UNITARY REPRESENTATIONS OF LIE GROUPS IN QUANTUM MECHANICS

by

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1. NON-RELATIVISTIC CLASSICAL MECHANICS AND THE GALILEAN GROUP**

Let $S(3)$ denote Euclidean 3-space. A Cartesian observer of $S(3)$ is a mapping $s \in S(3) \to \vec{x} \equiv (x_1 x_2 x_3) \in R^3$ for which the metric $\rho(s,s')$ of $S(3)$ may be written as

$$\rho(s,s') = \{(x_1 - x_1')^2 + (x_2 - x_2')^2 + (x_3 - x_3')^2\}^{\frac{1}{2}} \quad . \tag{1.1}$$

The group of transformations between Cartesian observers is the Euclidean group $E(3)$

$$x_a' = R_{ab} x_b + C_a \quad , \quad a = 1,2,3 \quad , \tag{1.2}$$

where R_{ab} is any real orthogonal matrix and C_a any real vector (independent of x).

Let t denote Newtonian time, which is simply a parameter assumed to be the same, up to a change of origin $t \to t' = t + t_0$, for all Cartesian observers. Note that in general R_{ab} and C_a are functions of t, i.e., Cartesian observers may be accelerating relative to each other.

Newtonian physics assumes that physical objects occupy volumes in $S(3)$ and vary their positions continuously with time, the variation of any body being determined by the others. The business of physics is to determine the laws of variation.

We shall be concerned mainly with a simplifying limiting case of physical objects, namely, Newtonian <u>particles</u>. A Newtonian particle is a physical object to which is attached an intrinsic label called its mass m (which will be

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** Throughout this paper an asterisk (*) used in a mathematical expression denotes complex conjugation and a dagger (†) passing to the adjoint operator.

discussed in more detail in a moment) and whose volume is so small (relative to its distance from other particles) that for practical purposes it can be neglected and shrunk to a point in S(3). Thus, a Newtonian particle is characterized at any time t by a point in S(3) and its mass.

In view of the importance of the mass of a particle for our later discussion, we consider in a little detail how it enters in Newtonian theory. Its existence is, of course, empirical and may, in principle at least, be established as follows: If any 2 particles interact in isolation (in practice, sufficiently far from other objects), then there exists a set of Cartesian observers such that the quantity

$$m_{12} = -\frac{d^2x_a^{(1)}}{dt^2} \Big/ \frac{d^2x_a^{(2)}}{dt^2} \tag{1.3}$$

(the ratio of the acceleration of the particles) is positive and is independent of a, t, $x^{(1)}$, $x^{(2)}$ and the nature of the interaction. In other words, m_{12} is an intrinsic property of the pair of particles 1 and 2. Furthermore, if α, β, γ are any 3 particles then (again empirically)

$$m_{\alpha\beta} = m_{\alpha\gamma} \cdot m_{\gamma\beta} \quad . \tag{1.4}$$

Equation (1.4), however, implies the existence of a set of intrinsic masses m_α, one for each particle, and unique up to a common scale factor, such that

$$m_{\alpha\beta} = m_\beta / m_\alpha \quad . \tag{1.5}$$

As the masses m_α are relatively positive, they are chosen by convention to be positive.

The result that $m_{\alpha\beta}$ is constant already lays the foundations for the law of variation of the positions of the particles with respect to time. The general law (Newton's law) is a linear generalization, namely, given a set of n isolated particles $(m_\alpha, x^\alpha, \alpha = 1,\ldots,n)$, there exists a set of Cartesian observers such that

$$\sum_{\alpha=1}^{n} m_\alpha \frac{d^2x^\alpha}{dt^2} = 0 \quad . \tag{1.6}$$

This law, in turn, brings out the importance of the force, defined by

$$F_\alpha = m_\alpha \frac{d^2x^\alpha}{dt^2} \quad , \tag{1.7}$$

as a basic physical concept. Forces are additive, from (1.6), and have additional good properties, which might be described as follows:

What we are looking for is a description of the interaction of particles which is as simple and as universal as possible. Now a description would be

provided by simply stating what each $\vec{x}^{\alpha}$ is as a function of t for each ensemble of particles, (this is what Kepler actually did for the planets), but such a description would be neither simple nor universal (as Kepler found to his cost). What Newton discovered is that there exists a quantity that is simple and universal, namely, F_{α}. The classic example of a simple universal F_{α} is in the Newtonian theory of gravitation, for which the simple inverse square law $F = m_1 m_2/r^2$ is sufficient to explain all (non-relativistic) effects. (Of course, one can reverse the logic and define gravitational effects to be those for which $F = m_1 m_2/r^2$. However, the point is that gravitational effects so defined cover a huge class of observed phenomena--falling bodies, projectiles, planetary motion, etc.)

From the group theoretical point of view, the interesting aspect of Newton's Equation (1.6) is its invariance group. Equation (1.6) does not hold for all Cartesian observers, but only for a subclass. Let us call the subclass Galilean observers. By noting that any Cartesian observer is related to a Galilean observer by a transformation of the form

$$x'_a = R_{ab}(t)x_b + C_a(t) \quad , \quad t' = t + t_0 \quad , \tag{1.8}$$

and inserting this result in (1.6), we see that the Galilean observers are those, and only those, for whom

$$R_{ab}(t) = R_{ab} \quad , \quad C_a(t) = C_a + V_a t \quad , \tag{1.9}$$

where R_{ab}, C_a, and V_a are independent of t. The subgroup G of (1.8) for which (1.9) holds is called the Galilean group G.

The geometrical significance of the Galilean group becomes clear if we note that it is formed exhaustively from the four subgroups:

1)	Time-translations	$t' = t + t_0$
2)	Space-translations	$x'_a = x_a + C_a$
3)	Rotations	$x'_a = R_{ab}x_b$
4)	Accelerations	$x'_a = x_a + V_a t$

(1.10)

The invariance of (1.6) under (1.10), 1) to 3), means that (1.6) does not prefer any origin in space or time or any direction in space, which is understandable. The invariance under 4) means that observers with different but constant velocities are equivalent. This is far less obvious, and was first discovered by Galileo. The invariance under 4) does have, however, a geometrical significance, namely, in the 4-space spanned by $S(3)$ and t, (1.6) does not prefer any slope for the t-axis.*

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* I am indebted to Henri Bacry for this remark.

The force defined by Equation (1.7) is clearly Galilean invariant, provided that the Galilean transformation is universal, i.e., it is a transformation of the coordinates of all the particles. Thus, in guessing the forces for any problem, one can restrict oneself to those that are Galilean invariant.

Let us now consider the Galilean group by itself. By definition, it is a 10-parameter Lie group, which is the semi-direct product of its connected part (det $R_{ab} = +1$) and the 2-element space reflexion (parity) group. Its Lie algebra $d\mathcal{G}$ has the basis:

1)	Time-translations	E
2)	Rotations	L_a
3)	Space-translations	P_a
4)	Accelerations	K_a

with commutation relations

$$\begin{aligned} &[E,M_a] = 0 \qquad [E,P_a] = 0 \qquad [E,K_a] = P_a \\ &[M_a,M_b] = \varepsilon_{abc}M_c \qquad [M_a,P_b] = \varepsilon_{abc}P_c \qquad [M_a,K_b] = \varepsilon_{abc}K_c \\ &[P_a,P_b] = 0 \qquad [P_a,K_b] = 0 \\ &[K_a,K_b] = 0 \ , \end{aligned} \tag{1.11}$$

where $a,b,c = 1,2,3$ and ε_{abc} is the Levi-Civita symbol. In words, $d\mathcal{G}$ is the semi-direct sum of the rotation algebra L and a 7-dimensional solvable algebra made up of the two abelian commuting vectors P and K and a scalar E which projects K onto P and commutes with P.

One of the most important properties of Galilean transformations is that they are a special case of contact transformations [1], namely, transformations $x \to x'(x,p)$, $p \to p'(x,p)$ which leave the symplectic form

$$\{A,B\} = \sum \left(\frac{\partial A}{\partial p}\frac{\partial B}{\partial x} - \frac{\partial A}{\partial x}\frac{\partial B}{\partial p}\right) \ , \tag{1.12}$$

where $p_\alpha = m_\alpha \frac{dx_\alpha}{dt}$, invariant.

Now a property of the group of contact transformations[2] is that if σ is the parameter of any 1-parameter simply connected Lie subgroup, then there exists a function $G(p,q)$ such that

$$\frac{\delta F}{\delta \sigma} = \{G,F\} \ , \tag{1.13}$$

where F is any regular function of p and q, and $\frac{\delta F}{\delta \sigma}$ is its rate of variation with respect to the group parameter σ. The function G is called the generator function for the 1-parameter subgroup.

Furthermore, for an n-parameter Lie subgroup of contact transformations with parameters α, β,...

$$\left(\frac{\delta}{\delta\alpha}\frac{\delta}{\delta\beta} - \frac{\delta}{\delta\beta}\frac{\delta}{\delta\alpha}\right) F = C^{\gamma}_{\alpha\beta}\frac{\delta F}{\delta\gamma} \quad , \tag{1.14}$$

where $C^{\gamma}_{\alpha\beta}$ are the structure constants of the group. Hence, inserting (1.13) into (1.14) and using the Jacobi relation for $\{A,B\}$, we obtain

$$\{\{G_{\alpha},G_{\beta}\}F\} = C^{\gamma}_{\alpha\beta}\{G_{\gamma},F\} \quad , \tag{1.15}$$

whence,

$$\{G_{\gamma},G_{\beta}\} = C^{\gamma}_{\alpha\beta}G_{\gamma} + \lambda_{\alpha\beta} \quad , \tag{1.16}$$

where the $\lambda_{\alpha\beta}$ have zero bracket with all F and hence are constants. Thus, under the bracket operation, the generator functions G_{α} of a Lie group of contact transformations form a representation (up to the constants $\lambda_{\alpha\beta}$) of the Lie algebra. The number of constants $\lambda_{\alpha\beta}$ can be minimized by transformations of the form $G_{\alpha} \to G_{\alpha} + \lambda_{\alpha}$, where the λ_{α} are constants, but whether the $\lambda_{\alpha\beta}$ can be eliminated entirely depends on the group structure.

The above results hold for <u>any</u> Lie group of contact transformations. Let us now return to the connected Galilean group G. For G, the generator functions corresponding to the generators in (1.11) can be seen to be

$$\begin{aligned} L &= \sum_{\alpha} \mathbf{x}_{\alpha} \times \mathbf{p}_{\alpha} \\ P &= \sum_{\alpha} \mathbf{p}_{\alpha} \\ K &= \sum_{\alpha} m_{\alpha}\mathbf{x}_{\alpha} - Pt \\ E &= \sum \frac{1}{2m_{\alpha}} p_{\alpha}^2 + \varphi \quad , \end{aligned} \tag{1.17}$$

where $p_{\alpha} = m_{\alpha}\frac{dx_{\alpha}}{dt}$, and φ is the potential from which the F_{α} can be derived, i.e.,

$$F_{\alpha} = -\frac{\partial\varphi}{\partial x_{\alpha}} \quad .$$

If we compute the brackets $\{L,E\}$, etc., for the generator functions (1.17), we obtain, as expected, the Lie algebra (1.11) up to constants. In fact, there is only one constant; namely, the relations (1.11) hold as they stand except that

$$[P_a,K_b] = 0 \to \{P_a,K_b\} = \delta_{ab}M \quad , \tag{1.18}$$

where $M = \sum_{\alpha} M_{\alpha}$ is the total mass. Further, the structure of the Galilean group is such that M cannot be eliminated (we shall be discussing this question again later).

Note that the generator for the time translations is just the Hamiltonian H for the system. Note further that $[H,K] \neq 0$, although $[H[H,K]] = 0$. Thus, although the Galilean group is an invariance group of Newton's Equations (1.6), it is not quite an invariance group of the Hamiltonian, or of Hamilton's equations of motion,

$$\frac{dp_\alpha}{dt} = -\frac{\partial H}{\partial x_\alpha} \qquad \frac{dx_\alpha}{dt} = \frac{\partial H}{\partial p_\alpha} \quad . \tag{1.19}$$

This is understandable since a choice of Hamiltonian forces a choice of direction for the t-axis in $S(3) \otimes R$ and thus destroys the Galilean invariance. Incidentally, the term $-Pt$, which is explicitly time-dependent, is inserted in the definition of K, so that in spite of the fact that $[H,K] \neq 0$, K can be a constant of the motion, i.e., so that

$$\frac{dK}{dt} = \frac{\partial K}{\partial t} + \{H,K\} = -P + P = 0 \quad . \tag{1.20}$$

2. NON-RELATIVISTIC QUANTUM MECHANICS

As is well known, Newton's laws, or the more general and sophisticated versions of them, such as Hamilton's, sufficed to explain all physical phenomena until the end of the last century. But after the turn of the century, the Newtonian framework was shattered both by the theory of relativity and by the quantum theory. In this lecture, we shall be concerned only with quantum theory. As is also well-known, the crux of the quantum theory is to replace the functions x and $p = m\frac{dx}{dt}$ needed to describe particles, by linear operators X and P on a Hilbert space, satisfying the relation

$$[X,P] = i\hbar \quad . \tag{2.1}$$

(This relation will be made mathematically more precise later.) For the moment, we shall only emphasize that the assumption (2.1) is the <u>only</u> new assumption made in the quantum theory. The old equations of motion

$$\frac{dX}{dt} = \frac{\partial H}{\partial P} \quad , \quad \frac{dP}{dt} = -\frac{\partial H}{\partial X}$$

are retained with $x \to X$, $p \to P$ (which is unambiguous since $H = \frac{p^2}{2m} + \varphi(x)$). There are four questions which we wish to discuss briefly:

1) How one arrives at the particular Ansatz (2.1)
2) How to make it mathematically precise
3) How to relate it to experiment
4) How the group structure of Newtonian theory is affected.

Let us begin with 1). The decision to replace x and p by operators

was based on a large number of empirical observations and on partial theories formed from these observations [1]. Since we could not even begin to describe the general picture in a part of one lecture, let us concentrate on one experimental result, namely, the discrete frequency of the light emitted from atoms, and try to sketch the motivation from that result. It was known at the time the quantum theory was founded that the atom consisted of a positively charged kernel of very small radius with negatively charged electrons circling it, about 10^{-8} cms out.

For such a system Newton's laws (extended to include Maxwell's) would predict a continuous emission of radiation from the circling (and therefore accelerating) electrons, leading to a continuous loss of energy on the part of the electrons (so that the atom would finally run down) and a continuous change in the frequency of the emitted radiation. The experimental situation, however, was quite the opposite. First, the atoms were quite stable (otherwise, our universe would not exist). Second, from spectroscopy it was known that the frequency of the radiation emitted from atoms, far from being continuous, could only have special sharp values (spectral lines) characteristic of the atom (yellow for sodium, green for copper, and so on). Hence, Newton's laws were incompatible with experiment on the atomic level. The question was: how to change them?

One worked backwards. If one assumes

1) Einstein's empirical law $E = h\nu$, where h is Planck's constant, ν the frequency of the emitted light, and E its energy, and
2) conservation of energy, i.e., energy lost by electron in the atom = energy of emitted radiation,

it follows from the discreteness of the frequency of the emitted radiation that the energy levels of the electron in the atom must be discrete. It follows that the Hamiltonian

$$H = \frac{1}{2m} p^2 - \frac{Ze^2}{r} \quad , \tag{2.2}$$

for an electron in an atom with nucleus of charge Ze, cannot take continuous values. This leaves one with three options:

1) Abandon the Hamiltonian (2.2)
2) Impose some conditions on it from outside
3) Change it so that it can naturally take only discrete values.

1) has the difficulty that it is almost impossible to think of a classical Hamiltonian which would take discrete values. 2) is what was done in the so-called "old quantum theory" (1900-25), and is very *ad hoc*. 3) is the option chosen by Schrödinger and Heisenberg. The choice they made was to interpret H as a linear operator, since H could then take discrete values naturally. This means interpreting x and p as linear operators X and P. To determine the kind of operators P and X should be, one must do more. Heisenberg analyzed the atomic

spectra in detail and concluded that P and X should be the matrices

$$P = \frac{\hbar}{i\sqrt{2}} \begin{bmatrix} 0 & \sqrt{1} & 0 & 0 & \cdot \\ -\sqrt{1} & 0 & \sqrt{2} & 0 & \cdot \\ 0 & -\sqrt{2} & 0 & \sqrt{3} & \cdot \\ 0 & 0 & -\sqrt{3} & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \qquad X = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & \sqrt{1} & 0 & 0 & \cdot \\ \sqrt{1} & 0 & \sqrt{2} & 0 & \cdot \\ 0 & \sqrt{2} & 0 & \sqrt{3} & \cdot \\ 0 & 0 & \sqrt{3} & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}, \tag{2.3}$$

where $\hbar = \frac{h}{2\pi}$. Schrödinger, on the other hand, built on a partial theory due to de Broglie. According to de Broglie, free particles should diffract like light from sufficiently small gratings and should therefore satisfy, in the relativistic case, a wave equation of the form

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 - m^2\right)\psi(x) = 0 \quad . \tag{2.4}$$

Comparing this with the classical energy moment relation,

$$\varepsilon^2 - p^2 - m^2 = 0 \quad , \tag{2.5}$$

Schrödinger concluded that P should be the operator

$$\frac{\hbar}{i}\frac{\partial}{\partial x} \quad , \tag{2.6}$$

on $L_2^3(-\infty,\infty)$, and went on to postulate that this identification should persist in the non-relativistic limit and in the presence of a potential.

One sees that the Schrödinger and Heisenberg Ansatz are equivalent by noting that they are special realizations of the Ansatz (2.1). Note, incidentally, that the Ansatz (2.1) need only be made at a single (initial) instant of time and it is therefore a kinematical Ansatz. Newton's laws then guarantee it for all times.

It might be wondered if the Ansatz (2.1) is absolutely necessary to obtain agreement with experiment, or whether one could get away with less. Wigner [2], for example, has proposed that (2.1) might be replaced by the weaker commutation relations

$$[H,P] = i\frac{\partial H}{\partial X} \quad , \qquad [H,X] = -i\frac{\partial H}{\partial P} \quad , \tag{2.7}$$

where H is the Hamiltonian, which would seem to be necessary from Heisenberg's analyses of the spectral lines. However, except in the case (2.1), the Ansatz (2.7) would make the commutation relations depend on H, i.e., on the dynamics.

Let us now turn to question 2), namely the question of putting the Ansatz $[X,P] = i\hbar$ on a better mathematical footing. For this we proceed as follows:

Let $\mathcal{H}$ be a Hilbert space, and let X and P be operators on it such that there exists for them a common invariant dense domain $\mathcal{D}$ on which

a) X and P are symmetric

b) $X^2 + P^2$ is essentially self-adjoint

c) $[X,P] = i\hbar$

d) the only bounded operator which commutes with X and P is a multiple of the unit operator.

Then X and P are uniquely and rigorously defined [3] on $\mathcal{H}$ up to a unitary transformation (which may depend on the time). They are essentially self-adjoint on $\mathcal{D}$. A realization of X and P, is the Schrödinger realization x and $\frac{\hbar}{i}\frac{\partial}{\partial x}$ on $L_2^3(-\infty,\infty)$, where the domain $\mathcal{D}$ could be, for example, the space K of all infinitely differentiable functions of compact support, or the space S of all infinitely differentiable functions of fast decrease (i.e., which decrease faster than any inverse power of x as $|x| \to \infty$). We shall see later (from Nelson's theorem) that conditions a) to d) are precisely the necessary and sufficient conditions, that X and P can be exponentiated to form a unique unitary irreducible representation of the Weyl-Heisenberg group W, i.e., that

$$e^{i\sigma X}e^{i\tau P} = e^{i\tau P}e^{i\sigma X}e^{i\sigma\tau\hbar} \quad , \text{ on } \mathcal{H} \quad . \tag{2.8}$$

Thus, an alternative definition of X and P is that they satisfy (2.8), i.e., that they are the generators of the unitary irreducible representation (UIR) of W, [4]. In fact, this definition of X and P was the starting point for von Neumann's celebrated proof [5] of the uniqueness of X and P up to a unitary transformation.

Having disposed of these mathematical points, we come to the experimental numbers. To extract the experimental numbers, we first put the self-adjoint operators A on $\mathcal{H}$ into a 1-1 correspondence with the measurable quantities (observables) which we shall then also denote by A. In practice, for the self-adjoint operators for which it is meaningful, the correspondence is [4]

$$A = f(P,X) = \frac{1}{(2\pi)^2}\int e^{i(P\nu+X\zeta)}d\nu d\zeta \int e^{-i(p\nu+x\zeta)}f(p,x)dpdx \quad , \tag{2.9}$$

where $f(p,x)$ are the corresponding classical functions. (The bounded subset of the operators for which (2.9) is meaningful form a dense set in the ring of bounded self-adjoint operators.)

Now let $P_\lambda(A)$ be the projection operator on the eigenspace of A belonging to the eigenvalue λ, where for the moment we assume λ to be discrete and the eigenspace finite dimensional. The numbers to be extracted are then

$$\text{trace } (P_\lambda(A)P_\mu(B)) \quad , \tag{2.10}$$

with appropriate modifications in the case that λ and μ are not discrete and that both eigenspaces are infinite dimensional. The meaning of the numbers (2.10) is that they are probabilities; namely, trace $(P_\lambda(A)P_\mu(B))$ is the probability of

finding the value μ from a measurement of B, having just previously found λ from a measurement of A, except points in spectra of self-adjoint operators, the probabilities are the only experimental numbers that quantum mechanics can predict.

In the particular case that the eigenvalues λ and μ are simple, i.e., that $P_\lambda(A)$ and $P_\mu(B)$ project onto 1-dimensional subspaces, (2.10) reduces to

$$|(f_\lambda(A), f_\mu(B))| \quad , \tag{2.11}$$

where $f_\lambda(A)$ and $f_\mu(B)$ are any unit vectors in the respective subspaces. This is the case which will be of most interest to us. (For future reference, we shall need for this case the concept of a quantum mechanical <u>state</u>. The state of a system after a measurement of A with result λ, where λ is simple, is defined to be the set of unit vectors in the 1-dimensional eigenspace. Such a set of unit vectors $e^{i\alpha} f_\lambda(A)$, $0 \leq \alpha < 2\pi$ is often called a <u>ray</u>. Thus, the states of a system are in 1-to-1 correspondence with the rays.)

Let us turn now to question 4), the group theoretical properties of non-relativistic quantum mechanics, and first consider the Hamiltonian

$$H = \frac{1}{2m} P^2 + \varphi(X) \quad , \tag{2.12}$$

for a single particle in an external potential.

In most cases of interest, H is essentially self-adjoint on the domain $\mathcal{D}$ above. Hence, by Stone's theorem [6], there exists a unique continuous 1-parameter group of unitary transformations $U(t)$ on $\mathcal{H}$, such that

$$\frac{dU(t)}{dt} = HU(t) \quad \text{on} \quad \mathcal{D} \quad . \tag{2.13}$$

We now show that $U(t)$ is the group of time translations. Since the Newtonian equations of motion are the same in classical and quantum theory, we have in both cases

$$\frac{dX}{dt} = \frac{1}{m} P \quad , \quad \frac{dP}{dt} = - \frac{\partial \varphi(X)}{\partial X} \quad . \tag{2.14}$$

In the quantum mechanical case, however, we have the extra condition

$$[X,P] = i\hbar \quad .$$

Inserting this equation into (2.14) and (2.12), we see that in the quantum mechanical case we have

$$\frac{dX}{dt} = \frac{i}{\hbar} [H,X] \quad , \quad \frac{dP}{dt} = \frac{i}{\hbar} [H,P] \quad \text{on} \quad \mathcal{D} \quad . \tag{2.15}$$

If we assume that the domain $\mathcal{D}$ is invariant with respect to $U(t)$, it follows at once that

$$X(t) = U(t)X(0)U^{-1}(t) \quad , \quad P(t) = U(t)P(0)U^{-1}(t) \quad \text{on} \quad \mathcal{D} \quad , \tag{2.16}$$

and, in general, for suitably defined $F(P,X)$ in (2.9)

$$F(P(t)X(t)) = U(t)F(P(0)X(0))U^{-1}(t) \quad . \tag{2.17}$$

Thus, $U(t)$ is the group of time translations. In quantum mechanics, therefore, the Hamiltonian H, like P and X, plays a dual role. It is a physical observable (energy) and it generates the group of time translations.

It may happen that H is not essentially self-adjoint on $\mathcal{D}$. In this case, there is usually a good physical reason, and the corresponding classical Hamiltonian also has bad properties, e.g., sends the particle off to infinity in a finite time [7].

Turning now to the Galilean group for a system of interacting particles, we find that, in analogy to P, X, and H, if we replace the classical generator functions of the Galilean group by their quantum mechanical counterparts to obtain

$$\begin{aligned} E &= H = \sum \frac{1}{2m} P_\alpha^2 + \varphi \\ L &= \sum X_\alpha \times P_\alpha \\ P &= \sum P_\alpha \\ K &= \sum m_\alpha X_\alpha - Pt \quad , \end{aligned} \tag{2.18}$$

then, in analogy to P, X and H, these ten operators (2.18) play a dual role. They are physical observables and at the same time they are the generators of unitary representations of the 1-parameter subgroups of the Galilean group G on $\mathcal{H}$, i.e., if σ is a parameter,

$$\frac{dF}{d\sigma} = \frac{i}{\hbar}[G_\sigma, F] \quad , \quad \sigma = 1 \ldots 10 \quad . \tag{2.19}$$

This is the quantum-mechanical analogue of the classical Poisson bracket relation

$$\frac{dF}{d\sigma} = \{\tilde{G}_\sigma, F\} \quad . \tag{2.20}$$

Using the quantum mechanical relation $[X,P] = i\hbar$, we can easily compute the commutators of the operators (2.18) amongst themselves. We obtain

$$\begin{aligned} &[M_a,M_b] = i\varepsilon_{abc}M_c \qquad [P_a,P_b] = 0 \qquad [K_a,K_b] = 0 \\ &[M_a,P_b] = i\varepsilon_{abc}P_c \qquad [P_a,K_b] = i\delta_{ab}M \qquad [K_a,H] = 0 \quad . \\ &[M_a,K_b] = i\varepsilon_{abc}K_c \qquad [P_a,H] = 0 \quad , \\ &[M_a,H] = 0 \end{aligned} \tag{2.21}$$

These relations are the analogue of the classical Poisson bracket relations for the generator functions amongst themselves. Note that (2.21) even contains the term $\delta_{ab}M$ which occurs in the classical Poisson bracket relations, but not in the Lie algebra of G.

Apart from the term $\delta_{ab}M$, (2.21) is just the Lie algebra of G. Hence, if the term $\delta_{ab}M$ were absent, the 1-parameter subgroups of G, generated by the G_α, would mesh together to form a unitary representation of G on $\mathcal{H}$ (modulo some domain restrictions which will be discussed later and which are normally satisfied). Thus, in quantum mechanics the generators G_α play the dual role of observables and generators (modulo $\delta_{ab}M$) of a unitary representation [8] of G on $\mathcal{H}$. This is true, of course, in classical mechanics also, where the generator functions are observables and generators of group transformations in the sense of Poisson brackets. But the relationship in quantum mechanics is more direct. In particular, the operation of commutation is simpler and more direct than the operation of forming Poisson brackets. In this sense, group theory, which plays a background role in classical theory, may be said to come into its own and play a central role in quantum mechanics.

Let us now consider the term $\delta_{ab}M$. Since it commutes with all the G_α, it cannot make a big difference to the representation of G on $\mathcal{H}$. It is easily checked that the difference it makes is that the 1-parameter subgroups of G, instead of meshing together to form a true unitary representation of G on $\mathcal{H}$, mesh together to form a unitary ray representation of G on $\mathcal{H}$, i.e., a representation by unitary operators $U(g)$ satisfying

$$U(g)U(g') = U(gg')e^{i\omega(g,g')} \quad , \tag{2.22}$$

where $g,g' \in G$ and ω is real. The reason for the name ray representation is that the factor $\exp i\omega(g,g')$ is irrelevant for rays, (where rays are defined as above to be sets of unit vectors related to a given unit vector f by $\exp(i\alpha)f$, where $0 \leq \alpha < 2\pi$). If we now recall that the experimental numbers which can be extracted from quantum theory are

$$|(f,g)| \quad , \tag{2.23}$$

where f and g are unit vectors, we see at once that they do not distinguish between vectors in the same ray. Thus, the experimental numbers do not distinguish between unitary ray representations and true unitary representations. We shall be returning in more detail to this point later, but for the moment we merely note that the failure of the experimental numbers to distinguish between true and ray representations means that the appearance of ray representations and hence, in particular, of the term $\delta_{ab}M$ in the Lie algebra (2.21), is quite natural in quantum mechanics.

In the case of a single free particle, the generators reduce to

$$\begin{aligned} M_a &= \frac{1}{2}\,\varepsilon_{abc}P_aX_c \\ P_a &= P_a \\ K_a &= mX_a - P_at \\ E &= \frac{1}{2m}\,P^2 \quad , \end{aligned} \tag{2.24}$$

where m is now the mass of the particle and E is both a generator of the Galilean group and the generator of time translations. Thus, a free particle "carries" a unitary ray representation of G. Furthermore, if the quantum mechanical commutation relation

$$[X,P] = i\hbar \quad ,$$

is irreducible on $\mathcal{H}$, then so is the representation (2.21) of G. A non-relativistic free particle may, therefore, be said to carry an irreducible unitary ray representation of G.

An interesting question is what would happen if we reversed our line of approach and demanded that a free non-relativistic particle carry a true unitary representation of G. This question has been investigated by Inönü and Wigner [9]. They showed that in a true irreducible unitary representation of G the quantum mechanical relation

$$[X,P] = i\hbar \quad ,$$

cannot be realized, which has the unpleasant physical consequence that X cannot be localized. The crucial point is that P^2 is a Casimir operator for G. Hence, in any unitary irreducible representation, it is a number, and the Fourier transform $\tilde{f}(X)$ of any $f(P)$ must therefore have a spread in X.

In a ray representation, the situation is saved by the ray relation

$$i[K_a,P_b] = \delta_{ab}m \quad , \tag{2.25}$$

or

$$i[K_a,P^2] = 2mP_a \quad . \tag{2.26}$$

The latter relation implies that P^2 assumes all values in the range $0 \leq P^2 < \infty$, which together with

$$[M_a,P_b] = i\varepsilon_{abc}P_c \quad , \tag{2.27}$$

implies that $\vec{P}$ takes all values in R^3, in which case the Fourier transform $\vec{X}$ is localizable.

In conclusion it might be worth remarking that the twin postulates of quantum mechanics, $|(f,g)|^2$ = probability, and $[X,P] = i\hbar$ are not entirely independent. The second can be deduced from the first, using group theoretical and other general arguments of a more or less plausible nature (see ref. 4, lecture 6).

3. INVARIANCE GROUPS IN NON-RELATIVISTIC QUANTUM MECHANICS

In the last two sections, we saw that the Galilean group G was the group of invariance of the non-relativistic equations of motion of an isolated system of n particles. Let us now consider a 2-particle system and "factor-off" the Galilean invariance by introducing center of mass and relative coordinates.

$$X = \frac{1}{M}(m_1x_1 + m_2x_2) \quad , \quad P = p_1 + p_2 \quad , \quad M = m_1 + m_2 \quad ,$$

and

$$y = x_1 - x_2 \quad , \quad \pi = \frac{1}{M}(m_2p_1 - m_1p_2) \quad , \tag{3.1}$$

respectively. Because of Galilean invariance the Hamiltonian splits into $H = H_{CM} + H_r$, where

$$H_{CM} = \frac{P^2}{2M} \quad , \quad [X,P] = i\hbar \quad ,$$

and

$$H_r = \frac{\pi^2}{2\mu} + \phi(y) \quad , \quad [y,\pi] = i\hbar \quad , \tag{3.2}$$

where $\mu = m_1m_2/M$ is called the reduced mass.

Clearly H_{CM} describes the motion of the centre of mass and H_r the relative motion of the particles.

The equations of motion derived from the "relative" Hamiltonian (3.2) will not, in general, retain any of the original Galilean invariance. However, in particular cases (i.e., for particular potentials $\phi(y)$) they may retain invariance under a subgroup of the Galilean group (e.g., the rotation group) or they may happen to be invariant under special groups which have nothing to do with Galilean invariance. In this lecture we wish to consider such cases. For this purpose, we define an invariance group.

Definition: An invariance group is defined to be any group of transformations on $\mathcal{H}$, the Hilbert space of y, π, which leaves invariant

a) the Hamiltonian H

b) the absolute values of the inner products $|(f,g)|$.

We first discuss the motivation for this definition. That the group should leave

the Hamiltonian invariant is practically self-explanatory since this is true of an invariance group even in classical mechanics. We only note that (in both classical and quantum mechanics) the invariance of H is slightly stronger than the demand that the group leave the equations of motion invariant. (For example, as we saw for an isolated system, the Galilean group left the equations of motion invariant but not the Hamiltonian.) However, for invariance groups of the relative Hamiltonian, the distinction between H and the equations of motion usually does not arise, and the invariance of H is used as the simplest and most compact was of defining invariance.

The more interesting question concerns b), namely the invariance of the inner products $|(f,g)|$ which are peculiar to quantum mechanics. The question is whether this demand is necessary, or at least reasonable.

For a group of transformations which have a passive interpretation, as is the case for the Galilean group G, the answer is yes. For if we change the observer of a system, without changing the system itself, the probability of the system making any particular transition $g \to f$ cannot change (since the system "does not know who is looking at it") and this is just another way of saying that $|(f,g)|$ is invariant.

For transformations which do not have a passive interpretation, i.e., for which we must change the system itself to implement them (these are usually transformations which have no geometrical interpretation), the argument is not so easy to establish. However, it is usual to demand the invariance of the probabilities in this case also, if only for simplicity and to preserve the analogy with the active case.

Demanding that the probabilities $|(f,g)|^2$ remain invariant, we come to a second question: Are unitary ray representations the most general group representations which leave the probabilities invariant?

To answer this, one first concentrates on a single transformation T and asks: What is the most general T such that

$$|(Tg,Tf)| = |(g,f)| \quad , \quad g,f \in \mathcal{H} \quad . \tag{3.3}$$

If T is linear, then the answer is simple: T must be unitary. In general, however, there is no need for T to be linear. In that case, we fall back on the following remarkable theorem due to Wigner [1].

Theorem

Let T be a transformation satisfying (3.3). Then there exists a unitary or anti-unitary transformation U such that for all $f \in \mathcal{H}$

$$(U^{-1}T)f = e^{i\delta(f)}f \quad . \tag{3.4}$$

Note that U is then unique up to a phase-factor, $\exp(i\delta)$, which is independent of f. [An anti-unitary transformation is defined to be a transformation such that

$$(Uf,Ug) = (g,f) = (f,g)^* \quad] \quad . \tag{3.5}$$

This theorem means that, for rays, T is equivalent to, and may be replaced by, a unitary or anti-unitary transformation.

This theorem was first stated by Wigner in his book on group theory in 1931. [1] However, the proof given in the book is not complete, and since then many papers [2] have been devoted to completing, simplifying and generalizing the proof.

The most definitive proof is that given by Bargmann [3] in 1964. This proof has the advantage of being basis-free and hence valid for non-separable as well as separable Hilbert spaces.

Wigner's theorem applies to any fixed transformation T. Consider now a group of transformations T(g). For each fixed g, T(g) can be replaced by a unitary or anti-unitary transformation U(g), unique up to a phase-factor $\exp i\delta(g)$. Using the group relation

$$T(g)T(g') = T(gg') \quad , \tag{3.6}$$

Equation (3.4), and the unitarity (or anti-unitarity) of U(g), one sees that

$$U(g)U(g') = U(gg')e^{i\omega(g,g')} \quad , \tag{3.7}$$

where $\omega(g,g')$ is a real number. It follows that any group of transformations T(g) preserving the probabilities (3.2) is equivalent to a set of unitary or anti-unitary transformations U(g) forming a ray representation of the group. In this sense, unitary of anti-unitary ray representations are the most general group representations preserving the probabilities.

In practice, only one anti-unitary transformation is used in physics. This is the time-reversal transformation. To keep the quantum mechanical equations of motion

$$\frac{dF}{dt} = \frac{i}{\hbar}[H,F] \quad , \tag{3.8}$$

invariant under time-reversal, it is necessary to let either $H \to -H$ or $i \to -i$ when $t \to -t$ · $H \to -H$ is ruled out because $H \geq 0$. Hence, $i \to -i$, and this leads to an anti-unitary transformation.

We turn now to some examples of invariance groups in quantum mechanics. For this purpose, it is usual to consider the relative motion Hamiltonian

$$H = \frac{\pi^2}{2\mu} + \phi(y) \quad . \tag{3.9}$$

The problem is, given $\varphi(y)$, to find unitary groups of operators which commute with this H, and have a direct physical meaning. Indeed, in practice, it

is usually the physical meaning that enables us to find the groups. The advantages of finding such groups are:

1) Since for the group generators G,

$$[H,G] = 0 \quad , \tag{3.10}$$

the group provides in the G's at least some of the constants of the motion.

2) At the same time, the G's are natural operators to diagonalize simultaneously with H.

3) The group can be used to reduce enormously the labor involved in making a calculation with the Hamiltonian, e.g., calculating an energy level, an emission probability, or a scattering amplitude.

Note that Equation (3.10) can be looked at from two points of view: The group generated by G leaves H invariant (is an invariance group of the equations of motion). Conversely, the group generated by H leaves G invariant (G is conserved).

Let us illustrate points 1), 2), and 3) above with the most important special case of an invariance group; namely, the case when $\varphi(y)$ in (3.9) is central, i.e., depends only on r where $r^2 = y_1^2 + y_2^2 + y_3^2$. In this case, H commutes with the rotation group generated by the three operators $L = y \times \pi$, with Lie algebra $[L,L] = iL$, and which are at the same time identified with the relative angular momenta of the particles in the 1, 2, 3 directions. [The transition from the group to the algebra and back will be justified in the next section.] Now with respect to 1) above it is clear that L_1, L_2, and L_3 are conserved. With respect to 2) it is not difficult to show that the so-called total relative angular momentum $L^2 = L_1^2 + L_2^2 + L_3^2$ and any one of L_1, L_2, L_3 (usually L_3) can be added to H to form a complete set on $\mathcal{H}$ ($\mathcal{H}$ being assumed irreducible with respect to $[y,\pi] = i\hbar$). Thus, a convenient and physically relevant basis in $\mathcal{H}$ is $f(\varepsilon,\ell,m)$ where

$$\begin{aligned} Hf(\varepsilon\ell m) &= \varepsilon f(\varepsilon\ell m) \quad , \\ L^2 f(\varepsilon\ell m) &= \ell(\ell+1) f(\varepsilon\ell m) \quad , \\ L_3 f(\varepsilon\ell m) &= m f(\varepsilon\ell m) \quad , \end{aligned} \tag{3.11}$$

where, because the rotation group is compact, ℓ is a non-negative integer and $-\ell \leq m \leq \ell$.

With respect to 3), we see at once that in calculating the eigenvalues of H, which are the eigenvalues of the differential operator

$$-\frac{\hbar^2}{2m}\nabla^2 + V(r) \quad , \tag{3.12}$$

on L_2, the use of (3.11) reduces the partial differential operator (3.12) to the simple differential operator

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{d}{dr}\,r^2\,\frac{d}{dr}-\frac{\ell(\ell+1)}{r^2}\right]+V(r) \quad , \tag{3.13}$$

and so simplifies the calculation.

But the group does much more for us than that. For example, if we wish to calculate the probability of a particle in the state $f(\varepsilon,\ell,m)$ emitting a photon with momentum $\vec{k}$ and ending up in a state $f(\varepsilon',\ell',m')$, then, to lowest order in the EM coupling constant e, and provided the wavelength of the emitted photon is large compared with the size of the atom [1,4], the relevant inner products to compute are the multipole moments of the particle. A typical one of these is the dipole moment,

$$E_a = \int (f_{\varepsilon'\ell'm'}, y_a f_{\varepsilon\ell m}) \quad , \quad a = 1,2,3 \quad . \tag{3.14}$$

Now for even quite low value of ℓ and ℓ', the number of quantities (3.14) to be computed is quite large, since $-\ell' \le m' \le \ell'$, $-\ell \le m \le \ell$. But thanks to the group properties of y (it is a polar vector with respect to rotations and space reflexions), we can

a) show that the E_a vanish unless $\ell' = \ell \pm 1$, $m' = m$, $m \pm 1$,

b) for $\ell' = \ell \pm 1$, reduce the calculations in each case to <u>one</u> calculation. In fact, the group invariance implies that

$$E_a = \varphi(\ell'm',\ell m)\int_0^\infty r^2 dr F^*_{\varepsilon'\ell'}(r) r F_{\varepsilon\ell}(r) \quad , \quad \ell' = \ell \pm 1 \quad , \tag{3.15}$$

where $m' - m = 0, \pm 1$ for $a = 3, 1 \pm i2$ respectively and the $F_{\varepsilon\ell}$ are the eigenfunctions of the simple differential operator (3.13). The crucial point about (3.15) is that the m' and m dependence appears only in the coefficients (Clebsch-Gordon coefficients) <u>which are independent of</u> $V(r)$. Thus, these coefficients need only be calculated once and for all (Figure 3.1), and then they can be used for <u>any</u> central potential. (The functions in the integral will, of course, depend on $V(r)$.)

$\ell' =$	$m' = m + 1$	$m' = m$	$m' = m - 1$
$\ell + 1$	$\sqrt{\frac{(\ell + m')(\ell + m' + 1)}{(2\ell + 1)(2\ell + 2)}}$	$\sqrt{\frac{(\ell - m' + 1)(\ell + m + 1)}{(2\ell + 1)(\ell + 1)}}$	$\sqrt{\frac{(\ell - m')(\ell - m' + 1)}{(2\ell + 1)(2\ell + 2)}}$
ℓ	$-\sqrt{\frac{(\ell + m')(\ell + m' + 1)}{2\ell(\ell + 1)}}$	$\frac{m'}{\sqrt{\ell(\ell + 1)}}$	$\sqrt{\frac{(\ell - m')(\ell + m' + 1)}{2\ell(\ell + 1)}}$
$\ell - 1$	$\sqrt{\frac{(\ell - m')(\ell - m' + 1)}{2\ell(2\ell + 1)}}$	$-\sqrt{\frac{(\ell - m')(\ell + m')}{\ell(2\ell + 1)}}$	$\sqrt{\frac{(\ell + m' + 1)(\ell + m')}{2\ell(2\ell + 1)}}$

FIGURE 3.1. VALUES OF $\phi(\ell'm',\ell m)$

The labor saved by using one group to obtain the results a) and b) in this example is obviously immense. Furthermore, the use of the group gives a much deeper insight into what is going on. It isolates the group properties of a central potential (independence of the potential of the angular variables θ, φ) from the dynamical properties (form of the dependence of $V(r)$ on r). The results a) and b) for this example are, of course, a special case of the Wigner-Eckart theorem, which has already been mentioned by Louis Michel and will be formulated for completeness in the next chapter.

We conclude by considering two Hamiltonians which have special invariance groups. The first is the harmonic oscillator Hamiltonian

$$H = \frac{1}{2m}\pi^2 + \frac{1}{2\kappa}y^2 \quad , \tag{3.16}$$

where κ is a constant. This is centrally symmetric and has the angular momentum invariance group generated by L discussed above. But, in addition, H commutes with the six operators

$$M_{ab} = X_a X_b + P_a P_b \quad , \tag{3.17}$$

where $X = \alpha y$, $P = \alpha^{-1}\pi$, $\alpha^4 = m\kappa$, and these six operators, together with the corresponding L_a, form the Lie algebra

$$\begin{aligned} [L,L] &= iL \quad , \\ [L,M] &= iM \quad , \\ [M,M] &= iL \quad , \end{aligned} \tag{3.18}$$

of the compact, connected Lie group $U(3)$. Thus, the Hamiltonian (3.16) is $U(3)$ invariant and, in fact, is just $M_{aa}(4m\kappa)^{-1/2}$.

For 1 particle, this result is not particularly exciting because the Hamiltonian (3.16) is so simple that we can calculate its properties directly anyway. However, in nuclear physics, in the nuclear shell model, it is much more interesting. [5] In the nuclear shell model, it is assumed that the particles in the nucleus interact with each other in such a way that, for each particle, the total effect is the same as if it were in a strong central potential due to all the other particles, together with somewhat weaker potentials due to the effects of other individual particles. A special case of this model is the Elliott model, in which one assumes that

a) the central potential is the harmonic oscillator potential.

b) the smaller potentials, while not $U(3)$-invariant, have definite $U(3)$ tensor properties (like X in the dipole moment). (These

properties are guessed from the general nature of the individual potentials, e.g. that they are 2-body interactions.)

From a) and b), one can go ahead, apply the Wigner-Eckart theorem, and deduce some general properties of the nuclei (e.g., the spacing of the energy levels) without having specified the potential in detail.

The second Hamiltonian we consider is the more spectacular

$$H = \frac{P^2}{2m} - \frac{Ze^2}{R} \quad , \tag{3.19}$$

of a particle in an attractive $1/R$ potential, e.g., of an electron in a hydrogen atom, considered already by Louis Michel. As he points out, using the $SO(4)$ invariance with generators L and the Lenz vector

$$A = \frac{1}{\sqrt{-2H}} \left\{ \frac{\vec{R}}{R} + \frac{1}{2mZe^2} (L \times P - P \times L) \right\} \quad , \tag{3.20}$$

one can predict [6]

1) the $SO(3)$ (angular momentum) content of each energy level (Figure 3.2), and
2) the value of the energy for each level.

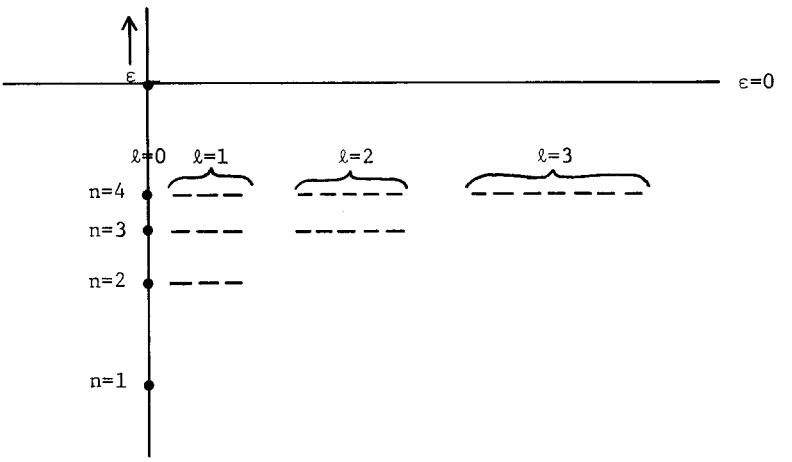

FIGURE 3.2. ANGULAR MOMENTUM CONTENT OF H-ATOM ENERGY LEVELS

The only thing one cannot predict is the multiplicity of the $SO(4)$ representation for each level. I should like to add just two comments to Michel's remarks.

a) The Lenz vector also has a meaning in classical physics; namely, for the planets in the gravitational field of the sun, it is a vector

directed along the major axis of the ellipse with length equal to the eccentricity. The fact that it is a constant of the motion is reflected in the fact that the ellipse does not precess. It is perhaps amusing to see that the absence of planetary precession and the degeneracy of the spectrum of the hydrogen atom have the same origin!

b) The second point is just a remark in defense of the groups $SO(4) \times T_H$ or $SO(4,1)$ which contain $SO(4)$ and have representations that can be used to describe all the bound states of the H-atom with the correct multiplicity. The remark is that these two groups can be used to simplify many spectroscopic calculations, and have even been used for calculations which were not feasible by direct methods [7].

4. GENERAL RESULTS ON REPRESENTATIONS OF LIE GROUPS

In this section, we will fill in some of the mathematical gaps which were left in the previous discussion. In particular, we wish to establish the connection between representations of the Lie algebras and the corresponding representations of Lie groups, to define unbounded tensor operators, and finally to formulate the Wigner-Eckart (WE) theorem.[1]

We begin with the case of 1-parameter continuous groups. From Stone's theorem, we know that to any 1-parameter continuous group of transformations $U(t)$ on $\mathcal{H}$, there corresponds a unique skew-adjoint generator G with a dense domain $\mathcal{D}$ on which

$$\frac{dU(t)}{dt} = GU(t) \quad , \tag{4.1}$$

and, conversely, to any skew-adjoint operator G with dense domain $\mathcal{D}$ there corresponds a unique continuous group of unitary transformations such that (4.1) is true on $\mathcal{D}$.

Furthermore, from the spectral resolution [2],

$$iG = \int \lambda dE(\lambda) \quad , \tag{4.2}$$

of G, we see at once that the vectors

$$[E(a) - E(b)]h \quad , \tag{4.3}$$

for all finite intervals $[a,b]$ and all $h \in \mathcal{H}$, form a dense domain A on which

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} G^n \quad , \tag{4.4}$$

converges to $U(t)$. A vector f for which (4.4) converges is said to be an analytic vector for G.

The question now is: What are the analogues, if any, of these results for general groups of continuous unitary transformations on $\mathcal{H}$? The answer is that for completely general groups, definitive results are not available. But for the important special case of simply connected finite parameter Lie groups, almost exact analogues of the above results have been established. We shall restrict ourselves to this case and let $U(g)$ denote henceforth a continuous unitary representation of a simply connected Lie group G on $\mathcal{H}$.

Since the Lie algebra of G contains r elements, where r is the number of independent parameters of G, the question in this case concerns the existence of common domains analogous to $\mathcal{D}$ and A above, for the r elements

$$G_\alpha = \left(\frac{dU(g)}{dx_\alpha}\right)_{x=0} \quad , \quad \alpha = 1 \ldots r \quad , \tag{4.5}$$

of the Lie algebra of G.

The existence of a common dense domain for the G_α was first established by Gårding [3] who in 1947 exhibited the domain $\mathcal{D}_g$ consisting of the vectors

$$\int d\mu(g')U(g')f(g')h \quad , \tag{4.6}$$

where $\mu(g')$ is the group invariant measure, $f(g')$ is any infinitely differentiable function of compact support over the group, and h is any vector in $\mathcal{H}$. (Note that $\mathcal{D}_g$ is not only a common dense domain G_α, but is invariant with respect to both G_α and $U(g)$.)

It was soon shown by Segal [4] that the G_α are actually essentially skew-adjoint on $\mathcal{D}_g$, i.e., their restrictions to $\mathcal{D}_g$ have unique skew-adjoint extensions. This is actually a special case of the following lemma which was later proved by Nelson. [5]

Lemma

The G_α are essentially skew-adjoint on any dense domain $\mathcal{D}$ which is invariant with respect to $U(g)$.

Proof. Let f be an eigenvector of $G_\alpha^\dagger$ with complex eigenvalue ζ. Then the function $\psi(g) = (f, U(g)d)$, $d \in \mathcal{D}$ both satisfies the differential equation

$\frac{\partial\psi(g)}{\partial x_\sigma} = \zeta\psi(g)$ and is bounded. Hence it is zero, in which case, since $\mathcal{D}$ is dense, $f = 0$. Thus, the deficiency indices of G_α are zero, i.e., G_α is essentially skew-adjoint [2].

The next question is whether there exists a common dense domain of analytic vectors for the G_α, i.e., a dense domain on which

$$\sum_{n=0}^{N} \frac{(t)^n}{n!} (G)^n \tag{4.7}$$

converges where G is any linear combination of the G_α. Here the spectral theorem does not help since, in general, the closures $\overline{G}_\alpha$ and $\overline{G}_\beta$ do not commute and so cannot be simultaneously resolved. Furthermore, the Gårding domain $\mathcal{D}_g$ does not help since that is not in general analytic. However, it has been shown by Cartier and Dixmier [6], Nelson [7] and Gårding [8] that for a unitary representation of a Lie group, a common dense analytic domain for the Lie algebra does in fact exist. Here we describe briefly a simplification of Nelson's proof due to Gårding. The point is to replace the infinitely differentiable functions of compact support $f(g)$ in the Gårding integral by a dense set of analytic functions $a(g)$ of sufficiently fast decrease to counter the (at most exponential) growth of the Haar measure and make the integral converge. Such a dense set of functions is given by

$$a(g) = e^{t\overline{\Delta}} f(g) \quad , \quad t > 0 \quad , \tag{4.8}$$

where $\overline{\Delta}$ is the unique self-adjoint extension of the operator

$$\Delta = 1 - G_1^2 - G_2^2 - \ldots - G_r^2 \quad , \tag{4.9}$$

on the Gårding domain $\mathcal{D}_g$ for the regular representation. The functions $a(g)$ have Gaussian decrease for $t > 0$.

It is interesting to note that the above results concerning the existence of a Gårding and analytic dense domain are not confined to unitary representations. They hold for any continuous representation by bounded operators. This is clear for the Gårding domain and follows for the analytic domain because, for a continuous representation, the growth of $U(g)$, like the Haar measure, is at most exponential. Even the result that the G_α are skew-adjoint on any group invariant domain $\mathcal{D}$ generalizes; namely, if superscript c denotes contragredient quantities, we have

$$(G_\alpha^c/\mathcal{D}_c)^\dagger = -\overline{(G_\alpha/\mathcal{D})}$$

So far, we have given the group representation $U(g)$ and asked questions about the Lie algebra. Now we ask the converse question: What is a necessary and sufficient condition that a Lie algebra G_α generate a unique unitary group $U(g)$ on $\mathcal{H}$? An answer was given by Nelson [7] in 1959, who established the following theorem:

Theorem

A necessary and sufficient condition that a Lie algebra of symmetric operators iG_α be the Lie algebra of a unique unitary Lie group $U(g)$ on $\mathcal{H}$ is that there exist in $\mathcal{H}$ a common dense invariant domain $\mathcal{D}$ for the G_α on which the iG_α are symmetric, and the operator

$$\Delta = -\sum_{\alpha=1}^{r} G_\alpha^2 + 1 \quad ,$$

is essentially self-adjoint.

In the course of the proof, Nelson has shown that the analytic domain for the self adjoint extension $\overline{\Delta}$ of the operator Δ is a common analytic domain for the Lie Algebra, and thus furnished an alternative proof of the existence of a dense analytic domain for the unitary representations. The essence of Nelson's proof is to obtain, from the general form of the commutation relations and the obvious bounds $\|G\| < \|\overline{\Delta}\|$, $\|G^2\| < \|\overline{\Delta}\|$, a bound $\|G^n\| < C_n\|\overline{\Delta}^n\|$, where $C_n \leq n!$. Then, if $\sum \frac{(t\overline{\Delta})^n}{n!} < \infty$ for all t, $\sum \frac{(tG)^n}{n!} < \infty$ for $t < t_0$, where $t_0 > 0$. Note that, in general, the entire vectors for $\overline{\Delta}$, i.e., $\sum \frac{(t\overline{\Delta})^n}{n!} < \infty$ all t, are not necessarily entire vectors for $\overline{G}$. Indeed, in general there do not exist any entire vectors for the Lie algebra of a unitary Lie group. The unitary representations of $SL(2,C)$ already provide a counter-example. Recently it has been shown by R. Goodman [10] that the analytic domain for the Lie algebra is exactly the analytic domain for the operator $\Delta^{1/2}$. Goodman has also discussed the question of the existence of entire vectors [11].

From the above results, namely the existence of an analytic domain for any continuous representation, and the existence of a unique continuous unitary representation when Δ is essentially self-adjoint, it is evident that for continuous Lie groups the relationship between Lie algebra and Lie group representations is all that could be required. We can operate relatively freely with the algebra in spite of the unbounded nature of the operators, a circumstance we had anticipated earlier. We close with a few incidental remarks:

First, in the case of UIR's of semisimple Lie groups, there are some stronger results due to Harish-Chandra.[9] For example, the vectors in the (necessarily finite dimensional) subspaces, which are invariant with respect to the

maximal compact subgroup of the group, are analytic vectors for the whole group. Furthermore, the linear span of such vectors, which is dense in $\mathcal{H}$, can be generated from any one such vector using the enveloping algebra of the Lie algebra.

Second, there are still some outstanding problems. One is to find an analogue of Nelson's results (Δ essentially self-adjoint) for non-unitary representations. Another is to ask for statements concerning the analytic continuation of the functions $(h,U(g)a)$ to complex values of the group parameters. How close are the singularities? Are they poles or cuts? And so on.

We next consider briefly the domain question for tensor operators. For a set of operators T_α, $\alpha = 1...s$ to transform as a tensor under a unitary group $U(g)$, we need only a dense domain $\mathcal{D}$ with

1) the T_α essentially self-adjoint on $\mathcal{D}$,
2) $\mathcal{D}$ stable with respect to $U(g)$,
3) $U(g)T_\alpha U^{-1}(g) = \hat{D}_{\beta\alpha}(g)T_\beta$ on $\mathcal{D}$, where $\hat{D}(g)$ is a representation of $U(g)$.

$\hat{D}(g)$ is usually finite-dimensional ($r < \infty$), but the definition can be extended to cover infinite dimensional representations as well.

If the group $U(g)$ is compact, one is usually interested not in the full (generally unbounded) tensor components T_α, but only in the restrictions $P'T_\alpha P$, where P,P' are the projections onto finite dimensional subspaces of $\mathcal{H}$ which are invariant with respect to $U(g)$. For the restrictions $P'T_\alpha P$ to exist, one needs only the weaker condition that there exist a dense domain $\mathcal{D}$ for the T_α such that $P\mathcal{D} \subset \mathcal{D}(\overline{T}_\alpha)$, where $\overline{T}_\alpha$ is the unique self-adjoint extension of T_α. The physical conditions are usually enough to guarantee this.

For example, in the dipole radiation example of the last section, the relevant matrix elements were $(f_{\varepsilon'\ell'm'}, y_a f_{\varepsilon\ell m})$, i.e., they were the matrix elements of the restrictions of y_a to the finite spaces $f_{\varepsilon\ell m}$. One can see that these restrictions must exist from the physical point of view as follows: The dipole radiation is actually just the first coefficient in the expansion of $(f_{\varepsilon'\ell'm'}, e^{iy_a/\lambda} f_{\varepsilon\ell m})$ in powers of $1|\lambda$, where λ is the wavelength of the emitted radiation. Now the restriction $P' \exp iy_a/\lambda P$ certainly exists since $\exp iy_a/\lambda$ is a bounded operator, so the only question is the validity of the subsequent expansion in powers of $1|\lambda$. This expansion is justified on the physical grounds that the wavelength λ can be (and in practice usually is) large compared with the mean value of $|y|$ for the wavefunction $f_{\varepsilon\ell m}$, i.e., compared with the "size" of the atom.

Finally, we consider the WE theorem. Let $U(g)$ be a unitary representation of G on $\mathcal{H}$ and T_α a tensor component belonging to the representation $\hat{D}(g)$. Let $\mathcal{H}_1, \mathcal{H}_2$ be irreducible subspaces of $\mathcal{H}$ with respect to $U(g)$, let

$\mathcal{H}_\Lambda$ be the Hilbert space for $D^\Lambda(g)$, and let the product space $\mathcal{H}_\Lambda \otimes \mathcal{H}_2$ decompose into

$$\mathcal{H}_\Lambda \otimes \mathcal{H}_2 = \sum_\lambda \oplus \mathcal{H}_\lambda \quad , \tag{4.10}$$

with respect to $U(g)$. The WE theorem states that

$$(f_1, T_\alpha f_2) = \sum_\lambda (f_\lambda, f_\alpha f_2) \varphi(\mathcal{H}_1 T \mathcal{H}_2)_\lambda \quad , \tag{4.11}$$

where the sum is taken over all λ such that the representations $D^\lambda(g)$ and $(U(g)/\mathcal{H}_1)$ are equivalent and f_λ, f_α are vectors in the directions f_1 and T_α, respectively. In other words, the T-dependent tensor $(f_1, T_\alpha f_2)$ can be expanded linearly in terms of the T-independent tensors $(f_\lambda, f_\alpha f_2)$ with scalar coefficients $\varphi(\mathcal{H}_1 T \mathcal{H}_2)_\lambda$. In particular, if $U(g)/\mathcal{H}_1$ occurs only once in the decomposition (4.10), then

$$(f_1, T_\alpha f_2) = (f_1, f_\alpha f_2) \varphi(\mathcal{H}_1 T \mathcal{H}_2) \quad , \tag{4.12}$$

i.e., $(f_1, T_\alpha f_2)$ is parallel to $(f_1, f_\alpha f_2)$.

The coefficients $\varphi(\mathcal{H}_1 T \mathcal{H}_2)_\lambda$ are usually called reduced matrix elements, and the T-independent tensors $(f_\lambda, f_\alpha f_2)$ are called Clebsch-Gordon coefficients. Note that the $(f_\lambda, f_\alpha f_2)$ are just the matrix elements of the unitary (intertwining) operator which transforms the direct product basis in $\mathcal{H}_\Lambda \otimes \mathcal{H}_2$ into the basis in which $U(g)$ is diagonal.

5. SURVEY OF EXPERIMENTAL AND THEORETICAL BACKGROUND TO ELEMENTARY PARTICLE PHYSICS

The rest of these chapters will be devoted to the group theory of elementary particle physics. But before going on to the group theory proper, it might be worthwhile to fill in a little of the experimental and theoretical background. This we shall do in the present chapter.

First we consider the experimental background [1].

The non-relativistic quantum mechanics discussed up to now suffices to describe completely the greater part of modern physics--atomic, molecular, plasma, solid state, low temperature, etc., physics. It is built on the twin postulates of Newton's laws and $[X,P] = i\hbar$. The basic constituents of matter for all these branches of physics are the protons, neutrons, and electrons which form the atoms, and the photons, which carry the EM (electromagnetic) field. These constituents of matter, or particles, are regarded as elementary. In particular, the protons, neutrons, and electrons are regarded as indestructible.

As soon, however, as one wishes to inquire into the finer features of atomic phenomena or wishes to investigate the structure of the atomic nucleus or the structure of the protons, neutrons, and electrons themselves, then the situation changes drastically. First, the energies necessary for the investigation are relativistic. Second, the electrons, protons, and neutrons are found to be far from indestructible. They can be destroyed and created almost at will. Third, not only can these particles be destroyed and created, but new particles are created and destroyed along with them. The new particles include the anti-particles of the proton, neutron, and electron, the π-meson which keeps the protons and neutrons bound in the nucleus, and many other particles (along with their anti-particles). To date, the number of new particles which have been produced is of the order of 100.

It should, perhaps, be emphasized that the particles referred to here differ in some fundamental ways from the Newtonian particles defined in the first lecture; namely,

a) they can be created and destroyed.

b) Although they can be created and destroyed, their masses are not arbitrary but are fixed by nature to have definite values outside our control. For example, the electron has a mass 9.11×10^{-28} grams.

c) As well as an intrinsic mass, the particles have an intrinsic angular momentum. The Casimir operator of the intrinsic angular momentum group takes the values $J(J + 1)$, where J (the spin of the particle) is half-integer.

Thus, the particles appear to be particles in the sense of Democritus (fixed, ultimate constituents of matter) rather than of Newton (fictitious limits of small bodies). For this reason they are called elementary particles. Of course, it is difficult to believe that 100 particles can be elementary, but until something more elementary is discovered, they are regarded as such. (An analogy is provided by the chemical elements, all 92 of which were regarded as elementary until the advent of atomic theory.)

In Figure 5.1, a list of the particles is presented. They are grouped together into multiplets (so-called isospin multiplets) of particles with approximately the same mass and spin. Even so, the number of multiplets is very large and it might help to clarify the situation a little if we briefly classify them by word.

The broadest classification of the particles is in terms of their interactions. Apart from the gravitational interactions, in which all the particles participate, but which are so weak as to be negligible, the particles can interact in only three ways:

a) By electromagnetic interactions, with coupling constant $e^2/\hbar c \sim 1/137$

b) By weak interactions, with coupling constant $g^2 << e^2/\hbar c$

c) By strong interactions, with coupling constant $G^2 >> e^2/\hbar c$.

Name	$I^G(J^P)C_n$ estab. ?=guess
$\pi^{\pm}(140)$	$1^-(0^-)+$
$\pi^o(135)$	
$\eta(549)$	$0^+(0^-)+$
$\eta_{o^*}(700)$ "ε"$\rightarrow\pi\pi$	$0^*(0^+)+$
$\rho(765)$	$1^+(1^-)-$
$\omega(784)$	$0^-(1^-)-$
$\eta'(958)$ or X^o	$0^+(0^-)\pm$
$\delta(692)$ $\pi_N(1016)$ $\rightarrow K\bar{K}$	$\geq 1(\ )$
$\phi(1019)$	$0^-(1^-)-$
$\eta_{o^*}(1060)$ "S*"$\rightarrow K_S K_S$	$0^+(0^+)+$
A1(1070)	$1^-(1^+)-$
$\rho_N(1660)$ "g"$\rightarrow 2\pi$	$1^+(N)-$
$\rho(1710)$? $\rightarrow 4\pi$	$1^+(\ )-$
U(2375)	$1^+(\ )-$
$K^+(494)$	$1/2(0^-)$
$K^o(498)$	
K*(892)	$1/2(1^-)$
$K_A(1240)$ or C	$1/2(1^+)$
$K_A(1280$ to $1360)$?	$1/2(1^+)$
$K_N(1420)$	$1/2(2^+)$
$K_A(1775)$ or L	$1/2$(A)
B(1235)	$1^+(1^+)-$
f(1260)	$0^+(2^+)+$
D(1285)	0^+(A)+
$A2_L(1280)$	$1^-(2^+)+$
$A2_H(1320)$	$1^-(2^+)+$
E(1422)	$0^+(0^-)\pm$
f'(1514)	$0^+(2^+)+$
$\pi/\rho(1540)$ "F_1"~K*$\bar{K}$?	1(A)
$\pi_A(1640)$ $\rightarrow 3\pi$	1^-(A)+

Isospin 0

$\eta(550)$	0^-
$\omega(780)$	1^-
$\eta^*(960)$	0^- (?)
$\phi(1020)$	1^-
$\eta(1070)$	0^+
f(1260)	2^+
D(1285)	$P=(-1)^{J+1}$
E(1420)	0^- (?)
f*(1515)	2^+
$\eta(700)$	0^+

Isospin 1/2

K(490)	0^-
K*(890)	1^-
K(1320)	1^+
K**(1420)	2^+
K(1780)	$P=(-1)^{J+1}$

Isospin 1

$\pi(140)$	0^-
$\rho(760)$	1^-
$\pi(1016)$	0^+
$A_1(1070)$	1^+ (?)
$A_2(1270)$	$P=(-1)^J$
$A_2(1315)$	2^+ (?)
$\pi(1640)$	$P=(-1)^{J+1}$
$\rho(1650)$	$P=(-1)^J$
B(1235)	1^+ (?)

SU(3) multiplets

$\pi(140)$, K(490), $\eta(550)$, $\eta^*(960)$	0^-
$\rho(760)$, K*(890), $\omega(780)$, $\phi(1020)$	1^-
$A_2(1315)$, K**(1420), f(1260), f*(1515)	2^+

FIGURE 5.1b MESONS

The following bumps have also been observed, but their spins and parities are not yet known; $\sigma(410)$; H(990); $\eta_\nu(1080)$; A1.5(1170); $A2_2(1320)$; $\rho\rho(1410)$; $K_SK_S(1440)$; $\phi(1650)$, R(1750); η or $\rho(1830) \rightarrow 4\pi$; ϕ or $\pi(1830) \rightarrow \omega\pi\pi$; S(1930); $\rho(2100)$; T(2200); $\rho(2275)$ $N\bar{N}_{I=0}(2380)$; $\kappa(725)$; $K_N(1080\text{-}1260)$; $K_{A(I=3/2)}(1175)$: $K_{A(I=3/2)}(1265)$; $K_{N(I=1/2)}(1660)$; $K^*(2240) \rightarrow \bar{Y}N$; $X^-(2500)$; $X^-(2620$; $X^-(2880)$.

	Particle or resonance[2]	I (J^P)
N	p	1/2(1/2+)
	n	
	N'(1470)	1/2(1/2+)
	N'(1520)	1/2(3/2−)
	N'(1535)	1/2(1/2−)
	N(1670)	1/2(5/2−)
	N(1688)	1/2(5/2+)
	N''(1700)	1/2(1/2−)
	N''(1780)	1/2(1/2+)
	N(1860)	1/2(3/2+)
	N(1990)	1/2(7/2+)
	N'''(2040)	1/2(3/2−)
	N(2190)	1/2(7/2−)
	N(2650)	1/2(?−)
	N(3030)	1/2(?)
Δ	Δ(1236)	3/2(3/2+)
	Δ(1650)	3/2(1/2−)
	Δ(1670)	3/2(3/2−)
	Δ(1890)	3/2(5/2+)
	Δ(1910)	3/2(1/2+)
	Δ(1950)	3/2(7/2+)
	Δ(2420)	3/2(11/2+)
	Δ(2850)	3/2(?+)
	Δ(3230)	3/2(?)
Λ	Λ	0(1/2+)
	Λ(1405)	0(1/2−)
	Λ'(1520)	0(3/2−)
	Λ'(1670)	0(1/2−)
	Λ''(1690)	0(3/2−)
	Λ(1815)	0(5/2+)
	Λ(1830)	0(5/2−)
	Λ(2100)	0(7/2−)
	Λ(2350)	0(?)
Σ	Σ	1(1/2+)
	Σ(1385)	1(3/2+) P_{13}
	Σ(1670)	1(3/2−) D_{13}
	Σ(1750)	1(1/2−) S_{11}
	Σ(1765)	1(5/2−) D_{15}
	Σ(1915)	1(5/2+) F_{15}
	Σ(2030)	1(7/2+) F_{17}
	Σ(2250)	1(?)
	Σ(2455)	1(?)
	Σ(2595)	1(?)
Ξ	Ξ	1/2(1/2+)
	Ξ(1530)	1/2(3/2+)
	Ξ(1820)	1/2(?)
	Ξ(1930)	1/2(?)
	Ξ(2030)	1/2(?)
	Ξ(2250)	1/2(?)
	Ξ(2500)	1/2(?)
	Ω−	0(3/2+)

SU(3) multiplets

p	$\frac{1}{2}^+$
n	"
Λ	"
Σ	"
Ξ	"
Δ(1236)	$\frac{3}{2}^+$
Σ(1385)	"
Ξ(1530)	"
Ω−(1686)	" (?)
N(1525)	$\frac{3}{2}^-$
Λ(1520)	"
Σ(1660)	"
Ξ(1815)	" (?)
N(1688)	$\frac{5}{2}^+$
Λ(1820)	"
Σ(1910)	"
Ξ(1930)	" (?)

Regge Recurrences

$\frac{1}{2}^+$-multiplet	N(1525)	$\frac{3}{2}^-$	Λ(1520)	$\frac{3}{2}^-$
$\frac{3}{2}^+$-multiplet	N(2190)	$\frac{7}{2}^-$	Λ(2100)	$\frac{7}{2}^-$

FIGURE 5.1a. BARYONS

Data are taken from A. Rosenfeld *et al.*, *Rev. Mod. Phys.* (January, 1970). The numbers in brackets are masses in millions of electron volts. J is the spin (half-odd-integer and integer for baryons and mesons respectively), and P is the parity.

Apart from the photon, which carries the EM field and interacts only electromagnetically, there are three main classes of particles:

1) The leptons: These do not interact strongly. There are four of them; the electron e, the μ-meson, and the two neutrinos ν_e, ν_μ. All have spin 1/2.
2) The baryons: The particles which interact strongly and obey Fermi-Dirac statistics (i.e., have half-odd integer spin).
3) The mesons: The strongly interacting particles which obey Bose-Einstein statistics (have integer spin).

The mesons and baryons can, of course, also interact weakly and electromagnetically, both with each other and with the leptons. The collective name for all strongly interacting particles is hadrons.

Anti-particles are omitted in Figure 5.1 because they have the same masses and spins as the particles. Further subdivisions of the particles have already been considered by Michel and will be touched on again in later lectures. An important property of the particles is their stability, or lack of it, (when left alone). The only really stable ones are the photon, neutrinos, electron, and proton. However, many others are metastable, i.e., have relatively long (10^{-13} sec) lifetimes. These include the leptons, n, Σ, Λ, Ξ, and Ω in class 2), and π, K, η in class 3). The rest of the particles are unstable. They have lifetimes of $\sim 10^{-23}$ secs and are usually not observed directly but as resonances in the scattering cross-sections for metastable particles.

It should, perhaps, be emphasized at this point that the experimental information that we can get on the elementary particles is very limited. The particles are so tiny and so unstable that essentially all one can do is scatter them and watch them decay.

In particular, one can only build particles with masses up to the energies available in the accelerators. Figure 5.1 is based on the present energies (pending the building of the 200 Gev Weston machine and Super-Cern). This table may not be, and probably is not, sufficient to let us see the true picture. For example, ten years ago only the part of Figure 5.1 above the Ω-line was available, and it is now clear that this would have been insufficient to predict today's picture.

Further, one gets information for weak and electromagnetic interactions only when these interactions are not swamped by the strong ones and, for the weak interactions in particular, the information is limited to decay.

For the strong interactions themselves, the information is limited not only by the energies available, but by the particles which are available as targets and projectiles for the scattering. Essentially the only available ones are:

Target: Protons, neutrons (and electrons)

Projectiles: Protons, neutrons (and electrons), photons and the

metastable mesons π and K, together with their anti-particles.

What is actually measured in the strong collisions is the <u>scattering amplitude</u> $A(p_A; p_B; p_{C_1}\ldots p_{C_N})$ for the processes

$$A + B \to C_1 + C_2 + \ldots + C_N \quad ,$$

(Figure 5.2), which is a function of the momenta $p_A \ldots P_{C_N}$ of the particles and whose absolute value squared is the probability for A and B to scatter into particles $C_1 \ldots C_N$ with these momenta.

Similarly, what is measured in electromagnetic interactions is the <u>form-factor</u> $F_{AB}(t)$ whose square is the probability for the particle A with momentum p_A to interact with the EM field, lose momentum k, and emerge as particle B (possibly the same as A) with momentum $p_B = p_A - k$ (Figure 5.3). On account of Lorentz invariance, $F_{AB}(t)$ is essentially a function of

$$t = k^2 = (p_B - p_A)^2 \quad ,$$

only. (It may have some polynomial dependence on P_A and P_B through the spins of A and B.) Actually, at present $F_{AB}(t)$ is known reasonably well (up to $t \simeq$ proton-mass) only for the electron (for which it is trivial), the proton, and the neutron. For some other metastable particles, notably π, Σ, Λ, K, a little is known about it for $t \to 0$.

Thus, to sum up, what has been established experimentally is the existence of a large number ($\sim$100) of particles of definite masses and spins and various lifetimes, most of them short. What can be measured, essentially, are their electromagnetic form-factors $F_{AB}(t)$, their strong scattering amplitudes, and their weak decays, all subject to strong experimental limitations.[1]

The business of elementary particle physics is to construct a theory which will

1) explain the interactions (form-factors, scattering amplitudes, decays) of the particles, and
2) predict their masses and spins.

This is a tall order since it combines 1) solving Newton's problem at a subnuclear level with solving 2) the problem of the structure of matter.

Not surprisingly, one has at present nothing like a complete theory of the elementary particles, though one does have some ideas and a workable, if not yet mathematically rigorous, theory of electromagnetic interactions. Almost all the ideas one has can be traced back to the theory of quantized fields introduced by Pauli, Heisenberg, and Dirac [2] in the heroic days of quantum mechanics, 1925-28. Because they lie at the root of most later developments and because they are necessary later as background for relativistic group theory, we conclude this lecture with a

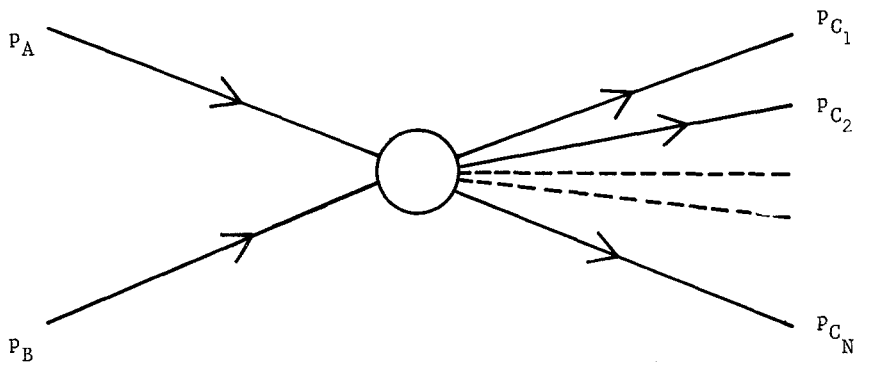

FIGURE 5.2

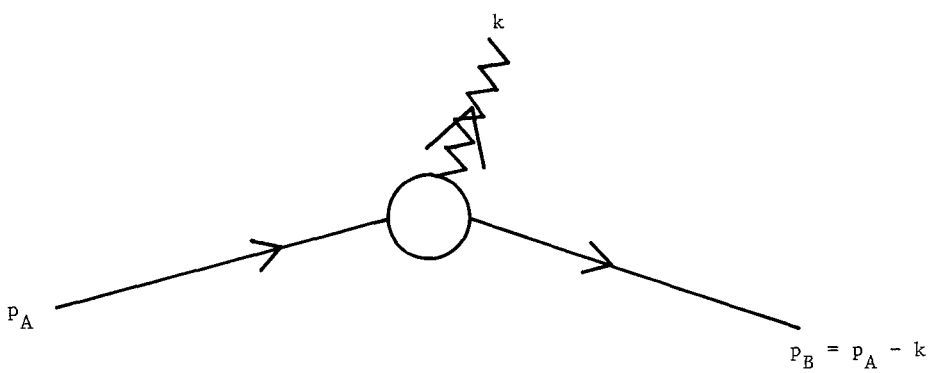

FIGURE 5.3

brief review of the ideas underlying the theory of quantized fields.

To begin with, we return to the Hamiltonian, which describes a non-relativistic classical particle in a potential

$$H = \frac{p^2}{2m} + \varphi(x) \quad . \tag{5.1}$$

Generalizing to describe interactions with the EM field and the field equations for the EM field itself, we have

$$H = \frac{p^2}{2m} + e[\varphi(x) + \frac{1}{c}\vec{v}\cdot\vec{A}(x)] + \frac{1}{2}\int d^3y[\pi(y)^2 + (\nabla A(y))^2] \quad , \tag{5.2}$$

where $A = (\varphi,\vec{A})$ is the EM potential and the integral term is the Hamiltonian for the free EM field. (It is equivalent to $\frac{1}{2}\int d^2y[E(y)^2 + H(y)^2]$ where $(E,H) = F_{\mu\gamma} = \partial_\mu A_\gamma - \partial_\gamma A_\mu$, but the form (5.2) is better for later quantization.) We can also write the interaction term (with coupling coefficient, or charge, e) as

$$e\int d^3y j_\mu(y)A_\mu(y) \quad , \tag{5.3}$$

where

$$j_\mu(y) = \delta(x - y)[1,v/c] \quad . \tag{5.4}$$

If we now quantize the particle according to non-relativistic quantum mechanics, we obtain

$$H = \frac{P^2}{2m} + e\int d^3y j_\mu(y)A_\mu(y) + \frac{1}{2}\int d^3y[\pi^2 + (\nabla A)^2] \quad , \tag{5.5}$$

where

$$2j_\mu(y) = \delta(X - y)[1, \frac{P}{mc}] + [1, \frac{P}{mc}]\delta(X - y) \quad , \tag{5.6}$$

and P and X are now the usual quantum mechanical operators, satisfying $[X,P] = i\hbar$. This Hamiltonian is only

a) semirelativistic because the EM field is relativistic but the particle is not.

b) semi-quantized because the particle is quantized but the EM field is not.

To remedy these defects, one quantizes the EM field by the Ansatz

$$[A_\mu(x),A_\gamma(x')] = i\hbar g_{\mu\nu}D(x - x') \quad , \tag{5.7}$$

where $D(x)$ is a numerical function (or, more precisely, distribution) to be discussed in a moment, and one makes the particle relativistic by the substitution

$$\frac{1}{2m}P^2 \to \vec{\alpha}\cdot\vec{P} + \beta m \quad , \tag{5.8}$$

$$j_\mu(x) \to \delta(x - X)\gamma_\mu \quad , \tag{5.9}$$

where the γ_μ are the 4×4 Dirac matrices defined by

$$[\gamma_\mu,\gamma_\nu]_\pm = 2g_{\mu\nu} \quad , \tag{5.10}$$

β is γ_0, $\vec{\alpha}$ is $\gamma_0\vec{\gamma}$, $g_{\mu\nu}$ is the metric tensor, and, for simplicity, we have assumed that the particle in question has spin $\frac{1}{2}$ (e.g. is an electron). For other spins we use an appropriate generalization of the γ_0 (see Section 7).

The Ansatz (5.7) for the EM field is the analogue of $[X,P] = i\hbar$ for the particle. Indeed, one can expand the purely EM part of the Hamiltonian as a sum of formal harmonic oscillators

$$\frac{1}{2}\int d^3x[\pi(X)^2 + (\nabla A)^2] = \frac{1}{2}\int d^3k[P(k)^2 + \omega(k)^2Q(k)^2] \quad ,$$

where

$$\vec{Q}(k) = \int d^3k \sin kx\, \vec{A}(x) \quad ,$$
$$\vec{P}(k) = \dot{\vec{Q}}(k) \quad ,$$

and

$$\ddot{Q}(k) + \omega^2(k)Q(k) = 0 \quad ,$$

and then (5.7) amounts to the Ansatz

$$[Q(k),P(k')] = i\hbar\delta(k - k') \quad ,$$

for the formal oscillators. The important properties of the distribution $D(x,t)$ are that $D(x)$ is Lorentz invariant,

$$\begin{aligned} D(x) &= 0 \quad , \quad x^2 < 0 \quad , \\ D(\vec{x},0) &= 0 \quad , \\ \dot{D}(\vec{x},0) &= \delta^3(\vec{x}) \quad , \\ (\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2)D(x) &= 0 \quad . \end{aligned} \tag{5.11}$$

The Ansatz (5.8)(5.9) for the particle means that it is no longer described in the Hilbert space $L_2(-\infty,\infty)$ for $[X,P] = i\hbar$, but in a Hilbert space $L_2[-\infty,\infty) \times R_4$ where R_4 is the 4-dimensional Dirac space.

It turns out, however, that while the relativistic quantized Hamiltonian (5.5)(5.10) is sufficient to describe processes in which the relativistic particle is conserved, it cannot take account of the experimental fact that when the relativistic energies are large enough, the particle can be created or destroyed. To allow for this possibility, one must go further and <u>second quantize</u> the Hamiltonian. This means introducing for the <u>particle</u> a field $\psi_\alpha(x)$, which is quantized according

to the rule

$$[\psi_\alpha(x),\psi_\beta(x')]_\pm = i\hbar D_{\alpha\beta}(x - x') \quad , \tag{5.12}$$

where $D_{\alpha\beta}(x)$ is a function analogous to $D(x)$, the $\pm$ commutator is taken according as to whether the particle obeys Fermi-Dirac statistics (has half-odd-integer spin) or Bose-Einstein statistics (integer spin), and α,β are spin indices. (In the case of the electron, which is a spin $\frac{1}{2}$ particle, the $+$ sign is taken and the indices α,β are the Dirac indices.) Using the field $\psi_\alpha(x)$, one makes the substitutions

$$\vec{\alpha}\cdot\vec{P} + \beta m \to \psi^+(x)(\vec{\alpha}\cdot\vec{\partial} + \beta m)\psi(x) \quad , \tag{5.13}$$

$$j_\mu(x) \to \psi^+(x)\gamma_0\gamma_\mu\psi(x) \quad , \tag{5.14}$$

in the relativistic first-quantized Hamiltonian (5.5)(5.8) and (5.9) and obtains finally

$$H = \psi^+(\vec{\alpha}\cdot\vec{\partial} + \beta m)\psi + e\int d^3x\psi^+(x)\gamma_0\gamma_\mu\psi(x)A_\mu(x)$$

$$+ \frac{1}{2}\int d^3x[\pi(x)^2 + (\nabla A(x))^2] \quad . \tag{5.15}$$

This is the fully quantized, relativistic, Hamiltonian of Dirac, Heisenberg, and Pauli. Note that in this theory the particles and the EM field are on the same footing. Each is described by a field and the field has a particle interpretation (photon interpretation in the case of the EM field), which is obtained by analyzing the quantization Ansätze (5.7) and (5.12).

Without accepting the Hamiltonian H (and its generalization to include interactions between particle-fields other than the electron $\psi(x)$ and photon $A(x)$) too literally, one can extract from it most of the ideas which are used in the later theories. Let us summarize briefly the most important and relevant ideas: [3]

1) The particles are described in some way by fields $\varphi(x)$ ($\psi(x)$ and $A(x)$ above) which are <u>quantized locally</u>, i.e., whatever quantization rules are adopted for the interacting fields, they should at least satisfy the conditions

$$[\varphi(x),\varphi(x')]_\pm = 0 \quad , \quad (x - x')^2 < 0 \quad . \tag{5.16}$$

These conditions are dictated by the principle of strong microscopic causality; measurements which are separated by spacelike distances should not interfere. (The $+$ sign in (5.16) is taken for fermion fields for which only <u>bilinears</u> in the field are observables.) The locality assumption is usually strengthened by the demand that the fields, which, to make sense both mathematically and physically, are

not operators but operator-valued distributions, should not be too wild in the sense of distributions.

2) The fields interact locally. For example, if a Hamiltonian exists, the interaction term in it would be of the form

$$H_{int} = g \int d^3x \psi^+(x)\gamma_0\psi(x)\varphi(x) \quad ,$$
$$H_{int} = g \int d^3x \psi^+(x)\gamma_0\gamma_\mu\psi(x)\partial_\mu\varphi(x) \quad , \tag{5.17}$$

etc., but not of the form

$$H_{int} = g \int d^3x \int d^4x' d^4x'' \psi^+(x'')\gamma_0 f(x - x', x - x'')\psi(x')\varphi(x) \quad , \tag{5.18}$$

where f is some Lorentz invariant function which does not vanish for $x \neq x'$, $x \neq x''$.

3) Under Lorentz transformations, the fields transform according to the law

$$\varphi_\alpha(x) \overset{\Lambda, a}{\to} S_{\alpha\beta}(\Lambda)\varphi_\beta(\Lambda^{-1}(x - a)) \quad , \tag{5.19}$$

where Λ is a homogeneous Lorentz transformation, a is a translation, and $S_{\alpha\beta}(\Lambda)$ is a representation of Λ. The choice of representation $S_{\alpha\beta}(\Lambda)$ is determined by the masses and spins of the particles. For free fields, or in the free field limit of interacting fields, the above description can be made a little more exact. The fields can be expanded in the form

$$\varphi(x) = \int d\mu(p)[\varphi(p)a(p)e^{ipx} + \bar{\varphi}(p)b^\dagger(p)e^{-ipx}] \quad , \tag{5.20}$$

where the unquantized "wavefunctions" $\varphi(p)$, $\bar{\varphi}(p)$ carry the Lorentz properties of $\varphi(x)$, and the operators $a(p)$ and $b^\dagger(p)$, which satisfy quantization relations of the form

$$[a(p), a^+(p')]_\pm = \hbar\delta(p - p') \quad ,$$
$$[a(p), b(p')] = 0 \quad , \quad \text{etc.} \quad , \tag{5.21}$$

carry the quantization properties. An analysis of the algebra (5.21) in Hilbert space shows that the operators $a(p)$ and $b^\dagger(p)$ can be considered as creation and destruction operators for states which have the right properties to be identified with free particle states. Thus, the particle description of the field may be said to be embodied in the quantization relations.

To sum up, one is confronted with a huge number of elementary particles experimentally and one is looking for a theory which will explain the elementary particles and their interactions. For want of better alternatives, one tries to find such a theory by using general ideas derived from local field theory. The fields in local field theory have particle properties in the free field limit, have definite transformation properties with respect to the Lorentz group, and they interact and are quantized locally.

6. REPRESENTATIONS OF THE POINCARÉ GROUP IN HILBERT SPACE

In the last lecture, we sketched briefly the experimental background to elementary particle physics and the basic theoretical tool, namely the theory of quantized fields, which is used to attack it. We saw that one of the most important properties of the fields was that they transformed in a manifestly covariant manner,

$$\psi_\alpha(x) \overset{\Lambda, a}{\to} S_{\alpha\beta}(\Lambda)\psi_\beta(\Lambda^{-1}(x - a)) \quad , \tag{6.1}$$

under inhomogeneous Lorentz, or Poincaré, transformations. In this lecture we wish to consider the question of Poincaré covariance in a more general way, that is, divorced from any particular theory such as field theory, and using nothing but the most fundamental quantum mechanical ideas. Later we shall try to establish the link with field theory.

We begin, as usual, with the probabilities

$$|(f,h)| \quad , \tag{6.2}$$

where f and h are vectors in the Hilbert space $\mathcal{H}$. The assumption that apart from the spectra these, and only these, are the physical numbers to be extracted from the theory is made not only in non-relativistic but in relativistic quantum theory, and underlies all other assumptions. (For simplicity, we assume that all vectors in $\mathcal{H}$ represent physical states (no super-selection rules), but the argument can easily be generalized to the case where this is not so.)

Let us now suppose that, due to the geometry of space-time, we wish to impose an invariance principle on the quantum mechanical system--we wish to demand that the system be invariant under some group G of space-time transformations. Let us for the moment not specify the group although, in practice, it will be the Galilean group or the Poincaré group. How are we to impose the invariance principle? Following the arguments used earlier, namely that under a change of observer the probability of a system making a given transition remains unchanged (the old argument that "the system does not care who is looking at it"), we impose the invariance principle by demanding that, under the transformations of the group, the

inner products (6.2) remain invariant, i.e.,

$$|(T(g)f,T(g)h)| = |(f,h)| \quad , \tag{6.3}$$

$f, h \in \mathcal{H}$, $g \in G$. We also demand that the Hamiltonian transform under the group in a way appropriate for the energy. The latter demand generalizes the idea of invariance groups used in non-relativistic theory.

Using Wigner's theorem, it follows that the invariance group can be implemented on $\mathcal{H}$ by a set of unitary or anti-unitary operators $U(g)$, forming a ray representation

$$U(g)U(g') = e^{i\omega(g,g')}U(g,g') \quad , \tag{6.4}$$

of the group.

If the group is continuous, physical continuity demands that as $g \to 1$ in the group topology, $T(g)f$ should represent the same state as f, i.e.,

$$T(g)f \to e^{i\alpha}f \quad , \tag{6.5}$$

whence

$$U(g)f \to e^{i\gamma(g,f)}f \quad , \tag{6.6}$$

i.e., physical continuity demands that $U(g)$ be ray-continuous in the sense of (6.6).

We see, therefore, that from quite general principles the invariance of a quantum mechanical system under a geometrical group demands that the Hilbert space $\mathcal{H}$ of the system carry a unitary or anti-unitary ray representation of the group. If the group is continuous, the representation must be ray-continuous.

For connected Lie groups, such a representation can be shown [1] to be equivalent to (or can be "lifted" to) a true continuous unitary representation of the covering group of either the group itself or some continuous central extension of it.

Thus, without loss of generality, we can confine ourselves to continuous unitary group representations. Whether we can use continuous unitary representations of the geometrical group itself or of some central extension depends on the geometrical group in question.

To proceed further, we must therefore specify the geometrical group more precisely. We shall specify finally to the Galilean and Poincaré group, in particular to the Poincaré group, but before doing so it might be interesting to point out that we could first limit ourselves to <u>kinematical</u> groups, i.e., 10-parameter, continuous, connected space-time Lie groups with rotations, a scalar time translation, vector space translations, and vector accelerations, with the commutation relationships not mentioned left open. Under general conditions [2], it can be shown that there are, in fact, only eight such groups, four non-relativistic ($t' = t + t_0$) groups including the Galilean group, and four relativistic groups

including the Poincaré group. For the four relativistic groups, the phase-factors $\exp i\omega(gg')$ can be lifted completely. For the four non-relativistic groups, the lifting requires a 1-parameter central extension. We have already seen this in the case of the Galilean group for which the central extension is generated by the total mass M.

Let us now concentrate on the relativistic case and in particular on the connected Poincaré group. From what we have just said, the Hilbert space $\mathcal{H}$ must carry a true continuous unitary representation of its covering group, which we denote by

$$P_+^\uparrow = T_4 \circledS SL(2,C) \quad , \tag{6.7}$$

where T_4 is the 4-dimensional translation group, s denotes semi-direct product, and $\uparrow+$ mean that time-and space-inversions are not included. Group multiplication is to the left. In particular, if we use the conventional paramatrization (Λ,a) for $P_+^\uparrow$, we have $(\Lambda,a)(\Lambda',b) = (\Lambda\Lambda', a + \Lambda b)$.

Needless to say, the representation of $P_+^\uparrow$ carried by $\mathcal{H}$ will not, in general, be irreducible. However, $P_+^\uparrow$ is a type 1 group, which means that any continuous unitary representation decomposes uniquely into a direct sum and/or a direct integral of continuous unitary irreducible representations (CUIR's). It follows that, from the group theoretical point of view, the elementary objects to study are the CUIR's of $P_+^\uparrow$. Some of the CUIR's will, in fact, be identified directly (i.e., without summation or integration) with elementary particles. This point will be discussed in more detail later. For the moment, we merely remark that for the case of non-relativistic quantum mechanics, we have already seen that a free Newtonian particle carries a CUIR of the extended Galilean group.

The CUIR's of $P_+^\uparrow$ were first classified by Wigner [3] in 1939. However, they are most simply classified by Mackey's method [4] of induced representations, which generalizes and simplifies Wigner's approach. We, therefore, proceed using Mackey's method. We first describe the method for a general group G, and then specialize to $P_+^\uparrow$.

Let G be any separable locally compact group, H any closed subgroup, G/H the right coset space, and $\mu(s)$ the left invariant (or left quasi-invariant) measure on G/H. Let $W(h)$, $h \in H$ be any unitary representation of H on a Hilbert space N, and $f(g)$ the set of vector functions over G with values in N satisfying the

1) subsidiary condition

$$f(hg) = W(h)f(g) \quad , \tag{6.8}$$

2) square integrability condition

$$\int d\mu(s)(f(g),f(g)) < \infty \quad , \tag{6.9}$$

where the inner product in the integrand is with respect to W and, on account of 1), is a function over G/H only.

The representation $U(g)$ of G defined by letting G act transitively on $f(g)$, i.e.,

$$f(g) \overset{g'}{\to} f(gg') \quad , \tag{6.10}$$

is unitary and is called the unitary representation of G induced by the representation W of H.

Note that if $H = 1$, $W = 1$, U is just the regular representation. At the other extreme, if $H = G$, then $U = W$. Note also that to induce U, two choices are necessary: a choice of subgroup H and a choice of representation $U(H)$ of H. In general, there is no guarantee that U will be irreducible or that the set of all induced representations will be exhaustive.

Let us turn now to the special case of $P_+^\uparrow$. The question is how, in this case, we make our choice of H and W. To answer it, we first have to introduce the concept of optics.

Orbits. Consider T_4. Every unitary irreducible representation of T_4 is 1-dimensional and of the form $\exp ipa$, where $a_\mu \in R$, $\mu = 1...4$, are the group parameters, and $p \in R_4$ is the character. Now let $g \in P_+^\uparrow$ act on a. We have

$$\exp(ip.a) \longrightarrow \exp(ip.ga) = \exp(ipg.a) \quad . \tag{6.11}$$

where $pg \in R_4$, i.e., we have an associated action of $P_+^\uparrow$ on p. The orbit of p is defined to be the subset pg of R_4, $g \in P_+^\uparrow$. Clearly, R_4 breaks up disjointly into orbits, and there are six kinds:

a)	$p^2 = m^2$	$p_0 > 0$, $p_0 < 0$	(timelike)	$SU(2)$
b)	$p^2 = -m^2$		(spacelike)	$SU(1,1)$
c)	$p^2 = 0$	$p_0 < 0$, $p_0 > 0$	(lightlike)	$E(2)$
d)	$p = 0$		(trivial)	$SL(2,C)$

where m^2 is any fixed positive number.

We are now in a position to choose the subgroup H and its representation W. The rules are as follows:

1) Choose an orbit (e.g., $p^2 = -m^2$),
2) Choose any point $p = \alpha$ on the orbit,
3) Determine the stability (little) group of α, i.e., the maximal subgroup K of $SL(2,C)$, leaving α fixed,
4) Choose $H = T_4 \circledS K$,
5) Choose $W(H) = \exp i\alpha a \otimes V(K)$, where $V(K)$ is any unitary irreducible representation of K,
6) Induce with H and $W(H)$.

With this choice of H, the representations of $P_+^\uparrow$ obtained are irreducible and (using all possible $V(K)$) exhaustive.

One can gain an intuitive feeling why this is so by noting that the following three things coincide: the coset space G/H, the orbit O, and the simultaneous spectrum S of the infinitesimal generators of T_4. Thus

$$G/H = O = S \quad .$$

The irreducibility can then be seen intuitively as follows. From the subsidiary condition 1), $f(g)$ is essentially a function over G/H and the Hilbert space of $W(H)$ only. But $P_+^\uparrow$ acts irreducibly on O by definition. Hence, $P_+^\uparrow$ acts irreducibly on $G/H = O$. And $V(K)$ acts irreducibly on N. Hence, $P_+^\uparrow$ acts irreducibly on both G/H and N. Hence, $P_+^\uparrow$ acts irreducibly on $f(g)$, as required. To see why the induced representations should be exhaustive, we note that given <u>any</u> representation $P_+^\uparrow$, the infinitesimal generators of T_4 can be simultaneously diagonalized and hence the vectors in the representation space can be written as functions over S. Hence, these vectors can be written as functions over $O = S$. For a fixed point in $s \in S$, the only remaining freedom is to transform according to some representation of the group leaving S invariant. But since $S = O$, the group leaving $s \in S$ invariant is just the stability group for a point $p = \alpha$ in O. Thus, the representation of $P_+^\uparrow$ corresponds to an induced representation.

The little group corresponding to the orbits a) to d) above are written beside them. The invariant differential form is

$$d\mu(p) = \frac{d^3p}{p_0} \quad , \qquad d\mu(p) = \frac{d^3p}{p_1} \quad , \tag{6.12}$$

for a), c), and b), respectively. The continuous unitary irreducible representations of $SU(2)$, $E(2)$, $SU(1,1)$, and $SL(2,C)$ are all known. We are thus in a position to determine explicitly all the CUIR's of $P_+^\uparrow$. In the next section, we shall do this in some detail, at least for the physically relevant representations. In particular, we shall try to express the induced representations in a form which is immediately useful for physics. For the rest of the present lecture, we turn to the more general question of the physical interpretation of the CUR's of $P_+^\uparrow$ carried by $\mathcal{H}$.

First, according to the theorems of Nelson *et al.*, there exists in $\mathcal{H}$ a domain $\mathcal{D}$ on which it is permissible to work with the Lie algebra of $P_+^\uparrow$. A canonical basis for the Lie algebra is

$$\begin{aligned}
&[P_0,L] = 0 \qquad [P_0,P] = 0 \qquad [P_0,K] = P \\
&[L,L] = iL \qquad [L,P] = iP \qquad [L,K] = iK \\
&[P,P] = 0 \qquad [P,K] = iP_0 \\
&[K,K] = -iL \quad ,
\end{aligned} \tag{6.13}$$

on $\mathcal{D}$.

Following non-relativistic quantum mechanics, we identify P_0,P and L with the physical energy, 3-momentum, and angular momentum, respectively, and call K, by analogy, the relativistic angular momentum. Thus, once again the operators play a dual role--group generators and physical observables.

Note that the relations (6.13) differ from the Galilean relations in only two respects,

$$\begin{aligned} [P,K] = M &\to [P,K] = iP_0 \\ [K,K] = 0 &\to [K,K] = -iL \quad , \end{aligned} \tag{6.14}$$

the first of which means that $[P,K]$ maps back onto the algebra itself instead of onto a central extension.

We have already seen that the spectrum S of the generators of T_4 can be identified with the orbit O. More precisely, if we denote the four generators of T_4 by P_μ, $\mu = 0,1,2,3$, they take values p_μ, $\mu = 0,1,2,3$, in the orbit $(p^2 = \pm m^2, 0)$. The orbit in a unitary irreducible representation of $P_+^\uparrow$ is, therefore, precisely the energy momentum spectrum. (Note that it makes sense to talk about the simultaneous spectrum of the P_μ since they commute on a domain $\mathcal{D}$ on which they are essentially self-adjoint.)

The identification of the orbits with the energy-momentum spectrum means that we can use direct physical arguments to decide which orbits and hence which CUIR's $\mathcal{H}$ should carry. Since physical mass-squared and energy are not negative, one usually makes the following assumptions about the energy-momentum spectrum and, hence, about the orbits:

1) $\mathcal{H}$ contains a unique normalizable ray (the vacuum state), which is invariant under $P_+^\uparrow$.
2) If there are no massless particles, the energy-momentum spectrum contains at least one isolated hyperbola (Figure 6.1) plus a continuum beginning at twice the height of the lowest hyperbola.
3) If there are massless particles present, the energy momentum spectrum fills the closed forward light cone.

On the isolated hyperboloids

$$p^2 = p^2 = \text{constant} \quad . \tag{6.15}$$

Furthermore, for each CUIR on such a hyperboloid, $\vec{P}$ takes all values in R_3. Hence, in contrast to the case of true unitary representations of the Galilean group, a position operator (Newton-Wigner operator) [5] satisfying $[X,P] = i\hbar$ can be defined. Hence, the CUIR's on the isolated hyperboloids can be identified with stable 1-particle states. The CUIR of the little group K(= SU(2) in the case when $p^2 > 0$) used to induce the CUIR of $P_+^\uparrow$ is then identified with the spin group of the particle. Thus, the spin group, which in non-relativistic quantum mechanics is introduced empirically and forms a direct

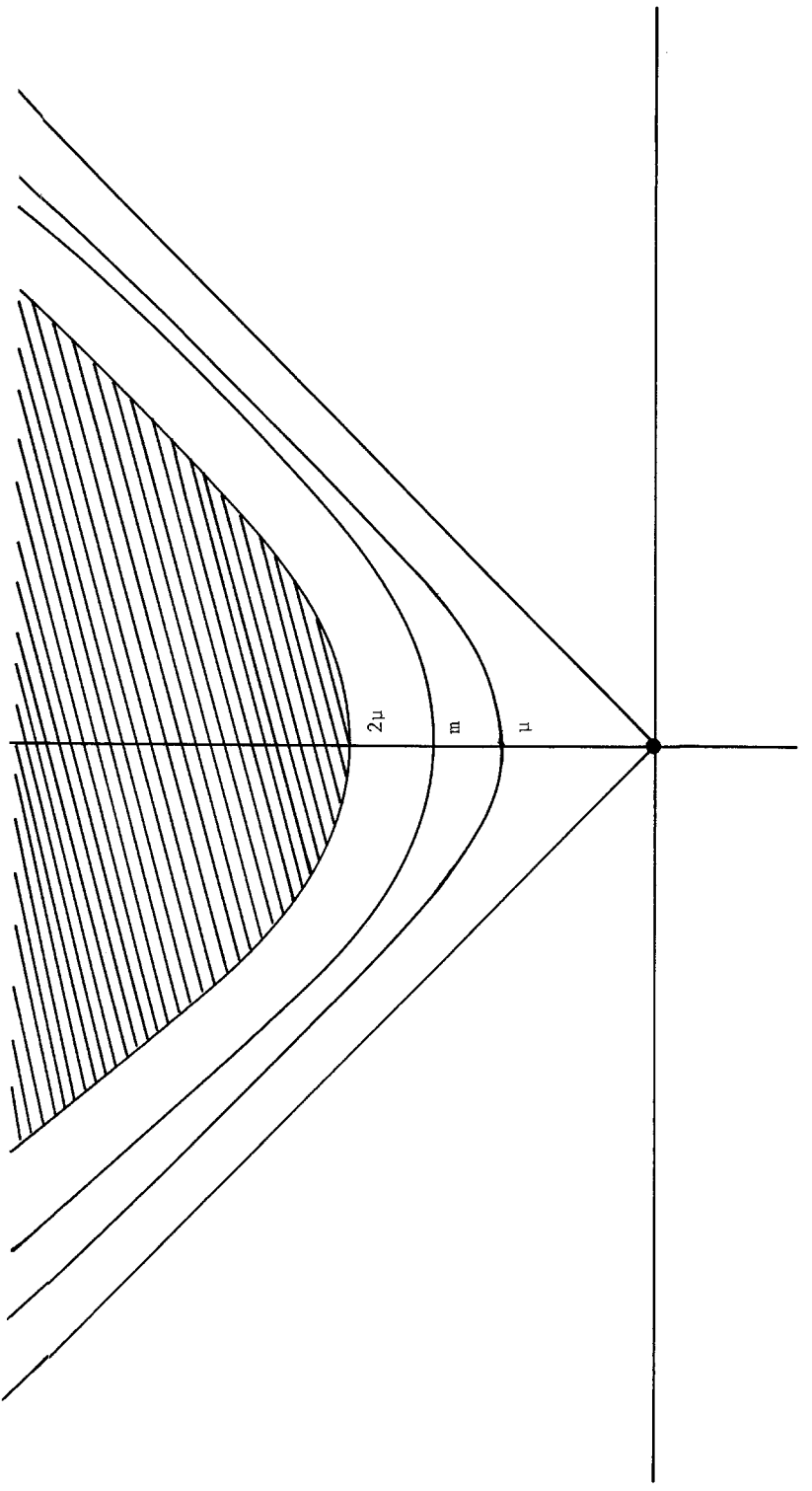

FIGURE 6.1. ENERGY-MOMENTUM SPECTRUM FOR MASSIVE PARTICLES

product with the Galilean group, is included automatically in the relativistic case.

Empirically, it is found that for any given mass there are only a finite number of elementary particles. Hence, the isolated hyperbolas are assumed to be finitely degenerate, i.e., to carry only a finite number of CUIR's of $P_+^\uparrow$.

The continuum states in the energy-momentum spectrum represent, in general, two or more particle states, but may include 1-particle states which happen to have a higher mass than the lowest two particle states. In general, the continuum states are infinitely degenerate. In the case that zero mass particles are present, the continuum is everywhere in and on the forward light-cone and there is a serious problem as to how one should identify the 1-particle states, including the zero-mass particle states themselves. One possibility would be to identify them with normalizable, non-isolated, eigenvalues of P^2. But this is by no means the only possibility and, within some of the postulated frameworks, it is even impossible. [6]

From the point of view of the orbits of $P_+^\uparrow$ allowed on $\mathcal{H}$, the energy-momentum spectrum conditions imposed are very strong. They reduce the six possible kinds of orbit which could be carried by $\mathcal{H}$ to the two kinds $p^2 \geq 0$, $p_0 \geq 0$. These orbits we shall call physical orbits. (They are actually characterized by $p_0 \geq 0$.)

The corresponding little groups are SU(2) and E(2). The CUIR's of SU(2) are well-known and require no comment. Those of E(2) are not so well-known, perhaps, but are actually simpler, as can be seen in the following way. The Lie algebra of E(2) is

$$[L_3, E_\alpha] = i\varepsilon_{\alpha\beta}E_\beta \quad , \qquad [E_\alpha, E_\beta] = 0 \quad , \tag{6.16}$$

where $\alpha,\beta = 1,2$. It follows at once that $\exp(2i\pi L_3)$ and E^2 are the Casimir operators of E(2). Assuming that $\exp(2\pi i L_3) = \pm 1$ (i.e., integer or half-odd-integer values for L_3), it is then easy to see that there are only two possibilities:

a) $E^2 \neq 0$. The CUIR is infinite-dimensional and L_3 takes all integer or half-odd-integer values.

b) $E^2 = 0$. The CUIR is 1-dimensional and L_3 takes one integer or half-odd-integer value.

Case a), the so-called continuous spin case, does not seem to be realized in nature. Case b) is realized (it describes the photons and neutrinos for $L_3 = \pm 1$ and $L_3 = \pm \frac{1}{2}$, respectively). When Case b) does occur, it is usual to use a 2-dimensional reducible CUR of E(2) with $L_3 = \pm m$, rather than the 1-dimensional CUIR. This is to accommodate the parity operator.

Since $\mathcal{H}$ can carry only the physical orbits $p^2 \geq 0$, $p_0 \geq 0$, it follows that only the CUIR's of $P_+^\uparrow$ corresponding to these orbits are directly related to

physics. This does not mean that the other orbits are completely irrelevant. As we shall see later, they play an important role in the analyses of scattering amplitudes. The reason is that, in practice, one uses not only the matrix elements of operators on $\mathcal{H}$ themselves, but also the analytic continuation of these matrix elements, considered as functions of p_μ, to points other than those in the physical spectrum.

7. REDUCTION OF REPRESENTATIONS OF $P_+^\uparrow$ TO MANIFESTLY COVARIANT FORM

In the last section, it was shown that on quite general grounds the Hilbert space $\mathcal{H}$ of a relativistic quantum mechanical system must carry a CUR of $P_+^\uparrow$, and the CUIR's which this CUR could contain were described from the point of view of Mackey's theory. For a complete description of the elementary particles (origin of the masses and spins, nature of the interactions, etc.), however, much more is needed. For example, in a field theory, as we saw in Lecture 5, we need not only the Poincaré transformation properties of the field, but its commutation relations and interaction laws as well. The next step, therefore, is to try to relate the CUIR's of $P_+^\uparrow$ to other aspects of relativistic particle physics.

The question is: How is the contact between the group theoretical properties and the other physical properties to be made?

Traditionally, following non-relativistic quantum mechanics, Maxwell's theory, and Dirac's (non-second-quantized) relativistic quantum mechanics, the contact is made through wave functions $\psi(p)$ or fields $\psi(x)$ which transform in a manifestly covariant way, i.e.,

$$\psi(p) \xrightarrow{g} S(\Lambda)\psi(p')e^{ip\cdot a} \quad , \quad p = p\Lambda \quad , \tag{7.1}$$

where $g = (\Lambda,a)$, $\Lambda \in SL(2,C)$, $a \in T_4$, and $S(\Lambda)$ is a finite-dimensional representation of the homogeneous part $SL(2,C)$ of $P_+^\uparrow$. In the second-quantized theory of free particles of Dirac, Heisenberg, and Pauli, we have, as mentioned in Lecture 5, also the relation

$$\psi(x) = \int \frac{d^3p}{p_0} \{e^{ip\cdot x}a(p)\,\psi(p) + e^{-ip\cdot x}b^\dagger(p)\bar{\psi}(p)\} \quad , \tag{7.2}$$

between $\psi(x)$ and $\psi(p)$ where $a(p)$ and $b^\dagger(p)$ are the creation and destruction operators and where the fields $\psi(x)$ have local commutation relations and, when interactions are introduced, local interactions.

We shall follow the above tradition to the extent that we shall try to relate Mackey's method to manifestly covariant wavefunctions. [1] As we shall see for the physical orbits, this can always be done, and so it implies no restrictions. (Restrictions come when we try to relate the manifestly covariant wavefunctions to

local fields, but that shall concern us only peripherally.)

We first recall Mackey's prescription for $P_+^\uparrow$ on $\mathcal{H}$:

a) Choose an orbit $p^2 = m^2 \geq 0$, $p_0 > 0$.

b) Choose a point $p = \alpha$ in the orbit.

c) Determine the little group K of α.

d) Let $H = T_4 \circledS K$.

e) Induce with $W(H) = e^{i\alpha a}V(K)$.

The induction procedure, we recall, is to choose the functions $f(g)$ over the group satisfying

1) $f(hg) = V(h)f(g)$

2) $\int d\mu(p)(f(g),g(g)) < \infty$

and letting the group act transitively on these functions,

3) $f(g_1) \xrightarrow{g_2} f(g_1 g_2)$.

We now make the transition from $f(g)$ to manifestly covariant wavefunctions in two steps.

First Step. We have a natural α-mapping $P_+^\uparrow \to$ Orbit given by $\alpha g = p$. We now define an inverse mapping $p \to P_+^\uparrow$ by introducing a representative Lorentz transform $\Lambda_0(\alpha,p) \in SL(2) \subset P_+^\uparrow$ for each p. The choice of $\Lambda_0(\alpha,p)$ is arbitrary but two standard ways of defining it are:

1) The canonical method: [2] Λ_0 is defined to be the unique Lorentz transformation in the 2-flat spanned by p and α.

2) The helicity method: [3] An arbitrary direction is chosen for the ζ-axis and Λ_0 is defined to be a pure Lorentz transformation in the ζ-direction to momentum $|\vec{p}|$, followed by a rotation from $(\varepsilon\ 0\ 0\ |\vec{p}|)$ to $(\varepsilon, \vec{p})$.

We then make the transformation

$$f(g) \to \phi(g) = V(\Lambda_0\Lambda^{-1})f(g) \quad , \tag{7.3}$$

where $g = (\Lambda,a)$, $\alpha\Lambda = \alpha\Lambda_0 = p$ and $V(\Lambda_0\Lambda^{-1})$ makes sense because $\Lambda_0\Lambda^{-1} \in K$. The point of this transformation is that, as is easily verified from Condition (1) and the relation $(k\Lambda)_0 = \Lambda_0$, which follows from the definitions of k and Λ_0, $\phi(g)$ satisfies the simpler subsidiary condition

1′) $$\phi(hg) = \phi(g)e^{i\alpha a} \quad ,$$

$$h = (k,a) \quad , \quad k \in K \quad , \quad a \in T_4 \quad .$$

Recalling that group multiplication is to the left, one sees at once from 1′) that $\phi(g)$ must be of the form

$$\phi(g) = \phi(\Lambda,a) = \theta(\Lambda)e^{i\alpha\cdot a}$$

where

$$\theta(\Lambda) = \theta(k\Lambda) \quad .$$

It follows that $\theta(\Lambda)$ is a function of p only, i.e.

$$\phi(g) = \theta(p)e^{i\alpha\cdot a} \quad .$$

Since V is unitary, the inner product remains unchanged.

$$2') \qquad \int d\mu(p)(f(g),f(g)) = \int d\mu(p)(\theta(p),\theta(p)) \quad . \tag{7.4}$$

The group multiplication law changes, however. In place of the simple transitivity 3), we obtain

$$3') \qquad \theta(p) \xrightarrow{g} V(\Lambda_0\Lambda\Lambda_0'^{-1})\theta(p')e^{ip\cdot a} \tag{7.5}$$

where

$$g = (\Lambda,a) \quad , \quad \Lambda_0 = \Lambda_0(\alpha,p) \qquad \Lambda_0' = \Lambda_0(\alpha,p') \quad , \quad p' = p\Lambda \quad .$$

Note that $V(\Lambda_0\Lambda\Lambda_0'^{-1})$ makes sense since $\Lambda_0\Lambda\Lambda_0'^{-1} \in K$. The rotations $V(\Lambda_0\Lambda\Lambda_0'^{-1})$ are called Wigner rotations. We see that, in effect, what we have done essentially is to change the "twist" introduced by $V(k)$ from the subsidiary condition to the group transformation.

For many purposes, the wavefunctions $\theta(p)$ are the most convenient. For example, the standard analysis of scattering amplitudes for general spin carried out by Jacob and Wick [3] is done in terms of $\theta(p)$. However, if we wish for manifest covariance, we must go farther. The transformation law (7.5) is not manifestly covariant on two counts:

1) It depends explicitly on p.

2) V is a representation of the little group, not $SL(2,C)$.

This brings us to Step 2.

Second Step. Elimination of the p-dependence from the transformation (7.14).

The basic idea underlying Step 2 is to modify $V(\Lambda_0\Lambda\Lambda_0'^{-1})$ so that it can be split into $V(\Lambda_0)V(\Lambda)V(\Lambda_0')^{-1}$. At the moment, $V(\Lambda_0)$, etc., make no sense since Λ_0, Λ, and Λ_0' are not separately in K. The modification is achieved by embedding $V(k)$ in any representation $S(\Lambda)$ of $SL(2,C)$ which is unitary when restricted to K. Letting $V_\lambda(K)$ be the representations of K occurring in $S(\Lambda)$, we define a set of wavefunctions $\theta_\lambda(p)$ (including $\theta(p)$) with the transformation law

$$\theta_\lambda(p) \xrightarrow{\Lambda,a} V_\lambda(\Lambda_0\Lambda\Lambda_0'^{-1})\theta_\lambda(p')e^{ip\cdot a} \tag{7.6}$$

In other words, we induce with the reducible representation $\sum \oplus V_\lambda$ of K (all on the orbit $p^2 = m^2$).

Now by definition

$$S_{\lambda\mu}(K) = \delta_{\lambda\mu} V_\lambda(K) \quad . \tag{7.7}$$

Hence, (7.6) can be written as

$$\theta_\lambda(p) \xrightarrow{\Lambda,a} S_{\lambda\mu}(\Lambda_0 \Lambda \Lambda_0'^{-1}) \theta_\mu(p') e^{ip\cdot a} \quad . \tag{7.8}$$

But since $S(\Lambda)$ makes sense, we then have

$$\theta_\lambda(p) \xrightarrow{\Lambda,a} [S(\Lambda_0) S(\Lambda) S^{-1}(\Lambda_0')]_{\lambda\mu} \theta_\mu(p') e^{ip\cdot a} \quad , \tag{7.9}$$

or

$$S^{-1}(\Lambda_0)\varphi(p) \xrightarrow{\Lambda,a} S(\Lambda) S^{-1}(\Lambda_0')\varphi(p') e^{ip\cdot a} \quad , \tag{7.10}$$

where

$$\varphi(p) = \sum \oplus \theta_\lambda(p) \quad .$$

Remembering that Λ_0 depends only on α and p, we see that (7.10) is equivalent to

$$\psi(p) \xrightarrow{\Lambda,a} S(\Lambda)\psi(p') e^{ip\cdot a} \quad , \tag{7.11}$$

where

$$\psi(p) = S^{-1}(\Lambda_0(\alpha),(p))\varphi(p) \quad . \tag{7.12}$$

Equation (7.11) has the required manifestly covariant transformation properties. Note that in the manifestly covariant formulation the Lie algebra of $P_+^\uparrow$ takes the simple form

$$P_\mu = p_\mu, \; L_{\mu\nu} = \frac{1}{i}\left(p_\mu \frac{\partial}{\partial p_\nu} - p_\nu \frac{\partial}{\partial p_\mu}\right) + S_{\mu\nu} \quad , \tag{7.13}$$

where $S_{\mu\nu}$ are the generators of $S(\Lambda)$ and $L_{\mu\nu} = (\vec{L},\vec{K})$.

Equation (7.13) shows that in the manifestly covariant formulation, $L_{\mu\nu}$ splits into the direct sum of an "orbital" part and a "spin" part $S_{\mu\nu}$.

For the manifest covariance, we have, however, to pay a heavy price:

1) The representation $S(\Lambda)$ of $SL(2,C)$ is arbitrary.

2) We have introduced the unwanted subsidiary fields

$$\theta_\lambda(p) \neq \theta(p) \quad .$$

3) Since $S(\Lambda)$ is, in general, not unitary, the inner product must be changed accordingly.

Let us discuss these points in turn:

1) The representation $S(\Lambda)$ in (7.10), which is usually called the spin group, is completely arbitrary. It is usually chosen to be a

finite-dimensional (non-unitary) representation of $SL(2,C)$ and as we shall be considering infinite dimensional spin groups in the next section, let us concentrate on the finite dimensional case. Even for the finite-dimensional representations, there is much arbitrariness. All choices of $S(\Lambda)$ will, of course, be the same from the point of view of the original CUIR of $P_+^\uparrow$. But they will not necessarily be the same from other points of view. For example, an interaction which involves no derivatives for one choice of $S(\Lambda)$ will have derivatives for another. Indeed for spin ≥ 1, the whole question of choosing the correct $S(\Lambda)$ is very much open. [4,5]

2) With regard to the subsidiary fields $\theta_\lambda(p)$, the point is that they should be eliminated in a manifestly covariant way. That this is possible for $P_+^\uparrow$ and finite-dimensional $S(\Lambda)$ follows from the following two properties of $SL(2,C)$: (a) The $\psi(p)$ for every irreducible finite-dimensional representation $D(n,m)$ of $SL(2,C)$ is of the form

$$\psi_{\dot\alpha_1\ldots\dot\alpha_n;\ \beta_1\ldots\beta_m}(p) \quad , \tag{7.14}$$

where the β are 2-valued indices belonging to the fundamental 2-dimensional representation, the $\dot\alpha$ are similar 2-valued indices for the conjugate representation, and ψ is completely symmetric in the α and β, respectively [6]. (b) p_μ is of the form $p_{\alpha\beta}^{\cdot}$, and hence if $p_{\alpha\beta}^{\cdot}$ is contracted with $\psi_{\dot\alpha_1\ldots\dot\alpha_n;\ \beta_1\ldots\beta_m}(p)$ to remove either all the undotted or dotted indices, the remaining indices carry an <u>irreducible</u> representation of $SL(2,C)$. These two properties can be used in an obvious way to project out, with polynomials in p, the parts of $\psi_{\dot\alpha_1\ldots\dot\alpha_n;\ \beta_1\ldots\beta_m}(p)$ which are irreducible with respect to $P_+^\uparrow$. The use of multispinors (7.14) is due originally to Fierz and Pauli [7].

3) With regard to the inner product, for finite dimensional-representations of $SL(2,C)$, which carry a parity operator, the situation is saved by the fact that although $S(\Lambda)$ is not unitary, it is pseudo-unitary, i.e., there exists a metric η in $S(\Lambda)$-space such that

$$\begin{aligned} S\dagger(\lambda)\eta S(\lambda) &= \eta \quad , \quad \lambda \in \Lambda \quad , \\ [S(k),\eta] &= 0 \quad , \quad k \in K \quad , \\ \eta &= \eta\dagger = \eta^{-1} \quad , \end{aligned} \tag{7.15}$$

where the adjoint is with respect to the $V(K)$ space. In fact, η is just the spinspace part of the parity operator. Hence the inner product

$$\bar\psi_1(p)\psi_2(p) = (\psi_1(p),\eta\psi_2(p)) \tag{7.16}$$

is SL(2,C) invariant and Mackey's inner product can be replaced by

$$\int d\mu(p)\bar{\psi}_1(p)\psi_2(p) \quad , \tag{7.17}$$

which is manifestly invariant i.e. invariant under $P_+^\uparrow$ and $S(\Lambda)$ separately. Note that

$$\int d\mu(p)\bar{\psi}(p)\psi(p) \quad , \tag{7.18}$$

is positive-definite on account of the subsidiary conditions.

We conclude this chapter with some examples of manifestly covariant fields.

a) On the orbit $p^2 = m^2$, we choose a D(n,n) representation of SL(2,C). The corresponding field $\psi_{\dot{\alpha}_1\ldots\dot{\alpha}_n;\ \beta_1\ldots\beta_n}(p)$ carries the spin $j = 0, 1, 2, \ldots n$ representations of $P_+^\uparrow$. We can eliminate the spins $j = 0, 1, 2, \ldots n-1$ by the manifestly covariant subsidiary conditions

$$p^{\dot{\alpha}_1\beta_1}\psi_{\dot{\alpha}_1\ldots\dot{\alpha}_n;\ \beta_1\ldots\beta_n}(p) = 0 \quad . \tag{7.19}$$

We usually see this field in its traceless symmetric tensor form $\psi_{\mu_1\ldots\mu_n}(p)$ with the subsidiary conditions $p^2 = m^2$,

$$p^{\mu_1}\psi_{\mu_1\ldots\mu_n}(p) = 0 \quad , \tag{7.20}$$

b) On $p^2 = m^2$ we choose a D(n,n+1) representation

$$\psi_{\dot{\alpha}_1\ldots\dot{\alpha}_n\beta_1\ldots\beta_n\beta_{n+1}} \quad . \tag{7.21}$$

This carries the spins $j = \frac{1}{2}, \frac{3}{2} \ldots n + \frac{1}{2}$, and we can eliminate the lower spins by the subsidiary condition

$$p^{\dot{\alpha}_1\beta_1}\psi_{\dot{\alpha}_1\ldots\dot{\alpha}_n;\ \beta_1\ldots\beta_n\beta_{n+1}} = 0 \tag{7.22}$$

Again, one can use vector notation and replace (7.21) by the field

$$\psi_{\mu_1\ldots\mu_n\alpha}(p) \quad , \tag{7.23}$$

with the subsidiary conditions

$$p^{\mu_1}\psi_{\mu_1\ldots\mu_n\alpha}(p) = 0 \quad ,$$

and

$$\tau^{\alpha\beta}_{\mu_1}\psi_{\mu_1\ldots\mu_n\beta}(p) = 0 \quad , \tag{7.24}$$

where the τ_μ are the Pauli and unit 2×2 matrices.

c) Because the field $\psi_{\mu_1\ldots\mu_n\alpha}(p)$ for $\alpha = 1,2$ does not accommodate parity, it is customary to replace $\alpha = 1,2$ by a Dirac index $\alpha = 1,2,3,4$. The subsidiary conditions then become

$$(\gamma^\mu p_\mu + m)\psi_{\mu_1\ldots\mu_n\alpha} = 0 \quad , \quad \gamma^{\mu_1}\psi_{\mu_1\ldots\mu_n\alpha} = 0 \quad . \tag{7.25}$$

These equations are known as the Rarita-Schwinger equations, [8] and describe spin $j = n + \frac{1}{2}$.

d) One can similarly use $\psi_{\alpha_1 \ldots \alpha_n}(p)$ where $\alpha_r = 1,2,3,4$ are Dirac indices, with the subsidiary conditions

$$(\gamma^{\mu}_{(r)} p_{\mu} + m)\psi_{\alpha_1 \ldots \alpha_n}(p) = 0 \quad , \quad r = 1 \ldots n \quad . \tag{7.26}$$

These fields carry spin $\frac{1}{2}(n + 1)$ and the subsidiary conditions are known as the Bargmann-Wigner [9] equations. The Rarita-Schwinger and Bargmann-Wigner equations automatically include the orbit condition $p^2 = m^2$.*

A simpler and somewhat more general approach to the results of Section 7 will appear in the Proceedings of the 1970 Istanbul Nato Summer School in Mathematical Physics.

8. INFINITE COMPONENT WAVE FUNCTIONS

In the last section, we saw that <u>any</u> representation of the Lorentz spin group $SL(2,C)$ whose restriction to the little group was unitary could be used to product a manifestly covariant unitary representation of $P^{\uparrow}_{+}$. We then devoted our attention to the finite-dimensional (non-unitary) spin groups. In the literature also, attention has been devoted almost entirely to finite-dimensional spin groups. In this section we wish to discuss why this is so.

In the first place, there are good historical precedents for using finite-dimensional spin groups, since the classical fields of Newton, Maxwell, Einstein, and Dirac are of this form (they use the finite-dimensional $D(00)$, $D(10) + D(01)$, $D(1,1)$ and $D(\frac{1}{2}0) + D(0\frac{1}{2})$ representations of $SL(2,C)$, respectively).

Secondly, in particle physics, each of the particles one wishes to describe is known empirically to have finite spin. Hence, it is natural to use a finite-dimensional spin group to describe it.

On the other hand, one could legitimately ask the question:

1) Since in the spin group a number of superfluous representations of the little group appear anyway and are eliminated by subsidiary conditions, why not use an infinite dimensional spin group plus infinite dimensional subsidiary conditions?

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* Note added in proof: In Step 1 of this chapter, if one wishes to avoid the explicit decomposition of g into (Λ,a) one can do so by defining $\theta(p)$ according to the equation

$$\theta(p) = \theta(g) = W(g_0 g^{-1}) f(g) \quad , \quad g_0 = (\Lambda_0, 0) \quad .$$

Also, if one is interested only in the final manifestly covariant form (7.11) and wishes to eliminate Step 1, one can do so by letting $f(g) \to \sum_{\lambda} f_{\lambda}(g) = F(g)$, and defining $\psi(p) = \psi(g) = S^{-1}(\Lambda) e^{-i\alpha\Lambda . a} F(g)$.

2) Since what we observe experimentally is, in any case, not just one particle but the infinite family of particles suggested by Figure 5.1, why not go the whole hog and try to describe all of the particles, or at least large sub-families of them, by means of a single covariant field. This field, in order to carry an infinite number of UIR's of $P_+^\uparrow$, would have to correspond to an infinite dimensional representation of $SL(2,C)$.

The possibility raised by Question 2) is even highly attractive. What we shall show, however, is that the attraction is deceptive and that infinite spin groups lead to difficulties which, at present at any rate, seem to be unsurmountable. We shall do this first for two special models, and then present a general no-go theorem which has been proved recently.

The difficulties come under two headings:

a) Violation of the spectral condition $p^2 \geq 0$

b) Violation of locality for quantized fields.

We first illustrate a) for two special models.

The first model we consider dates back to 1932 and was proposed by Majorana [1] as a possibility for avoiding the "negative energy" states of Dirac's theory, which were thought to be an embarrassment at that time. Majorana proposed that one use a wave function, with spin group corresponding to the $(j_0 = \frac{1}{2},\ c = 0)$ or $(j_0 = 0,\ c = \frac{1}{2})$ UIR of $SL(2,C)$ and satisfying the subsidiary condition

$$(\Gamma_\mu p_\mu - \kappa)\psi(p) = 0 \quad , \tag{8.1}$$

where κ is a positive number and Γ_μ is a p-independent $SL(2,C)$ vector. (The "Majorana representations" $(j_0 = \frac{1}{2},\ c = 0)$ and $(j_0 = 0,\ c = \frac{1}{2})$ are the only irreducible UR's of $SL(2,C)$ to carry a vector operator.)

The question then is: What UIR's of $P_+^\uparrow$ does Majorana's $\psi(p)$ carry?

To answer it, consider an orbit $p^2 = m^2 > 0$, $p_0 > 0$. Such an orbit would contain the vector $\alpha = (m000)$ whence from (8.1)

$$(\Gamma_0 m - \kappa)\psi(\alpha) = 0 \quad , \tag{8.2}$$

which is possible if and only if m is equal to one of the eigenvalues of κ/Γ_0. The eigenvalues of κ/Γ_0 turn out [2] to be $\kappa(j_0 + \frac{1}{2} + n)^{-1}$, $n = 0,1,2,3\ldots$ Thus, Majorana's $\psi(p)$ carries the orbits $p^2 = m^2$, $m = \dfrac{\kappa}{j_0 + \frac{1}{2} + n}$, $p_0 > 0$.

Furthermore, the little group of such an orbit is $SU(2)$ and, in the reduction of Majorana representations with respect to $SU(2)$, each representation $j = j_0, j_0 + 1,\ldots$ of $SU(2)$ occurs once and only once, with

$$\Gamma_0 = j + \frac{1}{2} = j_0 + \frac{1}{2} + n \quad . \tag{8.3}$$

Hence, each orbit $m = \kappa/\Gamma_0 = \kappa/(j + \frac{1}{2})$ carries exactly one UIR of $P_+^\uparrow$, and we have the mass-spin relationship

$$m = \frac{\kappa}{J + \frac{1}{2}} \quad . \tag{8.4}$$

Experimentally this mass-spin relationship is disastrous, but that is not a real problem as it could easily be modified. For example, by replacing κ by κp^2 in the subsidiary condition, it could be inverted, which would be very good experimentally.

The real difficulty comes from the non-physical orbits $p^2 = m^2 < 0$. These exist because they can be generated from vectors $\alpha = (000m)$ for which (8.1) is equivalent to

$$(\Gamma_3 m - \kappa)\psi(\alpha) = 0 \quad , \tag{8.5}$$

and this equation has non-trivial solutions since Γ_3 is self-adjoint. (Note that in this connection the unitarity of the Majorana representation of $SL(2,C)$ is actually a liability, since it implies that if Γ_0 is self-adjoint, then so is Γ_3; the above argument would have broken down for the finite-dimensional non-unitary Dirac representation of $SL(2,C)$, for which γ_0 is hermitian but γ_3 is not.)

The $p^2 < 0$ orbits are undesirable but are not an <u>immediate</u> catastrophe for the Majorana equation since they could simply be ignored. The trouble is that, in practice, one is interested not merely in the free Majorana particles, but also in their interactions. For example, if we try to introduce the EM interaction by means of the traditional minimum principle

$$(\Gamma_\mu p_\mu - \kappa)\psi(p) = 0 \rightarrow (\Gamma_\mu p_\mu - \kappa)\psi(p) = \Gamma_\mu A_\mu(k)\psi(p + k) \quad , \tag{8.6}$$

where A_μ is the vector potential and k the momentum transfer, one can show that for $k \neq 0$ the system makes transitions from $p^2 > 0$ to $p^2 < 0$ states, and similarly for any other interactions which are local in the Fourier transformed space. Now, of course, one might do better with some more complicated, non-local interaction. But since the purpose of the manifestly covariant wavefunctions is to provide a framework for introducing simple, local commutation relations and interactions, this would defeat the purpose. For this reason, the $p^2 < 0$ orbits are a real difficulty in Majorana's theory.

The second model we consider is a wavefunction $\psi(p)$ carrying a Dirac $\otimes$ unitary spin representation and satisfying the subsidiary condition

$$(\gamma \cdot p + M)\psi(p) = 0 \quad , \tag{8.7}$$

where M is a spin invariant, e.g.,

$$M = m_0 + m_1 \sigma_{\mu\nu} \Sigma_{\mu\nu} \quad ,$$

where m_0, m_1 are constants and $\sigma_{\mu\nu}$ and $\Sigma_{\mu\nu}$ are the generators of the Dirac

and unitary representations, respectively. This equation was first studied by Abers Grodsky and Norton [3] (AGN) in 1965 and has since been used in current algebra theory. An analysis of the equation, similar to that described above for the Majorana equation, for the case in which the unitary representation is $(j_0, C = 0)$, shows that for the $p^2 > 0$ orbits there is a mass-spin relationship

$$\pm m = m_1(J + \tfrac{1}{2}) \pm \{(m_0 - m_1)^2 + m_1^2[J(J+1) - j_0(j_0+1) - \tfrac{3}{4}]\} \quad , \qquad (8.8)$$

which can be drawn graphically as in Figure 8.1. The rising curve for $m > 0$ fits well with the observed particles (and with Regge theory, which we shall be describing later). However, the falling curve for $m > 0$ has no satisfactory interpretation. (The $m < 0$ curves can be identified with anti-particles.)

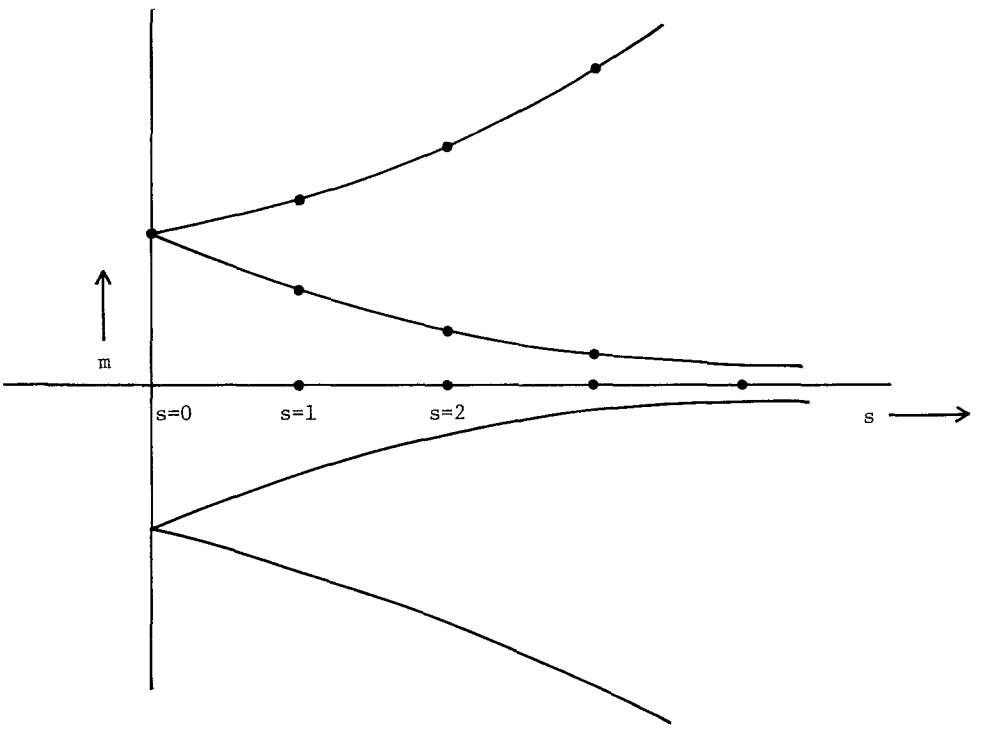

FIGURE 8.1. MASS-SPIN RELATIONSHIP FOR AGN EQUATION

Leaving aside the interpretation of the falling curve, we ask again whether $\psi(p)$ carries unphysical $p^2 < 0$ orbits. The answer is yes. The proof is perhaps worth giving.

Proof. Write the subsidiary condition (8.7) in the form

$$(\vec{\alpha} \cdot \vec{p} + \beta M)\psi(p) = p_0\psi(p) \quad , \qquad (8.9)$$

where $\vec{\alpha}$ and β are the self-adjoint Dirac matrices $\gamma_0\vec{\gamma}$ and γ_0, respectively. Now βM must be self-adjoint to provide a mass spectrum in the rest frame $\vec{p} = 0$, and $\vec{\alpha} \cdot \vec{p}$ is self-adjoint and bounded. Hence, for each $\vec{p}$, $\vec{\alpha} \cdot \vec{p} + \beta M$ is self-

adjoint. Hence, (8.9) may be regarded as an eigenvalue equation for the self-adjoint operator $\vec{\alpha} \cdot \vec{p} + \beta M$, i.e., p_0 is any point in the spectrum of $\vec{\alpha} \cdot \vec{p} + \beta M$.

The condition that there be no $p^2 < 0$ orbits is that $p_0^2 \geq \vec{p}^2$. But since p_0 is any point in the spectrum of $\vec{\alpha} \cdot \vec{p} + \beta M$, this implies that

$$(\vec{\alpha} \cdot \vec{p} + \beta M)^2 \geq \vec{p}^2 \quad , \tag{8.10}$$

or, since $\alpha^2 = 1$,

$$\vec{p} \cdot [\vec{\alpha}, \beta M]_+ + (\beta M)^2 \geq 0 \quad . \tag{8.11}$$

But since $\vec{p}$ varies over the whole Euclidean 3-space, this is possible if and only if

$$[\vec{\alpha}, \beta M]_+ = 0 \quad , \tag{8.12}$$

which on account of the anti-commutativity of $\vec{\alpha}$ and β, reduces to

$$[\vec{\alpha}, M] = 0 \quad . \tag{8.13}$$

But since

$$\sigma_{\mu\nu} = \vec{\alpha}, \ \vec{\alpha} \times \vec{\alpha} \quad ,$$

this means that $p^2 \geq 0$ is possible if and only if

$$[\sigma_{\mu\nu}, M] = 0 \quad , \tag{8.14}$$

i.e., if and only if M is a Dirac invariant, in which case equation (8.7) can be reduced to a direct sum of Dirac equations with M = constant.

Thus, the AGN equation, like the Majorana, is either trivial or contains unphysical orbits $\vec{p}^2 < 0$ and, once again, it can be checked that local interactions couple the physical orbits to unphysical ones.

Note that the $p^2 < 0$ difficulties arise whether or not the wavefunction $\psi(p)$ is quantized. If the field is quantized, then there are the further difficulties (b) concerning locality. To illustrate the point, consider an infinite component wavefunction $\psi(p)$ which has not yet been quantized, introduce a set of creation and destruction operators for particles satisfying Bose-Einstein or Fermi-Dirac statistics on a Hilbert space $\mathcal{H}$, i.e., satisfying

$$[a(p), a^+(p')]_{\pm} = \delta(p - p') \quad , \tag{8.15}$$

etc., and construct from $\psi(p)$ and $a(p)$ a quantized field in the standard way, namely,

$$\varphi(x) = \int d\mu(p) \{ e^{ip \cdot x} a(p) \psi(p) + e^{-ip \cdot x} b^\dagger(p) \tilde{\psi}(p) \} \quad , \tag{8.16}$$

where $\tilde{\psi}(p)$ transforms like $\psi(p)$. The locality difficulties can be subdivided into

a) locality proper

b) spin-statistics

c) CPT-invariance

d) analyticity.

Locality proper is the question whether the commutator

$$[\varphi(x),\varphi(x')] \quad , \tag{8.17}$$

vanishes for $(x - x')^2 < 0$. In the finite-dimensional case, the commutator does vanish for suitable choice of $\pm$ in (8.16). In the infinite dimensional case, however, in general no choice of sign in (8.15) and no simple modification will make (8.17) vanish. The possibilities for evading this difficulty have been investigated in some detail in the recent literature [4], but with no particularly attractive solution.

The spin-statistics difficulty is an extension of the problem: In the finite-dimensional cases, (8.17) vanishes for $\pm$ in (8.15), but the choice of $\pm$ is not arbitrary. It must be (+) (Fermi-Dirac statistics) if the field carries half-odd-integer spin and (-) (Bose-Einstein statistics) if the field carries integer spin, a correlation which is verified experimentally and is regarded as one of the most fundamental results of quantum field theory. But in the infinite dimensional case, since (8.17) does not vanish for either choice of sign, the spin-statistics correlation gets lost. (In the cases that (8.17) can be made to vanish, it can be made to vanish for either choice of sign, so the correlation becomes, at best, arbitrary.)

The other two difficulties, CPT invariance and analyticity, are special cases of the general result that for finite-dimensional spin groups, the Lorentz transformations can be continued to any complex values of the parameters whereas for infinite dimensional spin representations, this is not the case. (Infinite-dimensional representations of $SL(2,C)$ have dense sets of analytic vectors, but no entire vectors.) As a result, the EM form factors and the scattering matrix S have different analytic properties (as functions of the inner products of the momenta) in the finite and infinite-dimensional cases, and the analytic properties in the infinite-dimensional case do not seem to be the most desirable.

All models so far constructed using infinite-dimensional representations of $SL(2,C)$ have been found to be unsatisfactory in at least some of the above ways. This suggests that it might be possible to rule out infinite component fields on quite general grounds and, thus, restrict oneself to the finite-dimensional spin representations without any real loss in generality.

One such general set of conditions was found recently by Streater and Grodsky [5]. Their argument is as follows:

Let $\varphi(\sigma,x)$ be an infinite component field operating on a physical Hilbert space $\mathcal{H}$ with vacuum state h, and carrying a continuous bounded irreducible infinite dimensional spin group, $S_{\lambda\sigma}$. Rather than specify precisely how $\varphi(\sigma,x)$ is quantized, they assume only that it has been quantized in such a way that the vacuum expectation value

$$F(\sigma,\sigma',x,x') = (0,\varphi^{\dagger}(\sigma,x)\varphi(\sigma',x')0) \quad , \tag{8.18}$$

with unique vacuum state $0)$, has the following properties:

a) Translational invariance: $F(\sigma,\sigma',x,x') = F(\sigma,\sigma',x - x')$

b) Reasonable spectrum: $\tilde{F}(\sigma,\sigma',p) = 0$ for $p^2 < 0$, where $\tilde{F}$ denotes Fourier transform

c) Causality (locality): $F(\sigma,\sigma'x) = 0$ for $x^2 < 0$

d) Temperedness: $F(\sigma,\sigma',x)$ is a tempered distribution in x for all σ,σ'

e) Finite degeneracy of the lowest isolated mass-hyperboloid.

These are all assumptions that are made normally in quantum field theory. The temperedness assumption is a strengthening of locality (it implies that $f(\sigma,\sigma',x)$ is not too singular on the light cone) and, although this assumption can be relaxed, it cannot be relaxed very much if the correct analyticity properties are to be obtained for the S-matrix.

Grodsky and Streater now claim that these assumptions are incompatible. To prove this, they make use of a theorem due to Bogoliubov and Vladimirov [6] which states that if $f(x)$ is a tempered distribution with $f(x) = 0$ for $x^2 < 0$ and the Fourier transform $\tilde{f}(p) = 0$ for $p^2 < 0$, then $\tilde{f}(p)$ is a finite covariant, i.e., $\tilde{f}(p)$ has the representation

$$\tilde{f}(p) = \sum_{[n]} C_{[n]} p_0^{n_0} \ldots p_3^{n_3} \int dm^2 \rho_{[n]}(m^2)\delta(p^2 - m^2) \quad , \tag{8.19}$$

where the sum is finite, $[n] = [n_1 n_2 n_3 n_0]$ and $\rho_{[n]}$ is tempered. Applying this theorem to $F(\sigma,\sigma',x)$, which obviously satisfies the conditions, and smearing with a test function $f(x)\tilde{f}(p)$ with support only in the neighborhood of the lowest mass-hyperboloid in p-space, one obtains

$$(0,\varphi^{\dagger}(\sigma,f)\varphi(\sigma',f)0) = \text{Const. } \delta^4(p' - p) \sum_{[n]} C_{[n]}(\sigma,\sigma') p_0^{n_0} \ldots p_3^{n_3} \quad .$$

But since the spin-representation is assumed to be continuous, $C_{[n]}(\sigma,\sigma')$ is continuous in σ and σ'. Hence, $C_{[n]}(\sigma,\sigma')$ is the matrix element of a bounded linear operator in spin space V. Hence, for fixed σ', $C_{[n]}(\sigma,\sigma')$ may be regarded as a vector in V and since there are only a finite number of $C_{[n]}$, the linear span

$$\sum_{[n]} C_{[n]}(\sigma,\sigma') p_0^{n_0} \ldots p_3^{n_3} \quad , \tag{8.20}$$

for all $p_0 \ldots p_3$ and fixed σ', is finite dimensional. It follows that the expression (8.20) vanishes for an infinite number of values of σ. Referring back to (8.18), we see that there are, therefore, an infinite number of states $\varphi(\sigma,f)0)$ in $\mathcal{H}$, orthogonal to the state $\varphi(\sigma' f)0)$ for all $\vec{p}$ and $\vec{p}'$. Furthermore, since the spin group is irreducible, $\varphi(\sigma,f)0)$ vanishes if and only if $\varphi(\sigma',f)0)$ vanishes. It follows that the orthogonal states are not zero. Thus, the lowest mass-hyperboloid is infinitely degenerate. This is the result of Grodsky and Streater.

A corollary to their result, which has been pointed out by Grodsky and Streater, is that since any field $\psi(x)$ which is obtained by quantizing in the usual manner (8.16), a wavefunction $\psi(p)$ whose support is in $p^2 > 0$ and whose SL(2,C)-space projection on $p_0 > 0$ is polynomially bounded in p, will be automatically tempered and causal, it must belong to a finite dimensional representation of SL(2,C).

What does this result mean physically? It means that if we use infinite-dimensional representations of SL(2,C) one of two things must happen. Either the subsidiary conditions imposed on the wavefunctions are too weak, in which case there is an infinite number of spin states on each mass-hyperboloid (in gross contradiction to experiment), or else the subsidiary conditions are too strong (as in the Majorana and AGN cases discussed above). In that case, there is no spin degeneracy but the wavefunction cannot be quantized so as to describe a tempered local field with $p^2 > 0$.

Note that the temperedness of the distribution plays a critical role in the above arguments. It leads directly to the finiteness of the expansion (8.19), which leads in turn to the finiteness of the linear span (8.20) and hence to the infiniteness of the orthogonal complement. (Note added in proof: a generalization of the GS theorem which allows more general distributions, including Jaffe distributions, is now available [7].)

Perhaps the best way to summarize the results of this chapter is to say that while there are no group-theoretical reasons for excluding infinite spin groups, there appear to be other reasons to exclude them, namely, mass-spectrum, locality, and finite-spin degeneracy considerations. Thus, one can return, (with some relief!) to the finite dimensional spin representations.

9. LITTLE GROUP DECOMPOSITION OF THE SCATTERING AMPLITUDE

In the last couple of chapters we saw how the Poincaré group $P_+^\uparrow$ and its little group for $p^2 \geq 0$ could be used to characterize relativistic particles. In this chapter I should like to mention briefly how $P_+^\uparrow$ and its little group can be used to analyze scattering processes. One of the interesting features will be that, in spite of the spectral condition, the SU(1,1) little group for the orbits $p^2 < 0$ will also be relevant.

To put the role of the little groups into perspective, we consider the scattering amplitude (Figure 9.1) for 2-particles scattering into 2 particles (not necessarily the same), e.g. $\pi N \to \Sigma K$. The probability of the particles 1 and 2

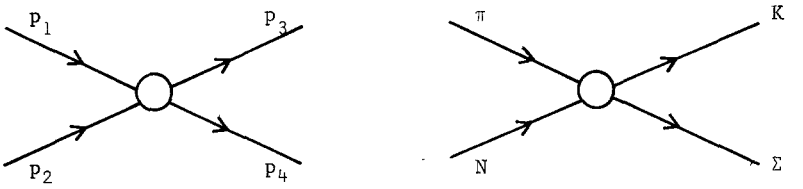

FIGURE 9.1. SCATTERING IN S-CHANNEL

with momenta p_1 and p_2 scattering into particles 3 and 4 with momenta p_3 and p_4 is given by

$$P(p_1p_2 \to p_3p_4) = |(p_3p_4,\ T\ p_1p_2)|^2 \quad , \tag{9.1}$$

where T is the scattering matrix. Because of Poincaré invariance, the scattering amplitude $(p_3p_4,\ T\ p_1p_2)$ is (apart from some kinematical factors, which we omit) a function of two invariant variables, s and t

$$(p_3p_4,\ T\ p_1p_2) = F(s,t) \quad , \tag{9.2}$$

where

$$s = (p_1 + p_2)^2 \quad , \quad t = (p_1 - p_3)^2 \quad . \tag{9.3}$$

For symmetry we can also define $u = (p_1 - p_4)^2$, but u is not an independent variable. In fact $u + s + t = \sum_{\alpha=1}^{4} m_\alpha^2$, where m_α are the masses. (In general, the scattering amplitude for 2 particles into $n-2$ particles depends on $3n - 10$ invariant variables, the $3n$ variables being the n 3-momenta of the n particles involved, the ten constraints coming from the conservation of the ten generators of $P_+^\uparrow$.) If the four particles involved in the scattering of Figure 9.1 are spinless (as we shall assume for simplicity) then F is a scalar function.

Now consider the process of Figure 9.2, namely the scattering of particles

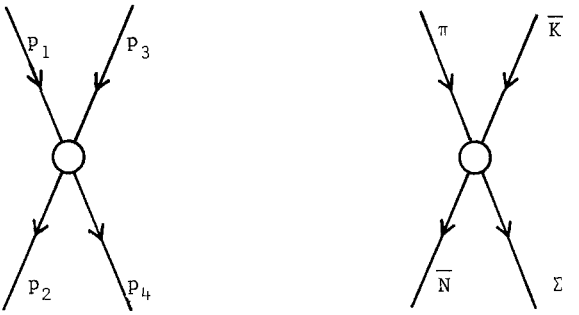

FIGURE 9.2. SCATTERING IN S-CHANNEL

1 and 3 with momenta p_1 and p_3 into particles 2 and 4 with momenta p_2 and p_4 (e.g. $\pi\bar{K} \to \bar{N}\Sigma$). The probability for this scattering is given by

$$P(p_1p_3 \to p_2p_4) = |(p_2p_4, T\, p_1p_3)|^2 \quad , \tag{9.4}$$

where

$$(p_2p_4, T\, p_1p_3) = F'(s',t') \quad , \tag{9.5}$$

and

$$s' = (p_1 - p_2)^2 \quad , \quad t' = (p_1 + p_3)^2 \quad . \tag{9.6}$$

One of the most basic and fruitful ideas to emerge in particle physics during the fifties was that the two scattering amplitudes $F(s,t)$ and $F'(s',t')$ are not only related, but are in fact the same analytic function [1]. That is to say, if one considered s' to be the analytic continuation of $s = (p_1 + p_2)^2$ to $p_2 \to -p_2$ and t' the analytic continuation of $t = (p_1 - p_3)^2$ to $p_3 \to -p_3$, then

$$F(s,t) = F'(s,t) \quad . \tag{9.7}$$

The process of Figure 9.1, for which $s > 0$, is called the s-channel and that of Figure 9.2, for which $t > 0$, the t-channel. The hypothesis (9.7) is based upon an analysis of Feynman diagrams and of axiomatic field theory [2]. It is related to the spectral condition, causality and the temperedness of the field-distributions.

Returning to the s-channel, an alternative pair of variables to (s,t) are $(s, \cos\theta)$,

$$F(s,t) = f(s, \cos\theta) \quad , \tag{9.8}$$

where θ is the angle between the three-momenta $\vec{p}_1$ and $\vec{p}_2$ in the center of mass frame of p_1 and p_2 (Figure 9.3). The relationship between t and $\cos\theta$ is

$$\cos\theta = \frac{s(t-u) + (m_3^2 - m_4^2)(m_1^2 - m_2^2)}{\{[s - (m_1 - m_2)^2][s - (m_1 + m_2)^2][s - (m_3 - m_4)^2][s - (m_3 + m_4)^2]\}^{\frac{1}{2}}} \tag{9.9}$$

This looks complicated unless the masses are equal. However, the important point is that $\cos\theta$ is linear in t. In the analyses of scattering data it is usual [3] to make a "partial wave decomposition" of $f(s, \cos\theta)$ i.e., to expand $f(s, \cos\theta)$ in terms of Legendre functions

$$f(s, \cos\theta) = \sum_{\ell} (2\ell + 1) a_\ell(s) P_\ell(\cos\theta) \quad . \tag{9.10}$$

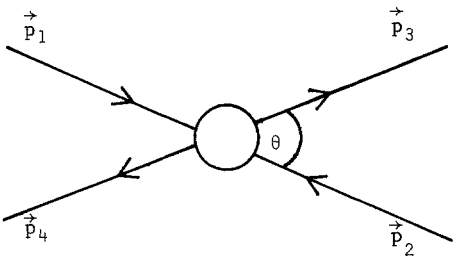

FIGURE 9.3. SCATTERING IN CM SYSTEM IN S-CHANNEL IN 3-SPACE

This is done for two reasons. (a) The unitary condition, which says that the total probability for scattering is unity, is diagonal in the P_ℓ basis. In fact, it reads

$$a_\ell(s) = \sin\delta_\ell(s) \exp i\delta_\ell(s) \quad ,$$

where $\delta_\ell(s)$ is real, and a scattering analysis is normally an analysis of the "phase-shifts" $\delta_\ell(s)$. (b) For low-energies, $s \sim (m_1 + m_2)^2$, the low values of $\ell(\ell = 0,1,2)$ dominate. (One can see this intuitively by noting that for low energy we have low relative angular momentum of the two particles, and as we shall see later, ℓ is the relative angular momentum.)

Regge Theory

One of the problems of scattering theory was how to combine the analyticity (9.7) with the expansion (9.10). As we go from the s channel, where $t < 0$ and $|\cos\theta| \leq 1$, to the t-channel, where $t > (m_1 + m_2)^2$ and $|\cos\theta| \geq 1$, the expansion (9.10) diverges. To overcome this difficulty, Regge [4] showed that, at

least for a class of non-relativistic potential scattering theories, the way to continue $\cos\theta$ was to express the expansion (9.10) in integral form. First, one writes

$$f(s, \cos\theta) = \frac{1}{2\pi i}\int_C \frac{(2\ell+1)d\ell}{\sin\pi\ell}\, a_\ell(s)P_\ell(\cos\theta) \quad , \tag{9.11}$$

where C is the contour of Figure 9.4, then divides the integrand into $+$ and $-$ signature parts

$$f^{\pm}(s, \cos\theta) = \frac{1}{2\pi i}\int_C \frac{(2\ell+1)d\ell}{\sin\pi\ell}\, a^{\pm}_\ell(s)[P_\ell(\cos\theta) \pm P_\ell(-\cos\theta)] \quad , \tag{9.12}$$

which have independent physical properties, and then, because each converges separately on the circle at infinity, opens up the contour to L, which is the furthest

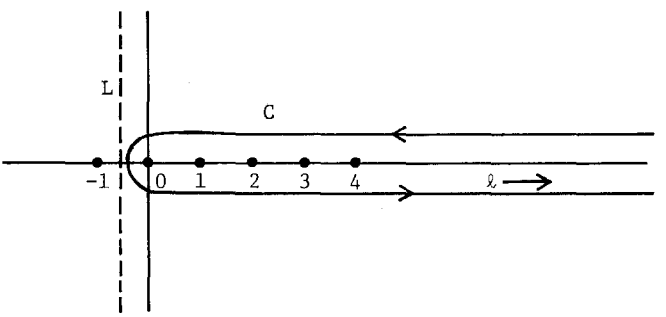

FIGURE 9.4. THE CONTOURS OF C AND L

line to the left allowed by the $P_\ell(\cos\theta)$. On the way, one picks up the poles of $a^{\pm}_\ell(s)$, which for the class of potentials considered is a meromorphic function of ℓ to the right of L, and obtains (simplifying for clarity to the case when $a^{\pm}_\ell(s)$ has only one pole to the right of L)

$$f^{\pm}(s, \cos\theta) = (2\alpha^{\pm}(s)+1)\frac{\beta^{\pm}(s)}{\sin\pi\alpha^{\pm}(s)}[P_{\alpha^{\pm}(s)}(\cos\theta) \pm P_{\alpha^{\pm}(s)}(-\cos\theta)]$$

$$+ \frac{1}{2\pi i}\int_L \frac{(2\ell+1)d\ell}{\sin\pi\ell}\, a^{\pm}_\ell(s)[P_\ell(\cos\theta)$$

$$\pm P_\ell(-\cos\theta)] \quad , \tag{9.13}$$

where $\alpha^{\pm}(s)$ is the position of the pole, and $\beta^{\pm}(s)$ the residue of $a^{\pm}_\ell(s)$ at the pole. The expression (9.13) can now be continued in $\cos\theta$ into the t-channel, and indeed to $t \sim \cos\theta \to \infty$.

What is the relevance of all this to relativistic scattering? The point is that one now makes the hypothesis [5] that although relativistic scattering may be quite different from non-relativistic scattering, it retains at least one feature of it, namely, the fact that $a_\ell(s)$ is meromorphic to the right of L.

This is quite an assumption, and indeed, has had to be modified. But it is at least within the general philosophy that nature is simple if looked at the right way--and here the postulate is that the right way to look at $f(s, \cos\theta)$ is from the point of view of its properties in the ℓ-plane to the right of L! In any case, let us investigate [6] the physical implications of (9.13).

The physical implications of (9.13) are best seen by noting that the pole $\alpha(s)$ is not fixed, but varies with s, and drawing the path of its real part as a function of s (Figure 9.5). There is good reason to believe, as we shall see in

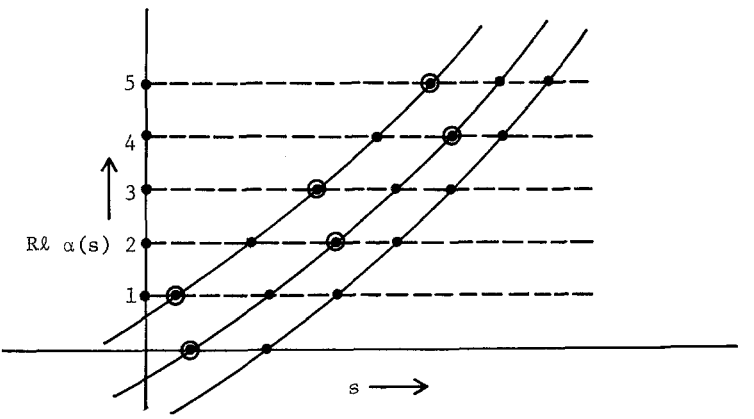

FIGURE 9.5. REGGE TRAJECTORIES

a moment, that its path is as in this figure. The physical implications are then two-fold:

(i) for $t \sim \cos\theta \to \infty$, $s < 0$, we have from (9.13)

$$f^{\pm}(s, \cos\theta) \to \frac{2\alpha^{\pm}(s) + 1}{\sin \pi\alpha^{\pm}(s)} \beta^{\pm}(s) (\cos\theta)^{\alpha^{\pm}(s)} \quad . \tag{9.14}$$

This means that in the t-channel, as $t \to \infty$,

$$f^{\pm}(s,t) \to A^{\pm}(s) t^{\alpha^{\pm}(s)} \quad , \tag{9.15}$$

i.e., we have an explicit statement about the behavior of the scattering amplitude as a function of the energy (t) for high energy. This is a result which could not be obtained experimentally and was not obtained theoretically before the advent of Regge theory. What was known theoretically before was that, because of the unitary condition for T, $f(s,t)$ was bounded, and probably decreased, as a function of t for $t \to \infty$. This is why $\mathrm{Rl}\ \alpha^{\pm}(s)$ is assumed to be less than 1 for $s < 0$ in Figure 9.5. But the explicit t-dependence was first obtained in Regge theory, and is clearly controlled by the Regge-pole at $\ell = \alpha^{\pm}(s)$.

(ii) If $\text{Im}\ \alpha^{\pm}(s)$ is small, then when $\text{Rl}\ \alpha^{\pm}(s)$ = integer, $1/\sin \pi\alpha^{\pm}(s)$ is large. Hence, remembering the factor $P_{\alpha^{\pm}(s)}(\cos\theta) \pm P_{\alpha^{\pm}(s)}(-\cos\theta)$, which is small for $\text{Rl}\ \alpha^{\pm}(s)$ = even/odd integer, we see that $f^{\pm}(s, \cos\theta)$ is large for $\text{Rl}\ \alpha^{\pm}(s)$ = even/odd integer. Returning to the s-channel, $s > (m_1 + m_2)^2$, we see that the s-channel amplitude therefore becomes large, or <u>resonates</u>, whenever $\text{Rl}\ \alpha^{\pm}(s)$ = even/odd integer. Furthermore, a simple analysis of how the amplitude resonates near $\alpha^{\pm}(s)$ = even/odd integer, shows that it behaves as if it were the contribution to the s-channel scattering of an unstable bound state particle or resonance of mass = $\sqrt{s}$, spin = $\text{Rl}\ \alpha^{\pm}(s)$, and life time $\propto [\text{Im}\ \alpha^{\pm}(s)]^{-1}$, Figure 9.5. This result clearly suggests that the $\text{Rl}\ \alpha^{\pm}(s)$ = even/odd integer points on the Regge-trajectory of Figure 9.5 should be interpreted as unstable particles of increasing mass and spin. And indeed, if one examines Figure 5.1a, one sees that the baryons for which it can be checked do indeed lie on Regge trajectories. The mesons do not have sufficiently well-determined spins and parities for a direct check but other considerations support the conjecture that they also lie on Regge trajectories. A typical conjecture [11] is shown in Figure 9.6.

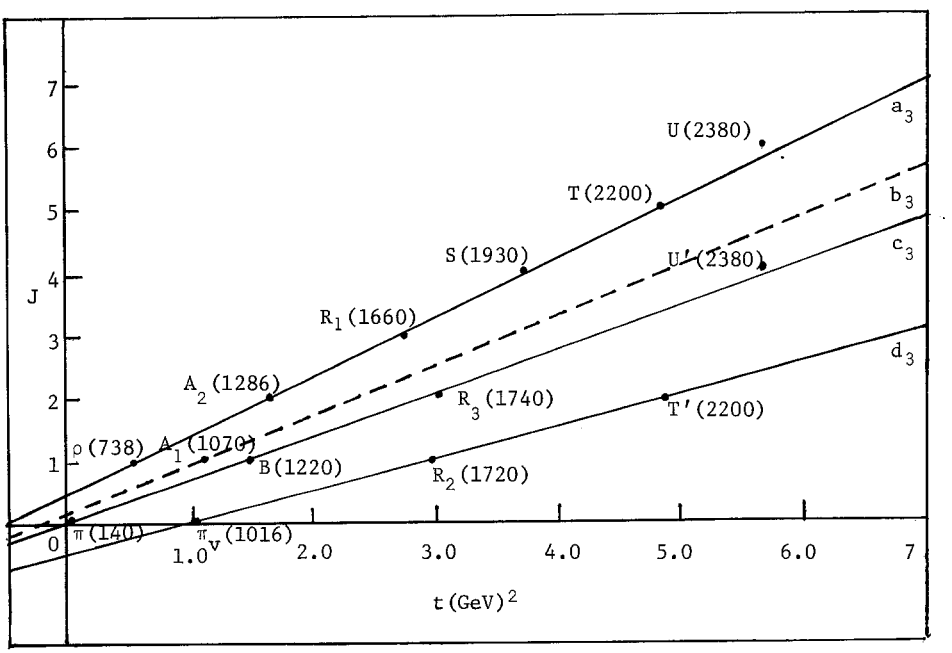

FIGURE 9.6.

The most beautiful part of the results (i) and (ii) lies in their combination. By combining them we see that the resonant states, or unstable particles, which are produced in the s-channel, dictate the high-energy behavior in the t-channel (and, of course, conversely). This unexpected relationship between these hitherto unconnected phenomena is a result that is almost certain to survive, no matter how the details of the Regge theory may have been modified.

A further beauty of the result is that it simultaneously solves a long-standing puzzle in scattering theory, namely, that if one were to continue the contribution to $F(s,t)$ of a particle with a fixed high spin (therefore high powers of $\cos\theta$, therefore high powers of t) from the s-channel to the t-channel, this contribution alone would violate the unitary condition for large t. The Regge result solves the problem by showing that the spin is really $\mathrm{Rl}\ \alpha^{\pm}(s)$, and hence is not fixed, but varies with s and becomes less than 1 for $s < 0$ in the t-channel.

After the above rather lengthy description of the background, let us turn at last to the little groups.

Consider first the two-particle state $|p_1p_2\rangle$ in the s-channel. This state can equally well be described by $|P,q\rangle$ where $P = p_1 + p_2$, $q = p_1 - p_2$. Since $p_1^2 = m_1^2$, $p_2^2 = m_2^2$, if we consider P as 4 independent variables there are two constraints on q. As a result, we can write $|P,q\rangle$ as $|P,R(\theta_1\phi_1)q_0\rangle = |s = P^2, \vec{P}, R(\theta_1\phi_1)q_0\rangle$, where q_0 is a fixed vector and $R(\theta_1\phi_1)$ is a rotation belonging to the little group $SO(3)$ of P. (The angle $(\theta_1\phi_1)$ is the angle between a fixed z-axis and the $\vec{p}_1 - \vec{p}_2$ line in the 3-dimensional diagram of Figure 9.3.) In a similar way the state $|p_3p_4\rangle$ can be written as $|s' = P'^2, \vec{P}', R(\theta_2\phi_2)q_0\rangle$, where $P' = p_3 + p_4$, $R(\theta_2\phi_2)$ is an element of the little group $SO(3)$ of P', and $(\theta_2\phi_2)$ is the angle between the fixed z-axis and $\vec{q}' = \vec{p}_3 - \vec{p}_4$. However, from energy momentum conservation we have

$$s = s' \quad , \quad \vec{P} = \vec{P}' \quad . \tag{9.16}$$

Hence $R(\theta_1\phi_1)$ and $R(\theta_2\phi_2)$ are elements of the <u>same</u> little group, namely that of $P = P'$. For the scattering amplitude, which is Poincaré, and therefore rotationally invariant, we then have

$$\begin{aligned} \langle p_3p_4|T|p_1p_2\rangle &= \langle s,\vec{P},R(\theta_2\phi_2)q_0|T|s,\vec{P},R(\theta_1\phi_1)q_0\rangle \\ &= \langle s,\vec{P},q_0|T|s,\vec{P},R^{-1}(\theta_2\phi_2)R(\theta_1\phi_1)q_0\rangle \\ &= \langle s,\vec{P},q_0|T|s,\vec{P},R(\theta_3,\phi_3)q_0\rangle \quad , \end{aligned} \tag{9.17}$$

where $(\theta_3,\phi_3) = \theta$ is the angle between the lines $\vec{p}_1 - \vec{p}_2$ and $\vec{p}_2 - \vec{p}_4$ in Figure 9.3. Hence

$$f(s,\cos\theta) = \langle p_3p_4|T|p_1p_2\rangle = \langle s,\vec{P},q_0|T|s,\vec{P},R(\theta)q_0\rangle \quad . \tag{9.18}$$

But since $R(\theta)$ is an element of the little group $SO(3)$ of P, this means that as a function of θ, $f(s, \cos\theta)$ is a function over the little group $SO(3)$. Hence the expansion

$$f(s, \cos\theta) = \sum_{\ell} (2\ell + 1)a_{\ell}(s)p_{\ell}(\cos\theta) \quad , \tag{9.19}$$

emerges as nothing but the expansion of the scattering amplitude as a function over the little group $SO(3)$ in terms of the irreducible representations of $SO(3)$ [7].

Once it is realized that the partial wave decomposition (9.19) which is restricted to the part of the $s - t$ plane which belongs to the s-channel, $s > 0$, is nothing but an expansion over the little group $SO(3)$ of $P = p_1 + p_2$, a method for extending the expansion to other channels immediately suggests itself, namely, to make a little group expansion in the same variable in the other channel also. In the other channel, $s = P^2 < 0$, since $P = p_1 + p_2$ with $p_2^0 < 0$. Hence the little group is $SO(2,1)$.

However, there is a snag. The snag is that whereas in the s-channel the unitary condition guarantees that $f(s, \cos\theta)$ will be square integrable over the little group $SO(3)$, in the other channels there is no guarantee that it will be square-integrable over $SO(2,1)$, and in general it is not.

At this point, however, one can return to Regge theory. Looking at the Regge expansion (9.13) one sees at once that the background integral is nothing but the expansion of the scattering amplitude in terms of the <u>principal</u> series of $SO(2,1)$. Thus what Regge theory says is that, although the full scattering amplitude is not square-integrable over $SO(2,1)$, when one removes the contribution of the Regge poles, the remainder <u>is</u> square-integrable. Thus equation (9.13) can be looked at from two points of view. From the Regge, or physical, point of view, the pole term is the important term and the integral just an incidental background term. From the group theoretical point of view, the integral is the interesting group expansion term, and the pole term just an incidental subtraction term to make the integral converge.

One may ask why only the principal series appears in the Regge formula. First there is a theorem due to Bargmann [8] which states that any square-integrable function over $SO(2,1)$ can be expanded in terms of the principal series and the discrete series with $\ell > -\frac{1}{2}$. This theorem explains why the supplementary, trivial and $D^{\pm}(-\frac{1}{2})$ representations do not appear. Secondly, the discrete series with $\ell > -\frac{1}{2}$ do not appear in our case because we have left out the spin of the particles [9].

In conclusion, I should mention that the further generalization to an expansion of $f(s, \cos\theta)$ in terms of the Lorentz group $SO(3,1)$ (to include $SO(2,1)$ for $t \sim \cos\theta_s$, and $SO(3)$ for s) has been considered. The $SO(3,1)$ expansion becomes particularly interesting and illuminating at the point $s = 0$ in the

continuation of s from the s to the t-channel, because we can choose $\vec{P} = 0$ as our reference frame, and then, for $s = 0$, $P = 0$, in which case $SO(3,1)$ is itself the little group. The fact that at $s = 0$ the little group expands to $SO(3,1)$ has physical consequences, notably that to every Regge trajectory crossing the line $s = 0$ in Figure 9.5 there is a family of trajectories with values $\alpha(0)$, $\alpha(0) - 1$, $\alpha(0) - 2$, ... corresponding to the representations $j = j_0, j_0 + 1, \ldots$ of $SO(3)$ occurring in an irreducible representation of $SO(3,1)$ [10].

10. INTERNAL SYMMETRIES

In the previous three chapters, we have considered the space time properties of relativistic Hilbert space in general, and of 1-particle states in particular. In the present lecture, we should like to consider some properties of the particles which are independent of space-time. Because they are independent of space-time, these properties, or symmetries, are called internal symmetries [1].

The first internal symmetries came to light when the structure of the atomic nucleus began to be investigated in the early thirties. The nucleus was found to consist of protons and neutrons (each about 2,000 times the mass of the electron, i.e., each about 10^{-25} grs.) and investigation of the forces that held them together (the nuclear forces) showed that

1) they were much stronger than the electromagnetic forces (they are the strong forces mentioned in Section 5) and
2) they were charge-independent. That is to say, apart from statistics, they did not distinguish between protons and neutrons--the force between two protons was the same as the force between two neutrons or the force between a proton and neutron. (This is in marked contrast to the electromagnetic forces which distinguish clearly between the proton and neutron, since the proton is charged and the neutron is not.)

To formulate charge-independence, it was convenient to introduce on the physical Hilbert space, an abstract invariance group. The group used was the $SU(2)$ group (isotopic spin group) and the idea was to assign the proton and the neutron, respectively, to two orthogonal vectors $|p)$ and $|n)$ in the 2-dimensional representation of $SU(2)$ and then to demand the invariance of the nuclear Hamiltonian H_N under the $SU(2)$ group

$$[SU(2), H_N] = 0 \quad . \tag{10.1}$$

The generator I of $SU(2)$ for which

$$I|p) = \frac{1}{2}|p) \quad , \quad I|n) = \frac{-1}{2}|n) \quad , \tag{10.2}$$

is called by convention I_3 or the third component of isospin. For many-nucleon states (nucleon = proton, neutron), one uses tensor products of the states $|p)$, $|n)$ in conjunction with (10.1).

The mathematics of the isospin group is the same as for the ordinary electron spin group $SU(2)$, but the physics is quite different. First, <u>every</u> vector in the ordinary 2-dimensional spin space is realizable in nature, but only $|p)$ and $|n)$ in isotopic spin space are realizable. (Nobody has ever succeeded in constructing a state which is a linear superposition $a|p) + b|n) ab \neq 0.$) Secondly, the operators in spin space transform non-trivially under space rotations, whereas the operators in isospace are independent of space-time.

Later in the investigation of nuclear structure, it was found that the nuclear forces between protons and neutrons were mediated by π-mesons, of which there are three, π^0 and $\pi^{\pm}$, the index referring to the charge. It turned out that the π's could be incorporated into the isospin scheme by assigning them to the 3-dimensional representation of $SU(2)$ with

$$I_3|\pi^{\pm}) = \pm|\pi^{\pm}) \qquad I_3|\pi^0) = 0 \quad . \tag{10.3}$$

Once higher (relativistic) energies became available, and the bombardment of nuclei with protons and neutrons of high energy resulted in the production of new particles, it was found that the forces producing the new particles were

a) of the same order of strength as the nuclear forces

b) still charge independent.

These forces are, therefore, called generically "strong" forces, and the associated Hamiltonian the strong Hamiltonian H_s.

From the charge independence of the strong interactions and the assignment of the creating particles (p,n) and $(\pi^{\pm 0})$ denoted by N and π, respectively, to irreducible representations of $SU(2)$, it follows that the created particles should also belong to irreducible representations of $SU(2)$. And this turns out to be the case. In fact, the metastable hadrons K^+, $\overline{K}^0$, η, Λ, $\Sigma^{\pm 0}$, Ξ^-, and Ω^- are found to belong to the 2,2,1,1,3,2 and 1-dimensional representations of $SU(2)$ and, hence, are denoted by K, $\overline{K}$, η, Λ, Σ, Ξ, and Ω, respectively.

For the unstable hadrons, the same results are found. All can be assigned to irreducible representations of $SU(2)$ with

$$[SU(2),H_s] = 0 \quad , \tag{10.4}$$

and, indeed, the number of unstable hadrons is now so large that one no longer refers to them individually but refers only to the isospin multiplets to which they belong. This method of referring to them has been anticipated in Table 5, in which

I (the total isospin) labels the representation of SU(2) to which the particles belong (dimension = 2I + 1). Note that all particles in the same SU(2) multiplet have the same mass (up to electromagnetic and weak interaction corrections) and the same spin, indeed have the same space-time properties in general. This is because the internal invariance group SU(2) and the space-time Poincaré group are independent, i.e.,

$$[SU(2), P_+^\uparrow] = 0 \quad . \tag{10.5}$$

In the production of new particles from $N,\bar{N},\pi$, a new invariance law became evident, namely, that when new particles are produced from $N,\bar{N}$ and π they come only in certain combinations. The simplest description of the allowed combinations is obtained by introducing a new operator Y on $\mathcal{H}$, assigning the SU(2) multiplets to integer eigenvalues of Y and demanding that

$$[Y, H_s] = 0 \quad , \tag{10.6}$$

i.e., that Y be conserved in the strong interactions. Because of its analogy to the electric charge, Q, which takes integer values on the particle states and is conserved in interactions, Y is called the <u>hypercharge</u>. Note, however, that whereas Q is conserved in <u>all</u> interactions, Y (like SU(2)) is conserved only in the strong ones. The hypercharges of $\pi,K,\bar{K},\eta$ and $N,\Lambda,\Sigma,\Xi,\Omega$ are (0,1,-1,0) and (1,0,0,-1,-2), respectively. In general, we have also the relation

$$Q = I_3 + \frac{1}{2} Y \quad , \tag{10.7}$$

which was first discovered by Gell-Mann and Nishijima.

The strong interactions are found to be invariant, therefore, under SU(2) and Y, and hence under the group $SU(2) \times U(1)$ where U(1) is generated by Y. But, as Louis Michel has pointed out, the fact that Q in (10.7) is integer, means that only those representations of $SU(2) \times U(1)$ occur, for which Y = D (modulo 2), where D is the dimension of the SU(2) representation. Since such representations of $SU(2) \times U(1)$ are exactly those which are representations of U(2), it follows that we can replace the invariance under $SU(2) \times U(1)$ by invariance under U(2).

A glance at Figure 5.1 will show that U(2) is the <u>maximal</u> group with the property $[U(2), P_+^\uparrow] = 0$, up to electromagnetic and weak corrections, since any other transformations among the particles will not commute with the mass and spin.

In spite of this, Figure 5.1 does suggest that one could go beyond U(2). The reason is that the different U(2) multiplets seem to fall into sets which, while they do not have the same mass, do have the same spin and have approximately the same mass. Examples of such sets are $(\pi, K, \bar{K}, \eta)$ and $(N, \Lambda, \Sigma, \Xi)$. For

this reason, it has been proposed that U(2) be enlarged to a group SU(3). The original choice SU(3) rested primarily on two factors:

1) SU(3) allows two and only two commuting generators, and two and only two additive operators (I_3 and Y) are necessary to label the particles.
2) SU(3) has an irreducible representation (the 8-dimensional adjoint representation) which can accommodate the two sets of eight particles (π, K, $\overline{K}$, η) and (N, Λ, Σ, Ξ) with the correct I, I_3, and Y values. ($I(I+1) = I_1^2 + I_2^2 + I_3^2$) See Figure 5.1.

(Actually, the η was not known experimentally at the time SU(3) was proposed and was predicted by SU(3).) Having adopted SU(3), one tries to assign the other, unstable, particles to SU(3) multiplets. Success has already been achieved with the $J^P = \frac{3}{2}^+$ multiplet (N^*, $\Sigma^* \equiv Y^*$, Ξ^*, Ω) which is assigned to the 10-dimensional representation (decimet) of SU(3) (Figure 5.1) and the $J^P = \frac{3}{2}^+, \frac{5}{2}^+, 1^-$, and 2^+ multiplets which are assigned to the 8-dimensional representation (octet). The higher J^P (spin-parity) multiplets look equally promising. The existence of Ω, like η, was actually predicted by the SU(3) assignment.

Thus, for particle assignments, SU(3) turns out to be as successful as the exact invariance group U(2) had been before it. Unlike U(2), however, SU(3) is not an invariance group, i.e.,

$$[X, H_s] \neq 0 \quad , \tag{10.8}$$

for those generators X of SU(3) which do not generate transformations within U(2). Indeed, if (10.8) were zero, SU(3) transformations would commute with the mass-operator and all the particles assigned to an SU(3) multiplet would have to have the same mass, which is manifestly not the case.

One might then ask: Of what use is SU(3) apart from classifying and predicting particles? The answer is that although SU(3) is not an invariance group, it is approximately an invariance group, and by stating how certain operators transform with respect to it, one can obtain physical predictions, correct to within the approximation (~20%).

The operators whose transformation properties are of most interest are

1) the Hamiltonian H_s
2) the mass operator M
3) the electric current $j_\mu(x)$
4) the weak current of the hadrons $j_\mu^\omega(x)$

The transformation properties of these operators are assigned on physical grounds and amount to a statement about the tensor character of the operators. The physical predictions are then extracted by using the WE theorem. Let us consider the operators 1) to 4) briefly:

1) The Hamiltonians: one usually assumes that

$$H_s = H_s^{(0)} + H_s^{(1)} \quad ,$$

where $H_s^{(0)}$ is SU(3) invariant and about five times as large as $H_s^{(1)}$. Hence, to within 20%, H_s can be regarded as SU(3) invariant. This allows us to obtain 20% estimates for the relations between strong decay processes such as

$$\begin{aligned} N^* &\to N\pi \quad , \\ \Sigma^* &\to \Sigma\pi \quad , \\ &\to \Lambda\pi \quad , \\ \Xi^* &\to \quad \pi \quad , \end{aligned}$$

for example,

$$\begin{aligned} \frac{N^* \to N\pi}{\Sigma^* \to \Lambda\pi} &= \frac{\rho}{\sigma} \frac{(N^*, H_s N\pi)}{(\Sigma^*, H_s \Lambda\pi)} \simeq \frac{\rho}{\sigma} \frac{(N^*, H_s^{(0)} N\pi)}{(\Sigma^*, H_s^{(0)} N\pi)} \\ &= \frac{\rho}{\sigma} \frac{C^{10\ 8\ 8}_{N^*\ N\ \pi}}{C^{10\ 8\ 8}_{\Sigma^*\ \Lambda\ \pi}} \frac{\left(\frac{3}{2}^-, H_s^{(0)} \frac{1}{2}^- 0\right)}{\left(\frac{3}{2}^-, H_s^{(0)} \frac{1}{2}^- 0\right)} \\ &= \frac{\rho}{\sigma} \frac{C^{10\ 8\ 8}_{N^*\ N\ \pi}}{C^{10\ 8\ 8}_{\Sigma^*\ \Lambda\ \pi}} \quad , \end{aligned} \tag{10.9}$$

where ρ, σ are kinematical phase-space factors, depending only on the masses (see Equation (10.15) below), the C's are Clebsch-Gordan coefficients, and $\left(\frac{3}{2}^-, H_s^{(0)} \frac{1}{2}^- 0\right)$ is a WE reduced matrix element. In a similar way, one can relate scattering processes such as

$$\begin{aligned} \pi N &\to \pi N \\ &\to K\Sigma \\ &\to K\Lambda \\ \bar{K}N &\to \pi\Sigma \\ &\quad K\Xi \quad , \end{aligned}$$

etc., for particles $(\pi, K, \bar{K})$ and $(N, \Sigma, \Lambda, \Xi)$ in the same SU(3) multiplets.

2) The Mass Operator M, like H_s, is assumed to be of the form

$$M = M^{(0)} + M^{(1)} \quad ,$$

where $M^{(0)}$ is SU(3)-invariant and $M^{(1)}$ is not. However, in the

case of the mass, one goes further and also makes a positive statement about $M^{(1)}$, namely, $M^{(1)}$ transforms as the Y-component of the eight-dimensional representation of SU(3).

Using the WE theorem or general tensor techniques, it is then possible to show that operating on the space of an irreducible representation of SU(3),

$$M = \alpha + \beta Y + \gamma[I(I+1) - \frac{1}{4}Y^2] \quad , \tag{10.10}$$

where $I(I+1) = I_1^2 + I_2^2 + I_3^2$ is the total isotopic spin and α, β and γ are SU(3) scalars. Thus, for any SU(3) multiplet, we have the mass-formula (10.10), where α is the mean value of M for the multiplet.

This formula agrees with experiment to within 4% for the metastable baryon octet, to within 0.5% for the $\frac{3}{2}^-$ baryon decimet (for which it makes <u>two</u> predictions, one of which is the prediction of the Ω at exactly the right mass value) and it agrees reasonably well (with some "representation mixing" modifications) for the remaining SU(3) multiplets.

At this point, we make a digression and use the SU(3) mass operator to illustrate the following point concerning tensor operators in finite-dimensional spaces. (There is a generalization to infinite dimensional spaces but it is more complicated in several respects.) Let $\mathcal{H}$ be a finite-dimensional Hilbert space, G a group implemented by unitary transformations on $\mathcal{H}$, and L(A) the space of linear operators on A. Then L(A) is itself a representation space for G. Hence any given operator $A \in L(A)$ can be expanded in terms of G. But this means that <u>any</u> operator is a tensor operator in the sense that it can be expanded as a series of <u>irreducible</u> tensor operators. Thus, the real content of the statement that an operator is a "tensor" is that in its expansion, one (or a few) <u>irreducible</u> tensor components dominate. For example, in the case of the mass operator M restricted to the octet space, M must be one of the 64 possible linear independent operators A on this space. With respect to SU(3), however, this 64-dimensional space (L(A)) splits into irreducible subspace of dimension $1,8,8,27,10,\overline{10}$ with respect to SU(3). Hence, a *priori*, M is limited to be

$$M = M^1 + M^8 + M^8 + M^{10} + M^{\overline{10}} + M^{27} \quad . \tag{10.11}$$

In addition, the <u>direction</u> of M within each irreducible subspace is completely determined by the condition that M be a U(2) invariant

$$[U(2), M] = 0 \quad , \tag{10.12}$$

and, in fact, this condition kills 10 and $\overline{10}$, and constrains M to be

$$M = M_0^1 + M_Y^8 + M_Y^8 + M_Y^{27} \quad , \tag{10.13}$$

where Y is the direction of the hypercharge. Thus, the only assumption that goes into the mass formula (10.10) which is not already implied by general considerations is that in (10.13) the component of M in the 27 is suppressed. This assumption is made for other tensor operators also (see below), in which case it is called explicitly "octet dominance". Incidentally, it might be worth remarking that the analogue of (10.13) for the decimet is $M = M_0^1 + M_Y^8 + M_Y^{27} + M_Y^{64}$, so that the octet dominance assumption kills <u>two</u> parameters (27 and 64) in this case. This is why the mass-formula yields <u>two</u> predictions for the decimet.

The strong decay rates and the mass formulae provide stringent cross-checks on the SU(3) particle assignments. In fact, since the mass of a particle is usually known experimentally long before its spin-parity, in practice one assigns according to the mass formula first and checks with spin-parity afterwards.

An interesting feature emerges in the case of the mesons 0^-, 1^-, and 2^+. These come not in octets but in nonets and, to fit the mass formula, one must assume that the two U(2) scalar particles (η,η^*),(ω,φ) and (f,f^*), respectively, which occur in the nonets, do not have definite SU(3) properties; rather, the linear combinations

$$\begin{aligned} \eta_8 &= \cos\theta\,\eta + \sin\theta\,\eta^* \\ \eta_0 &= \cos\theta\,\eta^* - \sin\theta\,\eta \quad , \end{aligned} \tag{10.14}$$

etc., belong to the SU(3) octet and scalar representations, respectively. This phenomenon is called "representation mixing" and it robs the mass formula of its direct predictions for these particles by adding the new unknown parameter θ. The best that the mass-formula can do is determine the various θ's. (They turn out to be $\theta_{0^-} \simeq 10^0$, $\theta_{1^-} \simeq 60^0$, $\theta_{2^+} \simeq 30^0$.) The interesting point, however, is that in spite of this, the mass formula is not empty. One can get indirect predictions, and indeed one of the most remarkable features of the decay rate analyses is that in two cases in which the experimental decay rates are in complete contradiction with phase space and SU(3) without mixing, the use of mixing accounts for the discrepancy. For example, experimentally

$$\frac{f \to 2\pi}{f \to K\bar{K}} \simeq 50 \quad , \quad \frac{f^* \to 2\pi}{f^* \to K\bar{K}} \leqslant \frac{1}{5} \quad , \tag{10.15}$$

whereas, on account of the much smaller mass of the pion, phase space would predict the ratio $\gg 1$ in both cases. With mixing, SU(3) predicts, using the WE theorem,

$$\frac{f \to 2\pi}{f \to K\bar{K}} \simeq \frac{\rho}{\sigma}\left[\frac{\cos\theta_2\left\langle\frac{8}{f_8}\middle|S^0\middle|\frac{8}{\pi}\right\rangle\left|\frac{8}{\pi}\right\rangle + \sin\theta_2\left\langle\frac{0}{f_0}\middle|S^0\middle|\frac{8}{\pi}\right\rangle\left|\frac{8}{\pi}\right\rangle}{\cos\theta_2\left\langle\frac{8}{f_8}\middle|S^0\middle|\frac{8}{K}\right\rangle\left|\frac{8}{\bar{K}}\right\rangle + \sin\theta_2\left\langle\frac{0}{f_0}\middle|S^0\middle|\frac{8}{K}\right\rangle\left|\frac{8}{\bar{K}}\right\rangle}\right]^2$$

$$\simeq \frac{\rho}{\sigma}\left[\frac{\cos\theta_2 C^{888_s}_{\pi\pi 8} + \alpha\sin\theta_2 C^{880}_{\pi\pi 0}}{\cos\theta_2 C^{888_s}_{KK8} + \alpha\sin\theta_2 C^{880}_{K\bar{K}0}}\right]^2$$

$$= \frac{\rho}{\sigma}\frac{3(2\alpha\sin\theta_2 + \cos\theta_2)^2}{4(\alpha\sin\theta_2 - \cos\theta_2)^2} \quad , \tag{10.16}$$

where S^0 is the scalar approximation to the S-matrix S,

$$\alpha = \frac{\langle 0\|S^0\|88\rangle}{\langle 8\|S^0\|88\rangle} \quad ,$$

and the phase space factor ρ/σ is given by

$$\frac{\rho}{\sigma} \simeq \left(\frac{p}{q}\right)^{2\ell+1} = \left[\frac{m_f(1 - 4m_\pi^2|m_f^2)^{1/2}}{m_f(1 - 4m_K^2|m_f^2)^{1/2}}\right]^5$$

$$\simeq [(1 - 4(500)^2|(1250)^2)^{-1/2}]^5 \simeq \left(\frac{5}{3}\right)^5 \simeq 15 \quad , \tag{10.17}$$

where m denotes mass, p and q are the final state π and K three-momenta, respectively, and $\ell = 2$ is the final state orbital angular momentum. Similarly,

$$\frac{f^* \to 2\pi}{f^* \to K\bar{K}} \simeq \frac{\rho'}{\sigma'} \cdot \frac{3}{4}\left[\frac{2\alpha\cos\theta_2 - \sin\theta_2}{\alpha\cos\theta_2 + \sin\theta_2}\right]^2 \quad , \tag{10.18}$$

where

$$\frac{\rho'}{\sigma'} = \left[\frac{m_{f^*}(1 - 4m_\pi^2|m_{f^*}^2)^{1/2}}{m_{f^*}(1 - 4m_K^2|m_{f^*}^2)^{1/2}}\right]^5$$

$$\simeq [(1 - 4(500)^2|(1500)^2)^{-1/2}]^5 \simeq \left(\frac{3}{\sqrt{5}}\right)^5 \simeq 4.5 \quad .$$

The values of θ_2 and α can be calculated from the mass formula and from f and A_2 decays, respectively. The values calculated in this way yield

$$2\alpha \simeq \tan \theta_2 \quad ,$$

and from (10.18) and (10.16) we see at once that this is exactly what is required to explain (10.15). Note that if there were no mixing ($\sin \theta_2 = 0$), SU(3) would predict

$$\frac{f^* \to 2\pi}{f^* \to K\bar{K}} = 3 \frac{\rho'}{\sigma'} \quad ,$$

which is even worse than phase space alone.

3) The Electromagnetic Current: Here one takes a lead from the charge. From the Gell-Mann-Nishijima result

$$Q = I_3 + \frac{1}{2} Y \quad , \tag{10.19}$$

we see that the charge is actually a generator of SU(3). Hence, it transforms like a component of an 8-tensor (octet). We call this the Q-component of the octet. Since the charge is constructed from the current according to

$$Q = \int j_0(x) d^3x \quad , \tag{10.20}$$

it is then natural to assume that the current $j_\mu(x)$ transforms in the same way, i.e., as the Q-component of an octet of currents. The assumption is not binding unless one uses other principles such as locality and minimal principle, but it is a good Ansatz. Algebraically, the Ansatz may be written

$$[J^\lambda, j^\mu_\mu(x)] = i f_{\lambda\mu\sigma} j^\sigma_\mu(x) \quad , \quad \lambda,\mu,\sigma = 1 \ldots 8 \quad , \tag{10.21}$$

where J^λ are the generators of SU(3) and the EM current is the Q-component of the octet tensor $j^\lambda_\mu(x)$. As an application of the Ansatz, one can consider the magnetic moments of the stable baryons. The magnetic moment operator is a linear function of the current so that if the current transforms like the Q-component of an octet, so does the magnetic moment operator μ^8_Q. Now for any octet member α, we have from the WE theorem

$$\left\langle {}^{8}_{\alpha} \middle| \mu^{\ 8}_{Q} \middle| {}^{8}_{\alpha} \right\rangle = C^{8\ 8\ 8}_{\alpha\ Q\ \alpha} (8,\mu^8_\mu,8)_c + d^{8\ 8\ 8}_{\alpha\ Q\ 8} (8,\mu^8_\mu,8)_d \quad , \tag{10.22}$$

where the c and d are Clebsch-Gordan coefficients and $(8,\mu^8_\mu,8)_c$

and $(8,\overset{8}{\mu},8)_d$ the corresponding reduced matrix elements. (The appearance of <u>two</u> reduced matrix elements is due to the fact that the $8 \otimes 8$ representation of SU(3) happens to contain the 8 twice.) From (10.22), it follows that all eight of the magnetic moments of the octet can be predicted from two of them. The two used as input are $\mu(p)$ and $\mu(n)$, which are well-known. The only predicted one which has been measured with good accuracy so far is $\mu(\Sigma^+)$ and the result agrees quite well with the prediction.

Similar considerations can be applied to the electromagnetic mass differences of the $\frac{1}{2}^+$ baryons, which are of order $(e/\hbar c)^2$, and the prediction obtained

$$m(\Xi^-) - m(\Xi^0) = m(\Sigma^-) - m(\Sigma^+) + m(p) - m(n) \ ,$$

agrees extremely well with experiment.

4) The Weak Current $j^\omega_\mu(x)$ of the Metastable Hadrons: $j^\omega_\mu(x)$ is assumed to determine their leptonic (e.g., $\Lambda \to p + e + \nu$) and non-leptonic (e.g., $\Lambda \to p + \pi$) decays through the Hamiltonians

$$H_{int} = \int d^3x j^\omega_\mu(x) j^\ell_\mu(x) \quad , \quad \int d^3x j^\omega_\mu(x) j^\omega_\mu(x) \quad , \qquad (10.23)$$

respectively, where $j^\ell_\mu(x)$ is the leptonic current. (Note the analogy between these interactions and the interaction

$$\int d^3x j_\mu(x) A_\mu(x) \quad , \qquad (10.24)$$

between the EM current $j_\mu(x)$ and the EM field $A_\mu(x)$.)

The weak current is actually a linear combination of a true vector current $v_\mu(x)$ and an axial (or pseudo) vector current $a_\mu(x)$

$$j^\omega_\mu(x) = v_\mu(x) + a_\mu(x) \quad , \qquad (10.25)$$

and $v_\mu(x)$ and $a_\mu(x)$, in turn, consist of parts that change the hypercharge eigenvalues by 0 and 1 units, respectively,

$$j^\omega_\mu(x) = v^0_\mu(x) + v^1_\mu(x) + a^0_\mu(x) + a^1_\mu(x) \quad . \qquad (10.26)$$

We have already seen that the EM current is assumed to be the $Q = I_3 + \frac{1}{2} Y$ component of an SU(3) octet. It is now assumed that $v^0_\mu(x)$ and $v^1_\mu(x)$ are the $I_\pm$ and $\Delta Y = \pm 1$ members of the <u>same</u> EM octet as $j_\mu(x)$. (The identity of the octet means that $v_\mu(x)$ and the EM $j_\mu(x)$ have the same reduced matrix elements.) The $a^0_\mu(x)$ and $a^1_\mu(x)$ are assumed to be the $I_\pm$ and $\Delta Y = \pm 1$ components of a new SU(3) octet. They cannot be components of the <u>same</u> octet as v and j since they have different space-time properties.

Using these transformation properties of $j_\mu^\omega(x)$ in the Hamiltonians (10.23), one can apply the WE theorem and obtain selection rules for the decays. One obtains the empirically observed $\Delta S \equiv \Delta(Y + B) = \Delta Q$ and $\Delta I = \frac{1}{2}$ rules for leptonic decays and (if one invokes also octet dominance, i.e., suppression of the 27-dimensional representation in $(8 \otimes 8)_{\text{symmetric}} = 1 + 8 + 27$), the empirically observed rules $\Delta S \neq 2$, $\Delta S = \Delta Q$, and $\Delta I = \frac{1}{2}$ rules for non-leptonic decays.

In sum, therefore, SU(3) is a group which is useful not only for classifying the elementary particles, but for predicting mass relationships between them and, because it is an approximate invariance group, it can be used for obtaining 20% estimates on the scattering, electromagnetic, and decay processes of the particles. The estimates are, of course, only on relative matrix elements. The dynamical content of the theory is hidden in the reduced matrix elements, which cancel out in the ratios.

At present, the origins of SU(3) symmetry and the 20% SU(3) symmetry-breaking are unexplained. Both are empirical discoveries which one has learned how to handle, but not to explain.

11. BEYOND SU(3)

It is natural to try to go beyond SU(3) and see if

1) the elementary particles have any further regularities

2) the SU(3) properties have any relation to space-time.

One regularity the particles certainly possess is the "Regge recurrence" mentioned in Section 9, namely, the property that each SU(3) multiplet of spin parity J^P reoccurs at higher masses with spin parity $(J + 2n)^P$ for $n = 1,2,3...$. Attempts to describe the Regge families $(J + 2n)^P$, $n = 1,2,3,...$, with infinite component wavefunctions do not seem successful, as we saw earlier.

Apart from the Regge recurrences and SU(3), the particles do not have any obvious regularities. However, the search for new regularities and the attempts to relate SU(3) to space-time have led to some interesting ideas. One of these is the use of new particles called quarks which are perhaps worth discussing.

The idea behind the quarks [1] is that the fundamental, 3-dimensional representation of SU(3) should, like the 8- and 10-dimensional representation, describe real particles (the quarks), and that since the 3-dimensional representation is fundamental, all the other particles should be bound states of the quarks. In particular, the 0^- and 1^- mesons should be bound states $q\bar{q}$ of

1 quark and 1 anti-quark, and the $\frac{1}{2}^+$ and $\frac{3}{2}^+$ baryons should be bound states qqq of 3 quarks. This would certainly be compatible with the SU(3) decompositions,

$$\begin{aligned} \bar{3} \times 3 &= 1 + 8 \quad , \\ 3 \times 3 \times 3 &= 1 + 8 + 8 + 10 \quad . \end{aligned} \tag{11.1}$$

On the other hand, the charges and hypercharges of the particles are additive quantum numbers because the corresponding operators are generators of SU(3). Hence, the charges and hypercharges of the component quarks would have to add up to those of the composite mesons and baryons, and one can check that for this to be true the charges and hypercharges of the quarks would have to be $\frac{1}{3}$ integer (the charge of the proton having been normalized to 1). This means that physically the quarks would be rather unusual objects.

Much experimental effort has been devoted to finding quarks, but so far without success. Nevertheless, the quark idea is used extensively. The reason is that even if the quarks are only fictitious, they provide a basis for making educated guesses about the real particles. Their use also simplifies many mathematical calculations.

The existence of quarks would not explain SU(3) itself, since an SU(3) triplet of quarks is assumed from the outset. But their existence would go far to explain the existence and mass-spin values of the J^P multiplets which are observed with ever increasing mass and spin.

In particular, for the lower multiplets 0^-, 1^-, $\frac{1}{2}^+$, and $\frac{3}{2}^+$ one has even been able to go beyond SU(3) to a larger group SU(6) [2] by using the quark model. The procedure is the following: The quarks are assumed to be spin $\frac{1}{2}$ particles. Hence, in their rest frames their wavefunctions $f^i_\alpha(x)$ are labeled by two sets of indices, $i = 1,2,3$ referring to SU(3) and $\alpha = 1,2$ referring to ordinary spin. One can now consider the group SU(6) of all unitary unimodular x-independent transformations on the 6-dimensional space $f^i_\alpha(x)$. This group contains SU(3) and the spin group SU(2) as subgroups in direct product form. If we now make the physical assumption that when the quarks bind together to form the 0^-, 1^-, $\frac{1}{2}^+$, and $\frac{3}{2}^+$ particles, the binding is in some sense SU(6) invariant, then we see that the bound state particles should belong to the $\bar{6} \times 6$ and $6 \times 6 \times 6$ representations of SU(6), respectively. To see whether this prediction agrees with experiment, one makes the SU(6) decompositions:

$$\begin{aligned} \bar{6} \times 6 &= 1 + 35 \quad , \\ 6 \times 6 \times 6 &= 70 + 56 + 20 \quad , \end{aligned} \tag{11.2}$$

and asks whether any of the irreducible representations obtained have the correct SU(3) and spin content to accommodate the observed 0^-, 1^-, $\frac{1}{2}^+$, $\frac{3}{2}^+$ multiplets. The answer is yes. Indeed, if one makes the SU(3) × SU(2) decompositions of the

SU(6) 35 and 56, one finds

$$35 = (1,1) + (8,1) + (8,0) \quad ,$$
$$56 = (10,\tfrac{3}{2}) + (8,\tfrac{1}{2}) \quad , \tag{11.3}$$

where the first figure refers to the dimension of the SU(3) representation and the second to the spin. This means that the 35 and 56 of SU(6) can accommodate the mesons 0^- and 1^- and the baryons $\frac{3}{2}^+$ and $\frac{1}{2}^+$, respectively. Thus, for the lower lying SU(3) multiplets, the assumption of SU(6) invariant binding for the quarks leads to the correct SU(3)-spin relationships. Attempts to extend the hypothesis of SU(6) invariant binding to the higher multiplets does not seem to work (presumably because the orbital angular momentum as well as the spin must be taken into account). For these, one falls back on more explicit dynamical quark models.

Having discovered that the lower lying SU(3) multiplets are predicted by SU(6) it is of interest to see whether SU(6) could be exploited farther. The investigation takes two forms, practical and principle. The practical investigations ask whether, following SU(3), we can make postulates about the SU(6) transformation character of H_s, M, $j_\mu(x)$ and $j^\omega_\mu(x)$ and obtain useful predictions.

For H_s, the answer is no. Although the quark-binding appears to be SU(6) invariant, the scattering matrix certainly is not. With the mass operator, one does a little better. By assuming that the mass breaking is additive with respect to SU(3) and spin, one can predict with 10% accuracy the mass spacing within the two higher SU(3) multiplets ($\frac{3}{2}^+$ and 1^-) in terms of the mass spacing within the two lower ones ($\frac{1}{2}^+$ and 0^-), respectively.

For the electric current $j_\mu(x)$, only the magnetic moment is considered, and this is assumed to transform like a member of an SU(6) 35. The matrix elements to be calculated are then of the form $\left\langle {}^{56}_{\alpha} \middle| \mu^{35}_{\beta} \middle| {}^{56}_{\gamma} \right\rangle$, where α and γ refer to members of the 56, i.e., $\frac{3}{2}^+$ and $\frac{1}{2}^+$ particles, and β refers to the magnetic moment member of the 35. But since for SU(6), $56 \times 35 (= 56 + 70 + 300 + 1134)$ contains the 56 only <u>once</u>, there is only <u>one</u> reduced matrix element

$$\left\langle {}^{56}_{\alpha} \middle| \mu^{35}_{\beta} \middle| {}^{56}_{\gamma} \right\rangle = C^{56\ 35\ 56}_{\alpha\ \beta\ \gamma} \left(56, \mu^{35}\ 56 \right) \quad ,$$

to be inserted in the WE theorem, and we obtain very strong predictions. In particular, using the proton magnetic moment $\mu(p)$ as input for the reduced matrix element, one obtains the SU(3) independent predictions

$$\mu(n) = -\frac{2}{3}\mu(p) \quad ,$$
$$\mu(10) = q\mu(p) \quad , \tag{11.4}$$
$$\mu_{N^* \to N+\gamma} = \frac{2\sqrt{2}}{3}\mu(p) \quad ,$$

where $\mu(n)$ is the magnetic moment of the neutron, $\mu(10)$ the magnetic moment, and q the charge, of any member of the $\frac{3}{2}^+$ decimet, and $\mu_{N^* \to N+\gamma}$ the magnetic moment contribution (which is the largest contribution) to the EM decay $N^* \to N\gamma$ The first equation is in unbelievably good agreement with experiment (~3%), the $\mu(10)$ have not yet been measured, and the third equation is in reasonable agreement with experiment (~30%). Thus, the assignment of SU(6) properties to the magnetic moment seems to be quite successful.

Finally, for the weak current, SU(6) has the advantage of being able to carry the vector and axial vector currents in the same representation. This is because in the SU(6) limit, which is assumed to be non-relativistic, $v_\mu(x) \to v_0(x)$ which is spin 0, $a_\mu(x) \to \vec{a}(x)$ which is spin 1, and we already know from the meson classification that the 35 of SU(6) has exactly the right content to carry spin 0 and 1 SU(3) octets. Assigning v_0 and $\vec{a}$ to the 35, one can make some interesting predictions, and they agree reasonably well with experiment.

In brief, therefore, the attempt to push SU(6) beyond a mere classification group for the particles, while not spectacularly successful, is not unsuccessful. It is only when SU(6) invariance is demanded for the scattering matrix that we get a complete breakdown.

The other kind of investigation into SU(6) is more a question of principle. The relationship between the space-time symmetries of particles and the internal SU(2),SU(3) symmetries has never been properly understood, and with the advent of SU(6), in which SU(3) and the spin group are simultaneously embedded, it looked as if one might have a handle on this problem. The question also arises as to how SU(6), whose formulation is completely non-relativistic, should be made relativistic. These two questions are related and hinge on the question as to how the spin group SU(2) in SU(6) is to be interpreted. Three possibilities suggest themselves

a) as the little group of $p = \alpha$ for $p^2 > 0$

b) as a subgroup of SL(2,C) in the manifestly covariant Lorentz transformations

$$\psi(x) \to S(\Lambda)\psi(\Lambda^{-1}x) \quad , \quad \Lambda \in SL(2,C)$$

c) as a subgroup of $P_+^\uparrow$.

Each of these possibilities suggest a way of making SU(6) relativistic.

In Case a), it is simply a question of expressing the SU(6) theory in a manifestly covariant formalism and this has been done explicitly in Ref. [3]. There are no new predictions.

In Case b), one takes the quark wavefunctions

$$f^i_\alpha(x) \overset{\Lambda,a}{\to} S_{\alpha\beta}(\Lambda) f^i_\beta(\Lambda^{-1}(x - a)) \quad , \tag{11.5}$$

where α is now a Dirac index and $S_{\alpha\beta}(\Lambda)$ the Dirac representation of SL(2,C), and considers the pseudo-unitary unimodular x-independent group SU(6,6) on the index space (β,i) [4]. This group contains SU(3) and SL(2,C) as subgroups in

direct product form, and thus replaces SU(6) directly. Using SU(6,6), one can proceed exactly as with SU(6). But one obtains very few new good predictions, and encounters a lot of trouble [5].

The difficulty stems from the fact that to relate the manifestly covariant wavefunctions to the physical particles, one must eliminate the auxiliary parts of the wavefunctions. This is done by means of the manifestly invariant projection operators

$$\frac{1}{2m}(\gamma^{\mu}p_{\mu} - m) \quad , \tag{11.6}$$

etc., discussed in Section 7. But while the operators (11.6) are manifestly Poincaré invariant, they are not SL(2,C) invariant (p_{μ} is an SL(2,C) scalar) and, hence, certainly not SU(6,6) invariant. Hence, the auxiliary components of the wavefunction cannot be eliminated in a way which is simultaneously SU(6,6) and $P_{+}^{\uparrow}$ invariant.

The problem becomes particularly acute in connection with probability conservation in scattering theory. Probability conservation is expressed through the unitarity condition

$$S^{\dagger}S = SS^{\dagger} = 1 \quad ,$$

for the scattering matrix. Now consider this equation in matrix notation,

$$\sum_{n} (i,S^{\dagger}n)(n,Sj) = S_{ij} \quad .$$

If in the sum $\sum_{n}$ we put in all the SU(6,6) states, then we have SU(6,6) invariance, but we do not have true probability conservation since the sum is not over the physical states. If, on the other hand, we include in the $\sum_{n}$ only the physical states, then we have true probability conservation, but we do not have SU(6,6) invariance since the projections on the physical states, as we have just seen, are not SU(6,6) invariant. Thus, for the scattering matrix, physical unitary and SU(6,6) invariance are mutually incompatible.

Of course, one might legitimately ask: Why should the S-matrix be SU(6,6) invariant? After all, it is not SU(6) invariant. The point is that by making SU(6) relativistic one had hoped to overcome the defect that SU(6) was not an invariance group. The failure to overcome that defect is a serious setback for SU(6,6), and together with the failure of SU(6,6) to provide useful new predictions, it has led to its abandonment.

The third attempt (c)) to make SU(6) relativistic is, in a sense, more ambitious than b). It rests on the observation that SU(3) cannot be completely independent of the space-time coordinates x since it does not commute with the space-time mass operator. So the attempt is to combine SU(3) and the full

Poincaré group $P_+^\uparrow$ in a larger group G. Of course, for the combination to be useful, some restrictions must be placed on G. (The group of all possible unitary transformations on Fock space obviously contains both SU(3) and $P_+^\uparrow$, but this observation contains no useful information.) The two main restrictions which have been suggested for G are:

1) In the limit of SU(3)-symmetry, G is an invariance group for the S-matrix.

2) Whether or not it is an invariance group, G is a Lie group.

Unfortunately, both suggestions run into trouble. In Case 1), one can show [6] that under very general conditions either S = 1 (no scattering) or $G = P_+^\uparrow \otimes G_0$, where $\otimes$ denotes direct product and G_0 contains SU(3), i.e., either S = 1 or the combination is trivial. In Case 2), one can show that in any irreducible representation of G the mass spectrum of $P_+^\uparrow$ has no gaps [7]. Hence, G would be unsuitable for classifying the hadrons. Even apart from this kind of trouble, the difficulties of combining SU(3) and $P_+^\uparrow$ in a larger group G can be seen by considering the action of G on $P_+^\uparrow$ and SU(3) space respectively. One can see that the action cannot make much physical sense unless the combination is trivial [8].

The failure of attempts b) and c) to make SU(6) relativistic make it appear that if SU(6) is to be regarded as anything other than a nonrelativistic accident, one must look elsewhere for a framework in which to embed it. Such a framework is provided by current algebra, which we shall discuss in the next chapter.

12. CURRENT ALGEBRA

In the last two sections, we saw that the elementary particles exhibit regularities or symmetries other than those demanded by Poincaré invariance. However, none of the symmetries is exact. U(2) symmetry becomes exact only in the limit that weak and electromagnetic interactions are neglected, SU(3) symmetry is broken to within about 20% by even the strong interactions, and SU(6) symmetry works at best in a haphazard and empirical way. The question is: Could one find a framework within which the U(2), SU(3), and SU(6) results could be understood in a coherent fashion? We have already seen that the idea of putting SU(3) and SU(6) into larger groups is rather unsuccessful. In the present lecture, we wish to discuss a more successful approach, namely, current algebra [1].

The starting point for the introduction of current algebra is the idea that the fundamental objects for strong interaction physics are not the fields $\psi(x)$ but the currents

$$j_\mu^X(x) \tag{12.1}$$

(which in a field theory would be constructed out of the fields). The role of the currents is to mediate the interactions. For example, the electromagnetic interactions of all the particles (strongly interacting or not) are assumed to take place via an interaction Hamiltonian of the form

$$H_e = e \int d^3x j_\mu(x) A_\mu(x) \quad , \tag{12.2}$$

where e is the electric charge, $j_\mu(x)$ the electric current, and $A_\mu(x)$ the electromagnetic potential. Similarly, the leptonic and non-leptonic weak decays of the (otherwise) strongly interacting particles are assumed to take place via interaction Hamiltonians of the form

$$H_{n\ell} = G \int d^3x j^\omega_\mu(x) j^\omega_\mu(x) \quad , \quad H_\ell = G \int d^3x j^\omega_\mu(x) j^\ell_\mu(x) \quad , \tag{12.3}$$

where G is the weak coupling constant, $j^\omega_\mu(x)$ is the weak current of the strongly interacting particles, and $j^\ell_\mu(x)$ is the weak current of the leptons.

From the currents $j^X_\mu(x)$, we can define charges

$$X(t) = \int d^3x j^X_0(x) \quad . \tag{12.4}$$

What current algebra assumes is that independent of the form or even the existence of an underlying field theory, the charges and currents satisfy simple algebraic relations among themselves (analogous to $[X,P] = i\hbar$ in quantum mechanics). The postulated relations are

charge-charge algebra
$$[X,Y] = iZ \quad , \tag{12.5a}$$

charge-current algebra
$$[X, j^Y_\mu(x)] = i j^Z_\mu(x) \quad , \tag{12.5b}$$

current-current algebra
$$[j^X_0(x), j^Y_0(x')] = i j^Z_0(x) \delta(x - x') \quad , \tag{12.5c}$$

$$[j^X_0(x), \vec{j}^Y(x')] = i \vec{j}^Z(x) \delta(x - x') + S(x,x') \tag{12.5d}$$

(12.5b), (12.5c), (12.5d): At equal times

where the structure constants of the algebra in question are in practice those of SU(2), SU(3), SU(2) × SU(2), SU(3) × SU(3) (and, with a modification to be discussed later, SU(6) × SU(6)). "At equal times" means that the time variable in X, Y and Z, etc., has the same value. The term $S(x,x')$ in the last equation is called a Schwinger term [2]. It is inserted because it can be shown that without it the equation would not be consistent. $S(x,x')$ is unknown, but it is usually assumed to be purely symmetric in X and Y, so that at least the antisymmetric part of the last equation is not empty.

Note particularly that since the algebraic relations (12.5) are nonlinear, they normalize the currents and hence make it meaningful to say that the coupling constants in (12.2), (12.3) are small, large, universal, etc. In fact,

the need to normalize the weak currents was one of the motivations for current-algebra [3].

In general, it is not assumed that the charges are time independent. However, we have the equivalence relations

$$\frac{dX(t)}{dt} = 0 \Leftrightarrow [H,X(t)] \Leftrightarrow \partial_\mu j^X_\mu(x) = 0 \Leftrightarrow X(t)|0\rangle = 0 \quad , \tag{12.6}$$

where H is the Hamiltonian under consideration and $|0\rangle$ is the vacuum state. The first two relations are fairly obvious. The last follows from a theorem due to Colemen [4].

The question now is: How are physical consequences to be extracted from this formal algebra?

Let us first consider the exact symmetry limit, e.g., SU(2) with weak and electromagnetic interactions neglected or SU(3) with the 20% SU(3) breaking interaction neglected. In that limit (12.6) holds for all the charges and the charge × charge algebra becomes the usual SU(2) or SU(3) symmetry algebra, with the charges as generators. In particular, if the physical Hilbert $\mathcal{H}$ is decomposed with respect to the charge algebra (12.5a), the mass degenerate particles can be, and are, assigned to irreducible subspaces of the algebra. We then obtain the usual SU(2) or SU(3) theory. In particular, the charge × current algebra (12.5b) then becomes the assignment of tensor properties to the current as described in the last two chapters.

The real advantage of the current algebra appears when the symmetry is not exact. In that case, it is assumed that the current algebra relations (12.5) are exact, but that (12.6) does not hold and, hence, that the assignment of particles to SU(3) subspaces of $\mathcal{H}$ is incorrect. However, it is assumed that there is at least a subalgebra of the charge algebra which is exact and is large enough to locate the particles in $\mathcal{H}$ relative to the algebra. (The subalgebra is that of U(2) for SU(3) and that of U(1) for SU(2).)

Having placed the particles relative to the algebra, the physical information is then extracted as follows: Consider the charge × charge relation $[X,Y] = iZ$. The presently measurable matrix elements of X, Y, Z are their values between 1-particle states. It is, therefore, suggestive to sandwich the equation $[X,Y] = iZ$ between 1-particle states. Let us denote 1-particle states by (n) and 2-or-more-particle states by (c). We obtain

$$\sum_{n'} (n, X\, n')(n', Y\, n'') + \sum_{c} (n, X\, c)(c, Y\, n'') - X \Leftrightarrow Y = i(n, Z\, n'') \, , \tag{12.7}$$

where the sum $\sum_{n'}$ runs over all the 1-particle states and $\sum_{c}$ over all the many-particle states. Now if

$$\sum_{c} (n, X\, c)(c, Y\, n'') = 0 \quad , \tag{12.8}$$

we would be in a strong position with regard to experiment, since we would have a

direct algebraic statement about the measurable quantities $(n, X\, n')$. However, in general, (12.8) is not true. Indeed, (12.8) is true essentially only in the exact symmetry limit since (12.8) implies that at least one of X and Y leaves the 1-particle states invariant and, in general, this can only happen if they leave the vacuum invariant as well.

Thus, in general one cannot omit the c-summation in (12.7), and one must proceed otherwise. How one proceeds depends on the matrix elements to be calculated. We shall mention here only two well-known examples:

1) Adler-Weisberger calculation [5]: One uses $SU(2) \times SU(2)$ algebra, namely one assumes that the isospin charges T_i and the charges A_i belonging to the axial vector current $j^A_\mu(x)$ satisfy the relations

$$[T_i, T_j] = i\varepsilon_{ijk}T_k \quad ,$$

$$[T_i, A_j] = i\varepsilon_{ijk}A_k \quad , \qquad i = 1, 2, 3 \qquad (12.5a)'$$

$$[A_i, A_j] = i\varepsilon_{ijk}T_k \quad .$$

Then one chooses $n = n'' =$ proton, and $X, Y = A_\pm = (A_1 \pm iA_2)/2$. It follows that $Z = I_3$ and $n' =$ neutron. If

$$g_A = (n\, A^+\, p) \quad ,$$

denotes the weak coupling constant between the neutron and proton, (12.7) reduces to

$$|g_A|^2 + \sum_c (pA^+c)(cA^-p) - (pA^-c)(cA^+p) = 1 \quad . \qquad (12.10)$$

Thus, we would have a prediction for $|g_A|^2$ if we could evaluate $\sum_c$. (In particular, $|g_A|^2$ would be 1 if $\sum_c$ were zero.) To evaluate $\sum_c$, one makes the so-called PCAC (partially conserved axial current) hypothesis, namely, that

$$\partial_\mu A^\mu_\pm(x) = \kappa g_A \pi_\pm(x) \quad , \qquad (12.11)$$

where $\pi_\pm(x)$ is the field of the $\pi^\pm$-meson, and the constant κ is determined from the decays $n \to p +$ leptons and $\pi \to$ leptons. Integrating (12.11) to

$$\frac{1}{i}\frac{d}{dt} A^\pm(t) = \kappa g_A \int d^3x\, \pi_\pm(x) \quad , \qquad (12.12)$$

and inserting the result into $\sum_c$, one obtains

$$|g_A|^2 \left\{1 + \kappa^2 \sum_c \frac{(p\pi^+c)(c\pi^-p) - (p\pi^-c)(c\pi^+p)}{(E_c - E_p)^2}\right\} = 1 \quad . \qquad (12.13)$$

The point now is that the $\sum_c$-term can be directly related to the cross-sections $\sigma_\pm(c)$ for $\pi_\pm p$ scattering. Inserting the observed values for $\sigma_\pm(c)$, one obtains $|g_A|^2 \simeq 1.18$, in excellent agreement with experiment. Note that the entire departure of $|g_A|^2$ from unity comes from the non-conservation of $A_\pm(t)$ (Equation (12.12)).

2) The second example [6] uses $SU(6) \times SU(6)$ algebra, or at least that part of it in which

$$[\vec{A},\vec{A}] = T + \vec{A} \quad , \tag{12.14}$$

where T is the isospin charge (generator of the isospin group) and $\vec{A}$ is the spatial charge

$$\vec{A}(t) = \int d^3x\, \vec{a}(x) \quad , \tag{12.15}$$

where $\vec{a}(x)$ is the space-part of the $SU(3)$ axial vector current. The use of the spatial charge is peculiar to $SU(6) \times SU(6)$. Only the time-component charges (12.4) are used for $SU(2)$, $SU(3)$, $SU(2) \times SU(2)$, and $SU(3) \times SU(3)$. Inserting Equation (12.14) between $\frac{1}{2}^+$ and $\frac{3}{2}^+$ states, denoted by N, we obtain

$$(N\,\vec{A}\Big[\sum_n n)(n + \sum_c c)(c\Big]\vec{A}'\,N) - \vec{A} \Leftrightarrow \vec{A}' = (N, T + \vec{A}''\,N) \quad . \tag{12.16}$$

If one now makes the approximation of replacing the sum over n and c by a sum over N only, i.e.,

$$\sum_n n)(n + \sum_c c)(c \to \sum_N N)(N \quad , \tag{12.17}$$

in (12.16), one obtains

$$\sum_{N'} (N,\vec{A}N')(N',\vec{A}'N) - \vec{A} \Longleftrightarrow \vec{A}' = (N,T + \vec{A}''N) \quad , \tag{12.18}$$

and by choosing appropriate members of N and $\vec{A}$, one can derive from (12.18) practically all the interesting $SU(6)$ results. Thus, $SU(6)$ can be simply understood as a combination of the charge-algebra (12.14) and the saturation assumption (12.17). It should be emphasized that the masses of the particles N are not assumed to be the same and the charges $\vec{A}(t)$ are not assumed to be time-independent.

These examples and other applications of the charge algebra support the view that the correct way to understand $SU(3)$, $SU(3) \otimes SU(3)$, etc., is not as exact symmetry groups, but as exact charge algebras. Any approximations to be made are made in the saturation of the algebra (the sum over intermediate states).

So far, we have discussed only the charge × charge algebras (when the symmetry is not exact). However, the charge × current algebras can be similarly handled, and in recent years most of what is called current algebra theory has been devoted to systematically (and very successfully) exploiting the charge × current algebras.

Let us sketch very briefly the kind of idea involved for one of the most important applications [1] of current algebra, namely, the derivation of what are called low energy theorems. For an interaction involving an external π-meson, the matrix element of interest can be written as

$$M = (a,T \int e^{ipx+qy\cdots}\pi(x)\varphi(y)\ldots d^4(xy\ldots),b) \quad , \tag{12.19}$$

where a and b are initial and final states, p_μ is the meson 4-momentum, $\pi(x)$ is the meson field, $\varphi(y)$ any other typical field or current (possibly another π-meson field), and T is the time ordering operator $(T(\pi(x)\varphi(y)) = \varphi(y)\pi(x)$, $\pi(x)\varphi(y)$ for $x_0 < y_0$, $x_0 > y_0)$.

Replacing $\pi(x)$ by $\partial_\mu A_\mu(x)$ according to (12.11), we obtain

$$\begin{aligned} M &= (a,T \int \ldots \partial_\mu A_\mu(x)\ldots b) \\ &= p_\mu(a,T \int \ldots A_\mu(x)b) - \sum (a,T' \int \ldots [A_0(x),\varphi(y)]_{ET}\ldots b) \quad , \end{aligned} \tag{12.20}$$

where the second term comes from the fact that the time derivative does not commute with the time-ordering T. The non-commutativity of ∂_0 and T can be expressed in the form $\frac{d}{dt}\theta(t) = \delta(t)$ and hence leads to <u>equal-time</u> commutators such as the commutator $[A_0(x),\varphi(y)]_{ET}$ exhibited, together with a residual time ordering T' for the remaining unequal times.

Now because the mass of the pion is small, for processes for which the pion 3-momentum is small, it is legitimate to let $p_\mu \to 0$. Then the first term in (12.20) vanishes and in the second term $\int e^{ipx} d^3x\, A_0(x) \to \int d^3x\, A_0(x) = A_0$ where A_0 is the axial charge. Hence, in the "soft-pion limit" $p_\mu \to 0$, M is dominated by the second term in (12.20), and the second term, in turn, is determined by the equal time charge $\times$ current commutator $[A_0,\varphi(y)]$ of the charge $\times$ current algebra. In this way the charge $\times$ current algebra determines the low energy or soft pion limit of π-meson processes. The argument generalizes, of course, to processes with more than one π, e.g., π - p scattering (Figure 12.1).

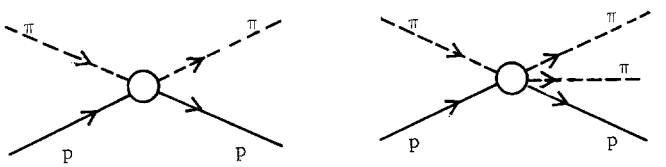

FIGURE 12.1. ELASTIC AND π-PRODUCING π-p SCATTERING

The success achieved with charge $\times$ charge and charge $\times$ current algebra tempts one to go farther and assume the current $\times$ current algebra. The current

× current algebra has not yet been severely tested experimentally, but its simplicity is appealing, as is the fact that it yields the charge × current and charge × charge algebras on integration. Note that neither the charge × current nor the current × current algebra is a Lie algebra, and a mathematical problem of some interest at present is to find all the unitary irreducible representations of an algebra of this form, i.e., an algebra of the form

$$[X_\alpha(x), X_\beta(y)] = f_{\alpha\beta\gamma} X_\gamma(x)\delta(x - y) \quad , \tag{12.21}$$

where the $f_{\alpha\beta\gamma}$ are the structure constants of a simple Lie group and $x \in R^3$.

An algebra of the form (12.21) would be particularly useful if the sum over all intermediate states to be inserted between the operators on the left hand side of (12.21) could be approximated (saturated) by a sum over a number of 1 particle states (not necessarily a finite number). This is because a saturation with 1-particle states would clearly yield algebraic relations for quantities of the form

$$\langle \text{1-particle}, X_\alpha(x) \quad \text{1-particle} \rangle \quad , \tag{12.22}$$

and such quantities have the property that their Fourier transforms with respect to x are the form-factors for the particles and so are within reach of experiment. Unfortunately, the saturation with 1-particle states raises some difficulties of principle. One can show, for example, that unless the current $j_\mu(x)$ is trivial, the current × current algebra (12.21) cannot be even approximately saturated with 1-particle states (even if an infinite number of 1-particle states are used) unless the masses are degenerate. However, it has been conjectured [7] that in the limit that P_z, the third component of the total momentum of all the states, becomes infinite, the saturation with 1-particle states may become exact and lead to predictions for the mass-spectrum and the form factors, or at least to correlations between the two. This conjecture, which is based on experience with the free-Dirac equation and the charge × current algebra, is still open. Preliminary investigations, using, for simplicity, the special case of a <u>factored</u> current

$$\begin{aligned} j^0_\alpha(x) &= \lambda_\alpha j^0(x) \quad , \\ [\lambda_\alpha, \lambda_\beta] &= i\varepsilon_{\alpha\beta\gamma}\lambda_\gamma \quad , \\ j^0(x)j^0(y) &= j^0(x)\delta(x - y) \quad , \end{aligned} \tag{12.23}$$

show that in the factored case the solutions can be written as infinite component wave equations. This result furnishes another link between conventional physics and infinite component wave equations, but since, as we have seen in Lecture 8, infinite component equations have some undesirable physical properties, the result may only be an indication that the factorization hypothesis (12.23) is too strong.

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