A remembrance of Hendrik Casimir in the 60th anniversary of his discovery, with some basic considerations on the Casimir Effect

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Abstract.   When the number and importance of the applications of the Casimir effect are flourishing, and on the occasion of the 60th anniversary of his beautiful discovery, as a tribute to the memory of Hendrik Brugt Gerhard Casimir I discuss here some fundamental issues related with the effect that need to be recalled from time to time, as well as on some of my personal impressions of Prof. Casimir. This article may also serve as an easy introduction for the non-specialist willing to learn something about the quantum vacuum.

1. Introduction: Hendrik Casimir

Though many of the things I will say in what follows may be well known to some of the readers, I cannot avoid recalling them here in order to put into context the different points I will address in the following sections. Let me here start with a short biographic introduction. Hendrik Brugt Gerhard Casimir was born in Den Haag (Holland) on July 15, 1909. Starting in 1928 he studied Physics at the University of Leiden, where he finished his PhD in 1931. In his PhD Thesis he dealt, from the point of view of Quantum Mechanics, with the problem of a rotating rigid body, as well as with the theory of the rotation group as applied to molecules. During this period Casimir spend a short time in Copenhagen, with Niels Bohr. I was really impressed to read in Casimir's autobiography the intensity of the effect that produced on him the first encounter with the great genius. He tells there the story that, in order to be able to convince his parents in Holland that Bohr, the professor he was visiting, was a really important person, he told them to write a letter to him [their son] in Denmark just putting on the envelope as address the following: Hendrik Casimir c/o Niels Bohr, Denmark, and nothing else. They did so and, in fact, the letter punctually traveled from Holland to Casimir's hands, in very few days. It is nice to recall that in those times the important physicists were recognized as great personalities, as are now movie stars or rap singers. They had wide recognition in their own countries, and some of them in the whole world. Physics itself was very highly regarded too.

After finishing his PhD, Casimir worked as an assistant of Wolfgang Pauli in Zürich. In 1938 he became a professor of physics in Leiden. During this time, he studied some problems on heat and electricity conduction and he contributed to projects aimed at reaching the lowest

temperatures known at that time, of the order of a few millikelvin. A low temperature lab was (as is also today) fundamental in order to carry out interesting research in material science, with a wide spectrum of aims. In 1942, during the second World War, Casimir moved to the Research Laboratories of Philips Incorporation in Eindhoven, Holland. In his new workplace que continued to be an active scientist, and in 1945 he wrote a paper, that became famous, on Onsager's principle of microscopic irreversibility. As soon as 1946, Casimir already became co-director of Philips' Research Laboratories, and a member of the Board of Directors of the Philips Incorporation in 1956, where he remained until 1972. In spite of having been working for the industry for most of his professional life, Casimir continues to be one of the more prominent theoretical physicists that have come out of his country, Holland that, let us remember, had not long ago a very important success with the 1999 Nobel Prizes obtained by Martinus Veltman and Gerardus 't Hooft.

Aside from the effect that now carries his name, as well as its relation with de van der Waals forces [1] (which I will discuss more extensively in what follows), Casimir published other important contributions to science, as to the theory of Lie groups, to hyperfine structure and the calculus of quadripolar models, the physics of low temperatures, magnetism and the thermodynamics of superconductors, and the already mentioned contribution to Onsager's theory. Casimir played also a role in the creation of the European Physical Society, of which he became president in 1972. In 1979 he was one of the main speakers in the celebration of the 25th Anniversary of CERN, the European Center for Nuclear Research, in Geneva (Switzerland). In September 1998, when he was almost 90, in which was finally to be the very last of his scientific activities, he gave the opening lecture under the title *"Some remarks on the history of the so called Casimir effect"*, in the 4th Meeting of the series QFEXT, that took place in Leipzig (Germany). He passed away on May 4, 2000, in Heeze (Holland).

I keep in my office, as a treasure, his self-biographical book *"Haphazard Reality: Half a Century of Science"*, that he kindly send to me together with a handwritten letter, in 1999, shortly before his dead. In the letter he gave an answer to my invitation to come to Barcelona and visit our Institute, that I had extended to him personally during the Leipzig meeting. Regretfully, the visit could never take place.

2. A few Historical Notes on the Casimir Effect

It is, at the very least, quite shocking that in the above named biography, Casimir makes almost no mention of the very important effect that bears his name [1]. He just writes on passing, in page 247, that his phenomenon *"is mentioned sometimes under the name Casimir effect"*, that *"it has been found experimentally"*, and that *"their generalizations have been seen to have certain theoretical importance"*. He describes, in a different paragraph, how in 1951, while both of them were participating in a conference in Heidelberg, he tried very hard to make understandable to the prominent physicist Wolfgang Pauli his arguments about the physical reality of the effect he had discovered, and how Pauli was unable to grasp them. All of Casimir's efforts, from one perspective, from another, were useless since Pauli kept building one objection after another until the point when, having Pauli realized that all of his counterarguments had no effect on the enthusiastic Casimir, Pauli called him, repeatedly, *"Stehaufmanderl"*, which is a very strong form of 'stubborn', and makes literal reference to a toy, very common in Germany at the time, with the form of a clown of wood that keeps all the way its vertical position, owing to its having a very heavy spherical base made of steel that does not allow it to fall down, even if you try it by any means! May this anecdote serve as a convincing example which makes perfectly clear that understanding the essence of the Casimir effect is far from easy. Some of the problems associated with this phenomenon still remain nowadays, as may have become clear to the reader from some of the contributions to these Proceedings.

During about ten years, starting towards the end of the 80's, Barcelona was surely the most

active place on earth working on (the theoretical aspects mainly) of the Casimir effect: Rolf Tarrach, Enric Verdaguer, Sonia Pabàn, August Romeo, Sergio Leseduarte, Klaus Kirsten, and the author of this paper himself, with some punctual contributions of Leipzig colleagues too, published a considerable number of papers, some of which have become, with the passage of years, fundamental and much loved references on the subject (judged from the several hundreds of citations they have collected since then). Several PhD thesis came out of this activity, not to speak of the researchers that got important promotions, to finish as professors inside and outside of Spain, thanks to this activity. Needless to say, it was a period of my scientific life of which I am very proud. Much more recently, in 2005, I had the pleasure and the honor to organize, in CosmoCaixa, Barcelona, the seventh meeting, QFEXT'05, of the already mentioned series on "Quantum Field Theory under the Influence of External Conditions". Casimir, for one, had participated in the fourth of these meetings, which took place in Leipzig, as previously mentioned.

3. On Heisenberg's Indeterminacy (or Uncertainty) Principle

Quantum Mechanics (QM), later extended to the Quantum Theory of Fields (QFT), constituted one of the great scientific revolutions of the XXth Century. Some of the fundamental principles in which it is based, as well as the consequences that follow from the axioms of these theories, in particular, the ones related with the collapse of the wave function and with the theory (and practice) of the quantum measurement, have not started to be well understood and successfully used until the first years of the XXIst Century. The situation will most probably improve in the years to come. The effect we are here dealing with refers precisely to one of these basic principles, and the very concepts associated to the same were the object of lively discussions among some of the founders of quantum physics (and some of its detractors, too).

The Casimir effect is usually 'defined' as "any measurable consequence of the quantum fluctuations of the vacuum state of a quantum system, as they are revealed by the imposition of boundary conditions, a specific topology, a background field, etc." In Heisenberg's version a quantum system has associated an energy operator, the system's Hamiltonian, which has a spectrum of eigenvalues and corresponding eigenvectors. The state of minimal energy (eigenvalue) is called the "vacuum state". It is important to understand that the quantum system is here, so to say, the 'universal domain' the playground or battlefield, in the same way as the 'universal set' and the 'probability space' one builds up in probability theory. In this way a classical similarity is closely established —in fact quantum mechanics has much to do with probability theory (it had even more directly to do in the case of the original formulation by E. Schrödinger, that was unsuccessful, as the reader will surely know). To sum up, the vacuum state of a quantum system is that of minimal energy, what can be ideally represented as the one of maximal 'quietness' an 'ocean of tranquility' in the most strict sense possible within the given quantum system. In order to exit from the vacuum state, the system needs a supply of energy. In the absence of this, how can there possibly be fluctuations of this vacuum state? The quantum fluctuations of the vacuum state are allowed by Heisenberg's uncertainty (or indeterminacy) principle, which is one of the pillars of the quantum world. In particular, one of its consequences is that a statement saying that "the energy of a quantum system in a given state has, at given time, a specific value'" has no strict sense. It is because of this reason that it turns out to be impossible to prevent the possibility of the vacuum state to fluctuate, the fluctuations having the order of magnitude given by Heisenberg's principle, that is, of Planck's fundamental constant. In this way, creation and anihilation of virtual pairs of a particle and an antiparticle occur, in a way such that all characteristics of the vacuum of our quantum system, as its charge, spin, etc., are kept unaltered during such processes. We should here stress the enormous importance of Heisenberg's principle. Let us recall, for instance, that thirty years ago now, that is, also thirty years after Casimir's discovery, Hawking formulated his nowadays very

famous theory on the quantum radiation of black holes taking as a fundamental point in his argument the uncertainty principle.

Let us now just do a further step, by choosing a specific model. It is very common in Quantum Mechanics to use the harmonic oscillator model in order to clarify any argumentation. This is also true, to some extent, in Classical Mechanics, but in the quantum paradigm it turns out to be almost unavoidable. The quantization of the simple harmonic oscillator leads to the result that the energy eigenvalues are given by: $E_n = (n + 1/2)\hbar\omega$, $n = 0, 1, 2, 3, ...$, where $\hbar$ is Planck's constant (over 2π) and ω the frequency of oscillation. The state of minimal energy corresponds to $n = 0$ and the vacuum to vacuum transition of the system turns into: $\langle\emptyset|H|\emptyset\rangle = (\hbar/2) \sum_{\omega} \omega$, where the sum must be interpreted in a general sense as a 'summation' over all possible frequencies: it turns into an infinite sum (a mathematical series) if the spectrum is discrete (as, for instance, when the quantum system is confined to a box, or if it is bound to have periodic boundary conditions); but it is an integral in the case of a continuous spectrum (as for an unbounded system); or, more generally, it is a multidimensional combination of both cases as, for instance, when the system is confined in one or more directions but unbounded in other directions. This is just a technical issue, of very little relevance in fact, but it turns out that some scholars not working on the subject but who got interested in it (this happens more and more often recently) get already stuck at this point and keep asking why we put a simple summation sign at that place.

What is indeed important is to ask the question (as already did so many physicists in the last century, as Albert Einstein, Wolfgang Pauli or Niels Bohr): has this 'vacuum energy', that is, the energy associated to the vacuum to vacuum transition and which is given in terms of the half-Planck-quantum, any physical meaning? That is, could it be seen in the lab? Can it have any observable consequences? In fact these question were not difficult to answer within the standard formalism of Quantum Field Theory. Continuing with the study of the simple harmonic oscillator in terms of creation and anihilation operators, it is a standard procedure in QFT to take the normal ordering prescription, which fixes as equal to zero the vacuum energy, by postulating that all creation operators are to be put to the right, while all anihilation operators should go to the left. Of course this is a prescription, and one could equally well take a different one, resulting in a different value for the vacuum energy. All this hints already to the answer of the questions above: within QFT the vacuum energy is an arbitrary quantity (it just means fixing an arbitrary origin of energies) and cannot be measured experimentally. This seems to be generally accepted by everybody now (see, however [2]). The situation radically changes when general relativity is involved: the vacuum itself gravitates and the vacuum fluctuation energy fulfills the equivalence principle [3, 4]), and here indeed it has an absolute meaning as a physically measurable quantity. Anyway, for a long time gravity considerations did not come into play and vacuum fluctuations remained a useless idea[1], until Casimir —who studied the problem with care under the suggestion of Bohr— came up with his brilliant proposal to compare two situations, one involving the vacuum alone and the other being a slightly modified situation, where just some boundary conditions were imposed on the vacuum field. But this is a very well known story to any potential readers of this article, so I will stop this description here.

In any case some word must still be uttered about an issue that is very important in respect to the specific calculations needed to produce a physically meaningful answer. Aside from the conceptual considerations before, it is clear from the very beginning that a regularization procedure will be needed all the time when dealing with this subject. Indeed, the very simple case of the harmonic oscillator already leads to an infinite and divergent sum. A part of the problem is in fact physically clear and natural. On putting some boundary conditions and comparing the two situations above, an infinite number of the modes appearing in the vacuum

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[1] The contribution of vacuum fluctuations to the Lamb shift must not be forgotten either.

sum go away when computing the difference, because they are compatible with the boundary conditions and appear on both terms (so they disappear when taking the difference). However, in a real generic situation this does not mean that the resulting value is already finite —an even if it is, it often happens that an additional, so-called finite renormalization is needed! This is indeed an elusive point, not fully appreciated in the literature, and thus the origin of different erroneous results. I cannot discuss this issue in more detail here, but the reader is warned that it is of paramount importance in this subject and makes it, in a way, quite far from trivial from a technical point of view.

Back to a historical context, the appearance of divergences in QFT lead some important physicists, as Albert Einstein or Paul Dirac, to reject it as a wrong theory (the right one not having been found yet). Now, a number of colleagues (as the already mentioned 't Hooft, Veltman, and many others) have developed a consistent renormalization theory that leads to extremely precise results that can be confronted with laboratory measurements to more than 14 digits in some cases. This cannot be mere coincidence. However, when dealing with the vacuum energy fluctuations in a cosmological context involving general relativity these same kind of calculations lead to an enormous value of the vacuum energy density which is in flagrant contradiction with what we observe. Again, this discrepancy has led some physicists (as Asim Barut and others) to the conviction that maybe vacuum fluctuations do not exist at all, and thus this big problem (ordinarily known as the cosmological constant problem) just disappears! From a fundamental viewpoint this basically leads to a new start in Quantum Mechanics and QFT: a non-linear theory where the second quantization is not need. The reader should be informed about this possibility, that has produced some striking results (as a god approximation in the order of magnitude to the Lamb shift and other quantum field effects), but it is a fact that this alternative does not have much support in the scientific community nowadays. On the other hand, the number of more standard approaches to solve the cosmological constant problem [5] are flourishing (just search for 'cosmological constant' in the ArXives).

4. The Casimir Effect and the Fulling-Davies Theory

The 1948 paper by H.B.G. Casimir [1] was the beginning of a whole branch of research which aims at answering, both from a fundamental and from a practical perspective, very profound questions on the vacuum structure of Quantum Field Theory. The interest of the subject is certified by the impressive number of papers published in recent years (see, e.g., [6]). To wit, fluctuations of the quantum vacuum appear practically everywhere, the key issue being to ascertain if these contributions are relevant or not for the physical (chemical or biological) process in question. They have been argued to be irrelevant for sonoluminiscence (a 10^{-5} contribution), but they are very important for accurate calculations in laser cavities, wetting phenomena of alkali compounds by Helium-3, the sticking of small plates in MEM and NEM devices, and so on [6].

When the plates move quickly, as with a high-frequency vibration, one has the dynamical Casimir effect, a phenomenon studied by S. Fulling and P. Davies in 1976 [7]. Actually, moving mirrors further modify the structure of the quantum vacuum, what manifests in the creation and annihilation of particles. Once the mirrors return to rest, a number of the produced particles may still remain which can be interpreted as radiated particles. This flux has been calculated in several situations by different procedures (averaging over fast oscillations multiple scale analysis, with the rotating wave approximation, numerical techniques, etc.). For a single, perfectly reflecting mirror, the number of produced particles as well as their energy diverge while the mirror moves. Several renormalization prescriptions had been devised to obtain a well-defined energy, however, for some trajectories this finite energy turned not to be a positive quantity and could not be identified with the energy of the produced particles (see e.g. [7]), a situation that was resolved with our approach [8], which relies upon two rather simple ingredients: (i) proper use of a Hamiltonian method and (ii) the consideration of partially transmitting mirrors, which

become transparent at sufficiently high frequencies. Realistic mirrors will always satisfy this condition and the result we have proven, in this way, is both that the number of created particles is finite and also that their energy is always positive for the whole trajectory corresponding to the mirrors' displacement. We have also calculated, in [8], the radiation-reaction force that acts on the mirrors owing to the emission and absorption of particles, and which is related with the field's energy through the energy conservation law, so that the energy of the field at any time t is equal, with the opposite sign, to the work performed by the reaction force up to time t. Such force is usually split into two parts: a dissipative force whose work equals minus the energy of the particles that remain, and a reactive force vanishing when the mirrors return to rest. We have shown for the first time that the radiation-reaction force calculated from the Hamiltonian approach for partially transmitting mirrors satisfies, all the way the energy conservation law and can thus naturally account for the creation of positive energy particles. Also, the dissipative part we obtain agrees with the one calculated by other methods, as using the Heisenberg picture or other effective Hamiltonians. To repeat, this is basically the result of a proper, physically meaningful renormalization prescription.

4.1. On the Nature of Quantum Vacuum Fluctuations
A fact that may be closely related with the considerations above, is that our universe appears to be spatially flat and to possess a non-vanishing cosmological constant (cc), as introduced by Einstein and with the same sign. Actually, for elementary particle physicists the cc constitutes a great embarrassment, calculations of it being off by the famous over-one-hundred orders of magnitude which one gets by setting a plausible cut-off of the order of the Planck length. The cc has to do with cosmology—through Einstein's equations and the FRW universe obtained from them—and also with the local structure of elementary particle physics, as the stress-energy density of the vacuum. Being precise, by the equivalence principle (wee more about that below) the vacuum expectation value of the stress-energy tensor $\langle T_{\mu\nu} \rangle \equiv -\mathcal{E} g_{\mu\nu}$ appears on the rhs of Einstein's equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G (\tilde{T}_{\mu\nu} - \mathcal{E} g_{\mu\nu}). \tag{1}$$

It affects cosmology: $\tilde{T}_{\mu\nu}$ contains excitations above the vacuum, and is thus equivalent to a cosmological constant $\Lambda = 8\pi G \mathcal{E}$. Recent observations yield [9]

$$\Lambda_{\text{obs}} = (2.14 \pm 0.13 \times 10^{-3} \text{ eV})^4 \quad \sim \quad 4.32 \times 10^{-9} \text{ erg/cm}^3$$

This idea is old and goes back to Zel'dovich [10], who already stated that the cc gets contributions from the zero point fluctuations

$$E_0 = \frac{\hbar c}{2} \sum_n \omega_n, \qquad \omega = k^2 + m^2/\hbar^2, \quad k = 2\pi/\Lambda. \tag{2}$$

Evaluating in a box and putting a cut-off at maximum k_{max} corresponding to reliable QFT physics (e.g., the Planck energy)

$$\rho \sim \frac{\hbar k_{\text{Planck}}^4}{16\pi^2} \sim 10^{123} \rho_{\text{obs}}. \tag{3}$$

The issue of the cc has got renewed thrust from the observational evidence of an acceleration in the expansion of our Universe, initially reported by two different groups [11]. As a consequence, many theoreticians have urged to try to explain this fact, and also to try to reproduce the precise

value of the cc coming from these observations, but it is more difficult to explain why the cc is so small but non-zero, than to build theoretical models where it exactly vanishes [5].

Before going on, it is interesting to observe that the Casimir effect dates back from the very same year, 1948, as the discovery of renormalized QED. Curiously enough, however, Feynman is reported to have argued that vacuum fluctuations could not be affected by gravity (for a nice account of this point, with historical remarks, see [12]). Thus, a very basic issue is if one is right in writing the equations above and thus assuming that the vacuum energy satisfies the equivalence principle of GR. In other words, how the renormalized Casimir energy of a pair of plates couples to gravity? The answer to this question is less straightforward than one might suspect. In fact very recently disparate answers have appeared in the literature: forces that depend on the orientation of the Casimir apparatus wrt the gravitational field of the earth, etc. [3, 4] Rigorous evidence seemed still to be lacking that the vacuum energy should be taken seriously in Gravity.

Essentially there are two ways to proceed with the calculations. A gauge-invariant procedure: as the energy-momentum tensor of the physical system must be conserved, one needs to include a physical mechanism holding the plates apart against the Casimir force, what leads in practice to a very complicated model-dependent calculation [3]. The alternative, and more practical, way is to find a physically natural coordinate system, more realistic than any other. A reasonable one, used for these purposes, is the Fermi coordinate system, a general-relativistic extrapolation of the concept of an inertial coordinate frame. This has been used in [3], a paper initially criticized in [4] (again, the reason one gets different answers in different coordinate systems is that the starting point is not gauge-invariant). However, in the end agreement seems to have been reached (at least in Fermi and Rindler coordinates), after renormalizing the mass of the plates, that the Casimir energy contributes with a gravitational mass in accordance with the equivalence principle. The end word is that, although a rigorous gauge-invariant proof is still lacking, the calculations performed until now hint towards the positive answer.

5. A Cosmological Imprint of the Casimir Effect?

Having clarified the important point above, in this section, rather than trying to understand the fine-tuned cancelation of the enormous contributions mentioned at the *local* level, we will elaborate on a quite simple idea (but, for the same reason, of potentially far reaching consequences), related with the *global* topology of the universe [13] and in connection with the possibility that fields of small mass pervading the universe most probably exist. They are indeed ubiquitous in inflationary models, quintessence theories, etc. We do not aim at solving the old cc problem, but just at nailing down some contributions to it which may be of the order of magnitude corresponding to the recent observations, as given above. We address here, so to say, the 'perturbative part' of the *new* cc problem [14]: we assume the existence of a quantum field background extending through the universe and calculate the contribution to the cc of the Casimir energy density for this field coming from some typical BCs. Ultraviolet contributions will be safely set to zero invoking some (to be fixed) mechanism of a fundamental theory.

Another hypothesis is the existence of both large and small dimensions (the total number of large spatial coordinates being always three), some of which may be compactified, so that the global topology of the universe will play an important role. There is an extensive literature both on what is the global topology of (spatial sections) of the universe and also on possible contribution of the Casimir effect as a source of some sort of cosmic energy, e.g., as in the case of the creation of a neutron star. Arguments favor different topologies, as a compact hyperbolic manifold, for the spatial section, that could have clear observational consequences. At a second stage it will have sense to consider all possibilities concerning the nature of the fields, the different models for the topology of the universe, and the possible BCs, with their effects on the sign of the force too.

Zeta function regularization techniques are used to this end with impressive success [15]. From previous experience [16], we know that the range of orders of magnitude of the vacuum energy density for the most common possibilities is not so widespread, and may only differ by at most a couple of digits. This has allowed us, both for the sake of simplicity and universality, to deal with simplified situations [17] as, to start, a space-time like $\mathbb{R}^{d+1} \times \mathbb{T}^p \times \mathbb{T}^q$, or $\mathbb{R}^{d+1} \times \mathbb{T}^p \times \mathbb{S}^q, \ldots$, which are very plausible models for the topology [13]. Here, $d \geq 0$ stands for a possible number of non-compactified dimensions. Recall the physical contribution to the vacuum or zero-point energy $\langle 0|H|0 \rangle$ is obtained by subtracting the vacuum energy corresponding to the situation with the only change that compactification is absent (in practice this is done by conveniently sending the compactification radii to infinity). It is its *difference* $E_C = \langle 0|H|0 \rangle|_R - \langle 0|H|0 \rangle|_{R \to \infty}$ (R a typical compactification length) what gives rise to the additional Casimir energy E_C contribution to the cc coming from the BC. Renormalization is to be carried out on the spectral parameter μ: in terms of the spectrum $\{\lambda_n\}$ of H, $\langle 0|H|0 \rangle = \frac{1}{2} \sum_n \lambda_n/\mu$, (the sum may involve several continuum and several discrete indices), and μ is needed to render the eigenvalues dimensionless [17]. The physical vacuum energy density in this case, where the contribution of a scalar field ϕ living in a partly compactified spatial section of the universe is considered, with

$$S = \frac{1}{2} \int d^4x \sqrt{-g} \left[g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + (m^2 + \xi R)\phi^2 \right], \tag{4}$$

will be denoted by ρ_ϕ (this is just the contribution to ρ_V coming from this field, there might be other, in general), $\rho_\phi = \frac{1}{2} \sum_n \lambda_n/\mu = \frac{1}{2} \sum_{\mathbf{k}} \frac{1}{\mu} \left(k^2 + M^2 \right)^{1/2}$, where the sum $\sum_{\mathbf{k}}$ is a generalized one and $M^2 = m^2 + \xi R$ an effective mass term [17].

An important issue is the specific *sign* of the resulting force. For scalar fields and the usual compactifications or BCs it seems not possible to get the right sign corresponding to the accelerated expansion of the universe. However, in worldbrane models and others, involving supergravitons and fermion fields, we have been able to prove that the appropriate sign can be obtained under particular but quite natural conditions. Specifically, in the case of the torus topology we have obtained [18] that the topological contributions to the effective potential have in fact a fixed sign, which depends on the BC one imposes. It is negative for periodic fermionic fields in a a supersymmetric theory, and positive for anti-periodic fields. Thus, topology provides a mechanism which, in a most natural way, permits to have a positive cc in a multi-supergraviton model with anti-periodic fermions [18]. Moreover, the value of the cc is regulated by the corresponding size of the torus (as is easy to see in the scalar case above). One can most naturally use in this case the minimum number, $N = 3$, of copies of bosons and fermions, and show that the order of magnitude of the observational values for the cc can be reproduced invoking quite natural assumptions.

6. Gravity Equations as Equations of State

Also in relation with the cosmological constant, different models have been proposed (T. Padmanabhan et al. [19], D. Blas, J. Garriga [20], etc.) where it appears under the form of an *integration constant*. This can be done, in particular, by modifying the gravity metric in a way that gives rise to the so-called unimodular gravity, but there are also other possibilities. In a seminal paper, in 1995, Ted Jacobson went even further, in a certain way, and obtained Einstein's equations from local thermodynamics arguments only [21]. He did this by way of generalizing black hole thermodynamics to space-time thermodynamics as seen by a local observer. This strongly suggest, in a fundamental context, that Einstein's equations are probably to be viewed as a sort of equation of state. As a consequence, Jacobson went further to suggest that EEs should probably not be taken as basic for quantizing gravity. More recently, C. Eling, R. Guedens, and T. Jacobson [22] extended these considerations to *polynomial* $f(R)$

gravity, with the important difference that the comparison only worked with *non-equilibrium* thermodynamics.

Modified gravity models, as $f(R)$ gravity, constitute a very important dynamical alternative to standard ΛCDM cosmology, in that they have the capability to describe the current accelerated expansion of our Universe (dark energy epoch) together with the initial de Sitter phase and inflation, and even some galaxy rotation curves for the dark (and ordinary) matter [23]. One could imagine that Jacobson's derivations can be generalized to cover all these more complicated theories of gravity that are extensively used nowadays. Indeed, we have been able to extend Jacobson's arguments (in a non-trivial way) to $f(R)$ gravities, being f a general function not just of the Ricci scalar but also of any covariant derivatives of the Ricci scalar of arbitrary order.

Moreover, we have obtained the corresponding field equations as an equation of state of *local* space-time thermodynamics [24]. This could be done by using (i) Iyer and Wald's more general definition of dynamic BH entropy [25] together with (ii) the concept of an effective Newton constant for graviton exchange (effective propagator) of Brustein et. al. [26]. After our work was issued, Wu et. al. [27] have obtained a direct extension of our results to Brans-Dicke and scalar-tensor gravities. Some other extensions of our arguments have also been issued [28].

Acknowledgments

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