9

Cyclotron Resonance Masers (CRM)

9.1 General Principles

From the viewpoint of principles of the stimulated radiation emission, any of the systems that emit radiation and contain an electron beam may be called free–electron lasers or masers. This implies that emitting elements are not bound in atoms or in a crystal lattice.

We now accentuate the general properties, inherent in all beam systems with distributed parameters. Classifying these systems by the resonance conditions (the conditions of synchronism), one can divide them into the two large classes. The systems, the operation of which is based on Cherenkov mechanism for the field–particle interaction, belong to the first class. To the second one belong the beam systems where charged particles are oscillators. In this case, the oscillators, moving with a relativistic velocity, can emit radiation at very high frequencies due to Doppler effect.

For operation of Cherenkov systems, electrodynamic structures that can slow down electromagnetic waves are required. Therefore, the mechanism for Cherenkov radiation emission is effective only when oscillations are excited at relatively low frequencies. Surely, the quasi–Cherenkov effects (e.g., Smith–Parcel radiation emission) can be used for stimulating the radiation emission within the optical range. However, the transverse size of the active area of the field–particle interaction is very small ($l_\perp \approx \lambda$).

The beam systems of the second class are much better adapted for stimulating the short–range radiation emission (up to the x–ray band). In the systems of this type, no slow–wave structures are required. Electromagnetic waves are free (either completely or almost completely). Cyclotron resonance masers and free electron lasers (in the traditional sense of these terms) belong to this class.

In the given section, we are to investigate CRM. For providing the radiation emission in CRM, oscillators serve as an energy source. These oscillators are produced by injecting the beam electrons into an external constant homogeneous magnetic field at a certain angle. As it is known, the condition for the prolonged synchronism between the field and oscillators (the cyclotron resonance conditions) have the form: $\omega - kv \pm \Omega_0/\gamma = 0$, where $\Omega_0 = qB/mc$. This condition indicates that excitation and amplification of fast waves ($\beta_{\text{ph}} > \beta$) is possible at the two frequencies:

$$\omega_{1,2} = \frac{\Omega_0}{\gamma\,(1 \pm \beta/\beta_{\text{ph}})} \,. \tag{9.1}$$

Here the sign "+" describes the low–frequency counterbeam propagating wave. The sign "−" determines the copropagating high–frequency wave. The energy of intrinsic waves of a passive electrodynamic structure is positive. At the same time, beam cyclotron waves can possess negative energy. It is the interaction between the positive–energy waves and the beam waves, characterized by negative energy, that is the origin of the development of cyclotron instabilities. In dispersion diagrams, parameters of the interacting waves are determined by the points of intersection of the corresponding partial branches. In Fig. 9.1, dispersion of the beam cyclotron waves and fast electromagnetic waves is plotted.

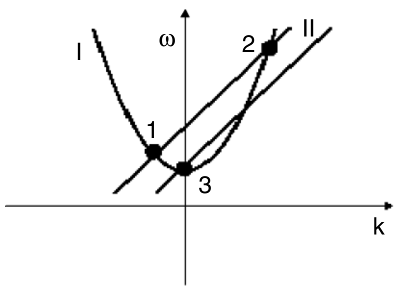

Fig. 9.1. Dispersion of the electromagnetic waves (I) and the beam cyclotron ones (II). Crossing point "1" corresponds to interaction of the beam waves with the backward low-frequency electromagnetic wave; the point "2" – with the high frequency wave; the point "3" – to the case of a gyrotron

There one can see the dispersion branch of fast electromagnetic waves $\omega^2 = k^2c^2 + \omega^2_{\text{c.o.}}$ propagating in a regular waveguide with a finite cutoff frequency $\omega_{\text{c.o.}}$. The branch of the beam cyclotron wave, which corresponds to the Doppler normal effect ($\omega = kv + \omega_0/\gamma$), is depicted there as well.

The first point of the intersection corresponds to the excitation of the low–frequency wave moving opposite the beam. The second point corresponds to the excitation of a high–frequency wave, copropagating with the beam.

As regards more or less compact microwave devices, the relativism of the beam particles is rather low so that, in fact, there does not exist any essential Doppler heightening of the frequency. The case $k = 0$ is of a special interest. In the first approximation, the frequency of operation is independent of the particle longitudinal velocity. This makes an advantage of CRM over the systems with Cherenkov interaction, where the longitudinal velocity of the beam particles must be maintained to a high precision. This scheme of CRM, suggested in the early sixties, has been called *the gyrotron* (see [47]).

The choice of $k = 0$ means that the electric field of the high–frequency wave is homogeneous all over along the beam at any moment. The waveguide, where the beam particles interact with the field, operates at the standing mode as a cavity. The cavity output edge can be made in the form of a corrugated structure, which is a Bragg mirror. This construction of the electrodynamic structure provides the mode selection. It also permits to heighten essentially the system output power. It is worth mentioning that the internal field strength in the cavity is $\sqrt{Q}$ times higher than the output field strength (here Q is the cavity quality factor). This permits to increase the efficiency of CRM operation. Really, the CRM efficiency is inversely proportional to the number of revolutions N, performed by the particle within the interaction region. At the end of the interaction path saturation of the amplitude of the wave is desirable. Therefore, by increasing the field strength in the interaction area, one can reduce its geometrical size. Respectively, this provides for diminution of the number of the particle revolutions and increases the system efficiency.

Besides, the condition $k = 0$ permits to diminish essentially the Doppler broadening $\Delta\,(kv)$ of the cyclotron resonance line if there exists the spread in values of the forward velocity of electrons Δv.

The second point of intersection corresponds to the excitation of waves at higher frequencies. However, to realize this type of interaction between electromagnetic and cyclotron beam waves, one needs the values of γ, essentially higher than the ones mentioned above. It is worth mentioning that the wave autoresonance excitation ($\omega = \Omega_0/\gamma\,(1-\beta)$) is possible within this pattern. As it is known [28, 29], the conditions for the cyclotron autoresonance are independent of changes in the particle energy values. In principle, there arises the possibility of the unlimited resonance acceleration of charged particles and of the complete transfer of the particle energy to the wave field. The operation of cyclotron autoresonance masers (CARM) is based on this principle.

Stimulation of the radiation emission in the millimeter–submillimeter wave ranges (or in a band of shorter wavelengths) and on heightening the power level of the oscillations excited has an essential feature. The point is that in CRM charged particles can interact not only with a single spatial mode but with a large number of these modes as well. And what is more, the particles can interact with a single spatial mode when there simultaneously exist several cyclotron resonances. There occurs the interaction of this type when the field strength of the wave excited reaches a value, sufficient for overlap of nonlinear cyclotron resonances. In all these cases, the particle motion becomes chaotic. There are both drawbacks and advantages inherent in this regime of exciting oscillations. On the one hand, a disadvantage is that the level of fluctuations in the characteristics of the field excited becomes higher. There also arises a broadening of the field spectrum. On the other hand, one can control the oscillation spectrum width, which is a rather attractive point. Besides, in this regime, the particle motion is not restricted within a single isolated cyclotron resonance. In principle, this holds out prospects for heightening effectiveness of the wave–particle energy interchange. This mechanism for the microwave excitation has been investigated in [48].

9.2 CRM in Small–Signal Approximation

In fact, the theory of CRM in the small–signal approximation has been presented in Sect. 7. And what is more, it has been proved that the kinetic theory must be applied for describing the process of the instability development in CRM. Within the framework of the fluid dynamics approximation, it is impossible to correctly describe the process of the electron beam instability development in an external magnetic field. However, in Sect. 7, the principal attention has been paid to solving the paradox of the absence of the beam instability within the framework of the fluid description approximation. Besides, we have grounded the fact that the use of the kinetic theory is necessary. We have also dwelled on applicability of the notion of Landau damping. At the same time, as a matter of fact, the process of development of the radiative instability has not been investigated yet. In addition, in Sect. 7 an unlimited homogeneous beam of electrons was considered. In the given subsection, we are going in detail to dwell on development of the radiative instability in the limited beam. The simplest model of the beam of this type is to be studied. That is, we now investigate a two–dimensional model, where a ribbon beam of electrons is propagating in parallel to an external homogeneous constant magnetic field. The area of the beam–field interaction is restricted by two ideally conducting planes, located at $x = \pm a$. The point is that the parameters of the beam and those of the electrodynamic system are independent of y–coordinate. Hence, in the simplest case, we will regard all physical processes as independent of y–axis. Longitudinal and transverse velocities of all of the beam particles are regarded as equal to one another. This model of CRM has been examined in [49, 50].

9.2.1 Dispersion Equation for Ribbon Beams

In discussing the CRM theory, our starting point is Maxwell's equations for the fields and Vlasov kinetic equation for the distribution function of the beam electrons (e.g., see (8.42)). The distribution function f_0 describes stationary undisturbed states of the beam particles. In general, it is an arbitrary function of the characteristics of Vlasov equation. In the presence of just a constant external homogeneous magnetic field, these characteristics take the form:

$$\gamma = \text{const} ; \qquad p_{\mathrm{z}} = \text{const} ;$$

$$z - vt = \text{const}\ ; \qquad \Omega_0 x - p_\mathrm{x} = \text{const}\ ; \qquad (9.2)$$
$$\arctan\,(p_\mathrm{x}/p_\mathrm{y}) - \frac{\omega_0}{\gamma} t = \text{const}\ .$$

Here $p_{\mathrm{x,y,z}}$ are corresponding momentum projections in mc units and γ is a particle energy in mc^2 units. It is easy to see that these characteristics are the solutions to the equation of motion of charged particles in a constant external homogeneous magnetic field. As we deal with a ribbon beam, the undisturbed distribution function is independent of the y coordinate. Consequently, it also does not depend on the azimuthal angle in the momentum space $\vartheta = \arctan\,(p_\mathrm{x}/p_\mathrm{y})$, for example, $\partial f_0/\partial\vartheta = 0$. Besides, we consider the beam density to be low so that beam stationary fields (electric and magnetic) are negligible. We also neglect the beam permeability.

In the electrodynamic system, formed by two parallel perfectly conducting planes, the waves of both the E– and H–types can be excited. We now choose the temporal and longitudinal–coordinate dependencies of these waves in the form $\exp\,[\mathrm{i}\,(kz - \omega t)]$. As well as the field undisturbed distribution function, the fields are independent of y–coordinate. For simplicity, we limit ourselves to the case of excitation just of H–waves. The components of these waves are E_y, B_x, B_z. And what is more, by making use of Maxwell's equations, one can express the components via the wave electric field E_y. As a result, expressions for all the components can be written as

$$\begin{aligned} E_\mathrm{y} &= E\,(x)\,\exp\,[\mathrm{i}\,(kz - \omega t)]\ ; \\ B_\mathrm{z} &= -\frac{kc}{\omega} E\,(x)\,\exp\,[\mathrm{i}\,(kz - \omega t)]\ ; \\ B_\mathrm{z} &= -\mathrm{i}\frac{c}{\omega}\frac{\partial E}{\partial x}\,\exp\,[\mathrm{i}\,(kz - \omega t)]\ . \end{aligned} \qquad (9.3)$$

To determine E_y, one has to use the wave equation:

$$\frac{\mathrm{d}^2 E_\mathrm{y}}{\mathrm{d}x^2} + \left(\frac{\omega^2}{c^2} - k^2\right) E_\mathrm{y} = \frac{4\pi}{c}\frac{\partial j_\mathrm{y}}{\partial t}\ . \qquad (9.4)$$

On the RHS of (9.4), the current density j_y drives the field E_y. This variable can be found by solving the linearized Vlasov equation. For the function $f\,(\mathbf{r}, \mathbf{p}, t)$, which represents a small deviation from the stationary distribution function f_0, this equation can be presented as

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \left[q\mathbf{E} + \frac{q}{c}\,[\mathbf{vB}]\right]\frac{\partial f_0}{\partial \mathbf{p}}\ . \qquad (9.5)$$

Substituting the expression for the fields (9.3) into (9.5), we shall use a cylindrical coordinate system in the momentum space ($p_\mathrm{x} = p_\perp \cos\vartheta$, $p_\mathrm{y} = p_\perp \sin\vartheta$). Thus, (9.5) can be rewritten as

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{q}{mc\omega}\left[(\omega - kv)\frac{\partial f_0}{\partial p_\perp} + \frac{kp_\perp c}{\gamma}\frac{\partial f_0}{\partial p_\mathrm{z}}\right] \times E\,(x)\sin\vartheta\,\exp\,[\mathrm{i}\,(kz - \omega t)]\ . \qquad (9.6)$$

The general solution of (9.6) can be found by direct integration:

$$f = \int \mathrm{d}t \frac{q}{mc\omega} \left[(\omega - kv) \frac{\partial f_0}{\partial p_\perp} + \frac{kp_\perp c}{\gamma} \frac{\partial f_0}{\partial p_\mathrm{z}} \right] \times E(x) \sin\vartheta \, \exp\left[\mathrm{i}(kz - \omega t)\right] \, . \tag{9.7}$$

In (9.7), the integral must be taken along the characteristics (9.2). Therefore, one may make transition from integrating over t to integrating over another variable (e.g., over ϑ):

$$f = \int \mathrm{d}\vartheta \frac{q\gamma}{mc\omega\Omega_0} \left[(\omega - kv) \frac{\partial f_0}{\partial p_\perp} + \frac{kp_\perp c}{\gamma} \frac{\partial f_0}{\partial p_\mathrm{z}} \right] \times E(x) \sin\vartheta \, \exp\left[\mathrm{i}(kz - \omega t)\right] \, . \tag{9.8}$$

We now suppose that Larmor radius and the ribbon beam thickness (a) are much smaller than the wavelength. Consequently, the electric field strength $E(x)$ in the integrand of (9.8) may be changed for the value $E(x_0)$ at the median plane of the beam and taken out from the integral. After that the integral in (9.8) can be easily calculated:

$$f = \frac{qE(x_0)}{2mc\omega} \left[(\omega - kv) \frac{\partial f_0}{\partial p_\perp} + \frac{kp_\perp c}{\gamma} \frac{\partial f_0}{\partial p_\mathrm{z}} \right] \times \left[\frac{\exp(-\mathrm{i}\vartheta)}{\omega - kv - \Omega_0/\gamma} - \frac{\exp(\mathrm{i}\vartheta)}{\omega - kv + \Omega_0/\gamma} \right] \exp\left[\mathrm{i}(kz - \omega t)\right] \, . \tag{9.9}$$

In the cylindrical system of axes, the expression for the current density in the momentum space can be submitted as

$$j_\mathrm{y} = -\frac{q}{\gamma} \int f \sin\vartheta \, p_\perp \, \mathrm{d}p_\perp \mathrm{d}p_\mathrm{z} \mathrm{d}\vartheta \, . \tag{9.10}$$

We now suppose that the equilibrium distribution function may be presented in the form of a product of several functions. One of them depends only on momenta, another one determines the beam structure along x–axis. Besides, we consider the beam to be cold and its thickness to be much shorter than the wavelength of the oscillations excited. In this case, the equilibrium distribution function may be presented in the form:

$$f_0 = n_0 2d\delta(x - x_0) \frac{\delta(p_\mathrm{z} - p_{\mathrm{z},0}) \, \delta(p_\perp - p_{\perp,0})}{2\pi p_{\perp,0}} \, , \tag{9.11}$$

Here n_0 is the beam equilibrium density; d is the beam half–width; $x = x_0$ determine the location of the plane of the beam axis (see Fig 9.2).

Making use of the distribution function (9.11), we integrate over ϑ, p_z, and $p_\perp$. After simple but bulky calculations, one gets the following expression for the disturbed component of the beam current:

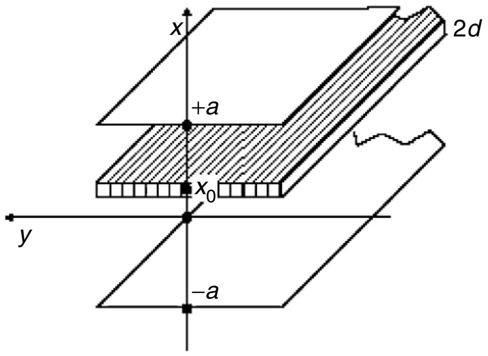

Fig. 9.2. Scheme of CRM with flat beam

$$j = \frac{\omega_{\mathrm{b}}^2 d}{4\gamma} E\left(x_0\right) \delta\left(x - x_0\right) \exp\left[\mathrm{i}\left(kz - \omega t\right)\right] G\left(\omega, k\right) \ , \tag{9.12}$$

Here

$$\begin{aligned} G = -\frac{1}{\pi} \Bigg\{ & \left[\frac{1}{\left(\omega - kv_{\mathrm{z},0} - \Omega_0/\gamma\right)} + \frac{1}{\left(\omega - kv_{\mathrm{z},0} + \Omega_0/\gamma\right)}\right] \\ & \times \left(\omega - kv_{\mathrm{z},0}\right) \\ & + \frac{v_\perp^2 k_\perp^2}{2} \left[\frac{1}{\left(\omega - kv_{\mathrm{z},0} - \Omega_0/\gamma\right)^2} + \frac{1}{\left(\omega - kv_{\mathrm{z},0} + \Omega_0/\gamma\right)^2}\right] \Bigg\} \ . \end{aligned}$$

We consider the beam narrow and its density sufficiently low. In a zeroth approximation over the beam density, one may not take into account the influence of this parameter on the field structure of the intrinsic wave of the electrodynamic system. All over the space where the beam is absent, we suppose that the field structure is the same as in the absence of the beam. The presence of the beam indicates itself by a jump of the microwave magnetic field component, tangential to the beam (H_z). There takes place this jump at the area where the beam is located. The magnitude of the jump can be found by integrating (9.4) over the beam small cross section:

$$\begin{aligned} & \lim_{\varepsilon \to 0} \left[\left(\frac{\partial E_{\mathrm{y}}}{\partial x}\right)_{x = x_0 + \varepsilon} - \left(\frac{\partial E_{\mathrm{y}}}{\partial x}\right)_{x = x_0 - \varepsilon}\right] \\ & \qquad = -2dk_\perp^2 E_{\mathrm{y}}\left(x_0\right) - \frac{\pi \omega_{\mathrm{b}}^2 d}{\mathrm{i}\gamma c \omega} E_{\mathrm{y}}\left(x_0\right) G \ . \end{aligned} \tag{9.13}$$

In addition, one must keep in mind that the electric field on conductive surfaces goes to zero: $E_{\mathrm{y}}\left(x = \pm a\right) = 0$. As it follows from (9.13), there is a discontinuity in the derivative of the wave electric field component with respect to the transverse coordinate. Therefore, all over the areas where the beam is absent, one can look for the solution to (9.4) in the form:

$$E_{\mathrm{y}} = \begin{cases} C \sin\left[k_{\perp}\left(x-a\right)\right] , & a > x > x_0 ; \\ D \sin\left[k_{\perp}\left(x+a\right)\right] , & -a < x < x_0 , \end{cases} \tag{9.14}$$

Here C and D denote constants; $k_{\perp,n} = \pi n/2a$.

The solution (9.14) satisfies the boundary condition on metallic planes ($E_{\mathrm{y}}\left(x = \pm a\right) = 0$). The constants C and D are related to one another by the continuity condition for the electric field (9.14) when $x = x_0$. Substituting the solution (9.14) into the boundary conditions (9.13) and excluding the constants, one gets the following dispersion relation:

$$\begin{aligned} L \equiv & \frac{k_{\perp}}{2}\left[\frac{\sin\left(2k_{\perp}a\right)}{\sin k_{\perp}\left(x-a\right)\, \sin k_{\perp}\left(x+a\right)}\right] - k_{\perp}d \\ = & -\frac{\omega_{\mathrm{b}}^2 d}{2\gamma c^2}\left\{\left(\omega - kv_{\mathrm{z},0}\right)\left[\frac{1}{\left(\omega - kv_{\mathrm{z},0} - \Omega_0/\gamma\right)}\right.\right. \\ & \left. + \frac{1}{\left(\omega - kv_{\mathrm{z},0} + \Omega_0/\gamma\right)}\right] \\ & - \frac{k_{\perp}^2 v_{\perp,0}^2}{2}\left[\frac{1}{\left(\omega - kv_{\mathrm{z},0} - \Omega_0/\gamma\right)^2}\right. \\ & \left.\left. + \frac{1}{\left(\omega - kv_{\mathrm{z},0} + \omega_H/\gamma\right)^2}\right]\right\} . \end{aligned} \tag{9.15}$$

If the thickness and density of the beam go to zero ($\omega_{\mathrm{b}} \to 0$, $d \to 0$), (9.15) describes intrinsic waves of the electrodynamic structure, formed by two conductive parallel planes. In addition, $k_{\perp} = \pi n/2a$.

The beam exerts a substantial influence on the waveguide modes under the resonance conditions (the conditions of synchronism): $\omega - kv_{\mathrm{z},0} \pm \Omega_0/\gamma = 0$. This relation describes the beam cyclotron modes. Thus, one can see that there takes place an effective energy interchange between the beam and electromagnetic modes at the points of intersection of the dispersion branches (see Fig. 9.1). The frequency values, under which the branch intersection becomes possible, are prescribed by the expression:

$$\omega_{1,2} = \frac{\Omega_0}{\gamma\left(1 - v_{\mathrm{z},0}^2/c^2\right)}\left[1 \pm \frac{v_{\mathrm{z},0}}{c}\sqrt{1 - \frac{k_{\perp}^2 c^2 \gamma^2}{\Omega_0^2}\left(1 - \frac{v_{\mathrm{z},0}^2}{c^2}\right)}\,\right] , \tag{9.16}$$

Here $k_{\perp}c$ is the minimum frequency that can propagate in the waveguide (the cutoff frequency).

As regards (9.15), it describes the relation of the system intrinsic frequencies to the intrinsic wave numbers. Generally speaking, this dispersion relation is transcendental, and it is difficult to give analysis to it. However, in the majority of the cases that are of practical interest, the analysis is possible. This equation can be solved with respect both to the frequency and to the wave numbers. Below we will investigate the three cases of the corresponding analysis. We consider them to be the most typical and interesting.

The First Case

Let us determine the conditions for the increase in the spatial disturbances when the waveguide intrinsic waves are synchronous with the beam cyclotron waves. Let us present the longitudinal wave number in the form $k = k_0 + h$ (here $k_0 = (\omega \pm \Omega_0)/v_{z,0} = \pm\omega/v_{ph}$; $v_{ph} = \pm\omega v_{z,0}/(\omega \pm \Omega_0)$ the magnitude h is a small disturbance of the wave number). In the expression for v_{ph}, the signs "±" determine the copropagating and contrary waves, respectively. For simplicity, let us suppose that the beam central plane is located in $x_0 = 0$. We now substitute the expression for the wave number into the dispersion relation (9.15). Furthermore, we expand the terms in h. The beam density is regarded as small. We also take into account that the left–hand side of (9.15) goes to zero in a zeroth approximation with respect to the beam density ($L(k_0) = 0$). Thus, one gets the following algebraic equation of the third degree for determining amendments to the wave number:

$$L'\left(\frac{\partial k_\perp}{\partial h}\right)h^3 - \frac{\omega_b^2 d}{2\gamma c^2}\left[\left(\frac{\Omega_0}{\gamma v_{z,0}} - \frac{k^2 v_{\perp,0}^2}{v_{z,0}^2}\right)h + \frac{k_\perp^2 v_{\perp,0}^2}{2v_{z,0}^2}\right] = 0\,. \qquad (9.17)$$

As regards the general case, in (9.17), we have preserved the terms of the same order of smallness. However, if the transverse velocity of the beam particles and the transverse wave number $k_\perp$ are high enough, the second term in the square brackets in (9.17) substantially exceeds the first one. Respectively, one gets the following equation for determining h:

$$h^3 = \frac{\omega_b^2 d k_\perp^2 v_{\perp,0}^2}{2ka\gamma c^2 v_{z,0}^2}\,. \qquad (9.18)$$

Here it is taken into account that if n is odd, $L'\,(\partial k_\perp/\partial k) \approx ka$.

As it follows from (9.18), under the conditions in question, there always occurs amplification of the microwave, the coefficient of amplification reaching its maximum:

$$\mathrm{Im}\,h = -\frac{\sqrt{3}}{2}\left[\frac{\omega_b^2 d k_\perp^2 v_{\perp,0}^2}{2ka\gamma c^2 v_{z,0}^2}\right]^{1/3}\,. \qquad (9.19)$$

Otherwise, the beam transverse velocity and the transverse wave number can take values, not too high so that the first term in the square brackets in (9.17) exceeds the second one. At the same time, if the beam transverse velocity is high enough to provide fulfillment of the inequality $\left(v_\perp^2/v^2\right) > (\Omega_0/\gamma k v)$, there also takes place the microwave amplification. However, the coefficient of amplification is smaller:

$$h = -\mathrm{i}\sqrt{\frac{\omega_b^2 d}{4k\gamma a c^2}\left(\frac{k v_\perp^2}{v^2} - \frac{\Omega_0}{\gamma v^2}\right)}\,. \qquad (9.20)$$

On the other hand, if the beam transverse velocity is so low that the inverse inequality is true $\left(v_\perp^2/v^2\right) < (\Omega_0/\gamma k v)$, it is easy to see that there does not take place any amplification.

The Second Case

One can be interested in tracing out the field amplitude evolution not only in space but also in time. For this purpose, the dispersion relation (9.15) must be solved not with respect to the wave number but with respect to the frequency ω. On the analogy of solving this equation with respect to k, we also suppose that the beam density is a small parameter. In (9.15), it is convenient to present the frequency ω in the form: $\omega = \omega_0 + \delta$. The undisturbed value of the frequency ω_0 corresponds to the point of intersection of the waveguide beam modes with the cyclotron ones; i.e., it is one of the frequencies, determined by (9.16); δ is a small disturbance of the frequency. Let us substitute this expression for ω into (9.15). It should be taken into account that the function $L'(\omega_0)\,(\partial k_\perp/\partial\omega)$ is equal to $-ka$ if n is odd. The equation for δ can be written as

$$\delta^3 - \frac{\omega_{\rm b}^2\omega_0^2 d\Omega_0}{2a\gamma^2}\delta + \frac{\omega_{\rm b}^2\omega_0^2 dk_\perp^2 v_\perp^2}{4a\gamma} = 0\,. \tag{9.21}$$

Equation (9.21) indicates the following. If the beam transverse velocity is sufficiently high so that the inequality $v_\perp^2 \gg \delta\, 2\omega_H/\gamma k_\perp^2$ holds, the increment of the instability development reaches its maximum, equal to

$$\operatorname{Im}\delta = \frac{\sqrt{3}}{2}\left(\frac{\omega_{\rm b}^2 dk_\perp^2 v_\perp^2}{4a\gamma\omega_0}\right)^{1/3}\,. \tag{9.22}$$

Otherwise, if the transverse velocity is low so that the opposite inequality $v_\perp^2 \ll \delta\, 2\Omega_0/\gamma k_\perp^2$ is true, there does not occur the instability development.

One could be interested in investigating the oscillations for which $k \to 0$. These waves are excited in gyrotrons. We now substitute $k = 0$ into (9.22). It should be taken into account that $k_\perp^2 = \left(\Omega_0^2/\gamma^2 c^2\right)$. Hence, one gets the following expression for the maximum increment:

$$\operatorname{Im}\delta = \frac{\sqrt{3}}{2}\left(\frac{\omega_{\rm b}^2 d\Omega_0 v_\perp^2}{4a\gamma^2 c^2}\right)^{1/3}\,. \tag{9.23}$$

The Third Case

In the beam system examined, there can develop instabilities which are not related to the synchronism of waveguide and beam modes. This occurs if the dispersion equation (9.15) describes only the beam modes – that is, when, under the condition that $\omega_{\rm b} \to 0$, $d \to 0$, the left–hand side of (9.15) does not go to zero, which means that $k_\perp \neq \pi n/2a$. In this case (under small values of the beam density), the solution to the dispersion equation (9.15) is located in the neighborhood of the point $\omega = kv + \Omega_0/\gamma \equiv \omega_0$. As above, let us substitute the solution in the form $\omega = \omega_0 + \delta$ into (9.15). For determining a small frequency addition δ, an algebraic equation of the second order has been derived. The solution to this equation can be written in the form:

$$\delta = -\frac{\omega_{\rm b}^2 d\Omega_0}{8\gamma c^2 L} \pm \sqrt{\frac{\omega_{\rm b}^2 d\Omega_0^2}{2\gamma^2 c^2 L}\left(\frac{\omega_{\rm b}^2 d}{32c^2 L} - \frac{k_\perp^2 v_\perp^2}{\Omega_o^2}\right)}\,. \qquad (9.24)$$

As it follows from (9.24), there does not take place any instability if $L > 0$. Otherwise, if $L < 0$, the instability is developing under a sufficiently high value of the beam transverse velocity $v_\perp^2 > \left(\omega_{\rm b}^2 \Omega_0^2 d/32k_\perp^2 c^2|L|\right)$.

It is worth tracing back to the physical reasons that induce the first and second terms in the round brackets under the root of (9.24). The first one is conditioned by the presence of the power bunching of the particles. In its turn, there arises the second term due to the inertial bunching. As one can see, the two mechanisms for bunching act in antiphase. If the beam density is rather low, the inertial bunching prevails. The bunching of this type causes the instability development, the instability increment being proportional to the square root of the beam density. As the beam density is increasing, the instability increment value is becoming higher as well. At the same time, the influence of the power bunching becomes more and more essential in this process. When $\omega_{\rm b}^2 = \left(32|L|c^2 k_\perp^2 v_\perp^2 \,/d\Omega_0^2\right)$, the two mechanisms for bunching start to compensate one another, which results in the instability derangement. If the beam density is $\omega_{\rm b}^2 = \left(16|L|c^2 k_\perp^2 v_\perp^2 \,/d\Omega_0^2\right)$, the increment reaches its maximum. The instability in question has the same nature as "the negative – mass effect," known in the theory of accelerators [51, 52]. There arises this effect as a result of bunching of nonisochronous oscillators, interacting with one another via the microwave field of the beam mode.

9.2.2 Bunching of Particles in CRM

They usually call two mechanisms of bunching of particles during the beam instability development in CRM: they are the forced and inertial ones. However, these notions are rather conditional. Actually, bunching is always conditioned by the reaction of the excited field on dynamics of the particles. The notion of the forced bunching implies the field direct influence on particle phases with respect to the wave. Let us go back to the system (9.30). There, in the second equation, which describes dynamics of the wave phase, the forced bunching is due to the terms proportional to the wave strength parameter (g). The inertial bunching is stimulated by the wave field influence on the particle energy and its longitudinal and transverse momenta. Respectively, this causes changes in the resonance conditions. In the overwhelming majority of cases, the inertial bunching prevails over the forced one. So, one may retain only the terms that determine the inertial bunching in the third equation of the system (9.33) (in the resonance phase equation).

We now focus on the physical mechanism of the inertial bunching. It is conditioned by the nonisochronous motion of electrons in the homogeneous magnetic field. The notion of nonisochronous motion implies the dependence of the rotational frequency on the electron energy. Suppose that electron motion around the field lines of the external constant magnetic field is purely

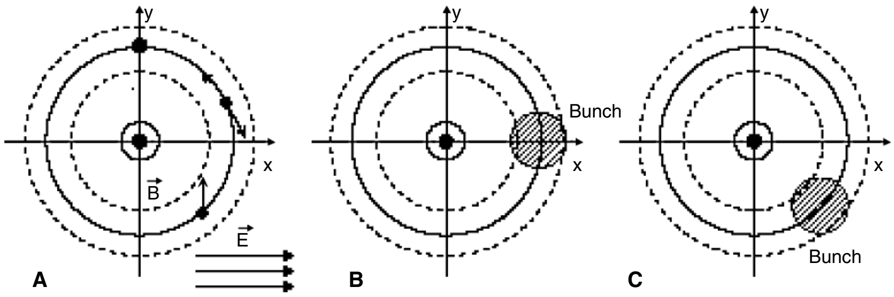

Fig. 9.3. Grouping of electrons into a bunch. (**A**) $t = 0$; $E_x = E_0 \cos \omega t$; (**B**) the case of the exact resonance $\omega = \Omega_0/\gamma$; (**C**) the case $\omega > \Omega_0/\gamma$

circular. We also suppose that the electrons are of the same energy (of the same Larmor radius), being uniformly distributed along the same circumference at the initial moment. The field of an external electromagnetic wave is switched on at this very moment. Besides, let Larmor radius be much smaller than the wavelength. In addition, we suppose that the electric field strength of the wave is directed along x–axis (see Fig. 9.3a).

As this graph indicates, the electrons, located in the half–space $y < 0$, are being decelerated by the wave field. At the same time, the electrons, located above the x–axis (i.e., where $y > 0$), are being accelerated by the same field. As the magnitude of Larmor radius depends on the particle energy ($r_L = \gamma v_\perp/\Omega_0$), the particles under deceleration pass over to a circumference of a smaller radius. The rotational frequency of these electrons is increasing. Respectively, the electrons under acceleration pass to a circumference of a larger radius, and their rotational frequency decreases. Thus, the electrons under deceleration come on in azimuthal direction to the electrons that have gained in energy. As a result, a bunch of electrons is formed. If $\omega = \Omega_0/\gamma$, the bunch is rotating synchronously with the electric field of the external electromagnetic wave. Under the condition $\omega = \Omega_0/\gamma$, the number of the decelerated electrons is approximately equal to the number of the accelerated ones. If $\omega > \Omega_0/\gamma$, the wave slips with respect to the particles. So the bunch gets into the wave decelerating phase; that is, the particles of the bunch transfer in the average their energy to the wave. This mechanism of bunching typical for cyclic accelerators is illustrated in Fig. 9.3.

As regards beams used in microwave devices, most often a somewhat different pattern of the particle bunching is realized. It is rather not bunching but phasing of rotation. Actually, the bunch geometrical sizes usually exceed Larmor radii of the electron rotation in the external magnetic field. In this case, each electron rotates with respect to its driving center. If no special conditions are prescribed, phases of rotation of the electrons are arbitrary (see Fig. 7.4). Therefore, the total current equals to zero because there always exists an electron rotating in antiphase.

We now suppose that, in addition to the external constant magnetic field, in the system also exists the field of an electromagnetic wave. In this case, electrons behave as it has been described above. That is, the rotation of the electrons that got into the accelerating phase is slowing down. On the contrary, the electrons in the decelerating phase rotate faster. This results in phasing of rotation of all electrons. As a consequence, there arises some nonzero total current (see Fig. 7.4). Exactly as in the previous case, to provide the total current energy transfer to the wave, the wave frequency has to be somewhat higher than the relativistic frequency of electron rotation in the magnetic field.

Generally speaking, electrons in real microwave devices, in addition to the rotational velocity, possess a longitudinal velocity. The resonance frequency ($\omega = kv + \Omega_0/\gamma$) depends not only on the energy and transverse velocity of the particles but on their longitudinal velocity as well. Under such conditions, there occurs the particle bunching both in the azimuthal direction, examined above, and in the longitudinal one. The process of bunching just slightly differs from the corresponding physical pattern, already described. However, one should take into account the existence of the longitudinal bunching, which can change the sign of the phase relations of the wave to the particle. In particular, it is evident that a certain condition exists, under which the wave frequency must be smaller than the particle rotational frequency in the magnetic field ($\omega < \Omega_0/\gamma$), which would provide the particle energy extraction. That is, the condition of particles bunching can change its sign (see Chap. 7).

9.3 Particle Interaction with Large Amplitude Wave

In what follows we discuss certain general features of the wave–particle interaction in a uniform magnetic field which are of importance for different types of CRM as well as for particle acceleration by a high-frequency field.

9.3.1 Averaged Equations of Motion

Let us consider a charged particle moving in an external constant magnetic field $\mathbf{B}_0$ directed along the z–axis. In addition, the particle is influenced by the wave field of an arbitrary polarization:

$$\mathbf{E}\exp(\mathrm{i}\mathbf{kr} - \mathrm{i}\omega t) \ ; \quad \mathbf{B} = \frac{c}{\omega}[\mathbf{kE}]\exp(\mathrm{i}\mathbf{kr} - \mathrm{i}\omega t) \ . \tag{9.25}$$

Not losing generality, one can suppose that only two components of the vector $\mathbf{k}$ (k_x and $k \equiv k_z$) are nonzero. In what follows we measure time in units of ω^{-1}, the velocity in units of c, the wave numbers in units of ω/c, and the momentum in units of mc. We also introduce the dimensionless field $\mathbf{g} = q\mathbf{E}/mc\omega$. Respectively, the equations of the particle motion are reduced to

$$\begin{aligned}
\dot{\mathbf{p}} &= \left(1 - \frac{\mathbf{kp}}{\gamma}\right) \mathrm{Re}\,(\mathbf{g} \exp \mathrm{i}\psi) + \frac{\Omega_0}{\gamma}\,[\mathbf{pe}] + \frac{\mathbf{k}}{\gamma} \mathrm{Re}\,(\mathbf{pg}) \exp \mathrm{i}\psi\ ; \\
\dot{\mathbf{r}} &= \mathbf{p}/\gamma\ ; \\
\dot{\psi} &= \mathbf{kp}/\gamma - 1\ .
\end{aligned} \tag{9.26}$$

Here $\mathbf{e} \equiv \mathbf{B}_0/B_0$, $\Omega_0 \equiv qB_0/mc\omega$, $\psi = \mathbf{kr} - t$.

The point to be made is that the field dimensionless amplitude g coincides with wave strength parameter [53]. They also call this value “the parameter of nonlinearity” or “the wave acceleration parameter.” By the order of magnitude, this parameter is equal to the ratio of the work performed by the wave on the particle within the distance equal to the wavelength to the particle rest energy. Being small, it is also equal to the ratio of the particle oscillation velocity in the wave field to the velocity of light.

We now multiply the first equation in (9.26) by $\mathbf{p}$, also taking into account that $p^2 = \gamma^2 - 1$. Thus, one gets the following equation for the particle energy:

$$\dot{\gamma} = \mathrm{Re}\,(\mathbf{vg}) \exp \mathrm{i}\psi\ . \tag{9.27}$$

Then (9.26) yields the integral of motion:

$$\mathbf{p} - \mathrm{Re}\,(\mathrm{i}\mathbf{g} \exp \mathrm{i}\psi) + \Omega_0\,[\mathbf{re}] - \mathbf{k}\gamma = \mathrm{const}\ . \tag{9.28}$$

The integral of motion (9.28) represents the generalized form of the integral, derived in [28, 29]. the direction between $\mathbf{k}$ and the external magnetic field is arbitrary and the field strength parameter g is taken into account.

For the further calculations, it is convenient to make transition to the new variables $p_\perp$, $p_\parallel$, ϑ, ξ, and, η. They are related to the former ones as

$$\begin{aligned}
p_\mathrm{x} &= p_\perp \cos\vartheta\ ; \\
p_\mathrm{y} &= p_\perp \sin\vartheta\ ; \\
p_\mathrm{z} &= p_\parallel\ ; \\
x &= \xi - \frac{p_\perp}{\Omega_0} \sin\vartheta\ ; \\
y &= \eta - \frac{p_\perp}{\Omega_0} \cos\vartheta\ .
\end{aligned} \tag{9.29}$$

Taking into account the integral (9.26), one can rewrite (9.28) in the new variables:

$$\begin{aligned}
\dot{p}_\perp &= (1 - kv) \sum_n \left(g_\mathrm{x} \frac{n}{\mu} J_n - g_\mathrm{y} J_n'\right) \cos\vartheta_n + k_\mathrm{x} v g_\mathrm{z} \sum_n \frac{n}{\mu} J_n \cos\vartheta_n\ ; \\
\dot{\vartheta} &= -\frac{\Omega_0}{\gamma} + \frac{(1 - kv)}{p_\perp} \sum_n \left(g_\mathrm{x} J_n' - g_\mathrm{y} \frac{n}{\mu} J_n\right) \sin\vartheta_n \\
&\quad + \frac{k_\mathrm{x} v_\perp}{p_\perp} g_\mathrm{y} \sum_n J_n \sin\vartheta_n + \frac{k_\mathrm{x} v}{p_\perp} g_\mathrm{z} \sum_n J_n' \sin\vartheta_n\ ;
\end{aligned} \tag{9.30}$$

$$\dot{p}_{\|} = \sum_n \cos\vartheta_n \left[g_{\mathrm{z}} J_n + (kv_{\perp} g_x - k_{\mathrm{x}} v_{\perp} g_{\mathrm{z}})\right] \frac{n}{\mu} J_n - kv_{\perp} g_{\mathrm{y}} J_n' \ ;$$
$$\dot{\xi} = -\frac{1}{\Omega_0} \sum_n J_n \sin\vartheta_n \left[g_{\mathrm{y}}(1-kv) + \frac{n}{\mu} k_{\mathrm{x}} v_{\perp} g_{\mathrm{y}}\right] \ ;$$
$$\dot{\gamma} = \sum_n \cos\vartheta_n \left[J_n \left(v_{\perp} g_{\mathrm{x}} \frac{n}{\mu} + v g_{\mathrm{z}}\right) - v_{\perp} g_{\mathrm{y}} J_n'\right] \ ;$$
$$\dot{z} = v \ .$$

In deriving (9.30), the use is made of the expansion:

$$\cos(x - \mu\sin\vartheta) = \sum_{n=-\infty}^{\infty} J_n(\mu)\cos(x - n\vartheta) \ . \tag{9.31}$$

Let us investigate the case of small amplitudes of the electromagnetic wave ($g \ll 1$). Respectively, the particle effectively interacts with the wave if one of the resonance conditions takes place:

$$\Delta_s(\gamma) \equiv kv + s\frac{\omega_0}{\gamma} - 1 = 0 \ . \tag{9.32}$$

Regarding (9.32) as fulfilled, we also introduce the resonance phase $\vartheta_s = s\vartheta - t$. After averaging, the system (9.30) yields the following equations of motion:

$$\begin{aligned}
\dot{p}_{\perp} &= \frac{1}{p_{\perp}}(1-kv) W_s g \cos\vartheta_s \ ; \\
\dot{p}_{\mathrm{z}} &= \frac{1}{\gamma} k W_s g \cos\vartheta_s \ ; \\
\dot{\vartheta}_s &= \Delta_s \equiv kv + s\frac{\Omega_0}{\gamma} - 1 \ ; \\
\dot{\gamma} &= \frac{g}{\gamma} W_s \cos\vartheta_s \ .
\end{aligned} \tag{9.33}$$

Here

$$W_s \equiv \alpha_{\mathrm{x}} p_{\perp} \frac{s}{\mu} J_s - \alpha_{\mathrm{y}} p_{\perp} J_s' + \alpha_{\mathrm{z}} p_z J_s$$

where $\alpha_{\mathrm{x,y,z}}$ are the components of the wave polarization unit vector. In (9.33, the last equation follows from the other ones. In deriving (9.33)), the terms proportional to $\Delta_s g$ have been neglected.

It is worth to note that the system (9.33) is derived after averaging over varying quickly phases (nonresonant ones). Resonances at various harmonics of the cyclotron frequency can take place depending on the wave and particle parameters. That is, generally speaking, s can be an arbitrary integer number. However, if the wave is propagating strictly along the constant external magnetic field, one can neglect its transverse structure and one should put $k_{\mathrm{x}} \to k_{\mathrm{y}} \to 0$. Respectively, $\mu \to 0$. Consequently, only the terms that describe the resonances $s = 0, \pm 1$ remain nonzero. They correspond to Cherenkov resonance and also to resonances with normal and anomalous Doppler effect. Thus, the cyclotron frequency harmonics are driven by the transverse inhomogeneity of the wave.

9.3.2 Qualitative Analysis

Giving analysis to the above–derived equations, which describe dynamics of the particle motion even in the simplified form (see (9.33)), is rather hampered. However, some information can be obtained by examining the integrals of motion (9.28) and the resonance conditions (9.32). Besides, in practice, the dimensionless amplitudes of the waves excited are usually small. Therefore, there exists the possibility of some substantial energy interchange between the wave and particles only under the conditions of their rather prolonged synchronous interaction. In this case, a particle phase $\vartheta_s = s\vartheta - t$ with respect to the wave is of the main interest. Actually, the phase relations, integrals of motion and resonance conditions are depicted by rather simple algebraic expressions.

The starting point is that in the space $\left(\gamma, p_{\|}, p_{\perp}\right)$ the particle can move only in the surface

$$\gamma^2 = p_{\|}^2 + p_{\perp}^2 + 1 \,, \tag{9.34}$$

which is a rotational hyperboloid. One should keep in mind that the particles cannot get into all areas of the surface and stay out of the areas, limited by the inequalities $\gamma < 0$ and $\mathrm{p}_{\perp} < 0$.

Integrals of Motion

The integral (9.28) is presented in the vector form. In reality, one deals with three algebraic relations, that is, with the projections of the integral (9.28) on to the axes of Cartesian system (x, y, z). During the wave–particle interaction, these projections keep on being constant (i.e, they are integrals as well). As regards these integrals, the third one is of especial importance (it is the projection of the integral (9.28) on to z–axis). It can be essentially simplified if we consider an electromagnetic wave propagating strictly along z–axis ($k_x = k_y = 0$, $k_{\perp} = 0$). Besides, averaging over the fast phase $\psi = \mathbf{kr} - t$, simplifies this integral as well. In both the cases, the integral takes the form:

$$p_{\|} - k = p_{\|,0} - k\gamma_0 \equiv C = \text{const} \,. \tag{9.35}$$

In (9.35) the subscript “0” designates the initial values of the longitudinal momentum and energy of the particle.

It is worth mentioning that (9.35) follows from the laws of conservation of energy and momentum at emission of a wave quantum. Really, these laws may be presented in the form:

$$\Delta\gamma = \gamma_0 - \gamma = \hbar\omega/mc^2 \; ;$$
$$\Delta\mathbf{p} = \mathbf{p}_0 - \mathbf{p} = \mathbf{e}\hbar\omega/mcv_{\rm ph} \; .$$

As it is easy to see, if the quantum is emitted along z–axis, one can derive the integral (9.35) by substituting $\hbar\omega$ from the first equation of this system into the second one. Note that the relation obtained does not contain Planck constant, that is, it is classic.

On the plane $\left(\gamma, p_{\|}\right)$, the integral (9.35) takes the form of an equation of parallel straight lines. They differ from one another in values of the constant C. Several of these lines are plotted in Fig. 9.4.

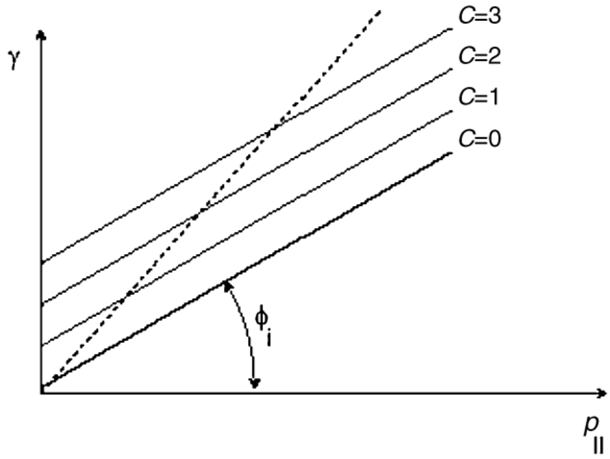

Fig. 9.4. Integral's straight lines in the plane $\left(\gamma, p_{\|}\right)$ for the case $k > 0$

Inclination of these lines of integrals with respect to $p_{\|}$–axis, prescribed by the longitudinal wave number k, is equal to $\arctan k^{-1}$. Running ahead, the resonances in the plane $(\gamma, p_{\|})$ also are represented by straight lines. The angle of their inclination with respect to $p_{\|}$–axis is equal to $\arctan k$. It is easy to see that for a wave propagating strictly along a constant external magnetic field in vacuum, $k = 1$. Respectively, the straight lines that depict these integrals are parallel to the resonance straight lines. In addition, if $C = s\Omega_0$, these straight lines coincide. These particular specificities make the conditions for autoresonance, which is of considerable independent interest.

It is worth depicting the integrals in the space $(\gamma, p_{\|}, p_{\perp})$. One should take into account that in this space the particle can move only over the surface of the rotational hyperboloid. Therefore, the integrals (9.35) can be presented in the form of a line of intersection of this plane with the hyperboloid. If the wave phase velocity along z–axis is lower than the velocity of light ($k > 1$), this curve of intersection takes the form of an ellipse (see Fig. 9.5).

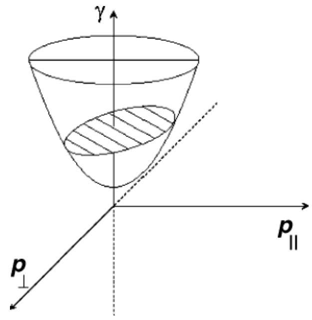

Fig. 9.5. Specific section of the hyperboloid $\gamma^2 = 1 + p_{\|}^2 + p_{\perp}^2$ by the resonant condition plane for the case $k > 1$. The same shape has the section of the hyperboloid by the integral plane for a fast wave ($k < 1$)

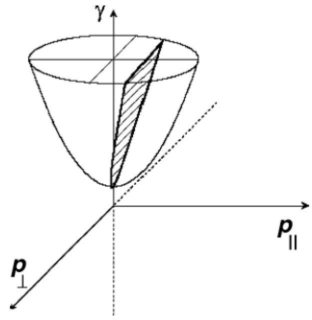

Fig. 9.6. Specific intersection of the hyperboloid (9.34) with a resonant condition plane for the case of particles interacting with a fast wave ($k < 1$). The same shape has the intersection of the hyperboloid with the integral planes for the case $k > 1$ (slow wave)

Otherwise ($k < 1$) the curve of intersection takes the form of a hyperbola (see Fig. 9.6). Even by examining the two plots, one can come to important physical conclusions. In particular, if the particle interacts with the fast wave ($k < 1$), the particle energy, not restricted by the integrals, can reach arbitrary positive values. This is the case when there principally exists the possibility of unlimited acceleration of charged particles. Transfer of a substantial amount of the particle energy to the wave also becomes possible. Surely, the problem of realization of the energy interchange of this type remains open. Below we will consider certain methods to do that. It should be noted that if the slow wave ($k > 1$) is interacting with the particle, the energy interchange is limited by the ellipse characteristics.

Let us examine the projections of the curves, located in the hyperboloid surface, on to the plane $(\gamma, p_{\perp})$. As it is easy to see, such projection is a second-order curve. Is is presented by the equation:

$$\frac{p_\perp^2}{A^2} + \frac{(\gamma - \gamma_*)^2}{B^2} = 1 \ . \tag{9.36}$$

Here

$$A^2 = \frac{C^2}{k^2 - 1} - 1 \ ; \quad B^2 = \frac{C^2 - k^2 + 1}{(k^2 - 1)^2} \ ; \quad \gamma_* = C\frac{k}{1 - k^2} \ .$$

If $A^2 > 0$ and $B^2 > 0$, (9.36) is an equation of an ellipse with its center being located at the point $p_\perp = 0$, $\gamma = \gamma_*$. In particular, this case is realized if the particle interacts with a slow wave ($k > 1$). One can be interested in determining the conditions under which the particle could transfer its total energy to the wave. If so, $\gamma \to 1$, $p_\parallel \to 0$, and $C = -k$. In this case, the following relation of the particle longitudinal momentum to the particle initial energy has to take place: $p_{\parallel,0} = k\,(\gamma_0 - 1)$. While the particle is transferring its energy to the wave, the particle transverse momentum increasing at the beginning then reaches its maximum, equal to $1/\sqrt{k^2-1}$. After that, it goes to zero.

If the particle interacts with a fast wave ($k < 1$), the curve of intersection of the integral with the hyperboloid is a hyperbola. In this case, the parameter A^2 in (9.36) is negative. The particle energy transfer to the wave is accompanied by the monotonous decrease in the particle transverse momentum. There occurs the total transfer of the particle energy to the wave ($\gamma \to 1$) under the same initial conditions, under which the particle interacts with the slow wave; that is, when $C = -k$ ($p_{\parallel,0} = (\gamma_0 - 1)\,k$).

As regards the wave–particle energy interchange, the dependence of the particle longitudinal momentum on the transverse one is of interest. Considering the case $k \neq 1$, one can use the equation of the hyperboloid (9.34) and the expression (9.35) for the integral. Correspondingly, this dependence can be presented as

$$\frac{p_\perp^2}{A^2} + \frac{\left(p_\parallel - p_*\right)^2}{B^2} = 1 \ , \tag{9.37}$$

where

$$A^2 = \frac{C^2}{k^2 - 1} - 1 \ ; \quad B^2 = k^2\frac{C^2 - \left(k^2 - 1\right)}{(k^2 - 1)^2} \ ; \quad p_* = -\frac{C}{k^2 - 1} \ .$$

If $k < 1$, (9.37) is the equation of a hyperbola. In this case the particle energy transfer to the wave is accompanied by the simultaneous and monotonous decrease in both the longitudinal and transverse momenta of the particle.

If the particle interacts with a slow wave ($k > 1$), (9.37) takes the form of an ellipse. In this case, the particle energy transfer to the wave can be accompanied by an initial increase in the particle transverse momentum, but after that its magnitude goes to zero.

It is also worth mentioning that the integral (9.35) takes the form of unlimited rays in the plane $\left(\gamma, p_\parallel\right)$ if the particle interacts with a fast wave ($k < 1$). Otherwise, if the particle interacts with a slow wave ($k > 1$), this process is depicted by limited segments of straight lines.

Resonances

In the plane $\left(\gamma, p_{\|}\right)$, the resonance conditions (9.32) as well as the integrals (9.34), take the form of the equations of straight lines (see Fig. 9.4). In this plot, the inclination angle prescribed by the longitudinal wave number k is equal to $\arctan(k)$. In contrast to the integral straight lines, the resonance lines take the form of rays if $k > 1$. If the particle is interacting with the fast wave ($k < 1$), the resonance lines take the form of limited segments of straight lines.

It is worth investigating the resonance conditions in the space $\left(\gamma, p_{\|}, p_{\perp}\right)$. There the resonance lines are the curves of intersection of the hyperboloid (9.34) with the planes of the resonances (9.35). In their typical form, these curves are analogous with the curves of intersection of the hyperboloid with the integrals. The difference is the following: if the particle interacts with the fast wave, they take the form of ellipses (it should be noted that if the hyperboloid intersects the integrals, they are hyperbolas). Otherwise, if the particle interacts with a slow wave, the curves are hyperbolas (if the hyperboloid intersects with the integrals, they are ellipses).

The analytical expression for the projection of these curves of intersection on to the plane $\left(p_{\perp}, p_{\|}\right)$ is analogous with (9.37):

$$
\begin{aligned}
&\frac{p_{\perp}^2}{A^2} + \frac{\left(p_{\|} - p_*\right)^2}{B^2} = 1 \,, \qquad k \neq 1 \,, \quad s^2 \Omega_0^2 \neq 1 - k^2 \,; \\
&p_{\perp}^2 + n^2 \Omega_0^2 \left[p_{\|} - \frac{k s \Omega_0}{1 - k^2}\right]^2 = 0 \,, \qquad s^2 \Omega_0^2 = 1 - k^2 \,.
\end{aligned} \tag{9.38}
$$

Here

$$
\begin{aligned}
A^2 &= \frac{n^2 \Omega_0^2}{k^2 - 1} - 1 \,; \\
B^2 &= \left(\frac{n^2 \Omega_0^2}{k^2 - 1} - 1\right) \Big/ \left(1 - k^2\right) \,; \\
p_* &= \frac{k n \Omega_0}{1 - k^2} \,.
\end{aligned}
$$

For $k < 1$, the first equation in the system (9.38) represents an ellipse. If $k > 1$, this is a hyperbola. If the particle interacts with a fast wave ($k < 1$), the frequency of which exceeds the particle Larmor frequency ($\Omega_0^2 < 1$), there exists a certain resonance number $s < s_{\mathrm{c}} = \sqrt{1 - k^2}/\Omega_0$, which would correspond to a negative value of the denominator of the first term in left-hand side of (9.38). That is, the resonance conditions (9.32) cannot be satisfied under any values of the particle momentum. One can choose certain values of the cyclotron frequency and the wave vector longitudinal component so that the parameter s_{c} would be an integer number. In this case, the plane of the resonance conditions (9.32) becomes tangential with respect to the hyperboloid if $s = s_c$. This case is described by the second equation in the system (9.38).

The analytical expression for the projection of the lines of the hyperboloid intersection with the resonances in the plane $(\gamma, p_\perp)$ can be presented in the form:

$$\frac{(\gamma - \gamma_*)^2}{A^2} + \frac{p_\perp^2}{B^2} = 1 \,, \tag{9.39}$$

where

$$B^2 = A^2 \frac{\left(1 - k^2\right)}{k^2} \,; \qquad A^2 = \frac{\left(s^2 \Omega_0^2 + k^2 - 1\right) k^2}{\left(1 - k^2\right)^2} \,.$$

The particle moves along the integral curves. As it is known, if the amplitude of the wave that interacts with the particle is small ($g \ll 1$), there takes place an effective energy interchange between the wave and the particle under the condition of synchronism, that is, under the resonance conditions (9.32). Therefore, it is worth examining a graph where the resonance and integral curves are presented simultaneously (see Fig. 9.7).

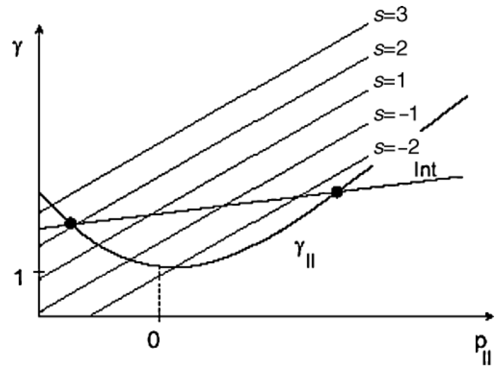

Fig. 9.7. Resonant conditions and the integral in plane $\left(\gamma, p_\parallel\right)$ for the case $k > 1$

For distinctness, this graph illustrates the case of the particle interaction with a fast wave. In this plot are the resonance curves, calculated for $s = 0, \pm 1, \pm 2, \pm 3$, and an integral of the system (9.28). In this very figure, the hyperboloid (9.34) is projected on to the plane $\left(\gamma, p_\parallel\right)$ under the supposition that $p_\perp = 0$ (the corresponding curve is denoted by $\gamma_\parallel$). In particular, Fig. 9.7 indicates that under the given conditions the wave–particle resonance interaction is possible just in the case when $s = 0, \pm 1, \pm 2$.

Suppose that at an initial moment of time the particle is located at the point 0 and then moves along the integral curve (9.35). As one can see, the area of the wave–particle resonance interaction (i.e., the area where the integral intersects the resonances) is small within the framework of the approximation of an isolated resonance. And what is more, Fig. 9.7 does not permit determining sizes of this area because there takes place intersection of the integral with the resonances at one point. However, in fact, if one takes into account nonlinearity, the resonances are characterized by a certain width (see below) proportional to $\sqrt{g}$. It should be mentioned that in the averaged integral (9.35) we have neglected the term proportional to g. The nonlinear resonance width is much larger ($\sqrt{g} \gg g$). Therefore, to determine this width, one can make use of the averaged integral (9.35). These simple qualitative arguments indicate that the synchronous (resonance) interaction between charged particles and the wave is possible only in a limited area of the plane $\left(\gamma, p_{\|}\right)$. Suppose that at the initial moment the particle is in a cyclotron resonance with the wave. As a result of the particle–wave interaction, the particle, moving along the integral curve, quickly leaves this resonance. Under such conditions, any substantial energy interchange between the wave and the particle is hardly possible. However, there exist exceptions: these are the cases of the autoresonance and of the stochastic wave–particle energy interchange. These cases will be investigated below.

Phase Relations

Let us suppose that particle dynamics is restricted by an isolated nonlinear resonance. As it has been demonstrated above, there occurs the synchronous wave–particle interaction in a relatively small area where the integral intersects the resonance. Here we are going to estimate qualitatively effectiveness of this interaction.

To provide the long–term synchronism between the wave and the particle, the resonance condition (9.35) has to be fulfilled. In fact, there occurs the wave–particle interaction in a limited area of the characteristic size L during a limited time interval $T \approx L/v$. The sign of the particle energy transfer to the wave (and v.v.) keeps on being the same all over the interaction area if the phase shift of the rotating electron with respect to the wave $\Delta = (\omega - kv - s\Omega_0/\gamma)\,T$ is smaller than 2π:

$$|\Delta| \leq 2\pi \; . \tag{9.40}$$

One can single out the two factors that cause the phase shift.

1. The phase shift can be conditioned by some initial deviation of the wave frequency from the frequency of the precise synchronism $\omega_r = kv + s\,\Omega_0/\,\gamma$. They call this a kinematic phase shift. It is easy to determine its value:

$$\Delta_{\mathrm{k}} \approx 2\pi N s \frac{(\omega - \omega_{\mathrm{r}})}{\omega_{\mathrm{r}}} \; . \tag{9.41}$$

   In (9.41), the parameter $N = \Omega_0 T\gamma/2\pi$ is introduced. It determines the number of revolutions, performed by the particle during its interaction with the field. Naturally, under the condition of the long–term synchronism this magnitude is large ($N \gg 1$).

2. Besides, the phase shift can be stimulated by the wave influence on the particle motion. The phase shift, conditioned by this effect, is called a dynamic phase shift. Under the influence of the wave, the velocity and energy of the particle change their values. Thus, there takes place deviation from the resonance conditions. As one can readily see, the shift magnitude is

$$\Delta_{\mathrm{d}} = k\left(v_{\mathrm{r}} - v\right) + s\Omega_0\left(\gamma_{\mathrm{r}}^{-1} - \gamma^{-1}\right) . \tag{9.42}$$

Making use of (9.35), one can relate the deviation of the particle velocity to the deviation of its energy. Respectively, the dynamic phase shift (9.42) may be rewritten as

$$\Delta_{\mathrm{d}} = 2\pi N s \frac{\Delta\gamma}{\gamma v_{\mathrm{ph}}} \frac{c^2 - v_{\mathrm{ph}}^2}{v_{\mathrm{ph}} - v_{\mathrm{r}}} . \tag{9.43}$$

In particular, if the wave phase velocity is equal to the velocity of light ($v_{\mathrm{ph}} = c$), then $\Delta_{\mathrm{d}} = 0$, that is, there takes place a total compensation of the phase shift. This is the case of the autoresonance. If at the initial moment the resonance conditions are precisely satisfied (i.e., there does not occur any kinematic phase shift), under the condition of the autoresonance there takes place no phase shift at all.

Knowing the magnitude of the dynamic phase shift and making use of the inequality (9.40), it is easy to determine the admissible change in the particle energy:

$$\frac{\Delta\gamma}{\gamma} = \frac{v_{\mathrm{ph}}}{sN} \frac{v_{\mathrm{ph}} - v_{\mathrm{r}}}{c^2 - v_{\mathrm{ph}}^2} . \tag{9.44}$$

It is worth mentioning that in some devices (e.g., in a gyrotron), the wave phase velocity substantially exceeds the velocity of light. Respectively, as it follows from (9.44), just small changes in the particle energy are possible [54]. At the same time, the wave phase velocity can be close to the velocity of light (i.e., the conditions are close to the conditions of the autoresonance). In this case, this very formula (9.44) indicates that one can essentially change the particle energy ($\Delta\gamma \approx \gamma$).

To characterize effectiveness of the wave–particle energy interchange, let us introduce a parameter of efficiency:

$$\eta = \frac{\Delta\gamma}{\gamma - 1} , \tag{9.45}$$

which is equal to the relative change in the electron kinetic energy. In practice, field interacts with a large number of charged particles, so one has to substitute $\langle\Delta\gamma\rangle$ for $\Delta\gamma$ in (9.45). Here the angular brackets designate averaging over the initial phases of the particles at the entrance to the interaction space. The magnitude η, determined in this way, is called a single–particle efficiency.

Making use of the equation of the rotational hyperboloid (9.34) and of the integrals (9.35), one can derive the following relation of the change in the particle energy to the change in the particle transverse momentum:

$$\Delta\gamma = \frac{1}{2}\frac{v_{\rm ph}\Delta p_{\perp}^2}{v_{\rm ph} - v} . \tag{9.46}$$

In the optimal regime of the wave–particle energy interchange, a particle should lose completely its transverse momentum, that is, $p_{\perp} \to 0$. Then, by substituting (9.46) into (9.45), one gets the following expression for the maximum available efficiency of a microwave device based on the emission of magnetic bremsstrahlung:

$$\eta_{\rm max} = \frac{v_{\perp}^2 v_{\rm ph}\gamma}{(v_{\rm ph} - v)} . \tag{9.47}$$

The maximum effectiveness of the particle energy transfer to the wave requires a certain value of the wave field strength. Really, passing through the region of interaction with the field, the particle gives away the maximum amount of its energy just when its dynamic shift (9.43) is of the order of 2π. If the field strength is lower than the optimal value, the dynamic shift is smaller than 2π and the energy transfer is small. Otherwise, if the field strength exceeds the optimal value, the particle is shifted from the phase of deceleration to the phase of acceleration. Thus, it starts to absorb the wave energy. To estimate the field strength optimal value, one can make use of the fact that the work performed by the wave field on the particle in the interaction region has to be equal to the optimal losses of the particle energy: $A \approx eE\,(2\pi r_{\rm L}N) \approx mc^2\Delta\gamma$, where $r_{\rm L}$ is Larmor radius. Substituting $\Delta\gamma$ from (9.44) into this relation, one can evaluate the optimal field strength:

$$g_{\rm opt} = \frac{\gamma\beta_{\rm ph}\,(\beta_{ph} - \beta)}{2\pi N^2\beta_{\perp}\,(\beta_{\rm ph} - 1)} . \tag{9.48}$$

If there takes place the wave–particle interaction in vacuum and the wave propagates strictly along the external magnetic field, then $v_{\rm ph} \to c$. In this case, the straight lines of the integrals (9.35) are parallel to the straight lines of the resonances (9.32). Under certain initial conditions, these straight lines coincide. Thus, infinitely long synchronous resonance interaction of the particles with the field becomes possible. Really, as (9.43) indicates, the magnitude of the dynamic phase shift goes to zero. Therefore, if the initial conditions are chosen to keep the kinematic phase shift, the total phase shift during the whole time of the wave–particle interaction is equal to zero as well.

9.3.3 Stochastic Regime

At present, microwave electronics tends toward heightening the power of the oscillations excited and toward shortening the wavelength of the waves generated. The two tendencies inevitably result in certain complications of the described processes. Particles interact not with a singled–out mode of the electromagnetic field but with a large number of modes. Really, the equation of balance between the power of the microwave losses in the cavity and the power transferred to the field in the cavity by the electron beam can be written as

$$E^2 V \frac{\omega}{Q} = \eta j U S \ . \tag{9.49}$$

Here E represents the microwave electric field strength in the cavity; Q and V denote the quality factor and volume of the cavity, respectively; j is the beam current density; η is the electron efficiency; the voltage is labeled by U; S reads the beam cross section. As this expression indicates, the increase in the microwave power is achievable in the two ways: either by heightening the current density of the beam or by enlarging the beam cross section.

As we have seen above (see (9.48)), the maximum efficiency is achievable only under certain optimal values of the field strength. Therefore, if one intends to rise the microwave oscillation power by heightening either the current density or the voltage of the beam, this inevitably causes an increase in the wave field strength in the area of the wave–particle interaction. However, this inevitably results in regrouping of the particles and in diminution of the electron efficiency. To avoid these phenomena, one has to heighten the microwave power by enlarging the geometrical sizes of the interaction area (V and S). However, if one enlarges the transverse sizes of the interaction area under a fixed value of the wave frequency, the beam particles interact with high spatial modes of the electromagnetic field. The field structure of the spatial modes, located close to one another, hardly distinguish from the field structure of the desirable wave. Therefore, these modes can be excited as well. Dynamics of the particle in the field of several waves essentially differ from the particle dynamics in the field of a single wave.

There also exists a qualitative difference in dynamics of charged particles if the conditions for several cyclotron resonances can be realized simultaneously. This happens under conditions when the field strength of the wave excited reaches a certain value high enough. As (9.48) indicates, the field strength can be heightened, for instance, when the wave phase velocity is approaching the velocity of light ($\beta_{\mathrm{ph}} \to 1$). Below it will be proved that an increase in the field strength value can cause overlapping of nonlinear cyclotron resonances. Then in the field of a single regular wave, the particle dynamics is determined by a large number of the resonances and becomes chaotic. This fact can play either a positive role or negative role. The advantage is that changes in the particle energy become unrestricted by the width of one resonance. Thus, in principle, there arises the possibility of a substantial heightening of the amount of the particle energy that could be transferred to the wave (in comparison with the case of interaction with an isolated resonance). However, chaotic particle dynamics causes phase scattering with respect to the wave. In its turn, this circumstance can be an additional factor of power stabilization of the excited wave. It is a negative aspect of the interaction with a large number of resonances. Besides, the spectrum of the excited oscillations becomes broader.

Now, we consider the conditions of overlapping of cyclotron resonances. Suppose for the moment that the wave–particle interaction does not influence essentially the particle energy: $\gamma = \gamma_0 + \tilde{\gamma}$, $\tilde{\gamma} \ll 1$. Besides, the resonance condition (9.32) is considered to be precisely satisfied for the particles of the energy γ_0. In this case, after expanding $\Delta_s(\gamma)$ in vicinity to γ_0, the last two equations in the system (9.33) yield a closed system of two equations for $\tilde{\gamma}$ and ϑ_s:

$$\frac{\mathrm{d}\tilde{\gamma}}{\mathrm{d}t} = \frac{g}{\gamma_0} W_s \cos\vartheta_s \, ; \qquad \frac{\mathrm{d}\vartheta_s}{\mathrm{d}t} = \frac{k^2 - 1}{\gamma_0} \tilde{\gamma} \, . \tag{9.50}$$

The system (9.34) describes a nonlinear mathematical pendulum. It yields the nonlinear resonance width:

$$\Delta\dot{\vartheta}_s = 4\sqrt{(k^2 - 1) g W_s / \gamma_0^2} \, . \tag{9.51}$$

It is handy to present this parameter in energy units:

$$\Delta\tilde{\gamma}_s = 4\sqrt{g W_s / (k^2 - 1)} \, . \tag{9.52}$$

To determine a distance between resonances, let us write the resonance conditions (9.32) and the averaged energy conservation law (9.28) for two neighboring resonances:

$$kp_{s+1} + (s+1)\,\Omega_0 - \gamma_{s+1} = 0 \, , \quad \gamma_{s+1} - p_{s+1}/k = C \, ;$$
$$kp_s + s\Omega_0 - \gamma_s = 0 \, , \quad \gamma_s - p_s/k = C \, .$$

One should keep in mind that the value of the constant C is the same for both resonances. Making use of these conditions, one gets the following value of the distance between the resonances:

$$\delta\gamma = \Omega_0 / \left(1 - k^2\right) \, . \tag{9.53}$$

The expressions (9.52) and (9.53) indicate the following. If the inequality

$$g > \frac{\Omega_0^2}{4\left(\sqrt{W_s} + \sqrt{W_{s+1}}\right)^2 (1 - k^2)} \tag{9.54}$$

holds, the sum of half–widths of the nonlinear resonances $(\Delta\tilde{\gamma}_s + \Delta\tilde{\gamma}_{s+1})/2$ is larger than the distance between the resonances $\delta\tilde{\gamma}$. In this case, there occurs overlapping of the resonances.

For practical applications, it could be convenient to rewrite (9.52), (9.53), and (9.54) in dimensional units:

$$\Delta\tilde{\gamma}_s = 4\sqrt{\frac{qEW_s}{mc\,(\omega^2 - k^2c^2)}} \, ;$$
$$\delta\gamma = \omega\Omega_0 / \left(\omega^2 - k^2c^2\right) \, ;$$
$$E > \frac{mc\omega\Omega_0^2}{4q\,(\omega^2 - k^2c^2)\left(\sqrt{W_s} + \sqrt{W_{s+1}}\right)^2} \, ,$$

where

$$W_s \equiv \left(\frac{\alpha_{\mathrm{x}} s p_\perp}{mc\mu + \alpha_{\mathrm{z}} p_{\mathrm{z}}/mc}\right) J_s(\mu) - \frac{\alpha_{\mathrm{y}} p_\perp}{mc} J_s'(\mu)\ ; \quad \mu \equiv \frac{k p_\perp}{m\Omega_0}\ .$$

The expression (9.52) for the nonlinear resonance width and the condition (9.54) for the emergence of the particle motion stochastic instability are rather general. They describe the most important cases of the wave–particle resonant interaction. Really, (9.52) yields the nonlinear resonance width in the cases of the field–particle Cherenkov interaction ($s = 0$), for the cyclotron resonances ($k_{\mathrm{z}} = 0$) and for nonlinear resonances Doppler normal ($s > 0$) and anomalous ($s < 0$) effect taken into account. Respectively, (9.54) describes the condition for the emergence of the stochastic instability, conditioned by overlapping of the corresponding nonlinear resonances.

We dwell now on certain specific cases.

1. Let us consider first of all the charged particle interaction with the longitudinal wave in a constant magnetic field. Under such conditions, a criterion of emergence of chaotic motion is determined in [55, 56, 57]. Corresponding expressions can be derived from (9.54). Surely, using (9.33) in the case of the longitudinal wave ($\alpha_{\mathrm{x}} = k_{\mathrm{x}}/k$, $\alpha_{\mathrm{y}} = 0$, $\alpha_{\mathrm{z}} = k_{\mathrm{x}}/k$) and taking into account the resonance conditions ($s\Omega_0 + kp_{\mathrm{z}} = \gamma$) one gets $W_s = \gamma J_s(\mu)/k$. Under the supposition that $\mu \gg 1$, (9.54) yields the following condition for emergence of the stochastic instability, stimulated by overlapping of the Cherenkov resonance ($s = 0$) with the neighboring Doppler-shifted resonances:
$$g > \frac{\Omega_0^2\sqrt{\mu}}{\gamma\,(1-k^2)}\,\frac{1}{16}\sqrt{\frac{\pi}{2}}\ . \tag{9.55}$$
2. Consider now a transverse electromagnetic wave propagating perpendicularly to the external magnetic field. In this case, overlapping of resonances is conditioned by relativistic effects only.

   For an E–wave with polarization $\{\alpha\} = (0, 1, 0))$, the criterion of overlapping is
$$g > \frac{\Omega_0^2}{16 p_\perp J_s'(\mu)}\ . \tag{9.56}$$
   Note that it is independent of the longitudinal velocity.

   For an H–wave ($\{\alpha\} = (0, 0, 1)$), (9.54) takes the form:
$$g > \frac{\Omega_0^2}{16 p_{\mathrm{z}} J_s(\mu)}\ . \tag{9.57}$$
   In contrast to the case of the E–wave, the value of the amplitude, required for the development of the stochastic instability, essentially depends on the magnitude of the particle longitudinal momentum.

3. Let us stay now on the condition (9.54) considering the particle motion in the field of a plane–polarized wave, propagating at the angle φ with respect to the external magnetic field in a medium, characterized by a dielectric constant $\varepsilon > 1$.

   As regards overlapping of Cherenkov resonance ($s = 0$) with neighboring cyclotron resonances in the E–wave field $\{\alpha\} = (\cos\varphi, 0, \sin\varphi)$, the condition (9.54) takes the form:

$$g > \frac{\Omega_0 v_z}{16 J_0(\mu)\,\gamma\,(1 - v_z^2)\sin\varphi} . \tag{9.58}$$

   In the case of the H–wave field (9.54) looks as

$$g > \frac{\Omega_0 v_z^2}{16 J_1(\mu)\,p_\perp\,(1 - v_z^2)} . \tag{9.59}$$

   The formulae (9.58) and (9.59) indicate that an increase in the particle longitudinal velocity heightens the amplitude sufficient for overlapping of resonances.
4. Particular attention should be given to the case of a longitudinal wave propagating in vacuum. Here $k = 1$, and there is no stochastic instability within the framework of the given approximation. The resonance condition now coincides with the integral of motion (see (9.53b)). Changes in the particle energy, which result from the wave–particle interaction, do not cause any violation of the resonance condition. That is, the conditions of autoresonance [48, 49] are realized. So, one may state that the stochastic instability of the particle motion does not develop under the conditions of autoresonance.
5. From the viewpoint of stochastic acceleration, one could be interested in the case of a high–energy particle ($\gamma \gg 1$) interacting with a plane E–wave ($\{\alpha\} = (0, 1, 0)$) propagating perpendicularly to the external magnetic field ($k = 0$). For simplicity, the particle longitudinal velocity is considered zero ($p_z = 0$). Besides, we suppose that there takes place the wave–particle interaction at high cyclotron resonances ($s \gg 1$). The last condition corresponds to the case of the particle stochastic acceleration in the wave field, the frequency of which substantially exceeds the cyclotron one ($\omega \gg \Omega_0$). Here the resonance condition has the form: $\Omega_0 = s/\gamma$. As $p_{\perp s} \approx \gamma$, one gets $\mu \approx s \gg 1$. Consequently, the use can be made of the Bessel function asymptotic $J_s(\mu) \approx 0.44 / s^{1/3}$. Then (9.54) yields

$$g > 0.28\,\Omega_0 s^{1/3} . \tag{9.60}$$

   As it follows from (9.60), the wave amplitude, necessary for overlapping of resonances, increases with an increase in the resonance number.

9.4 Nonlinear Regime of Operation

In the previous section, we have demonstrated that if the wave amplitude is high enough so that there takes place overlapping of nonlinear resonances, the motion of charged particles becomes stochastic. One can expect that the amplitude of the excited field can reach this level as a result of the collective instability development. In this case, the motion of charged particles becomes stochastically unstable. Consequently, the system passes on to the regime of exciting stochastic oscillations. Besides, it is quite possible that the noncorrelated chaotic motion of the particles could hamper the instability development. That is, the mechanism for the stochastic instability development could play the role of a mechanism stabilizing the output power level.

To consider the subject, our starting point will be the self–consistent nonlinear problem of exciting microwave oscillations by a system of "cold" in-phase rotators in the coordinate frame where their longitudinal momentum is equal to zero. Besides, at the initial moment, the oscillators possess just the transverse component of their momentum. As above, z–axis is directed along the strength line of a homogeneous constant external magnetic field.

So, the distribution function may be presented as

$$f = \frac{\varrho}{p_\perp} \delta \left(p_\perp - p_{\perp,0} \right) \, \delta \left(p_z \right) \, \delta \left(\vartheta - \vartheta_0 + \Omega_0 t / \gamma \right) \, , \tag{9.61}$$

where ϱ denotes the density.

The complete self–consistent system of Eq. (9.61) describes the electromagnetic radiation emission by the particles. It contains the equations of particles motion and Maxwell's equations for the electromagnetic field proportional to $\exp(\mathrm{i}kz)$:

$$\begin{aligned} &\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = q\mathbf{E} + \frac{q}{mc\gamma} \left[\mathbf{p} \left(\mathbf{B} + \mathbf{B}_0 \right) \right] \; ; \quad \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \frac{\mathbf{p}}{mc\gamma} \; ; \\ &\frac{\partial \mathbf{B}}{\partial t} = -\mathrm{i}c \left[\mathbf{k} \times \mathbf{E} \right] \; ; \quad \frac{\partial \mathbf{E}}{\partial t} = \mathrm{i}c \left[\mathbf{k} \times \mathbf{B} \right] \; ; \quad kE_z = -4\pi\varrho \, . \end{aligned} \tag{9.62}$$

Here $\mathbf{E}$ and $\mathbf{B}$ denote the electric and magnetic field strengths; ϱ is the charge density; q and m designate the charge and rest mass of particles, respectively. Note that the temporal dependence of the field is not singled out as a harmonic one in (9.62). This approach permits to describe temporal evolution of the fields and the motion of charged particles in the stochastic regime (i.e., when the excited fields are characterized by a broad frequency spectrum). The system (9.62) takes into account excitation of a longitudinal electric field, that is, the collective Coulomb field of charged particles.

Regarding the field strengths as harmonic functions of time, one deals with the problem of motion in a prescribed electromagnetic field. If $\varrho \to 0$, there arises the problem of motion of a single charged particle in the external constant magnetic field and in the field of an electromagnetic wave of a prescribed amplitude. This problem has been considered in the previous subsection. In particular, we have determined the conditions of appearance of a stochastic instability in the particles motion (9.54). The complete self–consistent system of Eq. (9.62) can be investigated only by numerical simulations. That has been carried out in [48]. Below we will briefly describe the most important results of this analysis.

The system (9.62) has been analyzed by numerical simulations under various values of the plasma frequency and fixed values of the cyclotron frequency $\Omega_0/\gamma = 0.5$. In this case, the resonance condition is fulfilled for $s = 4$. The field evolution in time, spectra of the excited fields, correlation functions, and evolution of the particles energy distribution have been displayed.

The result of the numerical analysis shows that the most important characteristics of particle dynamics and fields in a self–consistent system can be forecasted analyzing the single-particle dynamics in external electromagnetic fields.

If the density of charged particles is low ($\omega_b = 0.1$), the transverse component of the electric field is mainly excited. At the initial stage of the instability development, there takes place an exponential increase in the transverse electric field amplitude. Further, the amplitude heightening gives way to slow oscillations. These oscillations are stimulated by phase oscillations of particle bunches trapped in the wave field (see Fig. 9.8).

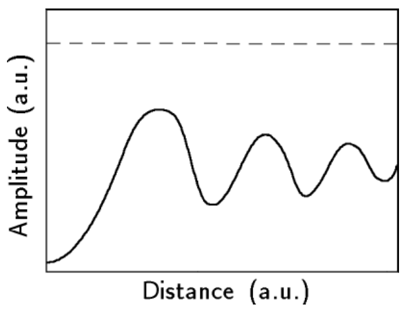

Fig. 9.8. Evolution of the envelope of the microwave transverse field amplitude. The resonances are not overlapping. The dashed line shows the threshold of stochastic instability

In this graph, the dash line designates the field power level, necessary for overlapping of the nonlinear resonances. In the electric field transverse component spectrum, there is a narrow peak at the basic frequency of the oscillations. In addition, there exist two satellites located on both sides of the peak (Fig. 9.9).

The presence of the satellites is conditioned by the wave modulation by phase oscillations of the bunches in the wave field. The correlation function of the transverse electric field oscillates with a slowly decreasing amplitude.

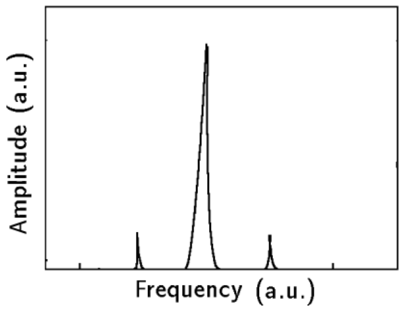

Fig. 9.9. Spectrum of the excited field. The case of an isolated resonance

The temporal dependence of the longitudinal electric field is somewhat more complicated. This is conditioned by superposition of the gyrofrequency harmonics. However, even in this case, there occurs an exponential increase in the wave field amplitude at the initial stage of the instability. Later on, the amplitude starts to oscillate at the frequency of the phase oscillations in the transverse wave field. In the longitudinal electric field spectrum, there exist several narrow peaks at the cyclotron frequency harmonics. The correlation function of the longitudinal field is a slowly decreasing periodic function of the frequency.

Thus, a beam of low density excites regular oscillations, characterized by a discrete spectrum. It is easy to see that the maximal amplitude of the transverse field is smaller than the field strength, necessary for overlapping of the resonances (9.54). Therefore, the particles are locked in an isolated resonance with the wave and their motion is practically regular. Analysis of the function of the particle distribution in energy indicates that the excitation of oscillations is accompanied by a broadening in the distribution function. However, this broadening remains within the limits of the nonlinear resonance width; that is, the particles keep on moving in an isolated cyclotron resonance, not passing on to the neighboring ones.

It is worth mentioning that the system efficiency, determined by the relation

$$\eta = \left(|E|^2 / 4\pi \right) q \Big/ \varrho_0 mc^2 \left(\gamma_0 - 1 \right) \ ,$$

turns out to be rather high. Under conditions above, it reaches 37%.

If one heightens the particle density up to the values that correspond to the condition of nonlinear resonances overlapping, the instability development substantially changes. At the beginning of the process occurs an exponential increase in the transverse electric field amplitude as in the case of an isolated resonance. This increase is limited by trapping of particles by the field of the excited field (see Fig. 9.10; $t \leq 50$).

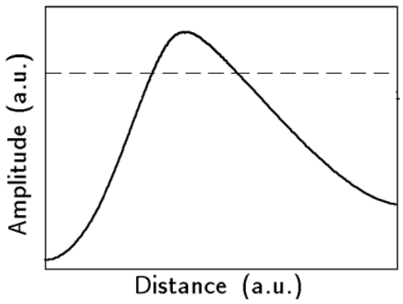

Fig. 9.10. Evolution of the transverse field amplitude envelope. The resonances are overlapping

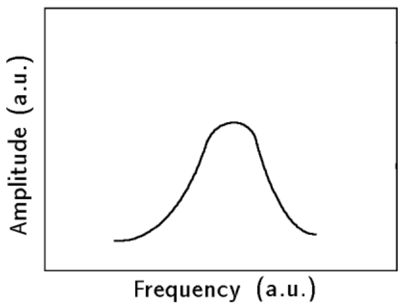

Fig. 9.11. Spectrum of the excited transverse field. The resonances are overlapping

The field power level is approximately twice as high as the level necessary for resonances overlapping. In consequence, the motion becomes chaotic. In its turn, the chaotic motion results in a chaotic modulation of the transverse field amplitude ($50 \leq t \leq 200$) and in appearance of a chaotic longitudinal field. The difference in a degree of their chaos can be explained in the following way. According to (9.62), the temporal evolution of the longitudinal Coulomb field is completely determined by the motion of charged particles. Respectively, the chaotic character of motion causes the self–consistent Coulomb field. The transverse electromagnetic field evolution is described by the inhomogeneous wave equation. Therefore, the beam chaotic current on the right–hand side of this equation can cause only irregular modulations of the transverse field complex amplitude.

The spectrum of the excited oscillations and the evolution of the distribution function correspond to this scenario of the instability development. Although the transverse field spectrum has a maximum at the basic frequency $\omega = 1$, it is substantially broadened. In contrast to the case of a low–density beam, the field correlation function quickly decreases in time. The longitudinal field spectrum is continuous, and it is much broader than the transverse field spectrum. The form of the distribution function indicates that until the moment $t \approx 40$ the instability behaves as in the case of an isolated resonance. However, starting from $t \approx 80$, the particle distribution function seizes several resonances. In addition to the slowed–down particles, there appears a group of stochastically scattered ones. Then the distribution function becomes more and more fuzzy. A number of the accelerated particles increases. However, generally speaking, the decelerated particles predominate over accelerated ones. This chaotic motion of oscillators accompanied by smearing of the energy distribution function causes a limitation of the self–consistent field. And what is more, starting from the moment $t \approx 225$, the field average amplitude corresponds to the value necessary for overlapping of the resonances. This means, in the long run, that the level of the field saturation turns out to be prescribed by the condition of overlapping rather then by trapping of the particles.

It is worth mentioning that even if the beam density is low so that the particles are under action of a single isolated resonance, their dynamics can also become chaotic. Really, the resulting self–consistent field is the wave field, the amplitude of which varies periodically in time. The particle motion in the field of this kind is equivalent to the motion in the field of three waves, the frequencies of which differ by frequency value of the bounce oscillations of the trapped particles. The wave amplitude values are large enough to provide nonlinear resonances of the three waves overlapping. Under these conditions, dynamics of the particle motion has to be chaotic. In its turn, the chaotic character of the particle motion has to cause smoothing down of the amplitude of modulation of the wave excited. However, numerical simulations indicate that the period of the wave modulation is much larger than the period of bounce oscillations. The reason of such prolonged maintenance of the regular modulation of the amplitude of the wave is the following. During the process of bunching, the particles mainly become bunched in the area of the phase space which corresponds to “an island” of stochastic stability of the particle motion. Finally, their motion does become chaotic but the amplitude of the excited oscillations decreases and is subjected just to small incidental modulations.

The principal attention has been paid above to the description of the interaction of charged particles with electromagnetic waves under the conditions of their cyclotron synchronism. No great attention has been paid to the operation of real microwave devices, which is a subject of extensive literature. Among them, gyrotrons and cyclotron autoresonance masers (CARMs) are of a special interest. As regards gyrotrons, the straight lines of the integrals are perpendicular to the resonance straight lines. As the particles move along the integral curves, there does not take place any energy interchange between the particles and waves within the framework of the small–signal approximation. The energy interchange is possible only when one regards finiteness of amplitudes of the waves. Under the conditions of an isolated resonance, the maximum amount of the particle energy transferred to the wave is of the order of magnitude of the nonlinear resonance width.

In CARMs, the resonance straight lines are parallel to the straight lines of the integrals. There can take place the infinitely long resonant field–particle interaction. Limitations on the magnitude of the energy transferred (either from the particles to the wave field or v.v.) are prescribed by the two reasons. It could be either depletion of an energy source or geometrical sizes of electrodynamic structures, where the field–particle interaction occurs. Notwithstanding the circumstance mentioned above, effectiveness of the gyrotron operation is all the same rather high reaching in practice tens of percent.

The above–studied physical mechanism of the field–particle cyclotron interaction permits to describe qualitatively new modes of CRM operation, that is, the stochastic regimes. The stochastic mechanisms indicate themselves more and more often while the power of the oscillations excited increases and one is advancing into a range of shorter wavelengths. Besides, this very approach can be used for deeper understanding of various processes, that is, particle acceleration and mechanisms of stochastic heating of an ensemble of charged particles [48]. In particular, not long ago, by making use of the above–described mechanisms, plasma heating up to high temperatures ($\approx 1.5\,\mathrm{MeV}$) has become possible, the effectiveness being rather high ($\approx 50\,\%$) [58, 59]. Probably, there exists no alternative to the described mechanism of stochastic heating. Really, there occurs a direct transformation of the regular wave energy to the energy of the particles chaotic motion without any intermediate stages.

An important conclusion, which can be made from the above-presented results, consists in the fact that, considering the dynamics of a single particle, one can describe correctly the entire physical picture of interaction of a flow of charged particles with the electromagnetic waves. That is, within the framework of the single–particle model, the levels of excited oscillations can be determined as well as thresholds for appearing the chaotic particle dynamics. Determined in this way the thresholds and levels are in fairy good agreement with the results of numerical simulation. In addition, having considered the case of an isolated resonance shows certain modulation at the bounce frequency. This modulation taken in mind, one can conclude that the analysis of single–particle dynamics produces not only correct qualitative results but also quantitative esteems of transition from the regular particle dynamics into the chaotic one. This enables one to determine the amplitude saturation level of the wave, the shape of the energy distribution function, and the main statistical characteristics of the excited field (spectra, correlation functions, dispersion, etc.).