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BLACK HOLES AND SOLITONS IN STRING THEORY AND M-THEORY

A LOW ENERGY LIMIT ANALYSIS

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A mi madre, mi padre y mi hermano

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Abstract

This thesis explores the effects of supersymmetry and higher-curvature corrections from superstring theory on black hole and soliton physics in different dimensions. The first-order $\alpha'-$corrections derived from heterotic string theory, up to a field redefinition, lead to Einstein-dilaton-Gauss-Bonnet theory. We construct the first-order correction of Schwarzschild black holes in arbitrary dimensions. In four dimensions, we show that the corrected backgrounds satisfy the first law of thermodynamics and allow the construction of slowly rotating black holes and accelerating black hole solutions in a perturbative approach. In five dimensions, we consider the α'-corrections to the black string background and the boosted black string. We construct the scalar perturbation of the metric field in the perturbative approach to assess the effect of α' on the Gregory-Laflamme instability. We found that in the regime of validity of our approach, the instability window enlarges and grows with α'.

In the context of the $D = 5$, $\mathcal{N} = 2$ gauged STU model, we construct supersymmetric solitons, establishing new asymptotically locally AdS_5 solutions with nontrivial boundary conditions for the gauge fields and their uplifts to type IIB supergravity. There exists a BPS limit for a specific relation between the boundary values of the gauge fields, leading to 1/8-BPS configurations. We study the phase diagram of this sector of the theory and found that, generally, there are two solitons for a given value of the boundary values of the gauge fields. Finally, we identify the states in supergravity that are dual to certain vacua in the dual field theory in the weak coupling and large N limit, labeled by the boundary conditions on the fields. This is done by computing the Casimir energy in the field theory in the weakly coupled limit and comparing it with the energy of the supergravity solution.

The four-dimensional STU model in maximal gauged supergravity is studied for static electric and magnetic black holes with general horizon geometries. Thermodynamic stability analysis reveals unstable electric black holes for temperatures below certain value, including Reissner-Nordström. While magnetic BPS black holes demonstrate quasi-stability. We find extremal non-BPS black holes, which are thermodynamically unstable.

These results deepen the understanding of the interplay between string-inspired corrections, black hole physics, and supersymmetric stability criteria in different dimensions.

Keywords – Black holes, Solitons, Supergravity, Higher curvature corrections

Resumen

Esta tesis investiga cómo la supersimetría y las correcciones de curvatura superior provenientes de la teoría de supercuerdas restringen la física de agujeros negros y solitones en varias dimensiones. Las correcciones de primer orden en α' derivadas de la teoría heterótica, módulo una redefinición de campos, se puede escribir como Einstein-dilatón-Gauss-Bonnet. Construimos la corrección lineal en α' del agujero negro de Schwarzschild en dimension arbitraria. En cuatro dimensiones, mostramos que las configuraciones corregidas satisfacen la primera ley de la termodinámica y permiten la construcción de agujeros negros rotantes en el régimen de rotación lenta y soluciones de agujeros negros acelerados de forma perturbativa. En cinco dimensiones, consideramos las correcciones en α' de la configuración de cuerdas negras y la cuerda negra con momentum. Construimos la perturbación escalar del campo métrico en el enfoque perturbativo para evaluar el efecto de α' sobre la inestabilidad de Gregory-Laflamme. Encontramos que la ventana de inestabilidad se amplía y crece con α'.

También en $D = 5$, en el contexto del modelo STU, $\mathcal{N} = 2$ gauged, construimos configuraciones de solitones, estableciendo nuevas soluciones asintóticamente locales AdS_5 con campos de gauge no triviales en infinito y sus embebimientos en la supergravedad tipo IIB en diez dimensiones. Existe un límite BPS para una relación específica entre los valores en el borde de los campos de gauge, lo que conduce a configuraciones 1/8-BPS. Estudiamos el diagrama de fases de este sector de la teoría y encontramos que, en general, existen dos solitones para un valor dado de los campos de gauge en el borde. Finalmente, identificamos los estados en supergravedad que son duales a ciertos vacíos en la teoría de campos dual en el régimen de acoplamiento débil y N grande, caracterizados por las condiciones de bode en los campos. Esto se realizó mediante el cálculo de la energía de Casimir en la teoría de campos en el límite débilmente acoplado y su comparación con la energía de la solución de supergravedad.

Estudiamos el modelo STU en cuatro dimensiones de la supergravedad maximal en el contexto de agujeros negros estáticos eléctricos y magnéticos con geometrías de horizonte generales. El análisis de estabilidad termodinámica revela que los agujeros negros eléctricos con temperaturas por debajo de cierto valor, incluyendo Reissner-Nordström,son inestables; mientras que los agujeros negros BPS magnéticos muestran ser cuasi-estables. Encontramos agujeros negros extremales no-BPS, los cuales resultan ser termodinámicamente inestables.

Estos resultados profundizan en la comprensión de la intersección entre las correcciones inspiradas en la teoría de cuerdas, la física de agujeros negros y los criterios de estabilidad supersimétrica en varias dimensiones.

Chapter 1

Introduction

Great effort has been devoted to constructing a unifying theory for the four fundamental interactions. The Standard Model of particle physics (SM) gives a consistent and predictive description of the quantum behavior of electromagnetic, weak, and strong interactions. Notwithstanding, gravity does not belong to this picture. The reason is that the classical description of the gravitation interaction, governed by the Einstein's General Relativity, does not allow a perturbative quantum description: By doing perturbation theory around flat spacetime, one can arrange the expansion in powers of $\sqrt{8\pi G_N}$, which in natural units has length dimension +1 making the theory non-renormalizable in contrast with the SM. More precisely, General Relativity is two-loop non-renormalizable [1, 2]. Another difficultly in the construction of a theory involving all the four interaction is based on the fact there is a huge hierarchy between the characteristic energy scalars, in the case of the SM is consistently defined at energies of order of the electro-weak scale $M_W \sim 100\text{GeV}/c^2$, while the characteristic scale where the quantum gravity effects are supposed to be important at order of the Planck mass $M_P = \sqrt{\hbar c/8PiG_N} \sim 2.4 \times 10^{18} GeV/c^2$.

Different proposals have been made to construct a quantum theory of gravity. One such approach is based on pure gravity, leading to loop quantum gravity [3, 4]. The main drawbacks of this framework are that (i) it is not clear how to couple gravity to matter consistently, and it is generally accepted that there is no physical regime in which gravity can be considered completely isolated from the other fundamental interactions, and (ii) it is not know how to recover Einstein's gravity from this construction. Another approach consists of formulating another theory and recovering *a posteriori* particles from which one of them can be identified with the graviton. The most accepted proposal is along these lines, and is called Superstring theory. The fundamental objects are open and closed strings, and the only parameter of the theory is the α', which has length dimension $+2$. In this framework, particles emerge as oscillations of open and closed strings; the spectrum of the oscillating string contains the graviton within the closed strings spectrum and gauge vectors in the open strings spectrum. The features that make this theory reliable are that it is free of anomalies, and there are strong reasons to believe that it is UV finite. The theory is defined as a supersymmetric conformal theory on a 2-dimensional σ-model embedded in a 10-dimensional ambient space.

Supersymmetry — a symmetry between bosons and fermions — is a key ingredient of the theory as it allows us to rule out pathological features in the quantum regime. In general, the invariance under supersymmetry of a theory restricts the set of possible interactions. In the context of the Superstring theory, there are five consistent string models: Type IIA, Type IIB, $E_8 \times E_8$ Heterotic string, SO(32) Heterotic string, and Type I. From the point of view of the construction of a unique theory of fundamental interaction, having more than one possible theory is not completely satisfactory. In the 90s, non-perturbative dualities between different string theories formulated in different backgrounds were found [5, 6]. This fact suggests that all of them describe the same fundamental degree of freedom of a more fundamental theory. In particular, it was shown in [6] that the non-perturbative regime of Type IIA is described by an eleven-dimensional theory named M-theory. The fundamental degrees of freedom of the conjectured M-theory are not known yet, but in eleven dimensions there exists a unique supergravity theory constructed in [7], that should correspond to the low-energy limit of this theory.

In string theory models, including the bosonic theory, the quantum theory is free of Weyl anomaly for a special set of background fields. This is expressed as a set of differential equations for the background fields coming from the vanishing of the β-function. In the low energy limit, when the string tension goes to infinity or $\alpha' \to 0$, the running of the β-function can be computed, and the resulting effective field theory is a (super)gravity theory whose fields are the background field to which the σ-model couples. The leading order in the α' expansion is a supergravity theory containing an Einstein-Hilbert term coupled to the 10D dilaton scalar field, the Kalb-Ramond 2-form, and depending on the spectrum of the string theory from which one started, there is also a set of p-forms constituting the Ramond-Ramond sector, in addition to fermions.

The massless spectrum of type IIB, as can be done for any other string theory, can be written in terms of representations of the stabilizer group of a fixed massless momentum, that in this case for $D = 10$ is SO(8). The representations are the vectorial $\mathbf{8}_{\mathrm{V}}$, the spinorial $\mathbf{8}_{\mathrm{S}}$, and its conjugated $\mathbf{8}_{\mathrm{C}}$; tensor product of them will lead to the different sectors. The NSNS sector correspond to the product of two vectorial representations $\mathbf{8}_{\mathrm{V}} \otimes \mathbf{8}_{\mathrm{V}} = \mathbf{35}_{\mathrm{V}} \oplus \mathbf{28}_{\mathrm{V}} \oplus \mathbf{1}_{\mathrm{V}}$, leading to a metric (symmetric traceless) g_{MN}, 2-form Kalb-Ramond (anti-symmetric) B_{MN} and a dilaton (trace) Φ. The RNS sector is $\mathbf{8}_{\mathrm{S}} \otimes \mathbf{8}_{\mathrm{V}} = \mathbf{56}_{\mathrm{S}} \oplus \mathbf{8}_{\mathrm{C}}$, it carry a spinorial and a vectorial index $\Psi_{M\alpha}$, that can be decomposed using 8-dimensional Γ-matrices as $\lambda_{\dot{\alpha}} = (\Gamma^M)_{\dot{\alpha}}{}^{\beta}\Psi_{M\beta}$, transforming in the $\mathbf{8}_{\mathrm{C}}$ and the remaining part transforms in the $\mathbf{56}_{\mathrm{S}}$. Similarly for the NSR sector $\mathbf{8}_{\mathrm{V}} \otimes \mathbf{8}_{\mathrm{S}}$ yields to the same decomposition. One physical consequence of this fact is that the spinors are chiral; each spin 1/2 has a definite chirality, and each spin 3/2 has the opposite chirality to the spin 1/2. The RR sector corresponds to the product of the spinorial $\mathbf{8}_{\mathrm{S}} \otimes \mathbf{8}_{\mathrm{S}} = \mathbf{35}_{\mathrm{C}} \oplus \mathbf{28}_{\mathrm{C}} \oplus \mathbf{1}_{\mathrm{C}}$, that arrange into a (self-dual) 4-form C_{MNPQ}, 2-form C_{MN} and 0-form C_0. The explicit form of the low energy limit theory is called $\mathcal{N} = 2$ type IIB supergravity in $D = 10$ and was first constructed by John Schwarz [8]. The dilaton and the RR 0-form potential are coordinates of a coset manifold $\mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2)$, and the fermions transform under the isotropy group SO(2) in the fundamental representation. The potential catastrophe coming from the presence of chiral fermions, which leads to a non-trivial contribution to the diffeomorphism anomaly, is cured by the fact that the self-dual RR fields also have a contribution that cancels the fermionic one [9]. We give our convention and the explicit form of the equations in Chapter 4.

In the context of supergravity theories, a generic solution of the equations of motion does not preserve the symmetries of the theory – supersymmetry is no exception. Configurations that do preserve some amount of supersymmetry are said to be BPS states. They are typically described by short multiplets and enjoy special stability properties, such as being protected from quantum corrections. At the level of supergravity, the physical configurations can be thought of as macroscopical configurations whose fundamental degrees of freedom are the allowed string states. If a configuration is unstable under quantum fluctuations, it will transition to another state as soon as the instability manifests. Therefore, such unstable configurations cannot be considered suitable for studying physical properties. By this criterion, studying BPS configurations provides a stable and reliable starting point. Due to the mentioned features, supersymmetric configurations played a key role in the realization of the AdS/CFT correspondence.

The AdS/CFT correspondence is a conjectured duality between a gravitational theory with an AdS_{d+1} vacuum and a conformal field theory formulated on a fixed spacetime in d dimensions. The first hint of the duality relies on the Bekenstein-Hawking formula for the black hole entropy, which establishes that the black hole entropy is one-quarter of the horizon area per Planck length square [10, 11]. The first non-trivial connection between AdS spacetime and a conformal field theory (beyond global symmetries) was the Brown-Henneaux asymptotic symmetry analysis of AdS_3, which showed that it has as an asymptotic symmetry algebra two copies of the Virasoro algebra with a central extension given by $c = 3\ell/2G$. These ideas were condensed in the so-called holographic principle formulated in 1993/1994 by 't Hooft and Susskind. It stated briefly that in a quantum theory of gravity, the information within a volume can be characterized completely in terms of degrees of freedom of its boundary [12, 13]. Later, in 1996, Strominger and Vafa showed that by counting the degeneracy of BPS states in string theory for large charge, it is possible to recover the Bekenstein-Hawking entropy [14]. This was a great achievement of string theory, indicating an agreement between black hole physics and soliton states in string theory. The first concrete realization of the duality was proposed by Juan Maldacena in 1997 [15], which conjectured a duality between $\mathcal{N} = 4$ Super Yang-Mills in four dimensions and Type IIB string theory on $AdS_5 \times S^5$ background. Soon after, different results were found suggesting that the duality in the supergravity approximation holds in other dimensions. Another remarkable example where the duality is precisely stated is between ABJM model in 2+1 dimensions [16] dual to 11-dimensional supergravity on $AdS_4 \times S^7/\mathbb{Z}_k$ background in the large N limit. The 't Hooft limit corresponds to keeping the ratio N/k fixed in the large N limit. In this case, the theory is dual to type IIA on $AdS_4 \times \mathbb{CP}^3$.

One of the most remarkable consequences of the AdS/CFT duality is that it provides a concrete prescription to access the non-perturbative regime of quantum field theories at strong coupling. The conjecture can be rephrased as stating that the strongly coupled regime of a quantum field theory in the large N limit is described by a gravitational theory in one higher dimension with an AdS vacuum. Shortly after Maldacena's proposal, Witten presented a formal prescription for computing holographic quantities [17].

Part of this thesis, specifically Chapter 4, explores the holographic consequences of supersymmetric deformations of $AdS_5 \times S^5$. Regarding deformations of $AdS_4 \times S^7$, Chapter 5 focuses on assessing the stability of black holes in the $D = 4, \mathcal{N} = 2$ gauged STU model. The asymptotic behavior of the fields suggests that the instabilities found in these backgrounds may correspond to unstable states in the thermal ABJM.

While supergravity captures the leading behavior of the gravitational dual in the large

N and strong coupling limit, understanding finite N or finite coupling effects in the dual CFT requires incorporating corrections to the gravitational action. These corrections manifest as quantum and higher-derivative terms, specifically higher-curvature corrections to the Einstein-Hilbert action. Studying these corrections allows us to refine holographic computations, such as corrections to black hole entropy, hydrodynamic transport, and entanglement entropy.

Beyond the holographic perspective, higher-curvature corrections emerge more generally in quantum gravity frameworks, independent of spacetime asymptotics. These types of corrections to Einstein's General Relativity naturally arise in any consistent quantum gravity framework and are a robust prediction of string theory [18]. Early work in quantum field theory on curved spacetime already revealed effective actions containing higher-order contractions of the Riemann tensor [19]. These corrections generalize the Einstein-Hilbert action and modify GR at ultraviolet (UV) scales. Higher-curvature terms are also natural in dimensions greater than four [20, 21] and are widely expected in any UV-complete theory. However, the inclusion of higher-order curvature terms, even though it improves renormalizability, typically introduces massive spin-2 ghost modes with energy unbounded from below [22, 23]. A notable exception is Einstein-Gauss-Bonnet in $D > 4$, which is ghost-free and features second-order field equations [24]. Subsequently, Gross and Witten derived higher-curvature corrections to GR in type II string theory up to quartic order in curvature, confirming that such corrections are ubiquitous features of string effective actions [25]. Quadratic corrections appear in bosonic and heterotic strings [26], with their functional dependence on the dilaton, graviton, and antisymmetric tensor fields obtained through string scattering amplitudes and two-loop σ-model beta functions [27].

While higher-curvature corrections play a pivotal role in the context of holography — enabling access to finite coupling corrections in the dual field theory — this thesis focuses instead on their consequences in asymptotically flat spacetimes. In particular, we investigate how such corrections modify the physics of black holes, including their stability and thermodynamic properties, independently of holographic considerations.

The thesis is organized as follows. Section 1.1 provides a general overview of the mathematical tools and conventions necessary for the precise formulation of the ideas presented herein. Section 1.2 introduces the notion of black holes and their main semi-classical features. A detailed review of the Iyer-Wald method for computing conserved charges in GR and beyond GR is also presented.

We then present the original results developed in this thesis. Chapter 2 focuses on α'-corrected black holes in $D \geq 4$ within a string-inspired setup derived from the first α'-correction in heterotic string theory; this chapter is based on [28]. We consider spherical black holes, slowly rotating black holes, and accelerating black holes. In Chapter 3, we continue with the same setup but specialize to $D = 5$. Based on [29], we study the Gregory-Laflamme stability problem for α'-corrected black strings.

In Chapter 4, we shift to a supergravity context by constructing and studying supersymmetry-preserving solutions that provide a dual description of deformations of $\mathcal{N} = 4$ SYM on the Coulomb branch, formulated on 3-dimensional Minkowski times a circle. This work is based on [30].

In chapter 5 we start by revisiting the construction of the STU model from the gauged SO(8) $\mathcal{N} = 8$ supergravity in $D = 4$. Then, we study the supersymmetric limit of the Duff and Liu electric and magnetic black holes in the purely dilatonic sector of the STU model, in a clean way. For planar black holes, we derive the equation of state and analyze the thermodynamic quasi-stability of the electric and magnetic solutions. This chapter is based on [31].

Finally, Chapter 6 presents our concluding remarks.

1.1 Differential geometry and Group theory

Special relativity is a theory about space, time, and observers. It is based on the special principle of relativity that we announce as follows: *Given any observer for which the physical laws hold, then the physical laws hold for any other observer in relative uniform motion with respect to the first.* This implies that there is no physical experiment that we can perform to distinguish between inertial observers. A tension emerges once we assume the Maxwell equations to be the physical laws describing the electromagnetic interaction. The mathematical form of these equations is not covariant under the Galilean group, in contrast to Newton's second law and the Schrödinger equation, which are covariant under an appropriate action of the Galilean group on physical objects. This tension is resolved by constructing the symmetry group of the Maxwell equation, which is the Poincaré group $\mathrm{ISO}(3,1)$, that is, the semidirect product between the Lorentz group $\mathrm{SO}(3,1)$ and the four-dimensional translations $(\mathbb{R}^4, +)$. An Inönü-Wigner contraction of the Poincaré algebra leads to the Galilean algebra, making the framework emerging from the symmetries of Maxwell equations a generalization of non-relativistic physics. However, this reasoning implies that the speed at which the electromagnetic interaction propagates, i.e., the speed of light c, is the same for any inertial observer and consequently is an unreachable speed. The limit that allows us to obtain the Galilean group from the Poincaré group is $c \to \infty$.

Physical events are defined to be points in the Minkowski space $\mathbb{M}_4$, which as a set is isomorphic to $\mathbb{R}^4$ and does not have a definite origin; meaning that it is an affine space. In the special relativity framework, there is a maximum speed for the propagation of information, and consequently, there exists a set of pairs of events that are causally disconnected, on which all the inertial observers must agree. This set can be easily constructed by considering a map from the Minkowski space $\mathbb{M}_4$ to the Minkowski vector space $\mathbb{V}_4$ where we will define a scalar product. The map is defined as $\texttt{vec} : \mathbb{M}_4 \times \mathbb{M}_4 \to \mathbb{V}_4$ that satisfies $\texttt{vec}(p,q) + \texttt{vec}(q,r) = \texttt{vec}(p,r)$. As a consequence of this, $\texttt{vec}(p,p) = 0$ and $\texttt{vec}(p,q) = -\texttt{vec}(q,p)$. In the vector space $\mathbb{V}_4$ we define a scalar product $\mathbb{V}_4 \times \mathbb{V}_4 \to \mathbb{R}$, as $(v,w) \mapsto v \cdot w = v^T \eta w$ where η in its diagonal form is given by $\eta = \text{diag}(-1,1,1,1)$, and is called the Minkowski metric. This inner product is constructed in such a way that it is Lorentz invariant, i.e., invariant under change of frame. We define the square interval between to points in Minkowski spacetime as the map $\Phi : \mathbb{M}_4 \times \mathbb{M}_4 \to \mathbb{R}$, $\Phi(p,q) = \texttt{vec}(p,q) \cdot \texttt{vec}(p,q)$.

The set of causally disconnected events in $\mathbb{M}_4$ is the set of pairs of points $\{(p,q) \mid \Phi(p,q) > 0\}$, and each pair of events belonging to this set is called spacelike separated events. The set of pairs (p,q) for which $\Phi(p,q) < 0$ is the set of causally connected events: there is always one of them in the past of the other, and all the inertial frames agree on this. Finally, there is the set $\{(p,q) \mid \Phi(p,q) = 0\}$, which defines all pairs of events connected by a light ray. This set defines the light cone structure of the spacetime $\mathbb{M}_4$; studying its geometry will lead to the definition of a Carrollian structure, and analyzing the space of null direction naturally leads to the idea of twistors [32].

Under the paradigm of special relativity, constructing physical theories compatible with its principles amounts to devising action principles that are invariant under the action of the Lorentz group. One way to achieve this is by requiring that the physical fields transform in a representation of the Lorentz group. Then, the invariant tensors of this representation allow the construction of invariants that can serve as candidates for a Lagrangian density.

However, including gravity in a way that respects the principles of special relativity is not straightforward. Einstein's General Relativity achieves this by utilizing differential geometry and differentiable manifolds to describe spacetime, where the straight lines of

Minkowski spacetime for free particles are replaced by geodesics on a pseudo-Riemannian manifold. In the next section, we discuss these mathematical tools, mainly based on [33, 34, 35, 36]. Then, in Section 1.2, we return to the physical consequences of this framework.

1.1.1 Differential geometry

A differentiable space can be naturally characterised in terms of intrinsic geometric objects by a manifold structure. A manifold is essentially a space that, at each point, is similar to Euclidean space. This is realised through maps from subsets of the manifold to Euclidean space, which are called 'patches' (recalling the work of modistes). The way in which the patches are glued together defines the manifold structure. This structure is independent of the set of patches chosen — in other words, it is independent of the coordinate system used. We consider a map ϕ from an open set of $\mathscr{O} \subset \mathbb{R}^n$ to an open set of $\mathscr{O}' \subset \mathbb{R}^m$ to be of class C^k if the coordinantes of image point are k-times continuously differentiable functions of the coordiantes in $\mathscr{O}$.

A D-dimension C^r manifold is a set $\mathscr{M}$ with a C^r atlas $\{(\mathscr{U}_\alpha, \phi_\alpha) \mid \alpha \in I \subset \mathbb{Z}\}$, where $\phi_\alpha : U_\alpha \to \mathbb{R}^D$ are invertible maps to a open subsets of $\mathbb{R}^D$ such that

(i) $\mathscr{U}_\alpha$ cover $\mathscr{M}$, namely $\mathscr{M} = \bigcup_\alpha \mathscr{U}_\alpha$.

(ii) In any non-empty intersetion $U_\alpha \cap U_\beta$ the map $\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(\mathscr{U}_\alpha \cap \mathscr{U}_\beta) \to \phi_\alpha(\mathscr{U}_\alpha \cap \mathscr{U}_\beta)$ is a C^k map between open subset of $\mathbb{R}^D$.

Two atlases are said to be compatible with a given C^k atlas if their union is also a C^k atlas of $\mathscr{M}$. The first condition of the definition is automatically satisfied, while the second condition must be checked at the intersection of two patches belonging to different atlases. The union of all the compatible atlases is called the complete atlas of the manifold and corresponds to all possible coordinate systems that we can put on $\mathscr{M}$ with transition functions being C^k. The topology of $\mathscr{M}$ is defined as the open sets of $\mathscr{M}$ consisting of unions of the sets $\mathscr{U}_\alpha$ in the complete atlas.

A manifold is orientable if it has an atlas, denoted by $\{(\mathscr{U}_\alpha, \phi_\alpha)\}$, such that the Jacobian of the transition function is positive at every non-empty intersection.

A topological space $\mathscr{M}$ is said to be a Hausdorff space if for any $p, q \in \mathscr{M}$ there exist two disjoint open sets $\mathscr{U}, \mathscr{V} \subset \mathscr{M}$ such that $p \in \mathscr{U}$ and $q \in \mathscr{V}$. At first sight, it looks like any manifold is a Hausdorff space, but this is not true; there are manifolds that are not Hausdorff. To illustrate this, let us construct an example of a non-Hausdorff space: take two real lines with coordinates $x \in \mathbb{R}$ and $y \in \mathbb{R}$. Let $x \sim y$ iff $x = y < 0$ be an equivalence relation. The resulting space after the identification is not Hausdorff, as the points a in the first line with coordinate $x = 0$ and a' with coordinate $y = 0$ are different points and there are no disjoint opens $\mathscr{U}, \mathscr{V}$ such that $a \in \mathscr{U}$ and $b \in \mathscr{V}$. Intuitively, other spaces that are not Hausdorff are, for instance, a shoe with a partially detached sole, or a frayed-edge blanket.

Another property that most of the manifolds that we will deal with enjoy is *paracompactness*. The definition is as follows. An atlas $\{(\mathscr{U}_\alpha, \phi_\alpha)\}$ is said to be *locally finite* if every point $p \in \mathscr{M}$ has an open neighborhood which intersects only a finite number of the sets $\mathscr{U}_\alpha$. In other words, for any point p in the manifold, there is a finite number of patches that contain p. A manifold is *paracompact* if for every atlas $\{(\mathscr{U}_\alpha, \phi_\alpha)\}$ there exist a locally finite atlas $\{(\mathscr{V}_\beta, \psi_\beta)\}$ for which any $\mathscr{V}_\beta$ is contained in some $\mathscr{U}_\alpha$. One can prove that a Hausdorff manifold is *paracompact* if and only if it has a countable basis in the topological sense. Paracompactness of a Hausdorff manifold follows automatically once we impose the existence of an affine connection (see, for example, [33]).

Functions on a manifold.

A real function on a manifold is defined as a map of the form $f : \mathscr{M} \to \mathbb{R}$, that assigns a real number to each point in the manifold $f(p) \in \mathbb{R}$. The differentiability properties of a scalar function of a manifold are defined by its local form. Given a patch $(\mathscr{U}_\alpha, \phi_\alpha)$ of $\mathscr{M}$, the function $f \circ \phi_\alpha^{-1}$ is a map from an open subset of $\mathbb{R}^D$ to $\mathbb{R}$. We say that a real function on a manifold is of class C^k if $f \circ \phi_\alpha^{-1}$ is of class C^k for all α in the atlas. The local descriptions of the real functions, that is what we will usually give, are glued together following the their definition and using the transition function $\phi_\alpha \circ \phi_\beta^{-1}$ between a non-empty intersection between patches.

The set of smooth functions on an open subset of $\mathbb{R}^D$ is infinite-dimensional. However, the set of smooth functions that are globally defined on an arbitrary manifold is smaller. A simple example is the circle S^1, which can be defined as the coset space $\mathbb{R}/\mathbb{Z}$. Smooth functions that are globally defined are those that are smooth and respect the periodicity of the circle.

In order to study the space of function on a manifold, we define the space of *germs of smooth functions* at the point $p \in \mathscr{M}$ denoted by $C_p^\infty \mathscr{M}$ as follows: Consider all the open subsets $\{\mathscr{U}_\alpha \subset \mathscr{M}, \alpha \in I' \,|\, p \in \mathscr{U}_\alpha\}$ that contains the point. Then, take the smooth function on the opens $f \in C^\infty(\mathscr{U}_\alpha)$, $g \in C^\infty(\mathscr{U}_\beta)$ such that $p \in \mathscr{U}_\beta \cap \mathscr{U}_\alpha \neq \emptyset$. The space of germs of smooth functions at the point $p \in \mathscr{M}$ is defined as

$$C_p^\infty \mathscr{M} = \bigcup_{\alpha \in I'} C^\infty(U_\alpha)/\sim\,, \qquad f \sim g \in \Longleftrightarrow f|_{\mathscr{U}_\alpha \cup \mathscr{U}_\beta} = g|_{\mathscr{U}_\alpha \cup \mathscr{U}_\beta}\,. \tag{1.1.1}$$

Essentially, this definition implements the idea of analytic continuation. That is to say, two smooth functions f, g defined on opens with no-vanishing intersection $\mathscr{U}_\alpha, \mathscr{U}_\beta$, respectively, that contain p, are identified as the same element if they coincide in the intersection.

Curves in a manifold

A curve in the manifold is defined as a continuous and differentiable map from an interval in $\mathbb{R}$ to $\mathscr{M}$, $\gamma : [0,1] \to \mathscr{M}$. Closed curves are curves where the initial and final point coincides $\gamma(0) = \gamma(1) \in \mathscr{M}$.

Tangent and cotangent spaces

We can define a vector space at each point of a differentiable manifold of dimension equal to the dimension of the manifold. This vector space is called the *tangent space at p* and is denoted by $T_p\mathscr{M}$. To construct it let us consider a point $p \in \mathscr{M}$ and a curve $\gamma : [0,1] \to \mathscr{M}$ that starts at p, i.e. $\gamma(0) = p$, and a representative of the space of germ of smooth functions at p, $f \in C_p^\infty \mathscr{M}$ defined on a chart $(\mathscr{U}, \phi)$. The composite function $f \circ \gamma : [0,1] \to \mathbb{R}, \lambda \mapsto g(\lambda) = f \circ \gamma(\lambda)$ is a real function, and we can compute its derivative. Considering that in the local patch $\phi(p) = (x^1(p), x^2(p), \dots, x^D(p)) = (x^\mu(p)) \in \mathbb{R}^D$, then, to compute the derivative we consider the composite $f \circ \phi^{-1}(\circ\phi \circ \gamma)(\lambda)$ and the derivative reads

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} g(\lambda)\bigg|_{\lambda=0} = \frac{\partial f \circ \phi^{-1}}{\partial x^\mu} \frac{\partial x^\mu}{\partial \lambda}\bigg|_{\lambda=0}\,. \tag{1.1.2}$$

We see that the derivative of an arbitrary germ of a smooth function f at p is given in terms of the real constant coefficients $t^\mu = \frac{\partial x^\mu}{\partial \lambda}|_{\lambda=0}$, fixed by the parametrization of the curve γ. Consequently, we can write the derivative of the function as $t^\mu \left.\frac{\partial f \circ \phi^{-1}}{\partial x^\mu}\right|_p$, that can be understood as the action of a differential operator on the space of germ functions at p,

$$\mathbf{t}_p = t^\mu \frac{\partial}{\partial x^\mu}\bigg|_p\,, \qquad \mathbf{t}_p : C_p^\infty \mathscr{M} \to C_p^\infty \mathscr{M} \tag{1.1.3}$$

The set of all first-order differential operators on the germs of smooth functions $C_p^\infty\mathscr{M}$ forms a vector space over $\mathbb{R}$ of dimension D. We define the tangent space of the manifold $\mathscr{M}$ at the point p with this vector space and denote it by $T_p\mathscr{M}$. A basis of this space at the point p are certainly the operators $\{\partial_\mu = \partial/\partial x^\mu\}|_p$. Therefore, a generic vector $\mathbf{X}$ at the point p is denoted by $\mathbf{X} = X^\mu\partial_\mu|_p$.

Given a vector space, there always exists a dual vector space, which we call *cotangent vector space at the point* p and denote by $T_p^*\mathscr{M} = \{\omega : T_p\mathscr{M} \to \mathbb{R}\}$. Elements of this set are also called 1-forms and are real-valued linear functions on the tangent space $T_p\mathscr{M}$ at p. Given a basis for $T_p\mathscr{M}$, for instance $\{\mathbf{V}_a\}$ with $a = 1, \ldots, D$, one can define uniquely a basis $\{\mathbf{V}^a\}$ of the $T_p^*\mathscr{M}$ by requiring that for any vector $\mathbf{X} = X^a\mathbf{V}_a$, the action of $\mathbf{V}^a$ on $\mathbf{X}$ leads to the components X^a, namely $\mathbf{V}^a(\mathbf{X}) = X^a$. Due to the linearity of the map, this requirement implies immediately that $\mathbf{V}^a(\mathbf{V}_b) = \delta^a_b$. In the case of the coordinate basis $\{\partial_\mu|_p\}$, we can define a basis of $T_p^*\mathscr{M}$ as $\{\mathrm{d}x^\mu|_p\}$ such that $\mathrm{d}x^\mu(\partial_\nu)|_p = \delta^\mu_\nu$. Similarly one can define the vector space $T_p\mathscr{M} = \{\mathbf{X} : T_p^*\mathscr{M} \to \mathbb{R}\}$. Then, the action of vectors on forms is also well defined, and we can write that $\mathbf{V}^a(\mathbf{V}_b) = \mathbf{V}_b(\mathbf{V}^a) = \delta^a_b$.

The notation $\{\mathrm{d}x^\mu|_p\}$ for the basis of the co-tangent space in the coordinates chart $x^\mu(p)$ is motivated as follows. For any function f on $\mathscr{M}$, we can define a 1-form $\mathrm{d}f \in \{T_p\mathscr{M} \to \mathbb{R}\}$ by imposing the following rule: For any $\mathbf{X} \in T_p\mathscr{M}$, $\mathrm{d}f(\mathbf{X}) = \mathbf{X}f$. If we write them in the coordinates basis, provided that $\mathrm{d}x^\mu(\partial_\nu) = \delta^\mu_\nu$, the components of the 1-form $\mathrm{d}f$ in the coordinate basis are $\partial_\mu f|_p$, leading to $\mathrm{d}f = \mathrm{d}x^\mu\partial_\mu f|_p \in T_p^*\mathscr{M}$.

One can define the tensor product of tangent and cotangent vector spaces at the point p in order to construct higher rank tensors. For instance, a tensor of rank (m, n) is an element of the vector space $\otimes^m T_p\mathscr{M} \otimes^n T_p^*\mathscr{M}$. In a chart $x^\mu(p)$ the tensor is written explicitly as

$$\mathbf{T} = T^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\nu_n}\frac{\partial}{\partial x^{\mu_1}} \otimes \ldots \frac{\partial}{\partial x^{\mu_m}} \otimes \mathrm{d}x^{\nu_1} \otimes \cdots \otimes \mathrm{d}x^{\nu_n}|_p\,. \tag{1.1.4}$$

So far, we have considered a chart $(\mathscr{U}, \phi)$ to give local expressions for tensors at an arbitrary point $p \in \mathscr{U}$. However, by definition, geometric objects must be independent of the atlas that we pick. In particular, in the overlap of two charts, the geometrical objects must be the same in one chart or another. Let us consider a second chart $(\mathscr{V}, \psi)$, with $p \in \mathscr{U} \cap \mathscr{V}$ and $\psi(p) = (y^1(p), \ldots, y^D(p)) = (y^\mu(p))$. The composite function $\psi \circ \phi^{-1}$ is a invertible map from a subset of $\mathbb{R}^D$ to a subset $\mathbb{R}^D$ given by the function $y^\mu = y^\mu(x^\nu)$. An arbitrary tensor $\mathbf{T}$ at the point p is written in the chart $(\mathscr{V}, \psi)$ in terms of the basis $\{\partial/\partial y^\mu|_p\} \subset T_p\mathscr{M}$, $\{\mathrm{d}y^\mu|_p\} \subset T_p^*\mathscr{M}$, and tensor products of them. In the intersection

$\mathscr{U} \cap \mathscr{V}$, the components of the tensor (1.1.4) in the coordinate basis associated to $x^\mu(p)$ are related to the components in the basis $y^\mu(p)$ by

$$T'^{\mu_1\ldots\mu_m}{}_{\nu_1\ldots\nu_n} = \frac{\partial y^{\mu_1}}{\partial x^{\rho_1}} \cdots \frac{\partial y^{\mu_m}}{\partial x^{\rho_m}} \frac{\partial x^{\sigma_1}}{\partial y^{\nu_1}} \cdots \frac{\partial x^{\sigma_n}}{\partial y^{\nu_n}} T^{\rho_1\ldots\rho_m}{}_{\sigma_1\ldots\sigma_n} \,. \tag{1.1.5}$$

This ensures that the geometrical object $\mathbf{T}$ is the same in any chart. From the transformation rule of the local components of a tensor, it is immediate to see that the partial derivatives of the components of a tensor do not transform as the rule (1.1.5), since the derivative also acts on the transformation matrices. Hence, to compute the derivatives of a tensor, we have to do something else. The most economical way, which requires no extra structure, is the *exterior derivative* defined in terms of the so-called *differential forms*. This intrinsic derivative arises naturally from the differentiable structure of the manifold. Another notion of derivative is the *Lie derivative*, which requires an additional vector field. The most versatile derivative is the covariant derivative associated with a connection, which is typically present in our geometric setups.

Differential forms

Let us consider the anti-symmetric part of the k-times tensor product of the cotangent space at p. We denote this vector space of dimension $\binom{D}{k}$ by $\bigwedge^k T_p^*\mathscr{M}$. For simplicity, let us define it in a point p belonging to a chart $(\mathscr{U}, \phi)$ with $\phi(p) = (x^\mu(p))$. The basis $\{\mathrm{d}x^\mu|_p\}$ induces a basis of $\bigwedge^k T_p^*\mathscr{M}$ by considering the antisymmetric tensor product

$$\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_k} = \sum_{\sigma \in \mathrm{Sym}_k} (-1)^{|\sigma|} \mathrm{d}x^{\mu_{\sigma(1)}} \otimes \cdots \otimes \mathrm{d}x^{\mu_{\sigma(k)}} \,, \tag{1.1.6}$$

where Sym_k is the symmetric group of k elements and $|\sigma| \in \{\pm 1\}$ is the parity of σ. The set $\{\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_k}|_p\}$ defines a basis of $\bigwedge^k T_p^*\mathscr{M}$, and an elemement of this space is called k-form and is defined to be

$$A = \frac{1}{k!} A_{\mu_1\ldots\mu_k} \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_k}|_p \tag{1.1.7}$$

where the components $A_{\mu_1\ldots\mu_k}$ are completely anti-symmetric. At this point, we introduce the wedge product as a bilinear map

$$\begin{aligned} \wedge : \bigwedge^{k_1} T_p^*\mathscr{M} \times \bigwedge^{k_2} T_p^*\mathscr{M} &\to \bigwedge^{k_1+k_2} T_p^*\mathscr{M} \\ (A, B) &\mapsto A \wedge B \end{aligned}$$

Let us define the exterior derivative as a map from k-forms to $(k+1)$-forms $\mathrm{d} : \bigwedge^k T_p^*\mathscr{M} \to \bigwedge^{k+1} T_p^*\mathscr{M}$. In a chart, it is defined as follows: let $A \in \bigwedge^k T_p^*\mathscr{M}$ then

$$\mathrm{d}A = \frac{1}{k!}\partial_\rho A_{\mu_1\dots\mu_k}\mathrm{d}x^\rho \wedge \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_k}|_p\,, \tag{1.1.8}$$

This object does not depend on the chart if the components $A_{\mu_1\dots\mu_k}$ are differentiable functions. The key point to prove this is that the components are completely anti-symmetric in the indices $\rho, \mu_1, \dots \mu_k$. Then, as the partial derivatives commute for a differentiable function, the components of (1.1.8) transform as (1.1.5).

The exterior derivative has the following properties:

1. It is a linear map.
2. It satisfies a the graded Leibniz rule $\mathrm{d}(A \wedge B) = \mathrm{d}A \wedge B + (-1)^k A \wedge \mathrm{d}B$, for any A, B being a k-form and p-form, respectively.
3. It is nil potent $\mathrm{d} \circ \mathrm{d} = 0$.

The exterior calculus that we have presented is instrumental in most of the physical models studied in this thesis. In particular, the above properties allow for an efficient implementation of exterior calculus in Wolfram Mathematica, taking advantage of the software's powerful pattern recognition capabilities.

Fiber bundle

So far, we have defined vector spaces associated with each point p of the differential manifold. In the local description, it is natural to assume a sort of continuity between vector spaces defined at points that are infinitesimally close to each other. To make this notion precise, it is necessary to define a *fiber bundle.*

A fiber bundle is a geometrical structure that consists of the following elements: A differentiable manifold E called the *total space*, a differentiable manifold $\mathscr{M}$ called the *base space*, a differentiable manifold F called the *standard fiber*, a Lie group G called the *structure group* which acts on the stander fiber $g : F \to F, \forall g \in G$, a surjective map $\pi : E \to \mathscr{M}$ called the *projection map* such that $\pi^{-1}(p)$ is a differentiable manifold diffeomorphic to the standard fiber. Given an atlas of the base manifold $\{(\mathscr{U}_\alpha, \phi_\alpha) \mid \alpha \in I\}$ the definition of the fiber bundle requires the definition a homeomorphism called *local trivialization* $\psi_\alpha : \pi^{-1}(\mathscr{U}_\alpha) \to \mathscr{U}_\alpha \times F$ such that $\pi \circ \psi_\alpha^{-1} = \mathrm{Id}_{\mathscr{M}}$. In the overlap of two charts $\mathscr{U}_\alpha \cap \mathscr{U}_\beta$, one can construct the composite of local trivializations, that is called

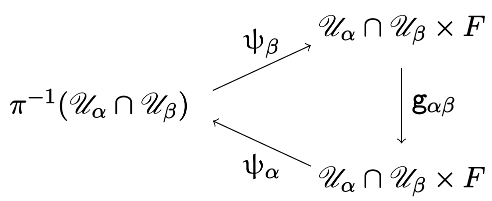

Figure 1.1.1: Transition functions on a fiber bundle.

transition function $\mathsf{g}_{\beta\alpha} = \psi_\beta \circ \psi_\alpha^{-1} : (\mathscr{U}_\alpha \cap \mathscr{U}_\beta) \times F \to (\mathscr{U}_\alpha \cap \mathscr{U}_\beta) \times F$. The last requirement is that for a given point p in the intersection, then $\mathsf{g}_{\beta\alpha}^{-1}(p)$ is in the structure group. The diagram of the transition function is depicted in Figure 1.1.1.

A special type of fiber bundle happens when the fiber is the structure group itself; in that case, it is called *principal bundle.* Another specialization that is fundamental for us is when the fiber is a vector space, and the structure group can act on the fiber in a representation of dimension equal to the dimension of the vector space. This type of fiber bundle is called a *vector bundle.*

Observe that when we defined $T_p\mathscr{M}$, the tangent space at the point $p \in \mathscr{M}$, as a vector space of dimension $D = \dim \mathscr{M}$. Once a basis of $T_p\mathscr{M}$ picked, it becomes isomorphic to $\mathbb{R}^D$. We can construct a vector bundle $T\mathscr{M} = \bigcup_{p \in \mathscr{M}} T_p\mathscr{M}$ by considering the base space to be the manifold $\mathscr{M}$ and the fiber at p to be $\mathbb{R}^D$. In a chart $(\mathscr{U}_\alpha, \phi_\alpha)$, with $\phi_\alpha(p) = (x^\mu(p))$ containing p we can consider a basis of $T_p\mathscr{M}$ to be $\partial/\partial x^\mu|_p$. The local trivialization is defined as

$$\psi_\alpha \left(v^\mu \left. \frac{\partial}{\partial x^\mu} \right|_p \right) = (p, v^\mu) \quad \text{, where} \quad v^\mu = (v^1, v^2, \ldots, v^D) \,. \tag{1.1.9}$$

At the intersection of the charts $(\mathscr{U}_\alpha, \phi_\alpha)$ and $(\mathscr{U}_\beta, \phi_\beta)$, considering $\phi_\beta(p) = (y^\mu(p))$, the transition function is $\mathsf{g}_{\beta\alpha} = \psi_\beta \circ \psi_\alpha^{-1}$

$$\mathsf{g}_{\beta\alpha}(p, v^\mu) = \psi_\beta \left(v^\mu \left. \frac{\partial}{\partial x^\mu} \right|_p \right) = \psi_\beta \left(v^\mu \frac{\partial y^\nu}{\partial x^\mu} \left. \frac{\partial}{\partial y^\nu} \right|_p \right) = \left(p, \frac{\partial y^\nu}{\partial x^\mu} v^\mu \right) \,. \tag{1.1.10}$$

If we fixed the point in the base, we see that the map between fibers is linear and the structure group is $\mathrm{GL}(D, \mathbb{R})$ since $\frac{\partial y^\nu}{\partial x^\mu} \in \mathrm{GL}(D, \mathbb{R})$.

Given a fiber bundle $\pi : E \to \mathscr{M}$. A map $s : \mathscr{M} \to E$, that assign to each point on the base, $p \in \mathscr{M}$, a point on the total space, $s(p) \in E$, such that $s(p)$ is above p, more precisely $\pi \circ s = \mathrm{Id}_{\mathscr{M}}$, is called a *section of the bundle.* A section is called continuous, differentiable, or smooth depending on the properties of the map s in each local trivialization of the bundle. This concept is crucial since most of the physical objects that we are interested in are sections of some fiber bundle over the spacetime. For instance, we define a vector field ξ on a smooth manifold $\mathscr{M}$ as a section of the tangent bundle $\xi : \mathscr{M} \to T\mathscr{M}$.

Naturally, we can define the vector bundle of the tensor product of the contanget and tanget space the point $p \in \mathscr{M}$, for instance $\otimes^m T_p\mathscr{M} \otimes^n T_p^*\mathscr{M}$ is the fiber at the point p of the vector bundle $\otimes^m T\mathscr{M} \otimes^n T^*\mathscr{M}$, and section of this bundle are tensor fields of rank (m, n). Of particular interest for us is the vector bundle of k-form denoted by $\bigwedge^k T^*\mathscr{M}$, and sections of this vector bundle are called k-form. The exterior derivative is a map between sections of $\bigwedge^k T^*\mathscr{M}$ to sections of $\bigwedge^{k+1} T^*\mathscr{M}$.

1.1.2 Metric on a manifold

The symmetric product of vector bundles $T^*\mathscr{M} \otimes_{symm} T^*\mathscr{M}$—namely, at each point $p \in \mathscr{M}$, the vector basis of the fiber is the symmetric tensor product of the cotanget space at p—is extremely relevant in geometry and physics since their global section allow to define a metric field $\boldsymbol{g}$. To be more precise, a pseudo-Riemannian metric on a manifold $\mathscr{M}$ is a $C^\infty\mathscr{M}$, symmetric, bilinear, non-degenerate form on sections of the tangent space. Namely, for any vector fields $\mathbf{X}, \mathbf{Y}, \mathbf{Z} : \mathscr{M} \to T\mathscr{M}$, the metric satisfies:

(i) $\boldsymbol{g}(\mathbf{X}, \mathbf{Y}) = \boldsymbol{g}(\mathbf{Y}, \mathbf{X}) \in C^\infty\mathscr{M}$.

(ii) For any smooth functions f_1, f_2 on $\mathscr{M}$, $\boldsymbol{g}(f_1\mathbf{X} + f_2\mathbf{Y}, \mathbf{Z}) = f_1\,\boldsymbol{g}(\mathbf{X}, \mathbf{Z}) + f_2\,\boldsymbol{g}(\mathbf{Y}), \mathbf{Z})$.

(iii) If $\boldsymbol{g}(\mathbf{X}, \mathbf{Y}) = 0$ for any $\mathbf{X}$, then $\mathbf{Y} = 0$ (non-degenerate).

In a coordinate patch, the metric is described by a symmetric rank 2 tensor

$$\boldsymbol{g} = g_{\mu\nu}(x)\mathrm{d}x^\mu\,\mathrm{d}x^\nu\,, \tag{1.1.11}$$

the symmetric tensor product will be usually denoted by juxtaposition.

Since the metric can be thought of as a symmetric matrix at each point p, it can always be diagonalized and normalized with entries being ± 1, through a point-dependent matrix $V \in \mathrm{GL}(D, \mathbb{R})$, hence

$$V^T \boldsymbol{g} V|_p = \mathrm{diag}(\underbrace{-1, \ldots, -1}_{\eta_-}, 1, \ldots, 1) = \eta \tag{1.1.12}$$

where η_- is the number of minus signs in the diagonal metric, and η is the usual name for the flat metric. A subgroup of $\mathrm{GL}(D, \mathbb{R})$ that leaves the flat metric invariant is $\mathrm{O}(\eta_-, D -$

η_-), usually we will work with $\mathrm{SO}(\eta_-, D-\eta_-)$ as the manifold that we are interested in are orientable and we would like to preserve the orientation. This invariance corresponds to an arbitrariness in the choice of the matrices V. To emphasize this fact, we introduce indices of $\mathrm{SO}(\eta_-, D-\eta_-)$ denoted by lower case latin letters $a, b, \ldots = 1, \ldots, D$, then the metric tensor at the point p can be written as

$$\boldsymbol{g} = \eta_{ab} V^a V^b \,, \qquad V^a = V^a_\mu \, \mathrm{d}x^\mu \,, \tag{1.1.13}$$

where the 1-forms V^a are called vielbein or frame and depend on the point. Given a metric, the choice of the vielbein is not unique, indeed $\tilde{V}^a = \Lambda^a{}_b V^b$ is another equally admissible vielbein basis whenever $\Lambda^a{}_b \in \mathrm{SO}(\eta_-, D-\eta_-)$, where Λ is a matrix that depends on the point of the manifold.

Let us introduce the completely anti-symmetric tensor $\epsilon_{a_1 \ldots a_D}$ of $\mathrm{SO}(\eta_-, D-\eta_-)$ with $\epsilon_{12\ldots D} = 1$. We define the *volume form* of a manifold $\mathscr{M}$ as the wedge product of the vielbein basis, which can be written in terms of the invariant tensor as follows

$$\mathrm{Vol}(\mathscr{M}) = V^1 \wedge \cdots \wedge V^D = \frac{1}{D!} \epsilon_{a_1 \ldots a_D} V^{a_1} \wedge \cdots \wedge V^{a_D} \,. \tag{1.1.14}$$

According to the definition of the vielbein, it can be written in terms of the determinant of the metric as

$$\mathrm{Vol}(\mathscr{M}) = \sqrt{|g|} \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^D = \sqrt{|g|} \mathrm{d}^D x$$

, where we used the fact that $\det(V^a{}_\mu) = \pm\sqrt{|g|}$, which is a direct consequence of (1.1.13), and we have picked the upper sign.

Given a vielbein basis $V^a = V^a{}_\mu \mathrm{d}x^\mu$, the inverse components are denoted by $V_a{}^\mu$ such that $V_a{}^\mu V^b{}_\mu = \delta^b_a$ and $V_a{}^\mu V^a{}_\nu = \delta^\mu_\nu$. Using them, we can construct a basis for the sections of the tangent and cotangent bundle. We denote $\{V^a \mid a = 1, \ldots, D\}$ a basis of the cotangent bundle. Any p-form $C = \frac{1}{p!} C_{\mu_1 \ldots \mu_p} \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p}$, can be written in the vielbein basis as follows

$$C = \frac{1}{p!} C_{a_1 \ldots a_p} V^{a_1} \wedge \cdots \wedge V^{a_p} \,, \tag{1.1.15}$$

where the components are related to the components in the coordinate basis as $C_{a_1 \ldots a_p} = C_{\mu_1 \ldots \mu_p} V_{a_1}{}^{\mu_1} \ldots V_{a_p}{}^{\mu_p}$.

The fact that the vector space of k-form at the point p, $\bigwedge^k T_p \mathscr{M}$, has the same dimension as $\bigwedge^{D-k} T_p\mathscr{M}$ means that they are isomorphic as vector spaces. This isomorphism is realized by the Hodge star $\star$ which maps sections of $\bigwedge^k T\mathscr{M}$ into sections of $\bigwedge^{D-k} T_p\mathscr{M}$. Let us consider a p-form (1.1.15), the Hodge star is defined as

$$\star C = \frac{1}{p!(D-p)!} C^{a_1\dots a_p} \epsilon_{a_1\dots a_p b_1 \dots b_{D-p}} V^{b_1} \wedge \dots \wedge V^{b_{D-p}} , \tag{1.1.16}$$

where the indices of the components where raised with the inverse flat metric η^{ab}. The Hodge star is an invertible operator, as the following relation holds

$$\star^2 C = (-1)^{p(D-p)+\eta_-} C . \tag{1.1.17}$$

This relation restricts the possibility of having (anti-) self-dual p-forms in a D-dimensional manifold. Indeed if the p-form C is (anti-)self dual $C = \pm \star C$, immediately implies $p = D/2$, namely D must be even. Acting with the Hodge star on the defining property and using the defining property again, we find that $[1-(-1)^{\frac{D^2}{4}+\eta_-}]C = 0$. Assuming that C is non-zero, then the above consistency condition is true only when $\frac{D^2}{4}+\eta_-$ is even. For Lorentzian spaces, e.g. $\eta_- = 1$, this is true for dimensions $D = 2(2n-1), n > 0$, the first ones are $D = 2, 6, 10, \dots$. Here we see that the self-dual 5-form field strength in $D = 10$ in type IIB is consistent as well as 3-form field strength in $D = 6$ as in the $\mathcal{N} = (2,0)$ theory demands (see, e.g., [37]). For Euclidean spaces, it is possible to have self self-dual form in dimensions $D = 4n, n > 0$, which includes the possibility of having self self-dual Yang-Mills field strength in $D = 4$.

Connections

At this point, we have introduced the exterior derivative as a built-in differential operator on the manifold. Notwithstanding, we can define a further object called an *affine connection* defined as follows. Let $\mathscr{M}$ be a D-dimensional differentiable manifold. Let us denote by $\mathfrak{X}(\mathscr{M})$ the set of sections of the tangent bundle, e.g., vector fields. An affine connection ∇ is a map of the form $\nabla : \mathfrak{X}(\mathscr{M}) \times \mathfrak{X}(\mathscr{M}) \to \mathfrak{X}(\mathscr{M})$, satisfying the following properties for any $X, Y, Z \in \mathfrak{X}(\mathscr{M})$.

(i) $\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$, linear in the second argument.

(ii) $\nabla_{(X+Y)} Z = \nabla_X Z + \nabla_Y Z$, linear in the first.

(iii) For any $f \in C^\infty(\mathscr{M})$, $\nabla_{fX} Y = f \nabla_X Y$.

(iv) For any $f \in C^\infty(\mathscr{M})$, $\nabla_X(fY) = X(f)Y + f\nabla_X Y$.

Spacetime covariant derivative

We can make a concrete realization of the definition by consider a local chart $(\mathscr{U}, \phi)$ with $\phi(p) = (x^\mu(p))$, leading to a basis of the space sections $\{\partial_\mu\}$ in the chart. Considering $X = X^\mu\partial_\mu, Y = Y^\mu\partial_\nu \in \mathfrak{X}(\mathscr{M})$. We can define the connection with indices as

$$\nabla_X Y = X^\mu \nabla_\mu Y^\nu \partial_\nu \,. \tag{1.1.18}$$

In the local patch, the connection can be thought of as a differential operator carrying an index ∇_μ that acts on vector fields. In particular, we can act on the basis itself, and by definition, the result will be a vector field

$$\nabla_\mu(\partial_\nu) = \Gamma^\rho{}_{\mu\nu}\partial_\rho \,, \tag{1.1.19}$$

we define $\Gamma^\rho{}_{\mu\nu}$ as the coefficient of this expansion. Consistency with the points (i), (ii), (iii) of the definition implies that the form of the derivative acting on the components of the vector must be

$$\nabla_\mu Y^\nu = \partial_\mu Y^\nu + \Gamma^\nu{}_{\mu\rho} Y^\rho \,. \tag{1.1.20}$$

This is known as the covariant derivative, which is defined in terms of the coefficients $\Gamma^\nu{}_{\mu\rho}$, called the connection. The connection does not transform as a tensor; instead, it transforms in a particular way to make $\nabla_\mu Y^\nu$ a tensor. The covariant derivative of a 1-form ω can be defined consistently by requiring that the covariant derivative of the scalar $X(\omega) = X^\mu\omega_\mu$, with $X \in \mathfrak{X}(\mathscr{M})$, is the partial derivative. This implies that

$$\nabla_\rho \omega_\mu = \partial_\rho \omega_\mu - \Gamma^\lambda{}_{\rho\mu}\omega_\lambda \,. \tag{1.1.21}$$

From here the generalization is straightforward for any section T of the bundle $\otimes^p T\mathscr{M} \otimes^q T^*\mathscr{M}$.

Lorentz covariant derivative

In addition to the above basis of $\mathfrak{X}(\mathscr{M})$, we have a orthonormal basis $\{\hat{e}_a \equiv e_a{}^\mu\partial_\mu\}$, whose inverse coefficients $e^a{}_\mu$ diagonalize the metric. The latin lower-case indices are $\mathrm{SO}(D - \eta_-, \eta_-)$ indices. On this basis, the vector fields read $X = X^a\hat{e}_a$. We can consider the same argumentation and define another connection D, such that for any $X, Y \in \mathfrak{X}(\mathscr{M})$, $\mathrm{D}_X Y = X^a \mathrm{D}_a Y^b \hat{e}_b$. In this way, the connection can be thought of as a differential operator carrying a Lorentz index D_a and acts on the basis as

$$\mathrm{D}_a \hat{e}_b = \omega^c{}_{b|a} \hat{e}_c \,. \tag{1.1.22}$$

Under this consideration, the Lorentz covariant derivative is defined to act on a vector in the vielbein basis as

$$\mathrm{D}_a X^b = \partial_a X^b + \omega^b{}_{c|a} X^c \,, \tag{1.1.23}$$

contracting with the basis $\{e^a\}$ of sections of the co-tanget bundle and defining $\mathrm{D} = e^a \mathrm{D}_a$, the exterior derivative $\mathrm{d} = e^a \partial_a$ and $\omega^b{}_c = \omega^b{}_{c|a} e^a$, the above equation reads

$$\mathrm{D} X^b = \mathrm{d} X^b + \omega^b{}_c X^c \,. \tag{1.1.24}$$

The 1-form $\omega^a{}_b$ is called *1-form spin connection* and is instrumental in the definition of spinors on a manifold. From the above condition, the covariant derivative can also be defined for 1-forms by lowering the index with the flat metric

$$\mathrm{D} X_b = \mathrm{d} X_b + \omega_b{}^c X_c \,. \tag{1.1.25}$$

Observe that if we contract the above equation with $\wedge e^b$, we will obtain a 2-form

$$\mathrm{D} \wedge X^\flat = \mathrm{d} \wedge X + \omega_b{}^c X_c \wedge e^b \,, \tag{1.1.26}$$

where $X^\flat = X_a e^a$ is a 1-form. Then, the generalization of the action of D on the components of any section of the form $\otimes^p T^*\mathscr{M} \otimes^q T\mathscr{M}$ is natural. In particular we can consider the flat metric itself $\eta = \eta_{ab} e^a \otimes e^b$. This gives an interesting property:

$$\mathrm{D}\eta_{ab} = \mathrm{d}\eta_{ab} + \omega_a{}^c \eta_{cb} + \omega_b{}^c \eta_{ac} = \omega_{ab} + \omega_{ba} \,. \tag{1.1.27}$$

If we impose that the flat metric is *compatible* with the spin connection, namely $\mathrm{D}\eta_{ab} = 0$, then

$$\omega_{ab} = -\omega_{ba} \,, \tag{1.1.28}$$

the spin connection is antisymmetric. We consider this assumption throughout this thesis.

Vielbein postulate

From the point of view of the definition of the connection, we are forced to use one basis of $\mathfrak{X}(\mathscr{M})$ or another. Hence, one can impose that the connections acting on vector fields coincide. To be precise, let us consider a section of $T^*\mathscr{M} \otimes T\mathscr{M}$, $\mathrm{D}X = \mathrm{D}X^a \otimes \hat{e}_a$ and $\nabla X = \nabla_\mu X^\nu \mathrm{d}x^\mu \otimes \partial_\nu$. The fact that the two connections acting on vector fields coincide is expressed simply as

$$\mathrm{D}X = \nabla X\,. \tag{1.1.29}$$

Since the above condition must be true for any vector field, it implies a relation between the vielbeins and the connections

$$\partial_\mu e^a{}_\nu + \omega^a{}_{b\mu}\, e^b{}_\nu - \Gamma^\lambda{}_{\mu\nu} e^a{}_\lambda = 0\,. \tag{1.1.30}$$

This is known as the *vielbein postulate* and encodes the equivalence between the two connections for vector fields. We have written the spin connection 1-form in the coordinate basis as $\omega_{ab} = \omega_{ab\mu}\mathrm{d}x^\mu$.

1.1.3 Curvature and torsion

The curvature tensor is defined as the commutator of two covariant derivatives acting on a vector. Let us start by computing the commutator of two Lorentz covariant derivatives:

$$\begin{aligned}\mathrm{D}\wedge\mathrm{D}X^a &= \mathrm{d}\wedge\mathrm{D}X^a + \omega^a{}_b \wedge \mathrm{D}X^b\,,\\ &= \mathrm{d}\wedge(\mathrm{d}X^a + \omega^a{}_c X^c) + \omega^a{}_b\wedge(\mathrm{d}X^b + \omega^b{}_c X^c)\,,\\ &= (\mathrm{d}\omega^b{}_c + \omega^a{}_b\wedge\omega^b{}_c)X^c\,.\end{aligned} \tag{1.1.31}$$

In arriving at the last equation, we have used the graded Leibniz rule of the exterior derivative. We define the curvature 2-form in this way

$$\mathrm{D}\wedge\mathrm{D}X^a \equiv \mathsf{R}^a{}_b V^b \implies \mathsf{R}^a{}_b = \mathrm{d}\omega^a{}_b + \omega^a{}_c\wedge\omega^c{}_b\,. \tag{1.1.32}$$

The 2-form $\mathsf{R}^a{}_b$ is called the curvature 2-form of $\mathrm{SO}(D-\eta_-,\eta_-)$. The last important definition is the torsion 2-form, defined as

$$\mathsf{T}^a \equiv \mathrm{D}e^a = \mathrm{d}e^a + \omega^a{}_b \wedge e^b\,. \tag{1.1.33}$$

Interestingly, there are no more tensors that we can generate independently from the torsion and the curvature. Indeed, the equations that we have defined:

$$\mathsf{T}^a = \mathrm{D}e^a\,, \tag{1.1.34}$$

$$\mathsf{R}^a{}_b = \mathrm{d}\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b\,, \tag{1.1.35}$$

regarded as equations for the vielbein an the spin connection given the torsion and the curvature, are called the *Maurer-Cartan structure equations.* They generalize the Frenet-Serret equation for curves to higher-dimensional manifolds equipped with a moving frame.

These tensors can be expanded in the basis of the 2-forms as follows

$$\mathsf{R}^a{}_b = \frac{1}{2} R^a{}_{b\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\,, \tag{1.1.36}$$

$$\mathsf{T}^a = \frac{1}{2} T^a{}_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\,. \tag{1.1.37}$$

Using the vielbein postulate (1.1.30) and the definitions (1.1.32)-(1.1.33), it is straightforward to show that in the coordinate basis the components of (1.1.36)-(1.1.37) read

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho{}_{\nu\sigma} - \partial_\nu \Gamma^\rho{}_{\mu\sigma} + \Gamma^\rho{}_{\mu\lambda}\Gamma^\lambda{}_{\nu\sigma} - \Gamma^\rho{}_{\nu\lambda}\Gamma^\lambda{}_{\mu\sigma}\,, \tag{1.1.38}$$

$$T^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu}\,. \tag{1.1.39}$$

If we compute the commutator of two covariant derivatives using the affine connection in the coordinate basis, we find that it is written in terms of the above tensors as follows

$$[\nabla_\mu, \nabla_\nu]X^\rho = R^\rho{}_{\sigma\mu\nu}X^\sigma - T^\lambda{}_{\mu\nu}\nabla_\lambda X^\rho\,. \tag{1.1.40}$$

Metric compatible connection

A metric-compatible connection can be defined by imposing that the corresponding covariant derivative acting on the metric is zero. For example, if we impose that the connection ∇ is metric compatible with $g_{\mu\nu}$. This means that

$$\nabla_\mu g_{\rho\sigma} = 0 \iff \partial_\mu g_{\rho\sigma} - \Gamma^\lambda{}_{\mu} g_{\lambda\sigma} - \Gamma^\lambda{}_{\mu} g_{\rho\lambda} = 0\,. \tag{1.1.41}$$

The latter is an equation for the connection with curve indices. The equation can be solved by making three copies of it, interchanging the indices cyclically, and then summing the first two and subtracting the last. The result is the well-known Levi-Civita connection

$$\Gamma^\lambda{}_{\mu\nu} = \frac{1}{2} g^{\lambda\delta}(\partial_\mu g_{\nu\delta} + \partial_\nu g_{\mu\delta} - \partial_\delta g_{\mu\nu})\,, \tag{1.1.42}$$

which is symmetric in the lower indices, implying that the torsion tensor is identically zero. In this case, the curvature tensor (1.1.38) is named *Riemann curvature tensor.*

We remark that imposing the metric-compatible conditions for the general affine connection $\Gamma^\rho{}_{\mu\nu}$ is not equivalent to requiring the same for the spin connection. For the latter, imposing metric compatibility in (1.1.27) did not eliminate the torsion. Indeed, the torsion is given in that case by (1.1.33). Imposing that the torsion is equal to zero in the latter case leads to:

$$0 = \mathrm{d}e^a + \omega^a{}_b \wedge e^b\,. \tag{1.1.43}$$

The spin connections $\omega^a{}_b$ satisfying the above equation are called *torsionless spin connection.* The above equation is an equation for the torsionless spin connection in terms of the vielbeins. A convenient and useful way to solve it is by using the contraction operator ι_a that maps p-forms into $(p-1)$-forms with a Lorentz index. Acting twice on the equation (1.1.43), applying the same trick as for the Levi-Civita connection, and then contracting with a 1-form basis, we find that

$$\omega_{ab} = \frac{1}{2}\iota_a \iota_b \mathrm{d}e_c e^c - \iota_{[a}\mathrm{d}e_{b]}\,. \tag{1.1.44}$$

This equation is useful as everything is expressed in terms of the exterior derivatives of the vielbeins. In general, we will work in a torsion-less setup, for which the spin connection and the Levi-Civita connection are equivalent as a consequence of the vielbein postulate. However, in a more general setup, where the torsion is in general non-zero, it is useful to split the general connection as $\omega_{ab} = \Omega_{ab} + \kappa_{ab}$, where Ω_{ab} is a torsion-less connection and κ_{ab} is Lorentz tensor 1-form called *contorsion.* Under these considerations, the torsion becomes determined completely in terms of the contorsion $\mathsf{T}^a = \kappa^a{}_b \wedge e^b$.

1.1.4 Complex manifolds

Up to now, we have dealt with real manifolds: in each chart, there exists an invertible map to a subset of $\mathbb{R}^D$. However, in supergravity and in the geometry of compact manifolds, complex manifolds play an important role. We define a *complex manifold* $\mathscr{M}$ as a differentiable manifold equipped with a holomorphic atlas. We define an holomorphic atlas on $\mathscr{M}$, $\{(\mathscr{U}_i, \varphi_i)\}$ with $\mathscr{U}_i \subset \mathscr{M}$ open and $\varphi_i : \mathscr{U}_i \to \varphi_i(\mathscr{U}_i) \subset \mathbb{C}^n$ such that in the intersections the functions

$$\varphi_{ij} = \varphi_i \circ \varphi_j^{-1} : \varphi_j(\mathscr{U}_i \cap \mathscr{U}_j) \to \varphi_i(\mathscr{U}_i \cap \mathscr{U}_j)$$

are holomorphic. We will say that two atlas

$$\{(\mathscr{U}_i, \varphi_i)\} \qquad \text{and} \qquad \{(\mathscr{U}'_j, \varphi'_j)\}$$

are equivalent if and only if $\varphi_i \circ \varphi_j'^{-1}$, defined in the appropriated intersection, are holomorphic $\forall i, j$.

Equivalently, we can define a complex manifold $\mathscr{M}$ of dimension n as a real differentiable manifold $\mathscr{N}$ of real dimension $2n$ with an equivalence class of holomorphic atlas. Essentially, the complex manifold $\mathscr{M} = (\mathscr{N}, \{\mathscr{U}_i, \varphi_i\})$ is the space $\mathscr{N}$ in addition to a complex structure.

An holomorphic function on a complex manifold $f : \mathscr{M} \to \mathbb{C}$ is a function such that $f \circ \varphi_i^{-1} : \varphi_i(\mathscr{U}_i) \to \mathbb{C}$ holomorphic for all charts of $\{(U_i, \varphi_i)\}$.

Example: Complex projective space

The projective space $\mathbb{CP}^n$ is defined as the set of lines in $\mathbb{C}^{n+1}$. Equivalently

$$\mathbb{CP}^n = (\mathbb{C}^{n+1} \backslash \{0\})/\mathbb{C}^* \,, \qquad \mathbb{C}^* = \mathbb{C} \backslash \{0\} \tag{1.1.45}$$

where $\mathbb{C}^*$ acts as multiplication on $\mathbb{C}^{n+1}$. The points of $\mathbb{CP}^n$ are denoted by $(z_0 : \cdots : z_n)$ such that

$$(\lambda z_0 : \cdots : \lambda z_n) \qquad \text{and} \qquad (z_0 : \cdots : z_n)$$

represent the same point in $\mathbb{CP}^n$ for $\lambda \in \mathbb{C}^*$. Because of this, we have a projection map

$$\texttt{proj} : \mathbb{C}^{n+1} \backslash \{0\} \to \mathbb{CP}^n \,. \tag{1.1.46}$$

We say that $\mathscr{U} \subseteq \mathbb{CP}^n$ is open if and only if $\texttt{proj}^{-1}(\mathscr{U})$ is open in $\mathbb{C}^{n+1}$.

In the case of $\mathbb{CP}^n$ we have a standard open covering. The only point that does not represent anything in $\mathbb{CP}^n$ is the origin. We can cover $\mathbb{CP}^n$ by using the following sets $\mathscr{U}_i = \{(z_0 : \cdots : z_n) \mid z_i \neq 0\}$. We will prove that $\{(\mathscr{U}_i, \varphi_i)\}$ defines an atlas of $\mathbb{CP}^n$ with

$$\begin{aligned}\varphi_i &: \mathscr{U}_i \to \mathbb{C}^n \\ \varphi_i(z_0 : \cdots : z_n) &= \left(\frac{z_0}{z_i}, \dots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \dots, \frac{z_n}{z_i}\right) \in \mathbb{C}^n \,.\end{aligned}$$

Let us show that this is a complex manifold. The transition functions are $\varphi_{ij} = \varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$. We want to see if they are holomorphic. Defining the inverse function as

$$\begin{aligned}\varphi_i^{-1} &: \mathbb{C}^n \to U_i \subset \mathbb{CP}^n \\ \varphi_i^{-1}(w_0, \dots, w_{i-1}, w_{i+1}, \dots, w_n) &= (w_0 : \cdots : w_{i-1} : 1 : w_i : \cdots : w_n) \,.\end{aligned}$$

We check that it is indeed the inverse:

$$\begin{aligned}&\varphi_i^{-1} \circ \varphi_i : \mathscr{U}_i \to \mathscr{U}_i \,, \qquad (1.1.47)\\ &(z_0 : \cdots : z_n) \mapsto \varphi_i^{-1} \circ \varphi_i(z_0 : \cdots : z_n) = \varphi_i^{-1}\left(\frac{z_0}{z_i}, \dots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \dots, \frac{z_n}{z_i}\right) \,, \\ &\qquad = \left(\frac{z_0}{z_i} : \cdots : \frac{z_{i-1}}{z_i} : 1 : \frac{z_{i+1}}{z_i} : \cdots : \frac{z_n}{z_i}\right) \,, \\ &\qquad = (z_0 : \cdots : z_n) \,.\end{aligned}$$

In the last equality, we pick another representative of the equivalence class, which is obtained by multiplying all the entries by z_i, since it is different from zero. Now, let us prove that the composite map is a holomorphic function. For this, we consider $\varphi_i : \mathscr{U}_i \to \mathbb{C}^n$ and $\varphi_j : \mathscr{U}_j \to \mathbb{C}^n$. Then,

$$\begin{aligned}&\varphi_{ij} = \varphi_i \circ \varphi_j^{-1} : \mathbb{C}^n \to \mathbb{C}^n \qquad (1.1.48)\\ &(w_1 \dots w_n) \mapsto \varphi_{ij}(w_1 \dots w_n) = \varphi_i \circ \varphi_j^{-1}(w_1 \dots w_n) \,, \\ &\qquad = \varphi_i(w_1 : \cdots : w_{j-1} : 1 : w_{j+1} : \cdots : w_n) \,,\end{aligned}$$

$$= \left(\frac{w_1}{w_i}, \ldots, \frac{w_{i-1}}{w_i}, \frac{w_{i+1}}{w_i}, \ldots, \frac{w_{j-1}}{w_i}, \frac{1}{w_i}, \frac{w_j}{w_i}, \ldots, z_n \right) .$$

It is a holomorphic function. The conclusion is that $\mathbb{CP}^n$ with the atlas $\{(U_i, \varphi_i)\}$ that we described is an complex manifold of dimension n.

1.1.5 Action of groups on a manifold

Holonomy

The definition of the holonomy group is the following. Let us consider a $m-$dimensional Riemannian manifold $(\mathscr{M}, \boldsymbol{g})$ with an affine connection ∇. The connection naturally defines the action of a group on the tangent space at $p \in \mathscr{M}$ into itself. To see this, consider a point $p \in \mathscr{M}$ and the set of closed loops at p, namely

$$L_p = \{c(t) : t \in [0,1],\, c(0) = c(1) = p\} .$$

Now, we take a vector $X \in T_p\mathscr{M}$ and parallel transport X along a curve $c(t)$. After the transport around the curve $c(p)$, we will end up with a new vector $X_c \in T_pM$. Therefore, the loop $c(t)$ and the connection ∇ induce a linear transformation $P_c : \; T_p\mathscr{M} \to T_p\mathscr{M}$. The set of these transformations

$$H_\nabla(p) = \{P_c : c \in L_p\} \tag{1.1.49}$$

is called the *holonomy group* at p. Let us see why $H_\nabla(p)$ is a group: (i) The identity element is the constant loop. (ii) The product $P_{c'}P_c$ corresponds to parallel transport along c and then along c'. (iii) The inverse of P_c is $P_{c^{-1}}$ which curve $c^{-1}(t) = c(1-t)$, the same as c but backwards.

Some properties of $H_\nabla(p)$ are the following: $H_\nabla(p)$ is a subgroup of $GL(m, \mathbb{R})$. Clearly $H_\nabla(p)$ depends on the curvature of the manifold, therefore if the Riemann tensor vanishes then $H_\nabla(p)$ is trivial. If $(\mathscr{M}, \boldsymbol{g})$ is parallelizable[1] we can make $H_\nabla(p)$ trivial.

Lie group acting on a manifold

Let us consider G a Lie group that acts on a manifold $\mathscr{M}$ by $\sigma : G \times \mathscr{M} \to \mathscr{M}$. The action σ is said to be

[1]If an $D-$dimensionall manifold $\mathscr{M}$ admits D vector fields which are linearly independent everywhere, M is said to be *parallelizable*. This implies that the tangent bundle and co-tangent bundle are trivial bundles [36].

- **Transitive:** if for any pair of points $p, q \in M$, there exist an element $\mathrm{g} \in G$ such that $\sigma(\mathrm{g}, p) = q$.
- **Free:** if any non-trivial element $\mathrm{g} \in G$ has no fixed points in $\mathscr{M}$. In other words, for any point $p \in \mathscr{M}$ and $\mathrm{g} \in G$ different form the identity, $\sigma(\mathrm{g}, p) \neq p$ or if there exist a $p \in \mathscr{M}$ such that $\sigma(\mathrm{g}, p) = p$ then $g = e$.
- **Effective:** if the unit element $e \in G$ is the unique element that defines the trivial action on $\mathscr{M}$. i.e. if $\sigma(\mathrm{g}, p) = p$ for all $p \in \mathscr{M}$, then g must be the identity.

A free action is effective but the coverse is not in general true. One aspect of a free action g acting on $\mathscr{M}$ is the following. Take two points $p, q \in \mathscr{M}$ then let us assume that there are two elements $\mathrm{g}, \mathrm{g}' \in G$ such that $\sigma(\mathrm{g}, p_1) = p_2$ and $\sigma(\mathrm{g}', p_1) = p_2$. Then $\sigma(\mathrm{g}, p) = \sigma(\mathrm{g}', p)$. Acting with $\sigma(\mathrm{g}'^{-1}, \cdot)$ leads to $\sigma(\mathrm{g}'^{-1}\mathrm{g}, p) = p$. Since we are assuming that the action σ is free, then the last equation implies that $\mathrm{g}' = \mathrm{g}$, and hence for a free action, there is a unique element that connects two points in $\mathscr{M}$.

Isotropy group

Let G be a Lie group that acts on a manifold $\mathscr{M}$. The isotropy group of $p \in \mathscr{M}$ is a subgroup of G defined as

$$H(p) = \{\mathrm{g} \in G : \sigma(\mathrm{g}, p) = p\}\,.$$

The group $H(p)$ is called the *little group* or *stabilizer* of p. It is a group because (i) the identity is in $H(p)$. (ii) From the previous point $\sigma(\mathrm{g}^{-1}\mathrm{g}, p) = p \iff \sigma(\mathrm{g}^{-1}, \sigma(\mathrm{g}, p)) = p$, thus if $\mathrm{g} \in H(p)$ then also $\mathrm{g}^{-1} \in H(p)$. (iii) The product of two elements in $\mathrm{g}, \mathrm{g}' \in H(p)$ is also in $H(p)$. (iv) The associativity is inherited from the group product.

If G acts freely on a manifold $\mathscr{M}$, then it is easy to see that $H(p) = \{e\}$ for any p.

In the next section, we give a brief review of General Relativity and black hole physics.

1.2 Gravity and Black Holes

General Relativity (GR) is a theory of gravity that describes gravitational interactions as the curvature of a 4-dimensional manifold. Formulated by Einstein in 1915, GR generalizes Newton's theory of gravity, which is incompatible with the Lorentz invariance of Maxwell's theory of electromagnetism and, consequently, with the special theory of relativity. As we discussed at the beginning of Section 1.1, in special relativity, the speed of light is a universal constant, and two inertial observers in relative motion agree on this by mapping their coordinates through a Lorentz transformation. In this framework, events of the spacetime are points of the Minkowski space $\mathbb{M}_4$, that is, an affine 4-dimensional space equipped with an inner product. This structure allows one to associate with each point $p \in \mathbb{M}_4$ three disjoint sets, the causal future of p, the causal past of p, and the set of causally unrelated events to p. This geometric structure is generalized by General Relativity, whose cornerstone lies in the so-called equivalence principle, which states that, locally, a gravitational field is indistinguishable from accelerated motion. As a consequence of this principle, a free-falling observer in a gravitational field cannot tell the difference between what they observe and what they would observe in Minkowski spacetime. This observation allows us to generalize the geometrical structure of Minkowski spacetime to any observer, not necessarily inertial.

Currently, GR has solid experimental evidence. Among the historical tests are the precession of the perihelion of Mercury, the bending of light rays in the presence of a gravitational field, and the gravitational redshift/blueshift of an electromagnetic wave traveling in a gravitational field. While the precession of the perihelion of Mercury was known at the time of the first formulation of General Relativity, the last two phenomena were genuine predictions of the theory that were tested experimentally in the first half of the 20th century. During the 70s, Irwin Shapiro formulated a new prediction of GR, which is called the Shapiro delay. It describes a relativistic delay on a signal passing through a gravitational field. It was tested in the weak field approximation at the level of the solar system, and in a strong gravitational regime using the famous Hulse-Taylor pulsar, showing agreement with the prediction of GR. For a review, see for example, [38].

One of the outstanding predictions of General Relativity is the existence of gravitational waves. For a long time, these waves were thought to be so tiny that their detection seemed beyond the reach of current technology. Nevertheless, on September 14, 2015, they were detected for the first time by the LIGO-Virgo collaboration [39]. The prediction of them remontes to Einstein's articles [40, 41], where he suggested the existence of excitation of the linearized theory that propagates at the speed of light. Since the theory is not linear, for a long time it was not clear whether these waves exist as a solution of the non-linear equations, or whether they are an artifact of the linearization. In a series of works by Bondi and Pirani they showed the existence of plane wave solutions to the non-linear equation [42]. Later on, Bondi, Van der Burg, Metzner [43], and Sachs [44] studied the boundary condition at the future null infinity on an asymptotically flat spacetime, and they showed that gravitational waves can transfer energy. In addition to this observation, they prove the existence of an infinite-dimensional symmetry group known as the BMS group, which corresponds to the semi-direct product between SO(3,1) and the real-valued smooth functions on the 2-sphere.

In the language introduced in the Section 1.1, GR is formulated as a theory for the spacetime itself described by a 4-dimensional pseudo-Riemannian manifold $\mathscr{M}$ with signature $(-1,1,1,1)$. Physical events in the spacetime are interpreted as points of the manifold. The manifold structure implements automatically the arbitrariness on the choice of coordinates, and physically meaningful quantities are quantities that are well defined on the manifold, i.e., quantities that are independent of the choice of the local coordinates. The dynamical field of the theory is the metric field $\boldsymbol{g} : \mathscr{M} \to \otimes^2 T^*\mathscr{M}$, that is, a symmetric section of the $\otimes^2 T^*\mathscr{M}$ bundle. The connection is considered to be the Levi-Civita affine connection ∇ associated with the metric defined in (1.1.42). Timelike geodesics defined on the manifold are interpreted as free-falling observers in the gravitation field, and null geodesics are interpreted as the trajectories of light rays. From now on, we will work in a local patch $(\mathscr{U}, \phi)$ with $\mathscr{U} \subset \mathbb{R}^4$, $\phi(p) = (x^0(p), x^1(p), x^2(p), x^3(p)) = (x^\mu(p))$ and the local basis of sections of the tanget and co-tanget vector space at $p \in \mathscr{M}$ are $\{\partial_\mu\}$ and $\{\mathrm{d}x^\mu\}$. The dynamics of the theory is governed by Einstein's field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \,, \tag{1.2.1}$$

where $R_{\mu\nu}, R$ are the Ricci tensor and Ricci scalar, respectively. The theory has two free parameters, the Newton's constant G and the cosmological constant Λ. The energy-momentum tensor $T_{\mu\nu}$ depends on the matter fields.

The equations (1.2.1) are highly non-linear and have a gauge redundancy associated with the invariance under diffeomorphisms. Even in the vacuum, namely when $T_{\mu\nu} = 0$, the equations are highly complicated and admit non-trivial solutions. The first non-trivial solution was found by Schwarzschild in 1916, one year after Einstein's article. Nowadays, we know that the solution represents a spherically symmetric black hole, but it took decades to understand its physical meaning. A key step was the introduction of Eddington-Finkelstein coordinates, which clarified the nature of the Schwarzschild horizon and removed the coordinate singularity in that region. The next section will be devoted to black holes and we will define them in a more accurate way. The equations (1.2.1) also admit cosmological solutions for $\Lambda > 0$, particularly when the energy-momentum tensor corresponds to that of an ideal fluid. In the absence of matter, the spacetime is de Sitter. These configurations play an important role in understanding the current state of our universe. Observations indicate that galaxies are moving away from Earth with a velocity proportional to their distance, known as Hubble's law. The current accelerated expansion of the universe is well modeled by a de Sitter-like phase, driven by a positive cosmological constant or dark energy (see e.g. [45]).

1.2.1 Black holes and no-hair theorems

A geometry that satisfies Einstein's field equations does not imply that it is possible to realize it in nature. For instance, there exist solutions of the equations that are naked singularities, namely, there is a timelike region where the curvature is infinite, and then there are points whose causal past always intersects the singularity. These kinds of spaces do not allow the formulation of the Cauchy problem, since we need to prescribe values for the fields in regions that are not part of the manifold, e.g., the singularity. The situation is dramatic when the field is considered to be the metric itself; in that case, the metric is divergent at the singularity.

Black holes are geometrical objects (manifolds) that contain a null surface called the event horizon. It separates the spacetime into two regions in such a way that at any point p on the horizon, the causal future of p intersects only with one of the regions, and the causal past of p intersects only with the other, making the horizon a one-way surface. This region is usually a smooth, traversable surface that is path-connected with an asymptotic region of the spacetime.

A more formal definition of a black hole is as follows. Let $\mathscr{I}^+$ be the future null infinity of a manifold $\mathscr{M}$ — the region where the future pointing null geodesics end for an asymptotic value of their affine parameter. We define the black hole region as $\mathscr{M} \backslash J^-(\mathscr{I}^+)$, where $J^-(\mathscr{I}^+)$ denotes the causal past of the future null infinity. The event horizon is defined as the boundary of the black hole region. This definition is teleological because determining the horizon requires knowledge of the full future evolution of the spacetime to identify $\mathscr{I}^+$.

Black holes are special because we can describe how they form. It is commonly understood that black holes are the product of the collapse of a star with mass above the Tolman-Oppenheimer-Volkoff (TOV) limit, which is about $2.2 - 2.9$ solar masses. Historically, the first bound regarding the collapse of a star was computed by Subrahmanyan Chandrasekhar. Nowadays, the Chandrasekhar bound, which is about 1.4 solar masses, constitutes the maximum mass of a collapsing star to form a white dwarf star. Stars with masses between the Chandrasekhar and TOV limit form a neutron star.

General Relativity admits black hole solutions. In 4-dimensions with $\Lambda = 0$, the topology of the horizon is fixed by the energy conditions: Hawking's theorem [46] states that in order to satisfy the energy conditions the topology of the horizon must be spherical, whence compact. The most general asymptotically flat and spherically symmetric solution is Schwarzschild black hole corresponding to a static eternal black hole, as it is asserted by Birkhoff theory. In pure gravity, the most general asymptotically flat black hole corresponds to the Kerr black hole. It is a statonary axisymmetric configuration characterized by its mass and angular momentum.

By coupling GR with Maxwell theory, the most general black hole solution is the Kerr-Newmann black hole characterized by its mass, angular momentum, and electric charge. Around the 70s, Israel and Carter had a collection of results [47, 48, 49] indicating that Kerr-Newman is the most general stationary solution that approaches to flat ((A)dS) spacetime in the asymptotic region. One consequence of this is that black holes are characterized by a small number of parameters, in the case of Kerr-Newman, just three real numbers. In this respect, from the GR description of a black hole, there is not much information that one can extract from it. In [46] Hawking showed that in General Relativity the area of black holes cannot decrease. These observations suggested a similarity between black holes' physical properties in GR and thermodynamic laws. This was stated in the paper *Four laws of black hole mechanics* by Bardeen, Hawking and Carter [50]. They establish that:

(i) The surface gravity κ of a black hole is constant at the event horizon, which is reminiscent to the zero law of thermodynamics,

(ii) a small variation of the mass leads to a variation on the horizon area $\mathcal{A}$ and a variation on the appropriate charges satisfying the following equation

$$\delta M = \frac{\kappa}{2\pi}\delta\mathcal{A} + \Omega\delta J + \Phi\delta Q\,, \tag{1.2.2}$$

which is reminiscent to the first law of thermodynamics if the surface gravity is related to the temperature and the horizon area with the entropy,

(iii) Hawking's results of non-decreasing area of the black holes, can be put into correspondence with the non-decreasing of the entropy—the second law of thermodynamics.

Due to the definition of black holes and uniqueness theorems in General Relativity, a problem with the laws of thermodynamics emerges, particularly with the second law, which states that the entropy of a closed system never decreases. The problem arises when considering a black hole and a small system outside the black hole with a certain amount of entropy, such as a glass of water. When you throw the glass into the black hole, the system becomes causally disconnected from you. Consequently, it is no longer possible to count its microstates, and the entropy of the universe seems to decrease. This observation leads Bekenstein [51] to conjecture that black holes must carry an entropy and, if so, must be proportional to the surface area of the black hole: $S \propto c^3 k_B \text{Area}/\hbar G$, where the proportionality constant must be dimensionless of order one. In [11], Hawking showed that classical black holes, considered as a background for a quantum field, radiate as a black body with a temperature proportional to the surface gravity. The precision of the computation allows us to assert that, indeed, the factor of order one missing in Bekenstein's formula is 1/4. In this way, the picture is complete, and nowadays we accept that black holes are thermodynamic objects with a temperature and entropy.

Black holes with scalar field

One natural question is whether one can construct black hole solutions supporting a non-trivial scalar field. Let us review the Bekenstein no-go theorem [10, 52], which can be announced as follows.

For an arbitrary theory of gravity coupled minimally to a scalar field with a potential with positive second derivative outside the horizon, a static black hole solution implies that the scalar field must be constant.

Proof. Consider a D-dimensional manifold $\mathscr{M}$ endowed with a static metric. The local form of the metric in a patch with coordinates $(t, x^{\mathbf{i}})$, $\mathbf{i} = 1, \ldots, d-1$, and a time-independent scalar field

$$\mathrm{d}s^2 = -N^2(x^{\mathbf{i}})\mathrm{d}t^2 + h_{\mathbf{ij}}(x^{\mathbf{k}})\mathrm{d}x^{\mathbf{i}}\mathrm{d}x^{\mathbf{j}}\,, \tag{1.2.3}$$

$$\phi = \phi(x^{\mathbf{i}})\,. \tag{1.2.4}$$

This metric will corresponds to a black hole, if there exist a codim-1 surface $\mathcal{H} = \{r(x^{\mathbf{i}}) - r_+ = 0\}$ such that the 1-form $\mathrm{d}r(x^{\mathbf{k}})$ is null at $\mathcal{H}$. In order to preserve the signature, this implies that $N|_{\mathcal{H}} = 0$. Considering the vielbeins $V^0 = N\mathrm{d}t$, $V^i = e^i{}_{\mathbf{j}}\mathrm{d}x^{\mathbf{j}}$ such that $h_{\mathbf{ij}} = e^i{}_{\mathbf{i}}e^j{}_{\mathbf{j}}\delta_{ij}$. We assume that the scalar field is minimally coupled, then the equation of motion of the scalar is the Klein-Gordon equation

$$\mathrm{d}\star\mathrm{d}\phi - V'(\phi)\star 1 = 0\,, \tag{1.2.5}$$

Using the definition of the Hodge dual $\star$ on $\mathscr{M}$, one can show that $\star\mathrm{d}\phi = -N\mathrm{d}t\wedge\underline{\star}\mathrm{d}\phi$, where $\underline{\star}$ is the Hodge star computed on the embedded manifold $t =$ constant. Therefore, evaluating the equations of motion

$$-\mathrm{d}(N\mathrm{d}t\wedge\underline{\star}\mathrm{d}\phi) - V'(\phi)\star 1 = 0\,. \tag{1.2.6}$$

Multiplying by V' and integrating by parts we find that

$$-\mathrm{d}(V'N\mathrm{d}t\wedge\underline{\star}\mathrm{d}\phi) + NV''(\phi)\mathrm{d}\phi\wedge\mathrm{d}t\wedge\underline{\star}\mathrm{d}\phi - V'^2(\phi)\star 1 = 0\,. \tag{1.2.7}$$

Let us integrate on the space Σ defined between $\mathcal{H}$ and the asymptotic region denoted by $\mathcal{A}$, such that $\partial\Sigma = \mathcal{H}\cup\mathcal{A}$. The Hodge star can be split as $\star 1 = N\mathrm{d}t\wedge\underline{\star}1$. Hence, the integral of (1.2.7) is given by

$$-\int_\Sigma \mathrm{d}(V'N\mathrm{d}t\wedge\underline{\star}\mathrm{d}\phi) + \int_\Sigma NV''(\phi)\mathrm{d}\phi\wedge\mathrm{d}t\wedge\underline{\star}\mathrm{d}\phi - \int_\Sigma V'^2(\phi)\star 1 = 0\,. \tag{1.2.8}$$

The first term has two contributions, one from the $\mathcal{H}$, which vanishes since $N|_{\mathcal{H}} = 0$, and the other from the asymptotic region $\mathcal{A}$. Assuming the scalar field approaches a constant quickly enough near the asymptotic region, the latter term also vanishes. This is a sensible assumption for having finite energy. The resulting expression is

$$\int_\Sigma N\mathrm{d}t\wedge\left(V''(\phi)\mathrm{d}\phi\wedge\underline{\star}\mathrm{d}\phi + V'^2(\phi)\underline{\star}1\right) = 0\,. \tag{1.2.9}$$

Implementing the assumption that the scalar potential has a positive second derivative everywhere, i.e., $V''(\phi) > 0$, the above equation is the sum of two positive terms, which can be zero if and only if each of them is zero. In this situation, the only solution is $\phi =$ constant. □

One possible way to circumvent Bekenstein's theorem is to consider scalar potentials whose second derivative is not necessarily positive everywhere; a Higgs potential is an example. However, in General Relativity, there are no-hair theorems for $\Lambda = 0$, that prohibit the existence of black holes with a minimally coupled scalar with an arbitrary potential. One of these results was proven by Sudarsky, which can be announced as follows: In General Relativity with a minimally coupled scalar field with an arbitrary potential with $\Lambda = 0$, it is not possible to have black holes with non-trivial profile for the scalar field [53]. It has been shown that it is possible to avoid the no-hair theorems by considering non-minimally coupled scalar fields. The first model that allows for finding hairy black holes was Einstein gravity with a conformally coupled scalar field. The part of the action containing the coupling is invariant under Weyl rescaling of the metric. In this setup, it is possible to construct hairy asymptotically flat black holes and cosmologies [54, 55], and a variety of black holes, solitons, and wormholes with cosmological constant in 4-dimensions [56, 57, 58]. These models have shown to be a fertile area to explore this kind of configuration (for a review see [59]).

In the context of supergravity theories, scalar fields play an instrumental role in their construction. Typically, there is a non-linear coupling between the scalars and the other fields, and for gauged supergravities, a scalar potential is also present (see, e.g., [60]). Under these conditions, it is generally non-trivial to consistently set the scalar fields to zero; the careful analysis of this issue falls under the framework of consistent truncations. Consequently, solutions of a supergravity theory generally feature non-trivial scalar fields. A concrete realization of a consistent truncation of the maximal SO(8)-gauged supergravity in $D = 4$ will be presented in Section 5.1. Examples of solutions with non-trivial scalar fields will be given in Chapters 4 and 5. Perturbative black hole solutions in the context of higher-curvature corrections, involving a non-minimal coupling between a scalar field and the Gauss-Bonnet density, will be discussed in Chapter 2.

So far, we have reviewed some of the black hole properties and their interpretations as thermodynamic objects. In the next section, we introduce a prescription to compute charges in General Relativity and beyond GR, known as Iyer-Wald prescription introduced in [61, 62, 63].

1.3 Conserved charges and black holes thermodynamics

1.3.1 Iyer-Wald charges

General Relativity is a theory for the metric field, and its action principle depends on the metric in the integration measure and through the trace of the Riemann tensor. Then, any generalization that involves only the metric field should be written in this way, e.g., power and traces of the Riemann tensor. To obtain a generic expression for the charges, we will consider a generic Lagrangian that depends on the metric explicitly and on the metric only through the Riemann tensor:

$$I\left[g_{\mu\nu}\right]=\int_{\mathscr{M}} d^{D}x\sqrt{-g}\mathcal{L}\left[R^{\mu\nu}{}_{\rho\sigma}\left(\boldsymbol{g}\right)\right], \tag{1.3.1}$$

where $R^{\lambda\rho}{}_{\mu\nu}=g^{\rho\sigma}R^{\lambda}{}_{\sigma\mu\nu}$ is the Riemann tensor. The metric is the only dynamical field, and the connection is the Levi-Civita connection. An arbitrary variation is given by

$$\begin{aligned}\delta I&=\int d^{D}x\sqrt{-g}\left(-\frac{1}{2}\delta g^{\mu\nu}g_{\mu\nu}\mathcal{L}+\frac{\partial\mathcal{L}}{\partial R^{\mu\nu}{}_{\rho\sigma}}\delta R^{\mu\nu}{}_{\rho\sigma}\right),\\&=\int d^{D}x\sqrt{-g}\left(-\frac{1}{2}\delta g^{\mu\nu}g_{\mu\nu}\mathcal{L}+E_{\mu\nu}{}^{\rho\sigma}g^{\nu\lambda}\delta R^{\mu}{}_{\lambda\rho\sigma}+E_{\lambda\nu}{}^{\rho\sigma}\delta g^{\mu\nu}R^{\lambda}{}_{\mu\rho\sigma}\right),\end{aligned} \tag{1.3.2}$$

where we define the following tensor, which has the same symmetries as the Riemann tensor:

$$E_{\mu\nu}{}^{\rho\sigma}=\frac{\partial\mathcal{L}}{\partial R^{\mu\nu}{}_{\rho\sigma}}. \tag{1.3.3}$$

The last term in (1.3.2), can be worked out by using the definition of the Riemann tensor

$$\begin{aligned}E_{\mu\nu}{}^{\rho\sigma}g^{\nu\lambda}\delta R^{\mu}{}_{\lambda\rho\sigma}&=2E_{\mu\nu}{}^{\rho\sigma}g^{\nu\lambda}\nabla_{[\rho}\delta\Gamma^{\mu}{}_{\sigma]\lambda}\,,\\&=2\nabla_{\rho}(E_{\mu\nu}{}^{\rho\sigma}g^{\nu\lambda}\delta\Gamma^{\mu}{}_{\sigma\lambda})-2\nabla_{\rho}E_{\mu\nu}{}^{\rho\sigma}g^{\nu\lambda}\delta\Gamma^{\mu}{}_{\sigma\lambda}\,,\\&=2\nabla_{\rho}(E_{\mu}{}^{\lambda\rho\sigma}\delta\Gamma^{\mu}{}_{\sigma\lambda}-\nabla_{\lambda}E^{\delta\rho\lambda\sigma}\delta g_{\sigma\delta})-2\nabla_{\lambda}\nabla_{\rho}E_{\delta}{}^{\lambda\rho}{}_{\sigma}\delta g^{\sigma\delta}\,.\end{aligned} \tag{1.3.4}$$

From the second equality to the third, we used the definition of the Christoffel symbol and integrated by parts. Then, the equation (1.3.2) can be written in the following way:

$$\delta I=\int_{\mathscr{M}} d^{D}x\sqrt{-g}\mathcal{E}_{\mu\nu}\delta g^{\mu\nu}+\int_{\mathscr{M}} d^{D}x\sqrt{-g}\nabla_{\rho}\Theta^{\rho}. \tag{1.3.5}$$

The field equations are given by the tensor

$$\mathcal{E}_{\mu\nu} = -\frac{1}{2}g_{\mu\nu}\mathcal{L} + E_{\lambda\nu}{}^{\rho\sigma}R^{\lambda}{}_{\mu\rho\sigma} - 2\nabla^{\lambda}\nabla^{\rho}E_{\nu\lambda\rho\mu} \approx 0\,, \tag{1.3.6}$$

and the boundary term can be read from the last equality in (1.3.4). It can be written as follows by replacing the variation of the Christoffel symbols:

$$\Theta^{\rho} = -2E^{\eta\lambda\rho\sigma}\nabla_{\eta}\delta g_{\sigma\lambda} + 2\nabla_{\lambda}E^{\lambda\sigma\rho\gamma}\delta g_{\sigma\gamma}\,. \tag{1.3.7}$$

Now, let us consider the variation of the action principle under a generic diffeomorphism $\xi = \xi^{\mu}(x)\partial_{\mu}$,

$$\delta_{\xi}I = \int_{\mathscr{M}} d^{D}x\sqrt{-g}\left(\frac{1}{2}\delta_{\xi}g_{\mu\nu}g^{\mu\nu}\mathcal{L} + \delta_{\xi}\mathcal{L}\right).$$

The quantity $\mathcal{L}$ is a scalar under a diffeomorphism, then its variation is the Lie derivative $\delta_{\xi}\mathcal{L} = \xi^{\mu}\nabla_{\mu}\mathcal{L}$. The variation of the metric under an infinitesimal diffeomorphism is the Lie derivative, which can be written in a convenient way as $\delta_{\xi}g_{\mu\nu} = 2\nabla_{(\mu}\xi_{\nu)}$. Therefore, the variation (1.3.1) reads

$$\delta_{\xi}I = \int_{\mathscr{M}} d^{D}x\sqrt{-g}\nabla_{\mu}\left(\xi^{\mu}\mathcal{L}\right)\,. \tag{1.3.8}$$

We prove that the action is quasi-invariant under an arbitrary diffeomorphism. Recall that the variation in (1.3.5) was arbitrary, let us specialize it to be an infinitesimal diffeomorphism, and then equate both variations. Hence, we find that

$$\int_{\mathscr{M}} d^{D}x\sqrt{-g}\left(\nabla_{\mu}\left[\Theta^{\mu}(\delta_{\xi}\boldsymbol{g},\boldsymbol{g}) - \xi^{\mu}\mathcal{L}\right] + \mathcal{E}_{\mu\nu}\delta_{\xi}g^{\mu\nu}\right) = 0. \tag{1.3.9}$$

The Noether current is defined as

$$J^{\mu} = \Theta^{\mu}\left(\delta_{\xi}g, g\right) - \xi^{\mu}\mathcal{L}\,, \tag{1.3.10}$$

$$= -2E^{\eta\lambda\mu\sigma}\nabla_{\eta}\delta_{\xi}g_{\sigma\lambda} + 2\nabla_{\lambda}E^{\lambda\sigma\mu\gamma}\delta_{\xi}g_{\sigma\gamma} - \xi^{\mu}\mathcal{L}\,, \tag{1.3.11}$$

We write the current as the divergence of a quantity plus the motion equations. Since $\mathcal{L}$ appears in the equations of motion, we can contract (1.3.6) with the vector ξ^{μ} and write its contraction with the Riemann tensor as the commutator of two covariant derivatives, leading to the following expression

$$2\mathcal{E}^{\mu}{}_{\nu}\xi^{\nu} = -4E^{\mu\nu\rho\sigma}\nabla_{\rho}\nabla_{\sigma}\xi_{\nu} - 4\nabla_{\nu}\nabla_{\rho}E^{\mu\nu\rho\sigma}\xi_{\sigma} - \xi^{\mu}\mathcal{L}, \tag{1.3.12}$$

Now, we proceed to write the Noether current as the divergence of an anti-symmetric rank-2 tensor. For this, we start from (1.3.11) and replace the Lie derivative of the metric, leading to

$$\begin{aligned} J^{\mu} &= -4\nabla_{\rho}\nabla_{(\nu}\xi_{\sigma)}E^{\mu\nu\rho\sigma} + 4\nabla_{(\nu}\xi_{\sigma)}\nabla_{\rho}E^{\mu\nu\rho\sigma} - \xi^{\mu}\mathcal{L}\,, \\ &= -4\nabla_{\rho}\nabla_{(\nu}\xi_{\sigma)}E^{\mu\nu\rho\sigma} + 4\nabla_{(\nu}\xi_{\sigma)}\nabla_{\rho}E^{\mu\nu\rho\sigma} + 4E^{\mu\nu\rho\sigma}\nabla_{\rho}\nabla_{\sigma}\xi_{\nu} + 4\nabla_{\nu}\nabla_{\rho}E^{\mu\nu\rho\sigma}\xi_{\sigma} \\ &\quad + 2\mathcal{E}^{\mu}{}_{\nu}\xi^{\nu}\,, \end{aligned} \tag{1.3.13}$$

Clearly, the third term combines with one of the terms from the first term. Integrating by parts the fourth term yields

$$\begin{aligned} J^{\mu} = &-2\nabla_{\rho}\nabla_{\nu}\xi_{\sigma}E^{\mu\nu\rho\sigma} + 2\nabla_{\rho}\nabla_{\sigma}\xi_{\nu}E^{\mu\nu\rho\sigma} - 2\nabla_{\nu}\xi_{\sigma}\nabla_{\rho}E^{\mu\nu\rho\sigma} + 2\nabla_{\sigma}\xi_{\nu}\nabla_{\rho}E^{\mu\nu\rho\sigma} + \\ &+ 4\nabla_{\nu}(\nabla_{\rho}E^{\mu\nu\rho\sigma}\xi_{\sigma}) + 2\mathcal{E}^{\mu}{}_{\nu}\xi^{\nu}\,. \end{aligned}$$

By integrating by parts the first and second terms, a simplification occurs, leading to a boundary term plus the field equations:

$$J^{\mu} = \nabla_{\rho}(4\nabla_{[\sigma}\xi_{\nu]}E^{\mu\nu\rho\sigma} + 4\nabla_{\nu}E^{\mu\rho\nu\sigma}\xi_{\sigma}) + \mathcal{E}^{\mu}{}_{\nu}\xi^{\nu}\,. \tag{1.3.14}$$

Therefore, if we evaluate a configuration that satisfies the field equations, then the Noether current can be written as $J^{\mu} = \nabla_{\nu}q^{\mu\nu}$ with

$$q^{\mu\nu} = -2\left(\nabla_{\rho}\xi_{\sigma}E^{\mu\nu\rho\sigma} + 2\xi_{\rho}\nabla_{\sigma}E^{\mu\nu\rho\sigma}\right). \tag{1.3.15}$$

where we have used the fact that $E^{\mu[\nu\rho\sigma]} = 0$, as it is heritated from its definition—once the derivative of the Lagrangian is computed. Now we proceed to write an expression for the charge.

Moving to the differential forms notation, we introduce the 1-form Noether current $\mathbf{J} = J_{\mu}\mathrm{d}x^{\mu}$. Observe that using the identity $\mathrm{d}\star\mathbf{J} = \nabla_{\mu}J^{\mu}\star 1$, we can rewrite the integral (1.3.9) on-shell, as follows:

$$\int_{\mathscr{M}} \mathrm{d}\star\mathbf{J} = 0 \tag{1.3.16}$$

To capture the idea of conservation law, we need to foliate $\mathscr{M}$ with (in general non-compact) spacelike surfaces $\Sigma_{+}(t)$ parameterized by a coordinate named $t \in [t_1, t_2]$. For clarity, we restrict t to take values in an interval. Then, $\mathscr{M} = \bigcup_{t\in[t_1,t_2]}\Sigma(t)$ and its boundary will corresponds to three regions $\Sigma_-(t_1), \Sigma_+(t_2), \bigcup_{t\in[t_1,t_2]} \partial\Sigma(t)$. If the fields go to zero fast enough in the asymptotic region — identified with $\partial\Sigma(t)$ at some t — then that last contribution drops. As a consequence of applying the divergence theorem to (1.3.16), the contribution coming from $\Sigma_-(t_1), \Sigma_+(t_2)$ will be the same, when they are compared in the same orientation, which is the manifestation of a conservation law. Let us define the embedding $X : \Sigma = \Sigma_+(t) \to \mathscr{M}$, for a given $t \in [t_1, t_2]$, the following quantity is conserved

$$Q[\xi] = \int_\Sigma X^* \star \mathbf{J}\,. \tag{1.3.17}$$

Defining $\mathbf{q} = \frac{1}{2} q_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu$, it is straightforward to show that

$$\star\mathrm{d}\star\mathbf{q} = (-1)^{\eta_- + D - 1} \nabla^\mu q_{\mu\nu} \mathrm{d}x^\nu = (-1)^{\eta_- + D - 1} \mathbf{J}\,. \tag{1.3.18}$$

The first equality is an identity of the Hodge star, while the second equality is what we showed in (1.3.14), assuming that we are on-shell. Using the identity for the square of the Hodge star (1.1.17) leads to $\mathrm{d}\star\mathbf{q} = \star\mathbf{J}$. By replacing it into (1.3.17) and using the fact that the pull-back commutes with the exterior derivative, we obtain the final expression for the conserved charge

$$Q[\xi] = \int_{\partial\Sigma} \tilde{X}^* \star \mathbf{q}\,. \tag{1.3.19}$$

where $\tilde{X} : \partial\Sigma \to \mathscr{M}$. This corresponds to the Komar expression for conserved charges. However, in [61], Wald showed that in the context of the phase space formalism to compute charges, the above quantity must be supplemented by a boundary term when dealing with the energy. In the cases that we are interested in, the boundary term is precisely the boundary term that makes the action principle well posed and the Euclidean on-shell action finite. We called this $(D-1)$-form $\mathbf{B}$. Then, the energy and the angular momentum are conserved charges associated with the time translation and rotation, and are given by

$$E = Q[t] - \int_{\partial\Sigma} t \cdot \mathbf{B}\,, \tag{1.3.20}$$

$$J = Q[\varphi]\,. \tag{1.3.21}$$

where t and φ are a timelike and spacelike Killing vector appropriately normalized. In the same article, Wald showed that the black hole entropy must be computed at the horizon

$\mathcal{H}^+$, as follows:

$$S = \beta \int_{\mathcal{H}^+} \bar{X}^* \star \mathbf{q} \tag{1.3.22}$$

where $\bar{X} : \mathcal{H} \to \mathscr{M}$ is the embedding of the horizon in a Cauchy slice on $\mathscr{M}$, and β is the periodicity of the Euclidean time, which in a general background can be computed by finding the surface gravity κ on the geodesic equation

$$\xi^\mu \nabla_\mu \xi^\nu = \kappa \xi^\mu \tag{1.3.23}$$

evaluated at the horizon, and defining $\beta = 2\pi/\kappa$. The vector ξ is a geodesic null vector at the horizon satisfying the above equation.

For the sake of concreteness, we give an example in the next section.

1.3.2 Kerr-AdS black hole

As a prototypical example, let us consider a rotating black hole configuration which is asymptotically AdS_4. It is a solution of the field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \frac{3}{\ell^2} g_{\mu\nu} = 0\,, \tag{1.3.24}$$

and was constructed by Carter [64]. In the coordinate patch (t, r, θ, φ) with $t \in \mathbb{R}, r > r_+, \theta \in [0, \pi], \varphi \in [0, 2\pi]$ the metric reads

$$\mathrm{d}s^2 = -\frac{\Delta_r}{\rho^2}\left(\mathrm{d}t - \frac{a \sin^2\theta}{\Xi}\mathrm{d}\varphi\right)^2 + \frac{\rho^2 \mathrm{d}r^2}{\Delta_r} + \frac{\rho^2}{\Delta_\theta}\mathrm{d}\theta^2 + \frac{\Delta_\theta \sin^2\theta}{\rho^2}\left(a\mathrm{d}t - \frac{r^2+a^2}{\Xi}\mathrm{d}\varphi\right)^2\,, \tag{1.3.25}$$

where the functions ρ^2, Δ_r, Δ_θ and the constant Ξ are given by

$$\begin{aligned} \rho^2 &= r^2 + a^2\cos^2\theta\,, \qquad & \Delta_r &= (r^2+a^2)(1+r^2/\ell^2) - 2mr\,, \\ \Delta_\theta &= 1 - \frac{a^2}{\ell^2}\cos^2\theta\,, \qquad & \Xi &= 1 - \frac{a^2}{\ell^2}\,. \end{aligned} \tag{1.3.26}$$

Let us analyse the asymptotic behaviour of the metric in the limit $r \to \infty$. In this region, the spacetime metric takes the form

$$\mathrm{d}s^2\Big|_{r\to\infty} = -\frac{r^2}{\ell^2}\mathrm{d}t^2 + \frac{\ell^2}{r^2}\mathrm{d}r^2 + r^2\left(\frac{\mathrm{d}\theta^2}{\Delta_\theta} + \frac{\sin^2\theta}{\Xi^2}\mathrm{d}\varphi^2 + \frac{2a\sin^2\theta}{\ell^2\Xi^2}\mathrm{d}t\mathrm{d}\varphi\right) + \dots\,.$$

Observe that the asymptotic behavior of the metric depends on the rotation parameter a, hence, the boundary conditions depend on the choice of a. However, one can perform a *large gauge* transformation: $\varphi \to \varphi - 2a/\ell^2 t$, that makes the space static at infinity. For $a = 0$ the above space corresponds to the asymptotic form of AdS$_4$. The Riemann tensor $R^{\mu\nu}{}_{\rho\sigma}$ in this region approach to $-\ell^{-1}\delta^{\mu\nu}_{\rho\sigma}$ plus subleading terms. Thus, we can assert that the (1.3.25) is asymptotically locally AdS$_4$. In the interior, the spacetime has a Killing horizon at $r = r_+$, such that $g^{-1}(\mathrm{d}r, \mathrm{d}r)|_{r_+} = 0$, namely

$$g^{-1}(\mathrm{d}r, \mathrm{d}r)|_{r_+} = \frac{\Delta_{r_+}}{r_+^2 + a^2\cos^2\theta} = 0\,,$$

hence, the horizon is located at the first zero of the function Δ_r. It is convenient to write the parameter m in terms of the r_+ following the above equation, namely $m = \frac{1}{2r_+}(r_+^2 + a^2)(1 + r_+^2/\ell^2)$. We will go back to the discussion of the horizon when we discuss the Bekenstein-Hawking entropy of the black hole. Let us now move to the evaluation of the conserved charges.

Considering General Relativity with negative cosmological constant $\Lambda = -3\ell^{-2}$, the Lagrangian scalar is given by $\mathcal{L} = (16\pi G)^{-1}(R + 6\ell^{-2})$, then the tensor (1.3.3) is given by $E_{\mu\nu}{}^{\rho\sigma} = (16\pi G)^{-1}\delta^{\rho}_{[\mu}\delta^{\sigma}_{\nu]}$. The 2-form given by equation (1.3.15) is therefore

$$\mathbf{q} = -\frac{1}{16\pi G}\nabla_\mu \xi_\nu \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = -\frac{1}{16\pi G}\mathrm{d}\xi^\flat\,. \tag{1.3.27}$$

where $\xi^\flat$ the 1-form obtained from ξ through the inverse metric. The boundary terms have two pieces, the Gibbons-Hawking-York term that allows to have a well-posed action principle [65] and is proportional to the trace of extrinsic curvature. The second piece corresponds to boundary terms responsible for rendering the mass to be finite [66]

$$\int_{\partial\mathscr{M}} \mathbf{B} = \frac{1}{8\pi G}\int_{\partial\mathscr{M}} \mathrm{d}^3x\sqrt{-h}(\mathcal{K} - \frac{2}{\ell} - \frac{\ell}{2}\mathcal{R})\,. \tag{1.3.28}$$

To define the extrinsic curvature, we consider $\boldsymbol{n} = n_\mu \mathrm{d}x^\mu$ begin the normalized 1-form outgoing normal to the boundary $\partial\mathscr{M}$. In the present case, the boundary term is located at $r = r_\infty$ with r_∞ being a constant regulator that, after rendering each term finite, will be sent to infinity. The normal 1-form to this surface is dr and hence its normalized 1-form is $\boldsymbol{n} = \mathrm{d}r/\sqrt{g^{rr}}$. The projector of vectors to 1-forms in the hypersurface is defined to be $\boldsymbol{h} = \boldsymbol{g} - \boldsymbol{n}\otimes\boldsymbol{n}$. Its pullback to $\partial\mathscr{M}$ (which coincides with the pullback of $\boldsymbol{g}$) defines the induced metric on the boundary. The trace of the extrinsic curvature is defined to be $\mathcal{K} = \nabla_\mu n_\nu h^{\mu\nu}$. The intrinsic curvature $\mathcal{R}$ is computed with the induced metric. The Killing vectors of the background are defined to be

$$\boldsymbol{t} = \frac{\partial}{\partial t}\,, \qquad \varphi = \frac{\partial}{\partial \varphi} \tag{1.3.29}$$

The conserved charges are given by

$$M = \int_{\partial\Sigma} (\star \mathbf{q}[\boldsymbol{t}] - \boldsymbol{t}\cdot\mathbf{B}) = \frac{m}{\Xi G}\,, \tag{1.3.30}$$
$$J = -\int_{\partial\Sigma} \star\mathbf{q}[\varphi] = \frac{aM}{\Xi}\,. \tag{1.3.31}$$

which agrees with [67, 68] in their uncharged limit. The conserved charge at the horizon $\mathcal{H}$ is interpreted as the entropy, and the Killing vector used must be the horizon generator. In the case of (1.3.25), the coordinates (t, r, θ, φ) are not well defined at the horizon, and we need to change the patch. An appropriate coordinate patch for this task is the ingoing Eddington-Finkelstein coordinates. The implicit change of coordinates is given by

$$\mathrm{d}t = \mathrm{d}u - \frac{(r^2+a^2)\mathrm{d}r}{\Delta_r}\,, \qquad \mathrm{d}\varphi = \mathrm{d}\tilde{\varphi} - \frac{\Xi a \mathrm{d}r}{\Delta_r}\,. \tag{1.3.32}$$

In these coordinates, the generator of the horizon acquires the form

$$\xi = \frac{\partial}{\partial u} + \Omega\frac{\partial}{\partial\tilde{\varphi}}\,, \tag{1.3.33}$$

where Ω defines the angular velocity of the horizon and is fixed in such a way that ξ belongs to the tangent space of $\mathcal{H}$, namely $\boldsymbol{g}(\xi,\xi)|_{\mathcal{H}} = 0$. This leads to

$$\Omega = \frac{a\Xi}{r_+^2 + a^2}\,. \tag{1.3.34}$$

This is the angular velocity measured relative to a frame rotating at infinity [69]. Since the generator ξ is a linear combination of two Killing vectors, it is also a Killing vector and therefore the horizon $\mathcal{H}$ is named a Killing horizon. The temperature of the black hole is defined as $T = \kappa/2\pi$, where κ is the surface gravity at the horizon defined through the geodesic equation

$$\xi^\mu \nabla_\mu \xi^\nu = \kappa \xi^\nu\,, \tag{1.3.35}$$

evaluated at the horizon. The temperature obtained from the above equation for the background (1.3.25) in ingoing coordinates is given by

$$T = \frac{\ell^2 r_+^2 - \ell^2 a^2 + 3r_+^4 + a^2 r_+^2}{4\pi \ell^2 r_+ (r_+^2 + a^2)} \,. \tag{1.3.36}$$

Under these considerations, the entropy is directly computed by the Wald formula

$$S = \frac{1}{T} \int_{\mathcal{H}} \star \mathbf{q}[\xi] = \frac{\ell^2 \pi (r_+^2 + a^2)}{G(\ell^2 - a^2)} = \frac{\mathcal{A}}{4G} \,.$$

As expected, we recover precisely the quarter of the horizon area formula for the black hole entropy in General Relativity, where the area is defined as the integral of the volume form of the horizon induced by the embedding $\mathcal{H} \hookrightarrow \mathscr{M}$. The thermodynamic quantities satisfy the first law of thermodynamics, which reads

$$\delta(M/\Xi) = T\delta S + \tilde{\Omega}\delta J \,, \tag{1.3.37}$$

where $\tilde{\Omega}$ is the difference between the angular velocity at the horizon and at infinity $\tilde{\Omega} = \Omega + \frac{a}{\ell^2}$.

In the next chapter, we will start presenting the new results of this thesis. We will apply most of the framework presented in the last section, and the discussion will be based on the article [28].

Chapter 2

α'-corrected black holes in the heterotic string theory

Higher-curvature corrections to Einstein's general relativity (GR) are ubiquitous in any sensible approach to quantum gravity, and they are a solid prediction of string theory [18]. Even before the formulation of the latter, effective actions containing higher order contractions of the Riemann tensor were known to emerge in quantum field theory on curved spacetime [19] and in the semiclassical approach to quantum gravity. Such actions are the natural generalization of Einstein-Hilbert action, thus correcting GR in the ultraviolet (UV) regime. Also, from the mathematical point of view it was early understood that higher-curvature terms were natural in higher dimensions [20, 21, 70]; and, on general grounds, it is widely accepted that any attempt to formulate a sensible UV-complete theory will involve higher-curvature corrections in a way or another. In 1976, Stelle argued that gravitational actions which include terms quadratic in the curvature tensor are renormalizable [22]. This is due to the fact that non-linear renormalization of the graviton and the ghost fields suffices to absorb the non-gauge-invariant divergences that might arise. Stelle explained how these and other divergences may be eliminated in a way that simplifies the renormalization procedure, even when matter fields are coupled. Nevertheless, renormalizability is not the only issue: The inclusion of quadratic-curvature terms in the gravitational action typically introduces massive local degrees of freedom, apart from the massless graviton of GR [23]. These extra modes organize themselves as a massive spin-2 and a massive spin-0 excitations, yielding a total of 8 local degrees of freedom. The massive spin-2 part of the field has negative energy, and this is the reason why it is usually asserted that, with exception of a few remarkable cases [21, 70, 71, 72], augmenting the Einstein-Hilbert action with a finite set of higher-curvature terms yields ghosts when the theory is expanded about maximally symmetric vacua. The observation of [23] motivated, in the early 80s, the search for ghost-free higher-curvature theories and consistent UV completions. Since then, actions containing higher-curvature terms were considered in the context of cosmology [73], black hole physics [74], and string theory [24]. In 1985, Zwiebach studied the compatibility between the presence of curvature squared terms and the absence of ghost modes in the low energy limit of string theory [24]. He argued that the so-called Einstein-Gauss-Bonnet (EGB) action was a good candidate for string effective action as it yields a ghost-free non-trivial gravitational self-interacting theory in any dimension greater than four, $D > 4$. The EGB action is made out of a dimensionally extended version of the quadratic Chern-Gauss-Bonnet topological invariant, which, while being dynamically trivial in $D \leq 4$, does yield a UV correction to GR in $D > 4$ with a single massless spin-2 excitation and with field equations of second order. The latter property makes EGB theory free of Ostrogradski instabilities. Still, in [74] Boulware and Deser showed that the EGB model, proposed in [24] as a stringy action, contains, in addition to flat spacetime, a second non-perturbative anti-de Sitter (AdS) vacuum which turns out to be unstable due to the presence of ghosts. This is nothing but the fact that actions that are polynomials of degree k in contractions of the Riemann tensor generically yield k different vacua, many of them being artifacts of the truncation of the effective theory. In [75] the authors noticed that the inclusion of the dilaton field in the EGB effective action suffices to remove the spurious (A)dS vacuum permitted in its absence. They also showed that the spherically symmetric static solutions to the dilatonic EGB theory might have a well-defined asymptotic behaviour, being non-trivial, and being compatible with the existence of a regular event horizon at which the dilaton is well-behaved[1] This was later confirmed by explicit examples, and here we will also provide a concrete realization of it.

Soon after [24], in a foundational paper of string theory [25], Gross and Witten finally proved that the gravitational field equations of string theory actually contain higher-curvature corrections to GR. More precisely, they derived the modifications of the classical gravitational equations for the type II string theory by studying tree-level gravitational scattering amplitudes, and they determined the effective gravitational action up to quartic

[1]Higher-curvature black holes were also studied in the context of thermodynamics [76] and many other subjects, like the holography [77], the weak-gravity conjecture [78], among others. For related early works on this paper, see [79, 80, 81, 82, 83, 84, 84].

order in the curvature tensor, which corresponds to order $\mathcal{O}(\alpha'^3)$ string corrections. Unlike bosonic string theory, type II superstring theory in $D = 10$ dimensions does not contain quadratic corrections to GR, and the cubic ones can be set to zero by fields redefinition – although quadratic corrections can actually appear in Calabi-Yau compactification of the quartic actions, with the moduli playing the role of the couplings cf. [85, 86, 87]–. In contrast, quadratic corrections do appear in critical bosonic and heterotic string theories. They were studied in [26, 88] by Metsaev and Tseytlin, who checked the equivalence of the string equations of motion and the σ-model Weyl invariance conditions at order $\mathcal{O}(\alpha')$. They obtained the functional dependence on the dilaton, the graviton, and the antisymmetric tensor. To do so, they first determined the $\mathcal{O}(\alpha')$ terms in the string effective action starting from the expressions for the 3- and 4-point string scattering amplitudes; then, they computed the 2-loop β-function in the worldsheet σ-model. This results in an effective gravity action with quadratic-curvature (R^2) corrections coupled to the other massless fields of the theory; see also the important works [27, 89], and for modern developments on α' corrections in relation to T-duality and Double Field Theory see [90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100] and references therein and thereof.

In recent years, with the advent of AdS/CFT correspondence and its ramifications, higher-curvature terms were reconsidered in the context of holography and the interest on them was revived. Probably the best-known example of this is the discussion of the higher-curvature terms in relation to the Kovtun-Starinets-Son (KSS) viscosity bound [101, 102], which showed that, for a class of conformal field theories (CFT) with a gravity dual with the EGB action, the shear viscosity to entropy density ratio could violate the conjectured KSS lower bound. This proved that the presence of higher-curvature terms could result in qualitatively new phenomena; see also [103, 104]. Microcausality violation in the CFT was also studied in the same type of scenario [101], which was rapidly interpreted as evidence supporting the idea of a universal lower bound on the shear viscosity to entropy density ratio for all consistent theories. This triggered a long series of works devoted to check the consistency conditions of effective theories with higher-curvature modifications. For example, in [77] the authors discussed causality conditions in R^2 theories; they study causality violation in holographic hydrodynamics focusing on the EGB theory as a working example. In the latter theory, the value of the only R^2 coupling constant is related to the difference between the two central charges of the dual 4-dimensional CFT, and the authors of [77] showed that, when such difference is sufficiently large, causality is violated. This problem was also studied in [105], where the author discussed the relation between causality constraints in the bulk theory and the condition of energy positivity in the dual CFT. He specifically argued that special care is needed when solving the classical equations of motion in the higher-curvature gravity theory, for which the study of causality problems may be subtle. Holography in presence of EGB gravity actions have been further studied in [77] and in references thereof. The authors of [77] studied the problem in arbitrary number of dimensions D and established a holographic dictionary that relates the couplings of the gravitational theory to the universal numbers in the correlators of the stress tensor of the dual CFT, cf. [106]. This allowed the authors to examine constraints on the gravitational couplings by demanding consistency of the CFT, and this yielded a much more general set of causality constraints.

Both in the context of AdS/CFT and in other scenarios, the consistency conditions for higher-curvature theories were intensively studied in the last fifteen years. This line of research has continued and a much more general picture of the set of consistency conditions has been accomplished. Causality, locality, stability, hyperbolicity and other aspects were revisited. In [107], it was shown how causality constrains the sign of the stringy R^4 corrections to the Einstein-Hilbert action, giving a general restriction on candidate theories of quantum gravity. In [108], a special type of pathology that the truncated EGB theory exhibits was studied. This is a phase transition driven by non-perturvative effects that might take place in gravitational theories whenever higher-curvature corrections with no extra fields are considered. In [109], Maldacena et al. studied causality constraints on corrections to the graviton 3-point coupling. They considered higher-curvature corrections to the graviton vertex in a weakly coupled gravity theory and they derived stringent causality constraints. By considering high energy scattering processes, they noticed a potential causality violation that might occur whenever additional Lorentz invariant structures are included in the graviton 3-point vertex. They argued that such a violation could be cured by the addition of an infinite tower of extra massive higher-spin fields such as those predicted by string theory. This problem was later reconsidered by many authors, cf. [110].

Motivated by this renewed interest in higher-curvature gravity, in the last years there have been important developments in the subject, and many new higher-curvature models were proposed and studied. The list includes the quasitopological theories [111, 112, 113], the critical gravity theories in AdS [71, 72], the so-called Einsteinian cubic gravity [114, 115], and their generalizations [116, 117]. Black holes have recently been studied in all these setups [112, 118, 119, 120], as well as in string theory inspired scenarios [121, 122, 123];

see also [124, 125, 126] and references therein and thereof. Here, we will present and study analytic, static, spherically symmetric solutions to the α'-corrected gravity action in arbitrary dimension D and including a non-vanishing dilaton coupling. We will consider the graviton-dilaton sector of the low-energy effective action of string theory with R^2 terms in a specific frame that will enable us to solve the problem explicitly to order $\mathcal{O}(\alpha')$ in the entire spacetime. Our solutions manifestly show that the theory is compatible with static, spherically symmetric solutions which are asymptotically flat and exhibit a regular event horizon at which the dilaton is well-behaved. This chapter is organized as follows: In Section 2.1, we present the gravity theory in a convenient frame. We briefly discuss the field redefinition ambiguity to the relevant order, and we use it to solve the adequate ansatz. The field equations are written down and solved, and the black hole solution for $D = 4$ is presented. In Section 2.1.1, we study the black hole thermodynamics. This amounts to work out the Wald entropy formula, which, as usual in this type of setup, yields corrections to the Bekenstein-Hawking area law. The mass of the solution may then be inferred from the first law of black hole mechanics. In Section 2.1.2, we perform a consistency check of the previous formulae by explicitly computing the black hole mass by means of the Iyer-Wald method for conserved charges, which shows perfect agreement. We also show the agreement with the Euclidean action approach. In Section 2.1.3, we generalize our result by introducing angular momentum in the slowly rotating approximation. We derive a stationary metric that represents stringy modifications to the Kerr geometry. In Section 2.2, we obtained the α' correction to the C-metric, which accommodates accelerating black holes. While we work in the string frame, in Section 2.3 we discuss the frame transformation that maps out theory to the Einstein frame, including the higher curvature corrections. In the latter frame, rotating solutions have already been studied in the literature, and we discuss the precise relation between the two frames. In Section 2.4, we generalize the static solution by presenting the explicit form of the dilatonic black hole solution in arbitrary dimension D. We end the chapter with conclusive remarks 2.5.

2.1 Dilatonic black hole

We consider the low-energy effective action of string theory including α' corrections to the graviton-dilaton sector; namely [26, 88]

$$I\left[g_{\mu\nu},\phi\right]=\int_{M}\mathrm{d}^{D}x\sqrt{-g}e^{-2\phi}\left[R+4\left(\nabla\phi\right)^{2}+\alpha\ R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho}+\mathcal{O}\left(\alpha^{2}\right)\right]\ , \tag{2.1.1}$$

where we denoted $\alpha = \frac{1}{8}\alpha'$. We are not considering the dependence on the B-field here. Performing field redefinition $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$, $\phi \to \phi + \delta\phi$ with

$$\delta\phi = -\tfrac{\alpha}{2}\left(R + 4(2D-5)\partial_\mu\phi\partial^\mu\phi\right) , \tag{2.1.2}$$

$$\delta g_{\mu\nu} = -4\alpha\left(R_{\mu\nu} - 4\partial_\mu\phi\partial_\nu\phi + 4g_{\mu\nu}\partial_\alpha\phi\partial^\alpha\phi\right) , \tag{2.1.3}$$

one obtains the action in a frame that is convenient for the computation we want to undertake; namely

$$\begin{aligned} I\left[g_{\mu\nu},\phi\right] = \int \mathrm{d}^D x\sqrt{-g}e^{-2\phi}\Big[&R + 4\left(\nabla\phi\right)^2 \\ &+\alpha\left(R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} - 4R_{\mu\nu}R^{\mu\nu} + R^2 - 16\left(\partial_\mu\phi\partial^\mu\phi\right)^2\right) + \mathcal{O}\left(\alpha^2\right)\Big] , \end{aligned} \tag{2.1.4}$$

up to six-derivative operators of order $\mathcal{O}\left(\alpha'^2\right)$, cf. [127]. As the R^2-terms take the form of the $4d$ Euler characteristic, the field equations of the theory in this frame are of second order in an explicit manner. Let us consider first the case in $D = 4$. The field equations derived from (3.1.1) are given by

$$G_{\mu\nu} + 4\partial_\mu\phi\partial_\nu\phi - 2g_{\mu\nu}\partial_\rho\phi\partial^\rho\phi + 2S_{\mu\nu} - 2g_{\mu\nu}S^\rho{}_\rho + \alpha H_{\mu\nu} = 0 , \tag{2.1.5}$$

$$R + 4\partial_\rho\phi\partial^\rho\phi + 4S^\mu{}_\mu + \alpha L_{GB} - 32\alpha\left(\nabla^\mu\left(\partial_\rho\phi\partial^\rho\phi\right)\partial_\mu\phi + \left(\partial_\rho\phi\partial^\rho\phi\right)S^\mu{}_\mu + \frac{1}{2}\left(\partial_\rho\phi\partial^\rho\phi\right)^2\right) = 0 , \tag{2.1.6}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ is the Einstein tensor, and where

$$S_{\rho\sigma} \equiv e^{2\phi}\nabla_\rho\left(e^{-2\phi}\nabla_\sigma\phi\right) , \quad L_{GB} \equiv R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 4R^{\mu\nu}R_{\mu\nu} + R^2 , \tag{2.1.7}$$

and

$$\begin{aligned} H_{\mu\nu} \;=\;& S_{\mu\nu}R - 4S^\sigma{}_{(\mu}R_{\nu)\sigma} + 2S^\sigma{}_\sigma R_{\mu\nu} + 2S^{\sigma\lambda}R_{\mu\sigma\lambda\nu} - 8\left(\partial_\rho\phi\partial^\rho\phi\right)\partial_\mu\phi\partial_\nu\phi \\ &+g_{\mu\nu}\left(2\left(\partial_\rho\phi\partial^\rho\phi\right)^2 - S^\sigma{}_\sigma R + 2S^\sigma{}_\lambda R^\lambda{}_\sigma\right) . \end{aligned} \tag{2.1.8}$$

Lagrangian L_{GB} is the integrand of the 4-dimensional Chern-Gauss-Bonnet topological invariant which, in the absence of the dilaton and in $D = 4$, yields the Euler characteristic; this is the EGB quadratic gravity Lagrangian.

We are interested in solving the equations above for a static spherically symmetric spacetime with non-trivial dilaton profile (later, in Section 2.1.3, we will generalize the solution to the stationary, non-static case). In order to do so, we work perturbatively at order $\mathcal{O}(\alpha)$, and propose the ansatz

$$\phi(r) = \phi_0 + \alpha\phi_1(r) \ , \tag{2.1.9}$$

$$ds^2 = -\left(1+\alpha N_1(r)\right)^2 \left(1 - \frac{\mu}{r} + \alpha f_1(r)\right) \mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1-\frac{\mu}{r}+\alpha f_1(r)} + r^2 \mathrm{d}\Omega^2 \ , \tag{2.1.10}$$

where $\phi_1(r)$, $N_1(r)$, and $f_1(r)$ are functions of the radial coordinate r to be determined; μ is an arbitrary constant; $\mathrm{d}\Omega^2$ is the constant-curvature metric on the unit sphere. The solution we will find in this way will be valid up to order $\mathcal{O}(\alpha)$. Plugging this ansatz in the field equations and expanding up to first order in α, we obtain a remarkably simple system of equations which lead to the following general solution

$$\phi(r) = \phi_0 + \alpha\left(A + B\log\left(\frac{r-\mu}{r}\right) - \frac{2}{\mu r} - \frac{1}{r^2} - \frac{2\mu}{3r^3}\right) \tag{2.1.11}$$

with A and B being two arbitrary constants; the former constant appears merely as a shift of ϕ_0 which does not enter in the metric, and so it can be absorbed by redefining $\bar{\phi}_0 = \phi_0 + \alpha A$, which gives the value that the dilaton takes at infinity; notice that, at infinity, (2.1.11) goes like $\phi \simeq \bar{\phi}_0 + \mathcal{O}(1/r)$. Up to $\mathcal{O}(\alpha)$ terms, for the metric we find

$$g^{rr} = 1 - \frac{\mu}{r} + \alpha\left(-\frac{\mu B}{r}\log\left(\frac{r-\mu}{r}\right) + \frac{C}{r} + \frac{2}{r^2} + \frac{\mu}{r^3} - \frac{10}{3}\frac{\mu^2}{r^4}\right) \ ,$$

$$g_{tt} = \frac{\mu}{r} - 1 - \alpha\left(\frac{B\left(2r-3\mu\right)}{r}\log\left(\frac{r-\mu}{r}\right) + D + \frac{4}{r^2} + \frac{5\mu}{3r^3} + \frac{2\mu^2}{r^4} - \frac{\mu^2 D - \mu C + 2\mu^2 B + 8}{\mu\, r}\right)$$

where D and C are other two integration constants. The former can be eliminated by rescaling the time coordinate as $t \to t/(1+\alpha D)$.

If we define $r_+ = \mu + \alpha\mu_1$, we can easily find the α-corrected location of the event horizon by solving for μ_1 as a function of the integration constants. This amounts to demand $g^{rr}\left(r_+\right) = g_{tt}\left(r_+\right) = 0$, which is actually required for the horizon to be regular. Expanding up to first order in α, this yields

$$\left(B\log\left(\frac{\mu}{\alpha\mu_1}\right) + \frac{\mu_1 + C}{\mu} - \frac{1}{3\mu^2}\right)\alpha + O\left(\alpha^2\right) = 0 \ , \tag{2.1.12}$$

$$\left(B\log\left(\frac{\mu}{\alpha\mu_1}\right) - 2B + \frac{\mu_1 + C}{\mu} - \frac{1}{3\mu^2}\right)\alpha + O\left(\alpha^2\right) = 0 \ , \tag{2.1.13}$$

from which we consequently obtain that $B = 0$. Therefore, the α'-corrected black hole configuration reads

$$g^{rr}(r) = 1 - \frac{\mu}{r} + \alpha\left(\frac{C}{r} + \frac{2}{r^2} + \frac{\mu}{r^3} - \frac{10}{3}\frac{\mu^2}{r^4}\right) + \mathcal{O}(\alpha^2)\ , \tag{2.1.14}$$

$$g_{tt}(r) = \frac{\mu}{r} - 1 - \alpha\left(\frac{4}{r^2} + \frac{5\mu}{3r^3} + \frac{2\mu^2}{r^4} - \frac{8-\mu C}{\mu\, r}\right) + \mathcal{O}(\alpha^2)\ , \tag{2.1.15}$$

$$\phi(r) = \bar{\phi}_0 - \alpha\left(\frac{2}{\mu r} + \frac{1}{r^2} + \frac{2\mu}{3r^3}\right) + \mathcal{O}(\alpha^2)\ , \tag{2.1.16}$$

and, up to $\mathcal{O}(\alpha)$ corrections, the location of the horizon is

$$r_+ = \mu + \alpha\left(\frac{1}{3\mu} - C\right)\ . \tag{2.1.17}$$

The solution we have just derived is asymptotically flat, and it exhibits a smooth event horizon at r_+, where the dilaton remains finite:

$$\phi(r_+) = \bar{\phi}_0 - \frac{11}{3}\frac{\alpha}{r_+^2} + \mathcal{O}(\alpha^2)\,; \quad \phi(\infty) = \bar{\phi}_0 + \mathcal{O}(\alpha^2). \tag{2.1.18}$$

In the following sections, we will analyze the physical properties of this solution, we will compute its conserved charges, and, finally, we will generalize it to $D \geq 4$ dimensions.

2.1.1 Thermodynamics

The thermodynamics of higher-curvature black holes has been studied for a long time [76, 128, 129], and in a vast number of contexts. Here, we will focus on the properties of the black hole solution we just presented. The Wald formula gives the entropy as a Noether charge computed at the horizon. This is given by the following integral on the horizon $\mathcal{H}$

$$S = \frac{\beta}{4}\int_{\mathcal{H}} \sqrt{-g}\,\epsilon_{\mu\nu\rho\sigma}\, q^{\mu\nu} dx^\rho \wedge dx^\sigma, \tag{2.1.19}$$

in $D = 4$ spacetime dimensions, with β being the periodicity of the Euclidean time. The Noether pre-potential associated to this charge is given by (1.3.15). For the action (3.1.1), the tensor E (1.3.3) and the Noether pre-potential take the following form

$$E^{\mu\nu}{}_{\rho\sigma} = \frac{1}{2}e^{-2\phi}(\delta^{\mu\nu}_{\rho\sigma} + \alpha\delta^{\mu\,\nu\,\mu_3\mu_4}_{\rho\,\sigma\,\nu_3\nu_4}R^{\nu_3\nu_4}{}_{\mu_3\mu_4}) + \mathcal{O}(\alpha^2)\ , \tag{2.1.20}$$

$$q^{\mu\nu} = 2e^{-2\phi}T^{\mu\nu} + \alpha e^{-2\phi}(4T^{\mu\nu}R + 16T^{\sigma[\mu}R^{\nu]}_{\sigma} + 4T^{\rho\sigma}R^{\mu\nu}{}_{\rho\sigma}) + \mathcal{O}(\alpha^2)\ , \tag{2.1.21}$$

respectively, where we have defined $T^{\rho\sigma} \equiv 4\xi^{[\rho}\nabla^{\sigma]}\phi - \nabla^{[\rho}\xi^{\sigma]}$. Evaluating the Wald entropy (2.1.19) for our solution, we obtain

$$S = 16\pi^2 e^{-2\bar{\phi}_0}\mu^2 - 32\pi^2\alpha\, e^{-2\bar{\phi}_0}\,(C\mu - 8) + \mathcal{O}(\alpha^2)\ . \tag{2.1.22}$$

If we naturally identify

$$e^{-2\bar{\phi}_0} = \frac{1}{16\pi G}\ , \tag{2.1.23}$$

G being the 4-dimensional Newton constant, the leading term in (2.1.22) reproduces the Bekenstein-Hawking entropy, while the order $\mathcal{O}(\alpha)$ terms yield corrections to it. More precisely, we find

$$S = \frac{\pi\mu^2}{G} + \frac{16\pi\alpha}{G}\left(1 - \frac{1}{8}C\mu\right) + \mathcal{O}(\alpha^2)\,. \tag{2.1.24}$$

Notice that (2.1.24) depends both on μ and C. The dependence of C can be traced back to the fact that $\alpha(8 - C\mu)/\mu$ is the $\mathcal{O}(\alpha)$ correction to the parameter in front of the Newtonian piece $\sim 1/r$ in the component g_{tt} of the metric, cf. (2.1.15). Then, using (2.1.17), the entropy can also be written as

$$S = \frac{\pi r_+^2}{G} + \frac{46\pi\alpha}{3G} + \mathcal{O}(\alpha^2)\,. \tag{2.1.25}$$

Notice that the potential term linear in r_+ (i.e. the one that could come from the term linear in μ in (2.1.24)) has cancelled out. In fact, at order $\mathcal{O}(\alpha)$, in virtue of the field equations, the computation reduces to that of the full action evaluated on the undeformed GR solution $f_1 = N_1 = \phi_1 = 0$. This means that, at that order, the only correction to the area law $S = \frac{A}{4G}$ is given by a positive constant. On the same grounds, corrections of the form $\mathcal{O}(\alpha r_+^{D-4}/G)$ are expected in higher dimensions.

Next, let us compute the Hawking temperature. We can do this by resorting to the Euclidean formalism. However, it is convenient to first simplify the expressions a bit. We can write μ as a function of r_+ by simply inverting (2.1.17), which yields

$$g_{tt}(r) = -1 + \frac{1}{r}\left(r_+ + \frac{23\alpha}{3r_+}\right) - \alpha\left(\frac{4}{r^2} + \frac{5r_+}{3r^3} + \frac{2r_+^2}{r^4}\right) + \mathcal{O}(\alpha^2)\ , \tag{2.1.26}$$

$$g^{rr}(r) = 1 - \frac{1}{r}\left(r_+ - \frac{\alpha}{3r_+}\right) + \alpha\left(\frac{2}{r^2} + \frac{r_+}{r^3} - \frac{10r_+^2}{3r^4}\right) + \mathcal{O}(\alpha^2)\ , \tag{2.1.27}$$

$$\phi(r) = \bar{\phi}_0 - \alpha\left(\frac{2}{r_+ r} + \frac{1}{r^2} + \frac{2r_+}{3r^3}\right) + \mathcal{O}(\alpha^2)\ . \tag{2.1.28}$$

This gives the periodicity condition for the real section of the Euclidean geometry to be regular at $r = r_+$; namely

$$\beta = 4\pi r_+ + \frac{44\pi\alpha}{3r_+} + \mathcal{O}(\alpha^2) \ , \tag{2.1.29}$$

which results in the black hole temperature

$$T = \frac{1}{4\pi r_+}\left(1 - \frac{11}{3}\frac{\alpha}{r_+^2}\right) + \mathcal{O}(\alpha^2) \, . \tag{2.1.30}$$

This corrects the Hawking formula for GR at scales $r_+ \simeq \alpha^{1/2}$. This result, together with the expression (2.1.25) for the entropy, yields the first law type relation

$$\delta E \equiv T\delta S = \delta\left(\frac{r_+}{2G} + \frac{11\alpha}{6Gr_+}\right) + \mathcal{O}(\alpha^2) \, , \tag{2.1.31}$$

from which, up to subleading orders in α, we can obtain the gravitational energy

$$E - E^{(0)} = \frac{r_+}{2G}\left(1 + \frac{11}{3}\frac{\alpha}{r_+^2}\right) \tag{2.1.32}$$

E_0 being an integration constant that corresponds to the energy of the reference background. Below, we will confirm this result by rederiving the gravitational energy using the Iyer-Wald method for computing Noether charges. It is also worth noticing that, if we insist in extrapolating the formulae above for small values of r_+, which is not well justified as higher-order terms are expected to be relevant in that regime, then the formula obtained for the specific heat changes its sign and becomes positive within the range $\frac{11}{3}\alpha < r_+^2 < 11\alpha$; the black hole temperature (2.1.30) vanishes at the lower bound $r_+^2 = \frac{11}{3}\alpha$.

2.1.2 Conserved charges

In order to compute the gravitational energy of the solution, we have to supplement the bulk action with the appropriated boundary terms. In the case of the higher curvature action (3.1.1), the boundary term to be added reads

$$I_{BT} \equiv \int_{\partial M} \mathrm{d}^3x\sqrt{-h}e^{-2\phi}\left[2K + 4\alpha\delta^{\mu_1\mu_2\mu_3}_{\nu_1\nu_2\nu_3}K^{\nu_1}_{\mu_1}\left(\frac{1}{2}\mathcal{R}^{\nu_2\nu_3}{}_{\mu_2\mu_3} - \frac{1}{3}K^{\nu_2}_{\mu_2}K^{\nu_3}_{\mu_3}\right)\right] \tag{2.1.33}$$

$$\equiv \int_{\partial M} \mathrm{d}^3x\sqrt{-h}\mathcal{B} \tag{2.1.34}$$

where K is the trace of the extrinsic curvature K^{μ}_{ν}, and $\mathcal{R}^{\mu\nu}{}_{\rho\sigma}$ and $h_{\mu\nu}$ are the intrinsic curvature and the induced metric on ∂M, respectively; cf. [130]. The contribution (2.1.34) renders the variational principle well-posed. Then, the energy of the spacetime, which corresponds to the black hole mass, is given by the following integral on the sphere at infinity, S^2_∞; namely

$$M = \int_{S^2_\infty} \left(\boldsymbol{Q}[\boldsymbol{t}] - \boldsymbol{t}\cdot\boldsymbol{B}\right) \ , \tag{2.1.35}$$

where $\boldsymbol{Q}[\boldsymbol{t}]$ is the Hodge dual of the Noether pre-potential for the killing vector $\boldsymbol{t} = \partial_t$ and

$$\boldsymbol{B} = \frac{1}{3!}\,\mathcal{B}\,n^\sigma\,\sqrt{-g}\,\epsilon_{\mu\nu\rho\sigma}\,dx^\mu\wedge dx^\nu\wedge dx^\rho \ . \tag{2.1.36}$$

In flat spacetime, the trace of the extrinsic curvature is $K^{(0)} = \frac{2}{r}$ which gives a divergent piece in the action principle as the volume element contributes with r^2. To obtain a finite action principle and a finite energy definition, we have to subtract the extrinsic curvature of flat spacetime to each piece of the extrinsic curvature appearing in the formulae above. In other words, we have to define

$$\bar{K}_{\mu\nu} \equiv K_{\mu\nu} - K^{(0)}_{\mu\nu} \ ,$$

using flat space as a background reference; this corresponds to set $E^{(0)} = 0$ for Minkowski spacetime. According to this, the energy content of the spacetime, as defined in (2.1.35), precisely gives

$$M = \frac{r_+}{2G}\left(1 + \frac{11}{3}\frac{\alpha}{r_+^2}\right) \ , \tag{2.1.37}$$

which agrees with (2.1.32).

Another crosscheck for this result can be done by means of the Euclidean action formalism. In the saddle point approximation, the on-shell Euclidean action gives the partition function; namely

$$\log Z \simeq I^E + I^E_{BT} \ , \tag{2.1.38}$$

where the superscript E stands for Euclidean. It is worth emphasizing that, at order $\mathcal{O}(\alpha)$, the computation of the Euclidean action reduces to the evaluation of the full action $I^E + I^E_{BT}$ on the undeformed GR solution. Therefore, the energy of the configuration can be simply derived from (2.1.38) by computing

$$\bar{E} = -\frac{\partial \log Z}{\partial \beta} \ . \tag{2.1.39}$$

The on-shell action computed with $\bar{K}_{\mu\nu}$ for the configuration (2.1.26)-(2.1.28) with the

Euclidean time periodicity (2.1.29) turns out to be finite, and it reads

$$I^E + I^E_{BT} = -\frac{\pi r_+^2}{G} - \frac{10\alpha\pi}{3G} \,. \tag{2.1.40}$$

From this expression, we easily find

$$\bar{E} = \frac{r_+}{2G}\left(1 + \frac{11}{3}\frac{\alpha}{r_+^2}\right) , \tag{2.1.41}$$

which, again, exactly reproduces (2.1.32) at the right order. This results in an $\mathcal{O}(\alpha)$, r_+-dependent correction to the GR Smarr formula; namely

$$TS - \frac{1}{2}\bar{E} = \frac{2\alpha}{Gr_+} \,. \tag{2.1.42}$$

At order $\mathcal{O}(\alpha)$ this is equivalent to an additive constant in the entropy.

2.1.3 Adding angular momentum

Black hole solution (2.1.10) can be generalized to the stationary non-static case, and the analytic expression in the slowly rotating approximation can also be found following the similar perturbative method as before. At first order in α and including the rotation parameter in linear and quadratic terms as well as in terms of the form $a\alpha$, the solutions reads

$$\begin{aligned} ds^2 = &-\left[1 - \frac{\mu}{r} + \frac{\mu a^2 \cos^2\theta}{r^3} + \alpha f_1(r)\right]\mathrm{d}t^2 + 2a\left[-\frac{\mu}{r}\sin^2\theta + \alpha\, h_{t\varphi}\,(r,\theta)\right]\mathrm{d}t\mathrm{d}\varphi + \\ &+ \left[\frac{1}{1 - \frac{\mu}{r} + \alpha g_1(r)} - a^2\frac{(\mu - r)\cos^2\theta + r}{(r-\mu)^2 r}\right]\mathrm{d}r^2 + \\ &+ (r^2 + a^2\cos^2\theta)\mathrm{d}\theta^2 + \left[(r^2 + a^2) + \frac{a^2\mu\sin^2\theta}{r}\right]\sin^2\theta\mathrm{d}\varphi^2 \,, \end{aligned} \tag{2.1.43}$$

with

$$f_1(r) = \frac{2\mu^2}{r^4} + \frac{5\mu}{3r^3} + \frac{4}{r^2} - \frac{8}{\mu r} \,, \tag{2.1.44}$$

$$g_1(r) = -\frac{10\mu^2}{3r^4} + \frac{\mu}{r^3} + \frac{2}{r^2} \,, \tag{2.1.45}$$

$$h_{t\varphi}(r) = \sin^2\theta\left(\frac{\hat{C}}{r} + \frac{2\mu^2}{r^4} + \frac{3\mu}{r^3} + \frac{6}{r^2}\right) , \tag{2.1.46}$$

and with $\hat{C}$ being a new integration constant that, at this order, enters in the angular momentum; see (2.1.50) below. The scalar configuration is

$$\phi(r) = \phi_0 - \alpha\left(\frac{2\mu}{3r^3} + \frac{1}{r^2} + \frac{2}{\mu r}\right) . \tag{2.1.47}$$

One can verify that, expanding both in the Gauss-Bonnet coupling, α, and in the rotation parameter, a, all the field equations are solved at the right order; namely

$$E_{\mu\nu} = \mathcal{O}\left(\alpha a^2, \alpha^2\right) . \tag{2.1.48}$$

The angular momentum can be computed by using the Wald formalism, which yields a form

$$J = -\int_{S^2_\infty} \star\boldsymbol{q}[\partial_\varphi] \tag{2.1.49}$$

with $\star\boldsymbol{q}[\partial_\varphi]$ representing the Hodge dual of the Noether pre-potential $\boldsymbol{q}[\partial_\varphi]$ for the Killing vector ∂_φ. The angular momentum of the spacetime is given by

$$J = \frac{a\mu}{2G}\left(1 - \frac{\alpha\hat{C}}{\mu}\right) . \tag{2.1.50}$$

The integration constant $\hat{C}$ is a spurious constant, that disappears when the angular momentum is replaced. Solution (2.1.43)-(2.1.46) gives string theory modification to Kerr geometry. In particular, we see order $\mathcal{O}(a\alpha)$ modifications to the off-diagonal term in the Boyer-Lindquist coordinates. This will result in deviations from the GR prediction of the Lense-Thirring precession. It will also induce modifications to the spheroidal shape of the shadow of a rotating black hole; see [131] and references thereof.

2.2 Accelerating black holes

Let us consider the following ansatz for the metric and for the dilaton

$$ds^2 = \frac{\Omega(x,y)}{A^2(x+y)^2}\left(-F(y)\,dt^2 + \frac{dy^2}{F(y)} + \frac{dx^2}{G(x)} + G(x)\,d\varphi^2\right) , \tag{2.2.1}$$

$$\phi = \phi(x,y) . \tag{2.2.2}$$

Assuming the expansion $\phi = \phi_0(x,y) + \alpha\phi_1(x,y) + \mathcal{O}(\alpha^2)$, $F = F_0(y) + \alpha F_1(y) + \mathcal{O}(\alpha^2)$, $G = G_0(x) + \alpha G_1(x) + \mathcal{O}(\alpha^2)$ and $\Omega(x,y) = 1 + \alpha\omega(x,y) + \mathcal{O}(\alpha^2)$. In General Relativity, the ansatz (2.2.1), leads to the C-metric which accommodates accelerating black holes (see

[132] for a modern interpretation as well as a historical review), even in the presence of minimally coupled, self interacting scalar fields [133]. Here A, stands for the acceleration and $\Omega(x,y)=1$ in General Relativity in vacuum.

To the lowest order on the string tension α, Einstein equations lead to

$$F(y)=F_0(y)=f_3y^3+f_2y^2+f_1y+f_0 \text{ and } G(x)=G_0(x)=f_3x^3-f_2x^2+f_1x-f_0\ , \tag{2.2.3}$$

fulfilling $G(\xi)=-F(-\xi)$. Here f_i with $i\in\{0,...,3\}$ are integration constants. The quadratic, the linear or the f_0 term in the polynomials (2.2.3) can be removed by a simultaneous, constant shift of the independent variables (x,y), maintaining the form of the metric (2.2.1). For future purposes, it is better to keep all the f_i as non-vanishing at the moment.

The field equations of the α'-corrected theory (3.1.1), at linear order in α are solved by

$$F_1(y) = d_3y^3+d_2y^2+d_1y+d_0\ , \tag{2.2.4}$$

$$G_1(x) = f_3h_1x^3+3f_3h_2x^2+\frac{(3d_3f_1-3f_1f_3h_1-6f_2f_3h_2+3d_1f_3-2d_2f_2)}{3f_3}x \tag{2.2.5}$$

$$-\frac{(-6f_0f_3h_1-3f_1f_3h_2+3d_0f_3-d_2f_1+6d_3f_0)}{3f_3}\ , \tag{2.2.6}$$

$$\omega(x,y) = 2\phi_1(x,y)+\frac{3f_3j_1x+3(2f_3h_1+f_3j_1-2d_3)y-6f_3h_2-2d_2}{3(x+y)f_3}\ , \tag{2.2.7}$$

leading to the following inhomogeneous, PDE for $\phi_1(x,y)$

$$0 = (x+y)\left(G_0(x)\frac{\partial^2\phi_1}{\partial x^2}+F_0(y)\frac{\partial^2\phi_1}{\partial y^2}\right)$$

$$+(f_3x^3+3f_3x^2y-2f_2xy-f_1x+f_1y+2f_0)\frac{\partial\phi_1}{\partial x} \tag{2.2.8}$$

$$+(f_3y^3+3f_3xy^2+2f_2xy+f_1x-f_1y-2f_0)\frac{\partial\phi_1}{\partial y}-6A^2f_3^2(x+y)^5\ . \tag{2.2.9}$$

Here the constants (d_i,f_j,h_k,j_l) are new integration constants that emerge from the integration of the field equations at linear order in α. Even though the equation (2.2.9) seems not to admit an analytic solution, it can be solved as a power series in the acceleration A, around $A=0$. In order to be able to take the limit $A=0$ in (2.2.1), it is useful to perform the following change of coordinates (see Chapter 14 of [132]):

$$x=-\cos\theta,\ y=\frac{1}{Ar},\ t=A\tau\ , \tag{2.2.10}$$

and choosing

$$f_2 = -f_0 = 1 \text{ and } f_3 = -f_1 = -2mA \ , \tag{2.2.11}$$

leads to the following parametrization for the C-metric in General Relativity

$$ds_0^2 = \frac{1}{(1-Ar\cos\theta)^2}\left(-Q_0(r)\,\mathrm{d}\tau^2 + \frac{\mathrm{d}r^2}{Q_0(r)} + \frac{r^2\mathrm{d}\theta^2}{P_0(\theta)} + P_0(\theta)\,r^2\sin^2\theta\mathrm{d}\varphi^2\right) \ , \tag{2.2.12}$$

with

$$Q_0(r) = \left(1-\frac{2m}{r}\right)\left(1-A^2r^2\right) \ , \tag{2.2.13}$$

$$P_0(\theta) = 1-2mA\cos\theta \ . \tag{2.2.14}$$

In terms of (r,θ), and choosing the constants $f's$ as in (2.2.11), the equation (2.2.9) is integrated, order by order in the acceleration A. For such purpose, it is convenient to choose

$$\phi_1(r,\theta) = (1-Ar\cos\theta)\,H(r,\theta) \ , \tag{2.2.15}$$

with

$$H(r,\theta) = \sum_{i=0} H_i(r,\theta)\,A^i$$

which leads to the following functions at the lowest orders

$$H_0(r,\theta) = -\frac{4m}{3r^3} - \frac{1}{r^2} - \frac{1}{mr} \ , \tag{2.2.16}$$

$$H_1(r,\theta) = \left(\frac{26m}{3r^2} - \frac{1}{r}\right)\cos\theta \ , \tag{2.2.17}$$

$$H_2(r,\theta) = \frac{2m\left(2\sin^2\theta - 23\right)}{3r} \ . \tag{2.2.18}$$

Other solutions are possible, but they lead to logarithmic or divergent behavior for the dilaton as $r \to \infty$.

In order to clarify the meaning of the plethora of integration constants which remain arbitrary on the metric functions, it is useful to reconstruct the full, corrected spacetime (2.2.1), in (r,θ) coordinates. The change of coordinates (2.2.10), induces the presence of A^{-1} terms in the metric coming from the terms (2.2.4)-(2.2.7) written in terms of (r,θ), which are removed by setting $d_3 = 0$. Imposing the absence of divergences at $\theta = 0$ and $\theta = \pi$ on the metric functions suffices to fix all the remaining integration constants, but j_1, leading to $d_2 = d_1 = d_0 = h_1 = h_2 = 0$, which in consequence leads to vanishing corrections of the function F and G, namely

$$F_1(r,\theta) = 0, \; G_1(r,\theta) = 0 \; , \tag{2.2.19}$$

and to a conformal factor $\omega(r,\theta)$ given by

$$\omega(r,\theta) = 2\phi_1(r,\theta) \; , \tag{2.2.20}$$

where we have also set $j_1 = 0$ since a non-vanishing value of j_1 can be absorbed on the dilaton's additive, arbitrary constant ϕ_0.

Putting all these ingredients together, lead to the corrected metric which is given by

$$ds^2 = \frac{1+2\alpha\phi_1(r,\theta)}{(1-Ar\cos\theta)^2}\left(-Q_0(r)\,\mathrm{d}\tau^2 + \frac{\mathrm{d}r^2}{Q_0(r)} + \frac{r^2\mathrm{d}\theta^2}{P_0(\theta)} + P_0(\theta)\,r^2\sin^2\theta\mathrm{d}\varphi^2\right) \; . \tag{2.2.21}$$

Here $\phi_1(r,\theta)$ is given by (2.2.15) and $Q_0(r)$ and $P_0(\theta)$ given by (2.2.13) and (2.2.14), respectively. One can check, that the metric (2.2.21), with $\phi_1(r,\theta)$ in (2.2.15), solve the field equations of the theory (3.1.1), disregarding terms of the form $\mathcal{O}(\alpha^2)$ and $\mathcal{O}(\alpha A^3)$, i.e. when evaluated on the corrected C-metric, the field equations vanish up to

$$E_{\mu\nu} = \mathcal{O}(\alpha^2) + \mathcal{O}\left(\alpha A^3\right) \; . \tag{2.2.22}$$

It is very interesting to notice that the regularity conditions lead us to move the whole effect of the α-correction to the conformal factor. The solution can be found to higher orders on the acceleration by performing the integration of the PDE (2.2.9), at the desired order on A, after moving to (τ, r, θ) coordinates via (2.2.10), in such a manner that the limit of vanishing acceleration is regular.

2.3 Map of Gauss-Bonnet to the Einstein frame

Recently in [123], the authors constructed the dimensional reduction of the Heterotic String on a flat torus, to dimension four, and constructed rotating solutions, perturbativelly in the rotation parameter, including the first α'-correction, in the Einstein frame. It is interesting to compare the setup we have considered here, defined by the action (3.1.1), with the one of reference [123], where disregarding the contribution of the $B_{\mu\nu}$-field, leads to an action of the form

$$I[g'_{\mu\nu},\tilde{\phi}] = \int \mathrm{d}^4x\sqrt{-g'}[R' - \frac{1}{4}\nabla_\mu\tilde{\phi}\nabla_\nu\tilde{\phi}g'^{\mu\nu} + \\ + \alpha e^{-\tilde{\phi}}(R'^{\mu\nu}{}_{\rho\sigma}R'^{\rho\sigma}{}_{\mu\nu} - 4R'^{\nu}{}_{\sigma}R'^{\sigma}{}_{\nu} + R'^2)] \ . \tag{2.3.1}$$

Considering a Weyl transformation of the form

$$g_{\mu\nu} \quad \mapsto \quad g'_{\mu\nu} = e^{\Phi}g_{\mu\nu} \ , \tag{2.3.2}$$

where Φ is some scalar function on the spacetime, the transformation of the quadratic scalars constructed with the Riemann tensor are given by

$$R'^{\mu\nu}{}_{\rho\sigma}R'^{\rho\sigma}{}_{\mu\nu} = e^{-2\Phi}\Big[R^{\mu\nu}{}_{\rho\sigma}R^{\rho\sigma}{}_{\mu\nu} - 4R^{\nu}{}_{\sigma}\nabla_\nu\Phi^\sigma + 2R^{\nu}{}_{\rho}\Phi^\rho\Phi_\nu - R\Phi^\lambda\Phi_\lambda \tag{2.3.3}$$
$$+ D_2\nabla_\sigma\Phi^\nu\nabla_\nu\Phi^\sigma + (\Box\Phi)^2 - D_2\nabla_\sigma\Phi_\nu\Phi^\sigma\Phi^\nu$$
$$+ D_2\Box\Phi\Phi^\lambda\Phi_\lambda + \frac{1}{8}D_2D_1(\Phi^\lambda\Phi_\lambda)^2\Big] \ ,$$

$$R'^{\nu}{}_{\sigma}R'^{\sigma}{}_{\nu} = e^{-2\Phi}\Big[R^{\nu}{}_{\sigma}R^{\sigma}{}_{\nu} - D_2R^{\nu\sigma}\nabla_\nu\Phi_\sigma - R\Box\Phi + \frac{1}{2}D_2R^{\nu\sigma}\Phi_\nu\Phi_\sigma - \frac{1}{2}D_2R\Phi^\lambda\Phi_\lambda$$
$$+ \frac{1}{4}D_2^2\nabla_\sigma\Phi^\nu\nabla_\nu\Phi^\sigma + \frac{1}{4}(3D-4)(\Box\Phi)^2 + \frac{1}{16}D_2^2D_1(\Phi^\lambda\Phi_\lambda)^2$$
$$-\frac{1}{4}D_2^2\nabla_\sigma\Phi_\nu\Phi^\sigma\Phi^\nu + \frac{1}{4}D_2(2D-3)\Box\Phi\Phi^\lambda\Phi_\lambda\Big] \ , \tag{2.3.4}$$

$$R'^2 = e^{-2\Phi}\Big[R^2 - 2D_1R\Box\Phi - \frac{1}{2}D_1D_2R(\partial\Phi)^2 + D_1^2(\Box\Phi)^2 \tag{2.3.5}$$
$$+\frac{1}{2}D_1^2D_2\Box\Phi(\partial\Phi)^2 + \frac{1}{16}D_1^2D_2^2(\partial\Phi)^4\Big]$$

where $\Phi_\lambda \equiv \nabla_\lambda\Phi$ and $D_p \equiv (D-p)$. These expressions lead to the following transformation of the Gauss-Bonnet density, which we had actually worked out in arbitrary dimension D

$$R'^{\mu\nu}{}_{\rho\sigma}R'^{\rho\sigma}{}_{\mu\nu} - 4R'^{\nu}{}_{\sigma}R'^{\sigma}{}_{\nu} + R'^2$$
$$= e^{-2\Phi}\Big[R^{\mu\nu}{}_{\rho\sigma}R^{\rho\sigma}{}_{\mu\nu} - 4R^{\nu}{}_{\sigma}R^{\sigma}{}_{\nu} + R^2 + 4D_3G^{\nu\sigma}\nabla_\nu\Phi_\sigma - 2D_3R^{\nu\sigma}\Phi_\nu\Phi_\sigma$$
$$-\frac{1}{2}D_4D_3R(\partial\Phi)^2 - D_3D_2\nabla_\sigma\Phi^\nu\nabla_\nu\Phi^\sigma + D_3D_2\nabla_\sigma\Phi_\nu\Phi^\sigma\Phi^\nu$$
$$+ D_3D_2(\Box\Phi)^2 + \frac{1}{2}D_3^2D_2\Box\Phi(\partial\Phi)^2 + \frac{1}{16}D_4D_3D_2D_1(\partial\Phi)^4\Big] \ ,$$
$$= e^{-2\Phi}(\mathcal{G} + \mathcal{P}) \tag{2.3.6}$$

where

$$\mathcal{G} = R^{\mu\nu}{}_{\rho\sigma}R^{\rho\sigma}{}_{\mu\nu} - 4R^{\nu}{}_{\sigma}R^{\sigma}{}_{\nu} + R^2 \tag{2.3.7}$$

and $\mathcal{P}$ stands for the remaining terms. Replacing these expression in (2.3.1) and choosing the scalar Φ as

$$\Phi = -\frac{4}{D-2}\phi \ , \tag{2.3.8}$$

and identifying the scalar field in (2.3.1) $\tilde{\phi}$ as

$$\tilde{\phi} = \frac{4}{D-2}\phi, \tag{2.3.9}$$

and setting $D = 4$, leads to

$$\begin{aligned} & I[g'_{\mu\nu} = e^{-2\phi} g_{\mu\nu}, \tilde{\phi} = 2\phi] \qquad (2.3.10) \\ & = \int \mathrm{d}^4x\sqrt{-g}e^{-2\phi}\left[R + 4\left(\partial\phi\right)^2 + \alpha\left(\mathcal{G} - 32\left(\partial\phi\right)^4 - 16G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + 24\Box\phi\left(\partial\phi\right)^2\right)\right] \end{aligned}$$

where we have disregarded boundary terms. Notice that this action belongs to the family of the most general String Theory actions α'-corrected, which after field-redefinitions lead to second order field equations, since each of the derivative terms for the scalar sector belongs to the Horndeski family [134]. Indeed, upon comparison with eq. (2.6) of reference [127] one can read from the action (2.3.10) that the coefficients (λ, μ, ν) of reference [127] are given by $\lambda = -32, \mu = -16$ and $\nu = 24$, and they indeed fulfil the consistency constraint $\lambda + 2\left(\mu + \nu\right) + 16 = 0$. The relation among the relative coefficients of the higher derivative operators of the scalar attests about the UV finiteness of the action. In consequence, using the results of reference [127], one can see that the action (2.3.1) of reference [123] in the Einstein frame and our action (3.1.1) in the string frame are related by a field redefinition, composed with a change of frame.

Therefore, our static and rotating solutions of Sections 2.1 and 2.1.2, correspond to a change of frame of the solutions found in [81] and [123], respectively, composed with a field redefinition. On the other hand, the solution corresponding to accelerating black holes we presented in Section 2.2, is completely new.

2.4 The D-dimensional solution

The analytic solution that we constructed in $D = 4$ can be generalized to arbitrary dimension D in a similar manner, although, as we will see, the form of the general case is a bit more involved.

Consider the ansatz

$$ds^2 = -\left(1+\alpha N_1(r)\right)^2\left(1-\left(\frac{\mu}{r}\right)^{D-3}+\alpha f_1(r)\right)\mathrm{d}t^2+\frac{\mathrm{d}r^2}{1-\left(\frac{\mu}{r}\right)^{D-3}+\alpha f_1(r)}+r^2\mathrm{d}\Omega^2_{D-2}\ ,$$

where now $\mathrm{d}\Omega^2_{D-2}$ is the constant-curvature metric on the unit $(D-2)$-sphere. For simplicity, let us define the quantity $X = \mu/r$. By plugging this ansatz in the field equations for D generic, we find the following general solution

$$\begin{aligned}N_1(r) &= -\frac{C_3}{\mu^2 f_0(r)}X^{D-3}+\frac{(D-3)(D-2)}{(D-1)\mu^2}X^{D-1}\left[F(r)+\frac{1}{f_0(r)}\left(X^{D-3}D-1\right)\right]\\&\quad+\frac{C_3}{(D-3)\mu^2}\log f_0(r)+\frac{C_2}{\mu^2}\ ,\\ f_1(r) &= -\frac{(D-3)}{(D-1)\mu^2}X^{2D-4}\left[(D-3)(D-2)F(r)+2(2D-3)\right]+\frac{C_1}{\mu^2}X^{D-3}\\&\quad-\frac{C_3}{\mu^2}X^{D-3}\log f_0(r)\ ,\end{aligned}$$

and

$$\phi_1(r) = \frac{(D-3)\,(D-2)^2}{2\,(D-1)\,r^2}X^{D-3}\left(F(r)-1\right)+\frac{(D-2)C_3}{2(D-3)\mu^2}\log f_0(r)+\frac{C_4}{\mu^2}$$

where $f_0(r) = 1-X^{D-3}$ and where $F(r)$ is given in terms of the hypergeometric function,

$$F(r)\ =\ {}_2F_1\left(1,\frac{D-1}{D-3},2\,\frac{D-2}{D-3},X^{D-3}\right)\ ;$$

C_1, C_2, C_3, and C_4 are integration constants, analogous to the constants A, B, C, and D of the $D=4$ case; for example, one can identify $C_3=\mu^2 B+2$, $C_4=\mu^2\bar{\phi}_0$, and so on. Some of these constants can be fixed as in the 4-dimensional case, i.e. by rescaling time coordinate t, shifting the zero-mode of ϕ, neglecting $\mathcal{O}(\alpha^2)$ remnants, imposing a globally flat asymptotic behavior as $r\to\infty$, and requiring regularity at the event horizon. One can easily check that the 4-dimensional solution studied in the previous sections is recovered in the case $D=4$. To see this, it is convenient to consider the relation

$${}_2F_1(1,3,4,z) = -\frac{3}{2z^3}\left[z(z+2)+2\log(1-z)\right]\ , \tag{2.4.1}$$

with $z = 1-f_0(r)$. The presence of logarithmic terms $\log f_0(r) = \log(1-z)$ in the functions $N_1(r)$, $f_1(r)$ and $\phi_1(r)$ is related to the fact that the third argument of the hypergeometric function, $c = 2\frac{(D-2)}{(D-3)}$, turns out to be an integer for some dimensions $(D=4,5)$. The logarithm, which in any case tends to zero at large r, disappears if one chooses C_3 appropriately.

2.5 Last remarks

Summarizing, the solutions we have presented in this chapter describe static, spherically symmetric configurations in the graviton-dilaton sector of the D-dimensional low-energy stringy effective action (3.1.1). This includes square-curvature terms and a non-vanishing dilaton coupling. We have used the freedom of field redefinitions to recast the action in a form that leads to second order field equations, while still working in the string frame. The set of solutions includes asymptotically flat black holes with regular event horizons, which behave as thermodynamic objects, just like expected. As a working example, we first focused on the the 4-dimensional case, which is given by (2.1.26)-(2.1.28). We derived the corrections to the thermodynamic variables introduced by the higher-curvature effects; we computed the Bekenstein-Hawking entropy (2.1.25), the Hawking temperature (2.1.30), and the mass formula (2.1.41) including the $\mathcal{O}(\alpha)$ effects. The computation of the Noether charges was shown to be in exact agreement with the first law of black hole mechanics as derived from the Wald entropy formula; the Euclidean action formalism also reproduces these results. We also obtained the correction to the C-metric, which contains accelerating black holes. We have show that regularity conditions imply that the whole modifications is contained within the conformal factor of the spacetime. We also integrated the equations of motion for the stationary, non-static solution in the slowly rotating approximation. This yields stringy corrections to the Kerr geometry in four dimensions. Although, in contrast to the 4-dimensional case, the field equations in arbitrary dimension D are more involved, we showed that in the static case they can still be solved explicitly in terms of hypergeometric functions. Static and rotating black holes in these theories have already been considered in the literature, when the theory is expressed in a different fashion, which is possible due to the freedom of field redefinition. Considering such freedom, we found the precise relation between our setup and the frames previously considered on the literature. In particular, we mapped our theory to the Einstein frame, including the higher curvature corrections, and we showed the equivalence of our setup and that of reference [123], where rotating solutions were already presented. These solutions may serve as working examples to investigate higher-curvature stringy effects in a concrete setup.

In the next chapter we implement the formulae developed in this chapter to explore α'-corrected higher dimensional black objects. We focus on the 5-dimensional case for black string configurations. In Einstein gravity this spaces are a direct product between a 4-

dimensional Einstein space, as Schwarzschild spacetime, times a circle. The presence of the Gauss-Bonnet density, motived by the field redefinition presented in this chapter, at first order in α' makes the space a warped product space between the circle and the 4-dimensional manifold. Black string configurations in Einstein gravity are well known to suffer of a pertubative instability known as Gregory-Lafflame instability. We study the correction of the spectrum of this instability and show that in the pertrubative regime, the instability region is enlarge and grows almost linearly in α'.

Chapter 3

α'-correction to the Gregory-Lafflame instability

Black strings and black branes can be defined as black hole spacetimes with horizons that have extended directions, which in the simplest case are planar [135]. These spacetimes have a very interesting dynamics, since in general they suffer from the Gregory-Laflamme instability [136, 137], which is triggered by a gravitational perturbation travelling along an extended direction with a wavelength above a given critical value. In a remarkable series of works [138, 139, 140], the authors were able to find strong numerical evidence in favour of the pinching-off of the horizons of black strings in a finite time. This phenomenon is compatible with the previous no-go results by Horowitz and Maeda [141], since it refers to a finite value of the time for asymptotic observers, providing an example of violation of cosmic censorship in dimension five for generic initial data. This result has been recently confirmed in [142], where the authors were able to numerically evolve the spacetime closer to the pinch-off, and provide evidence of a non-geometric progression for the time intervals of the appearance of new generations of black holes connected by black strings. The non-linear evolution of the system was also addressed in the context of the Large-D expansion of General Relativity (GR) [143, 144, 145], giving rise to a non-uniform black string as the final configuration after the GL instability is triggered, which is consistent with the existence of a critical dimension obtained by Sorkin in [146]. In references [147, 148] non-uniform black strings were constructed numerically and perturbatively, and in [149, 150, 151, 152] new numerical simulations of the fully non-linear Einstein equations provide evidence of violations of cosmic censorship triggered by GL instabilities for asymptotically

Minkowski spacetimes.

As the black string evolves, regions with higher curvature will be exposed, and it is natural to expect that higher curvature corrections to gravity may play a role. Remarkably, in the recent paper [153] a new gauge was found for initial value problem in Einstein-Gauss-Bonnet gravity, which leads to a strongly hyperbolic system for bounded curvatures. This may allow to evolve the black string in the presence of higher curvature terms, and study their effect on the dynamics of the system.

In this chapter we study the black string instability spectrum of backgrounds that are corrected at leading order in the Gauss-Bonnet parameter. This precise R^2 correction can be obtained from string theory as an expansion on the string tension. In such a framework, in order to construct solutions and study their stability beyond linear order in α in a consistent manner, one would have to consider higher powers of the curvature as corrections, therefore if one insists in interpreting our results as string theory corrections to the GL instability one can not go beyond linear terms in α, having always in mind that one is indeed working in an effective field theory setup. As mentioned below, this also affects the regime of validity of the gravitational perturbation. We also construct the boosted black string of this theory and obtain the first order corrections in α to the energy, momentum, entropy, temperature and tension.

3.1 The corrected, static, black string

We will consider the Einstein-Gauss-Bonnet action [21]

$$I[g] = \int d^5x\sqrt{-g}\left[R + \alpha(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2)\right] + \mathcal{O}(\alpha^2) \ . \tag{3.1.1}$$

as an effective field theory, to first order in the coupling α, which has mass dimension -2.

We start by re-obtaining the closed form static black string solutions, which were originally obtained in [154] to first order in α. Additionally, we provide thermodynamic quantities associated to this spacetime.

Let us consider a black string ansatz in dimension five, in the field equations of the theory defined by (3.1.1):

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{g(r)} + r^2\mathrm{d}\sigma^2 + b(r)\mathrm{d}z^2 \ , \tag{3.1.2}$$

where $\mathrm{d}\sigma$ is the line element of a two-sphere. We assume that the metric functions $X =$

$\{f, g, b\}$ are analytic in $\alpha = 0$, therefore they can be expanded as $X = X_0 + \alpha X_1 + \mathcal{O}(\alpha^2)$. The introduction of a dimensionful scale α forces us to consider a non-constant warp factor, $b(r)$ [155, 156]. Dropping terms $\mathcal{O}(\alpha^2)$, the system of equations for the functions X_0 and X_1 can be integrated in a closed manner. The general solution involves four new integration constants on top the integration constant of the X_0 functions, namely the mass parameter of the GR solution, as well as logarithmic terms in the radial coordinate. Using the freedom under coordinate transformations, the perturbative scheme in α and requiring the α-corrected spacetime to have an event horizon leads to the following expressions in terms of the radius of the horizon r_+:

$$f(r) = f_5(r) = 1 - \frac{r_+}{r} - \frac{(r - r_+)\left(6r_+^2 + 11rr_+ + 23r^2\right)}{9r_+ r^4}\alpha + \mathcal{O}(\alpha^2) \ , \tag{3.1.3}$$

$$g(r) = g_5(r) = 1 - \frac{r_+}{r} + \frac{(r - r_+)(r + 5r_+)(r + 2r_+)}{9r_+ r^4}\alpha + \mathcal{O}(\alpha^2) \ , \tag{3.1.4}$$

$$b(r) = b_5(r) = 1 + \frac{4\left(6r^2 + 3rr_+ + 2r_+^2\right)}{9r_+ r^3}\alpha + \mathcal{O}(\alpha^2) \ , \tag{3.1.5}$$

The temperature, mass and entropy of this black string are respectively given by

$$T = \frac{1}{4\pi r_+} - \frac{11}{36\pi r_+^3}\alpha + \mathcal{O}\left(\alpha^2\right) \ , \tag{3.1.6}$$

$$M = 8\pi r_+ L_z + \frac{88\pi}{9r_+} L_z \alpha + \mathcal{O}\left(\alpha^2\right) \ , \tag{3.1.7}$$

$$S = 16\pi^2 r_+^2 L_z + \frac{928\pi^2}{9} L_z \alpha + \mathcal{O}\left(\alpha^2\right) \tag{3.1.8}$$

where the former was computed from the surface gravity in Eddington-Finkelstein-like coordinates, and the latter two were computed using the Iyer-Wald method, as explained in Section 1.2. These expressions fulfil the first law of black hole thermodynamics

$$\delta M = T\delta S \ , \tag{3.1.9}$$

disregarding quadratic terms in α. Here, L_z is the length of the extended direction with coordinate z. Notice that both the correction to the mass and entropy are positive.

3.2 The perturbation

The generalized Lichnerowicz operator, namely, the linearized field equations around a generic background $\mathring{g}_{\mu\nu}$, read

$$\begin{aligned}0 = &\frac{1}{2}\left(-\delta^{\rho}_{\mu}\delta^{\lambda}_{\nu}\mathring{\Box} + 2\mathring{R}^{\rho}{}_{\mu\nu}{}^{\lambda} + g_{\mathring{\mu}\nu}\mathring{R}^{\rho\lambda} + 4\mathring{g}^{\lambda\eta}\mathring{G}_{\eta(\mu}\delta^{\rho}_{\nu)}\right) h_{\rho\lambda} - 2\alpha\mathring{R}_{\mu\xi\lambda\nu}\mathring{\Box}h^{\lambda\xi} \\ &+\alpha\left(-\mathring{R}^{\rho\eta\sigma}{}_{\lambda}\mathring{R}_{\xi\sigma\rho\eta}\mathring{g}_{\mu\nu} - 2\mathring{R}^{\sigma\rho}{}_{\lambda\nu}\mathring{R}_{\sigma\rho\xi\mu} + 4\mathring{R}_{\mu}{}^{\rho\sigma}{}_{\nu}\mathring{R}_{\lambda\rho\sigma\xi} + 4\mathring{R}_{\mu\rho\lambda\sigma}\mathring{R}_{\nu}{}^{\sigma\rho}{}_{\xi}\right) h^{\lambda\xi} \\ &+\alpha\left(4\mathring{g}_{\lambda(\mu}\mathring{R}_{\nu)\rho\sigma\xi}\mathring{\nabla}^{\sigma}\mathring{\nabla}^{\rho}h^{\lambda\xi} + 4\mathring{\nabla}^{\sigma}\mathring{\nabla}_{(\mu}h^{\xi\lambda}\mathring{R}_{\nu)\xi\lambda\sigma} - 2\mathring{g}_{\mu\nu}\mathring{R}^{\rho}{}_{\lambda\xi\eta}\mathring{\nabla}_{\rho}\mathring{\nabla}^{\eta}h^{\lambda\xi}\right) + \mathcal{O}(\alpha^2)\ ,\end{aligned}$$

where we have imposed transversality and tracelessness of the perturbation, i.e. $\mathring{\nabla}_{\mu}h^{\mu\nu} = 0$ and $h_{\mu}{}^{\mu} = 0$. We have also used the vacuum field equations, which allow to write the Ricci tensor and Ricci scalar in terms of an expression that is linear in α, that can be used in the linearization of the Gauss-Bonnet tensor to write every Ricci tensor and Ricci scalar in terms of the Riemann tensor plus $\mathcal{O}(\alpha^2)$ terms.

On the regime of validity of the perturbation: The black string is parameterized by the coordinates (t, r, x^i, z), where the x^i collectively denote the coordinates on a round sphere. We will focus on the s-wave, scalar mode, which is the responsible for the GL instability in GR. Therefore the metric perturbation that we are considering reads

$$h_{AB} = \varepsilon e^{\Omega t}e^{ikz}\begin{pmatrix} H_{tt}(r) & H_{tr}(r) & 0 & 0 \\ H_{tr}(r) & H_{rr}(r) & 0 & 0 \\ 0 & 0 & H(r)\sigma_{ij} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim e^{\Omega t}e^{ikz}H_{\mu\nu}(r)\ , \tag{3.2.1}$$

with $\varepsilon \ll 1$ and where σ_{ij} denotes the metric on the sphere.

Schematically, when evaluated on the perturbation, the bulk Lagrangian will have the form

$$R + \alpha R^2 \sim \text{Background} + \partial h\partial h + \alpha\left(\partial h\partial h\right)^2 + \mathcal{O}\left(\alpha^2\right)\ . \tag{3.2.2}$$

Due to the separation in modes, the terms $\partial h\partial h + \alpha\left(\partial h\partial h\right)^2$ will contain the following contributions

$$T_1 = k^2 + \alpha k^4 + \mathcal{O}\left(\alpha^2\right)\ , \tag{3.2.3}$$

$$T_2 = \Omega^2 + \alpha\Omega^4 + \mathcal{O}\left(\alpha^2\right)\ , \tag{3.2.4}$$

from the derivatives with respect to z and to t, respectively. In consequence, in order to ensure the validity of the perturbative approach, we impose

$$\alpha k^2 \ll 1 \qquad \text{and} \qquad \alpha\Omega^2 \ll 1 \ , \tag{3.2.5}$$

namely, a sufficient condition for the validity of dropping-off terms $\mathcal{O}\left(\alpha^2\right)$, is to keep attention on the modes with small k and small Ω as compared with $\alpha^{-1/2}$. We are interested in the existence and behavior of unstable modes, therefore, for a given k, after imposing the boundary conditions for the perturbations, we look for positive values of Ω that may allow to connect the regular asymptotic behavior of the perturbation both at the horizon and infinity.

As in GR, the dynamics of the scalar mode, defined by the functions $\{H_{tt}, H_{tr}, H_{rr}, H\}$ is completely controlled by the master variable $H_{tr}(r)$, and the remaining functions are given in terms of the master variable and its derivatives. The second order, linear, homogeneous ODE for $H_{tr}\left(r\right)$ has the form

$$A\left(r;\alpha\right)\frac{d^2H_{tr}}{dr^2}+B\left(r;\alpha\right)\frac{dH_{tr}}{dr}+C\left(r;\alpha\right)H_{tr}=0 \ , \tag{3.2.6}$$

where the coefficients depend on r and are linear in α, which is consistent with the perturbative approach we are considering. We do not provide the explicit expression for the coefficients since they are not illuminating.

The asymptotic expansions of (3.2.6) near the horizon and infinity lead to

$$H_{tr}(r) \sim \left(r-r_+\right)^{-1\pm\Omega\left(r_++\frac{11\alpha}{9r_+}+\mathcal{O}\left(\alpha^2\right)\right)}\left(1+\mathcal{O}\left(r-r_+\right)\right) \ , \tag{3.2.7}$$

$$H_{tr}(r) \sim \frac{e^{\pm r\sqrt{\Omega^2+k^2}}}{r^{\xi_\pm}}\left(1+\mathcal{O}(r^{-1})\right) , \tag{3.2.8}$$

where $\xi_\pm$ are constants which are not-relevant for recognizing the regular asymptotic behavior, dominated by the exponential growing/suppression. In consequence, we choose the $(+)$-branches in the near horizon expansion and the $(-)$-branches as r goes to infinity.

In order to find the spectrum, we proceed as follows: we introduce the function $F(r)$ such that

$$H_{tr}(r) = \left(r-r_+\right)^{-1+\Omega\left(r_++\frac{11\alpha}{9r_+}+\mathcal{O}\left(\alpha^2\right)\right)}F\left(r\right) \ , \tag{3.2.9}$$

and rewrite the equation for $F(r)$ in terms of the coordinate x such that $r = r_+/(1-x)$ which maps $r \in]r_+, +\infty[$ to $x \in]0,1[$. Then, we select the regular branch at the horizon by assuming that $F(x)$ has a Taylor expansion around $x = 0$. We therefore define the truncated series as

$$F_N(x) = 1 + \sum_{j=1}^{N} a_j(k, r_+, \Omega, \alpha)\, x^j \ , \tag{3.2.10}$$

and solve the equation, near the horizon ($x \to 0$) for the coefficients a_j. For a given wavenumber k, horizon radius r_+ and value of the coupling α, the frequencies are obtained by setting $F_N(1) = 0$, for a large enough N, such that a notion of numerical stability for the frequency Ω is attained. Actually, one can introduce the dimensionless quantities $\hat{\Omega} = r_+\Omega$ and $\hat{k} = r_+k$ and observe that (3.2.10) depends only on the pair $(\hat{k}, \hat{\Omega})$ and the dimensionless ratio α/r_+^2. We have also used shooting to validate our numerical results. Figure 1 depicts the spectrum of the scalar perturbation in GR (red lines), as

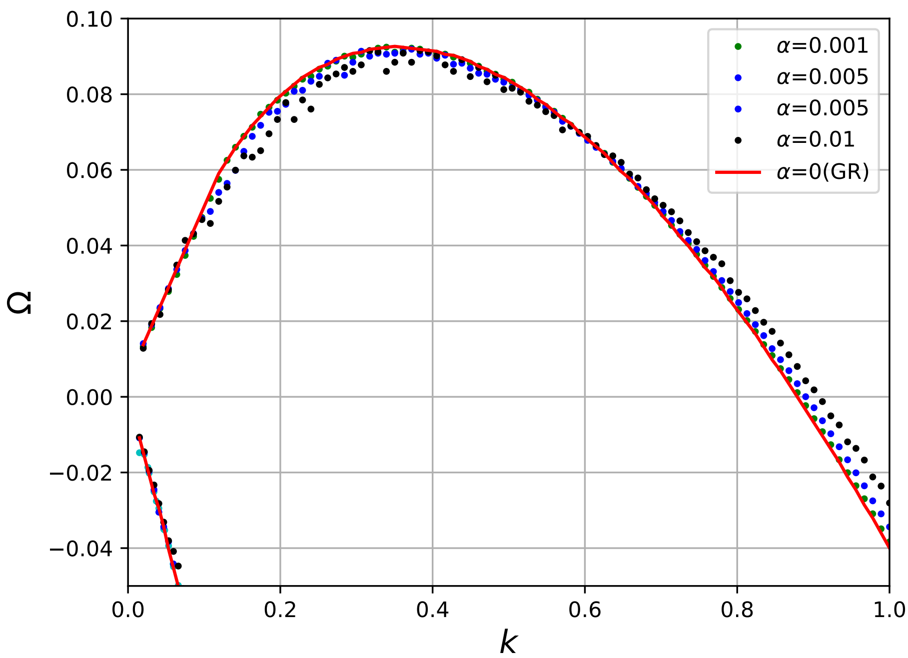

Figure 3.2.1: Spectra of the perturbation for different values of the α correction, and r_+=1.

well as its correction for different values of the coupling α. Aside from the unstable modes, with positive Ω, we also depict a second stable mode with $\Omega < 0$, which is present in GR and corrected due to the presence of the perturbative Gauss-Bonnet term. Our numerical resolution is not enough to discriminate the behavior of the corrected spectrum for $k \sim 0$, which may be attained for analytic treatment if one generalizes the approach of the AdS/Ricci-flat correspondence of [157] to the presence of higher curvature terms. It is interesting to notice that the critical value of k_c that may trigger the GL instability grows with the value of the Gauss-Bonnet coupling, namely the region of instability grows as the GB coupling is turned on, which can be seen in more detail in Figure 2.

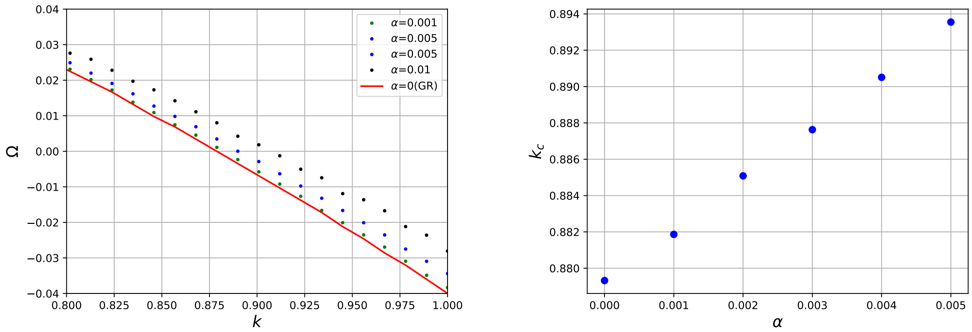

Figure 3.2.2: Behavior of the critical wavelength that triggers the black string instability on the scalar mode, as a function of the α correction, and r_+=1.

3.3 The α-corrected, boosted black string

A boosted black string, with momentum along the z direction, can be obtained by applying the transformation

$$\begin{aligned} t &\mapsto \cosh\beta t + \sinh\beta z \\ z &\mapsto \cosh\beta z + \sinh\beta t \ , \end{aligned} \tag{3.3.1}$$

to the metric (3.1.2). This transformation corresponds to a boost with rapidity $v = -\tanh\beta$, and it generates non-vanishing momentum along the z direction, which can be checked by computing the corresponding, α-corrected ADM momentum, as in GR [158]. Since the new configuration is characterized by a different set of global charges, it corresponds to a physically different state in the phase space of the theory in spite of being generated from the static solution (3.1.2) by a simple boost (3.3.1). Then, the α-corrected, boosted black string is

$$\mathrm{d}s^2_{\text{boosted}} = -b(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{g(r)} + r^2 d\sigma^2 + b(r)\mathrm{d}z^2 + (b(r) - f(r))\cosh^2\beta\,(\mathrm{d}t + \tanh\beta dz)^2 \ . \tag{3.3.2}$$

Notice that the solution is asymptotically flat on a static frame, since given b, f and g of equations (3.1.3)-(3.1.5), the g_{tz} component of the metric (3.3.2) vanishes as $r \to +\infty$.

Again, using the Iyer-Wald method, one can obtain the energy and the momentum of these α-corrected, boosted black strings, as conserved charges associated to the Killing vectors ∂_t and ∂_z, respectively. This yields

$$Q[\partial_t] = E = 4\pi r_+(\cosh^2\beta + 1)L_z + \frac{4\pi}{9r_+}(47\cosh^2\beta - 25)L_z\alpha + \mathcal{O}(\alpha^2) \; , \tag{3.3.3}$$

$$Q[\partial_z] = P = -4\pi r_+ \cosh\beta \sinh\beta L_z - \frac{188\pi}{9r_+}\cosh\beta\sinh\beta L_z\alpha + \mathcal{O}(\alpha^2) \; . \tag{3.3.4}$$

Going to Eddington-Finkelstein-like coordinates, one computes the surface gravity of the boosted black string, which leads to the following expression for the temperature

$$T = \frac{1}{4\pi r_+ \cosh\beta} - \frac{11}{36\pi r_+^3 \cosh\beta}\alpha + \mathcal{O}(\alpha^2) \; . \tag{3.3.5}$$

The entropy of the configuration is given by the Iyer-Wald formula

$$S = \frac{1}{T}\int_{\mathcal{H}_+} \star\mathbf{q}[\xi] \, , \tag{3.3.6}$$

where ξ is the horizon generator. As expected, the expression for ξ leads to the horizon velocity $v_h = -\tanh\beta$, and the entropy takes the form

$$S = \cosh\beta\left(16\pi^2 r_+^2 L_z + \frac{928\pi^2}{9}L_z\alpha\right) + \mathcal{O}\left(\alpha^2\right) \; . \tag{3.3.7}$$

As before, one recovers the GR expression of reference [158] when α goes to zero.

For black strings of a fixed length, the first law is fulfilled, namely

$$\delta M = T\delta S + v_h \delta P \; , \tag{3.3.8}$$

disregarding terms $\mathcal{O}\left(\alpha^2\right)$. On a more general ensemble, one can also consider variations of L_z, as in [158]. This leads to an extra work term in the first law, with the tension $\hat{\mathcal{T}}$ as the variable conjugate to L_z. In this case, the first law takes the form

$$\delta M = T\delta S + v_h\delta P + \hat{\mathcal{T}}\delta L_z \; , \tag{3.3.9}$$

where the tension acquires a correction with respect to that of black strings in GR, namely

$$\hat{\mathcal{T}} = 4\pi r_+ - \frac{100\pi}{9r_+}\alpha + \mathcal{O}\left(\alpha^2\right). \tag{3.3.10}$$

It is very interesting to notice that in contra-position to what occurs for boosted black strings in GR [159], the inclusion of the higher curvature correction α, spoils the validity of an standard Smarr Law, namely $2M$ is different from $3TS+\hat{\mathcal{T}}L_z+2v_hP$. As in the presence of a cosmological constant [160, 161], one can restore the validity of a relation between finite thermodynamics quantities, i.e. the validity of the Euler relation of thermodynamics for a homogeneous system, by including in the first law a work term proportional to variations of the dimensionfull coupling constant α. This approach leads to

$$\delta M = T\delta S + v_h \delta P + \hat{\mathcal{T}}\delta L_z + \mu_\alpha \delta\alpha \ , \tag{3.3.11}$$

with μ being the canonical conjugate of α and is defined as

$$\mu_\alpha = \frac{\partial M(S,P,L_z,\alpha)}{\partial\alpha} = -\frac{16L_z\pi}{r_+} \ . \tag{3.3.12}$$

With these expressions at hand, for the boosted black string characterized by the parameters v_h and r_+, it is direct to prove the following Smarr-like formula

$$2M = 3TS + \hat{\mathcal{T}}L_z + 2v_hP + 2\mu_\alpha\alpha \ , \tag{3.3.13}$$

which can also be obtained on dimensional grounds by a scaling argument. It would be interesting to have a physical understanding of the latter work term, which must be present if one insists on the validity of an Euler-like relation within this setup, and to explore its effect on the possible phase transitions that may be triggered by this new term. Notice also that the correction to the mass and the entropy are positive, while the contribution of the α term to the momentum of the boosted black string has the same sign as the uncorrected value[1].

[1]See the recent [162, 163] for a string theory setup that leads to negative contributions to the entropy, due to α' corrections at fixed global charges, for black strings and black branes. This is in tension with standard expectations from the weak gravity conjecture.

3.4 Further comments

In this chapter, we computed the effect of the Gauss-Bonnet term on the spectrum of the scalar mode that triggers the GL instability in a regime in which the Gauss-Bonnet coupling can be treated as a perturbation, which is necessary if we want to interpret this R^2 term as a higher curvature correction coming from string theory. In the previous chapter we construced the static, spherically symmetric black hole solution of this setup was obtained for arbitrary dimensions including the dilaton in a frame that leads to second order field equations (see also [81] for the original computation on the frame leading to fourth order field equations). Furthermore, the mentioned second order frame [127] allowed to identify the four-dimensional, rotating solution containing terms of order α, a, a^2 and αa. It would be interesting to explore effects of the rotation of the four-dimensional metric on the black string and evaluate the interplay between the superradiant instability and the GL instability as in [164, 165]. The presence of a^2 terms of the perturbative solution constructed in Chapter 2 allows the existence of an ergoregion and therefore a potential superradiant behavior. As with the dilaton, it is also known that the presence of fluxes and scalars with non-minimal couplings, permits the construction of closed form solutions of homogeneous black strings and black branes in presence of higher curvature terms [166, 167, 168, 169]. The effect of such terms on the GL instability still remains an open problem, but it is important to mention that some of the field theories that involve non-minimally coupled scalar fields may also admit strongly hyperbolic formulations as shown in [153].

As a simplified setup to evaluate the effect of the higher curvature corrections, one can consider the regime in which the higher curvature terms completely dominate over GR terms, which may be consistent if one goes beyond the perturbative regime. This approach was considered in pure R^2 [170, 171] and pure R^3 [172] Lovelock theories, exploiting the fact that these theories admit exact, homogeneous black strings [155, 156]. In these works it was shown that the GL instability persists, but in this regime the region of instability shrinks as one goes from R to R^2 and then to the R^3 theory. Recently, in the context of the Large-D expansion, these results were recovered analytically for the R^2 case [173] and also by the inclusion NLO terms in the $1/D$ correction it was shown that the critical dimension increases with the value of the coupling (see also [174]).

Finally, it is important to mention that in the context of M-theory, still at a perturbative level, the authors of [175] considered the R^4 correction in the analysis of the thermodynamic analogue of the GL instability of boosted black strings. More recently, the authors of [176] showed that the singularity of a two-dimensional black hole can be smoothed out by using a recent classification of the higher curvature corrections from a Bottom-Up approach via T-duality [95, 177, 178]. Furthermore, they elaborated on the regularity of the corresponding black string constructed out from this black hole. This setup is simple enough as to study the dynamics of black strings in three dimensions, considering higher curvature corrections of arbitrary high power.

In the following chapter, we change gear and move to study supergravity background at the leading order of the α' expansion of Type II string theory. More precisely, we consider type IIB supergravity compactified on a deformed S^5. The resulting model is known as the STU model in $D = 5$. In this 5-dimensional supergravity we construct supersymmetric configuration and asymptotically supersymmetric configurations. Those configurations, as we argued in the introduction, are usually described by short multiplets, making them stable under quantum corrections. The spacetime that we construct are deformations of the charged AdS soliton [179] — they have a contractible cycle in bulk —. From the holographic perspective, they describe a deformation of $\mathcal{N} = 4$ SYM in the Coulomb brunch formulated on 3-dimensional Minkowski spacetime times a circle. Next chapter is based on the article [30] writen in colaboration with Andrés Anabalón and Horatiu Nastase.

Chapter 4

Solitons in Supergravity and different vacua of $\mathcal{N} = 4$ Super Yang-Mills

4.1 Setup and motivations

In classical field theory, solitons are defined as field configurations that are different from the trivial vacuum and have finite energy. If the field theory is formulated on a non-compact space $\mathscr{M}$, the finiteness of the energy restricts the possible boundary conditions for the fields. This forces the field configuration to go to a vacuum at infinity. In other words, the fields denoted collectively by Φ are maps $\Phi : \mathscr{M} \to \mathscr{K}$, where $\mathscr{K}$ is the space where the field takes values, being generically a smooth manifold. For instance, in the case of N real scalar fields $\mathscr{K} = \mathbb{R}^N$, in the case of an algebra-valued gauge field $\mathscr{K} = \mathfrak{g} \otimes T^* \mathscr{M}$, where $\mathfrak{g}$ is the algebra of the gauge group. The dynamics of the fields are governed by the field equations, and the energy is defined as the Noether charge associated to time translations, that always exist in Minkowski. Let us take a generic point $p \in \mathscr{M}$, we define a vacuum configuration `Vac` in a neigborhood $\mathscr{U}_p$ of p as configurations in which the energy density vanishes identically on $\mathscr{U}_p$. In sensible theories, the trivial vacuum is defined as the configurations where the field takes the zero value. One necessary condition to have soliton configurations in a field theory is that the vacuum is degenerate, namely, there are more configurations belonging to `Vac` than the trivial one. Then, the condition of having finite energy but not being the trivial vacuum implies that a soliton configuration should interpolate between different elements in `Vac`.

This definition is broad enough to encompass what we will discuss in this chapter, that are soliton configuration in supergravity. We will study a particular set of configurations that are generically obtained from black holes by performing a double analytic continuation on the spacetime.

The first example of this type of configuration appeared in the context of the stability of the Kaluza-Klein (KK) vacuum in general relativity [180]. The KK vacuum, $\mathbb{R}^{1,3} \times S^1$, is perturbatively stable. The question that was studied and answered in [180] was whether the KK vacuum is stable at the semiclassical level. The argument is as follows: the analytic continuation of the KK vacuum is $\mathbb{R}^4 \times S^1$. A solution of the Euclidean Einstein equations with the same asymptotic behavior as $\mathbb{R}^4 \times S^1$ does exist, and is the analytic continuation of Schwarzschild-Tangherlini. After the analytic continuation of Schwarzschild-Tangherlini, the coordinate that was time becomes periodic and its period is related to the mass parameter that can be chosen in such a way that its periodicity is the same as the periodicity of the S^1 in the KK vacuum.

In this context, the Euclidean Schwarzschild-Tangherlini solution is a "bounce" solution of the KK vacuum, and it represents an instability of the KK vacuum if there are negative action modes in small fluctuations of it. However, the false vacuum will decay into the real Lorentzian vacuum if the latter agrees on the bounce solution on a 3-dimensional spacelike surface. This is precisely what happens in the present case: it is possible to Wick rotate a coordinate on the $S^3 \subset$ Euclidean Schwarzschild-Tangherlini to obtain the following Lorentzian configuration

$$\mathrm{d}s^2 = -r^2\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1-\frac{r_+^2}{r^2}} + \left(1-\frac{r_+^2}{r^2}\right)\mathrm{d}\varphi^2 + \cosh^2 t\,\mathrm{d}\Omega^2\,, \tag{4.1.1}$$

where r_+ is an integration constant, φ is the periodic coordiante with period $2\pi r_+$, t is the timelike coordinate and $\mathrm{d}\Omega^2$ is the metric of the unit 2-sphere. The periodicity of the coordinates φ is required to have a smooth manifold closed to the point p with coordinate $r = r_+$. Therefore, the radial coordinate takes values in $r \in [r_+, \infty[$ and the space is geodesically complete, as the coordinate φ shrinks smoothly at r_+. The space at $t = 0$, with topology $\mathbb{R}^2 \times S^2$, is isometric to a 3-dimensional surface of the Euclidean Schwarzschild-Tangherlini. Hence, the KK vacuum decays into the space (4.1.1).

The space (4.1.1) is at least peculiar; let us analyse why. For large values of r there is a 2-dimensional space spanned by (t, r) which is nothing but Minkowski space in Rindler coordinates with metric $-r^2\mathrm{d}t^2 + \mathrm{d}r^2$ that does not cover the full Minkowski. Indeed, they cover a wedge of it: let X, T be the coordinates of 2-dimensional Minkowski space with metric $-\mathrm{d}T^2 + \mathrm{d}X^2$, the change of coordinates $X = r\cosh t$, $T = r\sinh t$ leads to the Rindler metric mentioned above. This patch covers only the region $X^2 - T^2 = r^2 > 0$. Thus, an observer in the space (4.1.1)—for r large enough—is essentially in Minkowski space with one coordinate periodic of size $2\pi r_+$, which shrinks for smaller r. Furthermore, if the observer cannot detect the five-dimensional periodic coordinate, they will conclude that they are in four-dimensional Minkowski space with the region $X^2 - T^2 < r_+^2$ omitted.

The interpretation given by Witten is the following: the KK vacuum is unstable and decays into (4.1.1) by the formation of a bubble of size $r = \sqrt{r_+^2 + T^2}$ that grows exponentially with time. For this reason, the space (4.1.1) is called bubble-of-nothing. However, the bubble does not reach infinity since the KK vacuum (which is locally the same as the asymptotic region of bubble-of-nothing space) is unstable and will form more bubbles that will expand, and at some point, the boundaries of two bubbles will collide. The dynamics after this event are difficult to predict.

Our final observation about this decay process is that the topology of the initial state, e.g., the KK vacuum, has a non-contractible cycle, implying that there are nonequivalent ways to define spinors on it, for instance, with periodic or anti-periodic boundary conditions on the cycle. In contrast, the topology of the final state has no non-trivial cycles — topologically it is $\mathbb{R}^2 \times S^2$. Thus, there is a unique spin structure on it that corresponds to considering anti-periodic boundary conditions for the fermions. As a consequence of this, if we define spinors on the KK vacuum with periodic boundary conditions, the KK vacuum cannot decay into the bubble-of-nothing.

In Einstein gravity with $\Lambda = 0$, static black holes cannot have planar horizons, while by including negative cosmological constant, the topology of the horizon can be either planar, spherical, or hyperbolic. In that latter case, it is possible to construct higher genus horizons by making appropriate identifications. The explicit local form of these solutions of Einstein's equations $G_{\mu\nu} + \Lambda g_{\mu\nu} = 0$ are

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\mathrm{d}s^2(\mathscr{M}_{k,D-2}) \tag{4.1.2}$$

$$f(r) = -\frac{2\Lambda r^2}{(D-1)(D-2)} + k - \frac{2m}{r^{D-3}} \tag{4.1.3}$$

where $\mathscr{M}_{k,D-2}$ is a $(D-2)$-dimensional manifold of constant curvature k. Clearly, for $\Lambda < 0$, $m > 0$ and *any* choice of k, the metric function $f(r)$ is positive for r large enough and the metric asymptotes to AdS$_D$ spacetime. Additionally, the existence of solution of $f(r) = 0$ relies on the existence of an intersection between the parabola $\frac{r^2}{\ell^2} + k$, with $\ell^2 = -(D-1)(D-2)/2\Lambda$, and the hyperbola $2m/r^{D-3}$, in the region $r > 0$. This intersection always occurs, so there is always a zero of $f(r)$, making the space a black hole with a horizon of different topology depending on the choice of k.

Let us concentrate on the planar case $k = 0$ and let $x^1, \ldots, x^{D-2}$ be the coordinates on the $D-2$ planar space. Picking a the pair (t, x^{D-2}) and sending them to $(t, x^{D-2}) \mapsto (\mathrm{i}t, \mathrm{i}\varphi)$, the space becomes

$$\mathrm{d}s^2 = f(r)\mathrm{d}\varphi^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2(-\mathrm{d}t^2 + \mathrm{d}x^i\mathrm{d}x^i)\,, \tag{4.1.4}$$

$$f(r) = \frac{r^2}{\ell^2} - \frac{2m}{r^{D-3}}\,, \tag{4.1.5}$$

where x^i are the remaning coordinates on the planar space $i = 1, \ldots, D-3$. The space (4.1.4) is smooth everywhere, whenever there exist an $r = r_+$ such that $f(r_+) = 0$ and $f'(r_+) \neq 0$, that for our case, we showed that always exist. The following argument holds for any space of the form (4.1.4) with $f(r)$ satisfying the above conditions. Expanding the metric near $r = r_+$ yields

$$\mathrm{d}s^2 = f'(r_+)(r - r_+)\mathrm{d}\varphi^2 + \frac{\mathrm{d}r^2}{f'(r_+)(r - r_+)} + r_+^2(-\mathrm{d}t^2 + \mathrm{d}x^i\mathrm{d}x^i) + \ldots\,. \tag{4.1.6}$$

Considering the change of coordinates $r - r_+ = \rho^2 f'(r_+)/4$, the leading term in the metric reads

$$\mathrm{d}s^2 = \frac{f'(r_+)^2}{4}\rho^2\mathrm{d}\varphi^2 + \mathrm{d}\rho^2 + r_+^2(-\mathrm{d}t^2 + \mathrm{d}x^i\mathrm{d}x^i)\,. \tag{4.1.7}$$

The first part of the metric with coordinates (ρ, φ) corresponds to the 2-dimensional plane if and only if the range of the coordinate $\tilde{\varphi} = \frac{f'(r_+)}{2}\varphi$ has periodicity 2π. Namely, our coordinate is identified as follows $\varphi \sim \varphi + \frac{4\pi}{f'(r_+)}$. Under these considerations, the space close to the region $r = r_+$ is Minkowski space $\mathbb{R}^{1,D-1}$.

For the particular case of $f(r)$ given in (4.1.4), the region $r \to \infty$ is locally AdS$_D$, namely its Riemann tensor is asymptotically $R^{\mu\nu}{}_{\rho\sigma} = -\ell^{-2}\delta^{\mu\nu}_{\rho\sigma}$. However, the identification of the coordinate φ makes the asymptotic form of the spacetime only locally equivalent to AdS$_D$.

These types of spaces interpolate between Minkowski spacetime and a locally AdS_D and are called *AdS solitons*; they were first constructed in [181] in the context of AdS/CFT.

As we argue, spinors on these configurations must be anti-periodic, as the S^1 with coordinate φ is contractible. In particular, spinors defined on the boundary are anti-periodic. This configuration, in the context of minimal $\mathcal{N} = 2$ gauged supergravity in $D = 4$, is not supersymmetric. Therefore, the non-supersymmetric CFT is defined on a Minkowski space with one coordinate identified and fermions that are anti-periodic in the cycle: in other words, it is defined on the KK vacuum. However, the instability that we discussed at the beginning is triggered by gravitational instability, and in the dual CFT, there is no gravity.

Recently, a family of supersymmetric soliton configurations in the context of minimal gauge supergravity in $D = 4, 5$ was constructed [179]. The key point that allows for the preservation of some amount of supersymmetry is the presence of a non-trivial magnetic field. These configurations are constructed from the planar electrically charged Reissner-Nordström-AdS, that is a black hole with planar horizon and asymptotically AdS controlled by two parameters (m, Q) the mass and the charge. These black holes are bosonic solutions of the minimal gauged supergravity in $D = 4, 5$. The BPS equations in this context have solutions for certain relations between the parameters, but the resulting BPS configurations are naked singularities, as was pointed out in [182].

The construction of the solitons in [179] is obtained by considering an analytic continuation of the timelike coordinate t, one coordinate x along the plane, and the electric charge as follows $(t, x, Q) \mapsto (\mathrm{i}\varphi, \mathrm{i}t, -\mathrm{i}Q)$. Let us give the explicit expression for the $D = 5$ soliton as a solution of the $\mathcal{N} = 2$ minimal gauged supergravity for which the 5-form Lagrangian in the bosonic sector is given by

$$L = \frac{R}{2} \star 1 - \frac{1}{4} F \wedge \star F + \frac{6}{L^2} \star 1 + \frac{1}{2} A \wedge F \wedge F \tag{4.1.8}$$

where $F = \mathrm{d}A$ and L is the AdS_5 radius of the vacuum of the theory, and is the inverse of the gauge coupling. A solution to this theory constructed in [179], in a patch with coordinates (t, r, φ, y, z), is given by

$$\begin{aligned} &\mathrm{d}s^2 = f(r)\mathrm{d}\varphi^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2(-\mathrm{d}t^2 + \mathrm{d}y^2 + \mathrm{d}z^2)\,, \qquad f(r) = \frac{r^2}{L^2} - \frac{m}{r^2} - \frac{Q^2}{r^4}\,, \\ &A = -\sqrt{3}Q\left(\frac{1}{r^2} - \frac{1}{r_+^2}\right)\mathrm{d}\varphi\,. \end{aligned} \tag{4.1.9}$$

The existence of a solution $f(r_+) = 0$ allows us to identify the coordinate $\varphi \sim \varphi + \frac{4\pi}{f'(r_+)}$, making the space geodesically complete and smooth, as we discussed before. Regarding the gauged field, we observe that, in principle, one can perform a large gauge transformation with parameter $c\varphi$ for some constant c. This transformation will shift the component along $\mathrm{d}\varphi$ of the gauge field. However, the fact that the coordinate φ is not well defined when $r \to r_+$, makes $\mathrm{d}\varphi$ divergent in the region $r \to r_+$ as it shrinks. As a consequence, the regularity of the gauge field at $r = r_+$ requires that the function multiplying $\mathrm{d}\varphi$ must vanish fast enough when $r \to r_+$. This is precisely what happens in the above background and also has the consequence that $\lim_{r\to\infty} A \neq 0$ is constant along $\mathrm{d}\varphi$. Thus, global regularity of the background implies that the gauge field is non-zero at infinity.

The BPS equations of $\mathcal{N} = 2$ minimal gauged supergravity for the background (4.1.9) imply that $m = 0$. In contrast with the black hole configuration, in the present case, there still exists a zero of $f(r)$ when $m = 0$, making the BPS configuration regular. In that case, there exist 4 globally defined Killing spinors which are anti-periodic in the coordinate φ.

The boundary conditions that characterize this background are: the size of the S^1 at infinity $\beta_\varphi(m, Q)$, the value of the gauge fields at infinity, which can be captured by the circulation along the S^1

$$\mathscr{F}_\infty = \int F = \lim_{r\to\infty} \oint_{S^1} A \,, \tag{4.1.10}$$

which is also a function of m and Q. These quantities can be plotted in the plane (m, Q) as in Figure 4.1.1. The figures show that there exist values of β_φ and $\mathscr{F}_\infty$ such that their intersection in the parameter space is not unique. For the chosen values in the figure, one configuration at the intersection is BPS and the other is not; however, they share the same boundary conditions. Thus, they can be compared. We will study the stability of these configurations in this chapter.

The fact that these configurations are asymptotically locally AdS_5 and can be embedded in type IIB supergravity implies that the field theory description should be governed by $\mathrm{SU}(N)$, $\mathcal{N} = 4$ Super Yang-Mills formulated on $\mathbb{R}^{1,3} \times S^1$ in the large N limit. Deformations of this theory in the context of the AdS/CFT containing a contractible cycle has been studied before in the literature, for instance the Witten model [183], corresponding to a scaling $M \to \infty$ of a Schwarzschild-AdS black hole, or to a near-horizon near-extremal limit of D3-branes, is interpreted as dual to $\mathcal{N} = 4$ SYM at finite temperature or, after a double Wick rotation by replacing the periodic time t with a

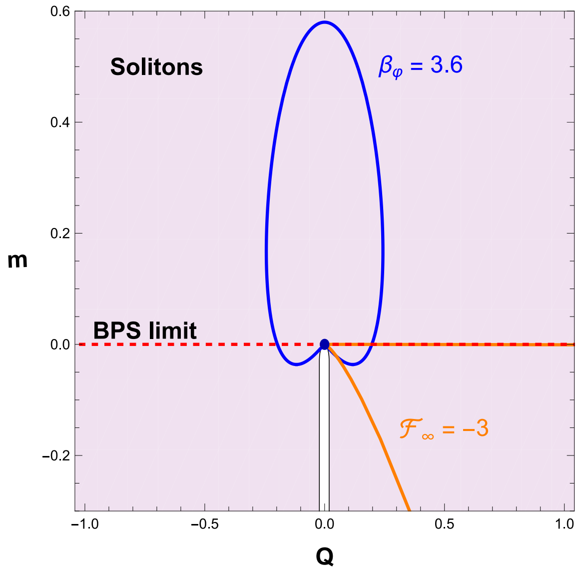

Figure 4.1.1: Parameter space (m, Q) of the background (4.1.9). The blue line represents backgrounds with the same β_φ. In orange, there are backgrounds which agree on the values of circulation at infinity $\mathscr{F}_\infty$. The dashed red line represents the BPS configurations. The purple region region indices where the solution exist in contracts with the semi-infinite line $Q = 0, m < 0$.

Kaluza-Klein (KK) angular coordinate ϕ, and a reduction on ϕ, as dual to 3-dimensional pure glue theory ($\equiv QCD_3$; fermions are antiperiodic, so massive and scalars gain a mass at one-loop, from the fermions), coupled to extra modes at the KK scale $T_{\text{KK}} = 1/R_\phi$, and one obtains a discrete spectrum of states. This is also similar to what one obtains by cutting off AdS space in the IR (the "hard-wall" model).

However, other behaviors are possible by deforming $\mathcal{N} = 4$ SYM. One such is the "Coulomb Branch (CB)" deformation of $\mathcal{N} = 4$ SYM, by a scalar operator of dimension $\Delta = 2$, studied in [184]. One obtains two possible metrics, described by dimensionless parameters $\pm\ell^2/L^2$, describing either discrete states (for minus sign) or continuous above a mass gap (for plus sign).

As we discussed, in asymptotically flat spacetime, the boundary conditions associated with the KK soliton [180] make the KK vacuum with antiperiodic boundary conditions for the fermions unstable towards decay, as the gravitational Hamiltonian is unbounded from below in this case. However, the AdS soliton [181], which also has antiperiodic conditions for the fermions on an S^1, is perturbatively stable, though susy-breaking. Recently, it was shown that supersymmetric AdS solitons exist [179, 185, 186, 187, 188] — with the configuration (4.1.9) discussed before being an instance — and the charged solitons (for the AdS Einstein-Maxwell theory) generate phase transitions in the dual field theory. These ideas have also been generalized to 10 dimensions, representing new models of holographic confinement [189, 190, 191, 192].

In this chapter, we will find AdS soliton-like solutions in the well-known STU model of type IIB supergravity. In general, its field content is that of 3 $U(1)$ gauge fields and 2 scalars. It is a consistent truncation of the 5-dimensional maximal gauged supergravity that one gets from type IIB supergravity compactified on an S^5. As such, this solution should describe a deformation of $\mathcal{N} = 4$ SYM, and we will find that there is a deformation of the Coulomb branch solution of [184], that interpolates between various possibilities for the spectrum, thus generating phase transitions in the field theory in 2+1 dimensions. For every possible value of the boundary sources, there are two possible AdS soliton-like solutions [179]. In the field theory, we find that there are two possible vacua in $\mathcal{N} = 4$ Super Yang-Mills when the fermions are anti-periodic on an S^1. Thus, the solitons nicely describe this degeneracy and holography yields the strongly coupled phase diagram of $\mathcal{N} = 4$ Super Yang-Mills in the large N limit.

The chapter is organized as follows. In Sections 4.1.1 and 4.1.2, we give our conventions for type IIB supergravity and describe the 5-dimensional model. In Section 4.2, we describe the general solutions, have a first go at a field theory interpretation, and the parametrization of the space of solutions. In Section 4.3, we describe holographic renormalization and describe the phase diagram from the point of view of gravity. Then, we show that when the fermions are anti-periodic on the S^1, one can have two possible states in the dual $\mathcal{N} = 4$ Super Yang-Mills, and we match these two results. In Section 4.4, we uplift the solution to 10-dimensions, describe the result in terms of deformations of distributions of D3-branes, and analyze the mass spectra in order to obtain a field theory interpretation of the phase transitions. In Section 4.5 we conclude. The Appendix A1 give details on the integrability conditions for the supersymmetry transformations. We also have the Appendix A2 on a possible interpretation of the solutions in terms of the Wick rotation of rotating D3-branes in 10 dimensions.

4.1.1 Conventions Type IIB supergravity

For any p-form F_p, we define

$$|\mathsf{F}_p|^2 = \frac{1}{p!}\mathsf{F}_{M_1\dots M_5}\mathsf{F}^{M_1\dots M_5}\,, \qquad |\mathsf{F}_p|^2_{MN} = \frac{1}{(p-1)!}\mathsf{F}_{MN_1\dots N_4}\mathsf{F}_N{}^{N_1\dots N_4}\,, \tag{4.1.11}$$

$$\star\mathsf{F}_p = \frac{\sqrt{-g}}{p!(D-p)!}\mathrm{d}x^{M_1\dots M_{D-p}}\epsilon_{M_1\dots M_{D-p}N_1\dots N_p}\mathsf{F}^{N_1\dots N_p}\,, \tag{4.1.12}$$

with $\epsilon_{12\ldots D} = 1$, and M, N denoting curve indices. Let us consider the 10-dimensional action principle of type IIB supergravity in the string frame. As we mentioned in the Introduction, the bosonic field content of type IIB supergravity is: the 10-dimensional metric g_{MN}, the dilaton scalar field Φ, and the Kalb-Ramond 2-form B_2. The RR fields are the even forms potentials C_0, C_2, C_4. The pseudo-action of IIB supergravity is

$$\begin{aligned}\mathcal{S}_{\text{IIB}} = \frac{1}{2\kappa^2}\int \mathrm{d}^{10}x\sqrt{-g}\left[e^{-2\Phi}\left(R + 4|\mathrm{d}\Phi|^2 - \frac{1}{2}|H_3|^2\right) - \frac{1}{2}|\mathsf{F}_1|^2 - \frac{1}{2}|\mathsf{F}_3|^2 - \frac{1}{4}|\mathsf{F}_5|^2\right] + \\ - \frac{1}{4\kappa^2}\int C_4 \wedge \mathrm{d}C_2 \wedge H_3 \,. \end{aligned} \tag{4.1.13}$$

The composite field strengths are defined by

$$\mathsf{F}_1 = \mathrm{d}C_0 \,, \qquad \mathsf{F}_3 = \mathrm{d}C_2 - C_0 \wedge H_3 \,, \tag{4.1.14}$$

$$\mathsf{F}_5 = \mathrm{d}C_4 + \frac{1}{2}(B_2 \wedge \mathrm{d}C_2 - C_2 \wedge H_3) \,. \tag{4.1.15}$$

It is convenient to define the higher degree forms

$$\mathsf{F}_7 = - \star \mathsf{F}_3 \,, \qquad \mathsf{F}_9 = \star \mathsf{F}_1 \,. \tag{4.1.16}$$

The equations of motion of NSNS sector are

$$\begin{aligned} R_{MN} + 2\nabla_M\nabla_N\Phi - \frac{1}{2}|H_3|^2_{MN} = \frac{e^{2\Phi}}{2}\Big(|\mathsf{F}_1|^2_{MN} + |\mathsf{F}_3|^2_{MN} + \frac{1}{2}|\mathsf{F}_5|^2_{MN} - \\ - \frac{1}{2}g_{MN}(|\mathsf{F}_1|^2 + |\mathsf{F}_3|^2)\Big) \,, \end{aligned} \tag{4.1.17}$$

$$2R - |H_3|^2 - 8e^{\Phi}\Box e^{-\Phi} = 0 \,, \tag{4.1.18}$$

$$\mathrm{d}(e^{-2\Phi} \star H_3) + \mathsf{F}_5 \wedge \mathsf{F}_3 + \mathsf{F}_1 \wedge \mathsf{F}_7 = 0 \,. \tag{4.1.19}$$

The equations of motion and Bianchi identities of the RR sector are

$$\mathrm{d}\mathsf{F}_1 = 0 \,, \qquad \mathsf{F}_5 = \star \mathsf{F}_5 \,, \tag{4.1.20}$$

$$\mathrm{d}\mathsf{F}_3 - H_3 \wedge \mathsf{F}_1 = 0 \,, \qquad \mathrm{d}\mathsf{F}_7 - H_3 \wedge \mathsf{F}_5 = 0 \,, \tag{4.1.21}$$

$$\mathrm{d}\mathsf{F}_5 - H_3 \wedge \mathsf{F}_3 = 0 \,, \qquad \mathrm{d}\mathsf{F}_9 - H_3 \wedge \mathsf{F}_7 = 0 \,. \tag{4.1.22}$$

The fermionic field content of IIB supergravity includes a pair spin 1/2 spinors (λ^1, λ^2), usually called dilatinos, transforming as a doublet under SO(2). There is also a pair of spin 3/2 spinors (Ψ^1, Ψ^2) called gravitinos transforming as a doublet under SO(2). They are both Majorana-Weyl spinors with opposite chirality. The supersymmetry parameters are a spin 1/2 doublet of SO(2), having the same chirality as the gravitinos. The explicit form of supersymmetry variations is

$$\delta\Psi_M = D_M\epsilon - \frac{1}{4\cdot 2}H_{MPQ}\Gamma^{PQ}\mathrm{i}\sigma_2\epsilon + \frac{e^\Phi}{8}(\mathsf{F}_1\mathrm{i}\sigma_2 + \mathsf{F}_3\sigma_1 + \frac{1}{2}\mathsf{F}_5\mathrm{i}\sigma_2)_{/}\Gamma_M\mathbb{P}_{10}\epsilon\,, \tag{4.1.23}$$

$$\delta\lambda = \partial\!\!\!/\Phi\epsilon - \frac{1}{2}H\!\!\!\!/_3\sigma_3\epsilon - e^\Phi(\mathsf{F}_1\sigma_1 + \frac{1}{2}\mathsf{F}_3\mathrm{i}\sigma_2)_{/}\mathbb{P}_{10}\epsilon\,. \tag{4.1.24}$$

where we have defined the chirality projectors $\mathbb{P}_{10} = \frac{1}{2}(\mathbb{1} + \Gamma_*)$ with $\Gamma_* = \Gamma^{01\ldots9}$. The Clifford map acting on any p-form F_p is defined as follows

$$F\!\!\!/_p = (\mathsf{F}_p)_{/} = \frac{1}{p!}F_{M_1\ldots M_p}\Gamma^{M_1\ldots M_p}\,. \tag{4.1.25}$$

The relation between the metric in the Einstein frame and string frame is $g^{(\text{string})}_{MN} = e^{\Phi/2}g^{(\text{Einstein})}_{MN}$.

4.1.2 The five dimensional model

We are interested in studying a truncation of type IIB supergravity compactified over the S^5 with action

$$\mathcal{S}_0 = \frac{1}{2\kappa}\int \mathrm{d}^5x\sqrt{-g}\left[R - \frac{(\partial\Phi_1)^2}{2} - \frac{(\partial\Phi_2)^2}{2} + \sum_{i=1}^{3}4L^{-2}X_i^{-1} - \right. \tag{4.1.26}$$
$$\left. -\frac{1}{4}X_i^{-2}(F^i)^2 + \frac{1}{4}\epsilon^{\mu\nu\rho\sigma\lambda}A^1_\mu F^2_{\nu\rho}F^3_{\sigma\lambda}\right]\,,$$

where F^i are 2-forms, related with gauge fields in the standard way, $F_i = d\bar{A}_i$, $X_i = e^{-\frac{1}{2}\vec{a}_i\cdot\vec{\Phi}}$, $\vec{\Phi} = (\Phi_1, \Phi_2)$ and

$$\vec{a}_1 = \left(\frac{2}{\sqrt{6}}, \sqrt{2}\right), \qquad \vec{a}_2 = \left(\frac{2}{\sqrt{6}}, -\sqrt{2}\right), \qquad \vec{a}_3 = \left(-\frac{4}{\sqrt{6}}, 0\right). \tag{4.1.27}$$

We remark that we have changed the standard coupling constant of the gauged supergravity by the AdS radius L through the relation $g = \frac{1}{L}$. We will be interested in purely magnetic solutions, in which case it is consistent to truncate the axions to zero. Einstein's field equations in 5 dimensions are

$$T^i_{\mu\nu} = F_{\mu\rho}{}^i F^{i\rho}_\nu - \tfrac{1}{4}g_{\mu\nu}F^i_{\rho\sigma}F^{i\rho\sigma}\,, \tag{4.1.28}$$

$$T^\Phi_{\mu\nu} = \partial_\mu\Phi_1\partial_\nu\Phi_1 + \partial_\mu\Phi_2\partial_\nu\Phi_2 - g_{\mu\nu}\left(\frac{(\partial\Phi_1)^2}{2} + \frac{(\partial\Phi_2)^2}{2} - \sum_{i=1}^{3}4L^{-2}X_i^{-1}\right)\,, \tag{4.1.29}$$

$$G_{\mu\nu} = \frac{1}{2} T^{\Phi}_{\mu\nu} + \sum_{i=1}^{3} \frac{1}{2X_i^2} T^i_{\mu\nu} \ , \tag{4.1.30}$$

The supersymmetry transformation of gravitino and the two dilatinos in the gauged $D = 5$ gauged supergravity (4.1.26) are [193]

$$\delta\psi_\mu dx^\mu = (\mathrm{d} + W)\,\Psi = 0 \ , \tag{4.1.31}$$

$$\delta\lambda_1 = \sum_i \Omega_i \frac{\partial X_i}{\partial \Phi_1} \Psi = 0 \ , \tag{4.1.32}$$

$$\delta\lambda_2 = \sum_i \Omega_i \frac{\partial X_i}{\partial \Phi_2} \Psi = 0 \ , \tag{4.1.33}$$

where

$$A^i = A^i_\mu dx^\mu \tag{4.1.34}$$

$$\Omega_i = -\frac{1}{8} (X_i)^{-2} \gamma^{ab} F^i_{ab} - \frac{i}{4} (X_i)^{-2} \left(\frac{\partial X_i}{\partial \Phi_1} \partial\!\!\!/ \Phi_1 + \frac{\partial X_i}{\partial \Phi_2} \partial\!\!\!/ \Phi_2 \right) + \frac{i}{2L} \ , \tag{4.1.35}$$

$$W = \frac{1}{4} \omega_{ab} \gamma^{ab} - \frac{i}{2L} \sum_i A^i + \frac{i}{4!} \left(\gamma_c \gamma^{ab} - 6 \delta^a_c \gamma^b \right) e^c \sum_i (X_i)^{-1} F^i_{ab} + \frac{1}{3! L} \gamma_c e^c \sum_i X_i \ .$$

The 1-form e^c stands for the vielbein basis, and ω_{ab} is the Levi-Civita spin connection 1-form. The complex spinor Ψ is defined in terms of the symplectic Majorana spinor ϵ^a as $\Psi = \epsilon^1 + i\epsilon^2$ (see for instance [194]). We use the following basis for the Clifford algebra:

$$\begin{aligned} \gamma^0 &= -i \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix} , \quad \gamma^1 = - \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} , \quad \gamma^2 = i \begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix} , \\ \gamma^3 &= \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix} , \quad \gamma^4 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3 \ . \end{aligned} \tag{4.1.36}$$

The 2-form integrability conditions is defined as

$$(\mathrm{d}W + W \wedge W)\Psi = 0. \tag{4.1.37}$$

This equation leads a non-trivial solution only when the determinant of the components of $\mathrm{d}W + W \wedge W$ is equal to zero.

Uplift to Type IIB supergravity

The Lagrangian (4.1.26) can be obtained from the compactification of ten dimensional type IIB supergravity over the five sphere with the ansatz [195]

$$\mathrm{d}s_{10}^2 = \tilde{\Delta}^{1/2}\mathrm{d}s_5^2 + L^2\tilde{\Delta}^{-1/2}\sum_{i=1}^{3} X_i^{-1}\left[\mathrm{d}\mu_i^2 + \mu_i^2\left(\mathrm{d}\phi_i + \frac{1}{L}A_i\right)^2\right], \tag{4.1.38}$$

$$F_5 = G_5 + \star G_5, \tag{4.1.39}$$

$$G_5 = \frac{2}{L}\epsilon_5\sum_{i=1}^{3}\left(X_i^2\mu_i^2 - \tilde{\Delta}X_i\right) - \frac{L}{2}X_i^{-1}\star_5\mathrm{d}X_i\wedge d\mu_i^2 \tag{4.1.40}$$

$$+ L^2\sum_i X_i^{-2}\mu_i d\mu_i\wedge\left(d\phi_i + \frac{1}{L}A_i\right)\wedge\star_5 F_i, \tag{4.1.41}$$

where $\star$ is the Hodge dual with respect to the ten-dimensional metric, $*_5$ is the Hodge dual with respect to the five-dimensional metric $\mathrm{d}s_5^2$, ϵ_5 is its volume form, and F_5 is the self-dual five-form field strength of type IIB supergravity. The ϕ_i are 2π periodic angular coordinates parametrizing the three independent rotations on S^5, $\tilde{\Delta} = \sum_i X^i\mu_i^2$ and $\sum_i \mu_i^2 = 1$. We will be interested in considering the higher-dimensional interpretation of some of our solutions using this uplift.

The equations of type IIB supergravity in the metric-dilaton-F_5 sector, discussed in Section 4.1.1 are given by

$$R_{\mu\nu} + 2\nabla_\mu\nabla_\nu\Phi - \frac{e^{2\Phi}}{4}\frac{1}{4!}\mathsf{F}_{\mu\rho_1\ldots\rho_4}\mathsf{F}_\nu{}^{\rho_1\ldots\rho_4} = 0\ . \tag{4.1.42}$$

$$R - 4\left(\partial\Phi\right)^2 + 4\Box\Phi = 0\ , \tag{4.1.43}$$

$$\mathrm{d}\mathsf{F}_5 = 0\ , \tag{4.1.44}$$

where Φ is the ten-dimensional dilaton. In addition to the self-duality condition $\mathsf{F}_5 = \star\mathsf{F}_5$. The lift of the solution has vanishing dilaton, and therefore the spacetime is Ricci flat, which is consistent with the trace of Einstein's equations.

Specializing the supersymmetry transformations of type IIB (4.1.23) and (4.1.24) for the case $\Phi = 0$ and $\mathsf{F}_1 = \mathsf{F}_3 = H_3 = 0$ leads to

$$\delta\psi_\mu\mathrm{d}x^\mu = d\epsilon + \frac{1}{4}\omega_{ab}\Gamma^{ab}\epsilon + \frac{1}{8}\frac{1}{2}\not{\mathsf{F}}_5 i\sigma_2\Gamma_a e^a\epsilon \equiv D\epsilon\ . \tag{4.1.45}$$

where e^a is an orthonormal basis of the 10D manifold. The 2-form integrability conditions obtained by computing the commutator of the derivative defined in (4.1.45), as it is explained in detail in Appendix A1, are

$$\Xi = \frac{1}{4}\mathsf{R}_{ab}\Gamma^{ab} + \frac{1}{16}\frac{1}{5!}i\sigma_2\mathcal{D}F_{b_1\ldots b_5}\Gamma^{b_1\ldots b_5}\Gamma_a e^a - \frac{1}{128}\frac{1}{4!}\not{F}_5 F_{ad_1\ldots d_4}\Gamma^{d_1\ldots d_4}e^a\wedge\Gamma_c e^c \ . \quad (4.1.46)$$

4.2 AdS soliton in Type IIB supergravity

AdS soliton type solutions with magnetic fluxes where found in the minimal gauged supergravity in five dimensions in [179]. We shall generalize these solutions now by including a non-trivial scalar profile. These solutions are double analytic continuations of a particular case of the electrically charged black hole solutions of the $U(1)^3$ truncation of the maximal gauged supergravity in five dimensions theory [195], which oxidize to spinning D3 branes in 10 dimensions. The vierbein and matter fields are

$$e^0 = \Omega(x)^{1/2}\mathrm{d}t \ , \qquad e^1 = \frac{\Omega(x)^{1/2}}{2x[(x-1)F(x)\eta]^{1/2}}\mathrm{d}x \ , \quad (4.2.1)$$

$$e^2 = \Omega(x)^{1/2}F(x)^{1/2}L\mathrm{d}\phi \ , \quad (4.2.2)$$

$$e^3 = \Omega(x)^{1/2}\mathrm{d}z \ , \qquad e^4 = \Omega(x)^{1/2}\mathrm{d}y \ , \quad (4.2.3)$$

$$\Phi_1 = \sqrt{\frac{2}{3}}\ln(x) \ , \quad \Phi_2 = 0 \ ,$$

$$A^1 = A^2 = q_1\left(x^{-1} - x_0^{-1}\right)L\mathrm{d}\phi \ , \qquad A^3 = q_2\left(x - x_0\right)L\mathrm{d}\phi \ , \quad (4.2.4)$$

with

$$\Omega(x) = \frac{x^{2/3}\eta}{x-1} \ ,$$

$$F(x) = L^{-2} + \frac{(-1+x)^2(q_1^2 - q_2^2 x)}{\eta x^2}. \quad (4.2.5)$$

As we shall see, the conformal boundary of the metric is located at $x = 1$. When the integration constant $\eta > 0$, then the range of the coordinate x is constrained to be $1 \leq x \leq x_0$, with $F(x_0) = 0$, the center of the spacetime (this would be a "horizon" if $F(x)$ would be in front of $-\mathrm{d}t^2$). When $\eta < 0$, then $x_0 \leq x \leq 1$. These two cases are not diffeomorphic to each other, as the scalar field is either everywhere positive or negative depending on which case one considers. Therefore the above configuration describe two physically inequivalent physical situations.

We should note that, in fact, there $F(x_0) = 0$ does not always have solutions:

-if $\eta < 0$, then $q_2 = 0$ means there is an x_0, but $q_1 = 0$ means there isn't. Note, however, that even a very small q_1 is enough to guarantee that there is an x_0.

-if $\eta > 0$, then $q_2 = 0$ means there is no x_0, but $q_1 = 0$ means there is one. Note, however, that even a very small q_2 is enough to guarantee that there is an x_0.

-if $|q_1| = |q_2| = q$, so, as we shall see, this is the supersymmetric solution, then if $\eta < 0$, there always is an x_0 (independently of q), but if $\eta > 0$, for large q there is a solution, but for small q (and in particular for $q_1 = q_2 = 0$) there isn't.

The canonical form of an asymptotically locally AdS_5 spacetime is achieved with the transformation (valid for $\eta > 0$, the other case corresponds to changing η into $-\eta$)

$$\begin{aligned} x &= 1 + \frac{\eta L^2}{\rho^2} + \frac{2\eta^2 L^4}{3\rho^4} + \frac{\eta^3 L^6}{3\rho^6} + O(\rho^{-8}), \\ \Omega(x) &= \frac{\rho^2}{L^2} + O(\rho^{-4}), \\ g_{\phi\phi} &= \Omega(x)F(x) = \frac{\rho^2}{L^2} - \frac{\mu}{\rho^2} + O(\rho^{-4}), \\ g_{\rho\rho} &= \frac{L^2}{\rho^2} - \frac{\frac{2}{9}\eta^2 L^6 - \mu L^4}{\rho^6} + O(\rho^{-8}), \\ \mu &= -\eta L^4 (q_1^2 - q_2^2). \end{aligned} \tag{4.2.6}$$

4.2.1 Supersymetric limit

We shall prove now that the configuration with $q_2 = -q_1$ is supersymmetric, using the 5-dimensional supersymmetry transformations. However, we show that once we uplift the configuration to type IIB supergravity, the configuration is also supersymmetric in the case $|q_1| = |q_2|$. One can see that when this is replaced in the integrability condition (4.1.37), its determinant is equal to zero. To integrate the equations for the Killing spinor, we introduce the radial coordinate r, which is the same one as the one we will introduce later for the uplift to 10 dimensions, through the change of coordinate

$$x = \left(1 + \epsilon \frac{\ell^2}{r^2}\right)^{-1} , \tag{4.2.7}$$

where $\epsilon = \pm 1$, and ℓ is related to η as $\eta = -\epsilon \ell^2 / L^2$. The 5-dimensional vielbeins that we will use are

$$e^0 = \frac{r}{L}\lambda(r)\mathrm{d}t \, , \qquad e^1 = \frac{\mathrm{d}r}{r\lambda(r)^2\sqrt{F(r)}} \, , \quad e^2 = \frac{r}{L}\lambda(r)\mathrm{d}y \tag{4.2.8}$$

$$e^3 = \frac{r}{L}\lambda(r)\mathrm{d}z \, , \quad e^4 = r\lambda(r)\sqrt{F(r)}\mathrm{d}\phi \, , \tag{4.2.9}$$

$$F(r) \;=\; \frac{1}{L^2} - \epsilon \frac{\ell^2 L^2}{r^4}\left(q_1^2 - q_2^2 \lambda(r)^{-6}\right) \;, \qquad \lambda(r)^6 = 1 + \epsilon \frac{\ell^2}{r^2} \;. \tag{4.2.10}$$

From the supersymmetry transformations (4.1.32), we integrate the Killing spinors when $q_1 = -q_2$, which gives two linearly independent complex spinors

$$\Psi_1 \;=\; e^{-\frac{i\pi\phi}{\delta}+\sigma(r)} \begin{pmatrix} 1 \\ \epsilon \frac{\lambda(r)^3 r^3}{\ell^2 L^2 q_1}\left(LF(r)^{1/2} - 1\right) \\ 0 \\ 0 \end{pmatrix} , \tag{4.2.11}$$

$$\Psi_2 \;=\; e^{-\frac{i\pi\phi}{\delta}+\sigma(r)} \begin{pmatrix} 0 \\ 0 \\ 1 \\ -\epsilon \frac{\lambda(r)^3 r^3}{\ell^2 L^2 q_1}\left(LF(r)^{1/2} - 1\right) \end{pmatrix} , \tag{4.2.12}$$

where

$$\sigma(r) = \int_1^r \frac{\left(1 + 2\lambda(u)^6\right)\left(3 - 2LF(u)^{1/2}\right)}{6u\lambda(u)^6 \, LF(u)^{1/2}} \mathrm{d}u \;, \tag{4.2.13}$$

and δ is the period of the coordinate ϕ, implying that the Killing spinor are anti-periodic. The presence of two complex Killing spinors means that the solution is 1/8 BPS. As a cross-check, we verify that in these conventions AdS_5 has four independent complex Killing spinors, constructed as given in section 3.1 of [196] within the $\mathcal{N} = 2$ theory. The Killing spinors are anti-periodic in the coordinate ϕ, with period δ, which make them globally defined as the coordinate ϕ is contractible. The most general Killing spinor is a linear combination of (4.2.11) and (4.2.12)

$$\Psi = c_1 \Psi_1 + c_2 \Psi_2 \;, \tag{4.2.14}$$

with complex coefficients c_1 and c_2. The Killing vector constructed from the Killing spinors (4.2.14) gives a combination of all the Killing vectors of the spacetime

$$\begin{aligned} \Psi^\dagger \gamma^0 \gamma^\mu \Psi \partial_\mu \;=\; & -L\left(|c_1|^2 + |c_2|^2\right)\partial_t + \left(|c_1|^2 - |c_2|^2\right)\partial_y - \left(c_1^* c_2 + c_1 c_2^*\right)\partial_z \\ & + i\left(c_1^* c_2 - c_1 c_2^*\right)\partial_\varphi \;. \end{aligned} \tag{4.2.15}$$

4.2.2 Dual interpretation: basic analysis

Below we will make clear that μ is proportional to the energy of the configuration. The expansion of the scalar field yields

$$\Phi_1 = \frac{\Phi_0}{\rho^2} + \frac{\sqrt{6}\Phi_0^2}{12\rho^4} + O(\rho^{-8}), \tag{4.2.16}$$

$$\Phi_0 = \frac{\sqrt{2}L^2\eta}{\sqrt{3}} \, . \tag{4.2.17}$$

Hence, these solitons excite a VEV of an operator of conformal dimension $\Delta = 2$ in the dual field theory, more precisely, in terms of $\mathcal{N} = 4$ SYM, the symmetric traceless operator in the $\underline{\mathbf{20}}'$ representation of $SO(6)$, $\mathrm{Tr}[X^I X^J - \frac{1}{6}\delta^{IJ}X^2]$, restricted to the neutral singlet $(1,1)_0$ under the decomposition of $SO(6) \to SO(2) \times SO(4) \simeq SO(2) \times SO(3) \times SO(3)$.

The case of operators with $\Delta = 2$ in $d = 4$ is very special and has to be treated separately, as considered in [197], for the case of the "Coulomb Branch" (CB) flow of [184] which, as we will shortly see, corresponds to our own solution (as our solution is a generalization of that flow).

In the standard case ($2\Delta - d \neq 0$, with d the dimension of the spacetime where the conformal field theory is defined), the expansion of the scalar of mass $m = \sqrt{\Delta(\Delta - d)}/R$ in terms of $z = R^2/\rho$ is [198]

$$\Phi = z^{d-\Delta}\left[\phi_{(0)} + z^2\phi_{(2)} + \ldots + z^{2\Delta-d}\left(\phi_{(2\Delta-d)} + \log z^2 \tilde{\phi}_{(2\Delta-d)}\right) + \ldots\right] \, , \tag{4.2.18}$$

where the independent coefficients are: the non-normalizable mode $\phi_{(0)}$, corresponding to the operator source in the dual, and $\phi_{(2\Delta-d)}$, sometimes also called $\phi_{(1)}$, corresponding to the operator VEV. $\phi_{(2)}, \ldots, \tilde{\phi}_{(2\Delta-d)}, \ldots$ are dependent on $\phi_{(0)}$, for instance

$$\tilde{\phi}_{(2\Delta-d)} = -\frac{1}{2^{2\Delta-d}\Gamma\left(\Delta - \frac{d}{2}\right)\left(\Delta - \frac{d-2}{2}\right)}(\partial_i\partial_i)^{\Delta-\frac{d}{2}}\phi_{(0)} \tag{4.2.19}$$

$$\phi_{(2)} = \frac{1}{2\left(2\Delta - d - 2\right)}\partial_i\partial_i\phi_{(0)} \, , \tag{4.2.20}$$

while $\phi_{(2\Delta-d)}$ gives the operator VEV by

$$\langle \mathcal{O} \rangle_{\phi_{(0)}} = -(2\Delta - d)\phi_{(2\Delta-d)} + F(\phi_{(0)}) \, , \tag{4.2.21}$$

with F a scheme-dependent function.

But when $\Delta = d/2$ like in our case ($\Delta = 2, d = 4$), one has to treat things separately, since there are several zero prefactors in the above, and as we see, $\phi_{(0)}$ and $\phi_{(2\Delta-d)}$ (normally the source and VEV) appear at the same order in the expansion. Another way to see this is that the mass formula has a double root, at the saturation of the BF bound ($m^2R^2 \geq -d^2/4$): $m^2R^2 = (\Delta - d/2)^2 - d^2/4$. The expansion in our case ($d = 4, \Delta = 2$) is, instead,

$$\Phi = z^2 \left[\log z^2 \left(\phi_{(0)} + z^2\phi_{(2)} + z^2 \log z^2 \psi_{(2)} + ...\right) + \left(\tilde{\phi}_{(0)} + z^2\tilde{\phi}_{(2)} + ...\right)\right] \,, \quad (4.2.22)$$

where now $\phi_{(0)}$ is the operator source, and $\tilde{\phi}_{(0)}$ is the operator VEV.

The expansion of the scalar in [197] coincides with our own (4.2.16) in the $\eta < 0$ case. That means that there is no source, only an operator VEV $\Phi_0 \propto \eta$, parametrizing the Coulomb Branch (in [197], the operator VEV was constant). In this $\Delta = d/2 = 2$ case, the prefactor $(2\Delta - d)$ of the operator is replaced by 2, so (since $\tilde{\phi}_{(0)} \equiv \Phi_0$ for us)

$$\langle \mathcal{O} \rangle = 2\tilde{\phi}_{(0)} \equiv 2\Phi_0 = \frac{2\sqrt{2}L^2}{\sqrt{3}}\eta. \quad (4.2.23)$$

The solution we have found is a generalization of the Coulomb Branch case in [184, 197] both by the VEV parameter η above, and by the parameters q_1, q_2 proportional to the boundary value of the gauge fields. Next we shall discuss the phase space of these solutions.

4.2.3 The space of solutions

A soliton solution is fully characterized in terms of its boundary conditions. Above we discused the solution in terms of the parameters (η, q_1, q_2), the last two of which do not have a direct physical meaning (η is the operator VEV in the field theory). A good set of physical variables are the boundary values of the gauge fields and the period $\phi \in [0, \delta]$. Indeed, solitons exist provided a regularity condition is imposed. This yields a boundary condition, namely the period δ of the S^1 is fixed by requiring the absence of conical singularities at x_0 (in the case when there is an $0 < x_0$ such that $F(x_0) = 0$, otherwise no soliton exists). In the Wick-rotated (in t) Euclidean black hole case, this would correspond to no singularities at the horizon, and would fix the temperature of the black hole. In the case at hand, one is actually working at zero temperature. Therefore, the scale is set by the KK scale δ, for compactification of the 4-dimensional theory onto ϕ, down to 2+1 dimensions. At this scale, the dimensionally reduced theory becomes just the 4-dimensional theory, KK expanded onto 2+1 dimensions.

The usual calculation, together with the condition $F(x_0) = 0$, gives the period of the angle $\phi \in [0, \delta]$, with[1]

$$\delta = \frac{2\pi x_0}{|-q_2^2 x_0^2 - q_2^2 x_0 + 2q_1^2|}\sqrt{\left|\frac{-q_2^2 x_0 + q_1^2}{-1 + x_0}\right|} = \frac{2\pi x_0^2 \sqrt{\eta/L^2}}{|-q_2^2 x_0^2 - q_2 x_0 + 2q_1^2||x_0 - 1|^{3/2}} \ . \tag{4.2.24}$$

Since $x_0 = x_0(q_1, q_2, \eta)$, it follows that, in the interpretation of the period of ϕ as inverse Kaluza-Klein temperature for compactification, we have, in terms of the previous set of parameters,

$$\frac{1}{\delta} \equiv T_{\text{KK}} = T_{\text{KK}}(x_0, q_1, q_2, \eta) = T_{\text{KK}}(q_1, q_2, \eta). \tag{4.2.25}$$

We want to understand how T_{KK} (governing the coupling of the KK reduced boundary 3 dimensional field theory) and η (governing its operator VEV) vary, as the parameters of the bulk solution (q_1, q_2, η) are varied.

It is difficult to calculate the general situation, so we restrict to the supersymmetric solution, with $|q_1| = |q_2| \equiv q$. Then

$$\frac{1}{T_{\text{KK}}} = \delta = \frac{2\pi x_0}{q|x_0^2 + x_0 - 2|} = \frac{2\pi}{\sqrt{\eta}}\frac{\sqrt{|1 - x_0|}}{x_0 + 2} \ , \quad q = \frac{x_0\sqrt{\eta}}{|1 - x_0|^{3/2}} \ , \tag{4.2.26}$$

which means that:

-$\delta \to \infty$, so $T_{\text{KK}} \to 0$, $\Leftrightarrow \eta \to 0$ at fixed x_0, or $x_0 \to \infty$ at fixed η, both of which imply $q \to 0$.

-$\delta \to 0$, so $T_{\text{KK}} \to \infty$, $\Leftrightarrow \eta \to \infty$ at fixed x_0, or $x_0 \to 0$ at fixed η, both of which imply $q \to \infty$.

Thus *the KK temperature T_{KK} is varied between 0 and ∞ by the variation of the q's, allowed by the solution.*

On the other hand, at fixed q, we have:

-$\eta \to 0$ gives $x_0 \to 1$, so in turn $\delta \to \infty$, so $T_{\text{KK}} \to 0$.

-$\eta \to \infty$ gives $x_0 \to 0$ (or ∞), so in turn $\delta \to 0$, so $T_{\text{KK}} \to \infty$.

Thus at fixed q, T_{KK} *tunes the VEV η, or the VEV η tunes* T_{KK}, which gives a *phase transition at* $T_{\text{KK}} = 0$ between the (no VEV, no "horizon") and (VEV, "horizon") phases.

[1]Note that since $F(x_0) = 0$ gives $\eta x_0^2 = (x_0 - 1)^2(q_1^2 - q_2^2 x_0)$, we have $x_0 = x_0(q_1, q_2, \eta)$.

We now note that, if we consider the 4-dimensional gauge coupling $g^2_{\rm YM}$ fixed, then $T_{\rm KK}$ can be exchanged for the 3-dimensional gauge coupling (the coupling of the dimensionally reduced theory), since

$$g^2_{\rm 3d,YM} = g^2_{\rm YM} T_{\rm KK}. \tag{4.2.27}$$

Then, instead of the interpretation of phase transition in KK temperature $T_{\rm KK}$, one has a phase transition in coupling, i.e., a *quantum critical phase transition*, happening at $g^2_{\rm 3d,YM} = 0$.

We will reinforce this interpretation later, when describing the mass spectrum coming from the gravity dual.

Finally, we find that it is more convenient to parametrize the system in terms of the normalized 5-dimensional (gravity dual) gauge invariant "Wilson lines" (ψ_1, ψ_2), integrated on a curve $C = \partial\Sigma_2$[2], parametrized by ϕ at the boundary $x = 1$, defined as

$$\begin{aligned}
&\lim_{x\to 1} \oint A^1 = q_1 \left(1 - x_0^{-1}\right) L\delta \equiv 2\pi L \psi_1 \ , \\
&\lim_{x\to 1} \oint A^3 = q_2 \left(1 - x_0\right) L\delta \equiv 2\pi L \psi_2 \ .
\end{aligned} \tag{4.2.28}$$

In terms of these sources it is very easy to see that the location of the supersymmetric solution discussed above, with $q_1 = -q_2$, yields

$$x_0 = \frac{\psi_2}{\psi_1}. \tag{4.2.29}$$

Hence, we find that for every value of the pair (ψ_1, ψ_2) there is one and only one supersymmetric soliton with $q_1 = -q_2$.

More generally, we can use the definition of (ψ_1, ψ_2) to eliminate the integration constants (q_1, q_2) in the definition of δ, (4.2.24). This determines x_0 in terms of the sources (ψ_1, ψ_2),

$$\psi_1^2 x_0^3 + (\psi_2^4 + 4\psi_1^4 - 4\psi_2^2\psi_1^2 - \psi_2^2 - \psi_1^2)x_0^2 - \psi_2^2(4\psi_1^2 - 2\psi_2^2 - 1)x_0 + \psi_2^4 = 0. \tag{4.2.30}$$

We see that the advantage of this parametrization is that the dependence on δ, which

[2]From the point of view of the 5-dimensional bulk; note that by using a 2-dimensional surface Σ_2 between infinity, $x = 1$, and the origin, $x = x_0$, and Stokes' law, we could write this as $\int_{\Sigma_2} B_{yz} d\Sigma^{yz} \equiv \int_{\Sigma_2} \epsilon_{yzt\rho\phi}\partial_\rho A_\phi d\rho d\phi$, so would be some 5-dimensional generalization of magnetic flux, but would not correspond in the boundary 4 dimensions to magnetic flux, unlike the AdS_4 case.

would be present in $F(x_0) = 0$ in terms of the boundary values of the gauge fields, and was also present in the previous form $x_0 = x_0(q_1, q_2, \eta)$, drops out. Hence, we have only a 2-parameter set (ψ_1, ψ_2) defining x_0. Note that $F(x_0) = 0$ in the previous form means $\eta = \eta(x_0, q_1, q_2)$,[3] but $x_0 = x_0(\psi_1, \psi_2)$ from (4.2.30), while $q_1 = q_1(\psi_1, x_0, \delta)$ and $q_2 = q_2(\psi_2, x_0, \delta)$ from their definition,[4] which finally means that, up to some possible discrete choices, $\eta = \eta(\delta, \psi_1, \psi_2)$. Indeed then, the general solution is completely characterized once we give 3 parameters (δ, ψ_1, ψ_2). Note that **in the** (ψ_1, ψ_2) **parametrization**, *we can cover both the* $\eta < 0$ *and the* $\eta > 0$ *solutions.*

The solutions of the cubic equation (4.2.30) have the form $x_{0i} = \lambda_1 \cos\left(\Theta + \frac{2\pi n_i}{3}\right) - \frac{\lambda_2}{3}$ with $n_i = 0, 1, 2$ for $i = 1, 2, 3$, and

$$\begin{aligned}
\lambda_1 =& \frac{2}{3\psi_1}\sqrt{8\psi_1^4\lambda_2 - \psi_1^2\lambda_2^2 - 8\psi_2^2\psi_1^2\lambda_2 + 2\psi_2^4\lambda_2 + 12\psi_2^2\psi_1^2 - 2\psi_2^2\lambda_2 - 2\psi_1^2\lambda_2 - 3\psi_2^2 - 6\psi_2^4}\,,\\
\lambda_2 =& \frac{1}{\psi_1^2}\left(-4\psi_2^2\psi_1^2 + \psi_2^4 + 4\psi_1^4 - \psi_2^2 - \psi_1^2\right)\,,\\
\Theta =& \frac{1}{3}\arccos\frac{4}{27\lambda_1^3\psi_1^2}\Big(3\psi_2^2\lambda_2^2 + 3\psi_1^2\lambda_2^2 + 9\psi_2^2\lambda_2 + 18\psi_2^4\lambda_2 - 12\psi_1^4\lambda_2^2 - 27\psi_2^4 + 12\psi_2^2\psi_1^2\lambda_2^2\\
& -3\psi_2^4\lambda_2^2 - 36\psi_2^2\psi_1^2\lambda_2 + \psi_1^2\lambda_2^3\Big)\,. \qquad (4.2.31)
\end{aligned}$$

Hence, we find that generically there are two solitons for each value of the pair (ψ_1, ψ_2). In Fig.4.2.1 we plot possibles x_{0i}, as a function of ψ_1, with ψ_2 fixed. Note that for a fixed, say, ψ_2, there is a maximum ψ_1 for which there is a solution (that is consistent with the existence of a $0 < x_0$, with $F(x_0) = 0$). The different branches intersect at infinity, $x = 1$, where they yield the soliton of the Einstein-Maxwell theory in five dimensions [179]. The plot points towards the existence of a non-trivial phase diagram in the canonical ensemble, as ψ_1 and ψ_2 are varied. Indeed, on the gravity side, we can find the energy of the solution by $E = E(\delta, \psi_1, \psi_2)$, which will allow us to study the phase diagram of these solutions.

As we saw in (4.2.27), $T_{KK} = 1/\delta$ defines the 3 dimensional gauge coupling, so fixing δ is like fixing the coupling constant in the UV. Then, since we are working at zero temperature, *all* the possible phase transitions are *quantum critical phase transitions.*

[3] Specifically, $\eta = (x_0 - 1)^2(q_1^2 - q_2^2 x_0)/x_0^2$.
[4] Specifically, $q_1 = 2\pi\psi_1/[(1 - x_0^{-1})\delta]$ and $q_2 = 2\pi\psi_2/[(1 - x_0^{-1})\delta]$.

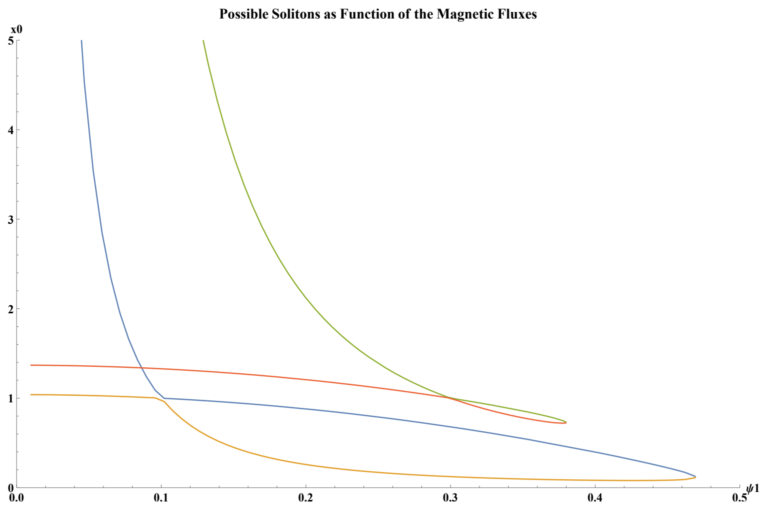

Figure 4.2.1: The different colors are different physical roots of (4.2.28). The x_0 in the y axis are plotted vs the dimensionless Wilson line ψ_1 in the x-axis. The blue and yellow lines have $\psi_2 = 0.1$ and the red and green line have $\psi_2 = 0.3$. Either both solutions have a positive scalar field VEV or both have a negative scalar field VEV. The only roots that contribute to the physics are x_{01} and x_{03} (see Appendix A for their definition).

4.3 Holographic renormalization and a phase diagram

Holographic renormalization

Here we will use holographic renormalization to compute the expectation value of the dual energy momentum tensor. The countertems to deal with this situation were constructed in [199, 197, 200],

$$S = S_0 + \frac{1}{\kappa}\int_{M^3\times S^1} K\sqrt{-h}\mathrm{d}^4x + \frac{1}{2\kappa}\int_{M^3\times S^1}\sqrt{-h}\left(-\frac{6}{L}+\frac{1}{2L}\left(\frac{1}{\ln(\rho/\rho_0)}-2\right)\Phi_1^2\right)\mathrm{d}^4x\ , \tag{4.3.1}$$

where S_0 is the action (4.1.26) truncated to $\Phi_2 = 0$, $g_{\mu\nu} = h_{\mu\nu} + N_\mu N_\nu$, and N_μ is the outward pointing normal to the boundary and $K_{\mu\nu} = \frac{1}{2}\nabla_\mu N_\nu + \frac{1}{2}\nabla_\nu N_\mu$ is the extrinsic curvature. The boundary integrals are over the D3-brane geometry. Namely, a three dimensional Minkowski spacetime times a circle,

$$\mathrm{d}s^2 = \gamma_{ab}\mathrm{d}x^a\mathrm{d}x^b = -\mathrm{d}t^2 + \mathrm{d}y^2 + \mathrm{d}z^2 + \mathrm{d}\phi^2\ , \tag{4.3.2}$$

which is the background spacetime for the quantum field theory. The scalar field has in general the asymptotic expansion

$$\Phi_1 = J_\Phi \frac{\ln(\rho^2/\rho_0^2)}{\rho^2} + \frac{\Phi_0}{\rho^2} + O\left(\frac{\ln(\rho^2/\rho_0^2)}{\rho^4}\right) , \tag{4.3.3}$$

with the on-shell variation

$$\frac{\delta S}{\delta J_\Phi} = \frac{1}{2\kappa L^5}\Phi_0. \tag{4.3.4}$$

Indeed, our soliton has no scalar sources and this relation provides the holographic interpretation of Φ_0 as a VEV, as already explained. The vacuum expectation value of the energy momentum tensor of the dual field theory is

$$\langle T_{ab}\rangle = \frac{-2}{\sqrt{-\gamma}}\frac{\delta S}{\delta\gamma^{ab}} \tag{4.3.5}$$

$$= \lim_{\rho\to\infty}\frac{\rho^2}{L^2}\frac{-2}{\sqrt{-h}}\frac{\delta S}{\delta h^{ab}} \tag{4.3.6}$$

$$= \lim_{\rho\to\infty}\frac{\rho^2}{L^2\kappa}\left(h_{ab}K - K_{ab} - \frac{3}{L}h_{ab} - \frac{1}{2L}h_{ab}\Phi_1^2\right) , \tag{4.3.7}$$

which yields

$$\langle T_{tt}\rangle = -\frac{\mu}{2L^3\kappa} , \qquad \langle T_{zz}\rangle = \langle T_{yy}\rangle = \frac{\mu}{2L^3\kappa} , \qquad \langle T_{\phi\phi}\rangle = -\frac{3\mu}{2L^3\kappa} . \tag{4.3.8}$$

Phase diagram from $E = E(\delta, \psi_1, \psi_2)$

The free energy of the solitons in the canonical ensemble is just the energy. Hence we will be interested to see how the the energy changes when we vary the sources (ψ_1, ψ_2). A convenient normalization of the energy is that of the AdS Soliton [181],

$$E_0 = -\frac{L^3\pi^4}{2\kappa\delta^3}V_2 , \tag{4.3.9}$$

where V_2 is the volume of the $y - z$ plane. So we plot the energy of the solution with the running scalar $E_\Phi = \langle T_{tt}\rangle\, V_2\delta$ divided by the absolute value of the energy of the AdS soliton in five dimensions,

$$F_\Phi \equiv \frac{E_\Phi}{|E_0|} = \langle T_{tt}\rangle\, V_2\delta\frac{2\kappa\delta^3}{L^3\pi^4 V_2} \tag{4.3.10}$$

$$= -16\frac{(\psi_1^2 x_0^2 - \psi_2^2)(\psi_1^2 x_0 - \psi_2^2)}{x_0(x_0-1)^2} , \tag{4.3.11}$$

We note that for the supersymmetric solution with $q_1 = -q_2$, we have $\psi_1^2 x_0^2 - \psi_2^2 = 0 = 0$, so $F_\Phi = 0$, as expected.

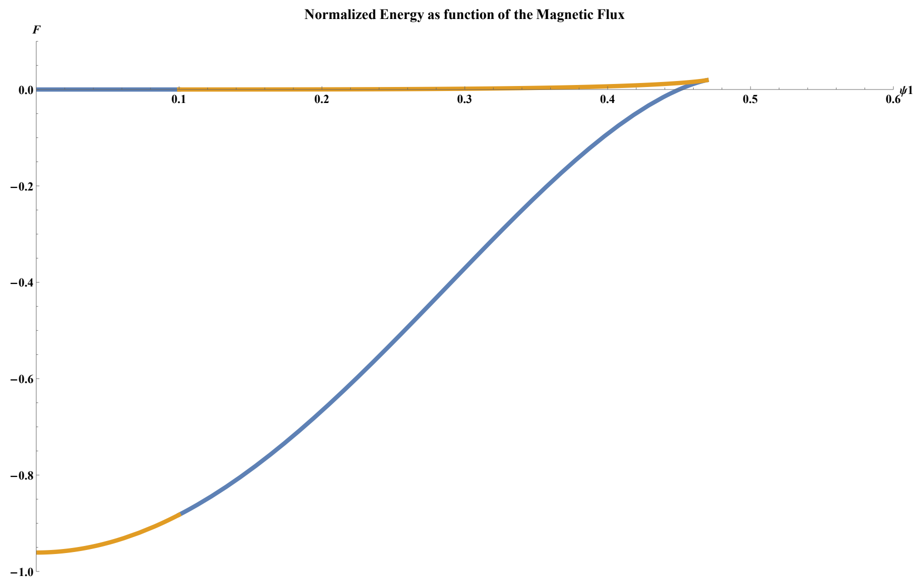

Figure 4.3.1: The normalized energy as a function of the Wilson line ψ_1 when $\psi_2 = 0.1$. The phase diagram is composed by the different D3-brane distributions. They turn out to be continuously connected on the gauge theory side due to the introduction of the Wilson lines in 5 dimensions. The different roots of the polynomial (4.2.30) have different colours.

The free energy F_Φ changes its color in Fig.4.3.1 in a continuous way. At this point is possible to see that the scalar VEV continuously goes to zero indicating a redistribution of the D3-branes in the sense of [184]. This happens when $\psi_2 = \pm\psi_1$. It is possible to see that the energy and all its derivatives are continuous at this point. That means that there is continuous phase transition at $\psi_2 = \pm\psi_1$, generically followed by the phase transition at ($q_1 = \pm q_2$, so) $|\psi_2| = x_0|\psi_1| < |\psi_1|$.

From $F(x_0) = 0$, meaning $\eta x_0^2 = (x_0 - 1)^2(q_1^2 - q_2^2 x_0)$, it is clear that we have the scalar VEV $\eta = 0$ only for $q_1^2 = q_2^2 x_0$, meaning for $\psi_1^2 x_0 = \psi_2^2$, or for $x_0 = 1$, or for both, in which case we have $\psi_1 = \pm\psi_2$ and $x_0 = 1$, where the horizon disappears (so there we transition between the "horizon" and "no horizon" phases, as we already explained). This is *the "quantum phase transition" at $T_{KK} = 0$ or $g^2_{3\mathrm{d,YM}} = 0$* described before.

The solution on the lower branch (with $F_\Phi < 0$) increases ψ_i until at some ψ_i, one reaches $F_\Phi = 0$, corresponding to the supersymmetric solution ($q_1 = -q_2$). There we have a phase transition to the phase dominated by the D3 brane distributions of [184] (which has zero energy), with anti-periodic boundary conditions for the fermions in ϕ. From the point of view of the dual field theory, reduced on ϕ to 3 dimensions, *this is another "quantum phase*

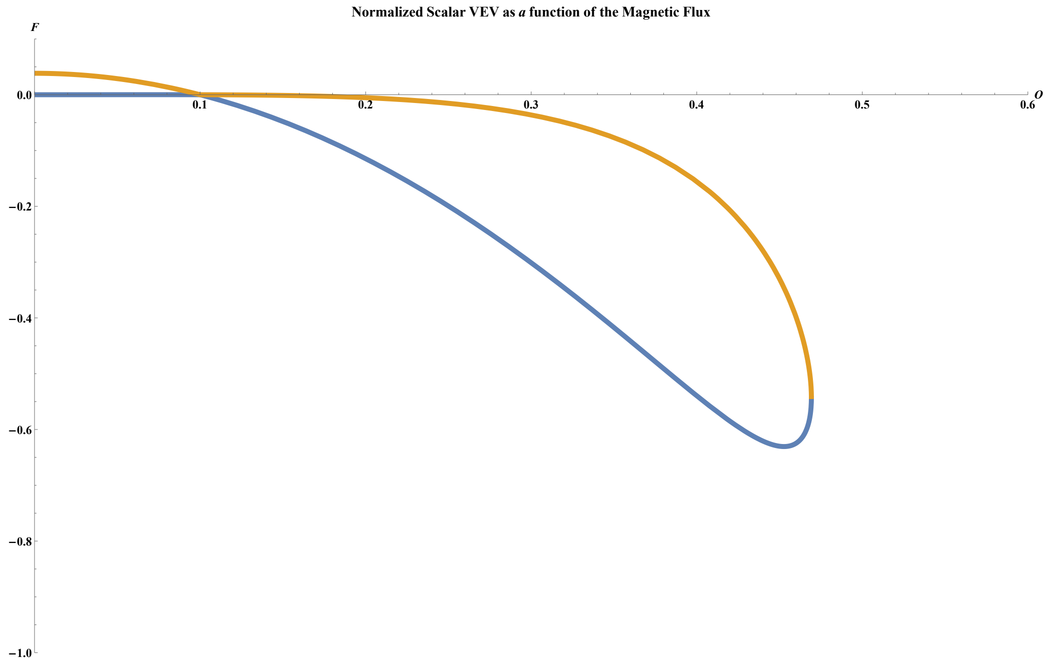

Figure 4.3.2: The normalized scalar field vacum expectation value as a function of the Wilson line ψ_1, when $\psi_2 = 0.1$. Here we see that the VEV is negative for some solutions and positive for others. The negative VEV yields D3 brane distributions different than the positive VEV, as we will discuss below. There is a crossover between the different regimes.

transition", at nonzero $g^2_{3\text{d},\text{YM}}$. One should note however that the distributions of [184] are singular in the IR, so its inclusion in the phase diagram suppose that they actually become regular when quantum corrections are included.

QFT energy

Here we will discuss in greater detail how to understand the phase diagram from the QFT point of view. It is straightforward to compute the vacuum expectation value of the energy of a single scalar field in the background (4.3.2). The result is

$$\langle E_{\text{QFT}} \rangle = -\frac{\pi^2}{6\delta^3} V_2 X \,, \tag{4.3.12}$$

where X is a numerical factor that depends whether the scalar field is periodic or anti-periodic in the S^1. It comes from Riemann zeta-function regularization of the sum over the modes in the circle and it yields

$$X_{even} = \sum_{n=1}^{\infty}(2n)^3 = \frac{8}{120}\,, \tag{4.3.13}$$

$$X_{odd} = \sum_{n=1}^{\infty}(2n-1)^3 = -\frac{7}{120}\,. \tag{4.3.14}$$

The field content of $\mathcal{N}=4$, $SU(N)$ super Yang-Mills is 6 scalars and 4 Weyl fermions in the adjoint representation plus one gauge vector. For the fermions the signs of the periodic and antiperiodic energies get interchanged. At weak coupling, we get the total energy by multiplying the scalar field energy by the number of degrees of freedom associated to each field, with the corresponding numerical factor depending on whether the fields are periodic or anti-periodic on the S^1. So for the case where the scalars, the vectors and the fermions are antiperiodic, we get

$$\langle E_{\mathrm{SYM}}\rangle = -\frac{\pi^2 V_2}{6\delta^3}X_{odd}(N^2-1)(6+2-8) = 0. \tag{4.3.15}$$

Hence this energy is automatically zero on account of the matching of the bosonic and fermionic degrees of freedom, and the fact that all fields have the same boundary condition on the S^1.

When the fermions are anti-periodic but the scalars and the vectors are periodic we get

$$\langle E_{\mathrm{SYM}*}\rangle = -\frac{\pi^2 V_2}{6\delta^3}X_{even}(N^2-1)(6+2+8\frac{7}{8}) = -\frac{\pi^2 V_2}{6\delta^3}(N^2-1). \tag{4.3.16}$$

The AdS/CFT dictionary tell us that $\frac{L^3}{\kappa} = \frac{N^2}{4\pi^2}$. So the gravitational energy is

$$E_0 = -\frac{\pi^2 V_2}{8\delta^3}N^2 = \frac{3}{4}\langle E_{\mathrm{SYM}*}\rangle\,, \tag{4.3.17}$$

which is a well-known result valid at large N. Thus, we learn that in our phase diagram is possible to see the interplay of $\langle E_{\mathrm{SYM}*}\rangle$ and $\langle E_{\mathrm{SYM}}\rangle$, and that morevoer, the value of $\langle E_{\mathrm{SYM}}\rangle$ at strong coupling and vanishing sources is also zero, see Fig. 4.3.1. This explains from the field theory point of view the existence of two branches of gravity solutions.

4.4 Continuous distributions of D3-branes vs. rotating D3-branes

We start by reviewing some of the findings of [184] on distributions of D3-branes. We show that when the gauge fields vanish in our soliton solutions, we recover the two different distributions of D3-branes that break the isometries of the S^5 to $SO(4) \times SO(2)$. The distribution of D3-branes of [184] are solutions of the supergravity action

$$I = \frac{1}{2\kappa} \int \sqrt{-g} \left(R - 2\sum_{i=1}^{5} (\partial \alpha_i)^2 - V \right) \mathrm{d}^5 x \, , \tag{4.4.1}$$

with

$$\begin{aligned} V &= -\frac{1}{2L^2} \left[\mathrm{Tr}(M)^2 - 2\mathrm{Tr}(M^2) \right] , \\ M &= diag(e^{2\beta_1}, e^{2\beta_2}, e^{2\beta_3}, e^{2\beta_4}, e^{2\beta_5}, e^{2\beta_6}), \\ \vec{\beta} &= \frac{1}{\sqrt{2}} B \vec{\alpha}, \end{aligned} \tag{4.4.2}$$

and

$$B = \begin{pmatrix} 1 & 1 & 1 & 0 & 3^{-1/2} \\ 1 & -1 & -1 & 0 & 3^{-1/2} \\ -1 & -1 & 1 & 0 & 3^{-1/2} \\ -1 & 1 & -1 & 0 & 3^{-1/2} \\ 0 & 0 & 0 & \sqrt{2} & -\frac{2}{3^{1/2}} \\ 0 & 0 & 0 & -\sqrt{2} & -\frac{2}{3^{1/2}} \end{pmatrix} . \tag{4.4.3}$$

Here M is a representative of the coset $SL(6, \mathbb{R})/SO(6)$, and the action of $SO(6)$ on M is by conjugation. Note that $B^T B = 4 \mathbb{1}_{5\times 5}$. Hence, the Lagrangian (4.4.1) is manifestly $SO(6)$ invariant.

The case $n = 2$ of table 1 of [184] is recovered when $\vec{\alpha} = (0, 0, 0, 0, -\frac{1}{2}\Phi)$ in terms of a canonically normalized scalar field Φ, and then $\vec{\beta} = -\frac{\Phi}{2\sqrt{6}}(1, 1, 1, 1, -2, -2)$. When the gauge fields vanish, this theory exactly coincides with the theory 4.1.26 when $\Phi = \Phi_1$. In the conventions of [184], this flow has $\Phi < 0$, and therefore this corresponds in our coordinates to having $x < 1$ and $\eta < 0$.

The case $n = 4$ of table 1 of [184] corresponds to $\vec{\alpha} = (\frac{\sqrt{3}}{4}\Phi, 0, 0, 0, \frac{1}{4}\Phi)$ with the canonically normalized scalar field Φ, and then $\vec{\beta} = \frac{\Phi}{2\sqrt{6}}(2, 2, -1, -1, -1, -1)$. In this case we match their potential with $\Phi = \Phi_1$. This flow has $\Phi > 0$, which in our coordinates is $x > 1$ and

$\eta > 0$.

4.4.1 Uplift of the metric to 10 dimensions

For the purposes of top-down AdS/CFT (whose rules are *derived* from string theory), it is not enough to consider a 5-dimensional solution; rather, one has to have a 10-dimensional solution, moreover obtained from a D-brane configuration. This is possible in our case.

Indeed, using the uplift (4.1.38) we can write our solution, with non-vanishing gauge fields, as follows.

Considering the change of variable $x = (1+\epsilon\ell^2/r^2)^{-1}$ and using the uplift (4.1.38), we can write the 10-dimensional metric as

$$\begin{aligned}
\mathrm{d}s^2_{10} &= \frac{\zeta(r,\theta)r^2}{L^2}\left(\frac{L^2\mathrm{d}r^2}{r^4F(r)\lambda(r)^6}+\mathrm{d}x^2_{1,2}+F\left(r\right)L^2\mathrm{d}\phi^2\right) && (4.4.4)\\
&+\frac{L^2}{\zeta(r,\theta)}\left\{\zeta(r,\theta)^2\mathrm{d}\theta^2+\lambda(r)^6\sin^2\theta\left(\mathrm{d}\phi_3+L^{-1}A_3\right)^2\right.\\
&\left.+\cos^2\theta\left[\mathrm{d}\psi^2+\sin^2\psi\left(\mathrm{d}\phi_1+L^{-1}A_1\right)^2+\cos^2\psi\left(\mathrm{d}\phi_2+L^{-1}A_2\right)^2\right]\right\}\ , && (4.4.5)\\
F\left(r\right) &= L^{-2}+\frac{\ell^4}{\eta r^4}\left(q_1^2-q_2^2\lambda(r)^{-6}\right)\ , \qquad \lambda(r)^6=1+\epsilon\frac{\ell^2}{r^2}\ , \qquad \zeta(r,\theta)^2=1+\epsilon\frac{\ell^2}{r^2}\cos^2\theta\ ,\\
A_1 &= A_2=\epsilon q_1\ell^2\frac{r^2-r_0^2}{r^2r_0^2}L\mathrm{d}\phi\ , \qquad A_3=\epsilon q_2\ell^2\frac{r^2-r_0^2}{(r^2+\epsilon\ell^2)(r_0^2+\epsilon\ell^2)}L\mathrm{d}\phi\ , && (4.4.6)
\end{aligned}$$

where $\epsilon = \pm 1$ depending whether the scalar is positive or negative, and r_0 is the zero of $F(r)$. For consistency, η and ϵ must have opposite signs, hence we considered $\eta = -\epsilon\ell^2/L^2$. We use $\vec{\mu} = (\cos\theta\sin\psi, \cos\theta\cos\psi, \sin\theta)$. The field strength 5-form $\mathsf{F}_5 = G_5 + \star G_5$ is defined in terms of G_5 given by

$$\begin{aligned}
G_5 &= \frac{2r^3\epsilon}{L^4\lambda^6}\left[\sin^2\theta+\lambda^{12}\cos^2\theta-\zeta^2\left(1+2\lambda^6\right)\right]\mathrm{d}r\wedge\mathrm{d}t\wedge\mathrm{d}y\wedge\mathrm{d}z\wedge\mathrm{d}\phi && (4.4.7)\\
&-\frac{\epsilon\lambda' r^3}{\lambda L^2}\left[2r^2\lambda^6F\left(r\right)+3L^2\left(1-\lambda^6\lambda_0^{-6}\right)\left(q_2^2-q_1^2\lambda^6\lambda_0^6\right)\right]\sin\left(2\theta\right)d\theta\wedge\mathrm{d}t\wedge\mathrm{d}y\wedge\mathrm{d}z\wedge\mathrm{d}\phi\\
&+3r^3\epsilon\lambda^5\lambda'\left[\sin\left(2\theta\right)\mathrm{d}\theta\wedge\left(q_1\sin^2\psi\mathrm{d}\phi_1+q_1\cos^2\psi\mathrm{d}\phi_2+q_2\mathrm{d}\phi_3\right)\right.\\
&\left.-q_1\cos^2\theta\sin\left(2\psi\right)\mathrm{d}\psi\wedge\left(\mathrm{d}\phi_1-\mathrm{d}\phi_2\right)\right]\wedge\mathrm{d}t\wedge\mathrm{d}y\wedge\mathrm{d}z\ .
\end{aligned}$$

The field strength 5-form can be written explicitly as the exterior derivative of a 4-form as $\mathsf{F}_5 = d(\mathcal{C}_4 + \tilde{\mathcal{C}}_4)$, where

$$\mathcal{C}_4 = -\left[\frac{r^4}{L^4}\zeta(r,\theta)^2+\frac{\ell^4}{r_0^2}\epsilon\cos^2\theta(q_2^2\lambda(r_0)^{-6}-q_1)\right]\mathrm{d}t\wedge\mathrm{d}y\wedge\mathrm{d}z\wedge\mathrm{d}\phi \qquad (4.4.8)$$

$$
\begin{aligned}
&+\ell^2\epsilon(q_1\cos^2\theta(\cos^2\psi d\phi_2+\sin^2\psi d\phi_1)+q_2\cos^2\theta d\phi_3)\wedge \mathrm{d}t\wedge\mathrm{d}y\wedge\mathrm{d}z\\
\tilde{\mathcal{C}}_4 \;=\;& \frac{L^4r^2\lambda(r)^6\cos^4\theta\sin(2\psi)}{2r^2\zeta(r,\theta)^2}\mathrm{d}\phi_1\wedge\mathrm{d}\phi_2\wedge\mathrm{d}\phi_3\wedge\mathrm{d}\psi \qquad (4.4.9)\\
&-\frac{L^4q_2r^2(\lambda(r)^6-\lambda(r_0)^6)}{r^2\zeta(r,\theta)^2\lambda(r_0)^6}\cos^4\theta\cos\psi\sin\psi\mathrm{d}\phi\wedge\mathrm{d}\phi_1\wedge\mathrm{d}\phi_2\wedge\mathrm{d}\psi\\
&-\frac{\ell^2L^4q_1r\epsilon\cos(2\psi)\sin^2(2\theta)}{8r^4\zeta(r,\theta)^4}(1+\frac{\epsilon\ell^2}{r_0^2}\cos^2\theta)\mathrm{d}r\wedge\mathrm{d}\phi\wedge\mathrm{d}\phi_3\wedge(\mathrm{d}\phi_1-\mathrm{d}\phi_2)\\
&-\frac{L^4r^4q_1\ell^2\epsilon\sin(2\theta)}{4r^6\zeta(r,\theta)^4}\mathrm{d}\theta\wedge\mathrm{d}\phi\wedge\mathrm{d}\phi_3\wedge\\
&\left[-\zeta(r,\theta)^4(\mathrm{d}\phi_1+\mathrm{d}\phi_2)+r^2\lambda(r)^6(r^{-2}-r_0^{-2})\cos(2\psi)\cos^2\theta(\mathrm{d}\phi_1-\mathrm{d}\phi_2)\right]
\end{aligned}
$$

with $G_5=\mathrm{d}\mathcal{C}_4$, $\star G_5=\mathrm{d}\tilde{\mathcal{C}}_4$.

The flux of the F_5 on the S^5 with coordinates $[\theta,\psi,\phi_1,\phi_2,\phi_3]$ and ranges $\theta,\psi\in[0,\pi/2]$, $\phi_1,\phi_2,\phi_3\in[0,2\pi]$ is given by

$$
\int_{S^5}\mathsf{F}_5=\int_{S^5}\star\mathsf{F}_5=\epsilon 4\pi^3L^4\,. \qquad (4.4.10)
$$

Regarding the supersymmetry of the configuration, we show that the determinant of the components of integrability conditions (4.1.46) are all zero for $|q_1|=|q_2|$, which ensures the existence of a solution of the Killing spinor equation (4.1.45). Consequently, from the point of view of IIB supergravity, the Killing spinor equation admits a solution even in the case $q_1=q_2$, in addition to the case $q_1=-q_2$ that we found in $D=5$.

As we already mentioned, when the supergravity $U(1)$ gauge fields vanish, we recover the singular distributions of [184]. These singularities are considered to be "good" in the analysis of [201]. As remarked in [201] these Coulomb branch states do not seem to admit a finite temperature analogue (without $U(1)$ gauge fields). However, the singularities satisfy the more general Gubser-criterion that the evaluation of the scalar field potential on the solution should never yield $+\infty$. Indeed, this is a property of the STU-model of maximal supergravity which has a scalar field potential which is everywhere negative.

4.4.2 Mass spectrum and phase transitions

In the case $A^i=0$ of [184], it was noted that for the dilaton, one can reduce the 10-dimensional equation of motion onto the 5-dimensional one, if we have a warped product form,

$$
\mathrm{d}s_{10}^2=\Delta^{-2/3}(r,\mu_i,\phi_i)\mathrm{d}s_5^2(y,z,t,r,\phi)+\mathrm{d}s_K^2(\mu_i,\phi_i,r)\;, \qquad (4.4.11)
$$

so ds_K that can depend on ds_5, but ds_5 independent on ds_K, and *if the dilaton is independent on* K*, so* $\Phi = \Phi(t, y, z, r, \phi)$. Here $\Delta = \sqrt{\det g_K / \det g_K^{(0)}}$, where $g_K^{(0)}$ is the metric of the *undeformed by* ds_5 metric of K, i.e., in this case, the metric of the round S^5 sphere, and g_K is the full deformed metric on K.

That is so, since we can easily verify that the 10-dimensional d'Alembertian operator on Φ is

$$\Box_{10D}\Phi = \frac{\Delta^{2/3}}{\sqrt{-g}}\partial_\mu(g^{\mu\nu}\sqrt{-g}\partial_\nu\Phi) + ds_K\ terms\,, \tag{4.4.12}$$

where $g_{\mu\nu}$ is the 5-dimensional metric for ds_5. Hence, this is equivalent to solving the d'Alembertian (massless KG) equation in the 5-dimensional metric ds_5^2.

In our case, with $A^i \neq 0$, specifically $g_{\phi K} \neq 0$, we have the same situation, if we impose the additional constraint that Φ is independent on the circle (KK) coordinate ϕ, so $\Phi = \Phi(t, y, z, r)$ only, in which case we have the same 5-dimensional $\Box$ operator, but acting on a field that only depends on 4 dimensions, so on the zero mode for the KK expansion on S^1.

By comparing this form with our own uplift form (4.1.38), we see that

$$\tilde{\Delta}^{1/2} = \Delta^{-2/3} = \frac{\zeta(r,\theta)}{\lambda^2(r)}\,, \tag{4.4.13}$$

which means that the 5-dimensional metric in our case can be put into the form

$$\begin{aligned} ds_5^2 &= \frac{\lambda^2 r^2}{L^2}\left(\frac{L^2 dr^2}{r^4 F(r)\lambda^6} + d\vec{x}_{1,2}^2 + F(r)L^2 d\phi^2\right) \\ &= \frac{r^2}{L^2}\left(1+\epsilon\frac{\ell^2}{r^2}\right)^{1/3}\left[\frac{L^d r^2}{r^4 F(r)\left(1+\epsilon\frac{\ell^2}{r^2}\right)} + d\vec{x}_{1,2}^2 + F(r)L^2 d\phi^2\right]\,, \end{aligned} \tag{4.4.14}$$

and by using redefining $r/L = L/z$, we have

$$ds_5^2 = \frac{L^2}{z^2}\left(1+\epsilon\frac{\ell^2 z^2}{L^4}\right)^{1/3}\left[\frac{dz^2}{L^4 F(z)\left(1+\epsilon\frac{\ell^2 z^2}{L^4}\right)} + d\vec{x}_{1,2}^2 + F(z)L^2 d\phi^2\right]\,, \tag{4.4.15}$$

with

$$F(z) = 1 - \epsilon\frac{\ell^2 z^4}{L^6}\left(q_1^2 - \frac{q_2^2}{1+\epsilon\frac{\ell^2 z^2}{L^4}}\right)\,. \tag{4.4.16}$$

Then the spectrum of the scalar 0^{++} glueballs is given by the eigenstates of the d'Alembertian operator in this 5-dimensional background. Since

$$\Box\Phi = \frac{z^5}{\left(1+\epsilon\frac{\ell^2 z^2}{L^4}\right)^{1/3}}\partial_z\left[\frac{\left(1+\epsilon\frac{\ell^2 z^2}{L^4}\right)}{z^3}F(z)\partial_z\right]\Phi + \frac{z^2}{\left(1+\epsilon\frac{\ell^2 z^2}{L^4}\right)^{1/3}}\partial_i\partial_i\Phi\ , \tag{4.4.17}$$

under the redefinition of the variable, $dz = \sqrt{F(z)\left(1+\epsilon\frac{\ell^2 z^2}{L^4}\right)}du$, and of the function, with a $e^{i\vec{k}\cdot\vec{x}}$ plane wave in the y, z, t directions, and with $\vec{k}^2 = -M^2$,

$$\Phi = e^{i\vec{k}\cdot\vec{x}}\frac{z^{3/2}}{F(z)\left(1+\epsilon\frac{\ell^2 z^2}{L^4}\right)}\Psi(z)\ , \tag{4.4.18}$$

from $\Box\Phi = 0$ we get the one-dimensional Schrödinger equation,

$$-\frac{d^2\Psi(u)}{du^2} + V(u) = M^2\Psi(u)$$
$$V(z) = -\left[F(z)\left(1+\epsilon\frac{\ell^2 z^2}{L^4}\right)\right]^{1/4} z^{3/2}\frac{\mathrm{d}}{\mathrm{d}z}\left\{\frac{F(z)\left(1+\epsilon\frac{\ell^2 z^2}{L^4}\right)}{z^3}\frac{\mathrm{d}}{\mathrm{d}z}\frac{z^{3/2}}{\left[F(z)\left(1+\epsilon\frac{\ell^2 z^2}{L^4}\right)\right]^{1/4}}\right\}. \tag{4.4.19}$$

We see that we can redefine $\tilde{F}(z) \equiv F(z)\left(1+\epsilon\frac{\ell^2 z^2}{L^4}\right)$, in which case we are back to the case considered in [202].

We can now make the same analysis from before (in 5 dimensions, in terms of the x coordinate) for the z_0 solving $F(z_0) = 0$, but now also consider together with the one solving $\tilde{F}(z_0) = 0$, which is more relevant:

-if $\epsilon = +1$, $q_2 \to 0$ gives an z_0, but $q_1 = 0$ gives no z_0 (but $q_1 \to 0$, yet $q_1 \neq 0$, gives an z_0).

-if $\epsilon = -1$, $q_2 = 0$ gives no z_0 (but $q_2 \to 0$, yet $q_2 \neq 0$, gives an z_0), but $q_1 \to 0$ gives an z_0.

-if $q_1 = \pm q_2$ (in the susy case), there always is an z_0.

-if $\epsilon = -1$, $q_2 = 0$, there is no solution to $F(z_0) = 0$, *but there is a solution to* $\tilde{F}(z) = 0$, *namely* $z_0 = L^2/\ell$.

-if $\epsilon = +1$ and $q_1 = q_2 = 0$, we get no z_0.

In the UV, at $z \to 0$, we have $\tilde{F}(z) \simeq 1$, so we get $z \simeq u$ and the same potential for both $\epsilon = \pm 1$,

$$V(u \simeq 0) \simeq \frac{15}{4u^2} \Rightarrow \Psi(u) = \sqrt{Mu}\left[C_1 J_2(Mu) + C_2 Y_2(Mu)\right]. \tag{4.4.20}$$

In the IR, we can have a $z_0 \neq 0$, or not, as we discussed, depending on ϵ, ℓ and q_1, q_2.

If we have a $z_0 \neq 0$, then for $z \to z_0$, with $\tilde{F}(z) \simeq \tilde{F}'(z_0)(z - z_0)$, $u - u_{\max} \simeq 2\sqrt{(z - z_0)/\tilde{F}'(z_0)}$, and writing $u_{\max} = Kz_0$, we get

$$\begin{aligned} V(u \simeq Kz_0) &\simeq -\frac{\tilde{F}'(z_0)}{16|z - z_0|} \simeq -\frac{1}{4(u - Kz_0)^2} \Rightarrow \\ \Psi &\simeq \sqrt{Kz_0 - u}\left[C_1' J_0\left(M(u - Kz_0)\right) + C_2' Y_0\left(M(u - Kz_0)\right)\right] . \end{aligned} \quad (4.4.21)$$

The IR boundary condition puts $C_2' = 0$, so the J_0 solution continued to the UV (at $u = 0$) must give also $C_2 = 0$, which will give a quantization condition on $M(Kz_0) = Mu_{\max}$, as $M = M_n$. But, of course, the quantization condition will depend on the parameters (q_1, q_2, ℓ) of the solution, which will define $u_{\max}$, decoupling the scale of M_n, $u_{\max}$, from the KK scale, $T_{\rm KK} = 1/\delta$. In any case, the spectrum is discrete.

On the other hand, if there is no z_0 (so $z_0 = 0$) in the IR,

-if $\epsilon = +1, q_1 = 0$, then $\tilde{F}(z) \sim (q_2^2 \ell^2/L^6) z^4$, so

$$V(z \to \infty) \simeq -\frac{1}{4} z^2 \frac{q_2^2 \ell^2}{L^6} \simeq -\frac{1}{4(u - u_{\max})^2} , \quad (4.4.22)$$

so, despite the fact that we don't have a z_0, we obtain the same form of the potential in terms of u, since $\int du \simeq 1/\sqrt{(q_2^2 \ell^2/L^6) z^2}$. So again a discrete spectrum.

-if $\epsilon = -1, q_2 = 0$, then there is no z_0 for $F(z)$, but there is one for $\tilde{F}(z)$, so the solution is again the same as before, and we have a discrete spectrum.

-if $\epsilon = +1, q_1 = q_2 = 0$, then there is no z_0 for $F(z)$ or $\tilde{F}(z)$, and then $\tilde{F}(z) \simeq \ell^2 z / L^4$, so

$$V(z \to \infty) \simeq +\frac{\ell^2}{L^4} = V(u) , \quad (4.4.23)$$

so *we have a continuous spectrum above a mass gap at* $M^2 = \ell^2/L^4$.

In conclusion, this case of $\epsilon = +1, q_1 = q_2 = 0$ is the only one for which we have a qualitatively different spectrum.

We can then say that *the introduction of the* q_1, q_2 *charges induces a phase transition from the spectrum continuous above a mass gap, continuously connected to the discrete spectrum.* At $q_1 = q_2 = 0$, the two spectra seemed distinct, as they were obtained in the two separate cases, $\epsilon = +1$ and $\epsilon = -1$, respectively.

Finally, when we have the pure AdS space, obtained formally by putting $F(z)=1, \ell=0$, we obtain that the potential in the UV is valid everywhere, $u=z$ and $V(z)=\frac{15}{4u^2}$. In this case, there is no limit on $u=z$, it spans from $u=0$ to $u=+\infty$, which means that the spectrum is continuous without a mass gap.

Since, as we saw in section 4, we had two phase transitions, interpreted as quantum phase transitions from the point of view of the 3-dimensional dual field theory reduced on ϕ, one from "no horizon" (given by the singular distributions of D3 branes) to "horizon" (at $g^2_{\rm 3d,YM}=0$), and then to "AdS space" (at $g^2_{\rm 3d,YM}\neq 0$), these are: from continuous above a mass gap to discrete, to continuous without a mass gap.

4.5 Final discussion

In chapter we have found AdS solitons depending on three parameters, namely the two sources associated to the gauge fields, which were proportional to the charge parameters q_1, q_2, and the value of the periodicity of the circle S^1, δ. We have shown that it is possible to describe the phase space in terms of the dimensionless sources (ψ_1, ψ_2), together with $\delta = 1/T_{\rm KK}$. The solutions give a dual scalar VEV $\langle\mathcal{O}\rangle_{(1,1)_0}$ in 3+1 dimensions, proportional to $\eta = \pm\ell^2/L^2$. Among the solutions, a special role is played by the supersymmetric solutions, with $q_1 = \pm q_2$.

We have found two phase transitions from the (E, ψ_1, ψ_2) diagram, as ψ_1 is varied, one at $\psi_1 = \pm\psi_2, x_0 = 1$, and another the one at $\psi_2 = \pm\psi_1 x_0(\psi_1, \psi_2)$ and $E = 0$, to the previous solutions of [184].

Our set of solutions continuously connects all the possibilities described in [184]. The 10-dimensional uplift of the solutions was found to be a deformation of the D3-brane distributions of [184], and in the appendix below we hint towards its description as a system of D3-branes, obtained from the Wick rotation of the rotating D3-branes in 3 independent planes, so one expects that there is a good string theory interpretation of the results, though we have not found it so far.

In terms of the 2+1-dimensional interpretation, the supersymmetric solutions give a quantum critical phase transition, at $g^2_{\rm 3d,YM}=0$, between a phase with no VEV (and no horizon in the dual), and spectrum that is continuous above a mass gap, and a phase with VEV (and horizon in the dual), and discrete spectrum, and the transition to periodic AdS space is to a continuous and no mass gap spectrum, at nonzero $g^2_{\rm 3d,YM}$.

Remarkably enough, we have found that the phase diagram of these solutions should correspond to the strongly coupled description of the existence of two possible vacua of the large N $\mathcal{N}=4$ SYM when compactified on an S^1 in four dimensions and antiperiodic boundary conditions for the fermions on the S^1. Unexpectedly, we found that at finite values of the source the supersymmetry breaking vacuum gets its supersymmetry restored, corresponding to the BPS states existing in supergravity.

Hence, this should correspond to the existence to a non-perturbative object in the field theory, most likely the Q-ball [203], embedded into the supersymmetric theory, and extended to strong coupling (where its stability properties and mass value with respect to the ones fundamental fields are not currently understood). Indeed, we see that in the UV, at $x=1$, we have $A^1=A^2=q_1(1-x_0^{-1})Ld\phi$, $A^3=q_2(1-x_0)Ld\phi$.

In the case of the double Wick rotation of the solution, with $F(x)$ multiplying $-\mathrm{d}t^2$ instead of $d\phi^2$ in the metric, this would give $A^1=A^2=q_1(1-x_0^{-1})Ldt$, $A^3=q_2(1-x_0)Ldt$, which is the standard case for $\mu_1=q_1(1-x_0^{-1})L$, $\mu_2=q_2(1-x_0)L$, chemical potentials, or sources, for the corresponding $U(1)$ charges $\int d^3x\ \rho$, where $\rho\sim\mathrm{Tr}[\bar{Z}\partial^0 Z]+\mathrm{Tr}[\bar{\psi}\gamma^0\psi]$, with Z complex combinations of X^I's, and ψ complex fermions, both charged under the $U(1)$'s.

Therefore in our case, $A^1=A^2=q_1(1-x_0^{-1})Ld\phi$ and $A^3=q_2(1-x_0)Ld\phi$, μ_1 and μ_2 are sources for the $U(1)$ current components in the ϕ direction, $\sim\mathrm{Tr}[\bar{Z}\partial^\phi Z]+\mathrm{Tr}[\bar{\psi}\gamma^\phi\psi]$, so they are understood as $J^\phi=\rho v^\phi=\rho\frac{d\phi}{\mathrm{d}t}\equiv\rho\omega$ (if we had $\rho\vec{v}$, we would write $Z=Z(\vec{x}-\vec{v}t)$, with $\vec{x}=(y,z)$). We see that, *from the point of view of the reduced 2+1 dimensional theory in (t,y,z), in which ϕ is an internal direction*, ω might be understood as Q-ball [203] angular frequency (for effective potential $V_{\text{eff}}(Z)=V+\frac{1}{2}\omega^2|Z|^2$), for writing $Z(\phi=\omega t,\vec{x})=e^{i\omega t}Z(\vec{x})$, and J^ϕ is then Q-ball charge density (except, of course, that we don't have a time dependence of the phase ϕ, that was just assumed). Then μ_1,μ_2 would be chemical potentials for the Q-ball charges.

This might provide a generalization of the Coulomb Branch solution for $\mathcal{N}=4$ SYM, by the parameters η (operator VEV) and q_1,q_2 (related to the μ_1,μ_2, the "chemical potentials for Q-ball charges"), that contains both solutions with arbitrary (or no) periodicity of ϕ, better understood within $\mathcal{N}=4$ SYM, and solutions with periodic ϕ and cigar-type solution with an x_0 ("horizon"), understood either from the point of view of the reduction to 3 dimensions ((y,z,t)), or from the point of view of Euclidean version of 4 dimensions, at finite KK temperature T_{KK}. We expect to make this picture more concrete in a future work.

In the next chapter, we turn to black hole physics in the STU model in $D = 4$. More precisely, based on [31], we study the stability of 4-charge electric and magnetic black holes.

Chapter 5

Black holes in the STU model

Black holes in SO(8)-gauged $\mathcal{N} = 8$ supergravity yield concrete examples of the physics of the low energy limit of M-theory. The four-dimensional theory was originally constructed in [204] and proven to describe the massless sector in the spontaneous compactification of eleven-dimensional supergravity on S^7 [205]. The consistent truncation of this theory to the four vectors gauging the Cartan subalgebra of SO(8) is the $\mathcal{N} = 2$ $U(1)^4$-STU model, whose bosonic sector consists of the metric, four U(1)-vectors, three dilatons and three axions. The static-spherically symmetric black holes in this model were obtained with either four electric or four magnetic charges in [206]. These black holes were embedded in eleven dimensions and generalized to hyperbolic and planar horizons in [195]. The spinning solution, with two dilatons and two axions set to zero, was constructed in [207]. In [206], the BPS limit of the spherical black holes was found to yield naked singularities in the electric case and not to exist in the magnetic one. When the four U(1)-vectors are equal to one another, this theory reduces to the minimal gauged $\mathcal{N} = 2$ supergravity, with the bosonic sector corresponding to the Einstein-Maxwell theory with a negative cosmological constant. In this limit, spherical black holes exhibit naked singularities [182], while finite-area black holes must have a locally hyperbolic horizon [208] and it was unknown whether running scalars would allow for spherical and planar black holes. Eventually, the first regular, finite-area spherically symmetric and planar supersymmetric black hole in $\mathcal{N} = 8$ supergravity was constructed in [209]. The $\mathcal{N} = 8$ black holes are known to contain different kinds of instabilities which have been studied restricting the number of charges [210, 211, 212].

In this chapter, we provide a general and simple proof that the four-charge planar electric black holes of gauged $\mathcal{N} = 8$ supergravity are unstable when the temperature is low enough. Indeed, there is a finite temperature at which the black holes are no longer equilibrium states in the thermodynamic sense. To this end, we construct the equation of state of these black holes and compute the determinant of the Hessian of the energy, showing that, below a certain temperature, it is always negative. This means that there is a spinodal line for these black holes, and we construct it explicitly in the pure Einstein-Maxwell case.

This result is puzzling for the magnetic supersymmetric black hole. Indeed, from electromagnetic duality, one would expect that the same equation of state would apply in this case by replacing the electric charge squared with the magnetic charge squared. However, this would mean that the supersymmetric black hole is unstable. Following the results of [213, 214] we propose that the Hessian has to be computed on an energy that goes to zero in the BPS limit and that satisfies the topological twist condition ab initio, a condition necessary for the supersymmetric charge to exist asymptotically. Using this mass we find that the black hole is indeed meta-stable. Metastability of supersymmetric black holes should be expected as they are at the boundary of the allowed region of stability.

The organization of the chapter is as follows. In Section 5.1 we give a detailed explanation about the embedding of the STU model in the maximal SO(8)-gauged supergravity in four dimensions. Then, in Section 5.2 we specialize the discussion in the previous section for the purely dilatonic sector of the STU model. In Sections 5.2.2 and 5.2.3, we provide the general non-extremal solutions of [206, 195] written in a slightly different way. Eventually, we derive the BPS limits of the electric and magnetic black holes. In the Section 5.3, we compute the equation of state of electric black holes and show that there is a critical temperature at which the Hessian is always negative. There, we compute the spinodal line for the usual Reissner-Nordström black hole. Then we analyze the magnetic case, where there are supersymmetric black holes of finite area. We find that if the ensemble of black holes is required to have boundary conditions that allow for the existence of a finite supercharge asymptotically, and the mass is shifted, the Hessian is indeed positive definite. We also discuss the stability of the non-BPS extremal solutions, which admit a first-order description in terms of a fake-superpotential [215, 216, 217, 218].

5.1 The STU model from maximal supergravity

In this section we recall basic facts about the STU model and its embeddings in the SO(8)-gauged maximal supergravity [206]. This STU model describes $\mathcal{N} = 2$ supergravity coupled to 3 vector multiplet, with a suitable Fayet-Iliopoulos (FI) term. The four vector fields gauge a Cartan subalgebra of the $\mathfrak{so}(8)$ gauge algebra. The corresponding gauge group $\mathrm{SO}(2)^4 = \mathrm{SO}(2)_0 \times \mathrm{SO}(2)_1 \times \mathrm{SO}(2)_2 \times \mathrm{SO}(2)_3 \subset \mathrm{SO}(8)$ can be chosen so that each factor act on one of the four couples (A_I), $I = 0, \dots, 3$:

$$A_0 = (1,2); \quad A_1 = (3,4); \quad A_2 = (5,6); \quad A_3 = (7,8).$$

in which the R-symmetry index of the eight gravitini $\psi_{\mathtt{i}\mu}$, $\mathtt{i} = 1, \dots, 8$, of the maximal theory, can be split:

$$\Psi_{\mathtt{i}\mu} = \{\Psi_{A_I\,\mu}\}_{I=0,\dots,3}\,,$$

so that the couple of gravitini $\psi_{A_I\,\mu}$ transform as a doublet under $\mathrm{SO}(2)_I$. The bosonic sector of the STU model is defined by the fields of the $\mathcal{N} = 8$ theory which are singlets under $\mathrm{SO}(2)^4$. The 28 gauge fields $A^{\mathtt{IJ}}_\mu$, $\mathtt{I}, \mathtt{J} = 1, \dots 8$, of the parent model, transforming in the **28** of $\mathrm{SL}(8, \mathbb{R})$, together with their magnetic duals $A_{\mathtt{IJ}\,\mu}$, yield, upon truncation, the following singlet vector fields

$$(A^M_\mu) = \mathbf{\Omega} \cdot (A^{12}_\mu,\, A^{34}_\mu,\, A^{56}_\mu,\, A^{78}_\mu,\, A_{12\,\mu},\, A_{34\,\mu},\, A_{56\,\mu},\, A_{78\,\mu})\,. \tag{5.1.1}$$

The above fields transform as a symplectic vector in the $(\mathbf{2}, \mathbf{2}, \mathbf{2})$ of the classical global symmetry group $\mathrm{SU}(1,1)^3$ of the $\mathcal{N} = 2$ theory and $\mathbf{\Omega}$ is a constant 8×8 symplectic matrix encoding the freedom in the choice of the basis of the symplectic representation (*symplectic frame*). We shall discuss the relevant symplectic frames below.

The 70 scalar fields $\phi^{\mathtt{ijkl}}$ of the maximal theory, transforming in the **70** of $\mathrm{SU}(8)$,[1] upon reduction to $\mathrm{SO}(2)^4$-singlets, reduce to three complex scalars

$$(\phi_{(12)(34)}, \phi_{(12)(56)}, \phi_{(12)(78)}) = z^i = \chi_i + i e^{-\phi_i}\,,\ i = 1, 2, 3$$

spanning the scalar manifold of the STU model:

$$\mathscr{M}_{\text{scal.}} = \left(\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}\right)^3\,.$$

This is a *special Kähler* manifold[2] whose isotropy group $\mathrm{U}(1)^3$ being the subgroup of $\mathrm{SU}(8)$ commuting with the $\mathrm{SO}(2)^4$ residual gauge group. The isometry group $\mathrm{SU}(1,1)^3$ defines

[1]Recall that the $\phi^{\mathtt{ijkl}}$ satisfy the reality condition $\phi^{\mathtt{ijkl}} = \frac{1}{24}\,\epsilon^{\mathtt{ijkli'j'k'l'}}\,(\phi^{\mathtt{i'j'k'l'}})^*$.

[2]See [60], and references therein, for a review of special geometry and of the gauging of $\mathcal{N} = 2$ models in the embedding tensor formalism.

the on-shell global symmetry group of the classical theory, acting on the eight A^M_μ in the $(\mathbf{2},\mathbf{2},\mathbf{2})$ symplectic representation, as mentioned above.

The fermionic sector, on the other hand, depends on the factor $\mathrm{SO}(2)_I$, within $\mathrm{SO}(2)^4$, which is chosen to lie within the R-symmetry group of the $\mathcal{N}=2$ model. This sector of the corresponding STU model, which will be referred to as STU_I, is then defined as the singlet sector, within the maximal theory, with respect to the remaining group $\mathrm{SO}(2)^3 = \mathrm{SO}(2)_J \times \mathrm{SO}(2)_K \times \mathrm{SO}(2)_L$, $I \neq J \neq K \neq L \neq I$ and $\psi_{A_I\,\mu}$ are the spin-3/2 fields of the truncation. The consistency of this truncation is guaranteed by the fact that the STU_I model, so defined, is the singlet sector with respect to the subgroup $\mathrm{SO}(2)^3 = \mathrm{SO}(2)_J \times \mathrm{SO}(2)_K \times \mathrm{SO}(2)_L$ of SO(8). Choosing, for instance, the STU_0 model, whose gravitini are $\psi_{A_0\,\mu} = (\psi_{1\,\mu}, \psi_{2\,\mu})$, the spin-1/2 fields which are singlets with respect to $\mathrm{SO}(2)_1 \times \mathrm{SO}(2)_2 \times \mathrm{SO}(2)_3$, are:

$$(\lambda^{A_0 34}, \lambda^{A_0 56}, \lambda^{A_0 78}) \equiv \lambda^{iA_0}\,, \quad i = 1,2,3,\ A_0 = 1,2.$$

and enter the three vector multiplets together with the complex scalar fields and three of the vector fields.

Upon reducing the gauge group of the maximal theory to $\mathrm{SO}(2)^4$ and the electric/magnetic duality index to the eight $\mathrm{SO}(2)^4$-singlets, the gauge connection reduces to:[3]

$$A^M_\mu\, X_M = A^M_\mu\, \Theta_M{}^I\, t_I\,, \tag{5.1.2}$$

where t_I are the four generators of $\mathrm{SO}(2)^4$. In the STU_I model, only t_I has a non-trivial action on the fermionic fields and thus the only minimal couplings involve the spin-1/2 and 3/2 fields and the single vector combination $A^M_\mu\, \Theta_M{}^I$. The 8-component symplectic vector $\Theta_M{}^I$ defines the FI term of the model and the corresponding scalar potential. For the sake of concreteness, we shall work in the STU_0 model, which we shall simply refer to as the STU model, denoted by θ_M the corresponding FI term $\Theta_M{}^0$ and by $A, B = 1,2$ the R-symmetry indices A_0, B_0.
The geometry of the spacial Kähler manifold $\mathscr{M}_{\text{scal.}}$ is described in terms of a holomorphic symplectic section $\Omega^M(z_i)$ which, in the symplectic frame that we adopt and modulo multiplication by a non-vanishing holomorphic function, reads:

$$\Omega^M = (z_2 z_3, z_1 z_3, z_1 z_2, -1, z_1, z_2, z_3, -z_1 z_2 z_3)\,, \tag{5.1.3}$$

[3] We absorb the gauge grouping constant g in the embedding tensor.

The Kähler potential, for instance, reads:

$$\mathcal{K}(z,\bar{z}) = -\log(i\,\overline{\Omega}^T\,\mathbb{C}\,\Omega) = 8\,\mathrm{Im}(z_1)\mathrm{Im}(z_2)\mathrm{Im}(z_3) = 8\,e^{-\phi_1-\phi_2-\phi_3}\,. \tag{5.1.4}$$

where $\mathbb{C}$ is the $\mathrm{Sp}(8,\mathbb{R})$-invariant matrix:

$$\mathbb{C} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix} \tag{5.1.5}$$

The metric reads:

$$g_{i\bar{\jmath}} = \partial_i\partial_{\bar{\jmath}}\mathcal{K} = -\frac{1}{(z_i-\bar{z}_{\bar{\imath}})^2}\delta_{i\bar{\jmath}} = \sum_{\hat{\imath}=1}^{3} e_i^{\hat{\imath}}(e_j^{\hat{\imath}})^*\,, \tag{5.1.6}$$

where $e_i^{\hat{\imath}}$ is the complex vielbein matrix. We also define the covariantly holomorphic symplectic section $V^M(z,\bar{z}) \equiv e^{\mathcal{K}/2}\,\Omega(z)^M$, in terms of which the $\mathcal{N}=2$ central charge $\mathcal{Z}$ on a given solution and the gauge superpotential $\mathcal{W}$ induced by the FI term, read:

$$\mathcal{Z} \equiv \Gamma^T\cdot\mathbb{C}\cdot V\ ,\ \ \mathcal{W} \equiv V^T\cdot\theta\,, \tag{5.1.7}$$

$\Gamma = (\Gamma^M) = (P^\Lambda,\ -Q_\Lambda)$ being the symplectic vector of quantized charges.

The relationship between the different STU_I models within the maximal one can be inferred from inspection of the gravitino shift-matrix $\mathrm{A}_{1\mathtt{ij}}$ as a function of the singlet scalars, whose only non-vanishing entries are:

$$\begin{aligned} \mathrm{A}_{1\,A_0,A_0}\,(L^{-1}) &= \frac{1}{\sqrt{2}}\,\overline{\mathcal{W}} = \frac{1}{\sqrt{2}}\,\overline{V}^T\cdot\theta\,,\\ \mathrm{A}_{1\,A_i,A_i}\,(L^{-1}) &= \frac{1}{\sqrt{2}}\,\overline{\mathcal{W}}_{(i)} = \frac{1}{\sqrt{2}}\,\mathcal{D}_{\hat{\imath}}V^M\,\Theta_M{}^i\,,\ \ i=1,2,3\,, \end{aligned} \tag{5.1.8}$$

where $\mathcal{D}_{\hat{\imath}}V^M$ are the Kähler-covariant derivatives of V^M:

$$\mathcal{D}_{\hat{\imath}}V^M \equiv e_{\hat{\imath}}^i\,\mathcal{D}_iV^M = e_{\hat{\imath}}^i\left(\partial_i + \frac{1}{2}\,\partial_i\mathcal{K}\right)V^M\ ,\ e_{\hat{\imath}}^i = -(z_i-\bar{z}_{\bar{\imath}})\delta_{\hat{\imath}}^i\,.$$

If $\mathcal{W}_{(0)} \equiv \mathcal{W}$ is the gauge superpotential in the STU=STU_0 model, $\mathcal{W}_{(i)}$ is the corresponding function in the STU_i model. In the chosen symplectic frame, we have:

$$\begin{aligned} \theta_M \equiv \Theta_M{}^0 &= \frac{1}{\sqrt{2}L}\,(1,1,1,1,0,0,0,0)\,, \\ \Theta_M{}^1 = \mathcal{O}_{1\,M}{}^N\,\theta_N\,,\ \ \Theta_M{}^2 &= \mathcal{O}_{2\,M}{}^N\,\theta_N\,,\ \ \Theta_M{}^3 = \mathcal{O}_{3\,M}{}^N\,\theta_N\,, \end{aligned} \tag{5.1.9}$$

where the symplectic matrices $\mathcal{O}_i$, $i = 1, 2, 3$ read:[4]

$$\begin{aligned} \mathcal{O}_1 &= \text{diag}(1, -1, -1, 1, 1, -1, -1, 1)\,, \\ \mathcal{O}_2 &= \text{diag}(-1, 1, -1, 1, -1, 1, -1, 1)\,, \\ \mathcal{O}_3 &= \text{diag}(-1, -1, 1, 1, -1, -1, 1, 1)\,. \end{aligned}$$

Moreover, one can verify that:

$$\begin{aligned} \mathcal{O}_i \cdot V(z, \bar{z}) &= \overline{\mathcal{D}}_{\hat{\imath}} \overline{V}\Big|_{(z_i \to z_i\,,\; z_{j \neq i} \to -\bar{z}_{\bar{\jmath}})}\,, \\ \mathcal{O}_i \cdot \mathcal{D}_{\hat{\jmath}} V &= |\epsilon_{ijk}|\, \mathcal{D}_{\hat{k}} V\big|_{(z_i \to z_i\,,\; z_{j \neq i} \to -\bar{z}_{\bar{\jmath}})}\,. \end{aligned} \tag{5.1.10}$$

Using the first of the above properties, one can verify that:

$$\mathcal{W}_{(i)} = \overline{\mathcal{W}}\Big|_{(z_i \to z_i\,,\; z_{j \neq i} \to -\bar{z}_{\bar{\jmath}})}\,. \tag{5.1.11}$$

We then conclude that solutions to the STU $=$STU$_0$ model are mapped to solutions of the STU$_i$ one through the following transformation:

$$\text{STU}_0 \to \text{STU}_i \;\Leftrightarrow\; \begin{cases} \Gamma \to \mathcal{O}_i \cdot \Gamma\,, & \theta \to \Theta^i = \mathcal{O}_i \cdot \theta \\ z_i \to z_i\,, & z_{j \neq i} \to -\bar{z}_{\bar{\jmath}} \end{cases} \tag{5.1.12}$$

Under this transformation, a BPS solution of the STU model is mapped into a BPS solution of the STU$_i$ one, which is non-BPS in the original truncation but BPS in the maximal theory. The action of $\mathcal{O}_i$ for going from STU$_0$ to STU$_i$, at the level of R-symmetry indices, as an SU(8) compensating transformation, has the effect to exchanging the couples $(A_0) = (1, 2)$ with (A_i) and (A_j) with (A_k), $i{\neq}j{\neq}k{\neq}i$. For instance, $\mathcal{O}_1$ implies $(1, 2) \leftrightarrow (3, 4)$ and $(5, 6) \leftrightarrow (7, 8)$.

Action and supersymmetry transformations

The $\mathcal{N} = 2$ gauged STU model has the following bosonic action principle (we set $8\pi G_N = 1$):

[4] The matrices $\mathcal{O}_i$ $(i = 1, 2, 3)$,together with the identity matrix $\mathbb{1}_{8\times 8}$ define a Klein group $\mathbb{Z}_2 \times \mathbb{Z}_2$. They satisfy the relation

$$\mathcal{O}_i \cdot \mathcal{O}_j = \delta_{ij}\, \mathbb{1} + |\epsilon_{ijk}| \mathcal{O}_k\,.$$

$$\mathcal{S} = \int \mathrm{d}^4x\sqrt{-g}\left(\frac{R}{2} - g_{i\bar{\jmath}}\partial_\mu z^i\partial^\mu \bar{z}^{\bar{\jmath}} - V(z,\bar{z}) + \frac{1}{4}\mathcal{I}_{\Lambda\Sigma}F^\Lambda_{\mu\nu}F^{\Sigma\mu\nu} + \frac{1}{8e}\mathcal{R}_{\Lambda\Sigma}F^\Lambda_{\mu\nu}F^\Sigma_{\rho\sigma}\varepsilon^{\mu\nu\rho\sigma}\right). \tag{5.1.13}$$

where $i,j = 1,2,3$, $\Lambda,\Sigma = 1,\ldots,4$ and we are considering the symplectic frame described by a holomorphic section of the form (5.1.3). This frame is associated with a prepotential of the form

$$F(X) = 2i\sqrt{X^0X^1X^2X^3}\,. \tag{5.1.14}$$

Indeed $\Omega^M(z)$, modulo multiplication by a non-vanishing holomorphic function, can be written in the form $\Omega^M = (X^\Lambda, \partial F/\partial X^\Lambda)$ provided we identify:

$$z_1 = -i\sqrt{\frac{X^1X^2}{X^0X^3}}\,,\quad z_2 = -i\sqrt{\frac{X^0X^2}{X^1X^3}}\,,\quad z_3 = -i\sqrt{\frac{X^0X^1}{X^2X^3}}\,. \tag{5.1.15}$$

The Kähler potential and the metric are given in eqs (5.1.4) and (5.1.6), respectively.

The supersymmetry variations of the fermions in a bosonic background are given by[5]

$$\delta\Psi^A_\mu = D_\mu\epsilon^A + \frac{1}{4}L^\Lambda\mathcal{I}_{\Lambda\Sigma}F^{\Sigma+}_{\rho\sigma}\gamma^{\rho\sigma}\gamma_\mu\varepsilon^{AB}\epsilon_B + \frac{1}{2}\underbrace{i(\sigma^2)^A{}_C\varepsilon^{BC}}_{\delta^{AB}}\mathcal{W}\gamma_\mu\epsilon_B\,, \tag{5.1.16}$$

$$\delta\lambda^{iA} = -\partial_\mu z^i\gamma^\mu\epsilon^A + \frac{1}{2}g^{i\bar{\jmath}}\bar{f}^\Lambda_{\bar{\jmath}}\mathcal{I}_{\Lambda\Sigma}F^{\Sigma-}_{\mu\nu}\gamma^{\mu\nu}\varepsilon^{AB}\epsilon_B + W^{iAB}\epsilon_B\,. \tag{5.1.17}$$

where

$$D_\mu\epsilon^A = \partial_\mu\epsilon^A + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\epsilon^A + \frac{1}{2}A^M_\mu\theta_M\underbrace{i(\sigma^2)^A{}_B\epsilon^B}_{\varepsilon^{AB}\delta_{BC}\epsilon^C} + \frac{i}{2}\mathcal{Q}_\mu\epsilon^A\,, \tag{5.1.18}$$

$$\mathcal{Q}_\mu = \frac{i}{2}(\partial_{\bar{\imath}}\mathcal{K}\partial_\mu\bar{z}^{\bar{\imath}} - \partial_i\mathcal{K}\partial_\mu z^i) = \frac{1}{2}\sum_{i=1}^{3}e^{\phi_i}\partial_\mu\chi_i\,, \tag{5.1.19}$$

$$V^M = e^{\mathcal{K}/2}\Omega^M = (L^\Lambda, M_\Lambda)\,, \tag{5.1.20}$$

$$\mathcal{W} = V^M\theta_M\,, \tag{5.1.21}$$

$$V(z,\bar{z}) = g^{i\bar{\jmath}}\mathscr{D}_i\mathcal{W}\overline{\mathscr{D}}_{\bar{\jmath}}\mathcal{W} - 3\mathcal{W}\overline{\mathcal{W}}\,, \tag{5.1.22}$$

[5]We consider $(\sigma^2)^A{}_C = -i\delta^A_1\delta^2_C + i\delta^A_2\delta^1_C$, for which the following identities hold $i(\sigma^2)^A{}_C\varepsilon^{BC} = \delta^{AB}$, $i(\sigma^2)^A{}_B\epsilon^B = \varepsilon^{AC}\delta_{CB}\epsilon^B$, $i(\sigma^2)_C{}^B\varepsilon^{CA} = \delta^{AB}$. Requiring that $(\sigma^2)^1{}_2 = (\sigma^2)_1{}^2$, meaning that they are the same matrices, implies that $(\sigma^2)_B{}^A = -i\delta^1_B\delta^A_2 + i\delta^2_B\delta^A_1$ leading to the identity $i(\sigma^2)_C{}^B\varepsilon^{CA} = \delta^{AB}$.

$$W^{iAB} = \underbrace{i(\sigma^2)_C{}^B \varepsilon^{CA}}_{\delta^{AB}} g^{i\bar{\jmath}} \mathscr{D}_{\bar{\jmath}}(\overline{V}^M \theta_M)\,. \tag{5.1.23}$$

and the embedding tensor θ_M was given in (5.1.9).

Further truncations of the STU model are summarized in the Figure 5.1.1. There are two truncations that are well studied. One of them is called the T^3 truncation, obtained by identified the three vector multiplets. Then, the T^3 model is an $\mathcal{N}=2$ gauged supergravity with one vector multiplet and one gravitational multiplet. The T truncation corresponds to setting two out of three complex scalars to the origin of the scalar manifold, in our parametrization it means for instance setting the scalar of the first two multiplet to the origin $z^1 = z^2 = \mathrm{i}$. Consistency implies that the associated vectors must be identified $A^1 = A^2$ and the third must be identified with the gravi-photon $A^3 = A^4$. The resulting theory has two vectors and one complex scalar field with a scalar potential containing a negative cosmological constant term.

There is a third subsector of the STU model, called the purely dilatonic, where the axion fields are set to zero. However, this sector is not on the same footing as the T truncation and the T^3 truncation because setting the axions to zero is not a consistent truncation. This is the sector of the STU model that we will study trough out this chapter. In the next section we will analyze the consequences to setting the axion fields to zero and then we study black holes solutions to this model.

5.2 Purely dilatonic sector of the STU model

We are interested in studying the dilatonic sub-sector of the STU model, where the complex scalars parameterizing the special Kähler manifold are purely imaginary $z_i = \mathrm{i}e^{-\phi_i}$, $i = 1,2,3$, which leads to a substantial simplification of the model. The effective action principle, that we consider for practical proposes, is given by

$$\mathcal{S} = \int \mathrm{d}^4x\sqrt{-g}\left(\frac{R}{2} - \frac{1}{4}\sum_i \partial_\mu\phi_i\partial^\mu\phi_i - \frac{1}{4}\sum_\Lambda Y_\Lambda F^\Lambda_{\mu\nu}F^{\Lambda\mu\nu} + \frac{1}{L^2}\sum_i \cosh\phi_i\right)\,. \tag{5.2.1}$$

The field equations coming from the action principle (5.2.1) are given by

$$\mathrm{d}(Y_\Lambda \star F^\Lambda) = 0\,, \qquad \Lambda = 1,\dots,4$$

$$\frac{1}{2}\Box\phi_i + \frac{1}{L^2}\sinh\phi_i - \frac{1}{4}\sum_{\Lambda=1}^{4}\frac{\partial Y_\Lambda}{\partial\phi_i}F^\Lambda_{\mu\nu}F^{\Lambda\mu\nu} = 0\,, \qquad i = 1,2,3\,,$$

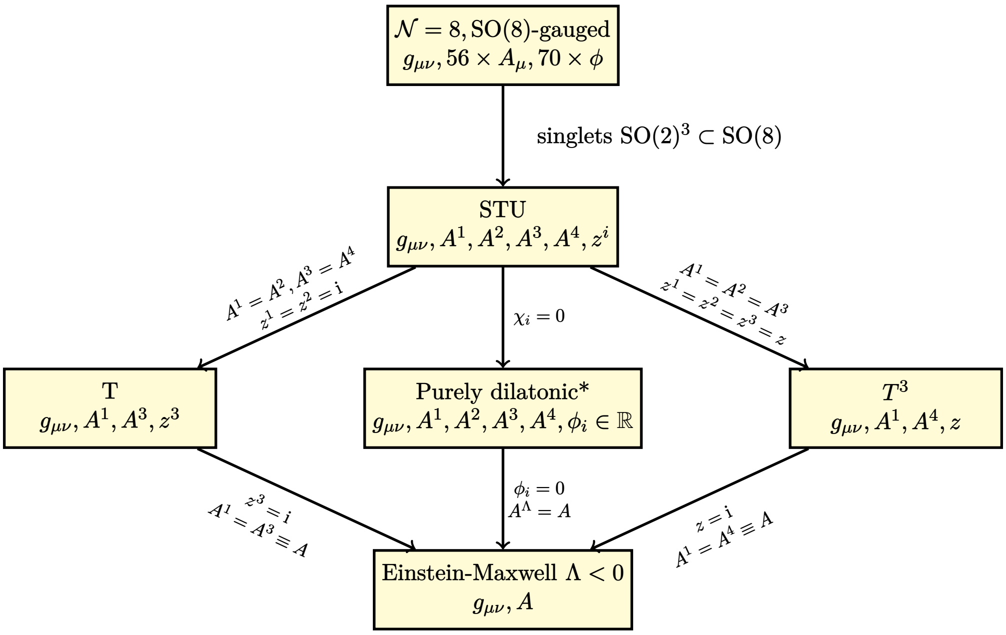

Figure 5.1.1: Summary of different truncations of STU model of the maximal SO(8)-gauged supergravity in $D=4$.

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=T^{(\phi)}_{\mu\nu}+T^{(F)}_{\mu\nu}\,.$$

The energy momentum tensors are

$$\begin{aligned}
T^{(\phi)}_{\mu\nu}&=\frac{1}{2}\sum_{i=1}^{3}\left[\partial_\mu\phi_i\partial_\nu\phi^i-g_{\mu\nu}\left(\frac{1}{2}(\partial\phi_i)^2-\frac{2}{L^2}\cosh\phi_i\right)\right]\,,\\
T^{(F)}_{\mu\nu}&=\sum_{\Lambda=1}^{4}Y_\Lambda\left(F^\Lambda_{\mu\rho}F^{\Lambda}{}_{\nu}{}^{\rho}-\frac{1}{4}g_{\mu\nu}F^\Lambda_{\rho\sigma}F^{\Lambda\rho\sigma}\right)\,.
\end{aligned}$$

They partially reproduce the field equations of the dilatonic sector of the STU model. It is required to impose also the following equations

$$\begin{aligned}
e^{2\phi_1}F^2\wedge F^3-F^1\wedge F^4&=0\,,\\
e^{2\phi_2}F^1\wedge F^3-F^2\wedge F^4&=0\,,\\
e^{2\phi_2}F^2\wedge F^2-F^3\wedge F^4&=0\,,
\end{aligned}\tag{5.2.2}$$

coming from the truncation of the axion fields to zero. These constraints are automatically fulfilled for purely electric and purely magnetic configurations. The quantities Y_Λ depends on the scalar fields and are equal to the diagonal components of the generalized couplings $\mathcal{I}_{\Lambda\Sigma}$ in the purely dilatonic sector, as see Section 5.1 for further details. They are given explicitly by

$$Y_i = e^{\phi_1+\phi_2+\phi_3-2\phi_i}\,, \qquad Y_4 = e^{-\phi_1-\phi_2-\phi_3}\,, \qquad i=1,2,3\,. \tag{5.2.3}$$

Reducing to the dilaton fields, some simplifications occur in the supersymmetry transformations: the Kähler connection vanishes $\mathcal{Q}_\mu = 0$, the first four components of the $U(1)$-section are real $L^\Lambda \in \mathbb{R}$, and the superpotential is real $\mathcal{W} \in \mathbb{R}$. Hence the Killing spinor equation coming from the supersymmetry variation of the gravitino reduces to

$$\begin{aligned}\delta\Psi^A_\mu \mathrm{d}x^\mu &\equiv \mathfrak{D}^A(\epsilon) \qquad\qquad (5.2.4)\\ &\equiv \mathrm{d}\epsilon^A + \frac{1}{4}\omega_{ab}\gamma^{ab}\epsilon^A + \frac{1}{2}A^M\theta_M\varepsilon^{AB}\delta_{BC}\epsilon^C + \frac{1}{4}L^T\mathcal{I}F_{ab}\gamma^{ab}\gamma\varepsilon^{AB}\epsilon_B + \frac{1}{2}\mathcal{W}\gamma\delta^{AB}\epsilon_B = 0\,,\end{aligned}$$

where we have defined the 1-form Clifford algebra valued $\gamma = \gamma_a e^a$. The supersymmetry variation for the gaugini are

$$\delta\lambda^{iA} = -\gamma^\mu\partial_\mu z^i\epsilon^A + \frac{1}{2}g^{i\bar{\jmath}}\overline{f}^\Lambda_{\bar{\jmath}}\mathcal{I}_{\Lambda\Sigma}F^\Sigma_{ab}\gamma^{ab}\varepsilon^{AB}\epsilon_B + g^{i\bar{\jmath}}\overline{\mathcal{U}}^M_{\bar{\jmath}}\theta_M\delta^{AB}\epsilon_B = 0\,, \tag{5.2.5}$$

where in this cases the quantities $z^i, \overline{f}^\Lambda_{\bar{\jmath}}$ and $\overline{U}_{\bar{\jmath}}{}^M\theta_M$ are purely imaginary in the dilatonic sector. The quantity $\overline{U}_{\bar{\jmath}}{}^M$ is the U(1) covariant derivative of the U(1) section $\overline{V}^M$ and $\overline{f}_{\bar{\jmath}}{}^\Lambda$ are the first four symplectic components of $\overline{U}_{\bar{\jmath}}{}^M$. The local supersymmetry parameters are chiral spinors satisfying $\gamma_5\epsilon_A = \epsilon_A$ and $\gamma_5\epsilon^A = -\epsilon^A$, and are related to each other through complex conjugation $(\epsilon^A)^* = \epsilon_A$. A^M are the symplectic gauge fields, and the embedding tensor is given by

$$\theta_M = \frac{1}{\sqrt{2}L}(1,1,1,1,0,0,0,0)\,. \tag{5.2.6}$$

The supercovariant derivative (5.2.4) is the object entering the Dirac bracket between supercharges that we outline in the next section. Since we are considering real γ-matrices, all the coefficients in the above equation are real. We can write an equation for the Majorana spinor $\psi^A = \epsilon^A + \epsilon_A$ by combining (5.2.4) with its complex conjugated. For simplicity, it is useful to define a complex spinor whose real and imaginary parts are the $SU(2)$ components of the Majorana spinor $\zeta = \psi^1 + \mathrm{i}\psi^2$. The Killing spinor equation for the complex Killing spinor is

$$\left(\mathrm{d} + \frac{1}{4}\omega_{ab}\gamma^{ab} - \frac{\mathrm{i}}{2}A^M\theta_M - \frac{\mathrm{i}}{4}L^T\mathcal{I}F_{ab}\gamma^{ab}\gamma + \frac{1}{2}\mathcal{W}\gamma\right)\zeta = 0\,. \tag{5.2.7}$$

The same can be done for the gaugino equations that implies the following equations for the complex spinor

$$\left(-\gamma^{\mu}\partial_{\mu}z^{i} - \frac{\mathrm{i}}{2}g^{i\bar{\jmath}}\overline{f}_{\bar{\jmath}}^{T}\mathcal{I}F_{ab}\gamma^{ab} + g^{i\bar{\jmath}}\overline{U}_{\bar{\jmath}}{}^{M}\theta_{M}\right)\zeta = 0\,, \tag{5.2.8}$$

See Appendix A3 for the explicit form of the γ-matrices that we are considering.

5.2.1 Dirac bracket between supercharges

In gauge theories, the conserved charges associated with large gauge transformations are obtained by the integration of a suitable conserved current on a co-dimension 2 spacelike surface, due to the fact that the Hodge dual of the Noether current is a closed form, up to imposing the field equations [219]. In [213] was developed a method to evaluate the superalgebra in backgrounds that have an asymptotic Killing spinor for $\mathcal{N} = 2$ gauged supergravity solutions. This prescription was applied to compute the BPS bound for configurations that asymptote to AdS or mAdS, resolving the tension between the BPS bound propose in [220] and the explicit BPS configurations found by Romans [182]. The same analysis was carried out in [214] for $D = 4$, $\mathcal{N} = 2$ gauged supergravity coupled to matter fields, where it was shown that the 3-form dual to the Noether supercurrent can be written as the exterior derivative of a 2-form up to imposing the field equations. This leads the supercharge to be expressed as an integral of a 2-form over a spacelike surface in the asymptotic region. The Dirac bracket between the supercharges is computed by acting with the supersymmetry transformation on the supercharge leading to

$$\{\mathcal{Q},\mathcal{Q}\} = -2\int_{\partial\Sigma}\left(\bar{\epsilon}^{A}\gamma\wedge\mathfrak{D}_{A}(\epsilon) - \bar{\epsilon}_{A}\gamma\wedge\mathfrak{D}^{A}(\epsilon)\right)\,. \tag{5.2.9}$$

where $\mathfrak{D}^{A}(\epsilon)$ is the supercovariant derivative defined in (5.2.4), and $\mathfrak{D}_{A}(\epsilon)$ is its complex conjugated. We defined the Clifford algebra valued 1-form $\gamma = \gamma_{a}e^{a}$. We consider $\mathcal{Q}$ being the product between the Grassmann-odd supercharge and the spinorial parameters ϵ^{A}, i.e. there are no free indices and it is a Grassmann-even quantity. If the superalgebra (5.2.9) is computed in a supersymmetric background, the spinor ϵ_{A} is chosen to be the Killing spinor of the background, leading to zero in the right-hand side. However, one can consider a configuration that breaks supersymmetry in the bulk but it has an asymptotic Killing spinor defined in the asymptotic region by imposing certain boundary conditions on the gravitino and the gaugini

$$\delta\Psi^A_\mu = o(1/r^n)\,, \qquad \delta\lambda^{Ai} = o(1/r^{n_i})\,, \tag{5.2.10}$$

where the constants $n, n_i > 0$ must be chosen in such a way that the conserved charges in (5.2.9) are finite. Configurations having the same boundary condition as a supersymmetric background can be understood as excitations of it. We will compute the superalgebra in backgrounds belonging to the dilatonic sector of the STU model. In that case (5.2.9) can be written in terms of the Majorana Killing spinor ψ^A as (see Appendix A4 for a detailed derivation.)

$$\begin{aligned}\{\mathcal{Q},\mathcal{Q}\} = 2\int \overline{\psi}^A\gamma_5\gamma\wedge\Big(\delta_{AB}\mathrm{d} + \frac{1}{4}\omega_{ab}\gamma^{ab}\delta_{AB} + \frac{1}{2}A^M\theta_M\varepsilon_{AB} + \\ + \frac{1}{4}L^T\mathcal{I}F_{ab}\gamma^{ab}\gamma\varepsilon_{AB} + \frac{1}{2}\gamma\mathcal{W}\delta_{AB}\Big)\psi^B\,.\end{aligned} \tag{5.2.11}$$

5.2.2 Electric dilatonic black holes and their singular supersymmetric limits

The four charges electric black hole in the dilatonic sector in the STU model, with a spherical horizon, was found in [206], and its generalizations to planar and hyperbolic horizons were constructed in [195]. Here, we present the solution for arbitrary horizon geometry controlled by the parameter $k = -1, 0, 1$ leading to hyperbolic, planar and spherical horizons. The BPS limit of the configurations can be analysed in a simple way by introducing a parameter q through a change of coordinates. The metric reads

$$\mathrm{d}s^2 = -\frac{f(r)}{\sqrt{H(r)}}\mathrm{d}t^2 + \frac{\sqrt{H(r)}}{f(r)}\mathrm{d}r^2 + r^2\sqrt{H(r)}\left(\frac{\mathrm{d}x^2}{1-kx^2} + (1-kx^2)\mathrm{d}y^2\right)\,, \tag{5.2.12}$$

$$f(r) = k + \frac{r^2}{L^2}H(r) - \frac{m}{r} - \frac{q}{r^2}\,, \qquad H(r) = H_1H_2H_3H_4\,, \qquad H_\Lambda = 1 + \frac{q_\Lambda}{r}\,. \tag{5.2.13}$$

The dilatons and the gauge fields are

$$\phi_1 = \frac{1}{2}\log\left(\frac{H_2H_3}{H_1H_4}\right)\,, \qquad \phi_2 = \frac{1}{2}\log\left(\frac{H_1H_3}{H_2H_4}\right)\,, \qquad \phi_3 = \frac{1}{2}\log\left(\frac{H_1H_2}{H_3H_4}\right)\,, \tag{5.2.14}$$

$$A^\Lambda = \left(\frac{Q_\Lambda}{\sqrt{2}rH_\Lambda} - \mu_\Lambda\right)\mathrm{d}t\,, \qquad Q_\Lambda^2 = q_\Lambda^2k + q_\Lambda m - q\,. \tag{5.2.15}$$

Where μ_Λ is related to the electric chemical potential which is fixed in terms of the rest of the parameters in such a way that the the gauge fields are regular in the Euclidean configuration. A necessary condition, and sufficient in this case, to preserve some amount of supersymmetry is that the matrices of the gaugino equations (5.2.8) are not invertible. Imposing that the matrices have zero determinant implies the following relation between the parameters

$$-kq = \frac{m^2}{4}\,. \tag{5.2.16}$$

The BPS configurations exist only in the spherical and planar case, even though they are naked singularities, they have a Killing spinor defined on the geometry. For the hyperbolic case the condition (5.2.16) leads to purely imaginary gauge fields, which only can make sense by Wick rotating the time coordinate leading to a Euclidean configuration without a Lorentzian limit.

The Majorana Killing spinor for the planar BPS configuration can be found and is given by

$$\psi^A_{\rm pl}(r) = \frac{f_{\rm pl}^{1/4}(r)}{2H^{1/8}(r)}(\alpha_{\rm pl}(r) - \beta_{\rm pl}(r)\gamma_1)(\delta_{AB} + \varepsilon_{AB}\gamma_0)\psi^B_0\,, \tag{5.2.17}$$

where ψ^B_0 is Majorana constant spinor and the relevant functions are

$$\alpha_{\rm pl}(r) = \left(1 - \frac{\sqrt{-q}}{r f_{\rm pl}^{1/2}}\right)^{1/2}, \qquad \beta_{\rm pl}(r) = \left(1 + \frac{\sqrt{-q}}{r f_{\rm pl}^{1/2}}\right)^{1/2}, \qquad f_{\rm pl}(r) = \frac{r^2}{L^2}H - \frac{q}{r^3}\,. \tag{5.2.18}$$

The projector $\delta_{AB} + \varepsilon_{AB}\gamma_0$ has matrix rank equal 2, which implies that planar BPS configuration preserves four real supercharges. Note also that the Killing spinor diverges at the curvature singularities of the manifold which are located at $H(r) = 0$.

The Majorana Killing spinor for the spherical BPS configuration is explicitly given by

$$\psi^A_{\rm sp}(t,r,x,y) = \frac{f_{\rm sp}^{1/4}(r)}{2H^{1/8}(r)} e^{\frac{it}{2L}} e^{\frac{i}{2}\gamma_{012}\arccos x} e^{-\frac{1}{2}y\gamma_{23}}(\alpha_{\rm sp}(r) + \beta_{\rm sp}(r)\gamma_1)(\delta_{AB} + \varepsilon_{AB}\gamma_0)\psi^B_0\,, \tag{5.2.19}$$

where the radial functions are

$$\alpha_{\rm sp}(r) = \left(1 + \frac{1-\frac{m}{2r}}{f_{\rm sp}^{1/2}}\right)^{1/2}, \qquad \beta_{\rm sp}(r) = \left(1 - \frac{1-\frac{m}{2r}}{f_{\rm sp}^{1/2}}\right)^{1/2}, \qquad f_{\rm sp}(r) = \frac{r^2}{L^2}H + \left(1-\frac{m}{2r}\right)^2 . \tag{5.2.20}$$

This background preserves four real supercharges, hence it is 1/2 BPS, and the Killing spinor diverges at the singularity. The spinor depends on all the coordinates, which is also the case in the purely AdS background.

5.2.3 Magnetic dilatonic black holes

The first example of, asymptotically globally AdS spacetime, supersymmetric static black holes was given by Cacciatori and Klemm [209] by considering extremal magnetically charged black holes. They also considered hyperbolic and planar horizon topology. The non-extremal version of the spherical black holes with an arbitrary number of vector multiplets and FI terms was constructed in [221]. In [218] a family of non-extremal solutions was constructed which contain, in certain limits, the solutions of [209] and of [195, 206]. Here we present the dilatonic, magnetic black hole configurations in a slightly different parametrization which allows us to straightforwardly connect, through a suitable BPS limit, the magnetic version of the four-dimensional black holes constructed in [195, 206] to the BPS configurations of [209], when the theory allows an embedding in the maximal gauged supergravity.

The metric of the configuration is

$$\mathrm{d}s^2 = -\frac{f(r)}{\sqrt{H(r)}}\mathrm{d}t^2 + \frac{\sqrt{H(r)}}{f(r)}\mathrm{d}r^2 + r^2\sqrt{H(r)}\left(\frac{\mathrm{d}x^2}{1-kx^2} + (1-kx^2)\mathrm{d}y^2\right), \tag{5.2.21}$$

$$f(r) = \frac{r^2}{L^2}H(r) + k - \frac{m}{r} - \frac{q}{r^2}, \qquad H(r) = H_1H_2H_3H_4, \qquad H_\Lambda = 1 + \frac{q_\Lambda}{r}, \tag{5.2.22}$$

and the matter fields are given by

$$\phi_1 = -\frac{1}{2}\log\left(\frac{H_2H_3}{H_1H_4}\right), \qquad \phi_2 = -\frac{1}{2}\log\left(\frac{H_1H_3}{H_2H_4}\right), \qquad \phi_3 = -\frac{1}{2}\log\left(\frac{H_1H_2}{H_3H_4}\right),$$
$$A^\Lambda = \frac{1}{\sqrt{2}}P_\Lambda x\mathrm{d}y, \qquad P_\Lambda^2 = q_\Lambda^2 k + q_\Lambda m - q, \tag{5.2.23}$$

Note that the magnetic charges P_Λ are related to the rest of the parameters in order to satisfy the field equations. This configuration corresponds to the magnetic black holes constructed in [206] with different topologies for the horizon. The parameter q

allow to analyze the BPS equations in a simple way, and it was introduced through a diffeomorphism by shifting the radial coordinate.

The vanishing of the determinant of matrices entering in the gaugini variations (5.2.8) implies a relation between the parameters that can be established in a simple way in terms of the variable p_Λ related to the q_Λ as

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix} . \tag{5.2.24}$$

The supersymmetry conditions are given by

$$m = -2kp_1 + \frac{8}{L^2} p_2 p_3 p_4 \,, \tag{5.2.25}$$

$$q = -\frac{1}{4} k^2 L^2 + k(-p_1^2 + p_2^2 + p_3^2 + p_4^2) - \frac{4}{L^2}(p_3^2 p_4^2 + p_2^2 p_3^2 + p_2^2 p_4^2 - 2p_1 p_2 p_3 p_4) \,. \tag{5.2.26}$$

These are sufficient conditions to have a non-trivial Killing spinor satisfying (5.2.4). In this limit, the magnetic charges and the metric function can be expressed as follows:

$$\begin{pmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{pmatrix} = \frac{1}{2L} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} kL^2 \\ 4p_3 p_4 \\ 4p_2 p_3 \\ 4p_2 p_4 \end{pmatrix} , \tag{5.2.27}$$

$$f(r) = \frac{1}{L^2 r^2} \left(\frac{k}{2} L^2 + (r + p_1)^2 - p_2^2 - p_3^2 - p_4^2 \right)^2 , \tag{5.2.28}$$

where p_Λ are real provided:[6]

$$(P_1 + P_2 - kL)(P_1 + P_3 - kL)(P_2 + P_3 - kL) < 0 \,.$$

The metric function factorizes in a perfect square, then if there is any horizon, it will be an extremal horizon. For the BPS configuration it follows that

$$\sum_\Lambda P_\Lambda = 2kL \quad \Longleftrightarrow \quad \frac{1}{\sqrt{2}} \Gamma^M \theta_M = k \,, \tag{5.2.29}$$

[6]If this condition is not met, we need to change the last three signs in the last column of the matrix on the right-hand side of eq. (5.2.27).

which was recast in a symplectic-invariant way by using Γ^M that is the symplectic vector of the topological charges $\Gamma^M = (P^\Lambda, Q_\Lambda)$, and Q_Λ are the electric charges that in the present case are zero. The condition (5.2.29) is known as the topological twist condition, and it was shown in [215] that (5.2.29) is a necessarily condition to have a BPS static configuration with dyonic topological charges in $D = 4$, $\mathcal{N} = 2$ supergravity with vector multiples and FI terms.

The configurations satisfying the BPS conditions (5.2.25) and (5.2.26) have a well-defined Majorana Killing spinor given by

$$\psi^A(r) = \frac{f(r)^{1/4}}{H(r)^{1/8}}\frac{1}{4}(1+\gamma_1)(\delta_{AB} - \varepsilon_{AB}\gamma_{23})\psi_0^B\,, \tag{5.2.30}$$

where ψ_0^B is a constant Majorana spinor. The projector $(1+\gamma_1)(\delta_{AB} - \varepsilon_{AB}\gamma_{23})$ rule out 6 of the 8 components of the doublet ψ_0^B. Hence, the magnetic BPS black holes for any geometry of the horizon have 2 real supercharges, which corresponds to 1/4 of the total supersymmetry.

5.3 Thermodynamic analysis for planar configurations

In this section, we will analyze the thermodynamic stability of the planar black holes in both the electric and magnetic cases. In this case, the equation of state can be written analytically in terms of the horizon "area-density", given by

$$A = r_+^2\sqrt{H(r_+)}\,. \tag{5.3.1}$$

From now on, with a slight abuse of notation, we will refer to it as the horizon area. In our units the length dimension of A is 2 and the length dimension of the electric charges Q_Λ is 1.

5.3.1 Stability of electric black holes

Considering the electric configurations with arbitrary parameters given in (5.2.13), we can solve the parameter m, which up to a numerical factor corresponds to the boundary term given by the on-shell value of the Hamiltonian density [222, 223], in terms of the horizon area and the electric charges Q_Λ,

$$E \equiv m = \frac{1}{A^{1/2}} \left[\prod_{\Lambda=1}^{4} \left(\frac{A^2}{L^2} + Q_\Lambda^2 \right) \right]^{1/4} . \tag{5.3.2}$$

The energy density given in (5.3.2) reproduces the Hawking temperature, up to a numerical factor, by computing its derivative with respect to the horizon area

$$T_H = \frac{m}{4\pi A} \left[\frac{A^2}{L^2} \sum_\Lambda \left(\frac{A^2}{L^2} + Q_\Lambda^2 \right)^{-1} - 1 \right] = \frac{1}{2\pi} \frac{\partial E}{\partial A} . \tag{5.3.3}$$

For the electric black holes, the regularity of the gauge fields at the horizon fixes the chemical potentials μ_Λ in (5.2.13). This can be also reproduced by the computation of the partial derivative of the energy (5.3.2) with respect to the electric charges

$$\mu_\Lambda = \frac{Q_\Lambda}{\sqrt{2} r_+ H_\Lambda(r_+)} = \frac{m Q_\Lambda}{\sqrt{2}(\frac{A^2}{L^2} + Q_\Lambda^2)} = \frac{1}{\sqrt{2}} \frac{\partial E}{\partial Q_\Lambda} . \tag{5.3.4}$$

Then, we can construct the first law for the electric black holes

$$\frac{1}{8\pi} \delta E = T_H \delta S + \frac{1}{8\pi\sqrt{2}} \sum_\Lambda \mu_\Lambda \delta Q_\Lambda , \tag{5.3.5}$$

where the factors can be reabsorbed in the definition of the energy density E and the electric charges Q_Λ.

Having an expression for the energy density as a function of the physical charges and the entropy of the electric black holes, we can analyse the stability of the system by studying the Hessian matrix

$$\mathcal{H}_{ab} = \frac{\partial E}{\partial l^a \partial l^b} , \qquad l^a = (A/L, Q_\Lambda) . \tag{5.3.6}$$

which is a 5×5 symmetric matrix, and hence there is a limitation to find the eigenvalues in a closed form for a generic value of the parameters. However, the determinant of the Hessian can be written in a simple form in terms of the temperature as follows

$$\det \mathcal{H} = \frac{\pi L^2}{4A^3} T_H - \frac{1}{16 E^3 L^4} \sum_\Lambda Q_\Lambda^2 . \tag{5.3.7}$$

Clearly, for all the extremal black holes, i.e. $T_H = 0$, the Hessian has at least one negative eigenvalue, indicating an instability. Consequently, all the extremal electrically charged black holes in this ensemble are thermodynamically unstable. Furthermore, even above extremality, there is a finite gap for which these black holes are all unstable.

Now we will specialize in the computation for electrically charged Reissner-Nordström black hole with a planar horizon, which is obtained by setting $q_\Lambda = 0$ and defining $Q^2 = -q$. The metric function and gauge fields reduce to its standard form

$$f(r) = \frac{r^2}{L^2} - \frac{m}{r} + \frac{Q^2}{r^2}\,, \qquad A^\Lambda = \left(\frac{Q}{\sqrt{2}r} - \mu_\Lambda\right) \mathrm{d}t\,, \tag{5.3.8}$$

and the temperature of the black hole becomes

$$T_H = \frac{3A^2 - L^2Q^2}{4\pi A^{3/2} L^2}\,. \tag{5.3.9}$$

In this case, it is possible to compute the eigenvalues of the Hessian matrix (5.3.6) which are given by

$$\lambda = \frac{A^2 - L^2Q^2}{2\sqrt{A}(A^2 + L^2Q^2)}\,, \tag{5.3.10}$$

$$\lambda_\pm = \frac{1}{8A^{1/2}}\left(5 + \frac{3L^2Q^2}{A^2} \pm \sqrt{1 + 22L^2\frac{Q^2}{A^2} + 9L^4\frac{Q^4}{A^4}}\right)\,, \tag{5.3.11}$$

where λ has multiplicity three. The eigenvalues $\lambda_\pm$ are always positive for any value of $A > 0$ and $Q \in \mathbb{R}$, while the triple eigenvalue λ is positive for large black holes compared with the AdS radius and the electric charge, namely $A > L|Q|$. Whence the spinodal line, which separates the stable from the unstable region, is located at $A = L|Q|$. Consistently with our previous discussion, the extremal black holes are obtained at $A = L|Q|/\sqrt{3} < L|Q|$, namely outside the stability region.

5.3.2 Stability of magnetic planar black holes

The planar magnetic black holes presented in (5.2.21) have a well-defined BPS limit that generically represents extremal BPS black holes with a globally defined Killing spinor (5.2.30). One can notice that for the magnetic black holes (5.2.21), it is also possible to solve the integration constant m in terms of the horizon area A, defined in (5.3.1), and the magnetic charges P_Λ as

$$m = \frac{1}{A^{1/2}}\left[\prod_\Lambda \left(\frac{A^2}{L^2} + P_\Lambda^2\right)\right]^{1/4}\,. \tag{5.3.12}$$

Then, if m is identified with the energy density of the configurations, we can run the same argument that we outline for the electric black holes and conclude that the extremal BPS black holes are unstable. This is in tension with the fact that the magnetic planar BPS black holes are vacuum states of the theory, and therefore are believed to be stable. In what follows, we will show that the configurations (5.2.21) with $k = 0$ asymptote to the BPS configurations, in the sense that they admit an asymptotic Killing spinor which leads to finite conserved charges, if and only if the topological twist condition (5.2.29) is satisfied. Then, we compute the Dirac bracket between the supercharges for the asymptotic Killing spinor, showing that the quantity that we would like to identify with the energy density should vanish in the BPS limit. We will take this fact into account to propose an energy density that leads to a semi-positive defined Hessian matrix on backgrounds that satisfy the topological twist condition imposed at the beginning of the analysis.

To simplify the analysis we consider the supersymmetry transformation for the complex spinors, and denote the complex gravitino and complex gaugini as the chiral one but erasing the SU(2) index. The leading order of the gaugini equations expanded at $r \to \infty$ goes as $1/r$ and is a matrix equation that can be solved by imposing the following projector

$$\zeta_\infty = \frac{1}{2}(1 + \gamma_1)\chi_\infty \,, \tag{5.3.13}$$

then the subleading term of the complex gaugini equations read

$$\delta\lambda^i = \frac{1}{2r^2}\left[\frac{(P_i^2 - P_4^2)^2 - (P_j^2 - P_k^2)^2}{2Lm^2} + \sum_\Lambda \Omega_{i\Lambda} P_\Lambda \mathrm{i}\gamma_{23}\right]\zeta_\infty + o(r^{-3}) \,, \qquad i{\neq}j{\neq}k{\neq}4 \,, \tag{5.3.14}$$

$$\Omega_{i\Lambda} = \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix} . \tag{5.3.15}$$

It is interesting to notice that the matrix in bracket in the subleading term is invertible unless the P_Λ and m satisfy the BPS conditions; the same will happen for the subleading terms in the gravitino equations. Using the projector (5.3.13) the gravitino equations for the complex spinor in the asymptotic region are given by

$$\delta\Psi_t = \partial_t\zeta_\infty + \frac{\mathrm{i}}{8Lr}\sum_\Lambda P_\Lambda \gamma_{023}\zeta_\infty + o(r^{-2}) \,, \tag{5.3.16}$$

$$\delta\Psi_r = \partial_r\zeta_\infty - \frac{1}{2r}\zeta_\infty + o(r^{-2})\,, \tag{5.3.17}$$

$$\delta\Psi_x = \partial_x\zeta_\infty + \frac{\mathrm{i}}{8r}\sum_\Lambda P_\Lambda\gamma_3\zeta_\infty + o(r^{-2})\,, \tag{5.3.18}$$

$$\delta\Psi_y = \partial_y\zeta_\infty - \frac{\mathrm{i}}{4}x\sum_\Lambda P_\Lambda\zeta_\infty + \frac{\mathrm{i}}{8r}\sum_\Lambda P_\Lambda\gamma_2\zeta_\infty + o(r^{-2})\,. \tag{5.3.19}$$

Note that the second term in the equation (5.3.19) is leading in the expansion on r, thus, a necessarily condition on the background to have asymptotic Killing spinors is that the topological twist condition (5.2.29) must be fulfilled. Solving the leading order of the above equations equal zero, and going back to the Majorana spinor, we find that the asymptotic Killing spinor on a background that fulfills the topological twist condition is given by

$$\psi^A_\infty(r) = \frac{1}{2}r^{1/2}(1+\gamma_1)\psi^A_0 + o(r^{-1/2}) \equiv r^{1/2}\mathbb{P}_{AB}\psi^B_0\,, \tag{5.3.20}$$

where ψ^A_0 is a constant doublet of Majorana spinors. Observe that the radial dependency agrees with both the expansion at infinity and the projection $1+\gamma_1$ of the Killing spinor of the BPS background given in (5.2.30), and there are no further projections of the asymptotic Killing spinor. Therefore, the asymptotic spinor (5.3.20) has 4 independent real components which represent an enhancement of the 2 independent real components with respect to the global Killing spinor defined in the vacuum (5.2.30). Indeed we can assert that the asymptotic Killing spinor can be split into two independent spinors obtained by projecting (5.3.20) with the projector

$$\mathbb{P}^\pm_{AB} \equiv \frac{1}{4}(1+\gamma_1)(\delta_{AB} \pm \varepsilon_{AB}\gamma_{23})\,. \tag{5.3.21}$$

Computing the right-hand-side of the algebra (5.2.9) on the background (5.2.21) satisfying the topological twist condition for the asymptotic Killing spinor (5.3.20) we get the following result

$$\{\mathcal{Q},\mathcal{Q}\} = -\frac{1}{2}\int_{\partial\Sigma}\sum_\Lambda H^{1/4}\left[\left(-\frac{\mathrm{i}r^2 f^{1/2}}{H_\Lambda} + \frac{\mathrm{i}r^3}{L}H_\Lambda\right)\overline{\psi}^A_0\gamma_0\mathbb{P}_{AB}\psi^B_0 - \right. \\ \left. -\frac{rP_\Lambda}{H_\Lambda}\overline{\psi}^A_0\gamma_5\varepsilon_{AB}\mathbb{P}_{BC}\psi^C_0\right]\mathrm{d}x\wedge\mathrm{d}y\,. \tag{5.3.22}$$

Note that the last term would be combined with the first term if the asymptotic spinor would satisfy the extra projection condition that we are lacking to reproduce from the asymptotic analysis. We proceed as follows, let us consider the two independent Killing spinors obtained by projecting the asymptotic Killing spinor (5.3.20) with (5.3.21) and then computing Dirac bracket between supercharges twice, one for each independent spinor. To emphasize this fact we include a subscript in the supercharge $\mathcal{Q}_\pm$. The resulting algebra is the following

$$\{\mathcal{Q}_\pm, \mathcal{Q}_\pm\} = \frac{\mathrm{i}}{2}\int_{\partial\Sigma}\sum_\Lambda H^{1/4}\left(-\frac{r^2 f^{1/2}}{H_\Lambda} + \frac{r^3}{L}H_\Lambda \pm \frac{rP_\Lambda}{H_\Lambda}\right)\overline{\psi}_0^A\gamma^0\mathbb{P}^\pm_{AB}\psi_0^B\mathrm{d}x\wedge\mathrm{d}y\,, \tag{5.3.23}$$

$$= \mathrm{i}\mathbf{M}_\pm\overline{\psi}_0^A\gamma^0\mathbb{P}^\pm_{AB}\psi_0^B\int\mathrm{d}x\wedge\mathrm{d}y\,, \tag{5.3.24}$$

where

$$\mathbf{M}_\pm = \frac{L}{2m^3}(m^2 \pm m_{\mathrm{BPS}}^2)(2m^2 \pm m_{\mathrm{BPS}}^2)\,, \tag{5.3.25}$$

$$m_{\mathrm{BPS}}^2 \equiv \frac{1}{2\sqrt{2}L}\prod_{i=1}^{3}\left(\sum_\Lambda \Omega_{i\Lambda}P_\Lambda^2\right)^{1/2}\,, \tag{5.3.26}$$

and $\Omega_{i\Lambda}$ is defined in (5.3.15). We will show that indeed, the charges are such that $m_{\mathrm{BPS}}^2 > 0$ in the region of the phase space where the black holes exist. As expected the BPS bound computed with the spinors that asymptote to the Killing spinor of the background (5.2.30), i.e. with the lower sign, leads to a non-trivial constraint on the parameter m

$$\mathbf{M}_- > 0 \implies m > m_{\mathrm{BPS}}\,. \tag{5.3.27}$$

While the BPS bound coming from the asymptotic Killing spinor with the upper sign is trivially fulfilled. Now, we go back to the issue of the definition of the energy density in this configuration.

Note that the derivative of (5.3.12) with respect to the horizon area correctly reproduces the Hawking temperature of the black hole. To avoid spoiling the above relation, any reasonable attempt to define a new thermodynamic energy density can only differ from (5.3.12) by the addition of a function of the magnetic charges.

For configurations satisfying the topological twist condition, we propose the following definition of energy density

$$E = m(A, P_\Lambda) - m_{\mathrm{BPS}}(P_\Lambda)\,, \tag{5.3.28}$$

where $m(A, P_L)$ is given in (5.3.12) and m_{BPS} is given by (5.3.26). Under these considerations, it is straightforward to prove that the 4×4 Hessian matrix is semi-positive defined on extremal configurations.

Now we move to the discussion on the existence of black hole configuration in the extremal limit that satisfies the topological twist condition. First of all, observe that the location of the horizon in terms of the horizon area and the location of singularities in terms of the integration constant m and q are

$$r_0 = \frac{A^2}{L^2 m} - \frac{q}{m}\,, \qquad r_\Lambda^{(\mathrm{sing})} = -q_\Lambda = -\frac{P_\Lambda^2}{m} - \frac{q}{m}\,, \tag{5.3.29}$$

respectively. The black hole configuration exist if $r_0 > r_\Lambda^{(\mathrm{sing})}$ which implies that

$$\frac{1}{m}\left(\frac{A^2}{L^2} + P_\Lambda^2\right) > 0\,. \tag{5.3.30}$$

The existence of a zero of the metric function $f(r_0) = 0$ implies the existence of a horizon with horizon area $A > 0$. The equation (5.3.30) implies that if such a condition is fulfilled, a horizon automatically covers the singularities.

Extremal configurations have Hawking temperature equal to zero. The Hawking temperature for the magnetic black hole configurations is obtained by computing the derivative of the energy (5.3.28) with respect to the entropy over 8π, leading to

$$T_H = \frac{m(A, P_\Lambda)}{4\pi A}\left[\frac{A^2}{L^2}\sum_\Lambda \left(\frac{A^2}{L^2} + P_\Lambda^2\right)^{-1} - 1\right]. \tag{5.3.31}$$

Replacing the topological twist condition on (5.3.31), we find that the right-hand-side gets factorized and consequently

$$T_H = 0 \qquad \Longrightarrow \qquad \mathrm{Pol}_+(A, P_\Lambda)\mathrm{Pol}_-(A, P_\Lambda) = 0\,, \tag{5.3.32}$$

where

$$\mathrm{Pol}_\pm(A, P_\Lambda) = (2\pm 1)\frac{A^4}{L^4} + \frac{A^2}{2L^2}\sum_\Lambda P_\Lambda^2 \mp \prod_\Lambda P_\Lambda \qquad \text{satifying} \qquad \sum_\Lambda P_\Lambda = 0\,. \tag{5.3.33}$$

The greater root for A coming from $\mathrm{Pol}_-(A, P_\Lambda) = 0$, which we call A_-, correctly reproduces the horizon area of BPS black holes. While the greater root for A from $\mathrm{Pol}_+(A, P_\Lambda) = 0$, which we call A_+, corresponds to the horizon area of non-BPS extremal black holes. In the surface with $T_H = 0$ for configurations satisfying the topological twist conditions, there are three magnetic charges as remaining free parameters. The horizon areas A_+ and A_- take values on this space. It is interesting to note that in the region where $A_- > 0$ we have $A_+ < 0$ and vice versa, therefore, the location where $A_- = 0$ coincides with $A_+ = 0$. This is depicted in Figure 5.3.1.

One can interpret this result as follows. For extremal black holes, there are certain boundary conditions that allow the existence of supersymmetric magnetic black holes, and its complementary region will lead to extremal non-BPS black holes. These two essentially different boundary conditions lead to the horizon area A_- and A_+, respectively, that can be written in terms of quartic invariant quantities constructed out of the embedding tensor θ_M, the topological charges symplectic vector Γ^M and K_{MNPQ} rank-4 completely symmetric tensor of $\mathrm{Sp}(8,\mathbb{R})$. The explicit form of the horizon area for the extremal BPS black holes is[7]

$$A_-^2 = \frac{3}{2}\frac{I_2}{I_0} + \sqrt{\left(\frac{3}{2}\frac{I_2}{I_0}\right)^2 - \frac{1}{4}\frac{I_4}{I_0}}\,, \tag{5.3.34}$$

since $I_2 < 0$ then the area is real an if $I_4 < 0$. The horizon area for extremal non-BPS black holes are

$$A_+^2 = \frac{1}{2}\frac{I_2}{I_0} + \sqrt{\left(\frac{1}{2}\frac{I_2}{I_0}\right)^2 + \frac{1}{12}\frac{I_4}{I_0}}\,. \tag{5.3.35}$$

again $I_2 < 0$ then the above area is real if $I_4 > 0$. We have defined

$$I_0 = -2\mathcal{I}_1(\theta,\theta,\theta,\theta) = \frac{1}{L^4}\,, \tag{5.3.36}$$

$$I_4 = -2\mathcal{I}_1(\Gamma,\Gamma,\Gamma,\Gamma) = 4\prod_\Lambda P_\Lambda\,, \tag{5.3.37}$$

$$I_2 = -\frac{2}{3}\sum_{i=1}^{3}\mathcal{I}_i(\Gamma,\theta,\Gamma,\theta) = -\frac{1}{6L^2}\sum_\Lambda P_\Lambda^2\,, \tag{5.3.38}$$

where the last relation was obtained provided that $\sum_\Lambda P_\Lambda = 0$, see Appendix A5 for the definition of the tensor $\mathcal{I}_i$.

5.3.3 First-order description and stable extremal non-BPS solutions

The BPS solutions discussed above admit a first-order description in terms of gradient-flow equations defined by a suitable black-hole superpotential. The general form of the latter was found, in the spherical horizon case, in [215]. A general discussion of the first-order

[7]For the expression of A_- in terms of $\mathrm{SL}(2,\mathbb{R})^3$-invariants see [224]. Here we also give the analogous expression for the horizon-area A_+, corresponding to new extremal non-BPS solutions.

description of extremal solutions in the STU model was performed in [218]. Using the following standard notation for the spacetime metric:

$$ds^2 = -e^{-2U}\, dt^2 + e^{-2U}\, dr^2 + e^{2(\psi-U)}\, d^2\Omega\,, \tag{5.3.39}$$

where $d^2\Omega$ is the metric on the horizon and $U = U(r)$, $\psi = \psi(r)$, the superpotential for the BPS case can be written in the form:

$$\mathscr{W}(U,\psi,z^i,\bar{z}^{\bar{\imath}}) = e^U\,(\mathcal{Z}+i\,e^{2(\psi-U)}\mathcal{W})\,. \tag{5.3.40}$$

where $\mathcal{Z} \equiv V^T\mathbb{C}\,\Gamma$ is the $\mathcal{N}=2$ central charge and $\mathcal{W} \equiv V^T\,\theta$ is the gauge superpotential. In our solutions:

$$U(r) = \frac{1}{2}\log\left(\frac{f(r)}{H(r)^{1/2}}\right)\,, \qquad \psi(r) = \log\left(r f(r)^{1/2}\right) \tag{5.3.41}$$

The scalar fields satisfy the gradient flow equations:

$$\frac{\mathrm{d}z^i}{\mathrm{d}r} = -2\,e^{-2\psi}\,g^{i\bar{\jmath}}\partial_{\bar{\jmath}}|\mathscr{W}|\,.$$

There is a non-BPS branch of extremal solutions whose first-order description was studied in [217, 216, 218]. The fake-superpotential for the dilatonic solutions has the form

$$\mathscr{W}_{\text{non-BPS}}(U,\psi,z^i,\bar{z}^{\bar{\imath}}) = e^U\left(\frac{1}{2}\left(\mathcal{Z}+\sum_{\hat{\imath}=1}^{3}\mathcal{D}_{\hat{\imath}}\mathcal{Z}\right)+i\,e^{2(\psi-U)}\mathcal{W}\right)\,, \tag{5.3.42}$$

where:

$$\mathcal{D}_{\hat{\imath}}\mathcal{Z} \equiv e^j_{\hat{\imath}}\left(\partial_i + \frac{1}{2}\,\partial_i\mathcal{K}\right)\mathcal{Z}\,, \tag{5.3.43}$$

are the three matter charges, $e^j_{\hat{\imath}}$ being the inverse vielbein matrix, see Appendix 5.1 for the relevant definitions related to the special geometry of the model.

The expressions for the relevant quantities for these non-BPS solutions are obtained from the corresponding ones derived above for the BPS black holes, upon changing $P_4 \to -P_4$. The topological twist condition (5.2.29), for instance, for the flat-horizon case, becomes:

$$P_1 + P_2 + P_3 - P_4 = 0\,. \tag{5.3.44}$$

Just as it happened for the BPS case, the above condition implies a factorization of the expression of the temperature T_H. The horizon area now corresponds to a root A'_- given

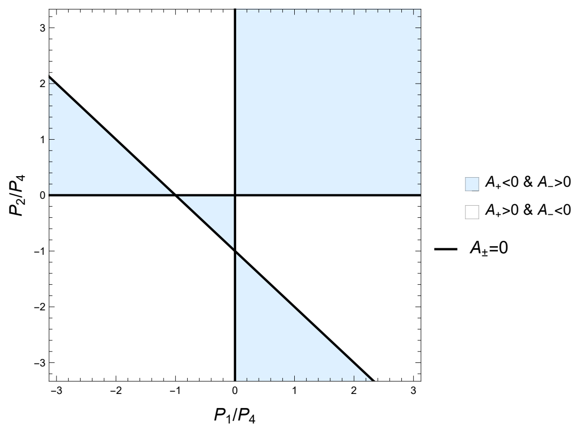

Figure 5.3.1: Considering $P_3 = -P_1 - P_2 - P_4$, we consider the plane $(P_1/P_4, P_2/P_4)$ where the coloured regions correspond to the existence of susy black holes with $A_- > 0$. We defined $A_\pm$ as the grater root for A of $\text{Pol}_\pm(A, P_\Lambda) = 0$

by that of A_- by changing $P_4 \to -P_4$:

$$A'^2_- = \frac{3}{2}\frac{I_2}{I_0} + \sqrt{\left(\frac{3}{2}\frac{I_2}{I_0}\right)^2 + \frac{1}{4}\frac{I_4}{I_0}}, \tag{5.3.45}$$

since $I_2 < 0$ then the area is real an if $I_4 > 0$. The expression of the Hessian, as a function of the charges, is the same as that of the BPS case. By the same token, we then conclude that also these non-BPS extremal solutions, described by first-order gradient flow equations, are stable. Stability seems then to be implied by the existence of a first-order description of the solution.

By the same token, we also find, for suitable values of the magnetic charges, extremal non-BPS solutions with area A'_+ whose expression is obtained from that of A_+ by changing $P_4 \to -P_4$:

$$A'^2_+ = \frac{1}{2}\frac{I_2}{I_0} + \sqrt{\left(\frac{1}{2}\frac{I_2}{I_0}\right)^2 - \frac{1}{12}\frac{I_4}{I_0}}. \tag{5.3.46}$$

Reality of A'_+ then requires $I_4 < 0$. We conclude that the condition $I_4 < 0$ does not uniquely define the BPS configuration. This is to be expected since, in the two cases, the linear conditions on the charges, (5.2.29) and (5.3.44), are different.

The stability of the magnetic configuration can be compared with perturbative stability analysis existent in the literature [225, 226] where the authors studied the perturbative stability of magnetic configurations in the context of the purely dilatonic sector of the

STU model. In particular, they studied the stability of magnetically charged $AdS_2 \times \mathbb{R}^2$ geometry, which corresponds to the near horizon region of the extremal magnetically charged black holes considered here.
Considering the extremal limit of the magnetically charged black hole solutions in (5.2.21)-(5.2.23), we can find analytically an expression for the value of the scalars at the horizon

$$\phi_i(r_+) = -\log\left[\frac{m^2 A}{(P_i^2 + A^2/L^2)(P_4^2 + A^2/L^2)}\right], \qquad i = 1, 2, 3. \tag{5.3.47}$$

In [226] they studied a perturbation of the STU model of $AdS_2 \times \mathbb{R}^2$ background in the T^3-truncation of the STU model, and gave an analytic expression for the eigenvalues of the mass matrix. These backgrounds can be obtained from our black holes by taking the extremal limit and then the near horizon geometry for $\pm P_1 = \pm P_2 = \pm P_3 \equiv \mathcal{P}$, for any choice of the relative signs. The smallest eigenvalue of the mass matrix of the perturbation is given by [226][8]

$$L^2_{AdS_2} m^2_{\min} = -\frac{1}{48}\frac{(5 + 3\overline{X}^4)^2}{2 + 3\overline{X}^4 + \overline{X}^8}, \tag{5.3.48}$$

$$\overline{X} \equiv \frac{m A^{1/2}}{(\mathcal{P}^2 + A^2/L^2)^{1/2}(P_4^2 + A^2/L^2)^{1/2}}, \tag{5.3.49}$$

where L_{AdS_2} is the AdS radius of the AdS_2 near the horizon and $\overline{X} = Y_1^{-1/2} = Y_2^{-1/2} = Y_3^{-1/2} = Y_4^{1/6}$ evaluated at the horizon. The BF bound on the AdS_2 is violated when $\overline{X} < (-1 + \frac{2}{\sqrt{3}})^{1/4}$. We found that for the BPS configuration $\overline{X}$ saturates the bound $\overline{X}|_{BPS} = (-1 + \frac{2}{\sqrt{3}})^{1/4}$ in agreement with [226], while for the non-BPS configuration $\overline{X}|_{non-BPS} = 1 > (-1+\frac{2}{\sqrt{3}})^{1/4}$, which is above the BF bound indicating that the instability is not triggered by the particular perturbation considered in [226]. For the non-BPS, thermodynamically stable configurations, we found that they also saturate the BF bound, as they are obtained from the BPS configurations by mapping $P_4 \to -P_4$, and the value of the scalars at the horizons depends on the P_Λ^2.

5.4 Last comments

In this work, we presented a simple argument to prove the thermodynamic instability of extremal planar 4-charges electric black holes in the STU model of the maximal theory in $D = 4$. This result constitutes a generalization of the instabilities of electric black

[8]The exact mapping between conventions is the following: $\phi_i^{\text{here}} = -\phi_i^{\text{there}}$, $A_\Lambda^{\text{here}} = 2A_\Lambda^{\text{there}}$, and $L = 1/\sqrt{2}$.

holes studied in [212]. The argument can be made sharper to conclude that there is a finite critical lower temperature for which electrically charged black holes are unstable, indicating that the mechanics that trigger the instability do not rely on particular features of extremal black holes. We note that indeed it is well known that extremal black holes are unstable under charged perturbations [227]. However, the phenomenon presented here is different because it does not require extremality and occurs at fixed charges. It seems, in other words, to somehow capture the non-linearity of gauged $\mathcal{N} = 8$ supergravity.

It would be interesting to investigate which kind of instability is at work. There are at least two candidates: a Gregory-Laflamme type of instability, due to the presence of flat directions; a superradiance effect that could appear in rotating black holes, as in our case, from the point of view of 11D supergravity, the system corresponds to rotating M2-branes along the $U(1)^4$ isometries of the S^7.

We also considered 4-charge magnetic black hole configurations with different horizon topologies constructed in [206, 195], and presented them in a slightly different form that allows us to prove that they do have a regular black hole BPS limit. This result shows that, when the embedding in the maximal theory exists, the non-extremal configurations which generalize [209], correspond to the black holes discussed in [206, 195] and, in the spherical case, they coincide with the non-extremal black holes presented in [221].

Within the context of extremal magnetically charged black holes with flat horizon geometry, we identified three families of extremal black holes classified by their boundary conditions. One of them corresponds to the family of extremal BPS black holes, whose fields can be written as a solution of a first-order system of equations controlled by a superpotential [215], and the quartic invariant in the charges turns out to be negative. The thermodynamic stability analysis of the Hessian matrix imposing the topological twist condition ab initio indicates these configurations are metastable. The remaining two families of black holes are extremal non-BPS and they differ in the sign of the quartic invariant; this suggests that the sign of the quartic invariant is not a sufficient condition to identify a black hole configuration as supersymmetric or not. One of these families is obtained from the extremal BPS family by flipping the sign of the magnetic charge of the graviphoton, and thus satisfies a topological twist condition with one sign flipped. This class is also described by a first-order system controlled by a fake-superpotential [217]. We assert thermodynamic quasi-stability for the black holes belonging to this family, and the perturbative stability analysis carried out in [226] shows that the mass of the perturbed scalars saturate the BF bound in the AdS_2. The last family corresponds to extremal non-BPS configurations that exist for certain choices of boundary conditions in a complementary region in the space of the magnetic charges where the two latter families exist. These black holes are not described by a first-order system and its Hessian has generically negative eigenvalues indicating thermodynamic instability. Applying the analysis in [226] to these backgrounds leads to scalars whose mass squared is above the BF bound of the AdS_2, indicating that the instability is not triggered by the perturbation considered in [226]. A more exhaustive perturbative analysis is required to explore this further. In general, all these instabilities open the question of what is the phase diagram relevant for supergravity and its dual field theory, something that we expect to explore in the future along the lines of [228].

Chapter 6

Summary and future directions

This thesis was mainly based on two facts coming from superstring theory. The first one is supersymmetry, a consequence of the framework of the theory. The second is a concrete prediction of the theory at low energies, which corresponds to the inclusion of higher curvature terms. Supersymmetry is a crucial ingredient of the formulation of the theory that allows for having sensible properties, such as identifying a vacuum state on the phase space of the theory by requiring that the configuration is supersymmetric. The prediction of higher curvature terms at low energies in lower dimensions is a very welcome fact as general relativity is a non-renormalizable theory and it is well accepted that higher curvature terms do improve the UV behavior of the theory.

In Chapter 1, we began the discussion by presenting general facts about differential geometry and General Relativity. In particular, we present in a detailed way the computations of charges following the Iyer-Wald method.

In Chapter 2, we study black hole physics in $D = 4$ by considering Einstein-dilaton-Gauss-Bonnet theory. This theory is obtained by performing a field redefinition of the α'-correction of the heterotic string theory in $D = 10$ and then making an analytic continuation on the spacetime dimension to arbitrary D. The correction of heterotic string in the lagrangian is proportional to $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ leading to four order equations of motion, by performing the field redefintion it is possible to map this theory, in string frame, to Einstein-dilaton-Gauss-Bonnet up to quadratic terms in α'. We construct the first correction to the Schwarzschild black hole in arbitrary dimensions. The metric functions are given by a Gaussian hypergeometric function. For the $D = 4$ case, we construct the conserved charges by considering the appropriate generalization of the Gibbons-Hawking-

York term for Gauss-Bonnet. The charges were finite and they satisfy the Smarr formula at first order in α'. Then, we construct a slowly rotating black hole starting from the stationary one by solving the equations, neglecting terms α'^2, $a\,\alpha'$, and compute the angular momentum. The only correction of order αa was in the off-diagonal term. This configuration was studied later by simulating its black hole shadow in [229]. After that, we constructed the α'-correction to the C-metric. In this case, for an arbitrary value of the accelerating parameter A, the resulting system of differential equations is a PDE for the first correction of the scalar field. Unfortunately, we did not manage to solve it analytically, but a perturbative solution in the acceleration parameter was provided.

In general, higher curvature theories have certain pathologies, such as a degeneracy of the vacuum, modes with unbounded energy from below, and higher derivative equations of motion, among others. However, string theory is a healthy theory regarding these issues. The evident tension is that if you take the first non-trivial α'-correction, e.g., in the Heterotic or type II theories, one lands in a higher curvature theory carrying the above-mentioned pathologies. A proposal to resolve this tension is to consider field redefinitions order by order in α', as we did in Chapter 2, where the higher curvature term is governed by the Gauss-Bonnet density belonging to the Lovelock family. However, it is possible to show that the quartic correction to type II theories cannot be mapped via field redefinition to the corresponding Lovelock density. In the last fifteen years, a new family of higher curvature theories, known as Quasi-topological families, defined by requiring the existence of a Birkhoff's theorem for generic values of the couplings, was constructed. They have second-order equations of motion around a spherically symmetric background, and their equations in this background are polynomial in the metric function [111, 230, 231, 232]. After that, a family of theories known as generalized quasi-topological gravities was proposed that inherits the properties of having second-order field equations around a spherically symmetric background, but the equations are no longer polynomial in the metric function. A remarkable example in $D = 4$ is Einsteinian cubic gravity [114]. Recently, it was shown in [233] that, considering type IIB reduced on a 5-sphere with its quartic correction, there exists a field redefinition in such a way that the quartic term can be recast as a series of QT terms. As a future direction, it would be interesting to implement this technique to prove that the statement holds for any 10-dimensional manifold, not only containing a 5-sphere factor.

In Chapter 3 we consider $D = 5$, α'-corrections to the black string with dilaton equal to zero. The correction was known since [154], we extended the analysis by studying the thermodynamics using Iyer-Wald charges and studying the perturbations around $\alpha/r_+^2 \ll 1$. For the latter, we study the scalar perturbation that in GR triggers the Gregory-Laflamme instability, showing that the instability window gets enlarged linearly with the perturbative coupling α'. At the beginning of Chapter 4, we discuss the semi-classical instability of the Kaluza-Klein vacuum proven by Witten in [180]. The black string configurations are asymptotically the Kaluza-Klein vacuum; hence, it would be interesting to understand the relation, if any, between the semi-classical instability of the boundary metric and the Gregory-Laflamme instability occurring at the black string horizon.

Then, we moved to a supergravity context where there exists a sharp way to assess the stability of a background by studying the presence of Killing spinors. If such spinors exist, then the energy of the background is bounded from below by the central charges as the supersymmetry algebra dictates. If there are no Killing spinors, stability is not warranted. In Chapter 4, we provide a new configuration asymptotically locally AdS_5 with a contractible cycle in the bulk. The dual interpretation corresponds to a deformation of the Coulomb branch of $\mathcal{N}=4$ SYM on $\mathbb{R}^{1,2}\times S^1$. The configuration that we presented is general enough, namely, the boundary values of the gauge fields at infinity are non-trivial, introducing a current in the dual theory, which allows us to write sections of the phase diagram and distinguish two branches of solutions for generic values of the boundary values of the gauge fields. We provide the uplift to type IIB, whose non-trivial field content is the self-dual 5-form and the metric. The configuration becomes BPS for a certain relation between the boundary values of the gauge fields, with a precise relation between their signs. While, from the point of view of the 10-dimensional field theory, one can see that there is no relation between the signs whatsoever. This is a consequence of the fact that in $D=5$ we are dealing with a $\mathcal{N}=2$ gauged supergravity, and as we explained for $D=4$ STU model in Section 5.1, there is no unique way to embed them in the maximal theory.

In Chapter 5 we construct the STU model in maximal SO(8)-gauged supergravity in $D=4$ and study the purely dilatonic sector of the theory. In this sector, we write the static magnetic and electric black hole configurations for arbitrary geometry of the 2-dimensional horizon constructed in [206, 195] in a slightly different way. We carry out the thermodynamic stability analysis for the electric black holes. For them, the BPS limit leads to naked singularities for spherical and planar horizons. For hyperbolic horizons, the configurations are not real. In the planar case, we obtained an equation of state that related the energy, the entropy, and the electric charges. Using this relation, we show that if the temperature of the black hole is below a certain value fixed in terms of the electric charges and the entropy, then the configuration is thermodynamically unstable. This instability also happens for the well-known Reissner-Nordström black hole. It would be interesting to understand the origin of these instabilities.

For the magnetic black holes, we prove that the BPS limit of magnetic configurations allows BPS states with a horizon protecting the curvature singularity, making the Duff and Liu magnetic black holes with arbitrary horizon geometry [206, 195] equivalent to those presented by Cacciatori and Klemm [209] ten years later. We perform a thermodynamic analysis for the planar configurations. We include an extra term in the definition of the thermodynamic energy in such a way that its Hessian matrix leads to quasi-stable configurations for the magnetic BPS black holes. Finally, we showed that extremal black holes do not imply BPS black holes, and they can be differentiated by looking at the $SL(2, \mathbb{R})^3$ invariants, which were constructed in Appendix A5. We assess the stability of the extremal non-BPS black holes by computing the eigenvalues of the Hessian matrix, and they have negative eigenvalues. By using the results provided in [226, 225] regarding the mass matrix for perturbations on $AdS_2 \times \mathbb{R}^2$ background in the STU model, we prove that at least they are not triggered by the near-horizon values of the scalar field in the extremal case. For the extremal non-BPS black holes, the values of the scalars at the horizon are above the BF bound. While for the extremal BPS black holes, the scalars precisely saturate the BF bound. This raises the question whether this is always the case: do all BPS extremal black holes have scalars saturating the BF bound at the horizon? It would be interesting in the future to go back to this problem and give a more general statement.

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A1 Integrability conditions of type IIB in the metric$-\Phi-\mathsf{F}_5$ sector

In this appendix we compute the integrability condition for IIB in the metric-F_5 sector. The supersymmetry transformations of the spin 3/2 field is

$$\delta\psi_\mu dx^\mu = d\epsilon + W\epsilon \equiv D\epsilon \ , \tag{A1.1}$$

where for our field content

$$W = \frac{1}{4}\omega_{ab}\Gamma^{ab} + \frac{1}{16}i\sigma_2 \not{F}_5 \Gamma_a e^a \ . \tag{A1.2}$$

We define integrability conditions 2$-$form as the commutator of the covariant derivative defined in (A1.1)

$$\Xi \equiv D \wedge D\epsilon \ . \tag{A1.3}$$

It is simple to show that

$$\Xi = dW + W \wedge W \ . \tag{A1.4}$$

Let us compute it term by term. The exterior derivative of W is

$$dW = \frac{1}{4}d\omega_{ab}\Gamma^{ab} + \frac{1}{16}i\sigma_2 d\not{F}_5 \Gamma_a e^a + \frac{1}{16}i\sigma_2 \not{F}_5 \Gamma_a de^a \ . \tag{A1.5}$$

Using the torsion-less condition $de^a + \omega^a{}_b \wedge e^b = 0$ and the definition of curvature 2$-$form $R^a{}_b = \omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$ we obtain

$$dW = \frac{1}{4}R_{ab}\Gamma^{ab} - \frac{1}{4}\omega_{ac} \wedge \omega^c{}_b\Gamma^{ab} + \frac{1}{16}i\sigma_2 d\not{F}_5 \Gamma_a e^a - \frac{1}{16}i\sigma_2 \not{F}_5 \Gamma_a \omega^a{}_c \wedge e^c \ . \tag{A1.6}$$

Note that in general one can write $W = W_A \otimes \Gamma^A$ where W_A is the tensor product of the 1-form space and 2×2 matrices, in general we suppress the tensor product symbol. The repeated indices A are sumed over all terms which defines W and encodes the index structure of the Γ matrices in each term. Using this, we have

$$W \wedge W = \frac{1}{2}W_A \wedge W_B \left[\Gamma^A, \Gamma^B\right] \ . \tag{A1.7}$$

A general identity of the Γ matrices that we will use are

$$\Gamma^{a_1\ldots a_p}\Gamma_{bc} = \Gamma^{a_1\ldots a_p}{}_{bc} - 2p\Gamma^{[a_1\ldots a_{p-1}}{}_{[b}\delta^{a_p]}_{c]} - \frac{p!}{(p-2)!}\Gamma^{[a_1\ldots a_{p-2}}\delta^{a_{p-1}}_{[b}\delta^{a_p]}_{c]}\ , \qquad (A1.8)$$

$$\Gamma_{bc}\Gamma^{a_1\ldots a_p} = \Gamma_{ab}{}^{a_1\ldots a_p} - 2p\delta^{[a_1}_{[b}\Gamma^{a_2\ldots a_p]}_{c]} - \frac{p!}{(p-2)!}\delta^{[a_1}_{[b}\delta^{a_2}_{c]}\Gamma^{a_3\ldots a_p]}\ . \qquad (A1.9)$$

In particular, we can derive from it

$$\left[\Gamma^{ab},\Gamma_c\right] = 4\Gamma^{[a}\delta^{b]}_c\ , \qquad \left[\Gamma^{ab},\Gamma_{cd}\right] = 8\delta^{[c}_{[a}\Gamma^{d]}{}_{b]}\ ,$$
$$[\Gamma^{a_1a_2a_3a_4a_5},\Gamma_{bc}] = -20\Gamma_{[b}{}^{[a_1a_2a_3a_4}\delta^{a_5]}_{c]}\ . \qquad (A1.10)$$

Replacing (A1.2) into (A1.7) we get

$$W\wedge W = \frac{1}{2}\frac{1}{4}\omega_{ab}\wedge\frac{1}{4}\omega_{cd}\left[\Gamma^{ab},\Gamma^{cd}\right] + \frac{1}{8^2}i\sigma_2\omega_{ab}\wedge e^c\frac{1}{5!}F_{d_1\ldots d_5}\left[\Gamma^{ab},\Gamma^{d_1\ldots d_5}\Gamma_c\right]$$
$$-\frac{1}{8^3}\left[\not{F}_5\Gamma_a,\not{F}_5\Gamma_c\right]e^a\wedge e^c\ . \qquad (A1.11)$$

Using the commutator relations, we obtain

$$W\wedge W = \frac{1}{4}\omega_{ac}\wedge\omega^c{}_b\Gamma^{ab} + \frac{1}{2}\frac{1}{8}i\sigma_2\omega^a{}_b\wedge e^b\not{F}_5\Gamma_a - \frac{1}{2}\frac{1}{8}i\sigma_2\omega^{ab}\wedge e^c\frac{1}{4!}F_{d_1\ldots d_4b}\Gamma^{d_1\ldots d_4}{}_a\Gamma_c$$
$$-\frac{1}{8^3}\left[\not{F}_5\Gamma_a,\not{F}_5\Gamma_c\right]e^a\wedge e^c\ . \qquad (A1.12)$$

Replacing (A1.6) and (A1.7) into (A1.4), the 2$-$form integrability conditions become

$$\Xi = \frac{1}{4}R_{ab}\Gamma^{ab} + \frac{1}{16}i\sigma_2\frac{1}{5!}dF_{b_1\ldots b_5}\Gamma^{b_1\ldots b_5}\Gamma_a e^a - \frac{1}{2}\frac{1}{8}i\sigma_2\omega^{ab}\wedge e^c\frac{1}{4!}F_{d_1\ldots d_4b}\Gamma^{d_1\ldots d_4}{}_a\Gamma_c$$
$$-\frac{1}{8^3}\left[\not{F}_5\Gamma_a,\not{F}_5\Gamma_c\right]e^a\wedge e^c\ . \qquad (A1.13)$$

Note that the second and third term form a Lorentz covariant derivative

$$\Xi = \frac{1}{4}R_{ab}\Gamma^{ab} + \frac{1}{16}i\sigma_2\frac{1}{5!}\mathcal{D}F_{b_1\ldots b_5}\Gamma^{b_1\ldots b_5}\Gamma_a e^a - \frac{1}{8^3}\left[\not{F}_5\Gamma_a,\not{F}_5\Gamma_c\right]e^a\wedge e^c\ . \qquad (A1.14)$$

The last term can be simplified by using $\left[\Gamma^{d_1\ldots d_5},\Gamma_a\right] = 2\Gamma^{d_1\ldots d_5}{}_a$, then

$$\left[\not{F}_5\Gamma_a,\not{F}_5\Gamma_c\right]e^a\wedge e^c = 4\not{F}_5\frac{1}{4!}F_{ab_1\ldots b_4}\Gamma^{b_1\ldots b_4}\Gamma_c e^a\wedge e^c - 2\not{F}_5\not{F}_5\Gamma_{ac}e^a\wedge e^c\ . \qquad (A1.15)$$

The last term of (A1.15) vanishes due to the fact that F_5 is self-dual,

$$\begin{aligned}(5!)^2 \not{F}_5 \not{F}_5 &= F_{a_1\ldots a_5} F_{b_1\ldots b_5} \Gamma^{a_1\ldots a_5}\Gamma^{b_1\ldots b_5} \ , \\ &\sim F^{d_1\ldots d_5}\epsilon_{a_1\ldots a_5 d_1\ldots d_5} F_{c_1\ldots c_5}\epsilon^{c_1\ldots c_5 b_1\ldots b_5}\Gamma^{a_1\ldots a_5}\Gamma_{b_1\ldots b_5} \ , \\ &= F^{d_1\ldots d_5} F_{c_1\ldots c_5}\delta^{c_1\ldots c_5 b_1\ldots b_5}_{a_1\ldots a_5 d_1\ldots d_5}\Gamma^{a_1\ldots a_5}\Gamma_{b_1\ldots b_5} \ , \\ &= F^{d_1\ldots d_5} F^{a_1\ldots a_5}\delta^{a_1\ldots a_5 b_1\ldots b_5}_{c_1\ldots c_5 d_1\ldots d_5}\Gamma_{a_1\ldots a_5}\Gamma_{b_1\ldots b_5} \ .\end{aligned} \tag{A1.16}$$

Now we can anti-symmetrize and construct a $\Gamma_{a_1\ldots a_{10}}$, and then use the fact that it is proportional to $\epsilon_{a_1\ldots a_{10}}\Gamma_{11}$,

$$\begin{aligned}(5!)^2 \not{F}_5 \not{F}_5 &= F^{d_1\ldots d_5} F^{a_1\ldots a_5}\delta^{a_1\ldots a_5 b_1\ldots b_5}_{c_1\ldots c_5 d_1\ldots d_5}\Gamma_{a_1\ldots a_5 b_1\ldots b_5} \ , \\ &\sim F^{d_1\ldots d_5} F^{a_1\ldots a_5}\Gamma_{d_1\ldots d_5 a_1\ldots a_5} \ , \\ &\sim F^{d_1\ldots d_5} F^{a_1\ldots a_5}\epsilon_{d_1\ldots d_5 a_1\ldots a_5}\Gamma_{11} \ .\end{aligned} \tag{A1.17}$$

Note that the last line vanishes since it is equal to minus itself. Replacing everything into (A1.14), we get the final form of the integrability conditions

$$\Xi = \frac{1}{4}R_{ab}\Gamma^{ab} + \frac{1}{16}\frac{1}{5!}i\sigma_2 \mathcal{D}F_{b_1\ldots b_5}\Gamma^{b_1\ldots b_5}\Gamma_a e^a - \frac{1}{128}\frac{1}{4!}\not{F}_5 F_{ab_1\ldots b_4}\Gamma^{b_1\ldots b_4}\Gamma_c e^a\wedge e^c \ . \tag{A1.18}$$

A2 Rotating D3-branes interpretation?

Regarding the configurations presented in chapter 4. We already saw that the 10-dimensional solution (4.4.6) is understood as a deformation of a solution described by a continuous distribution of D3-branes.

But we know [234] that an extremal RNAdS solution (double Wick rotation of the RNAdS soliton), with constant scalars $X^i = X$ =constant and equal gauge fields $A^i = A$ can be obtained as a limit from the 10-dimensional solution with angular momenta l_i, $i = 1, 2, 3$ in 3 different (non-intersecting) planes,

$$\begin{aligned}ds^2 = H^{-1/2}\left[-\left(1-\frac{2m}{r^4\Delta}\right)dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right] + H^{1/2}\Bigg[\frac{\Delta dr^2}{H_1H_2H_3 - 2m/r^4} \\ + r^2\sum_{i=1}^{3}H_i(d\mu_i^2 + \mu_i^2 d\phi_i^2) - \frac{4m\cosh\alpha}{r^4 H\Delta}dt\sum_{i=1}^{3}\ell_i\mu_i^2 d\phi_i + \frac{2m}{r^4 H\Delta}\left(\sum_{i=1}^{3}\ell_i\mu_i^2 d\phi_i\right)^2\Bigg]\end{aligned} \tag{A2.1}$$

where

$$\Delta = H_1H_2H_3\sum_{i=1}^{3}\frac{\mu_i^2}{H_i};\quad H = 1+\frac{2m\sinh^2\alpha}{r^4\Delta};\quad H_i = 1+\frac{\ell_i^2}{r^2}. \tag{A2.2}$$

So it is a reasonable question whether the current solution (4.4.6) cannot be obtained by a similar limit from the same.

At first, things seem plausible. With

$$\mu_1 = \cos\theta\sin\psi\ ,\quad \mu_2 = \cos\theta\cos\psi\ ,\quad \mu_1 = \sin\theta\ , \tag{A2.3}$$

and the rescaling (similar to, and inspired by the one in [234])

$$\begin{aligned} m &= \varepsilon^4 m'\ ,\ \ \sinh\alpha = \varepsilon^{-2}\sinh\alpha'\ ,\quad \ell_{1,2} = \varepsilon^2\tilde{\ell}'\ ,\quad \ell_3 = \varepsilon\ell'\ ,\\ r &= \varepsilon r'\ ,\ \ x^\mu = \varepsilon^{-1}x'^\mu\ , \end{aligned} \tag{A2.4}$$

followed by $\varepsilon \to 0$ and dropping the primes, one obtains

$$H_1 = H_2 = 1\ ,\ \ H_3 = 1+\frac{\ell^2}{r^2} = \lambda^6\Big|_{\epsilon=+1}\ ,\quad \Delta = 1+\frac{\ell^2}{r^2}\cos\theta = \zeta^2\Big|_{\epsilon=+1}\ , \tag{A2.5}$$

and so the coefficient of $d\vec{x}_{1,2}^2$ matches,

$$H^{-1/2}d\vec{x}_{1,2} \to \left(\frac{2m\sinh^2\alpha}{r^4\zeta^2}\right)^{-1/2} = \frac{\zeta r^2}{L^2}\ ,\quad L^4 \equiv 2m\sinh^2\alpha > 0\ , \tag{A2.6}$$

and one finds also matching for the coefficients of $d\phi_1^2, d\phi_2^2, d\phi_3^2$, which are $H^{1/2}r^2H_i\mu_i^2$ (note that $\frac{2m}{r^4H\Delta}\ell_i^2 \sim \varepsilon^6$ is subleading in ε with respect to $r^2H_i^2 \sim \varepsilon^2$, so is dropped), and of $\sum_i H_i d\mu_i^2 = \zeta^2 d\theta + \cos^2\theta d\psi^2$, which is $r^2H^{1/2} = L^2/\zeta$.

The problem comes in the interpretation of the terms with A_i and $d\phi$, and of the dr^2 term. Matching of the dr^2 coefficient results in the equation

$$2m = \ell^2L^2\left[q_1^2\left(1+\frac{\ell^2}{r^2}\right) - q_2^2\right] \Rightarrow \frac{\ell^2}{L^2}(q_1^2 - q_2^2) \simeq \frac{1}{\sinh^2\alpha}\quad \text{for}\ \ r \gg \ell\ , \tag{A2.7}$$

which could only be satisfied approximately, for $r \gg \ell$ and $q_2 < q_1$, due to the $1/r^2$ term (note that $q_1 = 0$ does not work, since it implies $m < 0$).

Matching of the terms with $A_i d\phi_i$, after the double Wick rotation, replacing dt from the rotating D3-brane solution with the $d\phi$ from the soliton solution, is only possible in some approximate sense as well, but now also with $r - r_0 \sim \varepsilon$ or $\sim \varepsilon^2$ fixed, since in the soliton $d\phi_i A_i$ is proportional to $q_1 \frac{\ell^2}{L}\frac{r^2-r_0^2}{r_0^2}$ or $q_2 \frac{\ell^2}{L}\frac{r^2-r_0^2}{r_0^2+4\ell^2}$, while the former has (at least) an extra power of ε, and so is proportional to $(\varepsilon^2\tilde{\ell})4m\cosh\alpha \simeq (\varepsilon^2\tilde{\ell})2L^4/\sinh\alpha$ or $(\varepsilon\ell)4m\cosh\alpha \simeq (\varepsilon\ell)2L^4/\sinh\alpha$, respectively, so one would have to consider some unusual simultaneous near-horizon limit, depending on the charge.

Moreover then, the coefficient of the $d\phi^2$ term, composed of $(\zeta r^2/L^2)F(r)L^2 = H^{-1/2}F(r)L^2$ and the $H_i\mu_i^2 A_i^2$ terms, would have to match $H^{-1/2}\left(1-2m/r^4\Delta\right) = H^{-1/2}\left(1-2m/(r^4\zeta^2)\right)$, which depends on the previous near-horizon limit.

In conclusion, the deformation found constructed here is a nontrivial deformation of the rotating D3-brane solution, that is not easily understandable within the same context, except maybe in some generalized

A3 Relation to other symplectic frames and spinor conventions

Let us now give the explicit relation between the symplectic frame used in the present work and other frames commonly used in the literature.

Frame 1. The first is the symplectic frame, which we shall refer to as *cubic frame*, in which the prepotential function $\tilde{F}(\tilde{X})$ has the following cubic form:

$$\tilde{F}(\tilde{X}) = \frac{d_{ijk}}{3!}\frac{\tilde{X}^i\tilde{X}^j\tilde{X}^k}{\tilde{X}^0} = -\frac{\tilde{X}^1\tilde{X}^2\tilde{X}^3}{\tilde{X}^0}\,. \tag{A3.1}$$

The corresponding holomorphic section $\tilde{\Omega}(z) = (\tilde{X}^\Lambda, \partial\tilde{F}/\partial\tilde{X}^\Lambda)$ reads, modulo multiplication by a non-vanishing holomorphic function:

$$\tilde{\Omega}(z) = (1, z_1, z_2, z_3, z_1z_2z_3, -z_2z_3, -z_1z_3, -z_1z_2)\,, \tag{A3.2}$$

where

$$z_i = \frac{\tilde{X}^i}{\tilde{X}^0} = \chi_i + i\,e^{-\phi_i}\ ,\ \ i = 1,2,3\,.$$

The two frames are related by the following symplectic matrix $\mathbf{E} = (E^M{}_N)$:

$$\Omega(z) = \mathbf{E}\cdot\tilde{\Omega}(z)\ , \quad \mathbf{E} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}. \tag{A3.3}$$

The charges $\Gamma^M = (P^\Lambda,\, Q_\Lambda)$ in our frame are then related to those $\tilde{\Gamma}^M = (\mathfrak{p}^\Lambda,\, \mathfrak{q}_\Lambda)$ as follows:

$$\Gamma^M = (-\mathfrak{q}_1, -\mathfrak{q}_2, -\mathfrak{q}_3, -\mathfrak{p}^0, \mathfrak{p}^1, \mathfrak{p}^2, \mathfrak{p}^3, -\mathfrak{q}_0)\,. \tag{A3.4}$$

The quartic invariant, see Appendix A5, in the cubic frame, reads:

$$I_4(\tilde{\Gamma}) = -(\mathfrak{p}^\Lambda\mathfrak{q}_\Lambda)^2 - 4\,\mathfrak{q}_0\,\mathfrak{p}^1\,\mathfrak{p}^2\,\mathfrak{p}^3 + 4\,\mathfrak{p}^0\,\mathfrak{q}_1\,\mathfrak{q}_2\,\mathfrak{q}_3 + 4\left(\sum_{i<j}\mathfrak{p}^i\mathfrak{q}_i\mathfrak{p}^j\mathfrak{q}_j\right). \tag{A3.5}$$

In light of eq. (A3.4), we can write the quartic invariant in our frame, in the magnetic case $Q_\Lambda = 0$, in terms of the charges in the cubic one as follows

$$I_4(\Gamma) = 4P^1P^2P^3P^4 = 4\mathfrak{p}^0\,\mathfrak{q}_1\,\mathfrak{q}_2\,\mathfrak{q}_3\,.$$

Frame 2. The second symplectic frame is the one which naturally arises from direct truncation of the SO(8) gauged maximal theory. It is a special coordinate frame with a prepotential function $\hat{F}(\hat{X})$ of the form:

$$\hat{F}(\hat{X}) = -2\sqrt{\hat{X}^0\hat{X}^1\hat{X}^2\hat{X}^3}\,. \tag{A3.6}$$

The holomorphic section $\hat{\Omega} = (\hat{X}^\Lambda,\, \partial\hat{F}/\partial\hat{X}^\Lambda)$, modulo multiplication by a non-vanishing holomorphic function, can be written in the form:

$$\hat{\Omega}(z) = (z_1z_2z_3, z_1, z_2, z_3, -1, -z_2z_3, -z_1z_3, -z_1z_2)\,, \tag{A3.7}$$

where:

$$z_i = \sqrt{\frac{\hat{X}^0 \hat{X}^i}{\hat{X}^j \hat{X}^k}}\ , \quad i{\neq}j{\neq}k{\neq}i\,. \tag{A3.8}$$

The symplectic transformation relating this frame with the cubic one is straightforward:

$$\hat{X}^0 = \frac{\partial \tilde{F}}{\partial \tilde{X}^0}\,,\ \hat{X}^i = \tilde{X}^i\,,\ \frac{\partial \hat{F}}{\partial \hat{X}^0} = -\tilde{X}^0\,,\ \frac{\partial \hat{F}}{\partial \hat{X}^i} = \frac{\partial \tilde{F}}{\partial \tilde{X}^i}\,.$$

We use the Majorana basis for the Clifford algebra

$$\gamma^0 = -i\begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}\,, \quad \gamma^1 = -\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}\,, \quad \gamma^2 = i\begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix}\,, \quad \gamma^3 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}\,. \tag{A3.9}$$

The charge conjugation matrix and γ^5 matrix are given by

$$C = \gamma_0\,, \qquad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3\,. \tag{A3.10}$$

We use $\mathcal{N} = 2$ chiral supersymmetry parameters ϵ^A, ϵ_A with $A = 1, 2$, satisfying

$$\gamma^5\epsilon^A = -\epsilon^A\,, \qquad \gamma^5\epsilon_A = \epsilon_A\,, \tag{A3.11}$$

that are defined as the chiral components of doublet of Majorana spinors $\psi^A = \epsilon^A + \epsilon_A$. The relation between the chiral spinors is $\epsilon_A = (\epsilon^A)^*$. It is useful to define the complex spinors

$$\zeta = \psi^1 + \mathrm{i}\psi^2 \tag{A3.12}$$

that allows one to write down a set of differential equations. The general rule to go from an equation with real coefficients for the Majorana spinor ψ^A to an equation for the complex spinor ζ is by replacing $\delta_{AB} \to 1$ and $\varepsilon_{AB} \to -\mathrm{i}$, and vice-versa.

A4 Dirac bracket between supercharges

The author of [214] showed that the Dirac bracket between the supercharges is expressed as

$$\{\mathcal{Q}, \mathcal{Q}\} = \int_{\partial\mathcal{M}} \mathrm{d}\Sigma_{\mu\nu}\epsilon^{\mu\nu\rho\sigma}\left(\bar{\epsilon}^A\gamma_\rho\mathfrak{D}_{A\sigma}(\epsilon) - \bar{\epsilon}_A\gamma_\rho\mathfrak{D}^A{}_\sigma(\epsilon)\right)\,, \tag{A4.1}$$

where $\mathfrak{D}^{A}{}_{\sigma}(\epsilon)$ is generically defined by the variation of the gravitino Ψ^A_σ. Since we are interested in the purely dilatonic sector, $\mathfrak{D}^{A}{}_{\sigma}(\epsilon)$ is given by (5.2.4) and $\mathfrak{D}_{A\sigma}(\epsilon)$ is its conjugated, explicitly

$$\mathfrak{D}_A(\epsilon) = \mathrm{d}\epsilon_A + \frac{1}{4}\omega_{ab}\gamma^{ab}\epsilon_A + \frac{1}{2}A^M\theta_M\varepsilon_{AB}\delta^{BC}\epsilon_C + \frac{1}{4}\overline{L}^T\mathcal{I}F_{ab}\gamma^{ab}\gamma\varepsilon_{AB}\epsilon^B + \frac{1}{2}\overline{\mathcal{W}}\gamma\delta_{AB}\epsilon^B\,, \tag{A4.2}$$

we used the fact that the Kähler connection vanishes. The 2-form volume is defined as

$$\mathrm{d}\Sigma_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\mathrm{d}x^\rho\wedge\mathrm{d}x^\sigma\,. \tag{A4.3}$$

First, we simplify the contraction appearing in the integral, namely

$$\mathrm{d}\Sigma_{\mu\nu}\epsilon^{\mu\nu\rho\sigma} = \frac{1}{2}\epsilon_{\mu\nu\lambda\delta}\epsilon^{\mu\nu\rho\sigma}\mathrm{d}x^\lambda\wedge\mathrm{d}x^\delta\,. \tag{A4.4}$$

Considering the definitions of the symbolic with curved indices

$$\epsilon_{\mu\nu\rho\sigma} = e^{-1}e^a{}_\mu e^b{}_\nu e^c{}_\rho e^d{}_\sigma\epsilon_{abcd}\,,\qquad \epsilon^{\mu\nu\rho\sigma} = e e_a{}^\mu e_b{}^\nu e_c{}^\nu e_d{}^\rho\epsilon^{abcd}\,, \tag{A4.5}$$

where the symbol with flat indices is $\epsilon_{0123} = 1 = -\epsilon^{0123}$. Hence,

$$\epsilon_{\mu\nu\lambda\delta}\epsilon^{\mu\nu\rho\sigma} = e^c{}_\lambda e^d{}_\delta e_g{}^\rho e_h{}^\sigma\epsilon_{abcd}\epsilon^{abgh}\,. \tag{A4.6}$$

using the identity $\epsilon_{abcd}\epsilon^{abgh} = -4\delta^g_{[c}\delta^h_{d]}$, it follows that $\epsilon_{\mu\nu\lambda\delta}\epsilon^{\mu\nu\rho\sigma} = -2(\delta^\rho_\lambda\delta^\sigma_\delta - \delta^\sigma_\lambda\delta^\rho_\delta)$, which leads to the following simplified form of (A4.4)

$$\mathrm{d}\Sigma_{\mu\nu}\epsilon^{\mu\nu\rho\sigma} = -(\delta^\rho_\lambda\delta^\sigma_\delta - \delta^\sigma_\lambda\delta^\rho_\delta)\mathrm{d}x^\lambda\wedge\mathrm{d}x^\delta = -2\mathrm{d}x^\rho\wedge\mathrm{d}x^\sigma\,. \tag{A4.7}$$

Replacing this result into the bracket between supercharges (A4.1) we obtain (5.2.11). The expression (5.2.11) can be obtained by defining a Majorana spinor

$$\psi^A = \epsilon_A + \epsilon^A \tag{A4.8}$$

which implies the following relations

$$\epsilon_A = \mathtt{P}_+\psi^A\,,\qquad \epsilon^A = \mathtt{P}_-\psi^A\,,\qquad \mathtt{P}_\pm = \frac{1}{2}(1\pm\gamma_5)\,. \tag{A4.9}$$

We recall that the conjugated of the chiral spinors is defined as $\bar{\epsilon}_A = \mathrm{i}(\epsilon^A)^\dagger\gamma^0$ and $\bar{\epsilon}^A = \mathrm{i}(\epsilon_A)^\dagger\gamma^0$, in order to preserve the chirality. In our basis $\gamma_5^\dagger = \gamma_5$, so one can check that

$$\bar{\epsilon}_A = \overline{\psi}^A \mathtt{P}_+ \,, \qquad \bar{\epsilon}^A = \overline{\psi}^A \mathtt{P}_- \,. \tag{A4.10}$$

Now, we use the fact that the electric components of the U(1) section and the superpotential are real functions, i.e. $L^\Lambda, \mathcal{W} \in \mathbb{R}$. Then, the supercovariant derivatives $\mathfrak{D}^A(\epsilon), \mathfrak{D}_A(\epsilon)$ can be written in terms of the Majorana spinor as follows

$$\begin{aligned}\mathfrak{D}^A(\epsilon) = {}& \mathtt{P}_-\mathrm{d}\psi^A + \frac{1}{4}\omega_{ab}\gamma^{ab}\mathtt{P}_-\psi^A + \frac{1}{2}A^M\theta_M\varepsilon^{AB}\delta_{BC}\mathtt{P}_-\psi^C + \\ & + \frac{1}{4}L^T\mathcal{I}F_{ab}\gamma^{ab}\gamma\varepsilon^{AB}\mathtt{P}_+\psi^B + \frac{1}{2}\mathcal{W}\gamma\delta^{AB}\mathtt{P}_+\psi^B \,, \end{aligned} \tag{A4.11}$$

$$\begin{aligned}\mathfrak{D}_A(\epsilon) = {}& \delta_{AB}\mathtt{P}_+\mathrm{d}\psi^B + \frac{1}{4}\omega_{ab}\gamma^{ab}\delta_{AB}\mathtt{P}_+\psi^B + \frac{1}{2}A^M\theta_M\varepsilon_{AB}\mathtt{P}_+\psi^B + \\ & + \frac{1}{4}L^T\mathcal{I}F_{ab}\gamma^{ab}\gamma\varepsilon_{AB}\mathtt{P}_-\psi^B + \frac{1}{2}\mathcal{W}\gamma\delta_{AB}\mathtt{P}_-\psi^B \,, \end{aligned} \tag{A4.12}$$

The 2-form appearing in the integral of the Dirac bracket are

$$\begin{aligned}\bar{\epsilon}^A\gamma\wedge\mathfrak{D}_A(\epsilon) = {}& \overline{\psi}^A\gamma\wedge\mathtt{P}_-\left(\mathrm{d}\psi^A + \frac{1}{4}\omega_{ab}\gamma^{ab}\psi^A + \frac{1}{2}A^M\theta_M\varepsilon^{AB}\delta_{BC}\psi^C + \right. \\ & \left. + \frac{1}{4}L^T\mathcal{I}F_{ab}\gamma^{ab}\gamma\varepsilon^{AB}\psi^B + \frac{1}{2}\mathcal{W}\gamma\delta^{AB}\psi^B\right) \,, \\ \bar{\epsilon}^A\gamma\wedge\mathfrak{D}_A(\epsilon) = {}& \overline{\psi}^A\gamma\mathtt{P}_+\wedge\left(\delta_{AB}\mathrm{d}\psi^B + \frac{1}{4}\omega_{ab}\gamma^{ab}\delta_{AB}\psi^B + \frac{1}{2}A^M\theta_M\varepsilon_{AB}\psi^B + \right. \\ & \left. + \frac{1}{4}L^T\mathcal{I}F_{ab}\gamma^{ab}\gamma\varepsilon_{AB}\psi^B + \frac{1}{2}\mathcal{W}\gamma\delta_{AB}\psi^B\right) \,, \end{aligned}$$

Clearly, its subtraction cancels the identity factor in $\mathtt{P}_\pm$ leading to (5.2.11).

A5 Quartic invariant of $SL(2,\mathbb{R})^3$

To construct the quartic invariants of $\mathrm{SL}(2,\mathbb{R})^3 \subset \mathrm{Sp}(8,\mathbb{R})$ we consider its generators given by

$$\mathtt{X}_i = \left.\frac{\partial\mathcal{M}}{\partial\chi_i}\right|_{\chi,\phi=0} \,, \qquad \mathtt{Y}_i = \left.\frac{\partial\mathcal{M}}{\partial\phi_i}\right|_{\chi,\phi=0} \,, \qquad \mathtt{Z}_i = [\mathtt{Y}_i,\mathtt{X}_i] \,, \qquad (\text{no sum over } i) \tag{A5.1}$$

with $i = 1,2,3$. They span the three commuting $\mathfrak{sl}(2,\mathbb{R})$ algebras. It is convenient to consider the basis with nilpotent generators $\mathtt{e}_i^2 = \mathtt{f}_i^2 = 0$ and the generators $\mathtt{h}_i$ of the

Cartan subalgebra, given by

$$\mathtt{h}_i = \frac{1}{2}\mathtt{Y}_i\,, \qquad \mathtt{e}_i = \frac{1}{4\sqrt{2}}(\mathtt{Z}_i + 2\mathtt{X}_i)\,, \qquad \mathtt{f}_i = \mathtt{e}_i^T\,, \tag{A5.2}$$
$$[\mathtt{h}_i, \mathtt{e}_i] = \mathtt{e}_i\,, \qquad [\mathtt{e}_i, \mathtt{f}_i] = \mathtt{h}_i\,, \qquad [\mathtt{h}_i, \mathtt{f}_i] = -\mathtt{f}_i\,.$$

We collect all the generators as $t_{(i)\alpha} = \{\mathtt{e}_i, \mathtt{h}_i, \mathtt{f}_i\}$ with $\alpha = 1,2,3$. By definition the positions of the symplectic indices are $t_{(i)\alpha} = (t_{(i)\alpha})_M{}^N$ and we lower them by the symplectic matrix $\mathbb{C} = \mathbb{C}_{MN}$ defining $t_{(i)\alpha}\mathbb{C} = (t_{(i)\alpha})_{MN}$. We construct the Cartan-Killing form $\eta_{(i)\alpha\beta} = \mathrm{Tr}(t_{(i)\alpha}t_{(i)\beta})$ and its inverse, denoted by $\eta^{\alpha\beta}_{(i)}$. These allow us to define $(t^{\alpha}_{(i)})_{MN} = \eta^{\alpha\beta}_{(i)}(t_{(i)\beta})_{MN}$ and construct the following tensors of $\otimes^4\underline{8}$

$$\mathcal{C}_{MNPQ} = \mathbb{C}_{MN}\mathbb{C}_{PQ}\,, \tag{A5.3}$$
$$(\mathcal{I}_i)_{MNPQ} = (t^{\alpha}_{(i)})_{MN}(t_{(i)\alpha})_{PQ}\,, \tag{A5.4}$$
$$(\mathcal{L}_{ij})_{MNPQ} = (t^{\alpha}_{(i)})_{M\bullet}(t^{\beta}_{(j)})^{\bullet}{}_N(t_{(i)\alpha})_{P\bullet}(t_{(j)\beta})^{\bullet}{}_Q\,, \tag{A5.5}$$
$$(\mathcal{Z}_{ijk})_{MNPQ} = (t^{\alpha}_{(i)})_{M\bullet}(t^{\beta}_{(j)})^{\bullet}{}_{\bullet}(t^{\gamma}_{(k)})^{\bullet}{}_N(t_{(i)\alpha})_{P\bullet}(t_{(j)\beta})^{\bullet}{}_{\bullet}(t_{(k)\gamma})^{\bullet}{}_Q\,. \tag{A5.6}$$

which are invariant under $\mathrm{SL}(2,\mathbb{R})^3$. Among all the above tensors only eight of them are independent, and one can pick these to be $\{\mathcal{C}, \mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3, \mathcal{L}_{12}, \mathcal{L}_{13}, \mathcal{L}_{23}, \mathcal{Z}_{123}\}$. Nevertheless, when these tensors act on two arbitrary symplectic vectors the eight invariants reduce to seven independent functions.

In our case we have two symplectic vectors Γ^M and $\theta^M = \mathbb{C}^{MN}\theta_N$, hence we can construct the following independent invariants

$$\mathcal{I}_1(\Gamma,\Gamma,\Gamma,\Gamma) = -2\prod_{\Lambda} P_\Lambda\,, \tag{A5.7}$$
$$\mathcal{I}_1(\theta,\theta,\theta,\theta) = -\frac{1}{2L^2}\,, \tag{A5.8}$$
$$\mathcal{I}_i(\Gamma,\theta,\Gamma,\theta) = \frac{1}{16L^2}\left(\sum_{\Lambda}\Omega_{i\Lambda}P_\Lambda\right)^2\,. \tag{A5.9}$$

The rest are zero or functionally dependent on the above ones. Note that the BPS mass can be expressed in terms of the invariants as

$$m^4_{\mathrm{BPS}} = 64L^3\prod_i \mathcal{I}_i(\Gamma,\theta,\Gamma,\theta). \tag{A5.10}$$

Other invariants that are dependent on the above are the following.

$$\mathcal{C}(\Gamma,\theta,\Gamma,\theta) = \frac{1}{2L^2}\left(\sum_{\Lambda} P_\Lambda\right)^2 , \tag{A5.11}$$

$$\mathcal{I}_1(\Gamma,\Gamma,\theta,\theta) = -\frac{1}{2L^2}(P_2P_3 + P_1P_4)\,, \tag{A5.12}$$

$$\mathcal{I}_2(\Gamma,\Gamma,\theta,\theta) = -\frac{1}{2L^2}(P_1P_3 + P_2P_4)\,, \tag{A5.13}$$

$$\mathcal{I}_3(\Gamma,\Gamma,\theta,\theta) = -\frac{1}{2L^2}(P_1P_2 + P_3P_4)\,, \tag{A5.14}$$

$$\mathcal{Z}_{123}(\Gamma,\Gamma,\theta,\theta) = -\frac{1}{128L^2}\prod_{\Lambda\geq\Sigma} P_\Lambda P_\Sigma\,, \tag{A5.15}$$