative matrix-form games, such as the well-known Prisoner's Dilemma [28]. One of the most pioneering works on this topic [11] has studied new non-trivial equilibrium occurring due to the quantization of strategies and full entanglement between players' decision states. Though it has been an object of criticism [3] (the considered strategy sets have no physical interpretation), this work has established the fundamental basis of contemporary research in quantum games [4, 13]. General quantum games can be characterized as conflicts that include superposition of players' strategies and entanglement of players' decisions [14].

In Eisert–Wilkens–Lewenstein (EWL) quantization protocol [11], the decisions of the player to cooperate or deflect cooperation have been assigned to basis states of the qubit (quantum analogy of the classical bit [7]). In this work, we extend this idea to describe the cooperative game of n-players with transferable utility (TU). In TU-games, players can freely distribute benefits produced by the coalition across its members since this profit is equally attractive to all players and can be described by one number [25]. Our idea is to represent TU-games by assigning basis states of the n–qubit to coalitions, where 1 on ith position means that ith player participates in this coalition, while 0 on the same position identifies that player is absent in the coalition. Whereas the proposed idea has the potential to be generalized into the domain of all cooperative games, in this work, we study quantization protocols only for two- and three-player games.

The main interest is to assess the potential of our quantization approach to cooperative games and determine if it will allow for non-trivial results in utility distribution due to quantum phenomena. All the necessary calculations will be performed using geometric algebra [8, 9, 18, 19, 23, 27]. The recent papers [1, 6] demonstrate an increasing number of applications of geometric algebras to quantum computing. In particular, we represent all the underlying quantum circuits using the language of Quantum Register Algebra (QRA) [21], a real form of complex Clifford algebra, [13, 20]. It should be emphasized that complex Clifford algebra can be seen as a viable and intuitive symbolic alternative to the well-known fermionic quantum computation model [5, 32]. This alternative enables us to study the proposed quantization schemes' properties and perform the automated proof of their equivalence. To validate the proposed approach, we 'solve' instances of two- and three-player weighted majority cooperative games. The QRA representations of the corresponding 2– and 3–qubit decision states and quantum gates on them will be programmed in GAALOP [1, 17], which makes it possible to directly obtain the final state of the game from which the probabilities will be extracted.

The rest of the paper is structured as follows. In section 2, we briefly discuss cooperative game theory and its solution concept of the Shapley value [31]. Then, section 3 is devoted to introducing the original quantization protocol for 2- and 3-player cooperative games. After that, in section 4, we demonstrate how QRA can be applied to conveniently represent quantum computing and study the properties of quantum circuits. Section 5 presents a detailed description of the QRA representations of the considered quantum games. Ultimately, we demonstrate how quantum cooperative games can be solved using GAALOP and discuss the results.

2. Cooperative games and Shapley value

In this section, we will define cooperative games, their subclass of weighted majority games, and the underlying solution concept of the Shapley value according to [26]. Throughout the paper, we focus solely on cooperative TU-games, where coalitions are assigned with a real number with a monetary equivalent that can be freely distributed in any feasible way. More precisely, a general cooperative TU-game is conventionally defined by a pair (N, v), where N is a set of players and $v : 2^N \to \mathbb{R}$ is the value function, which assigns to each non-empty subset $S \subseteq N$, $S \neq \emptyset$, called coalition, its utility represented by a real number $v(S)$ [25]. It is assumed that the so-called empty coalition $\emptyset$ produces no value, i.e. $v(\emptyset) = 0$.

One of the distinguished families of cooperative games is simple games [22]. A simple game $(N, \mathcal{W})$ is defined by a set $\mathcal{W}$, which is a 'winning' set of subsets of players' set N, i.e. $\mathcal{W} \subset 2^N$. The set $\mathcal{W}$ fulfills three main properties: $N \in \mathcal{W}$, $\emptyset \notin \mathcal{W}$, and

$$(S \subseteq T \subseteq N \,\text{and}\ \ S \in \mathcal{W}) \Rightarrow T \in \mathcal{W}. \tag{1}$$

This game can be represented as a standard cooperative game via simple identification

$$(N, \mathcal{W}) \sim (N, v)\,, \text{where } v(S) = \begin{cases} 1, & \text{if } S \in \mathcal{W} \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

For the purpose of this paper, we work with the class of the weighted majority games [24], where quota q and weights w_i are associated with each player $i \in N$. Then, we can identify weighted majority game $(N, q, (w_i)_{i \in N})$ with the simple game $(N, \mathcal{W})$ using the following relation

$$S \in \mathcal{W} \Leftrightarrow \sum_{i \in S} w_i \geqslant q. \tag{3}$$

To 'solve' a game often means to compute its solution. In terms of cooperative game theory, a solution is defined as a function σ that assigns each game (N, v) a subset (or just one) of allocations $\sigma(N, v)$ from the set $X^*(N, v)$

$$X^*(N, v) = \left\{ (x_i)_{i \in N} \,|\, \sum_{i \in N} x_i \leqslant v(N)\,, x_i \in \mathbb{R}\ \forall i \in N \right\}, \tag{4}$$

of all feasible allocations of the $v(N)$, where player's i payoff is denoted as x_i [26]. The Shapley value is one of game theory's main one-point solution concepts. The Shapley value for player $i \in N$ can be defined as

$$\phi_i(N, v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!\,(|N| - |S| - 1)!}{|N|!} \left(v(S \cup \{i\}) - v(S)\right). \tag{5}$$

The Shapley value is assigned to each player based on their expected payoff in the following situation. Assume players arrive randomly, and each order has an equal probability of $\frac{1}{|N|!}$. Then, when a player arrives, they obtain the marginal contribution to the coalition of the already arrived players [26].

Before the Shapley value concept for simple games is introduced, a player's property to be pivotal to the coalition must be explained. In a simple game $(N, \mathcal{W})$, a player $i \in N$ is pivotal to some coalition S, iff $S \notin \mathcal{W}$, but $S \cup \{i\} \in \mathcal{W}$. The analogy of the Shapley value for simple

Table 1. Marginal contributions of two players to sequential coalitions.

Sequential coalition	Player 1	Player 2
(1,2)	1	0
(2,1)	0	1

Table 2. Marginal contributions of three players to sequential coalitions.

Sequential coalition	Player 1	Player 2	Player 3
(1,2,3)	0	1	0
(1,3,2)	0	0	1
(2,1,3)	0	1	0
(2,3,1)	0	1	0
(3,1,2)	1	0	0
(3,2,1)	0	1	0

games is called the Shapley–Shubik power index [30]. It assigns to the player's percentual share of how many times the player is pivotal to a coalition of already arrived players in each sequential coalition (possible ordering of players). This power index reflects the player's power or pretension. Now, two particular instances of the weighted majority games will be presented and solved.

Example 1. Consider weighted majority game $(N,q,(w_i)_{i\in N})$, where $N=\{1,2\}, w_1=1, w_2=1$, and $q=1$. Table 1 demonstrates players' property of being pivotal (indicated by 1) for each possible order of arrival.

Thus, if we denote the Shapley–Shubik power index of player i as ϕ_i, we have the following results: $\phi_1(N,q,(w_i)_{i\in N})=50\%,\ \phi_2(N,q,(w_i)_{i\in N})=50\%$. In this game, players obtain the same distribution from creating the joint utility, though each exceeds the quota on one's own.

Example 2. Consider weighted majority game $(N,q,(w_i)_{i\in N})$, where $N=\{1,2,3\}$, $w_1=1, w_2=2, w_3=1$, and $q=2$. Table 2 demonstrates players' property of being pivotal for each possible order of arrival.

Thus, we have the following results: $\phi_1(N,q,(w_i)_{i\in N})=16.6\overline{6}\%$, $\phi_2(N,q,(w_i)_{i\in N})=66.6\overline{6}\%$, $\phi_3(N,q,(w_i)_{i\in N})=16.6\overline{6}\%$. Clearly, the second player dominates since he is pivotal four times out of six possible.

However, the results of both games are valid only for the ideal deterministic case, when a grand coalition N will always be formed, and all players want to cooperate and can come to an agreement. Conversely, what if we allow players to form bonds and pre-agreements or have doubts about cooperation? Moreover, what if this information affects the probabilities of coalitions' occurrence and we distribute players' expected utilities according to the restricted Shapley values? Will it lead to the redistribution of power indices and change the situation? We propose quantizing the weighted majority games to answer all these questions. This will enable us to fairly distribute payoffs based on additional information about players' relationships and intentions.

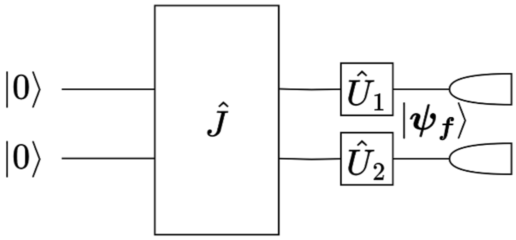

Figure 1. Two-player protocol scheme.

3. Cooperative games quantization

The main principle of the proposed quantization of cooperative games is the idea that the number of basis states representing n–qubit exactly corresponds to a cardinality of power set 2^N of players' set $N = \{1,\ldots,n\}$. Thus, we can easily identify each basis state with the particular coalition $S \subseteq N$ in the following way:

$$S \sim |a_1 \ldots a_i \ldots a_n\rangle \Leftrightarrow a_i = \begin{cases} 1, \ i \in S \\ 0, \ i \notin S. \end{cases} \tag{6}$$

Therefore, it is possible to associate the probability of occurrence of each basis state with the likelihood of occurrence of a particular coalition. Then, we propose to distribute players' payoffs (power indices) as an expected value of their Shapley values computed for the value functions restricted on the coalitions $S \subseteq N$ with respect to new quantum probabilities.

Our main objective is to study how information about players' initial agreement and satisfaction with this pre-agreement can redistribute the resulting payoffs through this quantum value. In the following subsection, we will introduce new parameters reflecting the additional information about the game setting and demonstrate how they affect the probabilities of the basis states in two-player games.

3.1. Quantization of two-player game

At first, we focus on two-player cooperative games to better demonstrate the principles of quantum cooperative games. Moreover, this simple instance will show that the proposed approach preserves and extends classical cooperative games. Figure 1 depicts the proposed game quantization scheme.

The scheme almost fully corresponds to the quantization of EWL protocol [11] but does not end with the disentanglement operator before the measuring device. The consideration that bonds created by the entanglement have to be preserved and considered during the payoff distribution explains the absence of the disentanglement operator. Moreover, for the particular choice of operators $\hat{U}_1(p_1)$ and $\hat{U}_2(p_2)$ considered in this work, entanglement does not play a role in the $|\psi_f\rangle$, if the disentanglement operator $\hat{J}^\dagger$ is applied.

The presence of each player from $N = \{1,2\}$ in the game is initially represented by the basis state $|0\rangle$. It describes the fact that the player does not cooperate. Thus, the initial 2–qubit state $|00\rangle$ corresponds to the empty coalition $\emptyset$, whereas for the remaining basis states the identification

$$|01\rangle \sim \{2\}, \ |10\rangle \sim \{1\}, |11\rangle \sim \{1,2\}, \tag{7}$$

holds. Then, the entanglement [10] operator $\hat{J}$ is applied on the 2–qubit state $|00\rangle$. In particular, we assume an entanglement operator

$$\hat{J}(\gamma) = \begin{pmatrix} \cos\frac{\gamma}{2} & 0 & 0 & i\sin\frac{\gamma}{2} \\ 0 & \cos\frac{\gamma}{2} & -i\sin\frac{\gamma}{2} & 0 \\ 0 & -i\sin\frac{\gamma}{2} & \cos\frac{\gamma}{2} & 0 \\ i\sin\frac{\gamma}{2} & 0 & 0 & \cos\frac{\gamma}{2} \end{pmatrix}, \tag{8}$$

presented in the EWL protocol [11]. The operator $\hat{J}(\gamma)$ depends on the entanglement measure $\gamma \in [0, \pi/2]$, where $\hat{J}(0)$ is an identity operator and $\hat{J}(\pi/2)$ creates one of maximally entangled two-qubit states $(|00\rangle + |11\rangle)/\sqrt{2}$, called the Bell's state. In this work, the entanglement parameter γ is interpreted as a measure of the initial agreement between players. When players are maximally entangled, the Bell's state is created and probability of occurrence of coalition N is $1/2$ and the same is valid for $\emptyset$. Thus, the full pre-agreement between players affects the game's possible outcome, such that they are either together or do not participate.

After that, the information about players' desire to change the initial state is incorporated into the game using tensor products of unitary operators $\hat{U}_i$. We interpret the operator $\hat{U}_i(p_i)$ as the player's $i \in N$ satisfaction with the outcome after the initial agreement and can be defined as follows

$$\hat{U}_i(p_i) = \begin{pmatrix} \cos(p_i) & \sin(p_i) \\ -\sin(p_i) & \cos(p_i) \end{pmatrix}, \tag{9}$$

where $p_i \in [0, \pi/2], \forall i \in N$. This strategy operator is inspired by the one considered in [12]. The greater p_i indicates the greater will to intervene in the game process and change the initial state. The final state of the game is then

$$|\psi_f\rangle = \left(\hat{U}_1(p_1) \otimes \hat{U}_2(p_2)\right) \hat{J}(\gamma) |00\rangle. \tag{10}$$

Thus, a two-player quantum cooperative game associated with the classical cooperative game (N, v), $|N| = 2$, can be defined as (N, v, γ, p_1, p_2), where $\gamma \in [0, \pi/2]$ and $p_i \in [0, \pi/2], \forall i \in N$. Now, we can proceed to the definition of the quantum Shapley value.

3.1.1. The quantum Shapley value of a two-player quantum cooperative game.

We define the quantum Shapley value $(\tilde{\phi}_i)_{i\in N}$ of the two-player quantum cooperative game (N, v, γ, p_1, p_2) as

$$\tilde{\phi}_i(N, v, \gamma, p_1, p_2) = \sum_{S\subseteq N: i\in S} p(S)\, \phi_i(S, v_S), \tag{11}$$

where v_S is a restriction of initial value function over coalition S and $p(S)$ is a probability of occurrence of coalition S defined as

$$p(S) = |\langle a_1 a_2 | \psi_f\rangle|^2, \tag{12}$$

where $S \sim |a_1 a_2\rangle$ and ψ_f is calculated according to (10). Tables 3–5 demonstrate how different boundary values of the newly introduced parameters will affect the final distribution provided by the quantum Shapley value for player $i = 1$. All numbers were rounded to three decimal places.

Table 3. Quantum Shapley value $\tilde{\phi_1}$ for $\gamma = 0$.

$p_1 \backslash p_2$	0	$\pi/4$	$\pi/2$
0	0	0	0
$\pi/4$	$0.5v(\{1\})$	$0.25v(\{1\})+0.25\phi_1(N,v)$	$0.5\phi_1(N,v)$
$\pi/2$	$v(\{1\})$	$0.5v(\{1\})+0.5\phi_1(N,v)$	$\phi_1(N,v)$

Table 4. Quantum Shapley value $\tilde{\phi_1}$ for $\gamma = \pi/4$.

$p_1 \backslash p_2$	0	$\pi/4$	$\pi/2$
0	$0.146\phi_1(N,v)$	$0.073v(\{1\})+0.073\phi_1(N,v)$	$0.146\phi_1(N,v)$
$\pi/4$	$0.427v(\{1\})+0.073\phi_1(N,v)$	$0.25v(\{1\})+0.25\phi_1(N,v)$	$0.073v(\{1\})+0.427\phi_1(N,v)$
$\pi/2$	$0.854\phi_1(N,v)$	$0.427v(\{1\})+0.427\phi_1(N,v)$	$0.854\phi_1(N,v)$

Table 5. Quantum Shapley value $\tilde{\phi_1}$ for $\gamma = \pi/2$.

$p_1 \backslash p_2$	0	$\pi/4$	$\pi/2$
0	$0.5\phi_1(N,v)$	$0.25v(\{1\})+0.25\phi_1(N,v)$	$0.5v(\{1\})$
$\pi/4$	$0.25v(\{1\})+0.25\phi_1(N,v)$	$0.25v(\{1\})+0.25\phi_1(N,v)$	$0.25v(\{1\})+0.25\phi_1(N,v)$
$\pi/2$	$0.5v(\{1\})$	$0.25v(\{1\})+0.25\phi_1(N,v)$	$0.5\phi_1(N,v)$

As we can see, the proposed approach recreates the original solution $\phi(N,v)$ for the choice $\gamma = 0, p_1 = p_2 = \pi/2$. Thus, it extends the classical Shapley value and brings other non-trivial outcomes. Expectedly, players are interested in cooperation with no pre-agreement ($\gamma = 0$) if and only if they are jointly able to obtain non-trivial benefits, i.e. when the game is strictly superadditive $v(N) > v(1) + v(2)$, as can be seen in table 3. However, depending on the definition of the underlying value function v, the player might achieve a greater or lesser payoff in the quantum setting compared to the canonical one.

Tables 4 and 5 show that, for strictly superadditive games, the pre-agreement ($\gamma > 0$) bonds only worsen the player's position, making the player more vulnerable to a possible betrayal (substantial change of the state after the entanglement). Indeed, players cannot achieve payoffs corresponding to their Shapley values under any circumstances. Table 4 demonstrates that, under the medium level of entanglement, it remains more rational to deviate from pre-agreement and change the initial state since the bond is not strong enough. Whereas in table 4, the player is able to obtain from $0.427v(\{1\})+0.427\phi_1(N,v)$ up to $0.854\phi_1(N,v)$, under strong bond, presented in table 5, players are no longer able to achieve such payoff. Moreover, it is better to stick to the pre-agreement under full entanglement and not change it.

On the contrary, if $v(N) \leqslant v(1) + v(2)$ holds, the player can gain an advantage and raise greater claims. In example 1, in case $\gamma = \pi/4$, $p_1 = \pi/2$, $p_2 = \pi/4$ player $i = 1$ achieves $\tilde{\phi_1}(N, v, \gamma, p_1, p_2) = 64\%$ instead of original $\phi_1(N,v) = 50\%$. Thus, it is of interest to study how the quantum game will behave when some coalitions are profitable whereas others are not. Therefore, three-player games will be quantized in the next subsection.

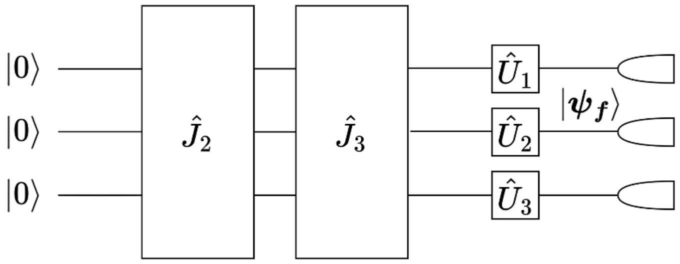

Figure 2. Three-player protocol scheme.

3.2. *Quantization of three-player game*

Three-player cooperative games can be quantized analogically to the two-player games. The proposed scheme is depicted in figure 2.

Compared to the previous scheme from figure 1, the main novelty is that two gates describe the entanglement: $\hat{J}_2$ and $\hat{J}_3$. The first entanglement gate creates entanglement between pairs of qubits and can be represented as

$$\hat{J}_2(\gamma_{12},\gamma_{13},\gamma_{23}) = (\mathrm{SWAP}\otimes\mathrm{Id})\left(\mathrm{Id}\otimes\hat{J}(\gamma_{13})\right)(\mathrm{SWAP}\otimes\mathrm{Id})\left(\mathrm{Id}\otimes\hat{J}(\gamma_{23})\right)\left(\hat{J}(\gamma_{12})\otimes\mathrm{Id}\right), \tag{13}$$

where parameter $\gamma_{ij}\in[0,\pi/2]$ describes entanglement between ith and jth player,

$$\mathrm{Id} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \tag{14}$$

is an identity operator, and

$$\mathrm{SWAP} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \tag{15}$$

interchanges the input states. Clearly, a more straightforward representation of this gate can be found and written. However, we believe that the SWAP gate representation provides a better image of the game process and how bonds between players are formed. This entanglement gate can be interpreted as the one that considers the pre-agreement between pairs of players but does not consider a potential bond that can be created between all three of them at once. For this reason, we assume a second entanglement gate.

Gate $\hat{J}_3(\gamma_{123})$ [10] is assumed to create the so-called GHZ state [16] when full entanglement between players is considered. To avoid the parametrization of this gate, we consider only discrete possibilities for entanglement measure: $\gamma_{123}\in\{0,1\}$. In case of zero entanglement, $\hat{J}_3(\gamma_{123})$ should act as an identity. Then, we define $\hat{J}_3(\gamma_{123})$ as follows:

$$\hat{J}_3(\gamma_{123}) = \begin{cases}\mathrm{Id}\otimes\mathrm{Id}\otimes\mathrm{Id}, \text{for } \gamma_{123}=0,\\ (\mathrm{Id}\otimes\mathrm{CNOT})(\mathrm{CNOT}\otimes\mathrm{Id})(\mathrm{H}\otimes\mathrm{Id}\otimes\mathrm{Id}), \text{for } \gamma_{123}=1,\end{cases} \tag{16}$$

where

$$\mathrm{H} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} \tag{17}$$

is a Hadamard gate and

$$\mathrm{CNOT} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{pmatrix} \tag{18}$$

is a controlled NOT gate. Thus, $\hat{J}_3(1)$ produces the GHZ state

$$(\mathrm{Id}\otimes\mathrm{CNOT})(\mathrm{CNOT}\otimes\mathrm{Id})(\mathrm{H}\otimes\mathrm{Id}\otimes\mathrm{Id})|000\rangle = \frac{|000\rangle + |111\rangle}{\sqrt{2}}. \tag{19}$$

Then, the final state of the three-player game is given as follows:

$$|\psi_f\rangle = \left(\hat{U}_1(p_1)\otimes\hat{U}_2(p_2)\otimes\hat{U}_3(p_3)\right)\hat{J}_3(\gamma_{123})\hat{J}_2(\gamma_{12},\gamma_{13},\gamma_{23})|000\rangle. \tag{20}$$

Thus, analogically to a two-player game, a three-player quantum cooperative game associated with the classical cooperative game (N,v), $|N| = 3$, can be defined as $(N,v,\gamma_{123},\gamma_{12},\gamma_{13},\gamma_{23},p_1,p_2,p_3)$, where $\gamma_{123}\in\{0,1\}$, $\gamma_{i,j}\in[0,\pi/2]$ and $p_i\in[0,\pi/2],\forall i,j\in N$. Then, the corresponding quantum Shapley value is

$$\tilde{\phi}_i(N,v,\gamma_{123},\gamma_{12},\gamma_{13},\gamma_{23},p_1,p_2,p_3) = \sum_{S\subseteq N:i\in S} p(S)\,\phi_i(S,v_S), \tag{21}$$

with $p(S) = |\langle a_1a_2a_3|\psi_f\rangle|^2$, $S\sim|a_1a_2a_3\rangle$ and ψ_f from (20).

The Shapley value is symmetric by its axiomatic definition. Therefore, to demonstrate that the proposed approach is reasonable, it is necessary to prove that the resulting entangled state $\hat{J}_2(\gamma_{12},\gamma_{13},\gamma_{23})|000\rangle$ does not depend on the ordering of the entanglement gates. However, direct calculations become rather extensive and complex in the three-player game. Therefore, in the next section, we will demonstrate how the formal language of QRA will help us to perform quantum computing and study the properties of the considered entanglement gate.

4. Automated proofs of circuit equivalence based on QRA

In this section, it is demonstrated that the ordering of the entanglements within $\hat{J}_2(\gamma_{12},\gamma_{13},\gamma_{23})$ is insignificant. We also indicate how the QRA apparatus can perform the automated proofs of this and other similar properties.

4.1. QRA

Consider a geometric algebra $\mathbb{G}_{2n}$ with the set of basis elements $\{e_1,\ldots,e_{2n}\}$. This algebra can be seen as a subalgebra of geometric algebra $\mathbb{G}_{2n+2}$ with the set of basis elements $\{e_1,\ldots,e_{2n},r_1,r_2\}$. Then, we define QRA($n$) [21] as a geometric algebra $\mathbb{G}_{2n}$ with the coefficients from $\tilde{\mathbb{C}} = \{a+b\iota | a,b\in\mathbb{R},\iota = r_1r_2\}$, i.e.

$$\mathrm{QRA(n)} = \left\{a_1g_1+\cdots+a_{2n}g_{2n} | a_i\in\tilde{\mathbb{C}}, g_i\in\mathbb{G}_{2n}\right\}. \tag{22}$$

An important part of this construction is the definition of QRA conjugation as a Hermitean-linear anti-automorphism that extends identity on vectors [20], i.e.

$$(a_1 g_1 + \cdots + a_{2n} g_{2n})^\dagger = \bar{a}_1 g_1^\dagger + \cdots + \bar{a}_{2n} g_{2n}^\dagger, \tag{23}$$

where $\bar{a}_i = \overline{a + b\iota} = a - b\iota \in \tilde{\mathbb{C}}$, $(ab)^\dagger = b^\dagger a^\dagger$ and $e_i^\dagger = e_i$. To use QRA to model quantum computing, we choose a different basis. This basis is called Witt basis [20] and is formed by elements

$$f_i = \frac{1}{2}\left(e_i + \iota e_{i+n}\right), \ f_i^\dagger = \frac{1}{2}\left(e_i - \iota e_{i+n}\right), \ i = 1, \ldots, n, \tag{24}$$

where the rules for computation with the Witt basis are given as follows:

$$\begin{aligned} &(f_i)^2 = \left(f_i^\dagger\right)^2 = 0, \quad f_i f_j = -f_j f_i, \quad f_i^\dagger f_j^\dagger = -f_j^\dagger f_i^\dagger, \\ &f_i f_i^\dagger f_i = f_i, \qquad f_i^\dagger f_i f_i^\dagger = f_i^\dagger, \qquad f_i^\dagger f_j = -f_j f_i^\dagger. \end{aligned} \tag{25}$$

There is the following straightforward identification of bra and ket vectors of Dirac formalism with elements of QRA:

$$\langle a_1 \ldots a_n | \longleftrightarrow I \left(f_n\right)^{a_n} \ldots \left(f_1\right)^{a_1}, \text{where } a_i \in \{0,1\}, \tag{26}$$

$$|a_1 \ldots a_n\rangle \longleftrightarrow \left(f_1^\dagger\right)^{a_1} \ldots \left(f_n^\dagger\right)^{a_n} I, \text{where } a_i \in \{0,1\}, \tag{27}$$

where $I = f_1 f_1^\dagger \cdots f_n f_n^\dagger$. To describe a three-player game, the space of 3–qubit states should be considered, i.e. we will work with the identification

$$\begin{aligned} |000\rangle &\longleftrightarrow \left(f_1^\dagger\right)^0 \left(f_2^\dagger\right)^0 \left(f_3^\dagger\right)^0 I = I, \\ |001\rangle &\longleftrightarrow \left(f_1^\dagger\right)^0 \left(f_2^\dagger\right)^0 \left(f_3^\dagger\right)^1 I = f_3^\dagger I, \\ |010\rangle &\longleftrightarrow \left(f_1^\dagger\right)^0 \left(f_2^\dagger\right)^1 \left(f_3^\dagger\right)^0 I = f_2^\dagger I, \\ |011\rangle &\longleftrightarrow \left(f_1^\dagger\right)^0 \left(f_2^\dagger\right)^1 \left(f_3^\dagger\right)^1 I = f_2^\dagger f_3^\dagger I, \\ |100\rangle &\longleftrightarrow \left(f_1^\dagger\right)^1 \left(f_2^\dagger\right)^0 \left(f_3^\dagger\right)^0 I = f_1^\dagger I, \\ |101\rangle &\longleftrightarrow \left(f_1^\dagger\right)^1 \left(f_2^\dagger\right)^0 \left(f_3^\dagger\right)^1 I = f_1^\dagger f_3^\dagger I, \\ |110\rangle &\longleftrightarrow \left(f_1^\dagger\right)^1 \left(f_2^\dagger\right)^1 \left(f_3^\dagger\right)^0 I = f_1^\dagger f_2^\dagger I, \\ |111\rangle &\longleftrightarrow \left(f_1^\dagger\right)^1 \left(f_2^\dagger\right)^1 \left(f_3^\dagger\right)^1 I = f_1^\dagger f_2^\dagger f_3^\dagger I. \end{aligned} \tag{28}$$

The space of 3–qubit bra vectors can also be described by QRA conjugation as $\langle a_1 a_2 a_3| = |a_1 a_2 a_3\rangle^\dagger$, for example

$$\langle 111| = |111\rangle^\dagger \longleftrightarrow \left(f_1^\dagger f_2^\dagger f_3^\dagger\right)^\dagger I = f_3 f_2 f_1 I. \tag{29}$$

The analogical identification on two-qubit states has been already presented in [13].

We conclude this subsection by answering how a circuit can be composed of individual blocks. According to [21], a serial circuit, formed by sequential application of the gates f_A and f_B on the same qubit, can be represented in QRA as $f_B f_A$. A parallel circuit [21], consisting of the gates f_A and f_B, acting on different qubits, can be represented as $f_A f_B$ up to a sign of individual monomials. When working with the two gates in a parallel circuit (their tensor product), it is sufficient to perform the following procedure, initially presented in [21].

(1) On the right side of multiplication, we assign artificial coefficient b to the monomials with the odd number of terms.
(2) On the left side of multiplication, a is assigned to monomials with the odd number of occurrences of elements of type $f_i^\dagger f_i$ or f_i.
(3) Then, after the multiplication, we perform simple reassignment: $ab \to -1,\ a, b \to 1$.

Since multiplication in QRA is an associative operation, applying the above-defined rule subsequently on pairs of quantum gates is sufficient. This technical step is necessary due to the nature of the problem. On the other hand, it allows us to implement things more straightforwardly. Examples of the serial and parallel circuits constructed via QRA can be found in [21].

4.2. General representation of SWAP gate

At first, to highlight the convenience of QRA notation, we use this algebra to find general representatives of SWAP(s,t) gates, describing the interchange of qubits s and t. At the end of this subsection, we apply QRA and GAALOP to perform the automated proof of the irrelevance of the entanglement gates ordering within $\hat{J}_2(\gamma_{12}, \gamma_{13}, \gamma_{23})$.

Lemma 4.1. *Let $|\psi\rangle$ be an n–qubit state and $1 \leqslant s < n$, $s \in \mathbb{Z}$. Then, the element*

$$SWAP(s, s+1) = f_s f_s^\dagger f_{s+1} f_{s+1}^\dagger - f_s f_{s+1}^\dagger + f_s^\dagger f_{s+1} + f_s^\dagger f_s f_{s+1}^\dagger f_{s+1} \tag{30}$$

acts on $|\psi\rangle$ as a SWAP between qubits s and $s+1$.

Proof. Let us note that the elements $f_s f_s^\dagger f_{s+1} f_{s+1}^\dagger$ and $f_s^\dagger f_s f_{s+1}^\dagger f_{s+1}$ act as identities if they act nontrivially (result is not zero). The elements $f_s f_{s+1}^\dagger$ and $f_s^\dagger f_{s+1}$ interchange two adjacent elements. We will use this property frequently throughout this section. The following direct computations are based on the fact that all elements of (30) have an even number of monomials.

$$\begin{aligned} \mathrm{SWAP}(s, s+1) \left(f_1^\dagger\right)^{a_1} \cdots \left(f_n^\dagger\right)^{a_n} I &= \left(f_1^\dagger\right)^{a_1} \cdots \left(f_{s-1}^\dagger\right)^{a_{s-1}} \mathrm{SWAP}(s, s+1) \left(f_s^\dagger\right)^{a_s} \cdots \left(f_n^\dagger\right)^{a_n} I \\ &= \left(f_1^\dagger\right)^{a_1} \cdots \left(f_{s-1}^\dagger\right)^{a_{s-1}} \left[\mathrm{SWAP}(s, s+1) \left(f_s^\dagger\right)^{a_s} \left(f_{s+1}^\dagger\right)^{a_{s+1}}\right] \\ &\quad \times \left(f_{s+2}^\dagger\right)^{a_{s+2}} \cdots \left(f_n^\dagger\right)^{a_n} I \end{aligned} \tag{31}$$

and SWAP$(s, s+1)$ acts on $(f_s^\dagger)^{a_s} (f_{s+1}^\dagger)^{a_{s+1}}$ as SWAP which completes the proof. $\square$

Theorem 4.2. *Let $|\psi\rangle$ be an n–qubit. Then, the element*

$$SWAP(s, t) = \left(f_s f_s^\dagger f_t f_t^\dagger - f_s^\dagger f_t - f_s f_t^\dagger - f_s^\dagger f_s f_t^\dagger f_t\right)$$

$$
\begin{aligned}
&\times\left(\sum_{\sum(a_i) is\ odd}\left(f^\dagger_{s+1}f_{s+1}\right)^{a_{s+1}}\left(f_{s+1}f^\dagger_{s+1}\right)^{b_{s+1}}\cdots\left(f^\dagger_{t-1}f_{t-1}\right)^{a_{t-1}}\left(f_{t-1}f^\dagger_{t-1}\right)^{b_{t-1}}\right)\\
&+\left(f_sf^\dagger_sf_tf^\dagger_t+f^\dagger_sf_t-f_sf^\dagger_t+f^\dagger_sf_sf^\dagger_tf_t\right)\\
&\times\left(\sum_{\sum(a_i) is\ even}\left(f^\dagger_{s+1}f_{s+1}\right)^{a_{s+1}}\left(f_{s+1}f^\dagger_{s+1}\right)^{b_{s+1}}\cdots\left(f^\dagger_{t-1}f_{t-1}\right)^{a_{t-1}}\left(f_{t-1}f^\dagger_{t-1}\right)^{b_{t-1}}\right)
\end{aligned}
\tag{32}
$$

acts as a SWAP gate between sth and tth qubit ($s<t$).

Proof. Because each part of the expression (32) has an even number of elements, it is easy to show that

$$
\mathrm{SWAP}(s,t)\left(f^\dagger_1\right)^{a_1}\cdots\left(f^\dagger_n\right)^{a_n}I=\left(f^\dagger_1\right)^{a_1}\cdots\left(f^\dagger_{s-1}\right)^{a_{s-1}}\mathrm{SWAP}(s,t)\left(f^\dagger_s\right)^{a_s}\cdots\left(f^\dagger_n\right)^{a_n}I
\tag{33}
$$

and, because of associativity, we have an expression

$$
\mathrm{SWAP}(s,t)\left(f^\dagger_s\right)^{a_s}\cdots\left(f^\dagger_n\right)^{a_n}I=\left[\mathrm{SWAP}(s,t)\left(f^\dagger_s\right)^{a_s}\cdots\left(f^\dagger_t\right)^{a_t}\right]\left(f^\dagger_{t+1}\right)^{a_{t+1}}\cdots\left(f^\dagger_n\right)^{a_n}I.
\tag{34}
$$

Thus, without loss of generality, we can only discuss the gate

$$
\begin{aligned}
\mathrm{SWAP}(1,n)=&\left(f_1f^\dagger_1f_nf^\dagger_n-f^\dagger_1f_n-f_1f^\dagger_n-f^\dagger_1f_1f^\dagger_nf_n\right)\\
&\times\left(\sum_{\sum(a_i)\text{is odd}}\left(f^\dagger_2f_2\right)^{a_2}\left(f_2f^\dagger_2\right)^{b_2}\cdots\left(f^\dagger_{n-1}f_{n-1}\right)^{a_{n-1}}\left(f_{n-1}f^\dagger_{n-1}\right)^{b_{n-1}}\right)\\
&+\left(f_1f^\dagger_1f_nf^\dagger_n+f^\dagger_1f_n-f_1f^\dagger_n+f^\dagger_1f_1f^\dagger_nf_n\right)\\
&\times\left(\sum_{\sum(a_i)\text{is even}}\left(f^\dagger_2f_2\right)^{a_2}\left(f_2f^\dagger_2\right)^{b_2}\cdots\left(f^\dagger_{n-1}f_{n-1}\right)^{a_{n-1}}\left(f_{n-1}f^\dagger_{n-1}\right)^{b_{n-1}}\right)
\end{aligned}
\tag{35}
$$

Let $|\psi\rangle$ be an n–qubit, if $a_i=b_i=1$ then $f^\dagger_if_if_if^\dagger_i=0$. Therefore, in the expressions (32) and (35), there are only such elements that $a_i+b_i\in\{0,1\}$. The gate

$$
\left(f^\dagger_2f_2\right)^{a_2}\left(f_2f^\dagger_2\right)^{b_2}\cdots\left(f^\dagger_{n-1}f_{n-1}\right)^{a_{n-1}}\left(f_{n-1}f^\dagger_{n-1}\right)^{b_{n-1}}
\tag{36}
$$

acts as the projection to the state

$$
\begin{aligned}
&\sum_{a_1,a_n\in\{0,1\}}\psi_{a_1\ldots a_n}\left(f^\dagger_1\right)^{a_1}\left(f^\dagger_2\right)^{a_2}\cdots\left(f^\dagger_{n-1}\right)^{a_{n-1}}\left(f^\dagger_n\right)^{a_n}I\\
&=\psi_{0a_2\ldots a_{n-1}0}\left(f^\dagger_2\right)^{a_2}\cdots\left(f^\dagger_{n-1}\right)^{a_{n-1}}I+\psi_{0a_2\ldots a_{n-1}1}\left(f^\dagger_2\right)^{a_2}\cdots\left(f^\dagger_{n-1}\right)^{a_{n-1}}f^\dagger_nI\\
&\quad+\psi_{1a_2\ldots a_{n-1}0}f^\dagger_1\left(f^\dagger_2\right)^{a_2}\cdots\left(f^\dagger_{n-1}\right)^{a_{n-1}}I+\psi_{1a_2\ldots a_{n-1}1}f^\dagger_1\left(f^\dagger_2\right)^{a_2}\cdots\left(f^\dagger_{n-1}\right)^{a_{n-1}}f^\dagger_nI\\
&=\psi_{0a_2\ldots a_{n-1}0}\left(f^\dagger_2\right)^{a_2}\cdots\left(f^\dagger_{n-1}\right)^{a_{n-1}}I+(-1)^{\sum a_i}\psi_{0a_2\ldots a_{n-1}1}f^\dagger_n\left(f^\dagger_2\right)^{a_2}\cdots\left(f^\dagger_{n-1}\right)^{a_{n-1}}I\\
&\quad+\psi_{1a_2\ldots a_{n-1}0}f^\dagger_1\left(f^\dagger_2\right)^{a_2}\cdots\left(f^\dagger_{n-1}\right)^{a_{n-1}}I+(-1)^{\sum a_i}\psi_{1a_2\ldots a_{n-1}1}f^\dagger_1f^\dagger_n\left(f^\dagger_2\right)^{a_2}\cdots\left(f^\dagger_{n-1}\right)^{a_{n-1}}I.
\end{aligned}
\tag{37}
$$

Thus, if $\sum a_i$ is odd, the middle elements of the SWAP gate must have a different sign with respect to the corresponding projections. Then, because of lemma 4.1, the element acts as a SWAP between the first and nth qubit which completes the proof. ☐

Now, when the general representation of the SWAP gate between any pair of qubits for an arbitrary n-qubit state is known, we can verify if gates used within $\hat{J}_2(\gamma_{12},\gamma_{13},\gamma_{23})$ operator commute or not.

4.3. Cicuit identities

Finally, with the help of GAALOP [2, 17], we demonstrate that the order of the entanglement operators within $\hat{J}_2(\gamma_{12},\gamma_{13},\gamma_{23})$ does not affect the outcome of the game. After the code from listing 1 in the appendix has been compiled using GAALOPWeb, we have obtained a MATLAB function from listing 2 in the appendix. Its empty output proves that, under every possible ordering of entanglement gates, $\hat{J}_2(\gamma_{12},\gamma_{13},\gamma_{23})$ is represented by the same element of QRA. Thus, implementing QRA in GAALOP can serve as an instrument to check the equivalence of the circuits. It can be seen as a viable alternative to symbolic calculations in other available languages. Moreover, the found QRA representation of SWAP(s,t) can be used to generalize the proposed approach into the domain of n-player quantum cooperative games in the future. In the next section, we will demonstrate another possible application of QRA to quantum cooperative game theory.

5. The QRA representation of two- and three-player quantum cooperative games

In this section, we use QRA to establish the parametric expressions describing the quantum Shapley values of the two- and three-player quantum cooperative games. The detailed deduction of QRA representations of 1– and 2–qubit gates can be found in [20].

5.1. Two-player game quantum Shapley value

The scheme is analogous to the two-player quantum non-cooperative game (except for the disentanglement operator) presented in [11]. Therefore, using the considerations established in [13], we can obtain the final 2–qubit state

$$
\begin{aligned}
|\psi_f\rangle = \left(\hat{U}_1(p_1)\otimes\hat{U}_2(p_2)\right)\hat{J}(\gamma)|00\rangle = \cos\frac{\gamma}{2}\Big(&\cos(p_1)\cos(p_2) f_1 f_1^\dagger f_2 f_2^\dagger \\
&-\cos(p_1)\sin(p_2) f_1 f_1^\dagger f_2^\dagger - \sin(p_1)\cos(p_2) f_1^\dagger f_2 f_2^\dagger \\
&+\sin(p_1)\sin(p_2) f_1^\dagger f_2^\dagger\Big) + i\sin\frac{\gamma}{2}\Big(\sin(p_1)\sin(p_2) f_1 f_1^\dagger f_2 f_2^\dagger \\
&+\sin(p_1)\cos(p_2) f_1 f_1^\dagger f_2^\dagger + \cos(p_1)\sin(p_2) f_1^\dagger f_2 f_2^\dagger \\
&+\cos(p_1)\cos(p_2) f_1^\dagger f_2^\dagger\Big).
\end{aligned} \tag{38}
$$

Thus, the quantum Shapley value of the two-player game can be represented as

$$
\begin{aligned}
\tilde{\phi}_1(N,v,\gamma,p_1,p_2) = &\left(\cos^2\frac{\gamma}{2}\sin^2(p_1)\cos^2(p_2) + \sin^2\frac{\gamma}{2}\cos^2(p_1)\sin^2(p_2)\right)\phi_i\left(\{1\},v_{\{1\}}\right) \\
&+\left(\cos^2\frac{\gamma}{2}\sin^2(p_1)\sin^2(p_2) + \sin^2\frac{\gamma}{2}\cos^2(p_1)\cos^2(p_2)\right)\phi_i(N,v).
\end{aligned} \tag{39}
$$

$$\begin{aligned}\tilde{\phi}_2\left(N,v,\gamma,p_1,p_2\right) = &\left(\cos^2\frac{\gamma}{2}\cos^2\left(p_1\right)\sin^2\left(p_2\right)+\sin^2\frac{\gamma}{2}\sin^2\left(p_1\right)\cos^2\left(p_2\right)\right)\phi_i\left(\left\{2\right\},v_{\{2\}}\right)\\ &+\left(\cos^2\frac{\gamma}{2}\sin^2\left(p_1\right)\sin^2\left(p_2\right)+\sin^2\frac{\gamma}{2}\cos^2\left(p_1\right)\cos^2\left(p_2\right)\right)\phi_i\left(N,v\right). \end{aligned} \tag{40}$$

Then, according to the definition of Shapley value, we have

$$\phi_i\left(\left\{i\right\},v_{\{i\}}\right) = v\left(\left\{i\right\}\right),\ \phi_i\left(N,v\right) = \frac{v\left(\left\{i\right\}\right)}{2}+\frac{v\left(N\right)-v\left(N\setminus\left\{i\right\}\right)}{2}. \tag{41}$$

Thus, we can obtain the following expression:

$$\begin{aligned}\tilde{\phi}_i(N,v) = &(\cos^2\frac{\gamma}{2}\sin^2(p_i)\cos^2(p_{N\setminus\{i\}})+\sin^2\frac{\gamma}{2}\cos^2(p_i)\sin^2(p_{N\setminus\{i\}})v(\{i\})\\ &+(\cos^2\frac{\gamma}{2}\sin^2(p_1)\sin^2(p_2)+\sin^2\frac{\gamma}{2}\cos^2(p_1)\cos^2(p_2))\left(\frac{v(\{i\})}{2}+\frac{v(N)-v(N\setminus\{i\})}{2}\right). \end{aligned} \tag{42}$$

It is easy to verify that the obtained expression is in full accordance with the results presented in tables 3–5. The QRA representation of the three-player quantum cooperative game will be described in detail.

5.2. *Three-player game quantum Shapley value*

The three-player game starts with the qubit

$$|000\rangle = f_1f_1^\dagger f_2f_2^\dagger f_3f_3^\dagger. \tag{43}$$

Then, the first entanglement gate $\hat{J}_2(\gamma_{12},\gamma_{13},\gamma_{23})$ is applied. The gate $\hat{J}_2(\gamma_{12},\gamma_{13},\gamma_{23})$ represents a series of gates $(\mathrm{SWAP}\otimes\mathrm{Id})\left(\mathrm{Id}\otimes\hat{J}(\gamma_{13})\right)(\mathrm{SWAP}\otimes\mathrm{Id})\left(\mathrm{Id}\otimes\hat{J}(\gamma_{23})\right)\left(\hat{J}(\gamma_{12})\otimes\mathrm{Id}\right)$, with each of them being tensor product of at least two gates. Further, we will use the notation

$$\left(\mathrm{SWAP}\left(1,2\right)\otimes\mathrm{Id}\left(3\right)\right)\left(\mathrm{Id}\left(1\right)\otimes\hat{J}\left(\gamma_{13}\right)\right)\left(\mathrm{SWAP}\left(1,2\right)\otimes\mathrm{Id}\left(3\right)\right)\left(\mathrm{Id}\left(1\right)\otimes\hat{J}\left(\gamma_{23}\right)\right)\left(\hat{J}\left(\gamma_{12}\right)\otimes\mathrm{Id}\left(3\right)\right) \tag{44}$$

to prevent possible ambiguity and specify which qubits the gates are applied to. The first gate in the series is $\left(\hat{J}(\gamma_{12})\otimes\mathrm{Id}(3)\right)$, where

$$\mathrm{Id}\left(3\right) = f_3f_3^\dagger+f_3^\dagger f_3. \tag{45}$$

This is a tensor product of one 2–qubit gate and one 1–qubit gate. However, when the identity operator is on the tensor product's right side, no sign change can occur, and it is sufficient to rewrite such gate as a multiplication directly. Thus, we directly obtain the expression

$$\begin{aligned}\hat{J}\left(\gamma_{12}\right)\otimes\mathrm{Id} = &\cos\frac{\gamma_{12}}{2}\left(f_1f_1^\dagger f_2f_2^\dagger+f_1^\dagger f_1f_2f_2^\dagger+f_1f_1^\dagger f_2^\dagger f_2+f_1^\dagger f_1f_2^\dagger f_2\right)\left(f_3f_3^\dagger+f_3^\dagger f_3\right)\\ &+i\sin\frac{\gamma_{12}}{2}\left(-f_1f_2+f_1f_2^\dagger-f_1^\dagger f_2+f_1^\dagger f_2^\dagger\right)\left(f_3f_3^\dagger+f_3^\dagger f_3\right). \end{aligned} \tag{46}$$

The next gate is $\mathrm{Id}(1)\otimes\hat{J}(\gamma_{23})$. However, the gate $\hat{J}(\gamma_{23})$ cannot affect signs of monomials. Thus, the following representation can be obtained:

$$\begin{aligned}\mathrm{Id}(1)\otimes\hat{J}\left(\gamma_{23}\right) = &\cos\frac{\gamma_{23}}{2}\left(f_1f_1^\dagger+f_1^\dagger f_1\right)\left(f_2f_2^\dagger f_3f_3^\dagger+f_2^\dagger f_2f_3f_3^\dagger+f_2f_2^\dagger f_3^\dagger f_3+f_2^\dagger f_2f_3^\dagger f_3\right)\\ &+i\sin\frac{\gamma_{23}}{2}\left(f_1f_1^\dagger+f_1^\dagger f_1\right)\left(-f_2f_3+f_2f_3^\dagger-f_2^\dagger f_3+f_2^\dagger f_3^\dagger\right). \end{aligned} \tag{47}$$

Then, the input states have to be interchanged using $\mathrm{SWAP}(1,2)\otimes\mathrm{Id}(3)$ to entangle the remaining pair of qubits. The $\mathrm{SWAP}(1,2)$ gate can be represented as

$$\mathrm{SWAP}\left(1,2\right)=f_1f_1^\dagger f_2f_2^\dagger+f_1^\dagger f_2-f_1f_2^\dagger+f_1^\dagger f_1f_2^\dagger f_2, \tag{48}$$

and, again, by straightforward multiplication, we obtain

$$\begin{aligned}\mathrm{SWAP}\left(1,2\right)\otimes\mathrm{Id}\left(3\right)=&f_1f_1^\dagger f_2f_2^\dagger f_3f_3^\dagger+f_1^\dagger f_2f_3f_3^\dagger-f_1f_2^\dagger f_3f_3^\dagger+f_1^\dagger f_1f_2^\dagger f_2f_3f_3^\dagger\\&+f_1f_1^\dagger f_2f_2^\dagger f_3^\dagger f_3+f_1^\dagger f_2f_3^\dagger f_3-f_1f_2^\dagger f_3^\dagger f_3+f_1^\dagger f_1f_2^\dagger f_2f_3^\dagger f_3.\end{aligned} \tag{49}$$

After that, the gate $\mathrm{Id}(1)\otimes\hat{J}(\gamma_{13})$ is applied, which completely copies the gate $\mathrm{Id}(1)\otimes\hat{J}(\gamma_{23})$ with changed entanglement parameter. At last, to preserve the initial identification of qubits with coalitions, we interchange the states back using $\mathrm{SWAP}(1,2)\otimes\mathrm{Id}(3)$ once more time. Thus, the whole effect of the $\hat{J}_2(\gamma_{12},\gamma_{13},\gamma_{23})$ on the initial state $|000\rangle$ can be described by the multiplication of the basis state $|000\rangle$ by the above-described gates in the corresponding order. Now, the entanglement gate $\hat{J}_3(\gamma_{123})$ has to be applied. Due to the discrete nature of the entanglement parameter γ_{123}, we have divided this section into smaller subsections describing each possible choice of the parameter γ_{123} separately.

5.2.1. *Case $\gamma_{123}=0$ and action of 1–qubit gates.*

In case $\gamma_{123}=0$, the operator $\hat{J}_3(\gamma_{123})$ collapses into $\hat{J}_3(0)=\mathrm{Id}(1)\otimes\mathrm{Id}(2)\otimes\mathrm{Id}(3)$. Thus, it can be described by the following element of QRA

$$\begin{aligned}\hat{J}_3\left(0\right)=&f_1f_1^\dagger f_2f_2^\dagger f_3f_3^\dagger+f_1f_1^\dagger f_2f_2^\dagger f_3^\dagger f_3+f_1f_1^\dagger f_2^\dagger f_2f_3f_3^\dagger+f_1f_1^\dagger f_2^\dagger f_2f_3^\dagger f_3\\&+f_1^\dagger f_1f_2f_2^\dagger f_3f_3^\dagger+f_1^\dagger f_1f_2f_2^\dagger f_3^\dagger f_3+f_1^\dagger f_1f_2^\dagger f_2f_3f_3^\dagger+f_1^\dagger f_1f_2^\dagger f_2f_3^\dagger f_3,\end{aligned} \tag{50}$$

which does not affect the state $\hat{J}_2(\gamma_{12},\gamma_{13},\gamma_{23})|000\rangle$.

Therefore, before the application of unitary operators, we obtain the state

$$\begin{aligned}\hat{J}_3\left(0\right)\hat{J}_2\left(\gamma_{12},\gamma_{13},\gamma_{23}\right)|000\rangle=&\left(\cos\frac{\gamma_{12}}{2}\cos\frac{\gamma_{13}}{2}\cos\frac{\gamma_{23}}{2}+i\sin\frac{\gamma_{12}}{2}\sin\frac{\gamma_{13}}{2}\sin\frac{\gamma_{23}}{2}\right)f_1f_1^\dagger f_2f_2^\dagger f_3f_3^\dagger\\&+\left(\sin\frac{\gamma_{12}}{2}\sin\frac{\gamma_{13}}{2}\cos\frac{\gamma_{23}}{2}+i\cos\frac{\gamma_{12}}{2}\cos\frac{\gamma_{13}}{2}\sin\frac{\gamma_{23}}{2}\right)f_1f_1^\dagger f_2^\dagger f_3^\dagger\\&+\left(\sin\frac{\gamma_{12}}{2}\cos\frac{\gamma_{13}}{2}\sin\frac{\gamma_{23}}{2}+i\cos\frac{\gamma_{12}}{2}\sin\frac{\gamma_{13}}{2}\cos\frac{\gamma_{23}}{2}\right)f_1^\dagger f_2f_2^\dagger f_3^\dagger\\&+\left(\cos\frac{\gamma_{12}}{2}\sin\frac{\gamma_{13}}{2}\sin\frac{\gamma_{23}}{2}+i\sin\frac{\gamma_{12}}{2}\cos\frac{\gamma_{13}}{2}\cos\frac{\gamma_{23}}{2}\right)f_1^\dagger f_2^\dagger f_3f_3^\dagger.\end{aligned} \tag{51}$$

The tensor product of three gates $\hat{U}_i, i=1,2,3$, is applied to the state described above. This tensor product can be represented as

$$\begin{aligned}\hat{U}_1\left(p_1\right)\otimes\hat{U}_2\left(p_2\right)\otimes\hat{U}_3\left(p_3\right)=&\left(\sin\left(p_1\right)\left(f_1-f_1^\dagger\right)+\cos\left(p_1\right)\left(f_1f_1^\dagger+f_1^\dagger f_1\right)\right)\\&\otimes\left(\sin\left(p_2\right)\left(f_2-f_2^\dagger\right)+\cos\left(p_2\right)\left(f_2f_2^\dagger+f_2^\dagger f_2\right)\right)\\&\otimes\left(\sin\left(p_3\right)\left(f_3-f_3^\dagger\right)+\cos\left(p_3\right)\left(f_3f_3^\dagger+f_3^\dagger f_3\right)\right),\end{aligned} \tag{52}$$

Whereas the serial circuit of more than two gates can be represented directly via multiplication, the sign-changing rule for the tensor product of two quantum gates cannot be generalized for the three gates. Thus, for the three-player cooperative game, it is necessary to establish the sign-changing rule for the general parallel circuit of three gates. When working with three gates, interactions between monomials become more complex and will have more possible effects.

Therefore, five parameters will be needed to define the sign-changing procedure instead of two artificial parameters.

(1) At first, we assign a to monomials from the left side of multiplication with an odd number of terms of type f_j and $f_j^\dagger f_j$.
(2) Then, in the middle term of multiplication, we assign b to monomials with odd number of terms of type f_j and $f_j^\dagger$, c to monomials, which have odd number of occurrences of terms of type $f_j^\dagger$ and f_j and, at the same time, odd number of occurrences of terms of type f_j and $f_j^\dagger f_j$, and d to monomials, which have odd number of occurrences of terms of type f_j and $f_j^\dagger f_j$.
(3) At last, we assign e to monomials from the right side of multiplication with an odd number of occurrences of terms of type $f_j^\dagger$ and f_j.
(4) Then, after performing the multiplication, we perform the reassignment:

$$a, b, c, d, e, ad, be, abe, ade \to 1, \tag{53}$$
$$ab, ac, ae, ce, de, ace \to -1. \tag{54}$$

Thus, before the multiplication and the reassignment, the tensor product can be represented as follows:

$$\begin{aligned}\hat{U}_1(p_1) \otimes \hat{U}_2(p_2) \otimes \hat{U}_3(p_3) = &\left(\sin(p_1)\left(af_1 - f_1^\dagger\right) + \cos(p_1)\left(f_1 f_1^\dagger + a f_1^\dagger f_1\right)\right)\\ &\left(\sin(p_2)\left(cf_2 - bf_2^\dagger\right) + \cos(p_2)\left(f_2 f_2^\dagger + d f_2^\dagger f_2\right)\right)\\ &\left(\sin(p_3)\left(ef_3 - ef_3^\dagger\right) +\ cos(p_3)\left(f_3 f_3^\dagger + f_3^\dagger f_3\right)\right).\end{aligned} \tag{55}$$

Alternatively, the representation of this gate in QRA can be obtained via subsequent application of the previously presented sign-changing rules for the pairs of gates. Since the considered tensor product has 64 non-zero elements, we omit its full representation and directly proceed to the case $\gamma_{123} = 1$.

5.2.2. Case $\gamma_{123} = 1$. In case $\gamma_{123} = 1$, the QRA representation of the gate

$$\hat{J}_3(1) = (\mathrm{Id}(1) \otimes \mathrm{CNOT}(2,3))(\mathrm{CNOT}(1,2) \otimes \mathrm{Id}(3))(\mathrm{H}(1) \otimes \mathrm{Id}(2) \otimes \mathrm{Id}(3)) \tag{56}$$

has to be found. The first part of the serial gate is $(\mathrm{H}(1) \otimes \mathrm{Id}(2) \otimes \mathrm{Id}(3))$, where

$$\mathrm{H}(1) = \frac{1}{\sqrt{2}}\left(f_1 f_1^\dagger + f_1 + f_1^\dagger - f_1^\dagger f_1\right). \tag{57}$$

Since the Hadamard gate is in a tensor product with identity operators on the right side, it can be represented as a straightforward multiplication

$$\mathrm{H}(1) \otimes \mathrm{Id}(2) \otimes \mathrm{Id}(3) = \frac{1}{\sqrt{2}}\left(f_1 f_1^\dagger + f_1 + f_1^\dagger - f_1^\dagger f_1\right)\left(f_2 f_2^\dagger + f_2^\dagger f_2\right)\left(f_3 f_3^\dagger + f_3^\dagger f_3\right). \tag{58}$$

Then, the $\mathrm{CNOT}(1,2)$ gate can be written down as

$$\mathrm{CNOT}(1,2) = f_1 f_1^\dagger f_2 f_2^\dagger + f_1 f_1^\dagger f_2^\dagger f_2 - f_1^\dagger f_1 f_2 - f_1^\dagger f_1 f_2^\dagger. \tag{59}$$

Thus, once more, we have

$$\mathrm{CNOT}(1,2) \otimes \mathrm{Id}(3) = \left(f_1 f_1^\dagger f_2 f_2^\dagger + f_1 f_1^\dagger f_2^\dagger f_2 - f_1^\dagger f_1 f_2 - f_1^\dagger f_1 f_2^\dagger\right)\left(f_3 f_3^\dagger + f_3^\dagger f_3\right). \tag{60}$$

At last, according to the sign-changing procedure for the parallel circuit of two gates, we have

$$\begin{aligned}\mathrm{Id}\,(1)\otimes\mathrm{CNOT}\,(2,3) &= \left(f_1f_1^\dagger+f_1^\dagger f_1\right)\otimes\left(f_2f_2^\dagger f_3f_3^\dagger+f_2f_2^\dagger f_3^\dagger f_3-f_2^\dagger f_2f_3-f_2^\dagger f_2f_3^\dagger\right)\\ &= \left(f_1f_1^\dagger+af_1^\dagger f_1\right)\left(f_2f_2^\dagger f_3f_3^\dagger+f_2f_2^\dagger f_3^\dagger f_3-bf_2^\dagger f_2f_3-bf_2^\dagger f_2f_3^\dagger\right)\\ &=f_1f_1^\dagger f_2f_2^\dagger f_3f_3^\dagger+f_1f_1^\dagger f_2f_2^\dagger f_3^\dagger f_3-bf_1f_1^\dagger f_2^\dagger f_2f_3-bf_1f_1^\dagger f_2^\dagger f_2f_3^\dagger\\ &\quad+af_1^\dagger f_1f_2f_2^\dagger f_3f_3^\dagger+af_1^\dagger f_1f_2f_2^\dagger f_3^\dagger f_3-abf_1^\dagger f_1f_2^\dagger f_2f_3-abf_1^\dagger f_1f_2^\dagger f_2f_3^\dagger\\ &=f_1f_1^\dagger f_2f_2^\dagger f_3f_3^\dagger+f_1f_1^\dagger f_2f_2^\dagger f_3^\dagger f_3-f_1f_1^\dagger f_2^\dagger f_2f_3-f_1f_1^\dagger f_2^\dagger f_2f_3^\dagger\\ &\quad+f_1^\dagger f_1f_2f_2^\dagger f_3f_3^\dagger+f_1^\dagger f_1f_2f_2^\dagger f_3^\dagger f_3+f_1^\dagger f_1f_2^\dagger f_2f_3+f_1^\dagger f_1f_2^\dagger f_2f_3^\dagger.\end{aligned}\tag{61}$$

Thus, we can obtain the complete representation of the gate $\hat{J}_3(1)$ by multiplying the above-presented terms in the corresponding order. Due to the extensive size of the gate $\hat{J}_3(1)$ and of the gates established in the previous subsections, the representation of the final state or the full parametric expression representing the quantum Shapley value can be non-informative and confusing. Therefore, we have decided to omit them. However, as already demonstrated, we can perform quantum computing using the GAALOP. The following section will describe how to measure the quantum states using GAALOP to assign the resulting probabilities to the quantum Shapley value.

6. Outcomes of the games and discussion

At first, we calculate the resulting probabilities for the two-player quantum cooperative game using GAALOP to demonstrate the quantization code of the most simple instance. The QRA code from listing 3 in the appendix provides a script describing the measurement of the resulting quantum states. It has been compiled as a MATLAB function using GAALOPWeb. After the symbolic substitution of ab with -1 and a,b with 1, the final probabilities can be easily obtained for any given γ, p_1, and p_2. For example, for the choice $\gamma=0, p_1=3\pi/8, p_2=\pi/8$, we obtain the results presented in listing 4 in the appendix. The correctness of this result can be easily verified using (38). The obtained probabilities can be straightforwardly substituted into (42) to compute the quantum Shapley values of players. The previous case can be generalized for the case of a 3-qubit system describing the three-player quantum cooperative game. The corresponding code can be found in listing 5 in the appendix. To demonstrate the functionality of our approach, we have computed the quantized version of the game from example 2 under different settings. The first instance is depicted in figure 3.

When there is no bond between players and the player with the most significant weight is indifferent to a game process, player 1 might benefit from cooperation with player 3 (3 also benefits due to the symmetric setting). The instance with the 'stronger' bond between players 1 and 3 is depicted in figure 4. The maximal change of the initial state remains the best option for the players 1 and 3. However, whereas the payoff in case $p_1=p_2=\pi/2$ has decreased, players begin to benefit from not changing state at all. The instance with the maximal bond between players 1 and 3 is depicted in figure 5. It can be seen that under $\gamma_{13}=\pi/2$ players' payoffs have decreased and now they maximally benefit from cooperation in a new sense: their actions have to be equivalent. Thus, both should completely change the initial state or not operate with it at all.

The case when all players are entangled via a 3-qubit gate is depicted in figure 6. This setting has an analogical effect, as the case depicted in figure 4. The last case that was considered is

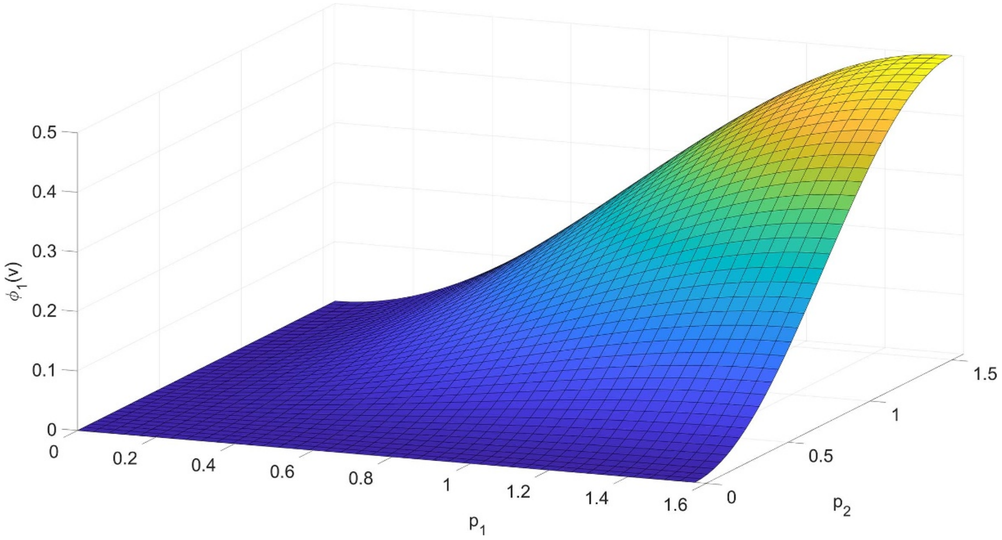

Figure 3. The quantum Shapley value $\tilde{\phi}_1$ for $\gamma_{123} = 0$, $\gamma_{12} = 0$, $\gamma_{13} = 0$, $\gamma_{23} = 0$, and $p_2 = 0$.

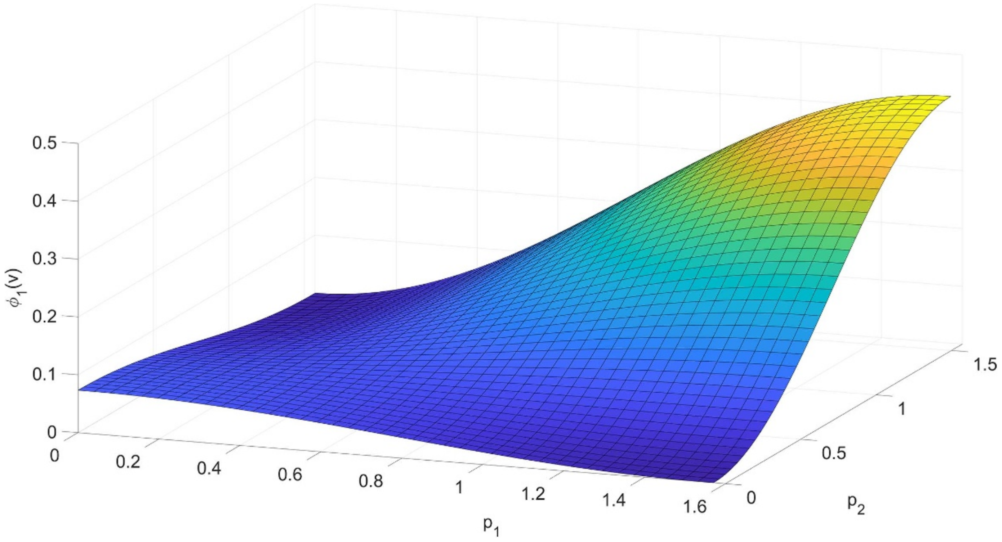

Figure 4. The quantum Shapley value $\tilde{\phi}_1$ for $\gamma_{123} = 0$, $\gamma_{12} = 0$, $\gamma_{13} = \pi/4$, $\gamma_{23} = 0$, and $p_2 = 0$.

presented in figure 7. Compared to all previously considered instances, this last setting demonstrates that the maximal possible payoff obtained by the player does not have to correspond to boundary decisions $p_1 = p_2 = \pi/2$ or $p_1 = p_2 = 0$, but can be found inside of the considered intervals.

Figures 3–7 have demonstrated that the quantum Shapley value redistributes the payoffs according to the pre-agreement between players and takes into consideration their 'acceptance' of the initial state. Indeed, players with smaller weights can benefit when the player with the

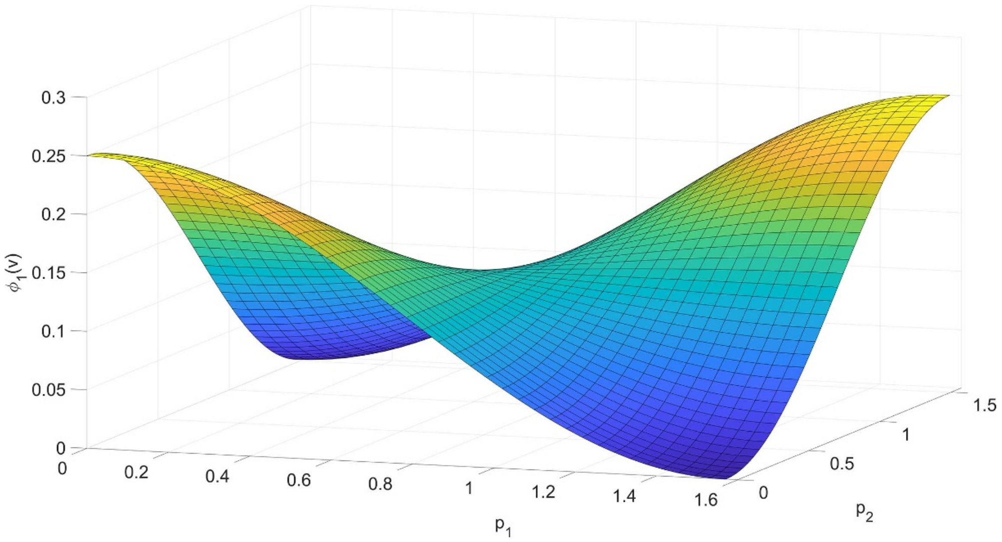

Figure 5. The quantum Shapley value $\tilde{\phi}_1$ for $\gamma_{123} = 0, \gamma_{12} = 0, \gamma_{13} = \pi/2, \gamma_{23} = 0$, and $p_2 = 0$.

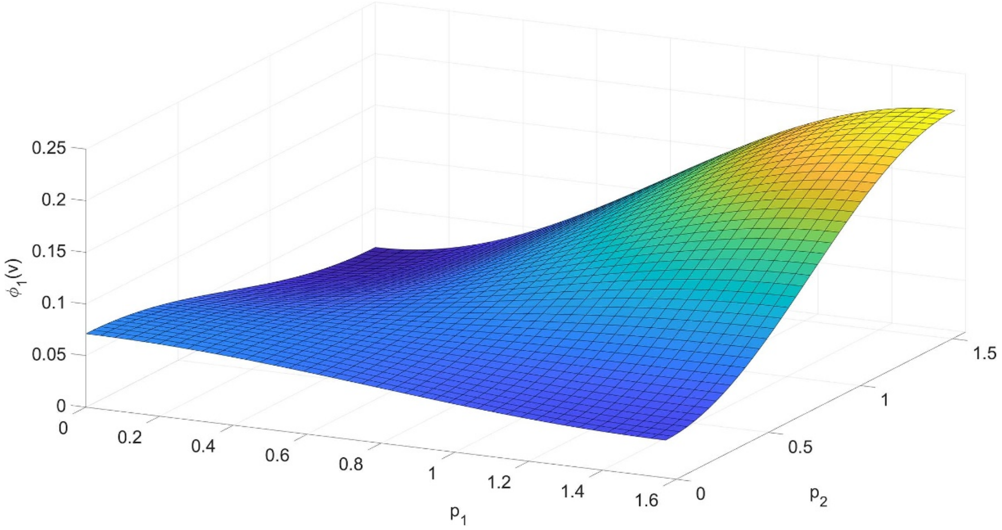

Figure 6. The quantum Shapley value $\tilde{\phi}_1$ for $\gamma_{123} = 1, \gamma_{12} = 0, \gamma_{13} = \pi/4, \gamma_{23} = 0$, and $p_2 = 0$.

most significant payoff is indifferent to a game process and has no strong bond between them. However, when equally strong players are maximally related (entangled), their distribution can only decrease, implying pre-agreements only damage their prosperity.

It can be concluded that the proposed quantization scheme for two-player and three-player cooperative games has allowed for non-trivial results. In particular, we have demonstrated that, depending on the properties of v, the strong bond between equally strong players not created within the negotiation process may decrease their payoffs (in case cooperation is beneficial).

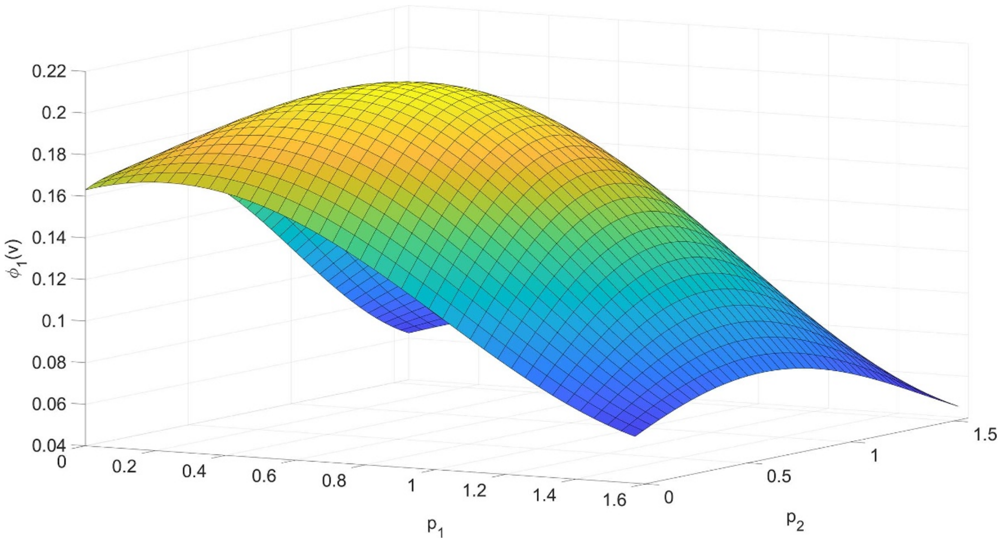

Figure 7. The quantum Shapley value $\tilde{\phi}_1$ for $\gamma_{123} = 1$, $\gamma_{12} = \pi/3$, $\gamma_{13} = 3\pi/4$, $\gamma_{23} = \pi/3$, and $p_2 = 3\pi/4$.

This peculiar outcome can be explained by an additional risk taken by the participants due to the existence of the pre-agreement, which can be interpreted as the initial probabilistic coalition structure. Alternatively, pre-agreement may improve players' payoffs when cooperation in the classical cooperative games is not the best option due to the definition of v. These results indicate the potential of the proposed quantization of the cooperative games. To the best of our knowledge, this is the first attempt at quantizing cooperative games, confirming that quantum cooperative games are a promising framework that might better help model human interactions. The further development of mathematical apparatus for such games might solve issues with the distribution of wealth in non-superadditive games, where the formation of the grand coalition can still be expected.

Our study has also demonstrated that QRA allows efficient computation within the quantum game theory framework. Moreover, QRA can even be used to perform automated proofs using the geometric algebra calculator GAALOPWeb. Thus, the language of QRA and its implementation in the GAALOP have provided us with a convenient tool to perform quantum computing and to study quantum cooperative games, in particular. The future implementation of the tensor product sign-changing rule within the GAALOP shall further simplify computations with the multiple qubit states using QRA. We believe that QRA has proved itself as a solid theoretically-based apparatus suitable for quantum computing and solution of quantum cooperative games in particular.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Appendix

Listing 1. Code for the automated proof of the entanglement symmetry.

```
i = er1 * er2 ;
f1 = 0.5*( e1 + i * e4 );
f1T = 0.5*( e1 - i * e4 );
f2 = 0.5*( e2 + i * e5 );
f2T = 0.5*( e2 - i * e5 );
f3 = 0.5*( e3 + i * e6 );
f3T = 0.5*( e3 - i * e6 );
I = f1 * f1T * f2 * f2T* f3 * f3T ;
psi=I;
J12 = cos(gamma12/2)*(f1*f1T*f2*f2T+f1*f1T*f2T*f2+f1T*f1*f2*f2T
+f1T*f1*f2T*f2) +i*sin(gamma12/2)*(-f1*f2+f1*f2T-f1T*f2+f1T*f2T);
Id3 = f3*f3T + f3T*f3;
J23 = cos(gamma23/2)(f2*f2T*f3*f3T + f2*f2T*f3T*f3+f2T*f2*f3*f3T
+f2T*f2*f3T*f3) +i*sin(gamma23/2)*(-f2*f3+f2*f3T-f2T*f3+f2T*f3T);
J13 = cos(gamma13/2)(f2*f2T*f3*f3T + f2*f2T*f3T*f3+f2T*f2*f3*f3T
+f2T*f2*f3T*f3) +i*sin(gamma23/2)*(-f2*f3+f2*f3T-f2T*f3+f2T*f3T);
Id1 = f1*f1T + f1T*f1;
SWAP12 =( f1 * f1T * f2 * f2T )+( f1T * f2 )
-( f1 * f2T )+( f1T * f1 * f2T * f2 );
J1= J12 * Id3;
J2= Id1 * J23;
J33=Id1*J13;
J3= SWAP12 * J33 *SWAP12;
S1=J1*J2*J3;
S2=J2*J3*J1;
S3=J2*J3*J1;
?X1=S1-S2;
?X2=S1-S3;
?X3=S2-S3;
```

Listing 2. Output for the automated proof of entanglement symmetry.

```
function [X1, X2, X3] = script()
end
```

Listing 3. Two-player cooperative game.

```
i = er1 * er2 ;
f1 = 0.5*( e1 + i * e3 );
f1T = 0.5*( e1 - i * e3 );
f2 = 0.5*( e2 + i * e4 );
f2T = 0.5*( e2 - i * e4 );
ket00 = f1 * f1T * f2 * f2T ;
J=cos(gamma/2)*(f1 * f1T * f2 * f2T+f1T * f1 * f2 * f2T+
f1 * f1T * f2T * f2+
f1T * f1 * f2T * f2)+
i*sin(gamma/2)*(-f1 * f2 -f1T * f2 +
```

```
f1*f2T+f1T * f2T);
U1tensorU2=(sin(p1)*(a*f1-f1T)+cos(p1)*(f1*f1T+a*f1T*f1))*
(sin(p2)*(b*f2-
b*f2T)+cos(p2)*(f2*f2T+f2T*f2));
psi_final=U1tensorU2*J*ket00;
?probability_final_0=4*abs(ket00*psi_final)*
abs(ket00*psi_final);
?probability_final_2=4*abs(ket00*f2*psi_final)*
abs(ket00*f2*psi_final);
?probability_final_1=4*abs(ket00*f1*psi_final)*
abs(ket00*f1*psi_final);
?probability_final_12=4*abs(ket00*f2*f1*psi_final)*
abs(ket00*f2*f1*psi_final);
```

Listing 4. Two-player cooperative game: output of the resulting function.

```
probability_final_0 =0.1250
probability_final_1 = 0.7286
probability_final_12 =0.1250
probability_final_2 = 0.0214
```

Listing 5. Three-player cooperative game.

```
i = er1 * er2 ;
f1 = 0.5*(e1 + i * e4);
f1T = 0.5*(e1 - i * e4);
f2 = 0.5*(e2 + i * e5);
f2T = 0.5*(e2 - i * e5);
f3 = 0.5*(e3 + i * e6);
f3T = 0.5*(e3 - i * e6);
ket000 = f1 * f1T * f2 * f2T*f3*f3T ;
Id1=f1*f1T+f1T*f1;
Id2=f2*f2T+f2T*f2;
Id3=f3*f3T+f3T*f3;
SWAP=f1 * f1T * f2 * f2T+f1T * f2-f1 * f2T +f1T * f1 * f2T * f2;
CNOT12=f1*f1T*f2*f2T+f1*f1T*f2T*f2-f1T*f1*f2-f1T*f1*f2T;
H1=1/sqrt(2)*(f1*f1T+f1+f1T-f1T*f1);
Id1tensorCNOT23=f1*f1T*f2*f2T*f3*f3T+f1*f1T*f2*f2T*f3T*f3-
f1*f1T*f2T*f2*f3-f1*f1T*f2T*f2*f3T+f1T*f1*f2*f2T*f3*f3T+
f1T*f1*f2*f2T*f3T*f3+f1T*f1*f2T*f2*f3+f1T*f1*f2T*f2*f3T;
J12=cos(gamma12/2)*(f1 * f1T * f2 * f2T+f1T * f1 * f2 * f2T+
f1 * f1T * f2T * f2+f1T * f1 * f2T * f2)+i*sin(gamma12/2)*
(-f1 * f2 -f1T * f2 + f1*f2T+f1T * f2T);
J13=cos(gamma13/2)*(f2 * f2T * f3 * f3T+f2T * f2 * f3 * f3T+
f2 * f2T * f3T * f3+f2T * f2 * f3T * f3)+i*sin(gamma13/2)*
(-f2 * f3 -f2T * f3 + f2*f3T+f2T * f3T);
J23=cos(gamma23/2)*(f2 * f2T * f3 * f3T+f2T * f2 * f3 * f3T+
f2 * f2T * f3T * f3+f2T * f2 * f3T * f3)+i*sin(gamma23/2)*
(-f2 * f3 -f2T * f3 + f2*f3T+f2T * f3T);
J2=SWAP*Id3*Id1*J13*SWAP*Id3*Id1*J23*J12*Id3;
```

```
J3=(1-gamma123)*Id1*Id2*Id3+gamma123*
Id1tensorCNOT23*CNOT12*Id3*H1*Id2*Id3;
U1tensorU2tensorU3=(-sin(p1)*sin(p2)*f1*f2+sin(p1)*sin(p2)*f1*f2T
+sin(p1)*cos(p2)*f1*f2T*f2+sin(p1)*cos(p2)*f1*f2*f2T-
sin(p1)*sin(p2)*f1T*f2+sin(p1)*sin(p2)*f1T*f2T
-sin(p1)*cos(p2)*f1T*f2T*f2-sin(p1)*cos(p2)*f1T*f2*f2T+
cos(p1)*sin(p2)*f1*f1T*f2-cos(p1)*sin(p2)*f1*f1T*f2T+
cos(p1)*cos(p2)*f1*f1T*f2T*f2+cos(p1)*cos(p2)*f1*f1T*f2*f2T-
cos(p1)*sin(p2)*f1T*f1*f2+cos(p1)*sin(p2)*f1T*f1*f2T
+cos(p1)*cos(p2)*f1T*f1*f2T*f2+cos(p1)*cos(p2)*f1T*f1*f2*f2T)*
cos(p3)*(f3*f3T+f3T*f3)+(-sin(p1)*sin(p2)*f1*f2+
sin(p1)*cos(p2)*f1*f2T*f2+sin(p1)*sin(p2)*f1T*f2T-
sin(p1)*cos(p2)*f1T*f2*f2T-cos(p1)*sin(p2)*f1*f1T*f2T+
cos(p1)*cos(p2)*f1*f1T*f2*f2T-cos(p1)*sin(p2)*f1T*f1*f2+
cos(p1)*cos(p2)*f1T*f1*f2T*f2)*sin(p3)*(f3-f3T)+
(-sin(p1)*sin(p2)*f1*f2T-sin(p1)*cos(p2)*f1*f2*f2T+
sin(p1)*sin(p2)*f1T*f2+sin(p1)*cos(p2)*f1T*f2T*f2-
cos(p1)*sin(p2)*f1*f1T*f2-cos(p1)*cos(p2)*f1*f1T*f2T*f2-
cos(p1)*sin(p2)*f1T*f1*f2T-cos(p1)*cos(p2)*f1T*f1*f2*f2T)*
sin(p3)*(f3-f3T);
psi_final=U1tensorU2tensorU3*J3*J2*ket000;
?probability_final_0=8*abs(ket000*psi_final)*
abs(ket000*psi_final);
?probability_final_1=8*abs(ket000*f1*psi_final)*
abs(ket000*f1*psi_final);
?probability_final_2=8*abs(ket000*f2*psi_final)*
abs(ket000*f2*psi_final);
?probability_final_3=8*abs(ket000*f3*psi_final)*
abs(ket000*f3*psi_final);
?probability_final_12=8*abs(ket000*f2*f1*psi_final)*
abs(ket000*f2*f1*psi_final);
?probability_final_13=8*abs(ket000*f3*f1*psi_final)*
abs(ket000*f3*f1*psi_final);
?probability_final_23=8*abs(ket000*f3*f2*psi_final)*
abs(ket000*f3*f2*psi_final);
?probability_final_123=8*abs(ket000*f3*f2*f1*psi_final)*
abs(ket000*f3*f2*f1*psi_final);
```

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