A study of the quantum-to-classical transition in gravity, and a study of the consequences of constraints in gauge theory path-integrals

by

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B.Sc., The University of Victoria, 2015
M.Sc., The University of British Columbia, 2017

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

in

The Faculty of Graduate and Postdoctoral Studies

(Physics)

THE UNIVERSITY OF BRITISH COLUMBIA

(Vancouver)

July 2021

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A study of the quantum-to-classical transition in gravity, and a study of the consequences of constraints in gauge theory path-integrals

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Abstract

In this thesis we discuss various aspects of low energy quantum gravity from a number of different angles. The ultimate goal we have in mind is to prepare ourselves for the upcoming wave of low-energy experiments which may test quantum gravity.

In the first part of this thesis we remain within "conventional" quantum theory. We start with a study of quantum decoherence via the emission of low energy gravitational radiation. We find that after sufficiently long times this radiation can completely decohere a matter system. In studying decoherence we needed a better understanding of gauge invariance and physical states in path-integrals with prescribed boundary data. We generalize the standard Faddeev-Popov procedure to fit this purpose, and in doing so we better understand the nature of electric fields around quantum charges. The analogous work is also done in linearized quantum gravity. This language is useful for analyzing the debate around a recently proposed gravitational-entanglement experiment. We do such an analysis, and ultimately agree that these experiments indeed test conventional quantum gravity. As a tangential project we study the interactions of quarks in a background gluon condensate, and show how this can cause confinement.

In the second part of the thesis we study an "alternative" quantum gravity theory, the Correlated WorldLine (CWL) theory. We study the theory perturbatively, and also make use of a large-N expansion to study it non-perturbatively. We apply our results to physical systems: verifying that two-path systems experience "path-bunching" which suppresses superpositions of massive objects. We also predict a frequency band in the microhertz range where tests of CWL involving massive objects are expected to see a signature.

Lay Summary

Einstein gave us a theory of gravity described by a flexible spacetime continuum. Quantum physics tells us that small objects are uncertain, fluctuating around in all possible ways. A theory of quantum gravity aims to unify these two theories, but it is quite hard to think about, "what does it mean for space and time to be fluctuating?"

Quantum gravity was long thought to be relevant only at lengths so small we could never test it, but there is another perspective. Perhaps the gravity of heavy objects prevents them from behaving quantum mechanically. Could this be why you and I are in definite locations but an electron is not?

We are currently at the start of an experimental era where these later theories of quantum gravity may actually be testable. In this thesis we study one of these "alternative" theories, and perform calculations to make predictions for these experiments.

Preface

Much of the body of this thesis has been published, or at least submitted for publication elsewhere, and in places we include the material here verbatim.

Chapter 3 is a modified version of J. Wilson-Gerow, C. DeLisle, and P.C.E. Stamp, *A functional approach to soft graviton scattering and BMS charges*, Class. Quantum Grav. **35** 16400 (2018). The initial idea for the paper was my own, arising from several conversations with the other two authors. I performed all of the calculations, except for certain parts of the section on the WKB approximation—this was initially calculated by C. DeLisle and checked by myself. C. DeLisle also identified that our results could be rewritten in terms of the gravitational memory function. P.C.E. Stamp wrote the first draft of the manuscript, and after input from myself and C. Delisle, and a referee report, rewrote much of the draft to make it more suitable for publication. All three authors edited the manuscript.

Chapter 4 is a modified version of J. Wilson-Gerow, P.C.E. Stamp, *Gauge Invariant Propagators and States in Quantum Electrodynamics*, arXiv:2011.05305 [hep-th]. There is some overlap between this work and my Master's thesis J.Wilson-Gerow, *Manifestly gauge invariant transition amplitudes and thermal influence functionals in QED and linearized gravity.* Master's thesis, University of British Columbia, 2017. This chapter is a significant extension and generalization of the earlier work in the Master's thesis. The initial idea, all calculations, and initial draft were all my own. P.C.E. Stamp helped to rewrite the initial manuscript to bring it to a form more suitable for publication. P.C.E. Stamp helped put the work into a context within the existing literature, and helped clarify the exposition in many places.

Chapter 5 contains some overlapping material with J.Wilson-Gerow, *Manifestly gauge invariant transition amplitudes and thermal influence functionals in QED and linearized gravity.* Master's thesis, University of British Columbia, 2017. The technical results reported in section 5.1 were also reported in the Master's thesis, however they have since been derived in a slightly different manner. The newer calculations are completely analogous to those presented in Chapter 4, and so they are not written explicitly in chapter 5.

Chapter 6 is a modified version of J. Wilson-Gerow, *Yang-Mills Gauss law and the heavy quark binding energy in the presence of a dimension-2 gluon condensate*, arXiv:2011.05312 [hep-ph]. This work was entirely my own.

Chapter 8 contained significant overlap with A.O. Barvinsky, J. Wilson-Gerow, and P.C.E. Stamp, *Correlated worldline theory: Structure and consistency*, Phys. Rev. D **103**, 064028, 2021. This work was initiated by P.C.E. Stamp and A.O. Barvinsky, who first performed the calculations. The manuscript was also written by those authors. My contribution was to re-do all of the calculations myself to ensure that we obtained consistent results. The work presented in Chapter 8 is not verbatim, it is my own re-telling of the work.

Chapter 9 has not appeared elsewhere, and is entirely my own work.

Chapter 10 has considerable overlap with the paper J.Wilson-Gerow and P.C.E. Stamp, *Paths and States in the Correlated Worldline Theory of Quantum Gravity*, arXiv:2011.14242 [gr-qc]. Although P.C.E. Stamp contributed to the version on the arXiv, the version presented here represents my original draft which was written without contribution from P.C.E. Stamp, except for the suggestion that the theory may be renormalizable.

Chapter 11 has not appeared elsewhere, and is entirely my own work.

Table of Contents

Appendices

List of Figures

Acknowledgments

Firstly, I thank my supervisor P.C.E. Stamp. His supervisory style is quite 'hands-off', and he offered me an incredible amount of space to pursue my own interests, while also being generous enough to invite me into his own collaborations.

I also thank my supervisory committee: Jess McIver, Gordon Semenoff, William Unruh, and Mark Van Raamsdonk, all of whom have taught one of my graduate courses and/or instructed a course for which I was a TA.

I have benefited enormously from conversations with many great physicists. I thank Abhay Ashtekar, Markus Aspelmeyer, Andrei Barvinsky, Daniel Carney, Laurent Chaurette, Yanbei Chen, Clifford Cheung, Colby DeLisle, Alexei Kitaev, Alex May, Dominik Neuenfeld, Saul Teukolsky, and Roger Penrose. My sincere apologies to anyone I have forgotten here.

This research was supported by a PGS-D award granted by NSERC, and also by an NSERC grant awarded to P.C.E. Stamp.

Dedication

Research is an exhausting task, though one I love. There have been uncountable moments during the last four years when I needed to draw strength, motivation, or inspiration to continue working. I've needed to start early, or work late, or skip a weekend, and sometimes all three for weeks on end. I spent the first 17 years of my life watching my mother Nicole do exactly this, at a job that she did not love, with the singular goal that I might one day have a chance to do better. She has profoundly inspired me, and I cannot stress that enough.

My partner Elle has tempered my jubilation in moments when I (falsely) thought that I had solved nature's greatest mysteries. She has also reminded me of my successes in moments where I doubted the validity of my undergraduate degree. Elle has been incredibly unselfish, and has supported me enumerable times throughout this process. Words fail to describe my appreciation for her as my partner.

Lastly I would like to thank my puppy Onyx. Regardless of how important I thought a project was he has consistently reminded me that: none of this matters, it is just symbols on paper, and we should really go outside and play fetch.

Chapter 1

Introduction

One of the greatest open challenges in physics is to have a single unified framework describing both gravitation and quantum mechanics. The most common approach is to incorporate gravitation into quantum mechanics. Gravitation, however, is quite different from the other fundamental interactions as it describes the structure of spacetime itself. There have then been speculations, originating with Feynman, that gravitation may not even fit within quantum theory, and will instead lead to a breakdown of quantum mechanics [4–11]. We'll refer to these two classes of approaches as "conventional" quantum gravity and "alternative" quantum gravity respectively, reserving 'quantum gravity' to broadly refer to both. Historically, these diametrically opposing attempts at unification have received a great deal of theoretical effort, but without experimental data there has been little to guide this effort. We are now, however, entering an era which has fantastic potential to resolve this issue—the era of low-energy experimental quantum gravity.

There is a variety of experiments, both existing and proposed for the near future, which are expected to be sensitive to signatures of conventional and alternative quantum gravity. Proposals have been made for both Earth-bound and space-based experiments. Just a few examples are: matter-wave interferometry, cavity optomechanics, small-scale Cavendish apparatuses, optical nanoparticle levitation, Bose-Einstein condensate tests, and even gravitational wave detectors [12–26]. The primary utility of these experiments for studying quantum gravity lies in their increasing sensitivity to, and control over, quantum states of increasingly more massive particles. To understand the results of these elegant experiments though, one must make theoretical predictions for physically realistic systems in low-energy quantum gravity.

The overarching goals in this thesis are to better understand quantum gravity. There is a plethora of research directions one can take in pursuit of this goal, ranging from the black-hole information problem to the finer details of "table-top" experiments. With the start of this new experimental era though, we have considered it to be particularly important to focus recent efforts towards the latter. Specifically, we have been working towards making predictions for various tests of quantum gravity. Despite the very 'grounded' goal, it has actually been essential in this program to complete various formal and technical projects relating to eg., gauge invariance, quantum decoherence, and infrared divergences in quantum field theory. Interestingly, this work also led to tangential projects which have implications for fields outside of experimental quantum gravity.

1.1 Conventional Quantum Gravity

1.1.1 Difficulties

Gravitation is described by a field theory, as are all other fundamental forces and particles. One would naturally expect that the framework of quantum field theory (QFT) can then be applied to gravitation, since it works so incredibly well for providing a quantum formulation of all other field theories. While one can do so in a perturbative sense, there is a host of conceptual issues which prevent this from being done for a full quantum gravity theory. All of these issues arise from the fact that gravitation is precisely the phenomenon of a having a dynamical spacetime, and so a quantum gravity theory is a theory where spacetime itself has a quantum description.

Here we'll mention just a few examples of the difficulty. We point to the review article written by S. Carlip for more detail [27].

i) *Microcausality:* In QFT one has a background spacetime on which fields can be defined, and the background metric defines notions of spacelike, timelike, and null separations between points. A fundamental tenet of QFT is that operators which are spacelike separated must commute/anticommute. This is meant to ensure that signals propagate causally in the QFT, and it is quite central to the structure of standard QFT [28]. If the spacetime itself is quantum mechanical though, there is no longer an objective statement about whether two points are spacelike, timelike, or null separated. The microcausality principle simply cannot be imposed as usual.

ii) *Observables:* General Relativity (GR) is invariant under diffeomorphisms (transformations which smoothly move around points on the spacetime manifold). Physical observables are then necessarily invariant under these transformations, and this generically requires observables to be completely nonlocal [29–33].

In classical GR one typically understand this in terms of *relations* between physical systems. Borrowing the example in [27]: we replace coordinate-dependent quantities such as 'the position of the Moon' with coordinate-independent physical quantities, 'the time it takes for a pulse of light to travel to the moon, reflect off a mirror, and return to our lab'. Often these coordinate-independent quantities reduce to measurements of proper time along a geodesic.

In the classical theory we are really using the proper time as an abstraction for a physical quantity. In reality we would have a physical system carried along our worldline, which we could reliably and repeatedly probe. The properties of this system would be chosen so that it functions as a clock. The "proper time along a worldline" is really a statement about observations of this hypothetical physical system following the worldline.

Furthermore, geodesics are useful in classical GR because of the 'probe' limit. If we assume the stress-energy of one object is much smaller than another, then the dynamics of the system is well described by the lighter object following a geodesic of the spacetime metric sourced by the heavy object. Specifying a geodesic is really an abstraction of a physical statement about the relative motion of gravitating objects.

In a quantum theory of gravity, away from the semi-classical limit, these two physical abstractions become incredibly complicated. Firstly, a quantum system which is meant to serve as a clock cannot be repeatedly and reliably probed without disturbing its state, so it is unclear what "proper time" would be referencing here. Furthermore, the dynamics of two gravitating quantum systems will not have a description in terms of definite paths and a definite metric, so it is unclear how one would even try to discuss geodesics.

One can to to make progress by restricting to observables 'at infinity', as is done in scattering theory for asymptotically flat spacetimes [34], or in holography for asymptotically AdS spacetimes [35, 36]. This then leaves the problem of 'bulk reconstruction', ie. to figure out how questions inside the spacetime are answered by the description at infinity [37, 38].

The upshot of this whole discussion is that the standard tools of local QFT do not seem to be useful for understanding a full quantum gravity theory.

1.1.2 Covariant quantization

Despite the difficulties, one can suspend one's disbelief and still try to treat gravity like a quantum field theory. This is the covariant quantization approach, pioneered largely by DeWitt [39–42]. In this approach one assumes the existence of a classical background metric, and treats perturbations about this metric as a quantum field. Ultimately, one attempts to make the approach self-consistent by using the resultant quantum effective action to determine the classical background.

The covariant approach does not evade the conceptual issues, and introduces a number of new technical challenges [27], but it at least provides a starting place to perform calculations.

In this approach one can use standard QFT tools to compute correlation functions of various operators. The resulting correlation functions contain divergences, as is typical in QFT, and one tries to renormalize the results by rescaling various parameters in the Lagrangian. Quantum General Relativity will not allow for such a procedure though, the theory is (at least perturbatively [43]) non-renormalizable. t'Hooft and Veltmann demonstrated that the 1-loop divergence in pure gravity vanishes, but including just a single scalar matter field will reintroduce a non-renormalizable divergence [44]. As a result one must introduce new higher-curvature counterterms into the Einstein-Hilbert Lagrangian to cancel these divergences. At two-loops new non-renormalizable divergences arise which require even more terms be added to the Lagrangian, and at higher loops one expects this to continue [45]. There seems to be no fundamental principle preventing this from continuing *ad infinitum*, introducing arbitrarily many new terms and undetermined parameters into the theory.[1] To gain any predictive power, it seems that a new fundamental principle is required which would specify all of these parameters.

In light of this, there has been enormous effort to go beyond the naive approach to quantization of General Relativity as a field theory. Two such candidates which have received the most attention are string theory and loop quantum gravity.

In loop quantum gravity, the smooth spacetime manifold is abandoned for a more fundamental description in terms of a discrete quantum graph-like structure [46]. General relativity emerges from this theory only after a large scale coarse-graining of the fundamental degrees of freedom. In recent years progress has been made to understand what this short distance structure implies for black holes. A picture is starting to emerge that singularities in quantum black holes may be propped open by the minimal eigenvalue of the area operator, the area gap $\Delta \propto \ell_P^2$ [47, 48].

In string theory, at least historically, one instead starts from a description of quantum mechanical strings propagating in flat spacetime [49]. One finds in the weak-coupling regime that the various modes of excitation of the string can be understood as different types of particle, and that the theory

[1] This is the usual story, however we expand on some subtleties in section 10.3.4.

necessarily contains a mode describing the graviton. Furthermore, one can perform consistency checks on the allowed coherent states in the theory to see that Einstein gravity emerges at low energies. Following this logic one can even compute "stringy-corrections" to predict the coefficients of higher-curvature terms in the effective gravity action [50]. At weak-coupling and high-energies though, examples such as high-energy scattering [49] and closed string T-duality suggest that there is also a fundamental non-locality in string theory [51]. In these examples, strings propagating in very compact regions are gauge-equivalent to strings propagating in very large regions—small length scales don't really exist in the theory.

In both of these theories, the description of nature at short distances seems to be nothing like a classical spacetime manifold.

It is fascinating to study these theories to gain insight into the fundamental nature of the world, but the unfortunate reality is that there is likely no hope of experimentally distinguishing the two. Furthermore, the effective field theory perspective extends this to say that it is very unlikely that we will ever be able to experimentally distinguish between any microscopic quantum theories of gravity, unless some drastically new physics intervenes at energies between current experiments and the Planck energy $E_P \approx 10^{19}\,\mathrm{GeV}$. [52–54],

1.1.3 Effective field theory

Roughly speaking, the first insight of gravitational effective field theory (EFT) is: any microscopic quantum theory, which generates a generally covariant gravity theory at large scales, can be described at low-energies as an *effective* quantum field theory with a Lagrangian containing an expansion in curvature invariants and derivatives thereof,

$$\mathcal{L}_{eff} = \sqrt{g}\left(\frac{1}{\ell_P^2}R + c_1 R^2 + c_2 R_{\mu\nu}R^{\mu\nu} + ...\right) + \mathcal{L}_{matter}. \tag{1.1}$$

This quantum field theory has infinitely many interaction vertices and modifications to the graviton propagator, all parameterized by unknown coefficients c_i.

The second insight of the effective field theory program is that this Lagrangian is already arranged as a derivative (energy) expansion. The coefficients c_1, c_2 above are dimensionless, and higher coefficients will have dimensions of increasing powers of length. One expects the dimensionless coefficients to be of order unity—this would be "natural"—but they may possibly have large numerical values. Given that the only length scale we know to be relevant in gravity is the Planck length, it is also natural to assume that the higher coefficients will be order unity numbers multiplying powers of the Planck length. With these "naturalness" assumptions on the size of numerical coefficients, and also with the assumption that no other relevant length scales emerge before the Planck length, one can straightforwardly conclude that the higher curvature terms will only contribute significantly at very high energies$\sim 10^{19}$ GeV.

It is thus no coincidence that General Relativity, and not some other gravity theory, emerges at low energies from the different microscopic quantum gravity theories—it is the universal low energy limit of quantum gravity.

The perspective shift here is important. First, one is giving up the quest to determine every one of the unknown parameters in the field theory. Furthermore, one accepts that quantum field theory is not even necessarily a fundamental description of nature, instead it is essentially the unique structure which emerges as a low energy description of many body quantum systems [55].[2] Additionally, the higher-derivative "non-renormalizable" divergences which arise in loop calculations are no longer viewed as a sickness of the theory, they merely represent that we have truncated the true theory to a finite order in the derivative expansion. One can confidently renormalize away these terms without worrying about introducing new undetermined coefficients into the Lagrangian—those higher curvature terms are already there, they are just irrelevant for low energy physics.

Given the general structure of the effective field theory one can only speculate about what happens at short distances—the theory loses predictive power when the infinite number of constants c_i become relevant. However, one is free to try and glean intuition from the nature of the higher curvature terms themselves.

The introduction of higher-curvature terms into the action can drastically change the behaviour of the theory at short distances [57]. This can be seen by very naive dimensional analysis. If we consider the just the following Lagrangian,

$$\mathcal{L} = \sqrt{g}\left(\frac{1}{\ell_P^2} R + c_1 R^2\right), \tag{1.2}$$

where R is the scalar curvature, $\ell_P^2 = 16\pi G\, c^{-4}$, and c_1 is a presumably order 1 coefficient, then is is clear that the second term is significant only in regions with curvature $R \sim \frac{1}{\ell_P^2}$. To further this intuition one can actually use this Lagrangian classically to find solutions for the gravitational field

[2]This idea was already clear in condensed matter physics [56].

outside of a static point mass, m. In the linear limit one finds an effective gravitational potential [57]

$$V(r) = -\frac{Gm}{r}(1 - e^{-r/(\ell_P \sqrt{2} c_1)}), \tag{1.3}$$

which is regular at the origin. This is further supported by the fact that there are some novel solutions to the full non-linear equation of motion which have a regular Kretschmann invariant at the origin [57]. One sees the analogous behaviour in the Born-Infeld higher-curvature electrodynamics theory: the electric field of a point charge is everywhere finite [58–60].

The classical theory is certainly not applicable at Planck scales, but from this discussion one still begins to suspect that a full theory of quantum gravity will resolve gravitational singularities. Because of the infinite number of higher curvature terms in the Lagrangian, the theory may even be non-local at small scales—making an interpretation in terms spacetime become quite strange. Loop quantum gravity and string theory already seem to provide explicit examples of this, and we refer to the work of Barvinsky and Vilkovisky for a more thorough discussion of non-locality in effective field theory [61].

The effective field theory approach allows for much more than wild speculation though, it provides a systematic framework for computing universal results for low-energy quantum gravity. In this sense quantum general relativity is a perfectly good theory of quantum gravity, provided one does not try to apply it to length scales of order the Planck length, $\ell_P \approx 10^{-35}\,\mathrm{m}$. Comparing this with the other quantum fields theories in the standard model, quantum GR is actually an exceptionally good theory, the standard model is not expected to be valid for anywhere near this small a scale.

One of the fascinating points highlighted by the EFT program is that not all predictions of quantum general relativity are local corrections to the classical theory, in fact one can find long-range quantum corrections to classical gravity. Indeed, a number of "low-energy theorems of quantum gravity" have been derived [see Donoghue and collaborators' works 62, 63, and refs. therein]. A select few examples include quantum corrections to: the Schwarzschild, Kerr, Reissner-Nordstrom, and Kerr-Newmann black hole metrics [64–66], the gravitational potential between two massive bodies [67], and the bending of light around massive bodies [68, 69]. One remarkable observation to come out of this all was that the leading quantum correction to the black-hole metrics looks precisely like a naive smearing of the black-hole's location over a region the size of its Compton wavelength.

1.2 Alternative Quantum Gravity

As we've mentioned above, the success of the gravitational effective field theory is bittersweet—it essentially guarantees that we will never experimentally distinguish between microscopic theories of quantum gravity. This fact alone would kill experimental efforts in quantum gravity if "conventional" quantum gravity theories were the only options, but that simply is not the case. There are a variety of "alternative" approaches to building a theory of both gravitation and quantum mechanics.

We've seen above that General Relativity does not fit perfectly within quantum theory, at short distances it must undergo drastic changes in order to remain compatible. Alternative quantum gravity theories take a contrary perspective and propose that it is quantum mechanics (QM) which must undergo drastic changes at macroscopic scales so that it stays compatible with General Relativity. One assumes that the theory uniting gravity and quantum mechanics will predict a breakdown of quantum mechanics for sufficiently massive objects.

This idea has quite a long history, with speculations dating back at least to Feynman [4, 70]. Rosen later recognized the potential importance of the Planck mass here, and proposed that perhaps QM becomes a non-linear theory for objects heavier than $m_P \sim 10^{-5}\,\mathrm{g}$ [71]. Shortly after, Károlházy used Heisenberg uncertainty arguments to suggest that spacetime was fundamentally fuzzy below certain length scales and that this would destroy quantum coherence in large systems [5, 6]. The first serious efforts to developing a theory of this type came from Kibble and Randjbar-Daemi's formulation of the Møller and Rosenfeld's semiclassical theory [72–75]. The other major figures in the history were Diósi and Penrose. Diósi built upon the suggestion of Károlházy and developed a model where gravitational "noise" serves to decohere quantum systems [9, 10]. Penrose put forth a separate set of arguments suggesting that, even in the absence of noise, an intrinsic gravitation-induced decoherence must occur [11]. Of course these ideas have since been refined in various ways [see 76–79, and refs. therein].

There are a variety of approaches to gravity-based QM breakdown, and it would take us far beyond our purposes here to accurately review each of their literatures. We refer the reader to some recent reviews for proper detail [76–79]. Broadly speaking we can classify these approaches as: stochastic theories, semiclassical theories, and the Correlated Worldline theory. The new wave of low-energy quantum gravity experiments may be sensitive to signatures of these theories, and in fact, a very recent experiment performed by Donadi et al. has lead the authors to claim to have ruled out stochastic collapse theories [80].

1.2.1 Stochastic Theories

Stochastic gravity theories come in two distinct types. The first type are often referred to as gravitational stochastic collapse models. In these models one retains conventional structures of linear QM, but forgoes the wavefunction for the density matrix and the Schrodinger equation for a Lindblad master equation. The evolution becomes non-unitary for large mass objects because of an introduced interaction with a stochastic "noise" field. The strength and statistical correlations of this noise field are motivated by gravitation [9, 10, 81, 82].

The idea of Penrose can also be fit into this category. Penrose did not actually provide an explicit mechanism, let alone a theory, which implemented his idea of gravitation-induced quantum decoherence. However, if one makes the simplifying assumption that his decoherence process is Poissonian, then one can demonstrate that it must be described by the same model as Diósi had proposed [80].

These models are successful in their quest, in that they generate decoherence which localizes massive objects. They are not actually theories of gravitation though, as they do not consider the back-reaction of the quantum matter onto spacetime. They are only trying to model a possible feature of a larger theory which incorporates gravitation and quantum mechanics.

There are various other limitations with these models, perhaps the most important being that they imply a transfer of energy into the quantum system. The noise field in these models is external to the universe, in the sense that it acts on objects but they do not back-react onto it. Generally speaking, it is completely possible to have quantum decoherence without a system gaining energy [83–85], however decoherence in the position basis will necessarily increase a particle's average kinetic energy—a wide-spread wavefunction has a lower average momentum than does a statistical mixture of very narrow wavefunctions. This is a violation of energy-momentum conservation, a fundamental concept in General Relativity, and it makes it unclear whether such models can really be incorporated into a theory of gravity.

Beyond the issues of energy conservation, the increase in $\langle p^2 \rangle$ as a function of time can be interpreted as a fluctuating acceleration of the particle, and this leads to photon emission [86, 87]. Donadi et al. performed a photodetection experiment, and their null result was used to claim such strong bounds on stochastic collapse models that the authors concluded that "gravity-related collapse of the wavefunction, in its present formulation, is ruled out".

The other type of stochastic gravity theory is referred to as just "stochastic gravity", and it is best to discuss it after introducing the semiclassical models.

1.2.2 Semiclassical Gravity

Semi-classical theories aim to retain classical General Relativity (GR), with the metric instead sourced by the quantum expectation value of the stress-energy tensor,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}\langle\psi|\hat{T}_{\mu\nu}|\psi\rangle, \tag{1.4}$$

and one attempts to self-consistently describe a quantum system evolving on this background. In conventional quantum gravity this equation is often taken as an approximation to gain insight into complicated problems [88], but in the semiclassical gravity theory eq. (1.4) is assumed to be fundamental. Despite this, useful reviews on semiclassical gravity have come from those who view it as an approximation to conventional quantum gravity [89–91].

The result is a non-linear quantum mechanics theory for the matter. For a single non-relativistic particle, the Newtonian approximation of this theory gives the Schrödinger-Newton equation for the wavefunction [92, 93],

$$i\hbar\frac{d\psi(x,y)}{dt} = -\frac{\hbar^2}{2m}\nabla^2\psi(x,t) - Gm^2\int d^3y\frac{|\psi(y,t)|^2}{|x-y|}\psi(x,t). \tag{1.5}$$

This theory has been heavily criticized. Indeed Kibble raised many issues right from its earliest days [75]. A few of the points which have been raised involve issues with: momentum conservation, the uncertainty principle, and superluminal signaling [94]. Some of these arguments are controversial [95, 96], but one serious issue which remains is the that there is an explicit protocol for performing superluminal signaling in the Schrödinger-Newton theory [97]. If this were not enough, it has even been claimed that eq. (1.4) has been falsified by the Page-Geilker experiment [98], though this too is not without disagreement [see 79, and refs. therein].

The issues here arise from the use of a non-linear quantum mechanics theory. Weinberg, Gisin, and Polchinski have each contributed to quite a general discussion about the how issues like this are generic in non-linear quantum mechanics theories [99–103]. The major problem is in retaining the conventional usage of measurements, operators, and states in Hilbert space.

The most pressing of these issues coming from instantaneous projective measurements. Indeed, Unruh has succinctly pointed out that wavefunction collapse would trigger an instantaneous change on the RHS of eq. (1.4) which would be inconsistent with the LHS [104].

We note that in the full theory of conventional quantum gravity one should also be forced to contend with the issue of instantaneous projective measurements as well. Diffeomorphism invariance effectively forces one into a description wherein systems are defined only relative to each other, the relative distances and angles themselves being quantum mechanical quantities [105][3]. This seems to preclude the notion of a projective measurement acting instantaneously on a global state vector. One may then need a genuine physical description of quantum systems wherein measurements do not play a central role.

There are theories which aim to preserve the semi-classical structure as much as possible while ameliorating the superluminal signaling issues. Though using different motivations, these theories converge on the result that one must introduce stochastic noise into semiclassical gravity to try and make it consistent [79, 97, 108–110]. The "stochastic gravity" theory which we previously alluded to is one such example [108, 109, 111].

1.2.3 Correlated Worldline theory

From Kibble's analysis, and the many others which followed, it became clear that a self-consistent theory describing the gravitational breakdown of QM would necessarily abandon much of the familiar structures of QM. The Correlated Worldline (CWL) theory, originating with P.C.E. Stamp, is an attempt at doing this in a controlled manner [112, 113].

The idea is to retain the path-integral from quantum mechanics, but allow for departures from the superposition principle. In a standard path-integral one considers all possible paths for an evolving system. Each of the paths is independent from one another, and their corresponding amplitudes sum together to produce a total amplitude for the process. In CWL theory, one introduces gravitational interactions between these paths. It is claimed that this causes a "path-bunching" to occur, wherein the various paths gravitate together. This mechanism is negligible for small mass objects, so CWL theory reduces to QM in the appropriate limit. For large masses however,

[3]We note that this is a part of a larger, and very important, question of what are the observables in quantum gravity. We'll avoid entering this discussion here, and refer to a few highlights of that literature [29, 30, 106, 107].

it is presumed that path-bunching will then dynamically select out only one classical trajectory for the system.

CWL theory is not a decoherence/wavefunction collapse theory. It was meant to dispense with the idea of operators, Hilbert space, projective measurements, etc., with measurement processes being described as dynamical interactions between small and large systems. The idea is that the large-system will already be "path-bunched" and upon correlating itself with the small system it will effectively cause path bunching in the small system.

The theory is quite new, being introduced by P.C.E. Stamp in 2012 [112], with technical details coming a few years later [113, 114]. Aside from the early work formulating of the theory, we have been involved in all of the recent work on CWL theory, and this will be presented in Part II of this thesis. For this reason we will reserve further discussion of CWL theory until later.

1.2.4 Summary

The majority of physicists believe that conventional quantum gravity is likely the correct description of nature, this author included, however it is scientifically important to seriously explore possible alternatives. There are a number of arguments for this. Firstly, there has been no experimental evidence for conventional quantum gravity, so to be pragmatic one must also consider alternative theories. Secondly, in this exercise one may find the structures of QM and QR so constraining that there is no sensible alternative to quantizing gravity. If instead a self-consistent alternative theory is found, then it deserves to be taken as seriously as the conventional theory. Thirdly, and most importantly in our opinion, alternative theories can provide important goals for experimental benchmarks.

It is of obvious interest to science to push technological limits by preparing and measuring larger and larger quantum systems. Also, the alternative gravity theories all predict a breakdown of quantum mechanics for large objects. With these alternative theories being on the horizon of current experimental capabilities, they serve as motivation to continue this quest, and provide concrete goals to aim for.

It is an extraordinarily exciting time for quantum gravitational physics because of the start of this new era of low-energy experimental tests. Some proposed experiments are even sensitive to signatures of conventional quantum gravity that cannot be mimicked by any alternative theory [13–15, 23, 115]. Although such tests would be completely unable to distinguish between superstring theory and loop quantum gravity, they may potentially determine whether or not gravity is even quantized in the conventional sense or if it instead leads to a breakdown of QM.

1.3 The layout of this thesis

Conventional quantum gravity (CQG) theories all flow to the same universal effective field theory (EFT) at low energies [52]. All CQG predictions for low-energy experiments are then described by one theory with no adjustable parameters. This makes comparison with alternative quantum gravity theories much more straightforward. The alternative theories all suggest a breakdown of standard QM behavior for sufficiently massive objects, either through dynamical wavefunction collapse, path-bunching, or quantum decoherence. In many cases this breakdown may resemble quantum decoherence. Since decoherence can also occur in CQG, it essential to quantify this decoherence to disentangle it from geniune breakdowns of QM.

A popular recent paper [116] claimed to do so, however we spotted a significant error in their work. They did not correctly treat the gauge invariance of quantum gravity, mistaking unphysical for physical variables. In response, we developed a novel method which kept gauge invariance manifest in such path-integral calculations and isolated the true physical degrees of freedom. An immediate application was to compute the correct gauge-invariant Feynman-Vernon influence functional which describes conventional gravitational decoherence. We demonstrated that very large accelerations and lengthy coherence times would be required for matter to decohere via graviton emission, correcting the claims of [116] that this could occur for stationary objects. This work was all reported in the author's Master's thesis [117].

Part I of this thesis is something of a continuation of this work. We had noticed that our previously developed framework was perfectly suited to address the problem of soft gravitons in scattering processes. Despite being experimentally irrelevant, the soft graviton problem had gained considerable attention in the literature after it was claimed that soft gravitons might be involved in resolving the black hole information problem [118]. We then applied our understanding of understand gravitational decoherence to scattering processes [119, 120]. Our results agreed with those derived by Carney et al. using diagrammatic methods [121], that the soft gravitons would completely decohere the out-state of scattered quantum matter. Furthermore, we provided a general framework for computing decoherence in systems with a large separation of scales between the system and environment. In this thesis, after we introduce some of the relevant formalism in chapter 2, the analysis of soft graviton decoherence will be the content of chapter 3.

In chapter 4 we proceed to refine our understanding of gauge-invariant path-integrals, and we develop a generalization of the textbook Faddeev-Popov trick to path integrals with boundary data. We apply this to address problems in QED regarding the nature of gauge-invariant electric field dressings, both for static charges, moving charges, and charge flux through null infinity [122]. In the later case we make connection again with the soft photon problem and clarify the nature of soft-photon dressed states.

In gravity the analogous discussion is quite timely, because of the debate over the relevance of physical states to an upcoming experiment proposing to test conventional quantum gravity [13, 14, 123, 124]. In chapter 5 we discuss the arguments of the various authors, and summarize their main points. We then use our path-integral framework to substantiate the claims of Anastopoulos and Hu [123, 124] that the experiments will not be sensitive to the true gravitational degrees of freedom. Despite this, we later provide a set of arguments suggesting that conventional quantum gravity is the only consistent theory which could produce a positive signature in these experiments.

After familiarizing ourselves with constraints in gauge theory we find a related, but tangential, question in Yang-Mills theory to be quite interesting. In chapter 6 we demonstrate a mechanism through which a vacuum gluon condensate can lead to quark confinement [125]. This work centers on a separation between the constrained and unconstrained degrees of freedom in Yang-Mills theory. Since this work is quite tangential to our original goal of understanding quantum gravity experiments, we soon return to quantum gravity.

In Part II of this thesis we spend quite a lot of time studying the Correlated Worldline theory of quantum gravity. We review some work done in direct collaboration with A.O. Barvinsky and P.C.E. Stamp [126, 127], however we also provide a lot of our own perspectives on the theory.

In chapter 7 we provide a very brief introduction to the CWL theory. In chapter 8 we then set-up a perturbative analysis of CWL to leading order in the gravitational coupling ℓ_P^2. The primary results are: i) conventional quantum gravity loop contributions are suppressed by undetermined parameters of the theory, ii) novel CWL contributions arise, but at this order they simply reproduce conventional quantum gravity tree-level results, and iii) the theory retains the diffeomorphism symmetry of conventional quantum gravity.

In chapter 9 we try to develop our own definition/understanding of the CWL theory in terms of the conventional structure of Hilbert space, operators, measurement, etc. This approach is quite anti-thematic from P.C.E. Stamp's intentions for the theory, however we felt that it was necessary to do in order to bring the theory closer to experimental test. We discuss how QM states and operators are embedded in CWL, fix the previously undetermined parameters, redefine the theory to simplify redundancies, define transition amplitudes, and ultimately attempt to discuss measurements.

In chapter 10 we make use of an enormous simplification of CWL theory—the fact that the theory has an intrinsic large-N limit. We use this to explicitly evaluate the gravitational path integral, and demonstrate the equivalence to an "in-out" semiclassical gravity theory. We use the exact expression for the generating functional to compute CWL corrections to various correlations functions, and discuss the fact that the theory actually seems to be renormalizable.

In chapter 11 we bring together the formal developments to try and make predictions for simple experimentally relevant systems. We show that simple probes of quantum mechanical oscillators will not reveal CWL signatures, and suggest possible alternatives. We then discuss the CWL modifications to a particle propagator. For a particle with a single dominant classical path, we find no non-trivial corrections. We then study a two-path system, and we find significant modification to the dynamics; the path bunching mechanism starts to reveal itself. This is followed-up by a discussion of the gravitationally-mediated-entanglement experimental proposals. We find that CWL will predict a null result for their experiment, in contrast with the nontrivial prediction of conventional quantum gravity. Finally, we apply CWL theory to a many-particle object such as a microscopic solid. The primary result is an estimate of the experimental timescales required to see signatures of CWL theory.

Throughout this thesis we will use units in which $\hbar = c = \epsilon_0 = 1$, and a $-+++$ metric signature.

Chapter 2

A brief technical introduction

In this thesis we are ultimately motivated by the theoretical side of experimental tests of quantum gravity, however we contribute to this from a variety of different angles. As discussed in chapter 1, we study: physical systems in conventional quantum gravity, formal questions in QED and quantum gravity, physical systems in Yang-Mills theory, and a range of formal and practical questions in the Correlated Worldline theory. It would be quite disjointed if we introduced here the relevant technical material for each of these studies. Instead, we will cover some basic ideas which appear multiple times throughout the thesis. The formalism relevant for each chapter will later be introduced as necessary.

2.1 Density matrices and Closed Time Path (CTP) evolution

2.1.1 Basics

In quantum mechanics, a pure state of a system is described by a state vector $|\psi\rangle$ in a Hilbert space. Observables are described by self-adjoint operators $\mathcal{O}_a$ on this Hilbert space. These operators can be decomposed into their respective eigenbases as $\mathcal{O}_a = \sum_j \lambda_a^j |\lambda_a^j\rangle\langle\lambda_a^j|$, where each eigenvalue λ_a^j corresponds to a potential outcome of measuring the observable $\mathcal{O}_a$, and the eigenstates are orthonormal $\langle\lambda_a^j|\lambda_a^k\rangle = \delta^{jk}$. One can compute the probability of obtaining the outcome λ_a^j, when measuring the observable O_a on a system in state $|\psi\rangle$, by computing the bilinear function $\langle\psi|\lambda_a^j\rangle\langle\lambda_a^j|\psi\rangle$. Using this one can compute the expected value (mean), $\langle\psi|\mathcal{O}_a|\psi\rangle$, for measurements of the observable $\mathcal{O}_a$.

If the state of a system is not known exactly, but the system is understood to be in any of the pure states $\{|\psi_k\rangle\}$ with probabilities p_k, then one says the system is in a mixed state. Mixed states are described using the density matrix

$$\rho = \sum_k p_k \, |\psi_k\rangle\langle\psi_k|. \tag{2.1}$$

It is straightforward to see that observables are appropriately computed as

$$\langle \mathcal{O}_a \rangle = \mathrm{Tr}\left(\rho \mathcal{O}_a\right). \tag{2.2}$$

One particularly useful feature of the density matrix is its connection with the probability density function in classical physics. Suppose we we're interested in an observable $\mathcal{O}$ for a single quantum particle. We can write eq. (2.2) in the position basis,

$$\langle \mathcal{O} \rangle = \int dx dx' \rho(x, x') \langle x' | \mathcal{O} | x \rangle, \tag{2.3}$$

where

$$\rho(x, x') \equiv \langle x | \rho | x' \rangle. \tag{2.4}$$

If $\mathcal{O}$ is a position space observable $\mathcal{O} = \mathcal{O}(\hat{x})$, then this simplifies considerably

$$\langle \mathcal{O} \rangle = \int dx \rho(x, x) \mathcal{O}(x), \tag{2.5}$$

and it is clear that the diagonal elements of the density matrix are just classical probabilities. Off-diagonal density matrix elements then encode some information about the quantum nature of the system. When the off-diagonal elements of a density matrix decay with time, we say that the system is undergoing quantum decoherence.

Another useful feature is for thermodynamics. Consider a system at temperature T described by the canonical ensemble. The thermodynamic average of an observable $\mathcal{O}$ is then

$$\langle \mathcal{O} \rangle = Z^{-1} \sum_n e^{-\beta E_n} \langle n | \mathcal{O} | n \rangle, \tag{2.6}$$

where $Z = \sum_n e^{-\beta E_n}$, $\beta = (k_B T)^{-1}$ and $\{|n\rangle\}$ are the energy eigenstates of the system. This can be conveniently rewritten in terms of the Hamiltonian,

$$\langle \mathcal{O} \rangle = Z^{-1} \mathrm{Tr}\big(e^{-\beta \hat{H}} \mathcal{O}\big), \tag{2.7}$$

from which it is clear that a thermal state in quantum theory is described by a density matrix

$$\rho_\beta = \frac{e^{-\beta \hat{H}}}{Z}. \tag{2.8}$$

2.1.2 Multi-partite systems

Consider two subsystems, A and B, and assume they are described by a pure state $|\Psi\rangle$. If it is possible to factor this state into a product of two states, ie.

$$|\Psi\rangle = |\phi\rangle \otimes |\chi\rangle, \tag{2.9}$$

then the systems are defined to be not entangled, otherwise they are entangled and are instead described by

$$|\Psi\rangle = \sum_i c_i |\phi_i\rangle \otimes |\chi_i\rangle, \tag{2.10}$$

where $\{|\phi_i\rangle\}$ and $\{|\chi_i\rangle\}$ are sets of orthonormal states for the subsystems A and B respectively. A canonical example of an entangled state is the Bell state for spins

$$|\psi\rangle = \tfrac{1}{\sqrt{2}}\big(|\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle\big). \tag{2.11}$$

Suppose we are interested in an observable for only subsystem A, $\mathcal{O} = O_A \otimes \mathbb{1}_B$. If the system is not entangled (2.9) then we have

$$\langle\mathcal{O}\rangle = \langle\phi|O_A|\phi\rangle\,\langle\chi|\chi\rangle = \langle\phi|O_A|\phi\rangle. \tag{2.12}$$

The state (2.9) is, in this sense, reducible: a complete description of A is contained in $|\phi\rangle$.

If the system is entangled however, then from eq. (2.10) we have

$$\begin{aligned}\langle\mathcal{O}\rangle &= \sum_{i,j} c_i^* c_j \langle\phi_i|O_A|\phi_j\rangle\,\langle\chi_i|\chi_j\rangle \\ &= \sum_i |c_i|^2 \langle\phi_i|O_A|\phi_i\rangle.\end{aligned} \tag{2.13}$$

The entangled state was an irreducible description of the two systems, and if we ignore system B then we must describe system A by the mixed state "reduced" density matrix

$$\rho_A = \mathrm{Tr}_B\big(|\Psi\rangle\langle\Psi|\big) = \sum_i |c_i|^2 |\phi_i\rangle\langle\phi_i|. \tag{2.14}$$

Entanglement and decoherence are intertwined concepts. If A and B are not entangled, then there is no decoherence. If however A and B are entangled, then we can say B has caused decoherence in A (or vice versa).

2.1.3 Time evolution

A pure state evolves, in the Schrödinger picture, as $|\psi(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}t}|\psi(0)\rangle$, and thus under Hamiltonian evolution a density matrix evolves as

$$\rho(t) = e^{-\frac{i}{\hbar}\hat{H}t}\rho(0)e^{\frac{i}{\hbar}\hat{H}t}. \tag{2.15}$$

If an initial state is prescribed at time $t = 0$, and an observable is measured at time t, then the expectation value is

$$\langle \mathcal{O} \rangle = \mathrm{Tr}\left(e^{\frac{i}{\hbar}\hat{H}t}\mathcal{O}e^{-\frac{i}{\hbar}\hat{H}t}\rho(0)\right), \tag{2.16}$$

where we've used the cyclic property of the trace to rearrange the expression.

If we read eq. (2.16) from right to left, it roughly looks as if: we start with a state, evolve forward to time t, measure the observable $\mathcal{O}$, and then evolve backwards to time $t = 0$. There is not actually any backwards in time evolution occurring here, but it is common terminology to say density matrix evolution has forward and backward/return time evolution, hence the name Closed Time Path (CTP) evolution. Additionally, if we start from an initial thermal state (the ground state is just a thermal state with inverse temperature $\beta \to \infty$) then this can be written as

$$\langle \mathcal{O} \rangle = Z^{-1}\mathrm{Tr}\left(e^{\frac{i}{\hbar}\hat{H}t}\mathcal{O}e^{-\frac{i}{\hbar}\hat{H}t}e^{-\beta\hat{H}}\right). \tag{2.17}$$

The thermal density matrix can be viewed as describing evolution though an imaginary time $\tau = -i\hbar\beta$. One can then view eq. (2.17) as evolution along a particular complex time loop, with the insertion of an operator $\mathcal{O}$ at time t.

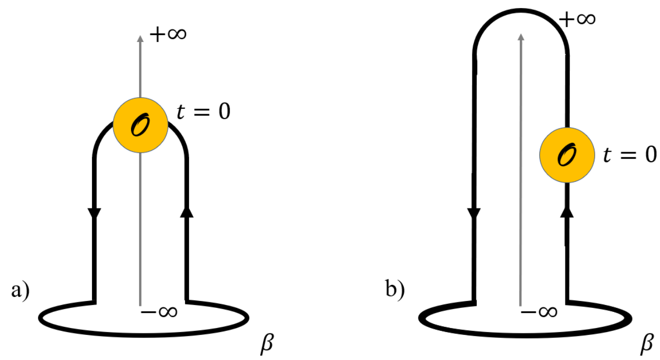


Figure 2.1: An illustration of the complex closed time contour describing density matrix evolution. Figure a) is a representation of eq. (2.17), whereas Figure b) is a completely equivalent contour corresponding to the insertion of $1 = e^{\frac{i}{\hbar}\hat{H}(\infty - t)} e^{-\frac{i}{\hbar}\hat{H}(\infty - t)}$ into eq. (2.17).

It is possible when using the density matrix to describe time evolution in a manner more generally than using a Hamiltonian, this is the enormous topic of open quantum systems [128]. For our purposes it is not necessary to go deeply into this topic. We will however discuss some basic aspects of its path-integral formalism. We will now set $\hbar = 1$.

Let us use a single particle for simplicity, but the following discussion generalizes trivially. The unitary time evolution of a pure state can be described via the Hamiltonian operator, or via the propagator,

$$\psi(x,t) = \int dy K(x,y|t)\psi(y,0), \tag{2.18}$$

where

$$K(x,y|t) = \int_{q(0)=y}^{q(t)=x} \mathcal{D}q(\tau)\, e^{iS[q(\tau)]}. \tag{2.19}$$

The integral here is over all paths $q(\tau)$ from y at $\tau = 0$ to x and $\tau = t$, and S is the action for the particle. To describe the unitary time evolution of a density matrix we would then also describe the return path

$$\rho(x,x',t) = \int dy dy' \mathcal{K}(x,x',y,y'|t)\rho(y,y',0), \tag{2.20}$$

$$\mathcal{K}(x,x',y,y'|t) = \int_{q(0)=y}^{q(t)=x} \int_{q'(0)=y'}^{q'(t)=x'} \mathcal{D}q\mathcal{D}q'\, e^{iS[q]-iS[q']}. \tag{2.21}$$

The density matrix propagator $\mathcal{K}$ in eq. (2.21) can obviously be factored into separate integrals for the forward and return paths, and the evolution of the system is just standard unitary evolution. One can describe an open quantum system by coupling these paths via a Feynman-Vernon influence functional $\mathcal{F}[q,q']$ [129]. This functional is often written as $\mathcal{F} = e^{i\Delta - \Gamma}$, for real functions Δ and $\Gamma > 0$,

$$\mathcal{K}(x,x',y,y|t) = \int_{q(0)=y}^{q(t)=x} \mathcal{D}q \int_{q'(0)=y'}^{q'(t)=x'} \mathcal{D}q' \, e^{iS[q]-iS[q']+i\Delta[q,q']-\Gamma[q,q']}. \quad (2.22)$$

The function Γ is of most interest to us, it is called the *decoherence functional*, as it suppresses certain density matrix elements.

This set-up is quite general, it can describe the influence on a system by an environment which is either classically stochastic or quantum mechanical. To understand decoherence via a quantum system, one has

$$\mathcal{F}[q,q'] = \int dX_1 \, dX_1' \, dX_2 \, \rho_0(X_1, X_1') \int_{X_1}^{X_2} \mathcal{D}X \int_{X_1'}^{X_2} \mathcal{D}X' \, e^{iS_X[X,q]-iS_X[X',q']}, \quad (2.23)$$

where ρ_0 is the initial density matrix for the environment X, and the trace over the environment is represented by tying the final state endpoints together.

Rather than studying specific elements of the density matrix propagator, one can instead use this formalism to directly evaluate expectation values. In this case there is a trace over both the system and environment, with an operator inserted to measure the system. An example of this would be

$$\langle \mathcal{O} \rangle = \oint \mathcal{D}q\mathcal{D}q' \oint \mathcal{D}X\mathcal{D}X' \, e^{iS[q]+iS[X,q]-iS[q']-iS[q',X']} \, \mathcal{O}(q). \quad (2.24)$$

The symbol $\oint$ denotes that we are taking the closed loop time contour, as in fig. 2.1. One can go a step further and produce a generating functional for these observables,

$$Z[J,J'] = \oint \mathcal{D}q\mathcal{D}q' \oint \mathcal{D}X\mathcal{D}X' \, e^{iS[q]+iS[X,q]+i\int dt J\mathcal{O}(q)-iS[q']-iS[q',X']-i\int J'\mathcal{O}(q')}, \quad (2.25)$$

and one can simply take functional derivatives with respect to J, J' to compute correlation functions of the desired observables. This is the starting point for the Schwinger-Keldysh, or "in-in" formalism [130–132] [4].

[4]We note that the Keldysh approach is actually more general than this. There one often allows the initial state to be an out-of-equilibrium state. In this case one still has the closed-time contour in the future, but the past boundary conditions are no longer specified using a "thermal circle".

Part I

Gauge theories and conventional quantum gravity

Chapter 3

A functional approach to soft graviton scattering and BMS charges

In this chapter we develop techniques for understanding decoherence in quantum gravity. There are a number of alternative quantum mechanics theories which posit that for sufficiently large masses gravitational effects will lead to either decoherence or some other breakdown of quantum mechanics. To progress towards any type of test of these theories, one must understand how decoherence occurs even in conventional quantum gravity, so that we can have a baseline for comparison. In the author's Master's thesis progress was made in this direction, showing how decoherence effects arise from unequal gravitational radiation being emitted from the various paths in a path integral [117]. In that work no explicit calculations were done which quantified the decoherence for specific examples. Here, we develop a general framework which incorporates the previous work and allows one to actually quantify the gravitational decoherence in various settings. Then, as an example, we apply this to quantify the soft-graviton induced decoherence in scattering events.

We consider the interaction between a matter system and long wavelength (soft) gravitons. Using a functional eikonal expansion, and a Feynman influence functional, we evaluate the effect of coupling to soft gravitons on the evolution of a matter system. We also introduce and compute a "composite generating functional" which allows us to calculate a decoherence functional for the time evolution of the system. These techniques allow us to formulate scattering problems in a way which deals consistently with infrared effects, and the expressions are also manifestly invariant under small diffeomorphisms. We show how the decoherence functional for scattering processes can be written in terms of the infinitely many conserved charges associated with asymptotic Bondi-Metzner-Sachs (BMS) symmetries, the soft-graviton factors, and the asymptotic gravitational memory. The results allows us to address the question of how much information is lost to the gravitational field during the scattering.

3.1 Introduction

Although the black hole information paradox [133, 134] has been with us now for over 4 decades, it is without any generally accepted resolution - recent reviews by Unruh and Wald [135] and by Marolf [136] indicate the depth of the issues involved. One idea that has emerged recently in this connection focuses on soft gravitons and soft photons, and the asymptotic charges associated with these [118, 137–144]. Insofar as black holes are concerned, the idea here is that information loss from the black hole will arise from both photon and graviton emission, and that this information is stored at the future boundary of the horizon. The information can, in this scenario, be described in terms of “charges” at future infinity; in the case of gravitons, these “BMS charges” are associated with BMS supertranslations. Extensive discussions of this point of view appear in the recent papers of Strominger et al. [139, 145, 146].

Quite apart from any implications for the physics of black holes, this work has raised important questions about the information loss associated with soft bosons coupled to matter fields: currently there is strong disagreement over whether there is any information loss at all, and if so how much.

One point of view argues that the emission of soft bosons, with its associated infrared catastrophe, must be associated with information loss - the information is carried off in the form of bremsstrahlung radiation, by an infinite number of soft bosons. According to this point of view, we must average over the soft bosons, noting that any information contained in them is only meaningful if one can access it using some measuring system, which will inevitably have a finite energy discrimination (typically formulated in terms of an IR cutoff on the boson excitations). This point of view goes back to early formulations of the IR divergence problem for QED [147, 148], which are now standard in many textbooks [55].

An opposing point of view argues instead that this information loss is illusory - that the IR modes are “carried along” with the relevant matter field [149–152]. This point of view goes back to Chung [153] and Kibble [154, and refs. therein] (see also Kulish and Faddeev [155]), who argued that any calculation of the IR properties should be formulated in terms of coherent states for the background radiation field, in which no IR cutoff should be involved. According to this point of view, we do not average over the very low-energy bosons when trying to describe any information loss, and in fact there is no information loss (one can however formulate a contrary point of view, also using coherent states [156], see also [157] for yet another perspective).

In order to address this we consider the concrete question: how can one describe and quantify the decoherence in a gravitational system, and what is the correct way to describe the information loss? The results we find are applicable way beyond the scattering problem - using a decoherence functional one can discuss any kind of information loss in the system, whether one deals with scattering or some quite different set-up.

In this chapter we will also argue that a correct answer to this question requires a non-perturbative formulation, and moreover one which does not rely on either IR cutoffs introduced by hand, or on some set of putative measuring systems acting at future infinity. Thus a second question asks - how can one formulate the problem of information loss non-perturbatively?

To deal with these various questions we introduce two new techniques in this chapter, viz.,

(i) we introduce a "composite generating functional" which, amongst other things, allows us to calculate the time evolution of the matter field reduced density matrix. This generating functional is a generalization of the decoherence functional well known in condensed matter physics [158]; in the present case we specialize to the case of a scalar matter field coupled to gravitons.

(ii) To formulate the infrared physics non-perturbatively, we adapt a technique originally devised by Fradkin and collaborators [159, 160], which takes the form of a WKB expansion about the eikonal limit. No coherent boson states or IR cut-offs are required in this formulation. This allows us to directly address the controversy, discussed from different points of view in refs. [118, 137–144] as well as refs. [121, 145, 146, 149–152, 156, 157], over information loss in graviton scattering.

Using these techniques we derive a functional eikonal expansion for the composite generating functional of a scalar field interacting with the gravitational field, written in terms of pairs of Feynman paths $T_{\mu\nu}(x), T'_{\mu\nu}(x)$ for the matter field stress energy - this is our principal new result. We then look at the scattering problem that has caused so much discussion. To do this one needs to further extend the composite generating functional technique, to calculate the scattering of a reduced density matrix for the matter field and its "in" and "out" states. We then discover that in the asymptotic limit where these states are very widely separated, the decoherence functional can be written in terms of the asymptotic BMS charges for the system, as well as in terms of a gravitational memory function. In this way we confirm that the information loss can be written in terms of these charges, as argued by Strominger and others [118, 137–145].

The rough plan of this chapter is as follows. In the next section (section 3.2) we describe the basic formalism used in this work. We introduce the composite propagators and generating functional used here, giving detailed expressions for a scalar matter field coupled to gravitons; and we then show how these can be used to derive the decoherence functional for the matter field. We also give a brief discussion of how one deals with diffeomorphism invariance in this formalism. Then, in section 3.3, we describe the eikonal expansion technique used here to isolate out the key infrared (IR) behaviour that we are interested in - this involves first making a formal separation between slow and fast variables, and then making a functional eikonal expansion for the graviton variables, to give expressions for quantities like the decoherence functional introduced in section 3.2.

In sections 3.4 and 3.5 we move on to discuss the scattering problem for the matter field. We first derive general results for the "composite S-matrix" of the reduced density matrix (this is *not* the scattering matrix for the fields themselves), in terms of our composite generating functional, and show how this can be written in terms of the decoherence functional $\Gamma[T, T']$. Finally, in section 3.5, we show how both of these can be written as a function of the BMS Noether charges and in terms of a gravitational memory function; and we summarize the extent to which these results answer the questions posed in this introduction.

3.2 Composite Generating Functional and Influence Functional

In this section we introduce the formal tools to be used, as well as establishing our notation. In particular, we

(i) describe "composite propagators" and the associated composite generating functionals. To make this clear we do it both for ordinary quantum mechanics, and for a matter field coupled to soft gravitons, after integrating out the soft gravitons.

(ii) introduce the Feynman influence functional for the matter field. For those unfamiliar with influence functionals and the related decoherence functional, we give a short introduction to these.

We also add brief remarks on the diffeomorphism invariance of the techniques used.

3.2.1 Propagators

We begin by recalling the usual definitions for propagators in quantum gravity. The following material is standard in quantum field theory [55]; we simply establish our notation here. We define ordinary propagators in the usual way as path integrals, so that, eg., a single particle has the propagator

$$\begin{aligned} K_2(x,x') &= \int \mathcal{D}g^{\mu\nu}\Delta(g)\, e^{iS_G[g^{\mu\nu}]} \int_{x'}^{x} \mathcal{D}q\; e^{iS_o[q,g^{\mu\nu}]} \\ &= \int \mathcal{D}g^{\mu\nu}\Delta(g)\, e^{iS_G[g^{\mu\nu}]}\; K_2(x,x'|g) \end{aligned} \tag{3.1}$$

where $K_2(x,x'|g)$ is the particle propagator in a fixed background metric $g^{\mu\nu}(x)$, $S_G[g^{\mu\nu}]$ is the Einstein action with appropriate gauge breaking terms added, and $\Delta(g)$ is a Faddeev-Popov determinant which provides the appropriate integration measure as the gauge breaking terms in the action serve to divide out diffeomorphism-equivalent metric configurations. At the moment, this path integral over metrics is only symbolic; we have not defined the measure, the Faddeev-Popov determinant, nor the "gauge-breaking" terms in the action. Throughout this chapter, and the entire thesis, we will only be quantizing a metric perturbation about Minkowski spacetime. In this limit, one can be precise about the objects mentioned above.

In the same way, for a field $\phi(x)$ which propagates between configurations $\Phi'(x)$ and $\Phi(x)$ we have

$$\begin{aligned} K_2(\Phi,\Phi') &= \int \mathcal{D}g^{\mu\nu}\Delta(g)\, e^{iS_G[g^{\mu\nu}]} \int_{\Phi'}^{\Phi} \mathcal{D}\phi\; e^{iS_M[\phi,g^{\mu\nu}]} \\ &= \int \mathcal{D}g^{\mu\nu}\Delta(g)\, e^{iS_G[g^{\mu\nu}]}\; K_2(\Phi,\Phi'|g) \end{aligned} \tag{3.2}$$

where $S_M[\phi, g^{\mu\nu}]$ is the scalar field action in the background $g^{\mu\nu}(x)$.

Our treatment of quantum gravity will be only as a low energy effective field theory of a metric perturbation about flat spacetime. Since it is an effective field theory, the energy scale of the problem will determine whether or not higher order interaction terms will be retained in the expansion on the action. The absence of fixed boundary data in the path integral for the metric perturbation implies, in the standard way, that we are considering vacuum-to-vacuum evolution for this quantum field. Of course, enforcing that the gravitational field remain in an unexcited state is an unphysical restriction in most cases—for many applications one should also consider specifying boundary data for the gravitational field. Indeed, in the following sections we will discuss exactly how one handles non-vacuum final states for the gravitational field, and what the implications are for the quantum decoherence of matter. Furthermore, in this chapter we will consider such low energies that linearized gravity is the appropriate description.

The gravitational action is specified by writing $g^{\mu\nu} = \eta^{\mu\nu} + 2\kappa h^{\mu\nu}$ in the usual way, with $\kappa^2 = M_P^{-2} = 8\pi G$. We can then write the total action in the form

$$\mathcal{S} \;=\; \mathcal{S}_M + \mathcal{S}_G + \mathcal{S}_{int} \tag{3.3}$$

in which $\mathcal{S}_M$ is the matter action in flat spacetime, and the graviton action is

$$\mathcal{S}_G \;=\; \mathcal{S}_{GHY} - \int d^4x h^{\mu\nu}(x)\bar{G}_{\mu\nu}(x) + S_{gb} \tag{3.4}$$

in which $\bar{G}_{\mu\nu}$ is the linearized Einstein tensor, viz.,

$$\begin{aligned}\bar{G}_{\mu\nu} = \frac{1}{2}\big(& - \partial^2 h_{\mu\nu} - \partial_\mu\partial_\nu h + \partial^\rho\partial_\mu h_{\rho\nu} \\ & + \partial^\rho\partial_\nu h_{\rho\mu} - \eta_{\mu\nu}\partial^\sigma\partial^\rho h_{\sigma\rho} + \eta_{\mu\nu}\partial^2 h\big)\end{aligned} \tag{3.5}$$

and $\mathcal{S}_{GHY}$ is the linearized Gibbons-Hawking-York boundary term [161, 162], and S_{gb} includes the gauge-breaking terms, and the matter-gravity coupling term is

$$\mathcal{S}_{int} \;=\; \kappa \int d^4x h^{\mu\nu}(x) T_{\mu\nu}(x) \tag{3.6}$$

in which $T^{\mu\nu}(x) \equiv T^{\mu\nu}(\phi(x))$ is the matter stress-energy tensor, viz.,

$$T_{\mu\nu} = -\partial_\mu\phi\partial_\nu\phi - \eta_{\mu\nu}\mathcal{L}_\phi. \tag{3.7}$$

for the scalar field.

3.2.2 Density Matrix Dynamics

We will be interested primarily in the direct calculation of probabilities for the matter field. We will thus be calculating reduced density matrices for the matter field, taken between two matter field configurations, having already integrated out the gravitons in a way which we will soon specify.

In chapter 2 we discussed the standard Schwinger-Keldysh formalism, [130, 131], and what follows will be very familiar. Both formalisms include a doubling of the degrees of freedom, with both forward in time and backward in time evolution. There is however one slight technical difference between our "composite" formalism and the Schwinger/Keldysh formalism, which we will describe later when necessary.

The dynamics of the reduced density matrix is written in terms of a propagator $\mathcal{K}(2,2';1,1')$ for the matter density matrix in the form

$$\rho_\phi(2,2') \;=\; \int d1 \int d1' \mathcal{K}(2,2';1,1')\; \rho_\phi(1,1') \tag{3.8}$$

where here the labels $1,1'$, and $2,2'$, refer abstractly to initial and final values states of the scalar matter fields $\phi(x), \phi'(x)$. These states will be described by inserting operators on the vacuum, so that, eg.,

$$|1\rangle \;\equiv\; \mathcal{O}_1(\phi)|0\rangle \tag{3.9}$$

and so on. The operators $\mathcal{O}_j(\phi)$ will be assumed to be products of field operators, eg. $\mathcal{O}_j(\phi) = \phi(x_{j_n})...\phi(x_{j_1})$. Thus $\rho^\phi(1,1') \equiv \rho^\phi(\mathcal{O}_1(\phi), \mathcal{O}_{1'}(\phi))$. We will sometimes refer to this forward/backward combination of paths as the "Keldysh" paths.

In what follows we aim to give useful expressions for the density matrix propagator $\mathcal{K}(2,2';1,1')$, in terms of a "composite generating functional", which can itself be written in terms of the Feynman influence functional [129]. We begin by giving formal expressions, and then explain their physical meaning [5].

Density Matrix Propagator

Let's start by just listing the main formal results we will require. The idea is to begin with a density matrix for the total "universe" (here this is the scalar matter field coupled to the gravitational field) and then trace over the gravitational environment to get the reduced density matrix for the matter field, so that

$$\hat{\rho}_\phi \;=\; Tr_G\; \hat{\boldsymbol{\rho}}_U \tag{3.10}$$

We will assume that in the distant past the universal density matrix begins in an uncorrelated product form, viz.,

$$\hat{\boldsymbol{\rho}}_U^{(in)} \;=\; \hat{\rho}_\phi^{(in)} \bigotimes \hat{\rho}_G^{(in)} \tag{3.11}$$

[5] There have been many formulations of non-equilibrium dynamics which can be applied in quantum gravity theory, and in which decoherence appears; important examples include the work of Barvinsky and Vilkovisky [61] as well as [163, 164]. There is some overlap between some parts of the work presented herein, and these papers. The principle difference between the current work and the earlier works is the development here of composite functionals, and their application to the scattering problem.

and that the gravitational density matrix $\hat{\rho}_G^{(in)}$ in the distant past can be described, in linearized gravity, by a thermal density matrix[6]; the matter state is initially a vacuum state for the matter field. Entanglement between the matter field $\phi(x)$ and the gravitons is then generated by the gravitational coupling. The assumption of an initial product state is typically made to simplify the formal development.

As mentioned in chapter 2, one can write a path integral expression for the propagator $\mathcal{K}(2,2';1,1')$ of the reduced density matrix as

$$\mathcal{K}(2,2';1,1') = \int \mathcal{D}\phi \int \mathcal{D}\phi' \;\; \mathcal{O}_2(\phi)\, \mathcal{O}_{2'}(\phi')\mathcal{O}_1(\phi)\mathcal{O}_{1'}(\phi') \times\; e^{iS_\phi[\phi]-iS_\phi[\phi']}\mathcal{F}[\phi,\phi'] \tag{3.12}$$

where $\mathcal{F}[\phi,\phi']$ is the Feynman influence functional, defined below. By a standard manouevre we rewrite the fields ϕ,ϕ' in this expression in terms of functional derivatives with respect to their corresponding external source variables $J(x), J'(x)$, allowing us to write eq. (3.12) in the form

$$\mathcal{K}(2,2';1,1') = \mathcal{O}_2\left(\frac{-i\delta}{\delta J}\right)\mathcal{O}_{2'}\left(\frac{i\delta}{\delta J'}\right)\mathcal{O}_1\left(\frac{-i\delta}{\delta J}\right)\mathcal{O}_{1'}\left(\frac{i\delta}{\delta J'}\right)\mathscr{Z}[J,J']\bigg|_{J=J'=0} \tag{3.13}$$

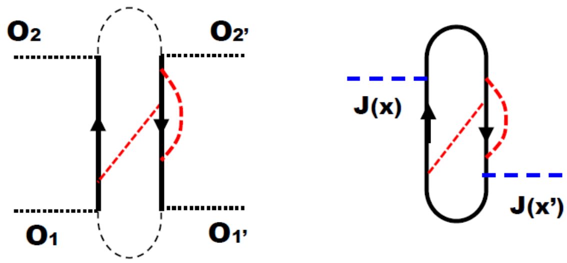
where the composite generating functional $\mathscr{Z}[J,J']$ is defined as

$$\mathscr{Z}[J,J'] \;=\; \int \mathcal{D}\phi(x)\int \mathcal{D}\phi'(x)\; e^{i\left[S_\phi[\phi]-S_\phi[\phi']+\int d^4x\,(J(x)\phi(x)-J'(x)\phi'(x))\right]}\mathcal{F}[\phi,\phi']. \tag{3.14}$$

The advantage of this manouevre is that we are no longer restricted to specific density matrix elements, specified by certain operators $\mathcal{O}_j$. We can now study, in a much more general framework, the physics of a matter system interacting with, and radiating, gravitons. Later, when we are concerned with computing specific density matrix elements, we will simply take the corresponding functional derivatives of our result for the composite generating functional.

It is here that we may remark on the difference between this "composite" formalism and the Schwinger-Keldysh formalism. The crucial difference is in the treatment of boundary conditions. In the latter, one "connects" the

[6]We assume a thermal state for the gravitons for two reasons: i) the framework here is not complicated by this generalization, and ii) it may serve as an approximate model of a stochastic gravitational wave background. In later sections we will set the temperature of the background to zero to understand purely radiative effects.



(a) (b)

Figure 3.1: Graphical representation of typical terms in (a) the propagator for the matter field density matrix $\mathcal{K}(2,2';1,1')$, in the absence of any external currents, and (b) the composite generating functional $\mathscr{Z}[\mathbf{J}]$ in the presence of external currents $J(x), J'(x)$. The matter fields are shown in heavy black, the graviton propagators in hatched red, and external currents in hatched blue; the effect of the field operators $\mathcal{O}_{\mathbf{j}}$ is shown in finely hatched black. The closed ends of the black ring in (b) may be misleading, it simply represents lines which run off to infinity. What they formally represent are circles in Euclidean time with infinite diameter that prepare the vacuum density matrix.

matter variables at future infinity, an operation corresponding to performing a trace. In comparison, in this 'composite" approach the forward and return integrals have independent boundary data. In the Schwinger-Keldysh formalism, one is then limited to computing expectation values, whereas in the "composite" formalism one can explicitly compute specific density matrix elements.

These explicit expressions are rather lengthy, and it is useful to introduce here a compact notation for the Keldysh paths involved [165], in which spacetime coordinates, fields and currents, etc., are all represented as 2-component boldface vectors, referring to the forward and backward segments of the paths. Then the equation of motion eq. (3.8) for the matter field reduced density matrix ρ^ϕ becomes

$$\rho_\phi(\mathbf{2}) \;=\; \sum_{\mathbf{1}} \mathcal{K}(\mathbf{2};\mathbf{1})\;\rho_\phi(\mathbf{1}) \tag{3.15}$$

so that $\rho_\phi(\mathbf{1}) \equiv \rho(1,1')$ and $\mathcal{K}(\mathbf{2};\mathbf{1}) \equiv \mathcal{K}(2,2';1,1')$. The result eq. (3.12) for the density matrix propagator is then written as

$$\mathcal{K}(\mathbf{2};\mathbf{1}) = \int \mathcal{D}\mathbf{\Phi}\,\mathcal{O}_{\mathbf{2}}\,\mathcal{O}_{\mathbf{1}} e^{iS_\phi[\mathbf{\Phi}]}\mathcal{F}[\mathbf{\Phi}], \tag{3.16}$$

where $\mathbf{\Phi} \equiv (\phi,\phi')$, where $S_\phi[\mathbf{\Phi}] \equiv S_\phi[\phi] - S_\phi[\phi']$, and where $\mathcal{F}[\mathbf{\Phi}]$ is the Feynman influence functional. The equivalent result eq. (3.12) for the density matrix propagator is written as

$$\mathcal{K}(\mathbf{2},\mathbf{1}) = \mathcal{O}_{\mathbf{2}}(\delta_{\mathbf{J}})\,\mathcal{O}_{\mathbf{1}}(\delta_{\mathbf{J}})\mathscr{Z}[\mathbf{J}]\Big|_{\mathbf{J}=0}. \tag{3.17}$$

and the composite generating functional is just

$$\mathscr{Z}[\mathbf{J}] = \int \mathcal{D}\mathbf{\Phi}\; e^{i[S_\phi[\mathbf{\Phi}]+\int d^4x\,\mathbf{J}\cdot\mathbf{\Phi}]}\,\mathcal{F}[\mathbf{\Phi}]. \tag{3.18}$$

where $\mathbf{J}\cdot\mathbf{\Phi} \equiv (J\phi - J'\phi')$, and $\delta_{\mathbf{J}} \equiv -i(\delta/\delta J(x),\ -\delta/\delta J'(x))$. One can give a graphical interpretation of the function $\mathcal{K}(2,2';1,1')$ appearing in eqs. (3.12) and (3.17) as shown in fig. 3.1(a); the equivalent graphical interpretation of the composite generating functional $\mathscr{Z}[\mathbf{J}]$ is shown in fig. 3.1(b).

We can see, that $\mathscr{Z}[\mathbf{J}]$ is acting as an analogue of the usual generating functional in ordinary quantum field theory, but now for computing reduced density matrix elements, in which the gravitons have already been integrated out, rather than computing the usual vacuum-to-vacuum correlation functions. This is shown in the graphical representation in fig. 3.1(b), in which an example Feynman graph is shown, with one graviton line representing the usual interaction of a system with itself in the forward time evolution, and another graviton line representing gravitational radiation which connects the forward and backwards lines.

Influence Functional

All the key physics in our problem is in the influence functional $\mathcal{F}[\phi,\phi']$; it not only describes the dephasing and relaxation of the matter field by the gravitational field, but also all reactive renormalization effects of gravitational interactions on the matter field.

Formally the influence functional is produced by integrating out the graviton and interaction terms (eqs. (3.4) and (3.6)) in the density matrix propagator, so that we have

$$\mathcal{F}[\mathbf{T}] \;=\; \int \mathcal{D}\mathbf{h}\; e^{\frac{i}{\hbar}(S_G[\mathbf{h}]+S_{\rm int}[\mathbf{h},\mathbf{T}])} \tag{3.19}$$

where we are again using our compact notation, and we have written $\mathcal{F}$ as a functional of $\mathbf{T}$ instead of $\mathbf{\Phi}$, using eq. (3.7). Hidden in this compact notation is the treatment of the initial and final data for the gravitons. The forward and return paths are connected in the future, which corresponds to a trace over the graviton degrees of freedom. The paths are also connected in the past, albeit now by an ring in imaginary time with circumference $1/k_BT$—recall the discussion from section 2.1.3. This prepares an initial thermal state for the gravitons at temperature T.

It is convenient to write

$$\mathcal{F} = e^{i(\Psi_o+\Psi)}, \tag{3.20}$$

where Ψ_o incorporates all static "self-gravity" effects (the analogue of the Coulomb contribution in a QED calculation), and where the complex phase functional $\Psi[T,T']$ contains all dynamic effects. In linearized gravity the gravitational action is quadratic in the field $h_{\mu\nu}$ and the interaction is linear, and it is then a theory of the type which has an exactly computable influence functional [129]. The details of this calculation can be found in the author's Master's thesis, [117]. Separating the real and imaginary parts as $\Psi[T,T'] = \Delta[T,T'] + i\Gamma[T,T']$ one has the explicit expressions [117, 166]:

$$\begin{aligned}
&\Delta[T,T'] \\
&= \frac{\kappa^2}{2}\int_{t_i}^{t_f} d^4x \int_{t_i}^{x^0} d^4\tilde{x}\,\left[T_{\mu\nu}(x) - T'_{\mu\nu}(x)\right] D_1^{\mu\nu\alpha\beta}(x,\tilde{x})\left[T_{\alpha\beta}(\tilde{x}) + T'_{\alpha\beta}(\tilde{x})\right] \\
&\Gamma[T,T'] \\
&= \frac{\kappa^2}{2}\int_{t_i}^{t_f} d^4x \int_{t_i}^{x^0} d^4\tilde{x}\,\left[T_{\mu\nu}(x) - T'_{\mu\nu}(x)\right] D_2^{\mu\nu\alpha\beta}(x,\tilde{x})\left[T_{\alpha\beta}(\tilde{x}) - T'_{\alpha\beta}(\tilde{x})\right]
\end{aligned} \tag{3.21}$$

where $D^{\mu\nu\alpha\beta}(x) = D_1^{\mu\nu\alpha\beta}(x) + iD_2^{\mu\nu\alpha\beta}(x)$ is just the finite-temperature graviton propagator:

$$D^{\mu\nu\alpha\beta}(x) \;=\; \int \frac{d^3q}{(2\pi)^3}\frac{e^{i\mathbf{q}\cdot\mathbf{x}}}{q}\,\Pi^{\mu\nu\alpha\beta}(q)\left(\sin qx^0 + i\cos qx^0 \coth\frac{\beta q^0}{2}\right) \tag{3.22}$$

defined at temperature T, where we write $\beta = 1/kT$, and where $\Pi_{\mu\nu\alpha\beta}(q)$ is the "TT projector", which projects out all but the transverse traceless modes. Note that eq. (3.21) is for the moment formal, since we have yet to specify how to deal with high-energy cutoffs, etc.

The imaginary part $\Gamma[T, T']$ of the influence functional phase is what is usually referred to as the "decoherence functional" [158], and is of primary interest to us here. Once exponentiated and inserted into the composite generating functional, its physical meaning is most easily understood by expanding the exponential. A 4th-order (in κ) term is shown in fig. 3.1; we see that it generates both "self-energy" graviton interactions on one or other of the matter lines, or an interaction between the forward and return lines.

The result of the interactions between lines is to cause dephasing in the dynamics of the matter field - this can happen even at $T = 0$, if accelerations are involved in the dynamics of the matter field - this will then lead to the emission of soft gravitons. At finite T, the matter field is interacting with a thermal bath, which has a well-defined rest frame - in this case one also has real relaxation processes caused by inelastic scattering of the gravitons.

Questions of Gauge Invariance

Let us briefly comment here on the use of the TT projector in eq. (3.21); see also ref. [117]. The need to satisfy both constraints and gauge/diffeomorphism invariance persists even in linearized gravity. In linearized gravity $h_{\mu\nu}$ is treated as a dynamical variable; however, not all components are independent. By linearizing, we break the full diffeomorphism invariance of GR; nevertheless small diffeomorphisms, for which $\kappa h_{\mu\nu}$ remains small, are still gauge symmetries of linearized gravity. As a consequence, not all components of $h_{\mu\nu}$ are physical. Likewise, in a Hamiltonian formalism not all components of $h_{\mu\nu}$ are independent canonical coordinates. The timelike components $h_{0\nu}$ do not have conjugate momenta since $\pi^{0\nu} = \partial\mathcal{L}/\partial(\partial_0 h_{0\nu})$ vanishes identically; and the timelike components of the linearized Einstein equation are not dynamical equations describing the time evolution of phase space data (h_{jk}, π^{jk}), since they involve no time derivatives of the canonical variables. Instead, the timelike linearized Einstein equations impose constraints on the phase space data which restrict what configurations can even exist on a time-slice.

In that subspace of phase space where the constraints are satisfied only the transverse-traceless components are independent. It is trivial to check that these components of the metric are invariant under gauge transformations; hence, if the constraints of linearized gravity are satisfied, then equations written in terms of the remaining variables will be gauge invariant. This is true in any quantum theory if the constraints are treated via Dirac quantization [166–168]. This is why the transverse-traceless projectors appear in the interaction kernels in eq. (3.21) - they project the influence functional onto the appropriate constrained subspace. This point has been mistaken in the literature, leading some to mistake pure gauge degrees of freedom as physical, and consequentially arriving at qualitatively incorrect predictions for gravitational decoherence [116].

Summary

It is helpful here to summarize our basic approach. In quantum field theory one typically starts from the generating functional of correlation functions, from which various transition amplitudes are obtained via functional differentiation. The evolution of the full system is unitary and described by the standard path integral.

Here we have introduced an analogous object for an open quantum system, the composite generating functional in eq. (3.18). We can then study the evolution of generic matter density matrices coupled to an unobserved background of gravitons in a way entirely parallel to typical quantum field theory computations, by taking functional derivatives with respect to external currents. This new formalism contains at its heart a decoherence functional, which describes both phase decoherence and relaxation processes. This "composite" formalism is somewhat more general than the related Schwinger/Keldysh formalism because it allows for a direct computation of density matrix elements, instead of limiting one to the computation of expectation values.

Clearly one should be able to extend this formalism to cover scattering processes, involving multi-particle "in" and "out" states, (in a way analogous to that LSZ formalism of standard quantum field theory). We develop these ideas in sections 3.4 and 3.5 below.

3.3 Functional Eikonal Expansion

As just noted, in sections 3.4 and 3.5 we will be applying our formalism to attack a problem of current interest, viz., information loss in scattering processes involving gravitons. However before doing so we must make a detour, because we require a method which can deal in a fully non-perturbative way with soft gravitons. A standard Feynman diagrammatic perturbation approach is not well suited to this problem, because the graviton is a massless particle, so that perturbation theory is plagued with infrared divergences.

In this section we will use a more appropriate non-perturbative treatment of the path-integral, a functional eikonal expansion, which is well suited to situations in which there is a separation of scales - in the present case provided by massive particles coupled to long wavelength gravitons. This will also allow us to give a more precise meaning to the formal expressions in the last section.

In what follows we begin by discussing how one makes a formal separation of scales, and then give the functional eikonal expansion for the matter propagator and for the composite generating functional.

3.3.1 Separation of Slow and Fast variables

The first thing we wish to do is make a formal separation between fast and slow variables in the composite functionals introduced above, in order to isolate out the interesting infrared behaviour. To do this we introduce a cutoff scale Λ_0 separating "soft" gravitons (with momentum $|q| \leq \Lambda_0$) from "hard" gravitons ($|q| > \Lambda_0$). In the course of our calculation we'll restrict the value of Λ_0, so that $\Lambda_0 \ll$ scalar particle masses.

Now since the interaction kernels D_1 and D_2 in the decoherence functional eq. (3.21) are given by a sum over contributions from each graviton mode, the influence functional can be factored into hard and soft parts, ie., we can write

$$\mathcal{F}[\mathbf{\Phi}] \;=\; \mathcal{F}_S[\mathbf{\Phi}]\mathcal{F}_H[\mathbf{\Phi}] \tag{3.23}$$

where Λ_0 serves as a UV cut-off in $\mathcal{F}_S[\boldsymbol{\phi}]$ and an IR cut-off in $\mathcal{F}_H[\boldsymbol{\phi}]$. We can use this to isolate the contributions from soft gravitons to the composite generating functional,

$$\begin{aligned}\mathscr{Z}[\mathbf{J}] \;&=\; \mathcal{F}_H[\delta_{\mathbf{J}}] \int \mathcal{D}\mathbf{\Phi} \;\; e^{i(S_\phi[\mathbf{\Phi}] + \int d^4x\, \mathbf{J}\cdot\mathbf{\Phi})} \; \mathcal{F}_S[\mathbf{\Phi}] \\ &\equiv \mathcal{F}_H[\delta_{\mathbf{J}}]\mathscr{Z}_S[\mathbf{J}]. \end{aligned} \tag{3.24}$$

where again we use the shorthand $\mathcal{F}_H[\delta_{\mathbf{J}}] \equiv \mathcal{F}_H[\mathbf{\Phi} \to -i\delta/\delta\mathbf{J}]$ introduced above in eq. (3.17); this transformation pulls the hard influence functional $\mathcal{F}_H$ outside the path-integral as a functional differential operator.

Since $\mathcal{F}_H$ has an IR cutoff Λ_0, one can simply expand it perturbatively without any issues - we will study this series in future work. In what follows we will focus only on the contributions to decoherence from soft gravitons, encapsulated in the soft composite generating functional $\mathscr{Z}_S[\mathbf{J}]$, which generates the propagators describing the evolution of the matter density matrix under the influence of soft gravitons.

We now rewrite the soft generating functional in a crucial way. Recall that it is always possible to rewrite a matter field propagator coupled to some dynamic field (here $h^{\mu\nu}(x)$) in the form of a propagator in some fixed or "frozen" background field configuration, with a subsequent functional integration over these field configurations. Accordingly we do this for the influence functional itself, pulling it outside of the path integral as a functional differential operator, to get

$$\begin{aligned}\mathscr{Z}_S[\mathbf{J}] =&\mathcal{F}_S[\delta_{\mathbf{h}}] \int \mathcal{D}\mathbf{\Phi}\, e^{i(S_\phi[\mathbf{\Phi}]+\int(\frac{1}{2}\kappa \mathbf{T}^{\mu\nu}\cdot\mathbf{h}_{\mu\nu}+\mathbf{J}\cdot\mathbf{\Phi}))}\Big|_{\mathbf{h}=0} \\ \equiv&\mathcal{F}_S[\delta_{\mathbf{h}}]\ Z[J|h]\ Z^*[J'|h']\Big|_{h,h'=0}\end{aligned} \tag{3.25}$$

where in the 2nd expression we write the forward and backward path variables explicitly. Here $\mathcal{F}_S[\delta_{\mathbf{h}}]$ is defined by the substitution $\mathcal{F}_S[\delta_{\mathbf{h}}] \equiv \mathcal{F}_S[\mathbf{T}_{\mu\nu} \to -2iM_P\delta/\delta\mathbf{h}^{\mu\nu}]$, ie., $T_{\mu\nu}$ and $T'_{\mu\nu}$ are substituted by their conjugate variables, and $Z[J|h]$ is the generating functional for a scalar field in the slowly-varying background metric perturbation $h^{\mu\nu}(x)$, ie., we have

$$Z[J|h] = \int \mathcal{D}\phi\, e^{iS_\phi[\phi]+i\int d^4x\,(\frac{1}{2}\kappa T^{\mu\nu}h_{\mu\nu}+J\phi)} \tag{3.26}$$

We see that in eq. (3.25) we have decoupled the primed and unprimed variables, so that from eq. (3.17) we have the density matrix propagator

$$\mathcal{K}_S(\mathbf{2},\mathbf{1}) = \mathcal{O}_{\mathbf{2}}(\delta_{\mathbf{J}})\,\mathcal{O}_{\mathbf{1}}(\delta_{\mathbf{J}})\ \mathcal{F}_S[\delta_{\mathbf{h}}]Z[J|h]\ Z^*[J'|h']\Big|_{h,h',\mathbf{J}=0} \tag{3.27}$$

Note that since the influence functional is a functional of only the transverse-traceless parts of the stress tensor, the auxiliary field $h_{\mu\nu}$ is also transverse-traceless in addition to being slowly-varying. We will use these properties many times throughout the following derivation.

We can now make the calculations much more physically concrete. We'll assume for now that aside from interactions with soft gravitons the scalar field is free, in which case

$$S_\phi[\phi] = -\frac{1}{2}\int d^4x\,\partial_\mu\phi\partial^\mu\phi + m^2\phi^2, \tag{3.28}$$

and

$$T_{\mu\nu} = -\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}\eta_{\mu\nu}(\partial_\lambda\phi\partial^\lambda\phi + m^2\phi^2). \tag{3.29}$$

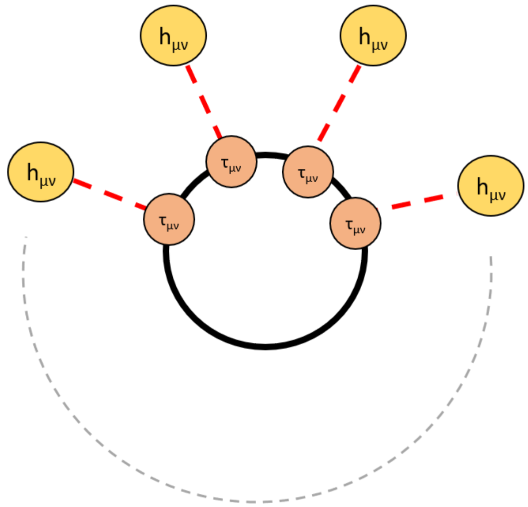


Figure 3.2: A Feynman diagram representation of one of the terms in the series expansion of the first exponential factor in eq. (3.30). The solid black lines represent flat spacetime particle Green's functions G_0, the red hatched lines leading to yellow circles indicate interaction with an external metric perturbation $h_{\mu\nu}$. The orange circles represent the interaction vertices.

The path integral for $Z[J|h]$ is then Gaussian and can be written, within a normalizing constant, as

$$Z[J|h] = e^{\frac{1}{2}\sum_{n=1}^{\infty} \mathrm{Tr}(\kappa G_0 h_{\mu\nu}\hat{\tau}^{\mu\nu})^n} e^{-\frac{i}{2}\int d^4x d^4y\, J(x)\mathcal{G}(x,y|h)J(y)} \tag{3.30}$$

in which $\hat{\tau}_{\mu\nu} \equiv \partial_\mu \partial_\nu - \frac{1}{2}\eta_{\mu\nu}(\partial^2 - m^2)$ is a differential operator corresponding to the stress-energy density of a point-particle, and $\mathcal{G}(x,y|h)$ is the scalar Green function on a fixed background $h^{\mu\nu}(x)$, ie.,

$$(\partial^2 - m^2 + \kappa h^{\mu\nu}\hat{\tau}_{\mu\nu})\,\mathcal{G}(x,y|h) = \delta^4(x-y). \tag{3.31}$$

We have already, and will continue to, make use of the flat-space limiting form $G_0(x,y) \equiv \mathcal{G}(x,y|h=0)$.

Notice that in eq. (3.30) the first exponential factor is just a rewriting of the functional determinant of the differential operator in eq. (3.31). If one thinks in terms of Feynman diagrams, a term in the series involving the trace of n factors of $G_0 h_{\mu\nu}\hat{\tau}^{\mu\nu}$ corresponds to a closed scalar loop with n insertions of the external field (see fig. 3.2). These diagrams describe vacuum polarization effects and will be suppressed by powers of $\Lambda_o/m \ll 1$. In this limit the polarization diagrams are then negligible. We can therefore omit the first exponential factor in eq. (3.30) and use the simpler result

$$Z[J|h] = e^{-\frac{i}{2}\int d^4x d^4y\, J(x)\mathcal{G}(x,y|h)J(y)} \tag{3.32}$$

This result for $Z[J|h]$, along with the result eq. (3.27) for the generating functional and our equation of motion for $\mathcal{G}(x,y|h)$, will then be the starting point for the formal eikonal expansion. Note that if we wish, we can also include non-gravitational interactions in this expression, by adding the usual $\exp(i\int \mathcal{L}_{\rm int}[\delta_J])$ factor as a prefactor on the right-hand side [7].

3.3.2 Functional Eikonal expansion for gravitons

The key intuition underlying any eikonal expansion is one of scale separation. In the absence of a background ($h_{\mu\nu} = 0$) the free Green's function $G_0(x,y)$ describes a single relativistic scalar particle propagating between spacetime points y and x. The dominant path is the straight-line classical solution, with corrections from quantum fluctuations about this path. In the presence of a slowly-varying background, the leading order eikonal approximation only modifies the propagator with an 'eikonal phase' accumulated along the classical path. From a perturbative standpoint this technique is very powerful - even the leading term sums an infinite class of diagrams, which here will capture in a non-perturbative way the leading IR-divergent effects from soft gravitons. Higher corrections capture sub-dominant contributions.

There are various ways to set up an eikonal expansion; in this paper we do this by adapting functional methods first introduced by Fradkin and collaborators [159, 160].

Equation of Motion

We begin by re-writing the equation of motion for $\mathcal{G}(x,y|h)$ in eq. (3.31) after a partial Fourier transform. To separate fast and slow variables we note that on a slowly-varying background the propagator is a rapidly varying function of the relative coordinate $(x-y)$ (on a scale $\sim 1/m$ but varies slowly with the "center of mass" coordinate $X \equiv (x+y)/2$. We define the partial Fourier

[7] The addition of interaction terms to the matter action will generally modify the matter stress tensor, but if the interactions are weak (in the sense that the coupling constant is small and perturbation theory is valid) then the additions to the stress tensor will be small compared to the leading order terms discussed here.

transform over $(x-y)$ according to

$$\mathcal{G}(x,y|h) = \sum_k e^{ik(x-y)} \mathcal{G}_k(X|h) \tag{3.33}$$

to move the fast modes into momentum space.

We then have the equation of motion

$$\left\{ G_0^{-1}(k) - \hat{\mathcal{H}}_h(x,k;h) \right\} \mathcal{G}_k(X|h) = 1, \tag{3.34}$$

where the inverse of the free propagator is just

$$G_0^{-1}(x) = -\partial^2 - m^2 \tag{3.35}$$

so that $G_0(k) = -(k^2+m^2)$, and we've also introduced the "Hamiltonian" operator

$$\hat{\mathcal{H}}_h(x,k;h) = -(\partial^2 - 2ik^\mu \partial_\mu) + \kappa h^{\mu\nu}(x) \left[k_\mu k_\nu + 2ik_\mu \partial_\nu - \partial_\mu \partial_\nu \right] \tag{3.36}$$

The sense in which this differential operator is a Hamiltonian will soon become clear.

We now introduce the Schwinger/Fock "proper time" representation for the propagators. The bare propagator is written as

$$\begin{aligned} G_0(k) &= -i \int_0^\infty ds\, e^{-is(k^2+m^2)} \\ &\equiv -i \int_0^\infty ds\, G_0(k,s), \end{aligned} \tag{3.37}$$

where we haven't written explicitly the small $i\epsilon$ convergence factor in the exponent. The Schwinger parameterized propagator $G_0(k,s)$ satisfies the equation of motion

$$i\partial_s G_0(k,s) = (k^2+m^2) G_0(k,s), \tag{3.38}$$

subject to the initial condition $G_0(k,s)|_{s=0} = 1$. To separate the fast variables from the slow variables we assume an explicit factorization in the Schwinger parameterization of the full propagator

$$\mathcal{G}_k(X|h) = -i \int_0^\infty ds\, G_0(k,s)\, \mathcal{Y}(k,s,X|h), \tag{3.39}$$

such that $\mathcal{Y}$ acts to weight the free propagator term under the proper time integral. This is completely analogous to the elementary application of the

WKB approximation in which one assumes that the Schrödinger equation with a slowly varying potential is solved by a plane wave with a slowly varying amplitude and wavelength. The equation of motion eq. (3.34) is satisfied if $\mathcal{Y}$ satisfies the "proper-time" Schrödinger equation

$$i\partial_s \mathcal{Y} = \hat{\mathcal{H}}_h \mathcal{Y}, \tag{3.40}$$

with the boundary condition $\mathcal{Y}|_{s=0} = 1$. We see that $\hat{\mathcal{H}}_h$ generates evolution in the time parameter s, hence the name Hamiltonian.

WKB Series Expansion

Now, to set up the eikonal expansion, we introduce a WKB representation for the function $\mathcal{Y}$, in the form

$$\mathcal{Y} \equiv e^{\chi}, \tag{3.41}$$

with χ expanded as a power series in the coupling

$$\chi \equiv \sum_{n=1}^{\infty} \kappa^n \chi_n. \tag{3.42}$$

In eq. (3.39) we've already separated off the free propagator factor, so χ should vanish when $\kappa = 0$ and hence the WKB series should indeed start at $\mathcal{O}(\kappa^1)$. It should then be clear that higher order derivatives of $\mathcal{Y}$ will bring down higher orders in the WKB expansion. Since the background is slowly varying, we then want to retain only the leading order terms in this series. To first order the equation of motion for χ_1 is

$$\left[i\partial_s + \partial^2 + 2ik^\mu \partial_\mu\right] \chi_1 \ = \ h^{\mu\nu}(x) k_\mu k_\nu, \tag{3.43}$$

and one can continue the hierarchy to find equations of motion for the higher order terms in the WKB series. We then have a systematic expansion suited for studying sub-leading soft-graviton effects.

If we now Fourier transform both sides of eq. (3.43) with respect to the center of mass coordinate X we have a simple linear ODE which has solution

$$\kappa\chi_1(q,k,s) = -i\kappa h^{\mu\nu}(q) k_\mu k_\nu \int_0^s ds'\, e^{-is'(2k\cdot q+q^2)}. \tag{3.44}$$

and dropping the ∂^2 term in the differential operator in eq. (3.43) - justified by the slowly-varying background assumption - allows us to drop the q^2 term in the exponent without altering our results to leading order.

Note that in the language of Feynman diagrams, a graviton-scalar vertex with a transverse-tracless $h_{\mu\nu}$ injecting momentum q is $\kappa h_{\mu\nu}(q)\tau^{\mu\nu}(k, k+q, q)$ where $\tau^{\mu\nu}(k_1, k_2, q) = \frac{1}{2}(k_1^\mu k_2^\nu + k_1^\nu k_2^\mu)$. When the incoming matter momentum k is much larger that of the graviton q, $k + q \approx k$ we recover the factor $\kappa h^{\mu\nu}(q)k_\mu k_\nu$ in the above expression. Recalling the preamble to this subsection, we see that this factor is a clear manifestation of eikonal physics—the particle momentum is not changing while interacting with the slowly-varying graviton background.

If we truncate the WKB series at leading order we have that

$$\mathcal{G}_k(X|h) = -i\int_0^\infty ds\, e^{-is(k^2+m^2)+\kappa\sum_q e^{iq\cdot X}\chi_1(q,k,s)}, \tag{3.45}$$

or, written in position space the leading order eikonal approximation to the full Green function is,

$$\mathcal{G}(x, y|h) = -i\sum_k e^{ik\cdot(x-y)}\int_0^\infty ds\, e^{-is(k^2+m^2)-i\kappa k_\mu k_\nu \int_0^s ds'\, h^{\mu\nu}(y-2s'k)}. \tag{3.46}$$

To best understand this expression we will use the method of stationary phase to evaluate the integrals. The eikonal phase $\exp(\kappa\chi_1)$ is a slowly varying function of s, k relative to both the bare propagator factor and the fourier factor. We can then pull it outside the integral and replace its dependence on s, k by those values extremize the oscillatory part of the integrand, *i.e.*

$$\begin{aligned} s &= \sigma/2m \\ k^\mu &= \frac{m(x-y)^\mu}{\sigma}, \end{aligned} \tag{3.47}$$

where the spacetime interval is $\sigma = \sqrt{-(x-y)^2}$. Substituting in these values, we obtain the intuitive expression for the full greens function

$$\mathcal{G}(x, y|h) = e^{\frac{i}{2}\kappa\int d^4z\,\tau^{\mu\nu}(z)h_{\mu\nu}(z)}G_0(x, y), \tag{3.48}$$

where

$$\tau^{\mu\nu}(z) = -\frac{p^\mu p^\nu \sigma}{m}\int_0^1 d\tau\, \delta^4(z - X_{\rm cl}(\tau)), \tag{3.49}$$

is precisely the stress-energy density for a classical massive relativistic point particle following the world line $X_{\rm cl}(\tau) = y + (x-y)\tau$. As promised, the leading order eikonal approximation for the greens function on a slowly varying background metric perturbation is given by the free Green function multiplied by an eikonal phase describing the phase accumulated along the classical worldline connecting the two spacetime points.

In summary, our main result is then that all effects of soft gravitons on a scalar matter system are described by the composite generating functional

$$\mathscr{Z}_S[\mathbf{J}] = \mathcal{F}_S[\delta_{\mathbf{h}}]\; Z[J|h]\; Z^*[J'|h']\Big|_{h,h'=0} \tag{3.50}$$

where

$$Z[J|h] = e^{-\frac{i}{2}\int d^4x d^4y\, J(x)\mathcal{G}(x,y|h)J(y)} \tag{3.51}$$

and the Green function has the simple form

$$\mathcal{G}(x,y|h) = e^{\frac{i}{2}\kappa\int d^4z \tau^{\mu\nu}(z)h_{\mu\nu}(z)} G_0(x,y). \tag{3.52}$$

This object, the eikonal approximated composite generating functional, generates density matrix propagators which provide expressions for the reduced density matrix of the matter system in which the state of the graviton field has been traced out.

There are a variety of potential applications of this general result. Although we have specified to a matter and gravity system, the formalism we have developed could in principle be applied to any quantum field theory with a large separation of scales where the low energy modes are unobserved. This formalism is in some sense the complement to standard renormalization theory. In renormalization theory one integrates out the high energy modes and focuses on the low energy modes, whereas here we have integrated out the low energy modes. The interesting difference here is that one can integrate out high-energy modes and still obtain a effective theory with unitary evolution. The high energy modes stay in their vacuum state because they are too difficult to excite, and thus the low energy modes interact with them only through their vacuum fluctuations. In contrast, when integrating out low energy modes, the high-energy system of interest can essentially excite arbitrarily large numbers of the low energy modes. As a consequence the end-state is not the vacuum, entanglement has developed, and decoherence can occur.

Although we have not yet put serious effort towards condensed matter applications of this formalism, one can certain imagine condensed matter models wherein some excitation of interest moves through an environment which has gapless (long wavelength/low energy) excitations.

Returning to the gravity focus, one application of this formalism was our work using computing various measures of entanglement and information transfer between the matter and soft gravitons [120]—content which was not included in this thesis. In the following section we will demonstrate another application of this formalism, to the study of scattering problems.

3.4 Scattering Problems

In this section we consider the specific example of scattering between matter fields and soft gravitons. Quite apart from the questions noted in the introduction, the interest here is that scattering problems are formulated over an infinite times, and so infinite wavelength gravitons are present. The methods we've developed in this chapter are actually perfectly suited to the problem of soft gravitons, as the eikonal approximation is more accurate as the graviton momentum goes to zero. In what follows we will recall the issues with soft gravitons in perturbation theory and then analyze the problem using our non-perturbative methods.

As we've mentioned previously, the presence of infinite wavelength gravitons causes infrared divergences in perturbation theory. One can see this quite clearly by considering a sample Feynman diagram for an $n \to m$ scattering process (see fig. 3.3). Since all external lines are "amputated" in scattering diagrams, the addition of a radiated soft graviton line will modify a scattering amplitude just by a multiplicative factor corresponding to: i) the new internal line coming from the blue circle to the graviton vertex, and ii) the coupling factor at the graviton vertex. One quite easily finds then (see eg. the work of Weinberg [148]) that the new diagram is related to the original by a factor

$$(8\pi G)^{1/2} \frac{(2k_m^\mu + q^\mu)(2k_m^\nu + q^\nu)\epsilon_{\mu\nu}}{k_m^2 + m^2 - i\epsilon} \tag{3.53}$$

where $\epsilon_{\mu\nu}$ is the polarization tensor for the graviton. Now, when the external particles are put "on-shell" we have $q^2 = 0$ and $(k_m - q)^2 + m^2 = 0$, which together imply that $k_m^2 + m^2 = 2k \cdot q$. It is then clear, that as we put the external particles on-shell and take the limit $q \to 0$ that the soft graviton factor is divergent as,

$$(8\pi G)^{1/2} \frac{k_m^\mu k_m^\nu \epsilon_{\mu\nu}}{k_m \cdot q - i\epsilon}. \tag{3.54}$$

It turns out that one can in fact re-sum the infinitely many diagrams including both these divergences and the completely analogous divergences

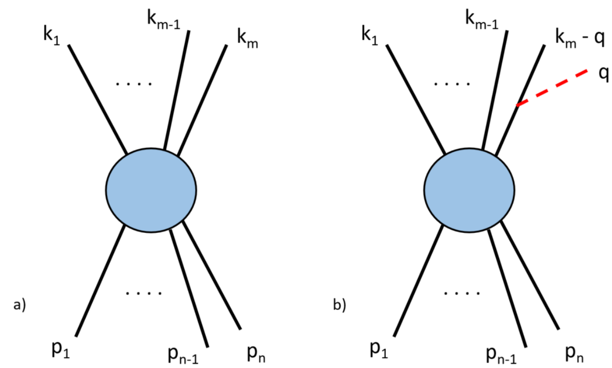


Figure 3.3: In a) we have a Feynman diagram for a general scattering process, wherein all of the internal lines/interactions are contained within the blue circle. In b) we have the related process, where the scattering process is the same but one of the external legs radiates a soft graviton with energy $q \to 0$.

coming from internal soft gravitons, but it requires abandoning the standard S-matrix and considering inclusive scattering cross-sections [148] (or the density matrix, as recently demonstrated in the work [121] which parallels our final results considerably).

We will now see how to apply the methods we've developed above to the soft-graviton problem, and will see that our formalism implicitly re-sums all divergences which occur in a perturbative approach, yielding finite results.

In conventional scattering problems, pure states evolve from the asymptotic past to the asymptotic future via the S-matrix. In what follows we will describe how reduced density matrices evolve over the same spatiotemporal region via a "composite S-matrix". Consider an initial product state of two systems Q and X written in the 'in' basis, $\rho_Q(\alpha,\alpha')\rho_X(a,a')$. The 'out' density matrix (which is generically not a product state) is related to the 'in' density matrix through the S-matrix,

$$\rho_{X,Q}(b,b';\beta,\beta') = \sum_{\alpha a,\alpha' a'} S_{\beta b,\alpha a} S^*_{\beta' b',\alpha' a'} \rho_Q(\alpha,\alpha')\; \rho_X(a,a') \tag{3.55}$$

If we then trace over system X (so that we consider X to be an "environment"), the evolution of the reduced density matrix for Q can be written

$$\rho_Q(\beta,\beta') = \sum_{\boldsymbol{\alpha}} \mathscr{S}_{\boldsymbol{\beta},\boldsymbol{\alpha}}\, \rho_Q(\alpha,\alpha'), \tag{3.56}$$

where the *composite S-matrix* is defined as

$$\mathscr{S}_{\boldsymbol{\beta},\boldsymbol{\alpha}} = \sum_b \sum_{a,a'} S_{\beta b,\alpha a} S^*_{\beta' b,\alpha' a'}\ \rho_X(a,a') \tag{3.57}$$

To understand the process of decoherence during any scattering process we must compute this object. In the following we will first derive a "composite scattering functional" $\mathscr{S}$ from the composite generating functional $\mathscr{Z}[\mathbf{J}]$. The computation is analogous to the standard LSZ procedure in quantum field theory for deriving the S-matrix generating functional [55]. From this we then derive the result we need for the composite S-matrix $\mathscr{S}_{\boldsymbol{\beta},\boldsymbol{\alpha}}$. We are then able to derive the leading eikonal result for the 'out' state predicted by the composite S-matrix, and compare this to recent results in the literature derived via diagrammatic methods.

3.4.1 Composite Scattering functional $\mathscr{S}$

Recall that the Lehmann-Symanzik-Zimmerman (LSZ) procedure [55] for computing the usual scattering operator S from a generating functional is compactly expressed by the formula

$$S =: e^{\int d^4x\ \phi_{\text{in}}(x) G_0(x)^{-1} \frac{\delta}{\delta J(x)}} :\ Z[J]\Big|_{J=0}, \tag{3.58}$$

where the colons denote normal-ordering of the operators. Again, $G_0^{-1}(x)$ is the inverse free Klein-Gordon Green function in eq. (3.35). The 'in' field ϕ_{in} satisfies the free Klein-Gordon equation

$$G_0^{-1}(x)\phi_{\text{in}}(x) = 0, \tag{3.59}$$

and is related to the full field ϕ via the weak asymptotic limits

$$\lim_{x^0\to-\infty} \left[\langle p|\phi(x)|q\rangle - \langle p|\phi_{\text{in}}(x)|q\rangle\right] = 0, \tag{3.60}$$

in which $|p\rangle, |q\rangle$ are arbitrary states of the system. We assume that the field is renormalized such that the pole in the two-point function is at $-p^2 = m^2$ and the residue is one, so we do not need to carry around factors of the field strength renormalization. The scalar field has the expansion in positive and negative frequency parts

$$\phi_{\text{in}}(x) = \phi^+_{\text{in}}(x) + \phi^-_{\text{in}} = \int d^3p\, \psi_p(x) a_p + h.c.\,, \tag{3.61}$$

where the creation/annihilation operators have the commutation relation normalized as $[a_p, a_k^\dagger] = \delta^3(p-k)$ and states are defined as $|p\rangle \equiv a_p^\dagger|0\rangle$, implying the wavefunctions are normalized as

$$\psi_p(x) = \frac{e^{-ipx}}{(2\pi)^{3/2}\sqrt{2E_p}}, \tag{3.62}$$

where $E_p = \sqrt{|\boldsymbol{p}|^2 + m^2}$ is the energy of a particle.

The elements of the S-matrix are given as usual by $S_{\beta,\alpha} = {}_{\text{in}}\langle\beta|S|\alpha\rangle_{\text{in}}$. The expression for the S-matrix we're using is convenient because of the explicit dependence on the generating functional, an object which we can write in path-integral expression form. It must be noted that, as written, eq. (3.58) can only generate S-matrix elements with ϕ particles in the in/out states. It will soon be clear that for our purposes this won't cause any problems.

If we wanted to compute the product of S-matrix elements $S_{\beta,\alpha}S^*_{\beta',\alpha'}$ we would consider two copies of the Hilbert space $\mathcal{H}\otimes\mathcal{H}$ and take matrix elements of $S\otimes S^\dagger$, where the operators in the first S commute with those in the second. Dropping the arguments of the functions we write this as

$$S\otimes S^\dagger = :e^{\int \phi_{\text{in}} G_0^{-1}\frac{\delta}{\delta J}} : \otimes : e^{\int \phi'_{\text{in}} G_0^{-1}\frac{\delta}{\delta J'}} : \, Z[J]Z^*[J']\Big|_{\mathbf{J}=0}. \tag{3.63}$$

Now we've already shown in the derivation of the composite generating functional that we can incorporate the effects of soft gravitons in some process by introducing a background metric perturbation $h_{\mu\nu}$, acting with the soft graviton influence functional operator $\mathcal{F}_S(\delta_h)$ on the forward and reverse Keldysh amplitudes, and then setting $h_{\mu\nu}\to 0$. The influence functional operator not only 'dresses' the process with soft internal gravitons, it also accounts for the eventual trace over soft graviton brehmstrahlung by introducing correlations between the forward and reverse Keldysh paths. Indeed we obtained the composite generating functional by doing precisely this on $Z[J]$, in eqs. (3.24) and (3.25).

The form of the composite scattering operator $\mathscr{S}$ then follows immediately; we have:

$$\begin{aligned} \mathscr{S} &= \mathcal{F}[\delta_{\mathbf{h}}]\, S[h]\otimes S^\dagger[h']\Big|_{\mathbf{h}=0} \\ &= :e^{\int \phi_{\text{in}} G_0^{-1}\frac{\delta}{\delta J}} : \otimes : e^{\int \phi'_{\text{in}} G_0^{-1}\frac{\delta}{\delta J'}} : \mathscr{Z}[\mathbf{J}]\Big|_{\mathbf{J}=0} \end{aligned} \tag{3.64}$$

where $S[h]$ is the scalar S-matrix evaluated on a background metric perturbation $h_{\mu\nu}$.

The scattering operator, or composite scattering functional $\mathscr{S}$, is in some ways a rather peculiar object - it describes scattering, but is non-unitary because it also incorporates the loss of information entailed by the averaging over the gravitons modes. To properly understand its properties we need to compute its matrix elements between in and out matter states.

3.4.2 Composite S-matrix elements

By taking matrix elements of the functional operator $\mathscr{S}$ in the double copied Hilbert space we obtain the desired matrix elements $\mathscr{S}_{\boldsymbol{\beta},\boldsymbol{\alpha}}$. We will start from $\mathscr{S}$ as given in eq. (3.64) to compute these matrix elements.

First we briefly review the derivation of the standard S-matrix as defined in eq. (3.58); and then we see how the parallel derivation works when deriving the composite S-matrix.

Bare S-matrix elements

In a typical scattering amplitude calculation in conventional field theory (see fig. 3.4) one considers "$n \to m$ scattering" between, in our case, scalar states $|\alpha\rangle = |p_1...p_n\rangle, |\beta\rangle = |k_1...k_m\rangle$. There are no other particles or fields considered in the problem - all scattering results from interaction potentials, and is unitary.

One accordingly assumes the existence of scattering states which are approximate eigenstates of the fully interacting Hamiltonian that look like eigenstates of the free Hamiltonian in the asymptotic future and past. The justification is that ultimately one should be working with wavepackets which are sufficiently well separated in the far future and past that they are essentially non-interacting. Scattering states then look like eigenstates of the free Hamiltonian—they are states with definite particle number.

Taking the matrix elements of the scattering operator in eq. (3.58), we immediately have

$$S_{\beta,\alpha} = \langle 0|a_{k_1} \dots a_{k_m} e^{\int \phi^- G_0^{-1} \frac{\delta}{\delta J}} e^{\int \phi^+ G_0^{-1} \frac{\delta}{\delta J}} a^\dagger_{p_1} \dots a^\dagger_{p_n}|0\rangle. \tag{3.65}$$

Commuting the creation/annihilation operators through the S-matrix we obtain the standard LSZ expression in terms of the amputation and on-shell

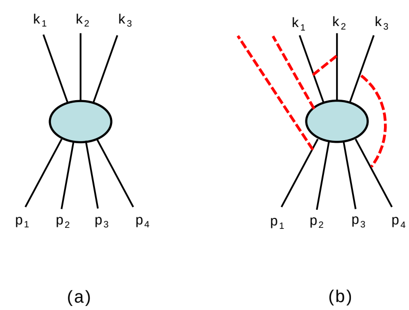


Figure 3.4: The scattering processes considered here. In (a) we see the scattering without gravitons, where scalar states $|\alpha\rangle = |p_1...p_n\rangle$, shown in black, scatter to $|\beta\rangle = |k_1...k_m\rangle$; the blue oval represents the scattering matrix $S_{\beta\alpha}$. In (b) gravitons are included, in red; the asymptotic graviton states are soft, with $|q| < \Lambda_o$.

restriction of the correlation function

$$\begin{aligned} S_{\beta,\alpha} &\equiv S_{\beta,\alpha}[\delta_J]Z[J]\Big|_{J=0} \\ &= \int d^3y_1 \dots d^3y_m\, \psi^*_{k_1}(y_1)\dots\psi^*_{k_m}(y_m) \int d^3x_1 \dots d^3x_n\, \psi_{p_1}(x_1)\dots\psi_{p_n}(x_n) \\ &\times G_0^{-1}(y_1)\dots G_0^{-1}(y_m)G_0^{-1}(x_1)\dots G_0^{-1}(x_n) \\ &\times \frac{\delta}{\delta J(y_1)}\frac{\delta}{\delta J(y_m)}\cdots\frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_n)}Z[J]\Big|_{J=0}. \end{aligned} \tag{3.66}$$

From this expression one can compute any S-matrix element between these massive particle states, given an expression for the generating functional (which is typically evaluated as a perturbative series in powers of the coupling constants).

However, let us now note that if one now includes soft gauge excitations like soft gravitons in the scattering calculations, as in/out states along with the massive particles, then our basic assumption of free particle asymptotic states no longer valid. The gravitons have arbitrarily long wavelength, and cannot then be disentangled from the asymptotic matter states. As is well known, this is an essential feature of the infrared divergences in the problem.

Composite S-matrix elements

Now let us consider composite S-matrix elements. To compute such composite S-matrix elements we take matrix elements of the operator eq. (3.64) in the basis of the double copy of the Hilbert space $|p_1 \ldots p_n\rangle \otimes |p'_1 \ldots p'_n\rangle$, to get

$$\mathscr{S}_{\boldsymbol{\beta},\boldsymbol{\alpha}} = \mathcal{F}[\delta_{\mathbf{h}}]\Big(S_{\beta,\alpha}[\delta_J] Z[J|h] \times S^*_{\beta',\alpha'}[\delta_{J'}] Z^*[J'|h'] \Big)\Big|_{\mathbf{J},\mathbf{h}=0} \tag{3.67}$$

Let us take a moment to properly understand this equation, a contribution to which is depicted in fig. 3.5. As noted above, we now have arbitrarily long-wavelength gravitons in the problem, which cannot properly be regarded as free particle states. Note, however, that a state describing a definite number of matter particles propagating on a very long-wavelength configuration of the metric perturbation is approximately free - not because the gravitons are well separated from the matter, but because there is very limited momentum exchange with the matter.

In our definition of the composite S-matrix eq. (3.57) the environment "X" is the metric perturbation field $h_{\mu\nu}(x)$. Its states, indexed by a, b, a', b', are not states of definite soft-graviton number; instead we assume a basis of Schrodinger states $\{|h_{ij}\rangle\}$ corresponding to states with definite slowly-varying field configuration $\hat{h}_{ij}(x)|h_{ij}\rangle = h_{ij}(x)|h_{ij}\rangle$. We are using this basis rather than a Fock basis.

We saw in eq. (3.32) that the generating functional for a non-interacting scalar field living on a slowly varying background metric perturbation can be be written as a simple Gaussian integral, in which the free scalar propagator is replaced by the propagator on a background metric perturbation. We can evaluate the action of $S_{\beta,\alpha}[\delta_J]$ on $Z[J|h]$ in eq. (3.67) in precisely the same way, as a generalization of what we would have if there were no background.

With these remarks in mind, let us now consider the term

$$S_{\beta,\alpha}[h] \;=\; S_{\beta,\alpha}[\delta_J] Z[J|h]|_{J=0}, \tag{3.68}$$

appearing in eq. (3.67); this describes the S-matrix in the presence of the slowly-varying field $h^{\mu\nu}(x)$. There will be both internal processes inside the "blue oval", coming from, eg., a ϕ^4 term in the Lagrangian, or perhaps be mediated by another field; and then there are external matter lines. We see that the effect of $h^{\mu\nu}(x)$ on any diagrams for the matter field will be to "dress" the scalar propagators according to the eikonal result eq. (3.52).

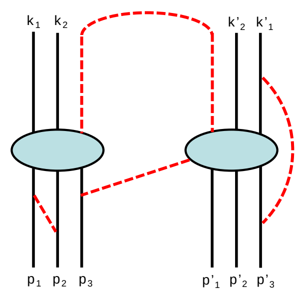


Figure 3.5: A process contributing to the composite S-matrix $\mathscr{S}_{\boldsymbol{\beta},\boldsymbol{\alpha}}$, where in the figure $\boldsymbol{\alpha} = (p_1, p_2, p_3; p'_1, p'_2, p'_3)$, and $\boldsymbol{\beta} = (k_1, k_2; k'_1, k'_2)$. The functional integration over the gravitons (shown in red) includes both graviton exchange between massive particles (shown in black) and gravitons emitted to/absorbed from infinity.

In what follows we will make a very simple approximation for the internal scalar propagators - we will assume they can be replaced by the bare propagator $G = \mathcal{G}|_{h=0}$. This apparently drastic simplification is actually equivalent to the assumption that is made in a diagrammatic IR treatment of soft graviton processes [148], where soft boson lines are assumed to attach to external legs but not to the internal "hard process". All of the dependence on $h_{\mu\nu}$ is then in the external legs of the diagram.

When acting with G_0^{-1} on outgoing external lines (thereby "amputating" them) we then have contributions from each (outgoing) leg, of form

$$\int d^4y \, e^{iky} G_0^{-1}(y) \mathcal{G}(y, z|h) \;=\; e^{ikz} e^{i\frac{\kappa}{2}\int d^4w \, h^{\mu\nu}(w)\tau_{\mu\nu}(w)}, \tag{3.69}$$

where again $\tau_{\mu\nu}(w) = -mU_\mu U_\nu \int_0^\infty ds \, \delta^4(w - z - sU)$ is the eikonal result for the stress-energy for a scalar excitation whose four-momentum is $p^\mu = mU^\mu$. Note that $\mathcal{G}(y, z|h)$ comes from the functional derivatives of $Z[J|h]$, with the relevant scattering vertex labeled by z. There is an analogous contribution for ingoing lines, viz.,

$$\int d^4x e^{-ikx} G_x^{-1} \mathcal{G}(z, x|h) \;=\; e^{-ikz} e^{i\frac{\kappa}{2}\int d^4w \, h^{\mu\nu}(w)\tau_{\mu\nu}(w)}, \tag{3.70}$$

where for ingoing lines the stress tensor is of the form $\tau_{\mu\nu}(w) = -mU_\mu U_\nu \int_0^\infty ds\, \delta^4(w - z + sU)$. In the absence of a slowly varying background metric the external leg would be straightforwardly amputated (*i.e.*, the inverse propagator acting on the propagator would yield a delta function), however when the background is treated via this eikonal approximation we obtain an additional eikonal phase for each leg. The analogous expressions for outgoing/ingoing lines on the return Keldysh path are obtained by taking the complex conjugates of these expressions.

Provided the scattering occurs in a region around the origin much smaller than Λ^{-1}, we can make the approximation that $z \approx 0$ within the eikonal phases since the slowly varying field $h_{\mu\nu}(w)$ is indifferent to such a translation. With this approximation the eikonal phases all factor out of the scattering amplitude. For a standard S-matrix computation we then obtain the eikonal result

$$S_{\beta,\alpha}[h] \;=\; e^{i\frac{\kappa}{2}\int d^4w\, h^{\mu\nu}(w)\sum_a \tau^a_{\mu\nu}(w)}\; S^{\Lambda_0}_{\beta,\alpha}, \tag{3.71}$$

where a runs over all external legs, and $S^{\Lambda_0}_{\beta,\alpha}$ is the S-matrix computed with IR cutoff Λ_0. This is an example of the well-known soft-factorization of scattering amplitudes.

If we now consider the full 'composite S-matrix' in eq. (3.67), then by taking the functional derivative and setting $h = 0$ we get the very simple result

$$\mathscr{S}_{\boldsymbol{\beta},\boldsymbol{\alpha}} \;=\; \mathcal{F}[\sum_a \tau^a, \sum_{a'} \tau'^{a'}]\; S^{\Lambda_0}_{\beta,\alpha} S'^{\Lambda_0}_{\beta',\alpha'}{}^*, \tag{3.72}$$

for $\mathscr{S}_{\boldsymbol{\beta},\boldsymbol{\alpha}}$, in which the entire effect of the soft gravitons has been reduced to the *sums*, $\sum_a \tau^a$ and $\sum_{a'} \tau'^{a'}$ over all scalar particles, of the the stress-energies from these particles at their asymptotic end-points - these 2 sums are then the arguments of the influence functional in eq. (3.72).

This is a remarkable simplification, given that the influence functional is usually a functional over all the paths in the particle path-integral. The reason is of course that the only effect of soft gravitons here, as incorporated in the influence functional, is to modify the phases of the 'classical' straight-line asymptotic paths of the particles follow straight lines

We have thus reduced the problem of the effect of real and virtual soft gravitons on scattering problems to the computation of the influence functional above. In the next section we compute this explicitly and investigate the consequences

3.5 Influence Functional, BMS Noether charges, & Gravitational Memory

In this section we will explore the properties of the composite scattering matrix, and see how they are influenced by the decoherence functional. We will simplify the results by setting the initial temperature of the graviton environment to zero. We find that the decoherence functional yields some previous results for this regime, and a new interpretation of them.

In what follows we use the expression derived in the last section for the composite S-matrix to first find the explicit form of the decoherence functional $\Gamma[T, T']$, and then show how it can be rewritten in terms of the asymptotic BMS charges and gravitational memory for the scattering of soft gravitons. Finally, we discuss the implications of these results for the information loss problem.

3.5.1 Form of Influence Functional

In eq. (3.21) we derived an explicit form for the influence functional eq. (3.19), which as we recall can be written as $\mathcal{F}[T, T'] = e^{i\Psi_0 + i\Delta} e^{-\Gamma}$, where Ψ_0, Δ, Γ are all real. The "self-gravity" and dissipation parts $\Psi_0[T, T']$ and $\Delta[T, T']$ merely lead to an overall phase shift for the composite S-matrix, which we will ignore here. The more interesting physics is in the decoherence functional $\Gamma[T, T']$ which suppresses coherence in the outgoing state.

It is useful to rewrite the general form of the decoherence functional given in eq. (3.21) as a momentum space integral, viz.,

$$\Gamma[T, T'] = \frac{1}{4M_P^2} \sum_{\sigma=+,\times} \int^{\Lambda_0} \frac{d^3q}{(2\pi)^3} \frac{1}{|\mathbf{q}|} |\epsilon^\sigma_{\mu\nu} \delta T^{\mu\nu}(q)|^2, \tag{3.73}$$

where $\delta T_{\mu\nu} = T_{\mu\nu} - T'_{\mu\nu}$ is the difference between the forward and return stress-tensors, and the on-shell Fourier transform is used

$$T_{\mu\nu}(q) = \int d^4z e^{i|\mathbf{q}|z^0 - i\mathbf{q}\cdot\mathbf{z}} T_{\mu\nu}(z). \tag{3.74}$$

Written this way it is clear that the decoherence functional is non-negative, and so the influence functional has modulus $|\mathcal{F}| \in [0, 1]$. It either leaves the composite amplitudes unchanged, or it suppresses them.

If we now specialize to the scattering problem discussed in the last section, things simplify drastically. The Fourier transform of the stress-tensor for scattering paths is just

$$\tau_a^{\mu\nu}(q) = i\eta_a m_a \frac{U_a^\mu U_a^\nu}{q \cdot U_a} \tag{3.75}$$

where $\eta_a = \pm 1$ depending on whether the index a refers to an outgoing or ingoing particle. Substituting this expression into our decoherence functional we obtain

$$\begin{aligned} \Gamma[\sum_a \tau^a, \sum_{a'} \tau'^{a'}] &= \frac{1}{2} \sum_{\sigma=+,\times} \int^{\Lambda_0} d^3q |\delta B^\sigma_{\beta,\alpha}(q)|^2 \\ &\equiv \frac{1}{2} \sum_{\sigma=+,\times} \int^{\Lambda_0} dq \int d\Omega(\hat{n}) \; |\delta B^\sigma_{\beta,\alpha}(q, \Omega(\hat{n}))|^2 \end{aligned} \tag{3.76}$$

where $d\Omega(\hat{n})$ is the infinitesimal solid angle in direction $\hat{n}$, and

$$B^\sigma_{\beta,\alpha}(q) = \frac{1}{(2\pi)^{3/2}\sqrt{2|q|}} M_P^{-1} \sum_a \eta_a \frac{p^a_\mu p^a_\nu \epsilon^\sigma_{\mu\nu}(q)}{p \cdot q}. \tag{3.77}$$

is the so-called "soft factor" we discussed previously (eq. (3.54))(see also eq. 2.29 in ref. [148]). The name comes from the statement that to leading order in the graviton momentum one can add a single soft graviton emission event to an S-matrix element $S_{\beta,\alpha}$ by simply multiplying the original S-matrix element by $B_{\beta,\alpha}$. This fact is commonly referred to as Weinberg's soft-graviton theorem.

The integral in eq. (3.76) is logarithmically divergent. This divergence means that unless $\delta B^\sigma_{\beta,\alpha}(q, \Omega(\hat{n})) = 0$ for every angle on the sphere $\hat{n}$ and for each polarization σ, the decoherence functional diverges and thus the influence functional as well as the composite S-matrix element will vanish.

This result was previously reported in a slightly less general form by Carney et al. [121]. These authors used the Weinberg diagrammatic approach to handle the IR divergences, and assumed that the matter in-state was a momentum eigenstate (*i.e.* only $\alpha' = \alpha$ was considered). If we choose to assume this initial condition as well, we recover their result.

It is actually illuminating to understand the relation between the WKB path integral result here and the derivations of Weinberg [148], and other similar recent discussions [121, 146]. Weinberg showed perturbatively that a specific infinite diagrammatic sum - of soft factors from all diagrams in which soft boson lines (both virtual and real) are inserted into a "hard" process - will exponentiate in a manner that renders scattering rates finite.

The WKB expansion - which is a natural and systematic approximation for quantum systems propagating on slowly varying backgrounds - already yields to lowest order a decoherence functional in which the soft factors are exponentiated. The next correction to leading WKB then leads to subdominant corrections to the Weinberg result; and so on. The decoherence functional also allows a useful interpretation of the diagrammatic expansions. For every momentum q with $|q| \ll \Lambda_0$, and for each polarization σ, the decoherence functional eq. (3.76) compares the soft factors for the forward and return Keldysh paths. If these factors are identical, ie., if an emitted soft graviton $|q, \sigma\rangle$ does not carry information discerning between the two processes, then that mode does not contribute to decoherence. Otherwise the soft-factor is different for the two paths, and by eq. (3.76) there is a contribution to the decoherence functional.

3.5.2 BMS Charges and Gravitational Memory

Let us now turn to several other ways of expressing the decoherence functional. The first will involve the so-called with the Bondi-Metzner-Sachs (BMS) charges, associated with the Bondi-Metzner-Sachs group of supertranslation symmetries [169–171]. The second will involve what is called gravitational memory. The connection between information loss, BMS symmetries, and gravitational memory has been the topic of a number of recent papers (see e.g. [139, 145, 152, 172, 173] and refs. therein), sometimes described in terms of an "infrared triangle". We will see that the connection to the decoherence functional gives further illumination of these relationships.

In what follows we discuss both BMS charges and gravitational memory, in each case by first briefly recalling what these terms refer to, and then showing how the decoherence functional can be understood in terms of them.

BMS Charges

There is a large literature on Bondi-Metzner-Sachs (BMS) charges and the BMS group (see refs. [34, 145] and refs. therein); here we will simply make the connection with our results on decoherence.

The BMS group is the group of diffeomorphisms whose actions on null infinity map one asymptotically flat solution to the Einstein equations to another, potentially physically inequivalent one. A subset of the generators of this group are the six Lorentz generators; their action is well understood in quantum field theory and will not be further discussed. The more interesting part of the group are the remaining supertranslation transformations, of which there are infinitely many.

Supertranslations are defined by functions $f(z,\bar{z})$ on the sphere. In retarded Bondi coordinates $(u,r,z,\bar{z})$ the supertranslation vector field on future null-infinity is

$$\zeta = f\partial_u - \frac{1}{r}(D^{\bar{z}}f\partial_{\bar{z}} + D^z f\partial_z) + D^z D_z f\partial_r, \tag{3.78}$$

where D_z is the covariant derivative with respect to the unit sphere metric $\gamma_{z\bar{z}} = 2(1+z\bar{z})^{-2}$. They are a generalization from the four standard global translations to a group of angle-dependent translations in the retarded time u. There is an analogous expression in advanced Bondi coordinates for the supertranslation vector field on past null infinity. In general one can perform independent transformations on future null infinity $\mathscr{I}^+$ and past null infinity $\mathscr{I}^-$ and thus the BMS group can be written as the direct product $BMS^+ \times BMS^-$. Recently it has been demonstrated that the "diagonal" subgroup, in which the same function $f(z,\bar{z})$ is used to simultaneously supertranslate both $\mathscr{I}^+$ and $\mathscr{I}^-$, is a symmetry of quantum gravity linearized about Minkowski space [137], and the associated Noether charges Q_f have been constructed [138]. Furthermore, Weinberg's soft graviton theorem [148] was shown to be a consequence of this symmetry, ie. a Ward identity in the quantum theory [138].

For pure gravity coupled to massless scalar matter the supertranslation charge on $\mathscr{I}^+$ is

$$Q_f = \frac{1}{4\pi G}\int_{\mathscr{I}^+} dud^2z\gamma_{z\bar{z}}f\left[T_{uu} - \frac{1}{4}(D_z^2 N^{zz} + D_{\bar{z}}^2 N^{\bar{z}\bar{z}})\right], \tag{3.79}$$

where

$$T_{uu} = \frac{1}{4}N_{zz}N^{zz} + 4\pi G \lim_{r\to\infty}\left[r^2 T^M_{uu}\right]. \tag{3.80}$$

The matter stress-tensor is denoted by $T^M_{\mu\nu}$, and N_{zz} is the Bondi news tensor describing outgoing gravitational waves. Again, there is an analogous expression in advanced coordinates on $\mathscr{I}^-$. It should be noted that since supertranslations are defined on null infinity $\mathscr{I}^+ \cup \mathscr{I}^-$ and massive particles never reach null infinity, some work must be done to obtain the correct expression for the hard supertranslation charge in a theory with asymptotically stable massive particles [174]. The supertranslation charge can be understood as the sum of "hard" and "soft" contributions. The first term in eq. (3.79) measures the weighted energy flux through $\mathscr{I}^+$ and is called Q_f^{hard} while the second term is linear in the zero frequency graviton creation operator and hence called Q_f^{soft}.

There is a particular choice of the function f which picks out a single angle $\hat{n}$ on the asymptotic sphere and a single graviton polarization σ (see [174]); with this choice, scattering states of the scalar field are eigenstates of the hard-matter part of the supertranslation charge

$$\hat{Q}^{\text{hard}}_{\hat{n},\sigma}|\alpha\rangle \;=\; Q^{\text{hard}}_{\hat{n},\sigma}(\alpha)|\alpha\rangle \tag{3.81}$$

with the eigenvalue $Q^{\text{hard}}_{\hat{n},\sigma}(\alpha)$ given by

$$Q^{\text{hard}}_{\hat{n},\sigma}(\alpha) \;=\; \sum_{a\in\alpha} \frac{-m_a \epsilon^{\sigma}_{\mu\nu}(\hat{n}) U^{\mu} U^{\nu}}{-U^0_a + \hat{n}\cdot\vec{U}_a} \tag{3.82}$$

We immediately see that we can connect all this with the decoherence functional - one simply rewrites the decoherence functional in eq. (3.76) in the form

$$\Gamma[\sum_a \tau^a, \sum_{a'} \tau'^{a'}] \;=\; \frac{1}{4M_P^2}\left(\int^{\Lambda_0} \frac{dq}{q}\right) \sum_{\sigma=+,\times} \int d\Omega(\hat{n}) \left|\Delta Q^{\text{hard}}_{\hat{n},\sigma} - \Delta Q^{\text{hard}\prime}_{\hat{n},\sigma}\right|^2, \tag{3.83}$$

where $d\Omega(\hat{n})$ is the solid angle area element, and where $\Delta Q^{\text{hard}}_{\hat{n},\sigma}$ denotes the difference between the in- and out-state values of $Q^{\text{hard}}_{\hat{n},\sigma}$, the hard charge eigenvalue defined in eq. (3.82), ie.,

$$\Delta Q^{\text{hard}}_{\hat{n},\sigma} = Q^{\text{hard}}_{\hat{n},\sigma}(\beta) - Q^{\text{hard}}_{\hat{n},\sigma}(\alpha) \tag{3.84}$$

where, as usual, we use the primed symbols refer to the return Keldysh path while the unprimed symbols refer to the forward Keldysh path. Eq. 3.83 expresses the decoherence functional in terms of the “BMS supertranslation charges”. We note that, as before, because of the logarithmic divergence of the integration over q, the decoherence functional here diverges unless the difference in hard supertranslation charges is the same on the forward and return Keldysh paths, ie., unless

$$\bar{\mathbf{Q}}^{\text{hard}}_{\hat{n},\sigma} \;\equiv\; (Q^{\text{hard}}_{\hat{n},\sigma}(\beta) - Q^{\text{hard}}_{\hat{n},\sigma}(\alpha)) - (Q^{\text{hard}}_{\hat{n},\sigma}(\beta') - Q^{\text{hard}}_{\hat{n},\sigma}(\alpha')) \;=\; 0 \tag{3.85}$$

which is a kind of “sum rule” for the scattering process - decoherence will suppress all scattering for which this identity is not satisfied.

Ward identities

The result eq. (3.83) begs for a physical explanation. Actually it follows from supertranslation charge conservation alone. Recall that the soft theorems are an expression of the Ward identity describing the conservation of supertranslation charge [138]. The Ward identity can be written in general as the statement that the charge commutes with the Hamiltonian and thus the S-matrix

$$\langle\beta|[\hat{Q}^{\sigma}_{\hat{n}}, S]|\alpha\rangle = 0, \tag{3.86}$$

where S is the usual S-matrix. Decomposing the supertranslation charge into hard and soft parts, assuming the initial state of the graviton field is the vacuum with zero soft charge, and noting that scalar scattering states are eigenstates of hard supertranslation charge, we can rewrite the Ward identity as the soft graviton theorem

$$\left[Q^{\text{hard}}_{\hat{n},\sigma}(\beta) - Q^{\text{hard}}_{\hat{n},\sigma}(\alpha)\right]\langle\beta|S|\alpha\rangle = \langle\beta|\hat{Q}^{\text{soft}}_{\hat{n},\sigma}S|\alpha\rangle. \tag{3.87}$$

Following a similar line of argument, we can instead write the following equation

$$\begin{aligned}\langle\beta|S|\alpha\rangle\langle\alpha'|[S^{\dagger}, \hat{Q}_{\hat{n},\sigma}]|\beta'\rangle &= \langle\beta|[\hat{Q}_{\hat{n},\sigma}, S]|\alpha\rangle\langle\alpha'|S^{\dagger}|\beta'\rangle \\ &= 0.\end{aligned} \tag{3.88}$$

Looking at the first equality and again decomposing the charge into hard and soft parts, assuming an initial state with zero soft charge, and using the hard charge eigenvalues we can then write

$$\begin{aligned}&\left[(Q^{\text{hard}}_{\hat{n},\sigma}(\beta) - Q^{\text{hard}}_{\hat{n},\sigma}(\alpha)) - (Q^{\text{hard}}_{\hat{n},\sigma}(\beta') - Q^{\text{hard}}_{\hat{n},\sigma}(\alpha'))\right] S_{\beta,\alpha}S^{*}_{\beta',\alpha'} \\ &= \langle\beta|\left[S|\alpha\rangle\langle\alpha'|S^{\dagger}, \hat{Q}^{\text{soft}}_{\hat{n},\sigma}\right]|\beta'\rangle\end{aligned} \tag{3.89}$$

As written, the right hand side is a matrix element between states β, β' of a commutator between operators on the full Hilbert space. We could instead factor the state into hard and soft parts $|\beta\rangle = |\beta_S\rangle|\beta_H\rangle$. If we do this and trace over the soft part of the outgoing state we obtain the composite Ward identity

$$\begin{aligned}&\left[(Q^{\text{hard}}_{\hat{n},\sigma}(\beta) - Q^{\text{hard}}_{\hat{n},\sigma}(\alpha)) - (Q^{\text{hard}}_{\hat{n},\sigma}(\beta') - Q^{\text{hard}}_{\hat{n},\sigma}(\alpha'))\right]\sum_{\beta_S} S_{\beta,\alpha}S^{*}_{\beta',\alpha'} \\ &= \sum_{\beta_S}\langle\beta_S|\left[\langle\beta_H|S|\alpha_H\rangle|\alpha_S\rangle\langle\alpha'_S|\langle\alpha'_H|S^{\dagger}|\beta'_H\rangle, \hat{Q}^{\text{soft}}_{\hat{n},\sigma}\right]|\beta_S\rangle.\end{aligned} \tag{3.90}$$

Since the RHS is the trace of a commutator, by the cyclic property of the trace the RHS vanishes. We have therefore derived the following identity for the composite S-matrix using the supertranslation Ward identity

$$\Big[(Q^{\text{hard}}_{\hat{n},\sigma}(\beta) - Q^{\text{hard}}_{\hat{n},\sigma}(\alpha)) - (Q^{\text{hard}}_{\hat{n},\sigma}(\beta') - Q^{\text{hard}}_{\hat{n},\sigma}(\alpha'))\Big] \sum_{\beta_S} S_{\beta,\alpha} S^*_{\beta',\alpha'} \;=\; 0. \tag{3.91}$$

In other words, for a given pair of processes, either the difference in hard supertranslation charges is the same on the forward and return Keldysh paths or the composite S-matrix element vanishes. This is precisely the result we already derived from the decoherence functional, in the form of the sum rule in eq. (3.85), but now demonstrated using a composite Ward identity.

Gravitational Memory

Another way of looking at our result for the decoherence functional in eq. (3.83) is in terms of "gravitational memory". Classically one can use the linearized Einstein equations to compute the evolution of a metric perturbation $h^{\mu\nu}(x)$ far from a source [175]. Thus we can then calculate the change $\Delta h^{\mu\nu}(x)$ in the metric, comparing at times well before and well after any change in the source, in a far field region at some distance r_0 from the source of the gravitational waves, which in our case will be from the scattering event. Any permanent change in the metric, ie., where $\Delta h^{\mu\nu}(x) \neq 0$, is called gravitational memory [176, 177].

The change $\Delta h^{\mu\nu}(x)$ has been worked out for a large variety of different sources; given the classical paths considered in the scattering setup here, then it is well-known that such a process will lead to a static change in the transverse-traceless part of the asymptotic metric given by the Braginsky-Thorne formula [177],

$$\Delta h^{TT}_{\mu\nu}(\vec{q}) \;=\; \frac{1}{r_0}\frac{1}{16\pi^2 M_P}\left(\sum_{j\in\alpha}\frac{p_{j\mu}p_{j\nu}}{q\cdot p_j} - \sum_{j\in\beta}\frac{p_{j\mu}p_{j\nu}}{q\cdot p_j}\right)^{TT} \tag{3.92}$$

Comparing this with our expression for the the hard supertranslation charges we see that we can rewrite the decoherence functional as

$$\Gamma[\sum_a \tau^a, \sum_{a'} \tau'^{a'}]$$
$$= 4\pi^2 r_0^2 \left(\int^{\Lambda_0} \frac{dq}{q} \right) \sum_{\sigma=+,\times} \int d\Omega(\hat{n}) \left| \epsilon^{\sigma}_{\mu\nu} \Delta h_{\mu\nu}(\hat{q}) - \epsilon^{\sigma}_{\mu\nu} \Delta h'_{\mu\nu}(\hat{q}) \right|^2, \quad (3.93)$$

where the difference here is taken between the latest time on future null infinity and the earliest time on future null infinity.

Since the asymptotic shift in the metric is in principle observable by considering the shifts in the relative positions of asymptotic detectors, we can understand this expression for the decoherence functional in the following way. Suppose we prepare an array of asymptotic detectors with given relative positions. During a scattering event information about the event is radiated away as soft gravitons, which will induce a static shift in the relative positions of the asymptotic detectors (see fig. 3.6). Since the information about the scattering event is stored as the shift in their relative positions, the scattered matter is entangled with the detectors. Attempts at demonstrating interference phenomena in the outgoing state of the scattered matter will be undermined by this, and the only states which can interfere will be those obtained by a scattering event which induces the same relative shifts in the asymptotic detectors. This explains the vanishing of most of the elements of the composite S-matrix.

3.5.3 Decoherence properties

Let us now discuss the implications of these results for the very physical question of what form the final state density matrix must take.

In fact it is clear that the vanishing of the composite S-matrix element, unless the "sum rule" in eq. (3.85) is satisfied, implies that the out-state density matrix must have a very restricted form. To see how this works, let us consider two kinds of in-state for the system. One will be a simple product of momentum eigenstates, whereas the other will be a "Cat state" in which we superpose two simple product states. We then have the following results:

(i) *Simple Product State*: Our first state will be the kind of state usually assumed in scattering calculations, in which there are no gravitons and where the initial matter state is a product over momentum eigenstates, ie., we have

$$|\alpha_1\rangle \; = \; \prod_j |p_j^{(1)}\rangle \quad (3.94)$$

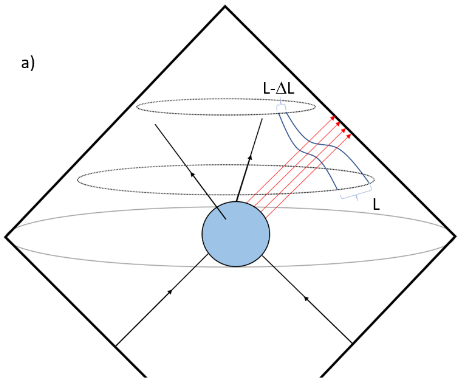


Figure 3.6: The solid black diamond represents null infinity. The solid black lines with arrows represent incoming and outgoing particles. The blue circles represent a scattering region. The red lines represent outgoing gravitational radiation. The solid blue lines represent two inertial "detectors" whose relative position is a witness of the the metric. We imagine that such detectors are located at all angles, this is just an example of two which receive the strongest part of the gravitational wave pulse. Both a) and b) have the same initial scattering state, but illustrate different branches of the outgoing wavefunction, which is a superposition of different scattering states. When gravitational radiation is accounted for, we see that the different branches a) and b) can generate angle dependent gravitational radiation and thus can have entanglement between the outgoing scattering states of the matter and the positions of inertial detectors very far away.

so that the incoming reduced density matrix for the matter has the simple form $\rho(\alpha,\alpha') = \delta_{\alpha,\alpha_1}\delta_{\alpha',\alpha_1}$, and then by eqs. (3.56) and (3.72), the outgoing reduced density matrix for the matter is

$$\begin{aligned}\rho(\beta,\beta') &= \mathscr{S}_{\beta\beta',\alpha_1\alpha_1} \\ &\sim S^{\Lambda_0}_{\beta,\alpha_1} {S'}^{\Lambda_0\,*}_{\beta',\alpha_1}\, \delta_{\bar{\mathbf{Q}}^{\rm hard}_{\hat{n},\sigma},0} \end{aligned} \tag{3.95}$$

where the Kronecker δ-function term imposes the BMS charge conservation sum rule. Thus the final state density matrix will vanish unless $Q^{\rm hard}_{\hat{n},\sigma}(\beta) = Q^{\rm hard}_{\hat{n},\sigma}(\beta')$ for both polarizations σ and all angles $\hat{n}$. As noted in refs. [121, 178] this condition is highly restrictive, and except in some pathological cases it is only satisfied when the two states are identical, $\beta = \beta'$. The trace over soft graviton emission has then rendered the outgoing matter density matrix completely diagonal. Contrasting this with standard unitary scattering in which states like eq. (3.94) can certainly evolve into superpositions of products of momentum eigenstates, ie., where $|\alpha_1\rangle \to \sum_\beta S_{\beta,\alpha_1}|\beta\rangle$, we see that the emission of soft gravitons has led to complete decoherence in the asymptotic limit where the states have moved off to infinity.

(ii) *Cat State*: Suppose the incoming state, instead of being a simple product of momentum eigenstates, is in a Schrodinger's Cat state, ie., a superposition of states like eq. (3.94). The simplest example would be a state of form

$$|\alpha\rangle = \frac{1}{\sqrt{2}}\,(|\alpha_1\rangle + e^{i\phi}|\alpha_2\rangle). \tag{3.96}$$

where the phase ϕ is a marker for the relative phase between the two components of this superposition (each being a simple product state like eq. (3.94)).

Then, by eq. (3.56), the outgoing matter density matrix is

$$\rho(\beta,\beta') = \;=\; \mathscr{S}_{\beta\beta',\alpha_1\alpha_1} + \mathscr{S}_{\beta\beta',\alpha_2\alpha_2} + e^{i\phi}\mathscr{S}_{\beta\beta',\alpha_1\alpha_2} + e^{-i\phi}\mathscr{S}_{\beta\beta',\alpha_2\alpha_1} \tag{3.97}$$

where, as before, the BMS charge conservation condition is built into these terms using the same δ-function as above. We'll assume that we are in a generic situation, where $\alpha_1 \neq \alpha_2$ implies that $Q^{\rm hard}_{\hat{n},\sigma}(\alpha_1) \neq Q^{\rm hard}_{\hat{n},\sigma}(\alpha_2)$. In this case, the BMS charge conservation condition

$$Q^{\rm hard}_{\hat{n},\sigma}(\alpha_2) - Q^{\rm hard}_{\hat{n},\sigma}(\alpha_1) = Q^{\rm hard}_{\hat{n},\sigma}(\beta) - Q^{\rm hard}_{\hat{n},\sigma}(\beta'), \tag{3.98}$$

has rather interesting implications for the various terms in section 3.5.3.

The first two "diagonal" terms on the RHS of section 3.5.3 will vanish unless $\beta = \beta'$. This leads to a rather general statement—diagonal density matrix elements scatter into diagonal density matrix elements. The last two "interference" terms on the RHS of section 3.5.3 will also vanish unless the BMS charge conservation condition is fulfilled. Thus, in the asymptotic limit, all interference terms are destroyed unless $\bar{\mathbf{Q}}^{\rm hard}_{\hat{n},\sigma} = 0$.

This result is actually quite extraordinary. Physically, for the interference term, the condition that $\bar{\mathbf{Q}}^{\rm hard}_{\hat{n},\sigma} = 0$ is precisely the condition required for the emitted graviton phases as (as encoded in the soft factors) to be the same for the 2 branches of the superposition. This of course is just the condition that the gravitons are not able to distinguish between the two states. This is what one might expect from standard considerations of measurement theory - an environment cannot cause decoherence between 2 states if it cannot distinguish between them. However it is remarkable that the condition for this to be the case is just the sum rule in eq. (3.85).

3.6 Conclusions

Let us now briefly summarize (i) the results we have found, and (ii) summarize the physical conclusions that emerge from these results, and how they compare with previous arguments.

Summary of Results: To give a unified discussion of soft graviton problems, we have chosen to use a non-perturbative formalism which allows us to calculate decoherence and information loss for an arbitrary process involving soft graviton emission. We have used this formalism to calculate the decoherence functional for the matter field, and then evaluated this functional for a scattering problem, in the asymptotic limit appropriate to the matter field S-matrix. This has allowed us to derive results for a "composite S-matrix", which encodes all information about decoherence in the scattering process.

The decoherence functional Γ encodes all information about information loss in any quantum-mechanical process. The functional Γ for asymptotic scattering, appearing in the composite S-matrix we have derived, can be written either in terms of the BMS asymptotic charges, or the gravitational memory associated with the scattering (compare eqs. (3.83) and (3.93)). This makes the connection to known results for the BMS asymptotic charges and the gravitational memory function, and shows how the BMS charge conservation condition operates in a very specific way to either impose (or not impose) decoherence on the final states.

(ii) Physical Implications: As we noted in the introduction, there has been widespread disagreement in the literature over the extent to which information loss occurs in scattering processes, with arguments both for [118, 121, 139, 146], and against [149–152] the existence of information loss from soft gravitons (or soft photons in QED). Part of the problem is that even when authors start from similar formal frameworks (typically either a coherent state approach, or a perturbative approach), they still do not necessarily arrive at the same conclusions. As an example, one may compare the discussions of Carney et al. [121, 156] and Kapec et al. [146] with those of Gabai and Sever, or Mirbabayi and Porrati [149, 150], which arrive at opposite conclusions about information loss starting from the same coherent state formulation of the problem.

Part of the disagreement between different groups stems from the following consideration. Clearly, BMS supertranslation charge conservation requires that classical brehmsstrahlung radiation is emitted when matter is scattered. A classical charged particle with momentum p, receiving an im-pulse which scatters it to momentum p', creates a gauge field disturbance of form

$$h^{\mu\nu}(k) \;\; \sim \;\; \frac{1}{|k|}\left[\frac{p'^{\mu}p'^{\nu}}{k\cdot p'} - \frac{p^{\mu}p^{\nu}}{k\cdot p}\right], \tag{3.99}$$

a sum of contributions from the incoming and outgoing matter momenta.

Now if we choose to dress the incoming matter with radiation which destructively interferes with the $p^{\mu}p^{\nu}/(k\cdot p)$ part of the radiation field, then because soft radiation simply passes through the scattering region [151, 152], the outgoing state will only contain the $p'^{\mu}p'^{\nu}/(k\cdot p')$ part of the radiation field. This outgoing state is also a dressed state, similar to the ingoing state but with different momentum. If one works entirely with dressed states then the outgoing radiation field knows nothing about the incoming matter state, and we get no decoherence; but that is because the incoming radiation field is specifically tuned to get this result! Those groups who do find finite decoherence [118, 121, 139, 146] do not make this assumption.

It is hardly surprising that changing the initial state changes what one finds for decoherence - depending on the couplings one has, this is a general feature of quantum mechanics. The question, of which conditions should be specified for the incoming matter and radiation fields for the present problem, is then a physical question about state preparation. Many possible scenarios can be imagined here, and in the last section we only considered two of these - we have no space to go through all the possibilities.

One purpose of setting up the formalism described here, which explicitly calculates a decoherence functional, is that questions about information loss can be answered just by looking at this functional, which depends only on the assumed ingoing and outgoing states, and the way in which the average over the gauge field is performed. In our calculations we did not use dressed incoming states; the outgoing radiation field then "knows" about both the in- and out-states of the matter. Tracing out the radiation then gives the sum rule eq. (3.91), in which the CHANGE in the hard charge must be the same on the forward and return Keldysh paths. If instead we had assumed dressed incoming states, then our sum rule would rather say that the FINAL hard charge must be the same on the forward and return Keldysh paths - a result consistent with those found in refs. [149–152]. Here we are agnostic as to whether nature prefers dressed scattering states or not, however in the next chapter we will arrive at a rather convincing argument for using dressed states which comes, surprisingly, only from gauge invariance considerations.

Finally, we emphasize that all our scattering calculations involve asymptotic states, ie., we have only been looking at what happens after decoherence has been given an infinite amount of time to take effect. This is why we get "all or nothing" results, ie., we find complete decoherence except for a few special states for which our BMS charge conservation condition $\bar{\mathbf{Q}}^{\text{hard}}_{\hat{n},\sigma} = 0$ is satisfied. If we look at the decoherence away from the asymptotic limit, before full decoherence sets in, one gets much more complex results, to be developed elsewhere.

Chapter 4

Gauge invariant propagators and states in Quantum Electrodynamics

In the previous chapter we discussed the soft-graviton problem in considerable detail, and made remarks on the disagreements in the literature around informations loss and the possibility that nature chooses "dressed states", where matter is accompanied by clouds of zero energy photons and gravitons. In this chapter we develop novel techniques for thinking about states in QED based on the path-integral. One application of techniques developed in this chapter is to show how, by enforcing gauge invariance, one can understand whether or not asymptotic states should be dressed. One of the motivations behind this chapter is to prepare us for the analogous study in quantum gravity—which we discuss in the subsequent chapter.

Here we study gauge invariant states in QED, where states are understood in terms of data living on the boundary of gauge invariant path-integrals. This is done for both scalar and spinor QED, and for boundaries that are either time slices, or the boundaries of a 'causal diamond'. We discuss both the case where the gauge field falls off to zero at the asymptotic boundaries, and the case of 'large gauge transformations', where it remains finite at the asymptotic boundaries. The dynamics are discussed using the gauge-invariant propagator, describing motion of both the particles and the field between the boundaries. We demonstrate how the path-integral naturally generates a 'Coulomb-field' dressing factor for states living on time-slices, and how this is done without fixing any gauge. We show that the form of the dressing depends only on the nature of the boundaries. We also derive the analogous dressing for states defined on null infinity, showing both its Coulombic parts as well as soft-photon parts.

4.1 Introduction

4.1.1 Background and Rationale

We can begin by recalling some of the motivations for giving path-integrals a more central place in the formulation of QM and QFT, and particularly in the definition of states. One key motivation has come from quantum gravity, where the difficulties in defining diffeomorphism-invariant physical quantities [29, 30, 107, 179–183] have led to approaches in which states are defined in terms of information residing on boundaries [184–190]. Much of modern quantum cosmology is also formulated using path integrals [for an introduction to quantum cosmology, see 191]. Another motivation for going beyond conventional "wavefunction & Hamiltonian" QM has been to deal with topological field theories, and in the theory of fractional statistics for many-particle systems [192–195]. On a technical level, that path integrals provide a more general formulation of QM has been known for a long time [196].

For more general classical spacetimes, and similarly for quantum gravity, one can argue that path integrals are actually unavoidable, because for any non-trivial spacetimes involving wormholes, in which there exist achronal regions [184–186, 197, 198], it is already known that one must employ path integrals to handle the dynamics of even simple particles. In these situations, the conventional Hamiltonian framework is then no longer applicable, and simple Hamiltonian evolution is undefined, whereas path integrals can still compute transition amplitudes/probabilities between states defined on time slices far from the achronal regions.

In a related recent development, it has been argued that non-trivial "wormhole" topologies in the quantum gravitational path-integral may play a substantial role in resolving the black hole information loss problem [see eg. 199–201]. The novelty here being that the gravitational path-integral seems to describe a structure more general than just "wavefunction & Hamiltonian" quantum mechanics [202, 203], instead resembling an ensemble of such theories. For these reasons, and those listed above, one may desire to have a "path-integral first" formulation of quantum theory.

Path integrals are commonplace in quantum field theory, however their use has historically been restricted mostly to the calculation of correlation functions of local operators, and by extension, of S-matrix elements. Recently though it been appreciated that path integrals have a more general utility, specifically their use in defining states and density matrices (in the same fashion as the seminal Hartle-Hawking proposal [204] for the wavefunction of the universe). This been appreciated most by the overlapping conformal field theory and quantum gravity communities [see eg. 205, 206]. In this chapter we will use path-integrals with this more modern view, albeit in a more down-to-earth setting, as we study the nature of states in QED. We will start from a "path-integral first" approach to QED, and use the path-integrals to understand various implications of gauge invariance for these states.

Often in quantum field theory one is interested in S-matrix elements, which relate asymptotic scattering states, and it is assumed (not always correctly) that these asymptotic states of the interacting field theory map onto free field states. The other objects of interest are typically the correlation functions of various local operators. Of course, one must appreciate that these quantities can often be more well defined than Schrödinger picture states, and that they have considerable utility for understanding important concepts such as eg. excitation spectra, or renormalization group flow and effective field theories. Furthermore, in conformal field theory the state-operator correspondence makes it that correlation functions are essentially the only quantities necessary to consider in such theories. In light of these facts, we still believe that there are some down-to-earth questions one can have in low energy effective quantum field theories which are best addressed by considering Schrödinger picture states.

One simple issue which we will have in mind here, which may soon have experimental relevance, is the following; if one is dealing with state superpositions involving a large spatial separation of charge or mass, real confusion arises in discussion of what are the correct physical variables, or how to test, eg., whether or not the gravitational field $g^{\mu\nu}(x)$ is quantized [13, 14, 123]. The related question of how to properly define notions like decoherence is also unclear, with different results being derived for decoherence rates by different authors [116, 117, 119, 166, 207]. It was observed in this author's Master's thesis [117] that the main source of this confusion in the literature was in how different authors handled gauge invariance in their calculations, and specifically whether or not they correctly identified the "physical states" in the gauge theory. This potentially subtle point becomes completely clear when one uses path-integrals to prepare states.

At a technical level, while integrating separately over gauge field and matter variables in a path integral, one needs to deal properly with both the constraints and the gauge redundancy. To deal with the latter one typically uses the Faddeev-Popov technique [208]. This still leaves the problem of properly isolating the physical states as one evaluates the "trace" part of the path-integral (ie. the parts where one connects the Keldysh paths).

For this, in a path-integral first approach, one needs an extension of the Faddeev-Popov technique to path integrals with prescribed boundary data. This is precisely what which we will develop and apply in this chapter.

In gauge theories a crucial role is played by constraints, and by the requirement of gauge invariance. This was recognized early on by Dirac, as part of his efforts to quantize constrained theories [167, 168]; he used operator representations of the constraints to annihilate physical states. Dirac was thereby led to introduce gauge-invariant "physical states" in Quantum Electrodynamics (QED) [209]; and the constraints were then the generators of the QED gauge transformations. One can also define gauge-invariant states prepared by a path integral. These states will satisfy the operator constraints provided that the action, the measure, and the set of summed paths are themselves invariant under the transformations generated by the constraints. A good example is provided in quantum gravity by the Hartle-Hawking "no-boundary" wave-function of the universe [204], where one has a Euclidean path-integral over four-dimensional metrics. This state then satisfies the Hamiltonian constraint of Einstein gravity, in the form of the Wheeler-DeWitt equation [210].

The main focus here will be to investigate and give explicit expressions for QED states, defined in the path-integral-first approach, for different kinds of boundary condition. We will work out the details for both scalar and spinor QED, and for 2 different flat spacetimes - in one case, boundary information is specified on two time slices, whereas in the other, it is given on a causal diamond. We will look at different kinds of boundary information, depending on whether the EM field A^μ vanishes at infinity or not, and whether or not one needs to allow for 'large' gauge transformations. At present we have not extended all of the results to linearized gravity although there seems to been no general obstruction to doing so. Here we focus on QED just because it is slightly more simple. One hopes in the future to see this technique to Yang-Mills theory and to quantum gravity beyond the linear limit.

4.1.2 Organization of the Chapter

This chapter is organized as follows. In section 4.2 we consider a quantum particle coupled to the electromagnetic field and derive the form of the gauge-invariant propagator between time slices. From the boundary terms of this propagator we can learn about properties of the quantum states in this theory. We use this simple example to highlight the gauge independence of the results, demonstrate how the boundary phases emerge without fixing a gauge beforehand, and then sketch an eikonal argument for the dressings coming from the remaining path-integral. We close by introducing the boundary Faddeev-Popov trick, and show how it gives the same results.

After this warm-up exercise, in section 4.3 we generalize the results to charged matter described by the Dirac field, ie. discuss how the results and methods of the previous section apply to full QED.

In section 4.4 we start on the much more complicated derivations required for general boundary hypersurfaces. In this section we derive the form of the propagator between states on the future and past regions of a large causal diamond, again for flat space.

Up to this point, all the discussion has been for gauge transformations which vanish at infinity. In section 4.5 we lift this restriction, and extend all of the previous results to the case of "large gauge transformations". This leads to an interesting connection with the soft-photon, large gauge transformation, and dressed state literature. From this work we discuss how one can understand whether or not nature would prefer to choose dressed states, connecting with ideas introduced in chapter 3.

4.2 Scalar Quantum Electrodynamics

In this section we discuss scalar QED; the next section will show how the results carry over to spinor QED. We wish to define gauge invariant physical states for scalar QED, starting from the path integral.

After briefly reviewing some of the salient issues, we define gauge invariant propagators for scalar QED, and show how they can be written so that a boundary term separates from the rest. This boundary term reveals features of the physical states of the theory. In this section we will assume the boundaries are defined by constant time slices in Minkowkski spacetime. The boundary term we find is reminiscent of Dirac's well known phase factor.

The calculation is manifestly gauge invariant throughout, and is carried out in two different ways. The first relies on a 'natural' transformation of the action for the system, whereas the second involves a generalization of the standard Faddeev-Popov technique [208] to include boundaries.

4.2.1 States in Quantum Electrodynamics

The question of how to define gauge invariant states in QED has a long history; here we recall some of the key arguments, and set up the calculations to follow.

States and Conservation Laws

In classical electrodynamics, Gauss' law $\vec{\nabla} \cdot \vec{E} = \rho$, obviously does not completely specify the electric field. One can add to the Coulomb solution any divergence-less field. In quantum theory the situation is analogous, the Gauss law operator constraint for physical states,

$$\left(\vec{\nabla} \cdot \hat{\vec{E}} - \hat{\rho}\right)|\Psi\rangle = 0, \tag{4.1}$$

has no unique solution. This becomes particularly important when one considers the construction of gauge-invariant states of charged particles in field theory.

Looking at the Schrödinger picture, in the $\hat{A}_j(x)$ field value basis the electric field operator is the (negative of the) conjugate momentum and is thus represented by a functional differential operator

$$\hat{E}^j(x) = i\frac{\delta}{\delta A_j(x)}. \tag{4.2}$$

If we consider a state with a single charged particle with charge e located at the origin, then any state of the gauge field of the form

$$\Psi[A] = U_{\mathcal{E}^j}\, \psi[A] = e^{-ie\int d^3x \mathcal{E}^j(x) A_j(x)}\, \psi[A], \tag{4.3}$$

with $\vec{\nabla} \cdot \vec{\mathcal{E}}(x) = \delta^{(3)}(x)$ and $\vec{\nabla} \cdot \hat{\vec{E}}(x)\, \psi[A] = 0$ is a valid physical state: provided that the gauge transformations vanish at spatial infinity[1]. These states correspond to coherent electric fields on top of the state $\psi[A]$,

$$[\hat{E}^j(y), e^{-ie\int d^3x \mathcal{E}^j \hat{A}^j}] = e\mathcal{E}^j(y)\, e^{-ie\int d^3x \mathcal{E}^j \hat{A}^j}, \tag{4.4}$$

and this is sometimes referred to as an electric field "dressing" the charge. Any solution to the classical Gauss law sourced by this point charge will be a valid function $\mathcal{E}^j$, so we have the same ambiguity in the quantum theory as in the classical theory.

An intuitive solution for the physical state dressing is just the Coulomb field

$$\mathcal{E}^j_C(x) = \frac{1}{4\pi}\frac{x^j}{|\vec{x}|^3}, \tag{4.5}$$

[1] This assumption is essential here, and later in the text we will consider the case with non-vanishing asymptotic "large" gauge transformations.

but one could also consider solutions such as the planar field, $\mathcal{E}^j_\perp$, or Faraday line, $\mathcal{E}^j_s$, given by

$$\begin{gathered}\mathcal{E}^z_\perp = 0 \\ \mathcal{E}^x_\perp = \frac{1}{2\pi}\delta(z)\frac{x}{x^2+y^2} \qquad\qquad \mathcal{E}^y_\perp = \frac{1}{2\pi}\delta(z)\frac{y}{x^2+y^2},\end{gathered} \tag{4.6}$$

and

$$\begin{gathered}\mathcal{E}^x_s = \mathcal{E}^y_s = 0 \\ \mathcal{E}^z_s = \theta(z)\delta(x)\delta(y),\end{gathered} \tag{4.7}$$

respectively. More generally one can consider any path Γ coming from spatial infinity and terminating on the charge, By integrating the one-form $\hat{A}_j dx^j$ along such a path one can construct a physical state dressing

$$\hat{U}_\Gamma = e^{-ie\int_\Gamma dx^j \hat{A}_j}, \tag{4.8}$$

corresponding to a general Faraday line of electric flux. Remarkably, the Coulomb and planar fields can be seen as certain smearings of these more general line fluxes, however this equivalence appears to be unique to Abelian gauge theory [211]. These Faraday line dressings have received considerable attention historically, and also recently, in both gauge theories and quantum gravity [32, 33, 203, 211–220].

This discussion is not restricted to states, one could similarly consider physical (gauge-invariant) operators. Indeed in Dirac's original paper on the gauge invariant formulation of QED it was noted that we can take the field operator for the Dirac field $\hat{\psi}(x)$ of a particle with charge e and modify it by a dressing of the form in eq. (4.3), $\hat{\psi} \to \hat{\psi}e^{-ie\hat{C}}$, with

$$\hat{C} = \int d^3x\, \mathcal{E}^j \hat{A}_j, \tag{4.9}$$

to obtain a gauge-invariant operator which creates Dirac particles with an accompanying electric field. In the recent literature, interest in Faraday line type dressings of field operators has come from inherent non-locality of these dressed operators. The recent work on this by Giddings and collaborators demonstrate that this inherent non-locality may have interesting implications for AdS/CFT, the black hole information problem, and the issues of soft graviton dressing [203, 217–220]. We do not however have the space here to review all of these recent developments.

In electrodynamics one can straightforwardly demonstrate that any solution $\mathcal{E}^j$ is just the superposition of a Coulomb solution and propagating electromagnetic waves. Computing the electromagnetic energy reveals that the Coulomb solution is that of minimal energy, but one could also just directly solve the field equations to see that the Faraday line will immediately dissolve into electromagnetic waves propagating to null infinity leaving behind a static Coulomb field [216]. When considering various dressings to make charged particle states gauge invariant, it then seems natural to pick the Coulomb solution. However, the situation is not so straightforward.

To highlight why it is unclear which electric field dressing to chose, first assume that the Coulomb dressing, call it $\hat{U}_C$, is the correct choice. Now, a position eigenstate $|x_1\rangle$ of a charged particle is dressed by a Coulomb field centered on position x_1, ie. $\hat{U}_{C_1}|x_1\rangle$. This state is an eigenstate of the longitudinal electric field operator, ie. it has a well defined value if one measures $\vec{\nabla}\cdot\hat{\vec{E}}$. Considering instead a simple superposition of position eigenstates: if we dress the particles accordingly, the state now describes a superposition of Coulomb fields centered on x_1 or x_2,

$$\frac{1}{\sqrt{2}}\big(|x_1\rangle + |x_2\rangle\big) \rightarrow \frac{1}{\sqrt{2}}\big(\hat{U}_{C_1}|x_1\rangle + \hat{U}_{C_2}|x_2\rangle\big), \tag{4.10}$$

and this state of course no longer has a well defined electric field. That is, the state is no longer an eigenstate of the longitudinal electric field operator. Intuitively it makes sense though that such a superposition would generate a superposition of Coulomb fields. But now consider a wavepacket $\psi_{\vec{p}}(x)$ peaked in position space at $\vec{x}=0$ and in momentum space at $\vec{p}$. If Coulomb dressing is the universally correct prescription for gauge invariant states then we expect this state to dress in the following way,

$$|\psi_{\vec{p}}\rangle \rightarrow \int d^3x \;\; \psi_{\vec{p}}(x)\, \hat{U}_{C_x}|x\rangle. \tag{4.11}$$

This superposition of Coulomb fields does not resemble the classical Liénard-Wiechert field of a moving charge—it doesn't even have a well defined electric field eigenvalue.

We might expect instead that the state $|\psi_{\vec{p}}\rangle$ should be dressed by an electric field $\mathcal{E}^j$ which describes the Liénard-Wiechert field, but this is not the case here. In hindsight, how could it? The Liénard-Wiechert solution makes explicit reference to the trajectory of the moving charge, not just its instantaneous state. This simple example then highlights i) that a universal choice of dressing (ie. one choice of $U_{\mathcal{E}}$ to apply to all states) can generate an unusual electric field which is contrary to our physical intuition, and ii) that any attempt to give state-dependent dressings will require a discussion of the history of the matter system, ie. a discussion of state preparation.

A natural starting point to understand the electric fields around charged particles in nature is then the path integral. The path integral will make explicit reference to trajectories of the matter system, and also has a transparent semi-classical limit which allows us to compare our results with our intuition. We will see that the correct answer to the questions posed above will come in a path-integral formulation from the fact that one can give a unique separation between constrained and unconstrained variables, which is manifestly gauge invariant.

Set-up

We know from, eg., the Hartle-Hawking work [204] that path-integrals can be used to prepare gauge-invariant vacuum states. The basic idea is to then generalize this idea to arbitrary states, maintaining gauge invariance throughout, and see what emerges. Some of the questions we are interested in answering include:

(a) What sort of electromagnetic dressing is "chosen" by states defined in this way?

(b) How do the physical states so defined depend on the geometry of the surface they live in?

(c) What are the physical degrees of freedom involved in spatial superpositions of charges, and what is the natural of entanglement between charges and the electromagnetic field?

(d) In defining decoherence and information loss, what states should we average over, how do we distinguish between real decoherence and "false" decoherence, and what is the correct way to calculate decoherence rates?

In all sections of this paper we will use manifestly gauge-invariant path-integrals to address them. We will introduce a novel 'boundary Faddeev-Popov' (bFP) trick to derive results - it is a straightforward generalization of the textbook Faddeev-Popov trick to path-integrals with prescribed boundary data. The path integrals we consider will be on the extended configuration space of the gauge field. That is, we prescribe data for all components A_μ, and do not a priori concern ourselves with the non-canonical nature of A_0, or the implementation of Gauss' law as a constraint.

However we find that, because all amplitudes are manifestly gauge invariant, the expected constraints are naturally implemented. One finds that boundary phases engender states which are eigenstates of certain parts of the electric field operator. We find that a natural separation of variables occurs, whereby a preferred solution to the constraint equation emerges kinematically, and the additional gauge invariant data is not inserted by hand, but rather it emerges dynamically from the path-integral.

In this section we will look at scalar electrodynamics, wherein a single charged particle and the quantum electromagnetic field propagate between two time slices (surfaces of constant t) in Minkowski space. We will assume here (but not later in the paper) that the gauge field A_μ, and thus the possible gauge transformations of it, will vanish sufficiently quickly at spatial infinity that surface terms generated by spatial integrations by parts can be ignored. In the final section we will consider "large" gauge transformations.

We therefore consider a non-relativistic quantum particle with position q, charge e, in an external potential $V(q,t)$, and coupled to the electromagnetic field A_μ. The extension to multiple particles is trivial. The action for the system evolving from an initial time t_i to a final time t_f is $S = S_M + S_{EM}$, where

$$S_M[q] = \int_{t_i}^{t_f} dt \left[\frac{1}{2}m\dot{q}^2 - V(q,t)\right] \tag{4.12}$$

describes the particle alone, and

$$S_{EM}[q, A_\mu] = \int_{t_i}^{t_f} d^4x \left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + A_\mu J^\mu\right] \tag{4.13}$$

describes the electromagnetic field along with the coupling to the matter; here $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor, and $J^0(x) = e\delta^3(x - q(t))$ and $J^j = e\dot{x}^j\delta^3(x - q(t))$ are components of the charge current. Note here that the current for a charged particle is conserved even when the equations of motion are not satisfied, ie. for a general path in the path integral.

We assume the path integral describing the propagator for this system resides within the region of spacetime shown in Fig. 4.1; we denote the surface of constant time $t = t_f$ by Σ_f, and likewise for t_i, Σ_i. An asymptotic timelike cylinder $S^2 \times \mathbb{R}$ at arbitrarily large radius will be denoted Σ_B. Our path integral is then over field configurations and particle trajectories in a region $\mathcal{V}$ bounded by $\partial\mathcal{V} = \Sigma_f \cup \Sigma_i \cup \Sigma_B$.

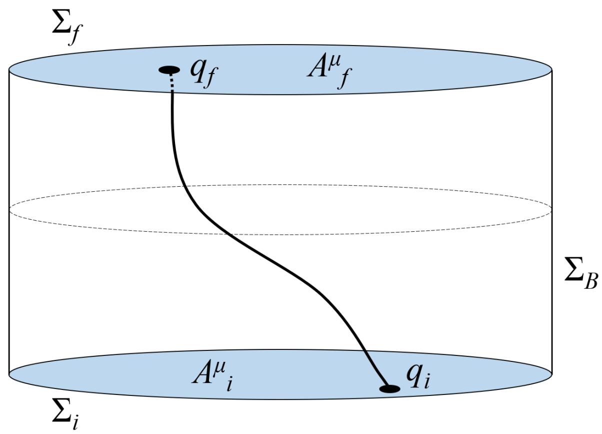


Figure 4.1: Depiction of the spacetime region through which the propagator K_{fi} in eq. (4.14) propagates, and thus, where the path-integration occurs. The initial configuration is on the timeslice surface $t = t_i$, the final configuration on the surface $t = t_f$. Particle paths propagate between q_i at t_i and q_f at t_f, an example of one such path is illustrated. Additionally, the gauge field propagates between prescribed configurations A_i^μ and A_f^μ on these same two timeslices.

We will choose to quantize the system on the extended configuration space, ie. considering all configurations of $A_\mu(x)$ before quantization rather than imposing constraints and gauge conditions at the classical level and quantizing the remaining degrees of freedom. The path integral describing the amplitude for transition between configurations $q_i, A_{\mu\,i}$ and $q_f, A_{\mu\,f}$ is then

$$\begin{aligned} K_{fi} &\equiv K(q_f, A_{\mu\,f}; q_i, A_{\mu\,i}) \\ &= \int_{q_i}^{q_f} \mathcal{D}q\, e^{iS_M} \int_{A_{\mu\,i}}^{A_{\mu\,f}} \mathcal{D}A_\mu\, e^{iS_{EM}}. \end{aligned} \tag{4.14}$$

Here and throughout this paper we will absorb field independent constants into the path integral measure.

Path integrals over gauge fields are usually handled using a gauge fixing procedure, to divide out the divergent volume from gauge equivalent field configurations. Here we will actually delay performing the FP procedure and perform some formal manipulations of the path-integral before proceeding to fix a gauge. The final result will ultimately be the same; but this order of operations turns out to be illuminating[8].

4.2.2 Gauge Invariant Propagator

The propagator in eq. (4.14) is manifestly gauge invariant under independent transformations of the initial and final data: provided we simultaneously transform the wavefunction of the particle and transform the gauge field.

To see this explicitly, consider the transformed propagator

$$\begin{aligned} K^{\Lambda}_{fi} &\equiv K^{\Lambda}(X_f, A_{\mu\,f}; X_i, A_{\mu\,i}) \\ &= e^{-ie\Lambda_f(q_f)} K(X_f, A^{\Lambda_f}_{\mu\,f}; X_i, A^{\Lambda_i}_{\mu\,i}) e^{ie\Lambda_i(q_i)}, \end{aligned} \tag{4.15}$$

where $\Lambda_{i(f)}$ is the gauge parameter on the initial (final) time slice, and the gauge field transforms as $A^{\Lambda}_{\mu} = A_{\mu} + \partial_{\mu}\Lambda$. The phases come from transforming the wavefunctions of the charged particle on the initial and final surfaces. The propagator with transformed boundary data can be expressed simply in terms of the original propagator. We can take the expression

$$K(q_f, A^{\Lambda_f}_{\mu\,f}; q_i, A^{\Lambda_i}_{\mu\,i}) = \int_{q_i}^{q_f} \mathcal{D}q\, e^{iS_M} \int_{A^{\Lambda_i}_{\mu\,i}}^{A^{\Lambda_f}_{\mu\,f}} \mathcal{D}A_{\mu}\, e^{iS_{EM}} \tag{4.16}$$

and perform a change of variables, $A_{\mu} = A'_{\mu} + \partial_{\mu}\Lambda$, for some time dependent function Λ which takes the value $\Lambda_{i,f}$ on $\Sigma_{i,f}$.

The boundary data for the new variable A'_{μ} is now just the original configuration, $A_{\mu\,i,f}$. The action is not invariant under this change of variables, but instead, it acquires a boundary term

$$\begin{aligned} \delta_{\Lambda} S_{EM} = \int_{\partial V} d^3x \Lambda\, n_{\mu} J^{\mu} &= \int_{\Sigma_f} d^3x \Lambda_f J^0 - \int_{\Sigma_i} d^3x \Lambda_i J^0 \\ &= e\left(\Lambda_f(q_f) - \Lambda_i(q_i)\right). \end{aligned} \tag{4.17}$$

Note that the contributions from the surface Σ_B will vanish as the radius is taken to infinity because of our initial assumptions, but the contributions on the space-like parts of the boundary, $\Sigma_f \cup \Sigma_i$, will not. The action is expressed in terms of A'_{μ} then as

$$S_{EM}[q, A] = S_{EM}[q, A'] + e\Lambda_f(q_f) - e\Lambda_i(q_i), \tag{4.18}$$

[8] One may object that the order of operations is not necessarily commutative, since the gauge group volume is infinite and thus the path-integral is divergent unless a gauge is fixed. To this we respond: we can always assume that our variables actually live on a finite spacetime lattice, so that the path-integral is not actually divergent before fixing a gauge.

so that the propagator with transformed boundary data then reads

$$K(q_f, A^{\Lambda_f}_{\mu f}; q_i, A^{\Lambda_i}_{\mu i}) = \int_{q_i}^{q_f} \mathcal{D}q\, e^{iS_M[q]} \int_{A_{\mu i}}^{A_{\mu f}} \mathcal{D}A'_\mu\, e^{iS_{EM}[q,A'] + e\Lambda_f(q_f) - e\Lambda_i(q_i)}$$
$$= e^{ie\Lambda_f(q_f)}\; K_{fi}\; e^{-ie\Lambda_i(q_i)}. \tag{4.19}$$

The boundary phases in eq. (4.19) generated by the gauge-field action will then precisely cancel the phases in section 4.2.2 arising from the $U(1)$ transformation of the matter wavefunctions, and therefore the propagator for the total system is gauge invariant.

We know from Hamiltonian dynamics that the Gauss law constraint is the generator of gauge transformations. The propagator (4.14) should therefore satisfy Gauss' law as an operator constraint on both Σ_f and Σ_i. To see this, consider a gauge transformation which vanishes on Σ_i but not on Σ_f, and rewrite the transformed propagator using a linear shift operator, as

$$K^{\Lambda}_{fi} = e^{-ie\Lambda_f(q_f) + \int_{\Sigma_f} d^3x \partial_\mu \Lambda_f \frac{\delta}{\delta A_{\mu f}}}\; K_{fi} \tag{4.20}$$

Since the propagator is gauge invariant, this implies the following simple functional differential equation

$$\begin{aligned} 0 \;&=\; \left[-ie\Lambda_f(q_f) + \int_{\Sigma_f} d^3x \partial_\mu \Lambda_f \frac{\delta}{\delta A_{\mu f}} \right] K_{fi} \\ &=\; \left[\int_{\Sigma_f} d^3x\, \partial_0 \Lambda_f \frac{\delta}{\delta A_{0\,f}} - i \int_{\Sigma_f} d^3x \Lambda_f \left(e\delta^3(q_f - x) - i\partial_j \frac{\delta}{\delta A_{j\,f}} \right) \right] K_{fi} \end{aligned} \tag{4.21}$$

The remaining functional derivative of the propagator with respect to $A_{j\,f}$ is just the electric field operator. This is seen in a "path-integral first" treatment by evaluating the functional derivative and using the standard expression for variations of the action endpoint in mechanics,

$$\begin{aligned} \frac{\delta K_{fi}}{\delta x_f} =& i \int_{x_i}^{x_f} \mathcal{D}q\, \frac{\delta S[q]}{\delta x_f}\, e^{iS[q]} \\ =& i \int_{x_i}^{x_f} \mathcal{D}q\, \frac{\partial \mathscr{L}}{\partial \dot{q}(t)}\bigg|_{t_f}\, e^{iS[q]}. \end{aligned} \tag{4.22}$$

Written in terms of the electric field operator we then get the constraint equation

$$0 = \left[\int d^3x\, \partial_0 \Lambda_f \frac{\delta}{\delta A_{0\,f}} - i \int d^3x \Lambda_f \left(e\delta^3(q_f - x) - \partial_j \hat{E}^j \right) \right] K_{fi}. \tag{4.23}$$

On the surface Σ_f, the functions Λ_f and $\partial_0\Lambda_f$ are independent, but arbitrary, functions which vanish at spatial infinity. As a result, the propagator then satisfies two separate local constraint equations

$$\frac{\delta}{\delta A_{0\,f}} K_{fi} \;=\; 0 \tag{4.24}$$

$$\left(\partial_j \hat{E}^j - \hat{J}^0\right) K_{fi} \;=\; 0, \tag{4.25}$$

so that, as expected, the gauge invariant propagator defined on the extended configuration space satisfies the Gauss law operator constraint. It is also independent of the prescribed data for A_0. We wont repeat the exercise, but the same logic demonstrates that identical constraints also apply on the initial surface Σ_i. These constraints are precisely what one imposes on physical states in Dirac's formalism for quantizing constrained Hamiltonian systems [167, 168, 221]. It is reassuring to see that they also emerge here from the path-integral first approach to defining states.

Since the propagator is independent of the data prescribed for A_0, up to normalization constants we can then freely integrate over the boundary data for A_0, to get

$$K_{fi} \;=\; \int_{q_i}^{q_f} \mathcal{D}q\, e^{iS_M} \int \mathcal{D}A_0 \int_{A_{j\,i}}^{A_{j\,f}} \mathcal{D}A_j\, e^{iS_{EM}}. \tag{4.26}$$

showing that A_0 is not a true dynamical variable. This is a well known point, but the way we demonstrated it will be useful for more general amplitudes.

No gauge-fixing is required to make the A_0 integral convergent, and the boundary data is unfixed, so we can directly go ahead and evaluate the integral. As we will see shortly, this rather uniquely determines how we should gauge fix the remaining A_j integral, and consequentially determines the form of the dressing for the states.

Notice that the boundary data for A_0 naturally fell out of the expression as a consequence of gauge invariance - there was no need for a detour through canonical Hamiltonian quantization, or a discussion of the missing conjugate momentum Π^0 to see this point. Geometrically this happens here because we chose to evolve between constant time slices, and the pullback of the 1-form $A_\mu dx^\mu$ to these boundaries is independent of A_0, making A_0 redundant, ie., not a true dynamical variable. We will see that in other boundary geometries, the redundant variable will again be the part of A_μ normal to the boundary.

4.2.3 Extracting the Dressing

We will now take the form (4.26) as a starting point, and look to evaluate the A_0 integral. Looking at the electromagnetic part of the action we can separate out the A_0 dependent terms

$$S_{EM} = \int_{t_i}^{t_f} d^4x \Big[-\frac{1}{4}F_{jk}F^{jk} + A_j J^j - \frac{1}{2}F^{j0}(\partial_j A_0 - \partial_0 A_j) + A_0 J^0 \Big] \quad (4.27)$$

Integrating the spatial derivatives by parts we obtain

$$S_{EM} = \int_{t_i}^{t_f} d^4x \Big[-\frac{1}{4}F_{jk}F^{jk} + A_j J^j + \frac{1}{2}F^{j0}\partial_0 A_j + \frac{1}{2}A_0\big(\partial_j F^{j0} + J^0\big) + \frac{1}{2}A_0 J^0 \Big] \quad (4.28)$$

The variable A_0 appears quadratically in the action, and since its end-points are being integrated over in eq. (4.26), it can be integrated out as a simple Gaussian integral. The result of evaluating the gaussian integral is to just substitute the saddle point solution $\tilde{A}_0$ back into eq. (4.28).

The saddle point equation for A_0 is just the Gauss law Maxwell equation, viz.,

$$(\partial_j F^{j0} + J^0) \;=\; -\partial_j \partial^j A_0 + \partial_0 \partial^j A_j + J^0 \;=\; 0. \quad (4.29)$$

for which the solution is

$$\tilde{\mathcal{A}}_0 = \nabla^{-2} J^0 + g + h, \quad (4.30)$$

where g is given by

$$g = \partial_0 \nabla^{-2}(\partial^j A_j), \quad (4.31)$$

where h is an undetermined homogeneous solution to the Laplace equation, and where ∇^{-2} is a symbolic representation of the Laplace Green's function,

$$\nabla^{-2} f(x) = -\frac{1}{4\pi}\int d^3y \frac{f(y)}{|\vec{x} - \vec{y}|}. \quad (4.32)$$

Regarding the homogeneous solution h, the only such solution which is regular at the origin while also vanishing at spatial infinity is the trivial solution, $h(x) = 0$, so we set $h = 0$. The solution (4.30) is then unique, without needing to impose further gauge fixing to eliminate the homogeneous solutions. If however we allow for large gauge transformations, then non-trivial expressions for $h(x)$ arise, and further gauge fixing is required. We will discuss this in section 4.5.

Notice that $\tilde{\mathcal{A}}_0$ is given in terms of a gauge invariant term $\nabla^{-2}J^0$ and a gauge variant term g. Under gauge transformation g transforms as $\delta_\Lambda g =$

$\partial_0\Lambda$, as it must, so that $\tilde{\mathcal{A}}_0$ transforms appropriately. Taking inspiration from this, we formally isolate the gauge invariant part of the remaining components A_j by defining

$$A_j = \mathcal{A}_j + \partial_j \Phi, \tag{4.33}$$

where $\delta_\Lambda \mathcal{A}_j = 0$, and Φ is a functional of A_j with the assumed transformation property $\delta_\Lambda \Phi = \Lambda$. The functions $(\mathcal{A}_j, \Phi)$ are just a new choice of field variables for the path-integration. To avoid introducing a field-dependent Jacobian into the integration measure, we will assume the *g-potential* Φ to be a linear functional of the A_j. Note that Φ is not given uniquely by the required transformation property: for now we leave it unspecified.

At this point, one who is familiar with the Hamiltonian quantization of QED might assume that $\mathcal{A}_j$ and $\partial_j\Phi$ are just the transverse and longitudinal parts of A_j. This is certainly a valid decomposition, but it is not the only choice—the freedom here exactly paralleling the freedom in the choice of $\mathcal{E}_j$ discussed in section 4.2.1. Rather than assuming the transverse/longitudinal decomposition, we will instead rewrite the path-integral in terms of the new variables $\mathcal{A}_j$ and $\partial_j\Phi$, and look for a natural decomposition of the path integral. We will see that the action, transformed to the new variables, ends up separating into a non-dynamical boundary term, plus terms which depend only on the charged particle and the new field variables $\mathcal{A}_j(x)$.

New Variables for the Action and Propagator

We begin by writing the propagator K_{fi} in terms of $\mathcal{A}_j$ in eq. (4.33) and the solution (4.30) for $\tilde{A}_0$, to get

$$K_{fi} \;=\; \int_{q_i}^{q_f} \mathcal{D}q \, e^{iS_M} \int_{\Phi_i}^{\Phi_f} \mathcal{D}\Phi \int_{\mathcal{A}_{j\,i}}^{\mathcal{A}_{j\,f}} \mathcal{D}\mathcal{A}_j \, e^{i\tilde{S}_{EM}}, \tag{4.34}$$

with a new electromagnetic field action $\tilde{S}_{EM}$ given by

$$\begin{aligned}\tilde{S}_{EM} = \int_{t_i}^{t_f} d^4x \Big[&-\frac{1}{4}F_{jk}F^{jk} + \mathcal{A}_j J^j + \partial_j \Phi J^j \\ &+ \frac{1}{2}\tilde{F}^{j0}\partial_0 \mathcal{A}_j + \frac{1}{2}\tilde{F}^{j0}\partial_0\partial_j\Phi + \frac{1}{2}J^0\nabla^{-2}J^0 + gJ^0\Big],\end{aligned} \tag{4.35}$$

where we've introduced the notation $\tilde{F}_{j0} = \partial_j \tilde{\mathcal{A}}_0 - \partial_0 A_j$. Note that F_{jk} is independent of Φ by antisymmetry.

We can now integrate by parts to strip the spatial derivatives off Φ, to get

$$\begin{aligned}\tilde{S}_{EM} \;=\; \int_{t_i}^{t_f} d^4x \Big[& \frac{1}{2}\tilde{F}^{j0}\partial_0\mathcal{A}_j - \frac{1}{4}F_{jk}F^{jk} + \mathcal{A}_j J^j \\ & + \frac{1}{2}J^0\nabla^{-2}J^0 - \Phi\partial_j J^j - \frac{1}{2}\partial_j\tilde{F}^{j0}\partial_0\Phi + gJ^0\Big]. \qquad (4.36)\end{aligned}$$

We then use the definition $\partial_j\tilde{F}^{j0} = -J^0$ along with the fact that $\partial_\mu J^\mu$ for an arbitrary trajectory of the particle, to rewrite the action as

$$\begin{aligned}\tilde{S}_{EM} \;=\; \int_{t_i}^{t_f} d^4x \Big[& \frac{1}{2}\tilde{F}^{j0}\partial_0\mathcal{A}_j - \frac{1}{4}F_{jk}F^{jk} + \mathcal{A}_j J^j \\ & + \frac{1}{2}J^0\nabla^{-2}J^0 + \Phi\partial_0 J^0 + \frac{1}{2}J^0\partial_0\Phi + gJ^0\Big], \qquad (4.37)\end{aligned}$$

This result reveals something remarkable—if we now make the choice $\partial_0\Phi = g$ for Φ, then the last three terms sum to a total time derivative. There is of course nothing forcing us to choose this form for Φ; since we are just making a change of path-integration variable, the final result for the propagator cannot depend on which variables we choose. We will make this choice, and since g itself is given as the time derivative of $\nabla^{-2}(\partial_j A^j)$, we can simply choose

$$\Phi = \nabla^{-2}(\partial_j A^j). \qquad (4.38)$$

so that our new field variable now becomes

$$\mathcal{A}_j = A_j - \partial_j\nabla^{-2}(\partial^k A_k) \qquad (4.39)$$

This is in fact just the standard transverse-longitudinal decomposition of A_j. We can see very clearly that $\partial^j\mathcal{A}_j = 0$, so it is indeed "transverse". Note however, that instead of assuming this from the start, we saw that this decomposition is simply dictated by the solution to the A_0 saddle point equation. This pattern of logic will be used again in later sections when we consider geometries for which it is much less clear *a priori* how to define a 'transverse part' of A_j.

Let us now complete the process of transforming to the new form for the field action. Note first that our choice of decomposition also simplifies the expression for the electric field, to

$$\begin{aligned}\tilde{F}_{j0} &= \partial_j\tilde{\mathcal{A}}_0 - \partial_0 A_j \\ &= \partial_j\nabla^{-2}J^0 - \partial_0\mathcal{A}_j, \qquad (4.40)\end{aligned}$$

ie., a manifestly gauge invariant form; and it renders $\mathcal{A}_j$ divergenceless.

We then find that a simple integration by parts gives

$$\int_{t_i}^{t_f} d^4x \frac{1}{2}\tilde{F}^{j0}\partial_0\mathcal{A}_j = \int_{t_i}^{t_f} d^4x \frac{1}{2}\partial_0\mathcal{A}^j\partial_0\mathcal{A}_j. \tag{4.41}$$

so that the field action takes the form

$$\tilde{S}_{EM} = \int_{\partial\mathcal{V}} \sigma\, d^3x J^0\nabla^{-2}(\partial_j A^j) + \tfrac{1}{2}\int_{t_i}^{t_f} d^4x \Big[-\partial_\mu\mathcal{A}^j\partial^\mu\mathcal{A}_j + 2\mathcal{A}_j J^j + J^0\nabla^{-2}J^0\Big] \tag{4.42}$$

with $\sigma = \pm 1$ for the future and past parts of the boundary respectively.

Let us now combine this field action, as written here in terms of the new variables, with the original matter action S_M in eq. (4.12), to get a complete form for the action

$$\tilde{S} \;=\; \tilde{S}_M + \tilde{S}_C + \tilde{S}_A \tag{4.43}$$

with the three new terms defined as follows:

(i) We incorporate the 'Coulomb self-energy' term from $\tilde{S}_{EM}$ into the matter action, to give

$$\tilde{S}_M \;=\; S_M + \tfrac{1}{2}\int_{t_i}^{t_f} d^4x\, J^0\nabla^{-2}J^0 \tag{4.44}$$

with S_M given by eq. (4.12) as before. We see that $\tilde{S}_M$ is gauge invariant.

(ii) The term $\tilde{S}_C$ is just a boundary term; we have

$$\begin{aligned}\tilde{S}_C &= \int_{\partial\mathcal{V}} \sigma\, d^3x\, J^0\nabla^{-2}(\partial_j A^j) \\ &= -\left[\int_{\Sigma f} d^3x\, A_{j\,f}(x)\frac{e}{4\pi}\frac{(y-q_f)^j}{|y-q_f|^3} - \int_{\Sigma f} d^3x\, A_{j\,i}(x)\frac{e}{4\pi}\frac{(y-q_i)^j}{|y-q_i|^3}\right]\end{aligned} \tag{4.45}$$

and we shall see shortly in what way this is related to Dirac's phase.

(iii) The dynamic part of the EM action - including both the free field term and the interaction with the matter current - is now

$$\tilde{S}_A \;=\; \int_{t_i}^{t_f} d^4x\big[-\tfrac{1}{2}\partial_\mu\mathcal{A}^j\partial^\mu\mathcal{A}_j + \mathcal{A}_j J^j\big] \tag{4.46}$$

and we see that, like $\tilde{S}_M$, this is also gauge invariant.

At the risk of future confusion, we will henceforth omit the σ and leave it implicit that a minus sign should be in front of the integral when integrating over the past part of the boundary. All of the variables in the bulk action are gauge invariant, while the boundary term transforms precisely as we determined it ought to in eq. (4.19). Also, the g-potential Φ is not at all present in the bulk action: it appears only in the boundary term.

We can now write the propagator K_{fi} in the form that we want. Since Φ does not appear in the bulk action, we can freely integrate over it to yield a harmless overall (divergent) normalization. Doing this, and continuing to absorb field independent constants into the measure, we arrive at our final expression for the propagator:

$$K_{fi} = e^{i\tilde{S}_C} \int_{q_i}^{q_f} \mathcal{D}q \, e^{i\tilde{S}_M} \int_{\mathcal{A}_i^j}^{\mathcal{A}_f^j} \mathcal{D}\mathcal{A}^j \, e^{i\tilde{S}_A}. \tag{4.47}$$

where $\mathcal{A}_i^j$ and $\mathcal{A}_f^j$ are the initial and final configurations of the transverse gauge field $\mathcal{A}^j(x)$.

This equation for K_{fi} is one of our key results - we have shown that the original form eq. (4.14) for K_{fi} can be rewritten as eq. (4.47). If we compare the boundary phase $e^{i\tilde{S}_C}$ with the expressions eqs. (4.3) and (4.5), we see that this boundary phase precisely describes a Coulomb field centered on the location of the charge. Furthermore, our result demonstrates that the transverse field will be determined dynamically by the remaining path integral. We emphasize that the arguments above were clearly independent of a gauge choice since we never explicitly chose a gauge. Thus the Coulomb form $\tilde{S}_C$ in eq. (4.45) arises naturally from boundary terms in the path-integral, and is not a consequence of choosing the Coulomb gauge.

Demonstrating how these boundary phases emerge was one of our primary goals here, however in doing so we've found that we can actually use the expression (4.47) to address one of the questions mentioned in the introduction. Specifically, we can address the proper way to describe decoherence from coupling to a gauge field. Since the transverse components degrees of freedom are the only independent degrees of freedom in A_μ, if we were to try and "trace" out the electromagnetic field to get a reduced density matrix for the particle we would only be able to integrate over the transverse components $\mathcal{A}_j$. This observation is not new, however it can be easily missed when using path-integrals to describe open quantum systems. Typically in a QED path-integral one simply inserts a gauge-breaking term into the action, accompanied by the appropriate ghost terms if necessary. If one uses a Schwinger-Keldysh path-integral formalism to describe decoherence though, naively inserting these terms into the action will not properly account for the fact that only the transverse field components are independent—one must instead use an expression such as eq. (4.47) (see [117] for details in QED and quantum gravity). This mistake was made recent popular publication [116], and unfortunately their mistake was propagated as their results were used by others to describe decoherence [222].

Example: Eikonal Approximation

Let us briefly discuss what one expects to find from evaluating the remaining path-integral over the transverse gauge field degrees of freedom, and the point charge. We will not, in this chapter, attempt to discuss detailed examples. However, the lowest-order eikonal approximation does serve to illustrate what one can expect. We give a very heuristic treatment here - more detail is found in, eg. the works of Fradkin or Fried [119, 159, 223], and also in chapter 3.

In this lowest-order eikonal approximation, fluctuations in the charge trajectory about the classical saddle point are neglected in the current. Starting from the effective field term $\tilde{S}_A$ in eq. (4.46), we write it as $\tilde{S}_A = \tilde{S}_A^0 + \tilde{S}_A^{int}$, where the interaction term is

$$\tilde{S}_A^{int} = \int_{t_i}^{t_f} d^4x \mathcal{A}_j(x) J^j(x) = e \int_{t_i}^{t_f} dt \dot{q}^j(t) \mathcal{A}_j(q(t)) \tag{4.48}$$

We then expand $q(t)$ as $q(t) = q_{cl}(t) + \delta q(t)$, where the classical trajectory q_{cl} is independent of A_μ, so that

$$\tilde{S}_A^{int} = e \int_{t_i}^{t_f} dt \left(\dot{q}_{cl}^j(t) + \delta \dot{q}^j(t) \right) \sum_{n=0}^{\infty} \frac{1}{n!} \partial_{k_1 \ldots k_n} \mathcal{A}_j(q_{cl}) \delta q^{k_1} \ldots \delta q^{k_n}. \tag{4.49}$$

If we were to isolate only the long wavelength parts of $\mathcal{A}_j$, we could truncate the above derivative expansion at $n = 0$; moreover, the high frequency trajectory fluctuations would not effectively couple to these long wavelength parts of $\mathcal{A}_j$ so the term linear in $\delta \dot{q}^j$ would also be negligible. Discarding these terms is of course only valid for the long-wavelength parts of the gauge field, but if we were to simply carry this through for the whole field then we will have effectively performed a lowest order eikonal approximation. The lowest-order contribution in the eikonal approximation for the path-integral in eq. (4.47) then comes simply from replacing the original interaction term in the action by one involving just the classical path of the particle

$$S_{int}^{eik} = e \int_{t_i}^{t_f} dt \; \dot{q}_{cl}^j(t) \mathcal{A}_j(q_{cl}) = \int_{t_i}^{t_f} d^4x \mathcal{A}_j(x) J_{cl}^j(x). \tag{4.50}$$

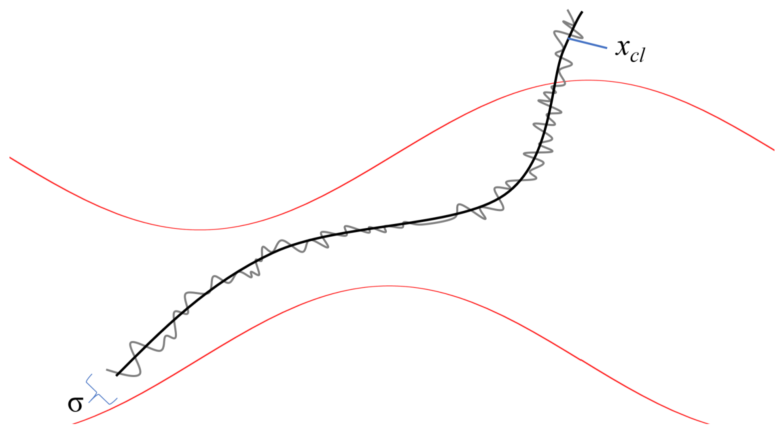


Figure 4.2: A representation of a quantum path (grey curve) in a path integral, where the corresponding classical path x_{cl} is indicated by the black curve. The characteristic scale of quantum fluctuations is σ. The long wavelength gauge field, represented here by the red waves is not sensitive to the high frequency quantum fluctuations, it only sees curves with approximately the same frequency, ie. only the classical path.

The functional integral for the gauge field coupled to an external classical source can be done exactly, and the resulting functional dependence on $\mathcal{A}_{j\,f}$ is Gaussian. Assuming the initial state of the gauge field is also some Gaussian state, eg. the vacuum, then the Gaussian form remains even after using the propagator as a kernel to evolve the initial state. The Gaussian functional dependence implies that the out state will generally be a coherent state of the electric field. The equivalent statement of this in the canonical quantization framework is textbook material, and it is easy to demonstrate because the Heisenberg equations of motion are still exactly solvable for a free field coupled linearly to a classical source. The upshot of this discussion is that without calculation, we can conclude that if the initial state is the vacuum state then, in the lowest-oder eikonal approximation, the resulting coherent state will be peaked on the classical electric field created by the source J^j_{cl}.

This particular eikonal approximation illustrates very nicely that, aside from the universal Coulomb part of the field, coherent dressed states can be understood in terms of quantum state preparation, as anticipated by Gervais and Zwanziger as well as by Carney et al. [224, 225]. It also gives a concrete method for computing the resulting long-wavelength parts of the dressing for a given state preparation mechanism. It would be very interesting to apply this to real world problems, away from the idealized infinity time S-matrix scattering theory, as it would demonstrate how time-dependent dressings emerge dynamically. We do not do this here, but we do consider it to be an interesting open project.

We can now address one of the questions raised in the introduction. If one considers the dynamics of a charged quantum particle, one will find that the expectation value for the long wavelength parts of the electric field operator will be precisely that which one finds in the corresponding classical problem, ie., the Liénard-Wiechert field of a moving charged particle [226]. The dressing can be characterized as follows: the resulting state of the electromagnetic field is an eigenstate of the longitudinal electric field operator with eigenvalue corresponding to the Coulomb field, and a coherent state of the transverse electric field which is peaked on a configuration determined by the classical limit of the history of the charged particle. This field has both longitudinal and transverse parts though, and by measuring fluctuations one can see that they behave quite different from one another in quantum theory.

Suppose one measures the fluctuations in the average strength of the electric field over a large region $\mathcal{V}$ which may contain a moving quantum charged particle,

$$\delta E^2 \equiv \frac{1}{V}\int_{\mathcal{V}} d^3x \left(\langle E_j E_j\rangle - \langle E_j\rangle\langle E_j\rangle\right), \tag{4.51}$$

where V is the volume of the region $\mathcal{V}$, and we take this to be large. One can use the fact that the transverse and longitudinal components of the field are orthogonal when integrated over all of space to obtain the decomposition of the expression for the fluctuations

$$\delta E^2 \equiv \frac{1}{V}\int_{\mathcal{V}} d^3x \left(\langle E_j^T E_j^T\rangle - \langle E_j^T\rangle\langle E_j^T\rangle + \langle E_j^L E_j^L\rangle - \langle E_j^L\rangle\langle E_j^L\rangle\right). \tag{4.52}$$

Now since the longitudinal part of the field is constrained and does not have dynamics independent of the particle, physical states are all eigenstates of $\hat{E}_j^L$, with an eigenvalue corresponding to the Coulomb field sourced by the charge. As a result, we see that the electric field fluctuations are given by

$$\begin{aligned}\delta E^2 = \frac{1}{V}\int_{\mathcal{V}} d^3x \Bigg(&\langle E_j^T E_j^T\rangle - \langle E_j^T\rangle\langle E_j^T\rangle \\ &+ \frac{e^2}{16\pi^2}\left\langle \frac{1}{(x-\hat{x})^4}\right\rangle - \frac{e^2}{16\pi^2}\left\langle \frac{(x-\hat{x})^j}{|x-\hat{x}|^3}\right\rangle \left\langle \frac{(x-\hat{x})_j}{|x-\hat{x}|^3}\right\rangle\Bigg),\end{aligned} \tag{4.53}$$

where the angled brackets in the first line denote an expectation value in the state of the photon field, whereas the angled brackets in the second line represent expectation value in the state of the charged particle. As a consequence of the constraint, the electric field strength fluctuations correspond to the sum of fluctuations of a quantum field with only two degrees of freedom at each point in space (the two independent transverse photon polarizations) and fluctuations from the fluctuating particle position.

We argued above that in the presence of a quantum charged particle, the long-wavelength parts of the transverse electric field will generically be in a coherent state. Coherent states will have the same fluctuations as the vacuum, and thus after suitable renormalization of the short distance behavior the first line of eq. (4.53) will just describe typical vacuum fluctuations of the two photon polarization modes. The second line will vanish for a localized particle, but will generically lead to additional fluctuations in the electric field strength.

4.2.4 Boundary Faddeev-Popov Trick

We can actually derive eq. (4.47) in a more interesting manner, without using off-shell current conservation. We will instead use a technique we refer to as the boundary Faddeev-Popov (bFP) trick. Similar manipulations have previously appeared in work of Rossi and Testa, [227–229], but we'll generalize the results to include quantum matter, and also make the gauge independence clear.

We start again from from the manifestly gauge-invariant expression for the propagator eq. (4.14), and now explicitly perform the Faddeev-Popov (FP) trick to fix a gauge. That is, we multiply the path integral by

$$1 = \int \mathcal{D}\Lambda \, \Delta[A^\Lambda] \delta\big(\mathcal{G}(A^\Lambda)\big), \tag{4.54}$$

where $\Delta[A^\Lambda] = |\det \delta_\Lambda \mathcal{G}(A^\Lambda)|$ is the FP determinant and $\mathcal{G}(A)$ is the gauge fixing function. This expression involves integration over gauge transformations in the region $\mathcal{V}$ over which the path-integral is occurring, and also over transformations on the boundary time slices $\Sigma_i \cup \Sigma_f$. Transformations residing on the boundary time slices are omitted in textbook applications of the FP trick, where one typically considers vacuum generating functionals with no explicit boundaries. The resulting integral is just

$$K_{fi} = \int \mathcal{D}\Lambda \int_{q_i}^{q_f} \mathcal{D}q \, e^{iS_M} \int_{A_{\mu\, i}}^{A_{\mu\, f}} \mathcal{D}A_\mu \, \Delta[A^\Lambda] \delta\big(\mathcal{G}(A^\Lambda)\big) \, e^{iS_{EM}[A]} \tag{4.55}$$

As before, we now consider a change of variables to $A'_\mu = A^\Lambda_\mu = A_\mu + \partial_\mu \Lambda$. The FP determinant is gauge invariant, and the action transforms by a boundary term

$$S_{EM}[A] = S_{EM}[A'] - \int_{\partial\mathcal{V}} d^3x \Lambda J^0. \tag{4.56}$$

The propagator can now be written as

$$K_{fi} = \int \mathcal{D}\Lambda\, e^{-i\int_{\partial\mathcal{V}} d^3x \Lambda J^0} \int_{q_i}^{q_f} \mathcal{D}q\, e^{iS_M}$$
$$\times \int_{A^{\Lambda_i}_{\mu\, i}}^{A^{\Lambda_f}_{\mu\, f}} \mathcal{D}A_\mu\, \Delta[A] \delta^{\mathcal{V}}\big(\mathcal{G}(A)\big) \delta^{\partial\mathcal{V}}\big(\mathcal{G}(A^\Lambda)\big)\, e^{iS_{EM}[A]} \tag{4.57}$$

where we now omit the primes in the notation, and use superscripts $(\mathcal{V})$ and $(\partial\mathcal{V})$ to denote quantities evaluated in the bulk and the boundary respectively.[9] Note that both the boundary data for the gauge field, and the delta function fixing the gauge on the boundaries, are still dependent on the gauge parameter Λ – this of course was not changed by a change of integration variables.

In the standard application of the FP trick one would note that there was no remaining dependence in the path-integral on Λ, and the integral over the gauge group would simply be divided out as overall normalization; but clearly we can't quite do that here.

To proceed, recall that the A_0 integral can be performed unambiguously without need for gauge-fixing. We therefore assume a gauge fixing function which does not involve A_0, and rewrite the transformed boundary data using a linear shift, using functional derivatives as we did in eq. (4.20). We define the operator

$$\hat{\mathcal{L}}_\Lambda \;=\; \int_{\partial\mathcal{V}} d^3x \big[\Lambda J^0 + i\partial_\mu \Lambda \frac{\delta}{\delta A_\mu}\big] \tag{4.58}$$

which now integrates over both past and future boundaries,

$$\int_{\partial V} = \int_{\Sigma_f} - \int_{\Sigma_i}, \tag{4.59}$$

[9]Although we are borrowing terminology commonly used in AdS/CFT, it should be clear from the context that the boundary here refers to the spacelike surfaces Σ_f and Σ_i. At no point here will we consider asymptotically AdS spacetimes.

and get

$$K_{fi} \;=\; \int \mathcal{D}\Lambda\, \delta^{\partial\mathcal{V}}\big(\mathcal{G}(A^{\Lambda})\big)\, e^{-i\hat{\mathcal{L}}_{\Lambda}} \int_{q_i}^{q_f} \mathcal{D}q\, e^{iS_M} \times \int_{A_{\mu\, i}}^{A_{\mu\, f}} \mathcal{D}A_{\mu}\, \Delta[A]\delta^{\mathcal{V}}\big(\mathcal{G}(A)\big)\, e^{iS_{EM}[A]} \tag{4.60}$$

The boundary delta function depends only on $\Lambda_{i,f}$ and not time derivatives thereof. The gauge transformations of the boundary data for A_0 are then completely decoupled from the transformations of the remaining A_j. In a time-sliced discretization of the path integral, the transformation involving Λ on the slices immediately after Σ_i and Σ_f will only affect the transformation of A_0 on the boundary. Additionally, there is no dependence in the integrand on Λ for any intermediate times. This "bulk" integration over the gauge group can be factored out as usual, leaving a residual integration over boundary gauge transformations.

The net result is that in eq. (4.60) we can rewrite $\hat{\mathcal{L}}_{\Lambda}$ as

$$\hat{\mathcal{L}}_{\Lambda} \;\;\rightarrow\;\; \int_{\partial\mathcal{V}} d^3x\, [\Lambda J^0 + i\partial_j \Lambda \frac{\delta}{\delta A_j}] \tag{4.61}$$

and omit the boundary data for A_0. The omission of A_0 boundary data dictates that its values on the boundary are integrated over.

We can use the delta function to evaluate the integral over the boundary gauge transformations, and this will fix the boundary phase. Assuming $\mathcal{G}$ is a good gauge fixing function, it will correspond to a unique gauge parameter $\Lambda = \Lambda_{\mathcal{G}}[A]$. Evaluating the integral over the boundary gauge transformation we then obtain

$$K_{fi} \;=\; e^{-i\hat{\mathcal{L}}_{\Lambda_{\mathcal{G}}}} \int_{q_i}^{q_f} \mathcal{D}q\, e^{iS_M} \int_{A_{j\, i}}^{A_{j\, f}} \mathcal{D}A_{\mu}\, \Delta[A]\delta^{\mathcal{V}}\big(\mathcal{G}(A)\big)\, e^{iS_{EM}[A]} \tag{4.62}$$

where now

$$\hat{\mathcal{L}}_{\Lambda_{\mathcal{G}}} = \int_{\partial\mathcal{V}} d^3x\, \Lambda_{\mathcal{G}}[A] \left[J^0 + i\partial_j \Lambda \frac{\delta}{\delta A_j} \right]. \tag{4.63}$$

The difference between the bFP trick and the usual FP technique is clear from eq. (4.62). While the path integral integrand itself is standard, the additional boundary phase effects a particular gauge transformation of the boundary data, which depends on the choice of bulk gauge fixing function $\mathcal{G}$. This boundary phase ensures that the resulting propagator remains independent of the choice of gauge fixing; it remains a gauge invariant object.

Since the propagator is independent of gauge choice, we can choose the most convenient gauge. The argumentation is then similar to what we did earlier. We first recall that after the A_0 integration, and the change of variables to the fields $\mathcal{A}_j$ and Φ, we're left with an effective action (4.36). Great simplification came if we then chose $\partial_0 \Phi = g$, where g given in eq. (4.31) was the unique gauge-dependent part of the saddle point solution $\tilde{A}_0$. Additionally, a few more terms in the effective action which involved g and the current summed to a total derivative after using off-shell current conservation.

We could actually skip the off-shell current conservation argument at this point, by simply choosing the Coulomb gauge $\mathcal{G}(A) = \partial^j A_j$. The particular usefulness of this gauge choice is that it sets $g = \Phi = 0$, considerably simplifying the action. It also makes the FP determinant irrelevant, and implies

$$\Lambda_{\mathcal{G}}[A] = -\nabla^{-2} \partial^j A_j, \tag{4.64}$$

for our boundary phases. The resulting expression for the propagator is

$$K_{fi} = e^{i \int_{\partial \mathcal{V}} d^3x \, \nabla^{-2} (\partial^k A_k) \left[J^0 - i \partial_j \frac{\delta}{\delta A_j} \right]} \int_{q_i}^{q_f} \mathcal{D}q \, e^{i \tilde{S}_M} \int_{\mathcal{A}_{j\,i}}^{\mathcal{A}_{j\,f}} \mathcal{D}\mathcal{A}_\mu \, e^{i \tilde{S}_A[\mathcal{A}]} \tag{4.65}$$

in which we now write things in terms of the effective actions $\tilde{S}_M$ and $\tilde{S}_A$, as in eq. (4.47).

We can show the equivalence of this result to (4.47) by noting that the remaining path-integral is independent of the longitudinal part of the gauge field. In the shift operator, the functional derivative $\partial_j \frac{\delta}{\delta A_j}$ then vanishes and we're left with an expression for the propagator K_{fi} in the precisely same form as eq. (4.47) above. Again, we find that the charge is dressed by a Coulomb field.

This concludes our analysis of the propagator K_{fi} for scalar particles evolving between time slices. We stress again that the propagator is gauge-independent, and thus so too are the states it defines on its endpoints. The propagator has clearly defined boundary phases which describe properties of the boundary states, and the form of these boundary phases is a simple consequence of the gauge invariance of the action, not a consequence of some *a priori* choice.

4.3 Spinor Quantum Electrodynamics

We have devoted considerable space to scalar electrodynamics; it is now fairly straightforward to generalize to real QED, with Dirac spinors coupled to the EM field. Again we will consider the manifestly gauge invariant path-integral for K_{fi}, and we will find the same Coulomb form for the dressing. The manipulations are similar to those for scalar electrodynamics, the only difference being that the matter field also changes under gauge transformation, and the $U(1)$ charge current in the boundary phase will become an operator.

The gauge invariant path integral representation of the transition amplitude is now

$$K_{fi} = \int_{\psi_i}^{\psi_f} \mathcal{D}\psi\mathcal{D}\bar{\psi} \int_{A_{\mu\, i}}^{A_{\mu\, f}} \mathcal{D}A_\mu \, e^{iS[A,\psi,\bar{\psi}]}, \tag{4.66}$$

where $\psi, \bar{\psi}$ are Grassmann fields, and the omission of boundary data for $\bar{\psi}$ indicates that this variable is to be integrated over on the boundary—necessary because the Dirac Lagrangian has a first-order form.[10] The action is the QED action with a single Dirac fermion field of charge e, viz.,

$$S[A,\psi,\bar{\psi}] = \int_{t_i}^{t_f} d^4x \Big[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \bar{\psi}\big(\gamma^\mu\partial_\mu - ie\gamma^\mu A_\mu + m\big)\psi \Big] \tag{4.67}$$

This action is completely invariant, without need to discard a boundary term, under the U(1) gauge transformation

$$\begin{aligned} A_\mu &\rightarrow A_\mu + \partial_\mu \Lambda \\ \psi &\rightarrow e^{ie\Lambda}\psi, \quad \bar{\psi} \rightarrow e^{-ie\Lambda}\bar{\psi}. \end{aligned} \tag{4.68}$$

One can easily verify that the propagator (4.66) is gauge invariant in the same way done in the previous section, ie. by transforming its data, undoing the transformation by a change of variables in the path integral, and using the invariance of the action.

The gauge invariance of the propagator implies that it satisfies the condition

$$\int_{\partial\mathcal{V}} d^3x \left[ie\Lambda\psi\frac{\delta}{\delta\psi} + \partial_\mu\Lambda\frac{\delta}{\delta A_\mu}\right] K_{fi} \;=\; 0. \tag{4.69}$$

By explicitly differentiating the path integral we can confirm that the functional derivatives are proportional to the conjugate momenta for the fields:

$$\frac{\delta}{\delta\psi_{i,f}} = \pm i\hat{\Pi}_{i,f} = \mp\hat{\psi}^\dagger_{i,f}, \tag{4.70}$$

[10] Typical formulations of the path integral in terms of fermionic coherent states end up instead with data for fixed ψ_i and $\bar{\psi}_f$. What we have here is equally valid, [55], differing only by a change of basis in the out state.

$$\frac{\delta}{\delta A_{j\,i,f}} = \mp i\hat{\Pi}^j_{i,f} = \pm i\hat{E}^j_{i,f}. \tag{4.71}$$

Together with the expression for the $U(1)$ charge density $J^0 = i\bar{\psi}\gamma^0\psi = -\psi\psi^\dagger$, the invariance condition (4.69) then implies that the propagator satisfies the operator constraints

$$\begin{aligned}\big(\partial_j\hat{E}^j - \hat{J}^0\big)K(A,\psi) &= 0 \qquad (4.72)\\ \frac{\delta}{\delta A_0}K(A,\psi) &= 0, \qquad (4.73)\end{aligned}$$

on both the future and past boundary time slices.

We now proceed to use the bFP trick to see exactly how this constraint is implemented, ie., how the electric field dressing of the states emerges. Starting from K_{fi} we again insert a gauge fixing function by multiplying by (4.54), but now we must change variables for both the gauge field and the Dirac field if the action is to be invariant:

$$\begin{aligned}K_{fi} = &\int \mathcal{D}\Lambda \int_{\psi_i^{\Lambda_i}}^{\psi_f^{\Lambda_f}} \mathcal{D}\psi\mathcal{D}\bar{\psi}\\ &\times \int_{A^{\Lambda_i}_{\mu\,i}}^{A^{\Lambda_f}_{\mu\,f}} \mathcal{D}A_\mu\,\Delta[A]\delta^{\mathcal{V}}\big(\mathcal{G}(A)\big)\delta^{\partial\mathcal{V}}\big(\mathcal{G}(A^\Lambda)\big)e^{iS[A,\psi,\bar{\psi}]}. \qquad (4.74)\end{aligned}$$

The A_0 integral can again be done without gauge fixing, and we can extract the transformations of the boundary data using exponentiations of the functional derivatives,

$$\begin{aligned}K_{fi} = \int \mathcal{D}\Lambda\,\delta^{\partial\mathcal{V}}\big(\mathcal{G}(A^\Lambda)\big)e^{\hat{\mathcal{L}}_\Lambda} &\int_{\psi_i}^{\psi_f} \mathcal{D}\psi\mathcal{D}\bar{\psi}\\ \times &\int_{A_{\mu\,i}}^{A_{\mu\,f}} \mathcal{D}A_\mu\,\Delta[A]\delta^{\mathcal{V}}\big(\mathcal{G}(A)\big)e^{iS[A,\psi,\bar{\psi}]}. \qquad (4.75)\end{aligned}$$

in which the operator $\hat{\mathcal{L}}_\Lambda$ now takes the form

$$\begin{aligned}\hat{\mathcal{L}}_\Lambda &= \int_{\partial\mathcal{V}} d^3x\big[ie\Lambda\psi\frac{\delta}{\delta\psi} + \partial_\mu\Lambda\frac{\delta}{\delta A_\mu}\big]\\ &= \int_{\partial\mathcal{V}} d^3x\big[\Lambda\big(\partial_j\hat{E}^j - \hat{J}^0\big) - i\partial_0\Lambda\frac{\delta}{\delta A_0}\big] \qquad (4.76)\end{aligned}$$

where the 2nd expression uses the relations (4.70), (4.71).

From this stage onwards, the manipulations are identical to those in the previous section except that the charge density in the boundary phase is an operator rather than a c-number. The resulting expression for the propagator is

$$K_{fi} = e^{i\hat{\mathcal{L}}_{\Lambda_{\mathcal{G}}}} \int_{\psi_i}^{\psi_f} \mathcal{D}\psi\mathcal{D}\bar{\psi} \times \int_{A_{j\,i}}^{A_{j\,f}} \mathcal{D}A_\mu\, \Delta^{\mathcal{V}}[A]\delta^{\mathcal{V}}\big(\mathcal{G}(A)\big)e^{iS[A,\psi,\bar{\psi}]} \tag{4.77}$$

where now

$$\hat{\mathcal{L}}_{\Lambda_{\mathcal{G}}} = \int_{\partial\mathcal{V}} d^3x\Lambda_{\mathcal{G}}[A]\big[\partial_j\hat{E}^j - \hat{J}^0\big] \tag{4.78}$$

The final expression for the propagator will of course be independent of choice of $\mathcal{G}(A)$. For formal manipulations the most convenient choice is the Coulomb gauge, because this sets the g-potential $\nabla^{-2}\partial^j A_j$ to zero, leaving only the invariant field components. With this choice we then get $\Lambda_G[A] = -\nabla^{-2}(\partial^j A_j)$. Since this choice eliminates the dependence of the integral on the longitudinal part of A_j, the shift operator $\exp\big(i\int_{\partial\mathcal{V}}\Lambda\partial_j\hat{E}^j\big)$ has nothing to shift, and the propagator is then

$$K_{fi} = e^{i\tilde{S}_C}\int_{\psi_i}^{\psi_f}\mathcal{D}\psi\mathcal{D}\bar{\psi}\, e^{i\tilde{S}_M}\int_{\mathcal{A}_{j\,i}}^{\mathcal{A}_{j\,f}}\mathcal{D}\mathcal{A}_j\, e^{i\tilde{S}_A}. \tag{4.79}$$

ie., of the same form as eq. (4.47) except that now the matter action is

$$\tilde{S}_M = \int_{t_i}^{t_f} d^4x\big[-\bar{\psi}\big(\gamma^\mu\partial_\mu + m\big)\psi + \frac{1}{2}J^0\nabla^{-2}J^0\big], \tag{4.80}$$

and the charge density in $\tilde{S}_C$ is now an operator which generates a $U(1)$ transformation on the fermion boundary data. The dynamic gauge field action is as before (*cf.* eq. (4.46)), except that now the matter current is $J^j = ie\bar{\psi}\gamma^j\psi$.

We emphasize that if we had chosen a different gauge fixing function $\mathcal{G}(A)$, the resulting gauge fixed action would look different, and so would the resulting boundary phase, but this difference would only be temporary; the shift operator in eq. (4.78) would no longer give zero in any other gauge, instead enacting a gauge transformation which would set the boundary term and action back into the form presented above. In this form the theory is not manifestly Lorentz invariant, but this is simply because we evaluated

K_{fi} between two constant t surfaces. In principle, one could choose a covariant gauge to compute the path-integral as long as one also evaluates the necessary shift of the longitudinal mode in the final expression.

Note, in this connection, that we could take the expression in eq. (4.79) one step further if we explicitly act with the $U(1)$ transformation effected by the boundary phase. This locally rotates the boundary data for the Dirac field by an angle which depends on the longitudinal part of the gauge field, giving our final expression for the gauge invariant QED propagator on the extended configuration space,

$$K_{fi} = \int_{e^{-ie\nabla^{-2}\partial^j A_{j\,i}}\psi_i}^{e^{-ie\nabla^{-2}\partial^j A_{j\,f}}\psi_f} \mathcal{D}\psi\mathcal{D}\bar{\psi} \int_{\mathcal{A}_{j\,i}}^{\mathcal{A}_{j\,f}} \mathcal{D}\mathcal{A}_j\, e^{i\tilde{S}_M + i\tilde{S}_A}. \tag{4.81}$$

Expanding the shorthand notation this reads

$$e^{-ie\nabla^{-2}\partial^j A_j}\psi = \exp\left(i\int d^3y A_j(y) \frac{e}{4\pi}\frac{y^j - x^j}{|y-x|^3}\right)\psi(x). \tag{4.82}$$

The gauge invariant propagator dresses every point excitation of the Dirac field by a Coulomb electric field sourced by the corresponding point charge. This is the central result of applying the bFP trick to QED. As with the particle considered in the first section, the transverse dressing will be determined dynamically by the remaining integral over gauge invariant variables. The expression here is precisely the form suggested by Dirac for a gauge invariant electron field [209]. Furthermore we have demonstrated that it is the Coulomb solution which naturally emerges from the path integral, not one of the exotic solutions such as the Faraday line (4.8).

4.4 Flat spacetime evolution in a causal diamond

Up to now we have dealt with the rather simple problem of QED on a flat background, defined between time slices. However it is clearly crucial to be able to discuss this for much more general kinds of boundary and boundary information. In principle this should extend to spacetimes including achronal regions; as noted in the introduction, discussions of this sort of problem began in the 1980's [184–186, 197, 198, 204, 230].

To give such a generalization has also been the goal of the "general boundary quantum field theory framework" [187–190], where one consider general spacetime regions $\mathcal{V}$ bounded by some boundary hypersurface $\partial\mathcal{V}$. A field configuration on $\partial\mathcal{V}$ is then mapped to an amplitude via a path integral over field configurations in $\mathcal{V}$.

In this formulation, as in the work of Hartle and Hawking [204], states can then be defined as non-local wave functionals, over configurations specified on all of $\partial\mathcal{V}$. If the general boundary hypersurface involves a union of future and past surfaces, then one can still interpret such "states" on $\partial\mathcal{V}$ as a transition amplitude [187, 188]. However, for more general spacetimes, such an interpretation is not valid, although one can obtain a probabilistic interpretation of the modulus squared of the state on $\partial\mathcal{V}$ in terms of a conditional probability to find a given field configuration on a subregion $\Sigma \subset \partial\mathcal{V}$, given another specified field configuration on the complementary region $\bar{\Sigma}$.

As we have seen, the approach in this chapter to defining states is in the same philosophy of the "general boundary" framework. In our view such an approach is essential for general spacetimes - one of the ultimate motivations in the present work is to set up a technique which can be used for achronal spacetimes, in which information fixed on just the past time slice is not always sufficient to predict quantum evolution [197, 230]. In our opinion such a technique may also be useful to properly address issues concerning black hole information loss.

In pursuit of this end, in the present section we apply the bFP trick to amplitudes for a more general boundary hypersurface. The region we will consider is a causal diamond in Minkowski spacetime, where the state is fixed on the null boundary hypersurface. Here there is still a natural splitting into past and future sections, and so we can define a propagator which represents a transition amplitude between states on the past and future null cones (which tend to null infinity as we take the limit of an infinitely large diamond).

A technical note—there is a subtlety here in the specification of boundary data for the path-integral. Because the conjugate momentum on a null surface involves a derivative along that surface, specifying the field configuration also specifies the conjugate momentum. Specifying this data on both the past and future boundaries would be an over-specification of boundary data for the classical evolution, and the corresponding interpretation as a quantum amplitude is then unclear.

One fixes this by specifying "half" of the field data in some chosen way [231]. We will assume throughout that it is only the positive frequency parts of the field which are specified: a choice which gives this amplitude an interpretation in terms of states in the Bargmann representation, ie. coherent states. In the following discussion we will avoid making this explicit, so as not to clutter the notation.

4.4.1 Formulation of the Problem

In the time-slice geometry, the variable A_0 was ultimately unphysical, and the remaining variables A_j split into purely physical transverse and pure gauge longitudinal parts - the transverse part being divergenceless, ie., $\partial^j \mathcal{A}_j = 0$. For more general boundary hypersurfaces, a natural idea would be to continue to decompose the field into parts with and without divergence. This is not possible, for 2 reasons. First, as before, there is still the issue of uniqueness – given a transverse-longitudinal decomposition of the vector field, one can freely add some transverse parts onto the longitudinal part and the result still transforms correctly under gauge transformation. Second, on null hypersurfaces there is no unique notion of divergence – the induced metric is degenerate, and so there is no unique inverse metric with which to define the divergence $h^{jk}\nabla_j A_k$.

For these reasons we again use a procedure whereby the path integral is used to generate a unique decomposition into pure gauge and gauge invariant parts of the field.

We recall that for flat timeslice boundaries, the boundary data of the component $A_0$was integrated over. The saddle point solution for this Gaussian integral determined the g-potential Φ, ie., the functional of A_j transforming as $\delta_\Lambda \Phi = \Lambda$; the pure gauge part of $\tilde{A}_0$ was the time derivative of Φ, and the longitudinal part A_j was the gradient of Φ. For more general boundaries we will then need to single out the component of A_μ normal to the boundary hypersurface. This component will play the same role as A_0, and the pure gauge part of its solution will yield a corresponding g-potential.

Coordination specification

To implement these ideas we need to choose coordinates appropriately. We pick hypersurface adapted coordinates $x^\mu = \{S, y^k\}$ such that $S =$ const. surfaces foliate the spacetime region $\mathcal{V}$, and the boundary hypersurface $\partial\mathcal{V}$ is described by particular values, $S = S_i, S_f$. Then, using a coordinate basis it is A_S which is the component generalizing A_0, because the pullback of $A_\mu dx^\mu$ to $\partial\mathcal{V}$ will be independent of A_S.

For a finite size causal diamond in Minkowski spacetime we then need to construct coordinates adapted to the boundary null cones. The coordinates we will use are rather intuitive. Consider a sphere of radius R at time $t = 0$, and from each solid angle send an inwards going radial null ray to the future and to the past. These null geodesics will converge at $r = 0$ at times $t = R$ and $t = -R$ respectively, and the surface generated by the null rays is the

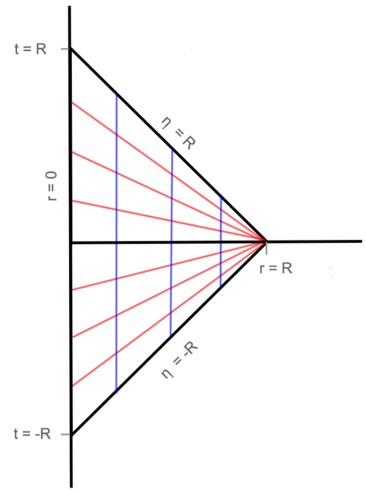


Figure 4.3: The r, η-coordinates. Each point represents a two-sphere of radius r. This is a standard Minkowski spacetime diagram, not a conformal diagram. The blue lines are lines of constant r, while the red lines are lines of constant η.

boundary of our causal diamond.

To construct coordinates in the interior we again start from the sphere $r = R$ at $t = 0$, and now send inwards going spacelike geodesics to the future and past. These spacelike rays converge at $r = 0$ but at times t dictated by their "velocities". The angles and radii of spheres are still useful coordinates, but now we will replace the time coordinate t with a coordinate parameterizing the "velocity" of each ray.

Each of the rays joining $r = 0$ to $r = R$ is described by a solid angle and t, r satisfying the simple relation

$$t = \eta f(r), \tag{4.83}$$

for

$$f(r) = 1 - \frac{r}{R}, \tag{4.84}$$

and for some $\eta \in [-R, R]$. From this relation we can quickly verify that the surfaces $\eta = \pm R$ are the future and past null boundaries of the causal diamond.

Inside the boundary, η parameterizes spacelike surfaces and thus serves as a useful time coordinate. Thus, as desired, we've found hypersurface adapted coordinates where certain values of "time" denote the boundary. We can straightforwardly compute the metric in these coordinates:

$$ds^2 = -f(r)^2 d\eta^2 + 2\frac{\eta}{R} f(r) d\eta dr + \left(1 - \frac{\eta^2}{R^2}\right) dr^2 + r^2 d\Omega^2 \tag{4.85}$$

where $d\Omega^2$ is the standard line element on the unit 2-sphere.

It is clear from this expression that $\eta = 0$ is just a standard time slice of Minkowski spacetime and that $\eta = \pm R$ are null hypersurfaces. Since $f(r)$ vanishes at $r = R$, there is a coordinate singularity. This is obvious from fig. 4.3, and indeed several components of the inverse metric will diverge as $r \to R$.

To deal with this we need to recall why we are interested in this geometry. Ultimately we wish to take R to be larger than all other length scales. The sphere $r = R$ then resembles spatial infinity, and the surfaces $\eta = \pm R$ resemble null infinity. As long as we don't take the strict limit $R \to \infty$, we can still specify data for massive fields on the boundary. The boundary considered here then plays a role similar to null infinity, but is not obtained via conformal compactification. Timelike worldlines will be able to connect all points interior to some point on the boundary.

Since the electromagnetic field is massless we expect field excitations to reach null infinity but we do not expect the same for spatial infinity. For this reason we make the assumption that all important quantities will vanish sufficiently fast for $r \to R$, while allowing for finite limits as $\eta \to \pm R$. We assume the fall-offs are sufficiently rapid that we can restrict $r \ll R$ throughout, and allow the metric to take the simple form

$$ds^2 = -d\eta^2 + 2\frac{\eta}{R} d\eta dr + \left(1 - \frac{\eta^2}{R^2}\right) dr^2 + r^2 d\Omega^2 \tag{4.86}$$

For reference, the non-zero inverse metric components are

$$\begin{aligned} g^{\eta\eta} &= -(1 - \frac{\eta^2}{R^2}), \qquad & g^{\eta r} &= \frac{\eta}{R}, \\ g^{rr} &= 1, & g^{AB} &= r^{-2} q^{AB} \end{aligned} \tag{4.87}$$

where x^A are sphere coordinates, and q^{AB} is the inverse metric on the unit 2-sphere.

We will formally "blow up" this surface $r = R$, that is, we excise the sphere $r = R$ from the boundary and consider the boundary as an open set where limits $r \to R$ can now be η dependent. Note that for all values of η,

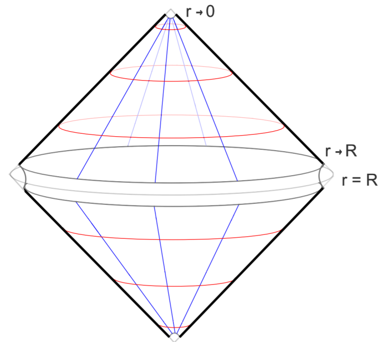


Figure 4.4: The blow-up procedure which treats the corners of the causal diamond geometry. The black lines show regions of the boundary, the lightest grey lines show the true causal diamond, and the darker grey lines show the boundaries of the boundary, which in the infinite limit coincide with the true causal diamond. Blue lines follow null generators of the boundary, while red lines denote constant t cuts.

the boundary region $r = R$ has relative measure zero. Thus when spatially integrating by parts, both $r = 0$ and $r = R$ will be zero volume surfaces, and we can then discard any spatial surface terms.

This deals with the singular behaviour of the spatial "corner" of the boundary hypersurface, but there are still the corners at the top and bottom of the causal diamond, $r = 0, \eta = \pm R$. We will also formally blow up these points to allow fields to take angle dependent limits as $r \to 0$ on the boundary; see fig. 4.4. In doing this, we assume nothing enters or leaves $\mathcal{V}$ through the strict points $r = 0$, $\eta = \pm R$.

If one now considers a QED propagator with information specified on the boundary of this causal diamond, the transformation of the component A_η involves $\partial_\eta \Lambda$, ie., a derivative normal to surfaces of constant η and thus independent of the actual pullback of Λ to the surface. Thus any boundary data specified for A_η in the path integral will be superfluous. In addition the QED Lagrangian will be quadratic in A_η, allowing it to be integrated out via Gaussian saddle point substitution.

Decomposition of the Action

For brevity we just consider the gauge field coupled to a conserved external source J^μ; this is easily generalized to scalar charged particles or to a Dirac field by promoting J^μ in the resulting boundary phase to an operator. As before, we first obtain results without explicitly fixing a gauge, then discuss how the bFP trick shortcuts the computation. Expanding the action so as to explicitly write A_η we have

$$\begin{aligned} S \;=\; & -\frac{1}{2}\int_{\mathcal{V}} d^4x\sqrt{g}\Big[F^{jk}\partial_j A_k - 2A_j J^j + F^{\eta j}\partial_\eta A_j \\ & \qquad - A_\eta\Big(\frac{1}{\sqrt{g}}\partial_j(\sqrt{g}F^{j\eta}) + J^\eta\Big) - A_\eta J^\eta\Big] \end{aligned} \tag{4.88}$$

where $\sqrt{g} = r^2\sin\theta$, and $j = \{r, \theta, \phi\}$. In writing this we've already freely integrated by parts in spatial directions. To integrate out A_η, we need to solve its saddle point equation, ie.

$$\frac{1}{\sqrt{g}}\partial_j(\sqrt{g}F^{j\eta}) + J^\eta = 0. \tag{4.89}$$

Since the metric is non-diagonal, the resulting equation is qualitatively different from the previous equation for A_0. In terms of A_η the equation of motion reads

$$\begin{aligned} &\partial_r\big(\sqrt{g}\partial_r A_\eta\big) - g^{\eta\eta}\partial_A\big(\sqrt{g}g^{AB}\partial_B A_\eta\big) \\ &\quad = g^{\eta r}\partial_A\big(\sqrt{g}g^{AB}F_{Br}\big) + \sqrt{g}J^\eta + \partial_r\big(\sqrt{g}\partial_\eta A_r\big) - g^{\eta\eta}\partial_A\big(\sqrt{g}g^{AB}\partial_\eta A_B\big) \end{aligned} \tag{4.90}$$

On the RHS the first two terms are obviously gauge invariant, and the last two terms together transform as required so that the solution to this equation, $\tilde{A}_\eta$, will transform as $\delta_\Lambda \tilde{A}_\eta = \partial_\eta \Lambda$.

Note that $\partial_\eta g^{\eta\eta} = 2\eta/R^2$, a dimensionful quantity of order R^{-1}. By our original assumptions, R is parametrically much larger than any other dimensionful quantity and thus this entire term is sub-leading. With R sufficiently large we can simply assume $\partial_\eta g^{\eta\eta} = 0$, allowing (4.90) to be written compactly as

$$D^j\partial_j A_\eta = \frac{1}{\sqrt{g}}g^{\eta r}\partial_A\big(\sqrt{g}g^{AB}F_{Br}\big) + J^\eta + \partial_\eta D^j A_j, \tag{4.91}$$

where we've defined the divergence-like differential operator D^j, acting as

$$D^j w_j = \frac{1}{\sqrt{g}}\partial_r(\sqrt{g}w_r) - g^{\eta\eta}\frac{1}{\sqrt{g}}\partial_A(\sqrt{g}g^{AB}w_B). \tag{4.92}$$

Now eq. (4.90) can be formally solved by assuming a Green's function G satisfying

$$D^j\partial_j G(x,x') = \frac{\delta^3(x-x')}{\sqrt{g}}, \tag{4.93}$$

that is,

$$\tilde{A}_\eta = \tilde{A}^I_\eta + g + h, \tag{4.94}$$

where

$$\tilde{A}^I_\eta = \int_{\Sigma_\eta} d^3x'\sqrt{g}G\left[\frac{1}{\sqrt{g}}g^{\eta r}\partial_A\big(\sqrt{g}g^{AB}F_{Br}\big) + J^\eta\right] \tag{4.95}$$

$$g = \partial_\eta \int_{\Sigma_\eta} d^3x'\sqrt{g}G\,D^jA_j, \tag{4.96}$$

and h is a homogeneous solution $D^j\partial_j h = 0$. The integration in these expressions is over Σ_η, the constant η hypersurface corresponding to the time η at which $\tilde{A}_\eta$ is being evaluated.

We don't have a general expression for this Green's function; however the results that we're interested in will ultimately only depend on its value on the null boundary, and one can find G on this boundary as well as at $\eta = 0$. At $\eta = 0$ we have $g^{\eta\eta} = -1$, and the differential operator simplifies to

$$D^j\partial_j f(x)\big|_{\eta=0} = \frac{1}{\sqrt{g}}\left[\partial_r\Big(\sqrt{g}\partial_r f(x)\Big) + \partial_A\Big(\sqrt{g}g^{AB}\partial_B f(x)\Big)\right] \tag{4.97}$$

which is of course just the standard Laplacian in spherical coordinates. This is because the hypersurface $\eta = 0$ is just the hypersurface $t = 0$. Thus at $\eta = 0$ the Green's function is given by

$$G(x,x')\big|_{\eta=0} = -\frac{1}{4\pi}\frac{1}{|x-x'|}. \tag{4.98}$$

At the boundary, the operator $D^j\partial_j$ simplifies considerably because $g^{\eta\eta}$ vanishes; we then have

$$D^j\partial_j f\big|_{\eta=\pm R} = \frac{1}{\sqrt{g}}\partial_r\big(\sqrt{g}\partial_r f\big), \tag{4.99}$$

which can be immediately integrated to find the Green's function

$$G(x,x')\big|_{\eta=\pm R} = \frac{\delta^2(x^A - x^{A\prime})}{\sin\theta}\theta(r'-r)\left[\frac{1}{r} - \frac{1}{r'}\right]. \tag{4.100}$$

which propagates along the null generators of the boundary. [11]

More progress can be made when looking at the homogeneous solutions. A general homogeneous solution, $D^j\partial_j h = 0$, will have the form

$$h(x) = \sum_{m,l} Y_l^m(\theta,\phi)\left[c^1_{ml}(\eta)r^{-\frac{1}{2}+\sqrt{\frac{1}{4}-g^{\eta\eta}l(l+1)}} + c^2_{ml}(\eta)r^{-\frac{1}{2}-\sqrt{\frac{1}{4}-g^{\eta\eta}l(l+1)}}\right] \quad (4.101)$$

with Y_l^m a spherical harmonic and $c^{1,2}_{ml}$ a set of time dependent coefficients.

We can immediately set $c^2_{ml} = 0$, since it is the coefficient of a term which will never be regular at the origin. The other term will either grow monotonically with r or be constant in r. With our assumptions that the fields vanish at large r, both situations are unacceptable and we can set c^1_{ml}=0. The solution (4.94) with $h = 0$ is then the unique solution satisfying the boundary conditions.

As an aside, note that if we relax the asymptotic spatial boundary conditions and simply demand for the fields to be finite as $r \to \infty$, we can accept solutions that are independent of r. Such solutions satisfy

$$-g^{\eta\eta}l(l+1) = 0. \quad (4.102)$$

For all spacelike slices, $g^{\eta\eta} < 0$, and the only solution is $l = 0$, ie. a constant function of θ, ϕ, r. These are the time dependent global $U(1)$ rotations. However on the null boundaries $g^{\eta\eta} = 0$, and the homogeneous solution space is enlarged to include any function on the sphere. This is interesting in the context of large gauge transformations, soft photons, etc, and we will return to this point in section 5.

Returning to the solution (4.94), note that the gauge-variant part g transforms as $\delta_\Lambda g = \partial_\eta\Lambda$. From (4.96) we see we can identify it as a g-potential of form $g = \partial_\eta\Phi$ with Φ given by

$$\Phi = \int_{\Sigma_\eta} d^3x'\sqrt{g}G\, D^j A_j. \quad (4.103)$$

For the causal diamond we can now decompose the gauge field into a gauge-invariant part $\mathcal{A}_j = A_j - \partial_j\Phi$, and a pure gauge part $\partial_j\Phi$; the subsequent development then parallels to that for the time slice. We substitute $\tilde{\mathcal{A}}_\eta$ into the action (4.88) and rewrite the action in the new variables $\mathcal{A}_j, \Phi$. Using current conservation, we then get an effective action

[11] The boundary condition for G is chosen so that influence propagates towards smaller radii, ie. causally on the future portion of $\partial\mathcal{V}$. When considering the past portion of $\partial\mathcal{V}$ one must flip the argument of the step function appropriately.

$$
\begin{aligned}
\tilde{S} \;=\; & \int_{\partial\mathcal{V}} d^3x\sqrt{g}\Phi J^\eta - \tfrac{1}{2}\int_{\mathcal{V}} d^4x\sqrt{g}\Big[\tilde{F}^{\mu j}\partial_\mu\mathcal{A}_j - 2\mathcal{A}_j J^j \\
& \qquad - J^\eta\int_{\Sigma_\eta} d^3x'\sqrt{g}G\Big(J^\eta + \frac{1}{\sqrt{g}}g^{\eta r}\partial_A(\sqrt{g}g^{AB}F_{Br})\Big)\Big],
\end{aligned}
\tag{4.104}
$$

with

$$
\tilde{F}^{\mu j} = \partial^\mu\tilde{\mathcal{A}}^j - \partial^j\tilde{\mathcal{A}}^\mu. \tag{4.105}
$$

Note that all of the terms involving Φ again summed to a total time derivative, and thus formed a boundary term in the action. The remaining bulk action is written in terms of explicitly gauge invariant variables.

We can actually take this expression further because the variable $\mathcal{A}_j = A_j - \partial_j\Phi$ is actually transverse in the sense that $D^j\mathcal{A}_j = 0$. Using this, and a few spatial integrations by parts, we expand the effective action in terms of the gauge invariant variables to get

$$
\begin{aligned}
&\tilde{S} = \int_{\partial\mathcal{V}} d^3x\sqrt{g}\Phi J^\eta \\
&+ \tfrac{1}{2}\int_{\mathcal{V}} d^4x\sqrt{g}\Big[\partial_\eta\mathcal{A}_r\partial_\eta\mathcal{A}_r - g^{\eta\eta}g^{AB}\partial_\eta\mathcal{A}_A\partial_\eta\mathcal{A}_B - 2g^{\eta r}g^{AB}(\partial_\eta\mathcal{A}_A)F_{rB} \\
&- F^{AB}\partial_A A_B + g^{AB}F_{rA}F_{rB} + 2\mathcal{A}_j J^j \\
&+ \Big(J^\eta + \frac{1}{\sqrt{g}}g^{\eta r}\partial_A(\sqrt{g}g^{AB}F_{Br})\Big)\int_{\Sigma_\eta} d^3x'\sqrt{g}G\Big(J^\eta + \frac{1}{\sqrt{g}}g^{\eta r}\partial_C(\sqrt{g}g^{CD}F_{Dr})\Big)\Big]
\end{aligned}
\tag{4.106}
$$

This is the expression we will work with - although we will not actually perform computations with this action. The purpose of the derivation was rather to demonstrate that when propagators are considered for different boundary geometries, by decomposing the gauge field into gauge-variant and gauge-invariant parts we can still unambiguously extract a boundary term from the action which describes the dressing required to make charged states gauge invariant.

As expected the action (4.106) is non-local in space. The "Coulomb" interaction term now contains not just the charge density J^η but also terms describing the magnetic field. These apparent interactions arise because our coordinates are no longer adapted to the isometries of Minkowski spacetime. One can set $\eta = 0$, and thus $g^{\eta r} = 0$, to verify that on a standard constant time slice, this gives the usual Lagrangian.

4.4.2 Form of the Propagator

We can now take all of this and consider the propagator

$$K(A_{\mu\,\partial\mathcal{V}}) = \int_{A_{\mu\,\partial\mathcal{V}}} \mathcal{D}A_\mu \, e^{i\int_{\mathcal{V}} d^4x\sqrt{g}\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+A_\mu J^\mu\right]} \tag{4.107}$$

for evolution of the gauge field coupled to a source J^μ, through a large causal diamond.

Since Φ doesn't appear in the integrand, the path integral over Φ can again be removed by a Faddeev-Popov procedure. Additionally, since the pullback of A_μ to the boundary of this causal diamond is independent of A_η, we know that the propagator does not depend on the value specified for A_η. We then conclude that the propagator is equal to

$$K(A_{\mu\,\partial_V}) = e^{i\int_{\partial\mathcal{V}} J^\eta\Phi} \int_{\mathcal{A}_j\,\partial\mathcal{V}} \mathcal{D}\mathcal{A}_j \, e^{i\tilde{S}[\tilde{\mathcal{A}}|J]}, \tag{4.108}$$

where the effective action $\tilde{S}$ is given by the bulk part of eq. (4.106), and the phase pre-factor describes the generalized Coulomb dressing, in which Φ is given by eqs. (4.100) and (4.103) evaluated on the boundary. The contribution from the future part reads

$$\Phi\big|_{\partial\mathcal{V}}(r', x'^A) = \int_{r'}^{\infty} dr\left(\frac{1}{r'} - \frac{1}{r}\right)\partial_r\big(r^2 A_r(r, x'^A)\big), \tag{4.109}$$

whereas on the past part the integration is over all r interior to r'.

This dressing describes the radial electric field at each point on $\partial\mathcal{V}$, with a strength determined by the total charge flux through $\partial\mathcal{V}$ at earlier times. This is one of our central results for the causal diamond geometry.

We emphasize again that this result is not the result of a specific gauge choice, and that the definition of gauge-invariant variables $\mathcal{A}_j$ again emerged naturally from the path integral. Remarkably, our procedure succeeded even though there is no unambiguous notion of the 'transverse' vector field, since one cannot define an intrinsic divergence on a null boundary.

If we now give the matter current J^μ its own dynamics, we can easily generalize the above derivation. This is possible because $U(1)$ charge current is conserved off shell for particles. Alternatively, as before, we can go back and skip the step which invokes current conservation by using the bFP trick. The derivations are as before; for Dirac fermions we then get the gauge invariant QED amplitude on the large causal diamond to be

$$K(A_{\mu\,\partial_V}, \psi_{\partial\mathcal{V}}) = \int_{e^{-ie\Phi}\psi_{\partial\mathcal{V}}} \mathcal{D}\psi\mathcal{D}\bar{\psi} \int_{\mathcal{A}_j\,\partial\mathcal{V}} \mathcal{D}\mathcal{A}_j\, e^{i\tilde{S}[\mathcal{A}|J]-i\int_{\mathcal{V}} d^4x\sqrt{g}\,\bar{\psi}\left(\gamma^\mu\partial_\mu+m\right)\psi}, \tag{4.110}$$

where Φ is given by eq. (4.109). Analogous to the time-slice amplitude we see a dressing of each Dirac excitation in the boundary state by a Coulombic electric field.

Since we have skipped the explicit derivation of eq. (4.110) and foregone the discussion of general boundaries in curved spacetime, we should at least mention that to do the bFP trick for more general boundaries one must necessarily use generalizations of canonical conjugate momenta and commutation relations. To highlight this, for a general path integral with data specified on boundary $\partial\mathcal{V}$, we can consider a variation of this boundary data, viz.,

$$\delta \int_{\phi_{\partial\mathcal{V}}} \mathcal{D}\phi\, e^{iS[\phi]} = i \int_{\phi_{\partial\mathcal{V}}} \mathcal{D}\phi\, e^{iS[\phi]}\delta S. \tag{4.111}$$

A general variation of the action is of the form

$$\delta S = \int_{\mathcal{V}} d^4x E(\phi)\delta\phi + \int_{\partial\mathcal{V}} d^3x\, (\partial_\mu S)\theta^\mu(\phi, \delta\phi), \tag{4.112}$$

where $E(\phi)$ is the scalar density equation of motion, the boundary is defined by a constant S hypersurface, and the symplectic potential current density θ^μ is given for a general Lagrangian in ref. [232]. For a Lagrangian density which is a function only of the fields and their first derivatives we have

$$\theta^\mu(\phi, \delta\phi) = \frac{\partial\mathcal{L}}{\partial\nabla_\mu\phi}\delta\phi. \tag{4.113}$$

For non-null boundaries $\sqrt{g}\partial_\mu S$ can be related to the normal covector and intrinsic volume element for the hypersurface, but the form in eq. (4.112) is more general and also applies to null boundaries.

For variations with support only on the boundary data we then have the functional derivative

$$\frac{\delta}{\delta\phi_{\partial\mathcal{V}}(x)} \int_{\phi_{\partial\mathcal{V}}} \mathcal{D}\phi\, e^{iS[\phi]} \;=\; i \int_{\phi_{\partial\mathcal{V}}} \mathcal{D}\phi\, e^{iS[\phi]} \left[\int_{\partial\mathcal{V}} d^3x'\, \frac{\delta\theta^S(\phi, \delta\phi)}{\delta\phi(x)} \right] \tag{4.114}$$

Defining $\frac{\delta}{\delta\phi(x)}\phi(x') = \delta^3(x-x')/\sqrt{g}$, the commutation relation between ϕ and $-i\delta/\delta\phi$ is obviously canonical. The functional derivatives $\frac{-i\delta}{\delta\phi_{\partial\mathcal{V}}}$ used in the bFP trick will then be operator representations of the generalized conjugate momentum

$$\Pi_{\partial\mathcal{V}}(x) = \int_{\partial\mathcal{V}} d^3x' \, \theta^S(\phi, g^{-1/2}\delta^3(x - x')). \tag{4.115}$$

This expression was used in deriving (4.110), and will be explicitly used in the following section.

4.5 Large Gauge Transformations and Additional Constraints

Up to now we have assumed that both A_μ and the gauge transformations on A_μ vanish sufficiently fast at spatial infinity that one can freely integrate by parts any expression with spatial derivatives. Energy-flux finiteness arguments lead one to expect the field strength $F_{\mu\nu}$ to obey such asymptotic fall-off conditions, at least in many physical situations. However, it is not clear why either A_μ, or gauge transformations of A_μ, should vanish at infinity.

Gauge transformations which don't fall off as quickly as required for the above manipulations are referred to as large gauge transformations. These have a long history, especially in gravity [169–171], and have also been widely discussed in recent years [see, eg. 145, 233–237, and refs. therein]. Many different choices of asymptotic fall-off conditions for A_μ have been made in the literature.

Invariance under the set of large gauge transformations implies a further set of constraints, in addition to Gauss' law and $E^0 = 0$. In this section we enlarge the set of allowed gauge transformations to those which are finite and non-vanishing at the spatial boundary, and generalize the techniques used above to handle these. The invariant propagators then shed light on the constraints implied by large gauge invariance; and the path integral gives explicit solutions to the operator constraint equations.

We will treat the spatial boundary as a large sphere or cylinder of radius $R \to \infty$, and we allow for gauge transformations which have finite asymptotic limits, viz.,

$$\lambda(t, x^A) \equiv \lim_{r \to R} \Lambda(t, r, x^A). \tag{4.116}$$

With finite asymptotic limits for Λ, we must also allow for finite asymptotic limits for the gauge field, viz.,

$$a_\mu(t, x^A) \equiv \lim_{r \to R} A_\mu(t, r, x^A). \tag{4.117}$$

We warm up by first discussing large gauge transformations for propagation between time slices; we then proceed to the causal diamond.

4.5.1 Large Gauge Transformations: Time Slicing

We would like to compute the propagator

$$K(A_{\mu\,\partial\mathcal{V}}) = \int_{A_{\mu\,\partial\mathcal{V}}} \mathcal{D}A_\mu\, e^{iS}, \tag{4.118}$$

where the region $\mathcal{V}$ over which we integrate is again part of Minkowski space, bounded by the constant t slices Σ_i, Σ_f and the large cylinder of radius $R \to \infty$, Σ_B, and the action is just (4.13). Again, for brevity we assume that the source is an external conserved current, but as was the case in the first section, the following manipulations easily generalize to dynamic matter fields. As just discussed, while we fix boundary data $A_{\mu\,\partial\mathcal{V}}$ on all of $\partial\mathcal{V}$, we now lift the restriction that A_μ vanishes at spatial infinity.

At the technical level, the new challenge is that we can no longer uniquely invert the Laplacian operator when solving the Gauss law equation as in eq. (4.30): there is now nothing restricting the homogeneous solutions.

To proceed with the integral we need to again use the boundary Faddeev-Popov trick. Suppose now that one tries to fix a Coulomb gauge in the FP path-integral, ie., write $\mathcal{G}(A) = \partial^j A_j$ in the expression (4.54). However in the enlarged gauge group this choice will leave the gauge under-determined, because there are homogeneous solutions, $\nabla^2\Lambda = 0$ which are non-vanishing at spatial infinity.

If however we restrict ourselves to gauge functions which are finite at spatial infinity, then the only remaining homogeneous solution is $\Lambda(x) = c(t)$. The only residual gauge transformations in the FP integral (4.54) are then time dependent global $U(1)$ rotations. These leave the spatial components A_j invariant, and only shift the spatially constant part of A_0. To properly implement the bFP trick we then must supplement the Coulomb gauge fixing delta function with another delta function which eliminates these residual transformations.

A sufficient choice is to gauge fix the $l = 0$ spherical harmonic mode of the asymptotic gauge function $\lambda(t, x^A)$. We refer to the $l = 0$ part of a function on the sphere using a superscript "(0)". Up to field-independent normalization we may then write

$$1 = \int \mathcal{D}\Lambda\, \delta(\partial^j A_j^\Lambda)\delta(a_0^{(0)\,\Lambda}). \tag{4.119}$$

in place of (4.54).

In what follows it is more clear if we explicitly separate out the asymptotic $l = 0$ part of all functions. The notation may seem heavier than necessary but it will allow for a much quicker generalization to the later treatment of the causal diamond amplitude. We will therefore write,

$$\Lambda(t, r, x^A) = \bar{\Lambda}(t, r, x^A) + \lambda^{(0)}(t), \tag{4.120}$$

where $\bar{\Lambda}$ has a finite asymptotic limit $\bar{\lambda}(t, x^A) = \lim_{r\to\infty} \bar{\Lambda}(t, r, x^A)$, but the function $\bar{\lambda}(t, x^A)$ has a vanishing $l = 0$ mode. We'll use this same notation for the gauge field,

$$A_\mu = \bar{A}_\mu + a_\mu^{(0)}, \tag{4.121}$$

in terms of which the action is simply

$$S[A] = S[A_j, \bar{A}_0] + \int_{t_i}^{t_f} dt a_0^{(0)} Q, \tag{4.122}$$

where $Q = \int d^3x J^0$ is the total charge.

With this, we can now multiply the propagator (4.118) by a carefully chosen factor of 1, from (4.119), to obtain

$$K(A_{\partial\mathcal{V}}) = \int \mathcal{D}\bar{\Lambda}\, d\lambda^{(0)} \times \int_{A_\mu\, \partial\mathcal{V}} \mathcal{D}\bar{A}_0\, \mathcal{D}a_0^{(0)}\, \mathcal{D}A_j\ \delta(\partial^j A_j^\Lambda)\, \delta(a_0^{(0)\,\Lambda})\ e^{iS[A_j, \bar{A}_0] + i\int_{t_i}^{t_f} dt\, a_0^{(0)} Q}. \tag{4.123}$$

Now, we implement the bFP trick by changing variables, as done before (cf. eqs. (4.55) to (4.57) and eq. (4.60)) to get

$$\begin{aligned} K(A_{\partial\mathcal{V}}) = \int \mathcal{D}\bar{\Lambda} d\lambda^{(0)}\, \delta^{\partial\mathcal{V}}(\partial^j A_j + \nabla^2\bar{\Lambda})\delta^{\partial\mathcal{V}}(a_0^{(0)} + \partial_0\lambda^{(0)}) \\ \times e^{-i\int_{\partial\mathcal{V}} \left[\bar{\Lambda} J^0 + \lambda^{(0)} J^0 + i\partial_0\bar{\Lambda}\frac{\delta}{\delta\bar{A}_0} + i\partial_0\lambda^{(0)}\frac{\delta}{\delta a_0^{(0)}} + i\partial_j\bar{\Lambda}\frac{\delta}{\delta A_j}\right]} \\ \times \int_{A_\mu\, \partial\mathcal{V}} \mathcal{D}\bar{A}_0 \mathcal{D}a_0^{(0)} \mathcal{D}A_j\, \delta(\partial^j A_j)\delta(a_0^{(0)})\, e^{iS[A_j, \bar{A}_0] + i\int_{t_i}^{t_f} dt\, a_0^{(0)} Q} \end{aligned} \tag{4.124}$$

In the bulk part of the path integral we have effectively inserted gauge fixing delta functions as desired. The additional gauge fixing delta function simply sets $a_0^{(0)} = 0$, reducing the action to its usual form. As always in the bFP trick, we've also obtained a number of delta functions and linear shift operators outside the path integral. The crucial observation here is that the delta functions constraining the boundary gauge transformations constrain only $\bar{\Lambda}$ and $\partial_0\lambda^{(0)}$, they do not constrain the other independent functions $\partial_0\bar{\Lambda}$ and λ^0.

In factoring out the bulk gauge group integral we are then left with residual integrals over $\partial_0\bar{\Lambda}$ and λ^0. The remaining boundary integrals over $\bar{\Lambda}$ and $\partial\lambda^{(0)}$ are trivially performed using the delta functions. The result is then

$$\begin{aligned}K(A_{\partial\mathcal{V}}) =& \left(\int d\lambda^{(0)} e^{-i\int_{\partial\mathcal{V}}\lambda^{(0)}J^0}\right) e^{i\int_{\partial\mathcal{V}}\nabla^{-2}(\partial^j A_j)J^0} \\ &\times \int_{\tilde{A}_j\,\partial\mathcal{V}} \mathcal{D}\bar{A}_0 \mathcal{D}\mathcal{A}_j\ \delta(\partial^j A_j) e^{iS[\mathcal{A}_j,\bar{A}_0]},\end{aligned} \tag{4.125}$$

where $\mathcal{A}_j$ is the transverse component of A_j. We can now perform the $\bar{A}_0$ integral and there is no ambiguity in its saddle point solution; it is again given by $\tilde{A}_0 = \nabla^{-2}J^0$ and the homogeneous solution is necessarily zero because by definition $\bar{A}_0$ has vanishing asymptotic $l = 0$ mode.

Note the remarkable feature, that the vestige of working on the configuration space for A^μ with non-vanishing asymptote is just the integral over $\lambda^{(0)}$ on the boundary. This does nothing other than add a delta function enforcing charge neutrality on the boundary state. In hindsight it is completely obvious that if we demand the amplitude to be invariant under global $U(1)$ transformations the state must be charge neutral - by enlarging the gauge group, we've simply imposed this new constraint.

If this constraint is physically unacceptable, then we can simply restrict the gauge group. Note however, that we do not need to eliminate all gauge functions which are finite asymptotically, only those which are constant on the sphere at spatial infinity. Gauge functions which approach $l \neq 0$ functions on the asymptotic sphere may still be allowed; however they do not affect time slice amplitudes. In the next subsection we see that allowing such gauge transformations actually has a nontrivial effect on the causal diamond amplitude.

4.5.2 Large Gauge Transformations: Causal Diamond Evolution

We would now like to consider the amplitude

$$K(A_{\mu\,\partial\mathcal{V}}) = \int_{A_{\mu}\,\partial\mathcal{V}} \mathcal{D}A_\mu\, e^{iS}, \tag{4.126}$$

where, as before, S is given by the sourced Maxwell action and the integration region $\mathcal{V}$ is the causal diamond of radius $R \to \infty$, but now we allow the gauge fields to be finite as $r \to R$. The story is very similar to the treatment of large fields in the time slice propagator, but with an interesting additional feature.

Looking back to (4.91) and its solution (4.94), we can see that without the assumption that the gauge field vanishes as $r \to \infty$, there are infinitely many possible homogeneous solutions. In the bulk, $\eta \in (-R, R)$, the only acceptable homogeneous solution (4.101) is a time-varying $h(\eta)$ which is constant in space.

The situation for these time-dependent global $U(1)$ transformations is identical to the case considered above for A_0 in the time slice amplitude, and the same remedy applies. We must separate off the asymptotic $l = 0$ part, $a_\eta^{(0)}$, and enforce an additional gauge fixing which sets $a_\eta^{(0)} = 0$ in the bulk. This will allow for a unique saddle point solution for the remaining field $\bar{A}_\eta = A_\eta - a_\eta^{(0)}$. The upshot is the same as the previous case; properly treating this asymptotic $l = 0$ part will just introduce delta functions on the boundary which enforce overall charge neutrality $\int_{\partial\mathcal{V}} d^3x \sqrt{g} J^\eta = 0$.

However there is another, more interesting, result in the causal diamond geometry. When we implement the bFP trick, we aim to introduce the Faddeev-Popov gauge fixing as in eq. (4.119) above; but the integrand here still does not uniquely fix the gauge. This is because when $\eta = \pm R$, ie. on the boundary of the causal diamond, there are homogeneous solutions $D^j \partial_j \Lambda = 0$ which are arbitrary functions on the sphere. The above delta functions will uniquely determine the gauge function Λ in the bulk, but on the boundary there is still a residual gauge freedom given by all functions Λ approaching a non-constant ($l \neq 0$) function on the sphere, $\lambda^{(l\neq 0)}(x^A)$, as $\eta \to \pm R$.

Such functions will be discontinuous at $r = 0, \eta = \pm R$, but this is allowed since these singular points have been formally "blown up", allowing for such angle dependent limits as $r \to 0$ on the boundary.

To uniquely fix the gauge we then append a further gauge fixing term on the boundary. Since the propagator is gauge invariant, the final result will not depend on this choice. To simplify many expressions, we choose $\nabla^B a_B^\Lambda = 0$, where $\nabla^B a_B$ is the vector divergence on the unit two-sphere. These choices together uniquely fix the gauge, so that we can write

$$1 = \int \mathcal{D}\Lambda \, \delta(D^j A_j^\Lambda) \delta(a_\eta^{(0)\,\Lambda}) \delta^{\partial\mathcal{V}}(\nabla^B a_B^\Lambda), \tag{4.127}$$

up to a field independent constant. We can now multiply this into our path integral representation for the causal diamond propagator. Doing this and implementing again the bFP trick, we obtain

$$K(A_{\partial\mathcal{V}}) = \int \mathcal{D}\bar{\Lambda} d\lambda^{(0)}\, \delta^{\partial\mathcal{V}}(D^j A_j + D^j \partial_j \bar{\Lambda}) \delta^{\partial\mathcal{V}}(a_\eta^{(0)} + \partial_\eta \lambda^{(0)}) \delta^{\partial\mathcal{V}}(\nabla^B a_B + \nabla^B \nabla_B \lambda)$$
$$\times\, e^{-i\int_{\partial\mathcal{V}} \left[\bar{\Lambda} J^\eta + \lambda^{(0)} J^\eta + i\partial_\eta \bar{\Lambda} \frac{\delta}{\delta \bar{A}_\eta} + i\partial_\eta \lambda^{(0)} \frac{\delta}{\delta a_\eta^{(0)}} + i\partial_j \bar{\Lambda} \frac{\delta}{\delta A_j}\right]}$$
$$\times \int_{A_\mu\, \partial\mathcal{V}} \mathcal{D}\bar{A}_\eta \mathcal{D}a_\eta^{(0)} \mathcal{D}A_j\, \delta(D^j A_j) \delta(a_\eta^{(0)})\, e^{iS[A_j, \bar{A}_\eta] + i\int_{-R}^{R} d\eta\, a_\eta^{(0)} Q} \tag{4.128}$$

The propagator (4.128) is structurally very similar to (4.124), except for the new factor we've introduced to gauge fix the residual transformations which are allowed on the causal diamond boundary. Again, we can freely evaluate the integral over $\partial_\eta \bar{\Lambda}$ and $\lambda^{(0)}$ in the boundary gauge transformations since they are not fixed by the boundary gauge fixing delta functions. We can evaluate the integrals over $\bar{\Lambda}$ and $\partial_\eta \lambda^{(0)}$ using the delta functions. The boundary delta functions now constrain $\bar{\Lambda}$ to be

$$\bar{\Lambda} \;=\; -\int_{\partial\mathcal{V}} d^3x \sqrt{g} G D^j A_j - \int d^2\Omega \bar{G} \nabla^B a_B \tag{4.129}$$

where, up to a minus sign, the first term is just Φ given in (4.109); $d^2\Omega$ is the area element on the unit two-sphere, and $\bar{G}$ is the Green's function for the Laplacian (less the $l = 0$ mode) on the unit two-sphere. Evaluating these integrals we obtain the final expression for the large gauge transformation invariant causal diamond amplitude

$$K(A_{\partial\mathcal{V}})$$
$$= \left(\int d\lambda^{(0)} e^{-i\int_{\partial\mathcal{V}} \lambda^{(0)} J^\eta}\right) e^{i\int_{\partial\mathcal{V}} d^3x \sqrt{g} \left[\int_{\partial\mathcal{V}} d^3x' \sqrt{g} G D^j A_j + \int_{\partial\partial\mathcal{V}} d^2\Omega' \bar{G} \nabla^B a_B)\right] J^\eta}$$
$$\times \int_{\mathcal{A}_j\, \partial\mathcal{V}} \mathcal{D}\bar{A}_\eta \mathcal{D}A_j\, \delta(D^j A_j)\, e^{iS[A_j, \bar{A}_\eta]} \tag{4.130}$$

The first factor in parentheses described charge neutrality for each of the states. If we deem that it is too strong a constraint, that the net charge flux through each of the past and future boundaries must vanish, then we can simply drop this factor and disallow gauge transformations which approach a constant function on the asymptotic sphere. Alternatively, we could just make this constant the same on the past and future parts of the boundary:

this would cause the delta function to enforce charge **conservation** on the propagator, not charge **neutrality** on the states. Ultimately, we have this direct relationship between allowed transformations in the gauge group and the constraints applied on our system. Nature ultimately decides whether certain constraints are enforced, and thus whether the gauge group includes the corresponding transformation, in the absence of an experimental indication all we can do is describe the possibilities.

We also see that in addition to the total charge flux constraint, we've found that invariance under large gauge transformations with higher spherical harmonics enforces a new constraint on the system. Since $\bar{G}$ and a_B are just angular functions, the new boundary phase in eq. (4.130) implies that a certain part of the electric field at each angle is determined solely by the net charge flux of charge through the boundary at each angle. Specifically, if we define the functional derivative

$$\frac{\delta}{\delta a_A(x^A)} a_B(x^{A\prime}) = \delta_B^A \, q^{-1/2} \delta^2(x^A - x^{A\prime}), \tag{4.131}$$

where q is the determinant of the metric on the unit two-sphere, then we see from eq. (4.130) that the gauge invariant amplitude satisfies

$$-i\frac{\delta}{\delta a_B(x^A)} K(A_{\partial\mathcal{V}}) = \left(\int_{\partial\mathcal{V}} d^3x' \sqrt{g} J^\eta(x') \nabla^B \bar{G}(x^{A\prime}, x^A) \right) K(A_{\partial\mathcal{V}}), \tag{4.132}$$

on each of the future and past parts of the causal diamond boundary.

It remains to understand what, physically, this functional differential operator represents. We can do so by using the relationship between functional derivatives and the symplectic current density in eq. (4.114). For the gauge field A_B we have

$$\theta^\eta(A_\mu, \delta A_B) = \frac{\partial \mathcal{L}}{\partial \nabla_\eta A_B} \delta A_B = -\sqrt{g} F^{\eta B} \delta A_B, \tag{4.133}$$

and if we separate the field as $A_B = \bar{A}_B + a_B$, where a_B is independent of r and $\bar{A}_B$ is vanishing at spatial infinity, then by linearity we have

$$\theta^\eta(A_\mu, \delta a_B) = \frac{\partial \mathcal{L}}{\partial \nabla_\eta A_B} \delta a_B = -\sqrt{g} F^{\eta B} \delta a_B. \tag{4.134}$$

Invoking (4.114) we then find

$$\begin{aligned} -i\frac{\delta}{\delta a_B(x^A)} K(A_{\partial\mathcal{V}}) &= \int_{A_{\partial\mathcal{V}}} \mathcal{D}A_\mu \, e^{iS[A]} \left(\int_{\partial\mathcal{V}} d^3x' \, \sqrt{g} F^{B\eta} q^{-1/2} \delta^2(x^A - x^{A\prime}) \right) \\ &= \int_{A_{\partial\mathcal{V}}} \mathcal{D}A_\mu \, e^{iS[A]} \left(\int_0^\infty dr \, r^2 F^{B\eta}(r, x^A) \bigg|_{\partial\mathcal{V}} \right). \end{aligned} \tag{4.135}$$

The bFP trick has then illustrated that physical (gauge invariant) states on the boundary of the large causal diamond satisfy the constraint equation

$$\left(\int_0^\infty dr\, r^2 \hat{F}^{B\eta}(r, x^A)\bigg|_{\partial\mathcal{V}}\right) K(A_{\partial\mathcal{V}}) = \left(\int_{\partial\mathcal{V}} d^3x' \sqrt{g} J^\eta(x') \nabla^B \bar{G}(x^{A\prime}, x^A)\right) K(A_{\partial\mathcal{V}}), \tag{4.136}$$

at every angle x^A on the sphere, independently, on each of the past and future parts of the boundary. This is an exact relation, irrespective of the data specified for the fields or the dynamics of the charged matter, ie. it is kinematically required. It is a direct consequence of gauge invariance for the causal diamond path-integral on the extended configuration space when the gauge fields are allowed to take finite values at spatial infinity.

This result bears a clear resemblance to results at null infinity which have been widely discussed in the literature, notably by Strominger [119, 145]. Indeed, since eq. (4.136) holds at each angle, we can multiply it by $\partial_B \varepsilon(x^A)$, for any function on the sphere $\varepsilon(x^A)$, and integrate over the sphere. We then obtain

$$\int_{\partial\mathcal{V}} d^3x \sqrt{q}\left((q^{AB}\nabla^A \varepsilon(x^A))\hat{F}_{Br} + \varepsilon(x^A)(r^2 J_r)\right) K(A_{\partial\mathcal{V}}) = 0. \tag{4.137}$$

If we go to complex stereographic coordinates $(z, \bar{z})$ on the unit sphere such that the metric is

$$d\Omega^2 = 2\gamma_{z\bar{z}} dz d\bar{z}, \tag{4.138}$$

with

$$\gamma_{z\bar{z}} = \frac{2}{(1+z\bar{z})^2}, \tag{4.139}$$

we obtain the operator constraint equation

$$\int_{\partial\mathcal{V}} dr d^2z \left(-\partial_z \varepsilon(z,\bar{z})\hat{F}_{\bar{z}r} - \partial_{\bar{z}}\varepsilon(z,\bar{z})\hat{F}_{\bar{z}r} + \varepsilon(z,\bar{z})\gamma_{z\bar{z}}(r^2 J_r)\right) K(A_{\partial\mathcal{V}}) = 0. \tag{4.140}$$

If we recall that in our coordinates r is the affine parameter on the null boundary, then we immediately recognize the operator above as the large gauge charge operator $\hat{Q}_\varepsilon$ in (compare ref. [145], section 2.5.11). The electric part of this operator creates soft photon states (with strictly zero energy). When the matter is quantum mechanical the computation can be carried though with no additional complications, and the result is to simply replace J_r by a functional differential operator representation of the $U(1)$ current operator $\hat{J}_r$.

From our analysis we can see clearly that if we blow up spatial infinity such that the value of the gauge field at spatial infinity, $\lambda(x^A)$, can be different whether approached from the future or past part of the boundary, then the amplitude satisfies (4.140) on the past and future null boundaries separately. As a consequence, in the same way that the boundary phases in eq. (4.47) describe a Coulomb dressing of the charges, the new boundary phase in eq. (4.130) indicates that the states the past and future parts are necessarily dressed in the way described originally by Kibble, Chung, and Faddeev and Kulish [153–155, 238–240], with zero energy photons.

If however, we do not blow up spatial infinity, then eq. (4.140) holds only when integrated over the whole null boundary of the causal diamond, and the dressing involving $\bar{G}\nabla^B a_B$ need only occur on the future boundary. This then implies the infinitely many conservation laws discussed by in the recent literature [119, 145], and is equivalent to Weinberg's soft photon theorem. It demonstrates how an initial state with no soft photon content evolves to a final state dressed with zero energy photons. We won't take a position here on whether such a condition must be imposed on $\lambda(x^A)$, rather we'd just like to highlight how, from the bFP trick on a configuration space with asymptotically finite A_μ, one finds either soft dressing or large gauge charge conservation.

4.6 Conclusions

In this chapter we have given a manifestly gauge invariant analysis of propagators describing QED amplitudes. A primary goal was to define and understand gauge invariant states in QED, where we use the modern understanding of states as data living the boundaries of path-integrals. Much of the analysis was done for the gauge field coupled to a conserved external current, but we also saw that all our results trivially generalize to the full dynamic theory, in which a gauge field couples to quantum charged particles or to the Dirac field.

We adopted a 'general boundary QFT framework', in which the path-integral allows us to go beyond the canonical quantization framework. This framework uses path-integrals, with data fixed on general closed hypersurfaces, to compute amplitudes - the interpretation in terms of states and transition amplitudes is then secondary, and only applies to particular geometries.

To treat the gauge redundancy in the QED path integral we introduced a boundary Faddeev-Popov trick, a natural generalization of the usual

Faddeev-Popov procedure to path-integrals with fixed boundary data. Although the bFP trick should be applicable for general boundaries, in this paper we considered two simple examples, viz., (a) when $\partial\mathcal{V}$ consists of two finitely separated constant time slices and a time-like cylinder at spatial infinity, and (b) when $\partial\mathcal{V}$ is the null boundary of a large causal diamond. The former case is then a conventional transition amplitude which has a representation in terms of a Hamiltonian operator, whereas the later is most easily described via the path-integral. In this limit the causal diamond boundary resembles null infinity and the amplitude resembles a scattering amplitude.

We worked in the extended configuration space of $U(1)$ gauge theory, in that we considered the amplitude to ostensibly be a functional of all field configurations A_μ prescribed on the boundary. Using a "path integral first" approach, we did not concern ourselves a priori with identifying canonical variables for quantization; instead we simply prescribed boundary data for the full four-vector potential. As a consequence of the gauge invariance of the QED action, the resulting amplitudes were gauge-invariant and independent of non-canonical variables. The resulting path integrals were written explicitly in terms of gauge invariant variables, and as a consequence of the bFP trick we obtained unique expressions for the dependence of the amplitudes on the gauge-variant parts of A_μ. The dependence arose only as a boundary phase.

The novel result here is that rather than solving the constraint equation, an equation which under-determines the state, we analyzed the path integral itself, and found unique expressions for boundary phases which indicated how the constraint equation ought to be satisfied. The amplitude's dependence on the gauge-variant parts of the field were determined kinematically, whereas the dependence on the gauge-invariant parts of the field remained to be determine dynamically, by a path integral over gauge invariant variables.

For each of the two geometries considered we considered both the case where gauge transformations vanish at spatial infinity, and the case where they have finite limits. In both cases, when gauge transformations were required to vanish at spatial infinity we obtained Coulombic dressing of the charges in the boundary state. When the gauge group was extended, only the causal diamond amplitude had noteworthy changes. The boundary states were annihilated by the "large-gauge charge" discussed previously in the literature on null infinity [145, 233–236]. Furthermore, just as the Coulomb field emerged naturally from the path integral in the previous scenario, from the large causal diamond path integral an explicit expression for the soft-photon dressing of states on the null boundary emerges very naturally. Our coordinate system, and the specific limit taken to null infinity, were sufficient to include both null and timelike matter. The resulting expressions were not novel, but the bFP technique used here was, and it provided a manifestly gauge invariant derivation of the result.

One of our main motivations in studying these questions was to develop methods which will allow us to study some rather concrete problems in quantum gravity - in particular, the ongoing debates about how one may test experimentally whether the gravitational field is quantized [13, 14, 123], and how to properly define and calculate decoherence rates [116, 117, 119, 166, 207]. The answer to both of these problems depends essentially on how one defines physical states for the metric field. The generalization of our methods to linearized gravity - which is all that is necessary to deal with these two problems - is straightforward if somewhat messy, and we will give our results in the following chapter.

On a more formal level, it is of considerable interest to generalize the bFP trick to gauge theories beyond QED, as well as to amplitudes in curved spacetime, and ultimately to full quantum gravity.

Chapter 5

Diffeomorphism invariance and gravity mediated entanglement

This chapter will be quite related to the previous chapter, however we will pivot back to the discussion of quantum gravity. We have not yet repeated all of the calculations of chapter 4 for linearized gravity, however we can report some gravitational results for the K_{fi} studied above. Some of the technical results reported in this chapter were already discussed in the author's Master's thesis [119], however in that previous formulation the gauge independence of the results was not as clearly demonstrated as in chapter 4. Nonetheless, since we have now properly demonstrated the gauge independence of these propagators, we can confidently quote our previously obtained results for linearized gravity.

In this chapter we first report our results for the gauge invariant propagator in linearized quantum gravity, and the related topic of constraints and physical states. We then discuss one of many related experimental proposals which aim to test conventional quantum gravity in a tabletop experiment [13, 14]. We review the proposal and the controversy surrounding it—ie. the debate over whether it can actually test quantum gravity. We finish by demonstrating how the discussion of physical states in quantum gravity introduced at the start of the chapter is actually crucial to understanding the different arguments, and we use the results to work towards a conclusion which neither side should be able to disagree with.

After preparing this chapter, we received a draft manuscript [241], which has some overlap with the content of this chapter. Except for some shared discussions before the manuscript preparation, our work and the work of [241] was done completely independently.

5.1 Gravitational Propagator

Let us consider quantum gravity linearized about Minkowski spacetime, with a single scalar matter field. The action, for this system propagating in the region $\mathcal{V}$ considered above (fig. 4.1), is then

$$S = \int_{\partial\mathcal{V}} d^3x\, h^{ij}\pi^{(1)}_{ij} - \int_{t_i}^{t_f} d^4x \left(h^{\mu\nu}G^{(1)}_{\mu\nu} - \mathcal{L}_M(\phi, \eta_{\mu\nu}) - \frac{1}{M_P}h^{\mu\nu}T_{\mu\nu} \right), \tag{5.1}$$

where

$$\pi^{(1)}_{ij} \equiv K^{(1)}_{ij} - \delta_{ij}K^{(1)} \tag{5.2}$$

is the linearized conjugate momentum to h_{ij},

$$K^{(1)}_{ij} = \frac{1}{2}(\partial_0 h_{ij} - \partial_i h_{0j} - \partial_j h_{0i}) \tag{5.3}$$

is the linearized extrinsic curvature of the boundary, and

$$G^{(1)}_{\mu\nu} = \frac{1}{2}(-\partial^2 h_{\mu\nu} - \partial_\mu\partial_\nu h + \partial^\rho\partial_\mu h_{\rho\nu} + \partial^\rho\partial_\nu h_{\rho\mu} - \eta_{\mu\nu}\partial^\sigma\partial^\rho h_{\sigma\rho} + \eta_{\mu\nu}\partial^2 h) \tag{5.4}$$

is the linearized Einstein tensor. The Planck mass is taken as $M_P = (8\pi G)^{-1/2}$. We use the shorthand notation for the trace $h = h^\mu{}_\mu$. The superscript '(1)' is used to emphasize that these quantities are first-order in $h_{\mu\nu}$, however in what follows we will drop the superscript '(1)' since all geometric objects are linearized. The stress-energy tensor $T_{\mu\nu}$ is defined as the right-hand side of Einstein's equation

$$T_{\mu\nu} = -2\frac{\partial\mathcal{L}_M(\phi, g_{\mu\nu})}{\partial g^{\mu\nu}}\bigg|_{g=\eta} + \eta_{\mu\nu}\mathcal{L}_M. \tag{5.5}$$

Under a linearized diffeomorphism, this action transforms by only a boundary term, and thus these are gauge symmetries of the classical theory. These transformations are of the form

$$\begin{aligned} x^\mu &\to x^\mu + \frac{2}{M_P}\xi^\mu \\ h_{\mu\nu} &\to h^\xi_{\mu\nu} = h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu \end{aligned} \tag{5.6}$$

where ξ is sufficiently small and slowly varying that $h^\xi_{\mu\nu}$ is of the same order as $h_{\mu\nu}$. Under a transformation of this form, which does not vanish on $\partial\mathcal{V}$, the linearized action changes by a boundary term

$$S \to S - 2\int_{\partial\mathcal{V}} d^3x\, \xi_0 \left(2G^{00} + M_P^{-1}\mathcal{L}_M\right). \tag{5.7}$$

For this system, the propagator on the extended configuration space is given by

$$K(h_f, \phi_f; h_i, \phi_i) = \int_{\phi_i}^{\phi_f} \mathcal{D}\phi \int_{h_i}^{h_f} \mathcal{D}h_{\mu\nu}\, e^{iS[h,\phi]}. \tag{5.8}$$

Using the same manipulations as in chapter 4, we find that the transformation (5.7) implies that the propagator transforms as

$$\begin{aligned} K(h_f^{\xi_f}, \phi_f^{\xi_f}; h_i^{\xi_i}, \phi_i^{\xi_i}) = \\ = e^{-2i\int_{\Sigma_f}\xi_0\left(2\hat{G}^{00}+\frac{1}{M_P}\hat{\mathcal{L}}_M\right)} K(h_f, \phi_f; h_i, \phi_i) e^{2i\int_{\Sigma_i}\xi_0\left(2\hat{G}^{00}+\frac{1}{M_P}\hat{\mathcal{L}}_M\right)}, \end{aligned} \tag{5.9}$$

under small diffeomorphisms of the boundary data.

We see that the propagator is invariant under small spatial diffeomorphisms, but not invariant under time-like diffeomorphisms which are non-vanishing on $\partial\mathcal{V}$. These time-like diffeomorphisms correspond to a "many-fingered" time evolution of the system, where different locations in space are time evolved differently. It would be quite unusual if the propagator were actually invariant under timelike diffeomorphisms, since simple time translation of the final state qualifies as one such transformation and we would then have a propagator describing a system with no dynamics![12]

One can perform manipulations completely analogous to those done for QED in section 4.2.2, and the results are that eq. (5.9) implies that the propagator satisfies

$$\pi^{0\nu} K_{fi} = -i\frac{\delta}{\delta h_{0\nu}} K_{fi} = 0, \tag{5.10}$$

and

$$\left(\hat{G}^{0\nu} - \frac{1}{2M_P}\hat{T}^{0\nu}\right) K_{fi} = 0, \tag{5.11}$$

on each of Σ_f and Σ_i, where the operator representations $\hat{G}^{0\nu}$ in the field basis are

$$\begin{aligned} \hat{G}^{00} &= \tfrac{1}{2}(\partial_j\partial_k - \delta_{jk}\nabla^2)h^{jk} \\ \hat{G}^{0k} &= -\frac{1}{2}\partial_j\pi^{jk} = \frac{i}{2}\partial_j\frac{\delta}{\delta h_{jk}}. \end{aligned} \tag{5.12}$$

[12]In the full gravitational theory this does indeed happen to some extent—the so called "problem of time" in gravity. See the discussions of Unruh and of Isham [242, 243]. However here in the linear limit we have a well defined background metric which defines a time variable. The small diffeomorphisms which remain are gauge symmetries of the system, and they do not act on this background time variable. Thus, we still have a description of the system evolving in time.

These are precisely the *first class constraints* one imposes on the wavefunction in the canonical "Dirac" formalism for quantizing General Relativity [167, 210, 221, 244–246], although here they have only been derived in the weak-field limit. A project of great interest would be to generalize this work to understand invariant propagators in full quantum gravity.[13]

There are two things of interest to note about the operator representation of the linearized Einstein tensor above. Firstly, for each of $k = 1, 2, 3$ the so-called "momentum constraints", involving $\hat{G}^{0k}$, has a completely analogous form to the Gauss law constraint from QED (recall eqs. (4.1) and (4.2)). Secondly the so-called "Hamiltonian" constraint, involving $\hat{G}^{00}$, depends only on the field h_{jk} and not its conjugate momentum. The Hamiltonian constraint has no analogy in QED, since it is a constraint on the canonical coordinates and not the canonical momenta it is a holonomic constraint; a better analogy for it would be the constraint on the relative distance between particles in the rigid rotor model.

Again, manipulations analogous to the those in chapter 4 show that the propagator (5.8) can be rewritten as

$$K(h_f, \phi_f; h_i, \phi_i) = \delta(\hat{\mathcal{H}})\, \hat{U}_G \int_{\phi_i}^{\phi_f} \mathcal{D}\phi\, e^{iS_M[\phi] + iS_{SG}[\phi]} \int_{h_i^{TT}}^{h_f^{TT}} \mathcal{D}h_{jk}^{TT}\, e^{iS_g[h^{TT}, \phi]}, \tag{5.13}$$

where the operator

$$\delta(\hat{\mathcal{H}}) \equiv \int d\xi_0\, e^{4\sigma i \int_{\partial\mathcal{V}} d^3x\, \xi_0 \left(G^{00} - \frac{1}{2M_P}\hat{T}^{00}\right)} \tag{5.14}$$

is the projector onto the kernel of $\hat{\mathcal{H}} \equiv \hat{G}^{00} - \frac{1}{2M_P}\hat{T}^{00}$. That is, it projects onto the subspace of the Hilbert space satisfying the Hamiltonian constraint $\hat{\mathcal{H}}\Psi = 0$, on both the initial and final surfaces. Here $\sigma = \pm 1$ on the Σ_f and Σ_i respectively.

The operator $\hat{U}_G$ ensures the "momentum constraints", ie. those involving G^{0k}, are satisfied: it is defined as

$$\hat{U}_G = \exp\left(i\sigma \frac{1}{M_P} \int_{\partial\mathcal{V}} d^3x\, h^{jk} \hat{B}_{jk}\right), \tag{5.15}$$

where

$$\hat{B}_{jk} = -\frac{1}{\nabla^2}\left(\delta_{jl}\partial_k + \delta_{kl}\partial_j - \frac{\partial_j\partial_k\partial_l}{\nabla^2}\right)\hat{T}^{0l}. \tag{5.16}$$

[13] Much of this proposed project has been done already by Mattei et al. [229], however they did not start from invariant propagators on the extended configuration space, nor did they include quantum matter.

In analogy with $\hat{U}_C = e^{i\tilde{S}_C}$ in QED (4.47), we see that all of the dependence on the longitudinal part of the metric field is contained in $\hat{U}_G$.

We have also defined here the instantaneous gravitational self-interaction term

$$S_{SG}[\phi] = \\ -\frac{1}{4M_P^2}\int_{t_i}^{t_f} d^4x \frac{1}{\nabla^2}\left(T^{00}T^{00} - 4T^{0j}P_{jk}T^{0k} + 2T^{00}P_{jk}T^{jk} + \frac{\partial_0 T^{00}\partial_0 T^{00}}{\nabla^2}\right). \tag{5.17}$$

The remaining path-integral is over the gauge invariant part of the metric, the transverse-traceless (TT) components h^{TT}_{jk}, which satisfy by definition $\partial^j h^{TT}_{jk} = \delta^{jk} h^{TT}_{jk} = 0$. The action describing these components is

$$S_g[h^{TT}, \phi] = \int_{t_i}^{t_f} d^4x\, \Pi^{jklm}\left(-\frac{1}{2}\partial_\sigma h_{jk}\partial^\sigma h_{lm} + \frac{1}{M_P}h_{jk}T_{lm}\right), \tag{5.18}$$

where we've utilized the TT projector $\Pi^{jklm} = \frac{1}{2}\left(P^{jl}P^{km} + P^{jm}P^{kl} - P^{jk}P^{lm}\right)$, with $P_{jk} = \delta_{jk} - \nabla^{-2}\partial_j\partial_k$. In momentum space the TT projector has a simple form in terms of the $+$ and $\times$ graviton polarization tensors[14],

$$\Pi^{jklm} = \epsilon_+^{jk}\epsilon_+^{lm} + \epsilon_\times^{jk}\epsilon_\times^{lm}. \tag{5.19}$$

Using this identity it is clear that we just have two independent massless fields,

$$S_g[h^{TT}, \phi] = \int_{t_i}^{t_f} dt \int d^3p \sum_{a=+,\times}\left(-\frac{1}{2}\partial_\sigma h_a \partial^\sigma h_a + \frac{1}{M_P}h_a T_a\right), \tag{5.20}$$

where we've defined $h_{+,\times} = \epsilon^{jk}_{+,\times}h_{jk}$.

The upshot of this whole discussion is that, as in QED, gauge invariance requires that nearly all of the variables in the metric are constrained. The only true dynamical variables in linearized gravity are the transverse-traceless components of the metric, and this is a gauge invariant statement. We reiterate that this is not a consequence of choosing to fix transverse-traceless gauge, rather it is because these are precisely the components of the metric which are invariant under gauge transformation.

[14] For a graviton plane-wave with momentum $\vec{p}$ along the x^3 axis, the polarization tensors are $\epsilon_+(p) = \begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{pmatrix}$ and $\epsilon_\times(p) = \begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$.

5.2 Gravitational entanglement experiments

Since all of this discussion of constrained variables, physical states, etc. has been somewhat abstract, it is useful to consider a specific example where this language is clearly important. The example we have in mind is (a slight modification of) an experiment recently proposed by Bose et al. [13] and independently by Marletto and Vedral [14], hereafter referred to as the 'BMV' experiment. This experiment aims to test whether the gravitational field is fundamentally quantum mechanical, like all other fields in nature. These authors have proposed an experiment which they claim is: i) within experimental limitations of the near future, and ii) sensitive to signatures of conventional low energy effective quantum gravity.

The claim that these experiments do indeed test quantum gravity has been very controversial, with arguments focused on which part of the gravitational field is actually being tested by the experiment. In what follows we will review their proposal as well as the controversy. We'll then analyze the proposal using the language introduced above, of constraints and physical states. We then use this analysis to discuss what exactly the experiment can say about the gravitational field.

5.2.1 Review of the BMV proposal

The central idea of the BMV proposal is to have gravitation serve to generate entanglement between two masses. Using a fact from quantum information theory, that local operations and classical communication (LOCC) cannot create entanglement between two systems[15] [248], they conclude that the generation of entanglement proves that the gravitational field is quantum mechanical.

The proposal of Bose et al. [13] differs slightly from that of Marletto and Vedral [14]; they are both matter-wave interferometry experiments, however with different read-out protocols. There has also been a number of subsequent experimental proposals which also use the idea of *gravitationally mediated entanglement* as a witness to the quantum nature of gravity. A few examples are experiments based on: the development of quantum squeezing in the light fields of gravitationally coupled optomechanical cavities [115], the development of non-gaussianities in Bose-Einstein condensate correlation functions [23], and atom interferometry in the vicinity of massive oscillators [15]. We'll review only the set-up of Bose et al., but the essential idea of all of these experiments is the same.

[15] This is true, provided there are no closed timelike curves in the spacetime [247].

The experimental set-up of Bose et al. is illustrated heuristically in figs. 5.1 and 5.2. The idea is to take a pair of particles and split them each into superpositions of two well-localized states. The particles then interact gravitationally and become entangled. The intuition behind the two-particle two-path set-up is actually quite simple. One would like to create and probe a superposition of the gravitational field. The first particle can be taken as the source, presumably generating such a superposition of gravitational fields. One must then determine how to probe such a gravitational superposition. One simple idea is to just allow a quantum particle to freely fall in this gravitational field—surely then the particle will itself evolve into a superposition (see fig. 5.3). It is argued that if the gravitational field were classical, then one does not expect the second particle to evolve into a superposition. Detection of a superposition of the probe particle which is appropriately correlated with the source particle then serves as evidence that the gravitational field can be superposed. This idea actually arises naturally from a thought experiment which had been analyzed earlier by Mari et al. [1]. In the BMV context it is noticed that the probe particle need not freely fall to be sensitive to the superposed gravitational field, the essential feature of the Mari et al. thought experiment is that the probe particle is in a superposition.

The two particles, labeled 1 and 2, are prepared such that each particle is in a superposition of two well-localized position states $|L\rangle$ and $|R\rangle$. The joint wavefunction is separable, ie. the particles are unentangled,

$$|\psi_0\rangle = \tfrac{1}{2}\big(|L\rangle_1 + |R\rangle_1\big)\big(|L\rangle_2 + |R\rangle_2\big), \tag{5.21}$$

and we can think of the system's wavefunction as having four "branches"

$$|\psi_0\rangle = \tfrac{1}{2}\big(|L\rangle_1|L\rangle_2 + |L\rangle_1|R\rangle_2 + |R\rangle_1|L\rangle_2 + |R\rangle_1|R\rangle_2\big). \tag{5.22}$$

In the branches where the particles are both L, or both R, they are separated by a distance d_s. In the branches where the particles are on opposite sides, they are separated by a distance $d_l = \sqrt{d_s^2 + (\Delta x)^2}$. The system is allowed to evolve for a time t before each of the two-path set-ups are recombined such that the particles are both back in their initial location $|C\rangle$.

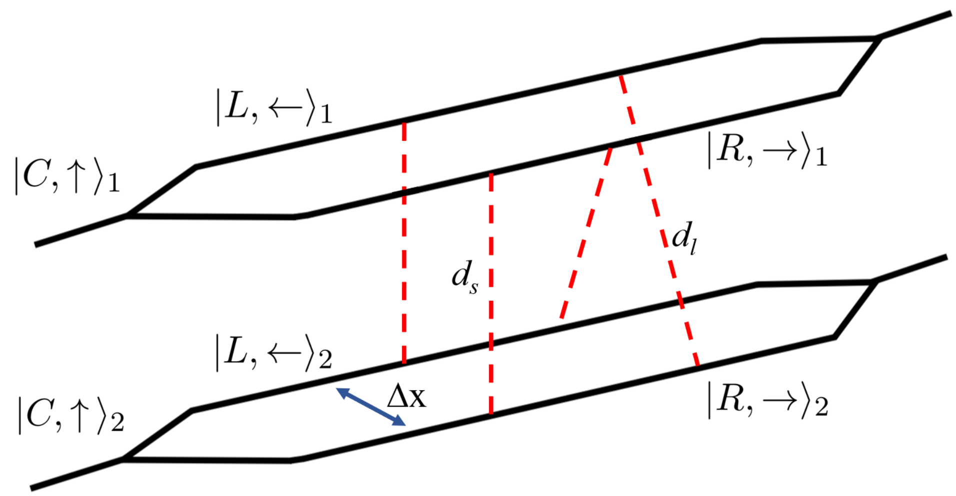


Figure 5.1: A schematic representation of the BMV experiment. Two particles are each made to pass through a two-path interferometry apparatus as in, eg. fig. 5.2. The spatial extent of the superposition is Δx and the initial distance between the two particles is d_s. The dashed red lines indicate each of the possible Newtonian gravitational interactions on the four branches of this system's wavefunction.

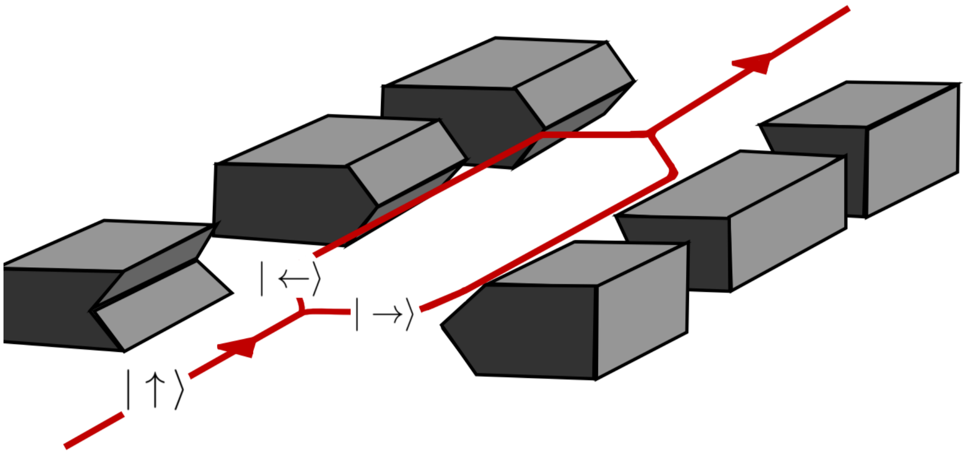


Figure 5.2: A rough illustration of a two-path thought experiment, where an arrangement of Stern-Gerlach magnets is set up such that a quantum particle is split into a superposition of two symmetric paths and then recombined. On each of the paths, left L and right R, the particle's spin is entangled with the particle's position. The intermediate state is then $|\psi\rangle = \frac{1}{\sqrt{2}}(|L\rangle|\leftarrow\rangle + |R\rangle|\rightarrow\rangle)$. Measurements are performed on the outgoing spin state, and deviations from $|\uparrow\rangle$ indicate that quantum decoherence has occurred.

Assuming that the superposition is created sufficiently slowly that the particles are always moving non-relativistically, then the gravitational interaction between them is just the Newtonian interaction. The state of the system just before recombination is then

$$|\psi(t)\rangle = \tfrac{1}{2}\big(|L\rangle_1|L\rangle_2 + e^{i\Delta Et}|L\rangle_1|R\rangle_2 + e^{i\Delta Et}|R\rangle_1|L\rangle_2 + |R\rangle_1|R\rangle_2\big), \quad (5.23)$$

where we've omitted an overall phase, and where the relevant energy scale is ΔE, defined by

$$\Delta E = Gm_1m_2\left(\frac{1}{\sqrt{d_s^2+(\Delta x)^2}} - \frac{1}{d_s}\right). \quad (5.24)$$

For generic values of the experimental parameters, the state eq. (5.23) is no longer separable, and the particles are entangled.

If the paths for the each of the particles are recombined, then the final state would just equal the initial state, $|C\rangle_1|C\rangle_2$, and the entanglement would be gone. Bose et al. [13] get around this issue by using the intrinsic spin of the particles. If the superposition is prepared using a Stern-Gerlach set-up (fig. 5.2), then for a each particle the initial states of its position is maximally entangled with its spin, ie. for particle 1

$$|\psi_0\rangle_1 = \tfrac{1}{\sqrt{2}} \left(|L\rangle_1 |\leftarrow\rangle_1 + |R\rangle_1 |\rightarrow\rangle_1 \right), \tag{5.25}$$

and likewise for particle 2. Initially there is maximal entanglement between each of the particles and their spin. The system evolves as discussed above, and entanglement is generated between the two particles. After recombining each of the particles' positions the entanglement remains only between the spin degrees of freedom, and the final state of the system is

$$\begin{aligned} &|\psi(t)\rangle = |C\rangle_1 |C\rangle_2 \\ &\times \tfrac{1}{2}\Big(|\leftarrow\rangle_1 |\leftarrow\rangle_2 + e^{i\Delta E t} |\leftarrow\rangle_1 |\rightarrow\rangle_2 + e^{i\Delta E t} |\rightarrow\rangle_1 |\leftarrow\rangle_2 + |\rightarrow\rangle_1 |\rightarrow\rangle_2 \Big), \end{aligned} \tag{5.26}$$

This is state is best written in the z basis,

$$|\psi(t)\rangle = |C\rangle_1 |C\rangle_2 \left(\cos\left(\frac{\Delta E t}{2}\right) |\uparrow\rangle_1 |\uparrow\rangle_2 - i \sin\left(\frac{\Delta E t}{2}\right) |\downarrow\rangle_1 |\downarrow\rangle_2 \right), \tag{5.27}$$

where as long as t is not an integer multiple of $\pi/\Delta E$, there is entanglement between the particles. It is completely obvious that when $t = \pi/(2\Delta E)$ the spins are actually in a maximally entangled Bell state. At this point, completely standard joint measurements on the two spins can verify the entanglement.

Bose et al., and the other groups proposing related experiments, then rely on the fact that local operations and classical communication (LOCC) cannot create entanglement between two systems [248]. The operations of preparing each particle is a superposition are obviously local to the respective particle. They claim that the gravitational interaction between the particles is then necessarily non-classical communication, and thus that the gravitational field is quantum mechanical.

5.2.2 Review of the controversy

These experimental proposals have been controversial for a number of reasons. Firstly there is the issue of whether the necessary experimental parameters will actually be within technical limitations, as the authors claim. Many believe that the authors may have overestimated the capabilities of current experimental technology [249]. Since this is outside of our expertise, we will not comment on this issue, instead deferring to those who are more familiar with the necessary experimental technology. Secondly, it has been argued that the authors' conclusions are incorrect, and that a positive observation of the gravitationally mediated entanglement in these experiments does not necessarily imply that gravity is quantum mechanical [124, 250][16] (see also [251, 252]); we'll focus on the debate initiated by [123, 124].

In the comment [123] and follow-up [124], both written by Anastopoulos and Hu, it is argued that the part of the gravitational field relevant for the BMV experiment, ie. the Newtonian part, is not a true degree of freedom of the gravitational field. Here we will try to faithfully paraphrase their argument. *Classically, the component of the metric perturbation, ϕ, which describes the 'Newton field' is pure gauge, and after fixing the gauge it is determined entirely by the Poisson equation $\nabla^2\phi = -4\pi G\mu$, where $\mu(x)$ is the mass density. The only freedom in the gravitational field is in the transverse-traceless graviton modes, not this 'Newtonian' part which is 'slaved' to the matter. Since the Newtonian interaction can be understood as simply another term in the matter's Hamiltonian, such an interaction is merely a property of matter. If one wants to quantize the gravitational field, they would then quantize the transverse-traceless graviton modes—only these would have a Hilbert space, non-commuting observables, etc.*

We are actually quite sympathetic to the point of view expressed by these authors. One can consider the simple analogy of the Coulomb field in electrodynamics. If one considers two free quantum particles, with charge, moving non-relativistically they have the Hamiltonian

$$\hat{H} = \frac{\hat{p}_1^2}{2m_1} + \frac{\hat{p}_2^2}{2m_2} - \frac{e_1 e_2}{4\pi}\frac{1}{|\hat{x}_1 - \hat{x}_2|}. \tag{5.28}$$

One can then apply the BMV logic to say that any entanglement between the two particles generated by the Coulomb interaction is "proof" that the electromagnetic field is quantum mechanical. We're then led to ask, "why then would physicists work so incredibly hard to test the quantum field theory predictions for QED? Wouldn't an observation of the reduced mass in the Rydberg spectrum for hydrogen be sufficient confirmation of the theory?" We say this only with tongue-in-cheek—one could hardly take this line of reasoning seriously as a *reductio ad absurdum*—but we do believe

[16] The authors of [124] immediately argued against the claims in the experimental proposals [13, 14] in the form of a short comment [123]. This work was later re-purposed within a large paper; see section 5.1 in [124].

that it serves to illustrate that there is considerable confusion around what exactly the BMV-type experiments tell us about quantum gravity. We will soon use our language of physical states etc. to substantiate the arguments of [123, 124] in better detail, but first let us finish the story up to present of the arguments around the BMV experiments.

There has been significant pushback to the criticisms in [123, 124], from the original authors in the Bose group and from Marletto and Vedral [253–255] and also from others [eg. 2, 3, 256**?**]. The original authors emphasize their assumption of locality for the mediator, our paraphrasing of their responses is as follows: *The Newtonian interaction emerges in the non-relativistic limit from the propagation of signals in a gravitational field. If we assume that this field ultimately has local interactions, as it does in conventional linear quantum gravity, then the whole gravitational field must be quantum mechanical.* To this we comment: "We agree that the locality assumption plays a central role here. However the 'whole' gravitational field doesn't see the mass density T^{00}, only certain components of the metric are sensitive to this source. One must be careful in thinking about the gravitational field as a single entity."

Christodoulou and Rovelli have a different argument [256, 257]. Our interpretation of their claim is: *In General Relativity one can understand the Newtonian interaction between the two particles as a mutual modification of each other's local metric, and therefore of their experienced proper time. As a consequence, in superpositions with different Newtonian interactions one must be describing superpositions of different spacetime geometries.* To this we have two comments: i) There is an inherent assumption here that General Relativity, with all of its symmetries and thus interpretations as the dynamics of a spacetime, persists as the description of gravity when the sources are quantum mechanics objects. It is certainly natural to assume this (up to higher curvature corrections at very small scales), however there is no experimental basis to believe this to be true. ii) The interpretation in terms of a mutual modification of each other's proper time arises only with a particular choice of coordinates, it is a gauge dependent interpretation and not necessarily the true description of the phenomenon. For example, one could choose the temporal gauge, where $g_{00} = -1$, and the interaction is described by the spatial components of the metric.

Belenchia et al. have, in our opinion, most clearly exposed the heart of the argument [2]; however we also think their claims in the follow-up article [3] are flawed. They identify that the entanglement is indeed mediated only by the Newtonian part of the field, and they discuss the issues this introduces with causality. They then show how these issues are resolved when one includes quantized gravitational radiation.

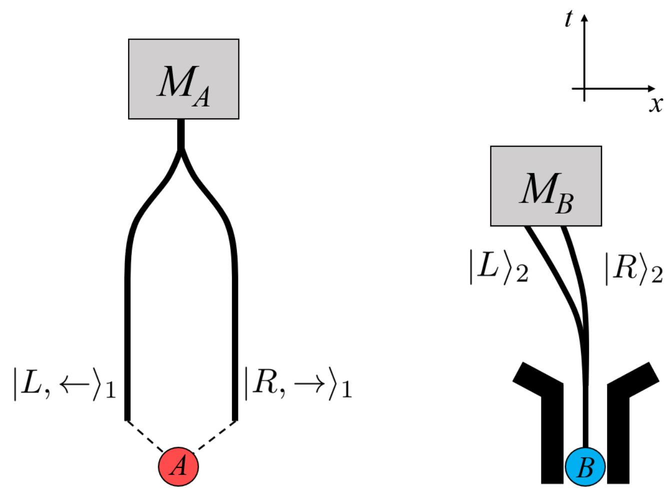


Figure 5.3: An illustration of the thought experiment considered by Mari et al.[1] and Belenchia et al.[2, 3]. Particle 1 was prepared in a spatial superposition long ago (denoted by the dashed lines), while particle 2 has been contained in a trap. Particle 1 sources a superposition of gravitational fields. When the trap is released, particle 2 freely falls in this superposed gravitational field. Alice and Bob later perform local measurements, M_A and M_B respectively, on their particles.

In their work Belenchia et al. reanalyze the thought experiments of Mari et al. [1], which we've described above. If Alice and Bob each have a particle, and are widely separated, then the particles can become entangled by the instantaneous Newton/Coulomb interaction if they are both in superpositions of two locations. Bob then seems to have the ability to generate entanglement with Alice's particle, even over spacelike separations, and this would be seen as anomalous decoherence if Alice performed local experiments on her particle. By choosing whether or not to let his particle freely fall and evolve into a superposition, Bob seems to have a protocol for superluminal communication. Belenchia et al. demonstrate how this apparent issue with causality is resolved within conventional low energy quantum gravity.

The essential point of their argument is that although the interaction between the particles is described by only the Newtonian part, there are two inescapable facts about the gravitational field: i) the graviton is massless and is therefore always radiated in some capacity during the experiment, and ii) quantum fluctuations of the graviton field can interfere with the sensitivity of position measurements. The upshot is that by including a quantum field describing quantized gravitational radiation, the protocol which would have allowed Bob to effectively communicate superluminally is now unusable because of an inability to sufficiently localize his particle and/or because of significant decoherence due to the radiation of gravitons.

Belenchia et al. provide a nice demonstration of how, in conventional quantum gravity, the radiative graviton degrees of freedom conspire to prevent superluminal signaling that one may attempt to perform using the Newtonian part of the field. The authors also consider a different parameter regime, where the particle motions are adiabatic and Bob is not trying to signal to Alice. This is the regime where the BMV experiments would operate. In this case they highlight that no gravitons are radiated but entanglement still develops between the particles. They then try to reconcile the fact that Bob is making a local operation on his particle, in deciding to release it from the trap, yet somehow entanglement is still generated between his particle and Alice's. The issue here being the 'local operations' part of the statement "LOCC cannot create entanglement".

In their follow-up [3], Belenchia et al. conclude that to make sense of this one must think of particles as being entangled with their own Newtonian fields. Their idea is then: *Alice's particle is entangled with her Newtonian field, releasing Bob's particle is a local operation acting on his particle and the field, but this is no issue because any entanglement with Alice's particle is now just transferred from the field to Bob's particle.* We have issues with this interpretation since it seems to completely disregard the point implied by Anastopolos and Hu, that the Newtonian part of the field is not a real degree of freedom. If the Newtonian part is not a real degree of freedom, how could it be entangled with something? We will soon address this question quantitatively.

The arguments of Belenchia et al. demonstrate how apparent issues with superluminal signaling are resolved within conventional quantum gravity, however their arguments are not comprehensive—they do not prove that conventional quantum gravity is the only possible theory with: i) a Newtonian interaction between quantum particles at low energies, and ii) a mechanism which precludes superluminal signaling. One would ultimately like to have a constructive argument, where one actually assumes i) and ii) above and uses these postulates to determine the structure of the theory. To do so would actually be somewhat of a quantum analog of Einstein's effort to unite Newtonian gravity with special relativity (which of course resulted in General Relativity). We will discuss this further in the next section.

5.2.3 Analysis of the BMV proposal

As mentioned previously, the understanding of gravity as a constrained theory takes back over 60 years to Dirac [245]. In a previous section we demonstrated that Dirac's constraints emerge when states are defined via the gauge invariant path integral, and we also showed that the path integral generates a solution to the constraint of a certain form. In this section we then take the this language which we have now familiarized the reader with, and demonstrate its utility for real experiments.

We'll aim, here, to substantiate the arguments of Anastopoulos and Hu [123, 124] regarding the nature of real gravitational degrees of freedom, and then address the points made by Belenchia et al. [3] about the entanglement between the particles and their Newtonian fields.

Let us consider eq. (5.13), except now with the matter replaced by two non-relativistic massive particles. In this limit, all components of the stress tensor except for $T_{00}(x) = \sum_j m_j \delta^{(3)}(q_j(t) - x)$ can be effectively neglected. The result is then,

$$K(h_f, q_{2\,f}, q_{1\,f}; h_i, q_{1\,i}, q_{2\,i}) = \int_{h_i^{TT}}^{h_f^{TT}} \mathcal{D}h_{jk}^{TT}\, e^{iS_g[h^{TT}]} \tag{5.29}$$
$$\times\, \delta(\hat{\mathcal{H}}) \int_{q_{2\,i}}^{q_{2\,f}} \int_{q_{1\,i}}^{q_{1\,f}} \mathcal{D}q_2 \mathcal{D}q_1\, e^{iS[q_1] + iS[q_2] + iS_{Ntn}[q_1, q_2]},$$

where the renormalized gravitational self-interaction (previously $S_{SG}[\phi]$) is now just given by Newtonian gravity

$$\begin{aligned} S_{Ntn} &= -\frac{1}{4M_P^2} \int_{t_i}^{t_f} d^4x T^{00} \nabla^{-2} T^{00} \\ &= \int_{t_i}^{t_f} dt\, \frac{G m_1 m_2}{|q_1(t) - q_2(t)|}. \end{aligned} \tag{5.30}$$

The transverse-traceless components are now effectively decoupled from the matter, since gravitational radiation is negligible in a non-relativistic limit.

The gravitational field now only sees the matter through the Hamiltonian constraint. If $\Psi[\phi, h_{jk}]$ denotes a physical state, ie. either of the endpoints of the propagator, the Hamiltonian constraint implies

$$\hat{\mathcal{H}}\Psi[q_2, q_1, h_{jk}] = \left[\frac{1}{2}(\partial_j \partial_k - \delta_{jk}\nabla^2) h^{jk} - \frac{1}{2M_P}\hat{T}^{00}\right] \Psi[q_2, q_1, h_{jk}] = 0. \tag{5.31}$$

This constrains a particular combination of the longitudinal parts of the metric $\partial^j h^L_{jk} \neq 0$, and the trace of the metric $\delta^{jk} h_{jk}$. These parts of the metric are clearly not dynamical degrees of freedom though; although they appear in the path-integral eq. (5.8), one ultimately finds (5.13) that these parts are completely constrained to the matter, leaving only the transverse-traceless parts as true dynamical degrees of freedom.

We also must consider the implications of the momentum constraints, eqs. (5.11) and (5.12). These constrained the way in which the state can depend on the longitudinal part of the field, h^L_{jk}. In particular, the result was that the state can only depend on the longitudinal part of the field through the operator $\hat{U}_G$, (eqs. (5.15) and (5.16)). When the momentum density parts of the stress tensor, T^{0j}, are negligible, we see that the momentum constraint implies that the state cannot depend on the longitudinal part of the metric. Taking the two constraints together we then find that a physical state for this system has the form

$$\Psi[q_2, q_1, h^{TT}_{jk}, \phi] = \delta\big(\nabla^2\phi + 4\pi G\,\hat{T}^{00}\big)\ \psi_M(q_1, q_2)\,\psi_g[h^{TT}_{jk}], \tag{5.32}$$

where we've defined the "scalar" part of the metric

$$\phi \equiv \frac{\delta^{jk} h_{jk}}{2M_P}. \tag{5.33}$$

The momentum constraint has ensured that the wavefunctional has support only on transverse field configurations $\partial^j h^T_{jk} = 0$, and the Hamiltonian constraint has ensured that the wavefunctional has support only on configurations where the trace of the transverse metric components is determined by a quantum Poisson equation. As anticipated, the remaining transverse-traceless components are the only independent gravitational degrees of freedom, and are described by free massless fields. Note that the constraint on ϕ is precisely that which was anticipated by Anastopulous and Hu [123]. Although their arguments were based on classical gravity, we've seen that in the quantum theory ϕ still precisely satisfies the Poisson equation.

Since the TT gravitons are decoupled, it is clear that the Newtonian interaction between particles is determined solely by the additional term in the matter action eq. (5.30). In terms of fields, only the scalar part of the metric, ϕ, is sensitive to mass density T^{00}. Observation of entanglement generated by the Newtonian field clearly does not directly imply the existence of the TT gravitons. In principle one could consider a gravity theory which describes only the Newtonian part of the interaction quantum mechanically. This could be done as we have here, using the field ϕ and the interaction term eq. (5.30). One could then also consider some wildly different treatment of the TT modes, such as a the semi-classical coupling (eg. [250, 251], see also [252] for a review of various schemes for coupling classical and quantum systems). In the previous section we discussed the likely issues with Lorentz invariance and causality in such a theory, but using this path integral framework it seems possible to at least formulate it.

From our discussion just above and in the previous section, we come to our final understanding of the arguments around the BMV experiment. Nobody is criticizing anyone else's calculation; ultimately it seems to boil down to an argument over semantics. The issue is the ambiguity in calling something "quantum mechanical"— to our knowledge, there is no unique mathematical expressions which go along with this statement.

As an example, consider a particle which is constrained to move on a sphere of radius a. One would certainly formulate a theory in Hilbert space for this particle, and enforce the constraint using the physical state condition $(\hat{r} - a)|\Psi\rangle = 0$. There is an operator $\hat{r}$ acting on the Hilbert space, but is the radial position of this particle really "quantum mechanical"?

Better yet, consider two non-relativistic particles moving in 1-dimension which are permanently joined by an indestructible and incompressible rod of length a. One would impose the constraint on the two-particle Hilbert space as $(\hat{x}_1 - \hat{x}_2 - a)|\Psi\rangle = 0$. Is it truly correct to say that both particle 1 and particle 2 are quantum mechanical? It seems so, but one must ask, "Are the particles really distinct entities?" Supposing it is widely agreed upon that particle 1 is indeed a quantum mechanical object, then there isn't anything interesting to say about particle 2 since it has no existence distinct from particle 1. One could describe the physics completely equivalently by saying that particle 1 feels forces which act at its location and also at the location a distance a to its right, and in this formulation is seems as if particle 2 merely describes a *property* of particle 1. It is likely best to think of the two particles as two aspects of a single object—the two ends of a rigid body, but this is merely a matter of interpretation, not of calculation/physics.

The analogy with the constraints in QED and gravity should be clear. The Newtonian field is quantum mechanical, but ultimately it is not distinct from the locations of the massive particles. One can consider the particles as quantum mechanical entities with the Newtonian field being a property of these particles, or equivalently think of the Newtonian field as quantum mechanical, with the locations of the particles describing properties of this field, or equivalently think of the particles and Newtonian field as different sides of the same coin. When Anastopolous and Hu [123, 124] criticize the BMV proposal, they are saying that there is nothing about quantum gravity to learn from the observing a superposition of the Newtonian field since it is just a property of matter, and obviously matter can be put in a superposition. To call the gravitational field quantum mechanical, they say, one would need to demonstrate non-classical properties of the actual degrees of freedom of the gravitational field, the TT gravitons.

Those supporting the BMV proposal consider the gravitational field more holistically, and they don't entertain the possibility that gravity violates Lorentz invariance at microscopic scale. If one does assume Lorentz invariance though, it seems to follow naturally then that the BMV experiment demonstrates that the whole gravitational field is quantum mechanical even in the stricter sense of Anastopoulos and Hu.

To see this, consider observers at a point $(\vec{x} = 0, t = 0)$ which are moving relative to one another. These observers will all have different timeslices representing their "now", and they will then specify states on different surfaces. As a consequence, these observers will identify different components of the metric as the transverse-traceless, longitudinal, and scalar[17] parts, and thus they will identify different parts of the metric as constrained. An observation of quantum properties of the scalar part of the metric in one frame is then a demonstration of quantum properties of the transverse-traceless components in another frame. If all uniformly moving observers are equivalent, it follows that the BMV experiment can indeed determine whether the whole metric behaves quantum mechanically.

From all of this discussion we arrive at our own perspective on this experiment. We believe that the BMV-type experiments can only reveal that there is a Newton interaction between the masses, and that this alone does not imply anything about the true radiative degrees of freedom in the gravitational field. In a related manner it does not necessarily imply anything about the quantum nature of the gravitational field in the relativistic limit, and therefore of its interpretation as quantum mechanical spacetime curvature. The possibility remains that relativistic gravitation at small length scales is quite unlike that at large length scales, and that an interpretation in terms of spacetime curvature only emerges as an effective description in a large-scale classical limit of an underlying theory.

The enumerable successes of relativistic quantum field theory for particle physics indicates that Lorentz symmetry is a (at the very least, incredibly good approximate-) symmetry of the local spacetime at microscopic scales. It follows that the appropriate description of the gravitational field should

[17] The term "scalar" here refers only to invariance under spatial rotations, it is not invariant under boosts.

be a relativistic field theory (at these scales). Of course we know this to be true of classical gravity, as it is described exceptionally well by General Relativity, but we could chose to play agnostic about whether this continues to be true for gravity at microscopic scales. If this microscopic relativistic field theory is to describe the Newtonian interaction quantum mechanically, then it should be some type of quantum field theory. From here one can follow the logical chain developed largely by Kraichnan, Feynman, Weinberg, Deser, and Boulware [70, 258–264] which essentially proves that a unitary Lorentz invariant theory of gravitation must equal Einstein gravity, up to higher curvature corrections. The logic of Deser actually extends even further, requiring only local Lorentz invariance. Deser argued that a consistent theory of gravitation, which takes place on a background spacetime described by a Lorentzian manifold [263], is necessarily Einstein gravity (again, up to higher curvature corrections). It then follows that if the BMV-type experiments yield a positive result (entanglement is detected), one would need to violate standard physical principles to come up with a theory other than conventional low energy effective quantum gravity which would mimic its predictions.

The interpretation of a positive result for these experiments seems to be only one two options: i) the full gravitational field is indeed quantum mechanical, and quantized gravitons exist or ii) there is an inherent *quantum gravitational* non-locality which exists in nature at a length scale much larger than the Planck length, Lorentz symmetry is likely broken, and, unless there is a completely novel mechanism which abhors superluminal communication, new quantum gravitational technologies should be able to exploit this non-locality for communication purposes. Occam's razor suggests that option i) is likely the correct interpretation of the experiment, but here we are trying to take the hypothetical BMV experiment results literally, to see what they can and cannot say about nature.

Entanglement with the Newtonian field

Let us now consider the claims of Belenchia et al. regarding entanglement with the gravitational field in the BMV experiment. We'll omit: the spin variables as they are just carried along and clutter the notation, and the TT gravitons as they are independent of the matter. As we saw above in eq. (5.32), the relevant degrees of freedom are then just the particle coordinates q_1, q_2 and the scalar part of the metric, ϕ. To further simplify the discussion, we'll consider time $t = 0$, where the superpositions have been created but the Newtonian energy has not yet led to an accumulation of phase on each of the branches of the wave function. Again, referring back to eq. (5.32) we see that the physical states describing the system is

$$\Psi[q_2, q_1, \phi] = \frac{1}{2}\Big(\delta(\phi - \phi_{LL})\psi_L(q_1)\psi_L(q_2) + \delta(\phi - \phi_{LR})\psi_L(q_1)\psi_R(q_2) + \delta(\phi - \phi_{RL})\psi_R(q_1)\psi_L(q_2) + \delta(\phi - \phi_{RR})\psi_R(q_1)\psi_R(q_2)\Big), \tag{5.34}$$

where $\psi_{L,R}$ are well localized wavefunctions on the L and R paths, and where the various configurations of ϕ are given by permutations of L and R in the expression

$$\phi_{LR}(\vec{x}) = \frac{Gm_1}{|\vec{x} - (\vec{q}_L + \vec{d}_s)|} + \frac{Gm_2}{|\vec{x} - \vec{q}_R|}. \tag{5.35}$$

The vectors $\vec{q}_{L,R}$ point to locations in the apparatus for particle 2, while $\vec{d}_s$ points in the direction up to the apparatus for particle 1 (fig. 5.1).

To simplify the discussion we could even consider just a single particle, where the physical state would be

$$\Psi[q, \phi] = \frac{1}{\sqrt{2}}\Big(\delta(\phi - \phi_L)\psi_L(q) + \delta(\phi - \phi_R)\psi_R(q)\Big). \tag{5.36}$$

From this expression it is clear that the physical state condition implies that the state of ϕ and of q is not separable. Does this then support the claim of Belenchia et al. that the particle is entangled with its own Newtonian gravitational field? We believe the answer is no.

Indeed, the state in eq. (5.36) is not separable, so there is certainly correlation between the field ϕ and the particle, but in the space of physical states for q and ϕ there are no separable states! Without a basis of separable states, it doesn't seem possible to quantify entanglement. We cannot even obtain a reduced density matrix for q. Suppose we naively tried to do so by tracing over ϕ field configurations,

$$\begin{aligned} \rho_{particle} &= \int d\phi(x)\, \langle \phi(x)|\Psi_{q,\phi}\rangle\langle\Psi_{q,\phi}|\phi(x)\rangle \\ &= \int dq\, dq'\, \rho(q, q')|q\rangle\langle q'|. \end{aligned} \tag{5.37}$$

Since there is an injective map from the configuration space of the particle's position to the configuration space of $\phi(x)$, upon tracing out the gravitational field we would find a completely diagonal density matrix, $\rho(q, q') =$

$p(q)\delta(q-q')$. If the state $|\Psi_{q,\phi}\rangle$ described a spatial superposition, then the resulting density matrix would be completely decoherent.

Decoherence of this type is known as "false decoherence" [265]. Indeed, Unruh mentioned the example of a spatially superposed charge and its Coulomb field in [265], however the only field theory involved in their calculations was a relativistic scalar field. If we were to take the superposition eq. (5.36) and recombine the particle into a position eigenstate at q_0, then upon tracing out the ϕ field we would simply obtain the pure state $\rho = |q_0\rangle\langle q_0|$, and we would have apparently reversed any decoherence.

Although this seems reasonable, we argue that the idea of false decoherence is not quite applicable in QED and linearized quantum gravity. The issue is that in writing eq. (5.37), we have used the states $|\phi(x)\rangle$ and $|q\rangle$ which are unphysical! Neither of these states satisfy the Hamiltonian constraint of quantum gravity. If we wanted to compute an observable, we would need to evaluate

$$\langle\hat{\mathcal{O}}\rangle = \mathrm{Tr}_{phys}\left(\hat{\mathcal{O}}|\Psi_{q,\phi}\rangle\langle\Psi_{q,\phi}|\right), \tag{5.38}$$

where the trace is only over the physical subspace of the Hilbert space.

The analogy of the particles joined by a rigid rod is again useful here. If we naively traced out one of the two particles, we would completely decohere the state of the other particle unless it was already in a position eigenstate. The naive calculation suggests if we ignored particle 2 that we could not demonstrate interference phenomenon with particle 1. This is certainly not true; the rigid rod system is completely quantum mechanical and in isolation it would evolve coherently. It is not possible to 'trace out' particle 2, because measurements on particle 1 would not involve marginalizing over all possible position of particle 2—a measurement of the position of particle 1 would immediately tell us the position of particle 2.

The upshot of all this discussion is that we cannot describe the state eq. (5.36) as an entangled state between the field ϕ and particle because there is no such thing as a particle without its field, the constraints of quantum gravity imply that they are actually two aspects of the same entity.

5.3 Conclusions

In this chapter we discussed the consequences of diffeomorphism invariance in linearized quantum gravity. We begun by reviewing our findings for the gauge invariant propagator in linearized quantum gravity. This was work already reported in the author's Master's thesis [117], however the subsequent developments discussed for QED in chapter 4 solidify and clarify the gauge independence of the previously reported results. We then used these results here to provide a technical context throughwhich we could analyze and interpret the recent experimental proposals of Bose et al. [13] and Marletto and Vedral [14].

The main technical results were that we could start from a path-integral describing a propagator on the extended configuration space in linearized quantum gravity, demonstrate its invariance under small diffeomorphisms, and rewrite it in such a way that an integral remained only over the gauge-invariant transverse-traceless components. Without imposing it by hand, the resulting form of the propagator was independent of the non-canonical variables $h_{0\mu}$. Furthermore, the dependence on the longitudinal and 'trace' parts of the metric emerged in a specific form as boundary phases outside of the path-integral. In a 'path-integral first' perspective, these boundary phases then tell us about the structure of "physical states" in the theory. What we observed, is that these boundary phases were particular solutions to the Dirac *first class* constraints on the states. The resulting constraints on the physical states were certainly not new, but the technique for finding their solutions from the path-integral is, to our knowledge, new.

After the discussion of physical states we were able to address the debate around the "BMV" experimental proposal. These experiments aim to use matter-wave interferometry to test whether the gravitational field is indeed quantum mechanical, but there has been considerable debate over whether the experiments do actually test this.

Our analysis supported the objections of Anastopoulos and Hu [123, 124]. Since the experiment is sensitive only to the Newtonian part of the gravitational field, and this part of the field is completely constrained to the matter, the experiment does not directly test whether the true dynamical degrees of freedom of the gravitational field are quantum mechanical. We did not agree with the conclusions of Anastopoulos and Hu [123, 124] however, that the experiment can teach us nothing about the proper gravitational degrees of freedom.

We discussed the possible issues with superluminal communication that can occur in a theory where only the Newtonian part of the field is quantum mechanical. Belenchia et al. [2, 3] discuss how these issues are resolved in conventional quantum gravity by considering the quantum mechanical effects of the "true" gravitational degrees of freedom. Based on this, we asked whether this is the only possible resolution, or whether some alternative theory of gravitation at microscopic scales would still be viable. We then presented the outline of an argument, based on Lorentz invariance and the uniqueness of Einstein gravity as a low energy effective field theory for gravity, which suggested that one would need to abandon deep physical principles such as the local Lorentz invariance of spacetime, causality, and/or unitarity at the microscopic level, if they wanted to believe that the BMV experiment does not reveal the quantum nature of the spacetime metric.

We did not discuss one interesting possibility though. Suppose a theory is a proper unitary relativistic theory of quantum gravity at the microscopic scale, but describes an intrinsic breakdown of quantum mechanics at larger scales, ie. for masses approaching the Planck mass. Since a theory of this type *is* conventional quantum gravity at the microscopic scale, the BMV experiment should not be able to distinguish it from conventional quantum gravity. A theory which is claimed to be of this type is the Correlated Worldline (CWL) theory [113, 114, 126]. In part II of this thesis, we will discuss our own perspective on the CWL theory, and address if it is indeed *equal to* conventional quantum gravity at microscopic scales or if we expect it to be distinguishable from conventional quantum gravity in the BMV experiment.

Chapter 6

Yang-Mills Gauss law and the heavy quark binding energy in the presence of a dimension-2 gluon condensate

We'll now take a tangent from our previous considerations of quantum gravity, before returning again in part II of the thesis. In chapters 4 and 5 discussed how one can see the importance of constraints in QED and quantum gravity path integrals. One result coming from that was a nice separation in the path-integral between parts associated with radiative degrees of freedom and parts describing the constrained piece of the gauge fields. The constrained part of the gauge field generated an interaction energy associated to the matter, even when it was static and not radiating on-shell photons/gravitons. We decided to take this observation and ask whether something similar would happen in Yang-Mills theory. We then posed the question, "Can we isolate the constraint equation in a Yang-Mills path-integral, and does this allow us to compute the binding energy between static quarks?". This chapter is our attempt at answering this.

In a sense we consider this as a warm-up exercise, to see how things work in a non-linear gauge theory, before we try to better understand gravity beyond the linear regime. A number of assumptions were made throughout this chapter which a proper expert in Yang-Mills theory should scrutinize. We hope that aspects of this calculation survive scrutiny so that ultimately some of this work can serve as a useful tool for particle physics.

In this chapter we study the binding energy of a heavy quark-antiquark ($q\bar{q}$) pair using the first-order path integral formalism. This makes the Yang-Mills constraint equation explicit, and highlights that it is valid without relying on a semiclassical approximation. A generalized "gauge-covariant

Coulomb gauge" is chosen to allow for a decomposition of the chromoelectric field into a gauge-covariant generalization of transverse and longitudinal parts. This decomposition makes it clear that the $q\bar{q}$ binding energy is determined solely by the solution to the constraint equation. Assuming that the low-energy physics is dominated by the existence of a dimension-2 gluon condensate, we develop an asymptotic series solution to the constraint equation and thus to the $q\bar{q}$ binding energy. We predict a short distance QCD string tension in terms of the condensate strength and quadratic Casimir eigenvalues, and match our result to results coming from OPE analyses.

6.1 Introduction

The discovery of asymptotic freedom in Yang-Mills theory by Gross and Wilczek and Politzer [266, 267] signaled a blessing and a curse for the theory. The discovery demonstrated that perturbative calculations could be reliable and experimentally testable at high-energies, allowing for high-energy experiments to confirm that $SU(3)$ Yang-Mills theory is the correct fundamental theory of the strong nuclear interaction. Simultaneously though, it confirmed that low energy phenomenon could not be described using weak coupling methods, such as perturbation theory or semi-classical approximations.

Perhaps the most important low energy phenomenon to understand in Yang-Mills theory is hadron formation. At low energies one never observes individual quarks; rather, these objects with non-trivial color charge are always bound together in colorless hadronic states. Despite having this empirical observation for over 55 years [268], we still do not have an accepted theoretical post-diction of the phenomenon, commonly referred to as quark (or color-) confinement.

Early analysis of the various mesonic states observed in nature found that they quite accurately fit into linear Regge trajectories [269–271], a hallmark feature of the spectrum of a relativistic quantum string [272–276]. Furthermore, one could read off the effective "QCD-string" tension from these trajectories. This observation narrowed the study for quark confinement. One then had the idea that some underlying low-energy Yang-Mills physics should lead to the emergence of an effective string of chromoelectric flux connecting two quarks [277]. It remained, however, to understand why and how the chromoelectric field lines in non-abelian Yang-Mills theory would bunch together to form a string-like object rather than spreading radially as they would in electrodynamics. A variety of proposals have been given for this mechanism, many of which involving a non-trivial vacuum structure arising from non-perturbative topological excitations such as magnetic monopoles, instantons, dyons, and center vortices [see eg. 278–280, 280–287].

The most useful tool we've had to understand low-energy Yang-Mills physics has been the lattice simulations. By discretizing the underlying spacetime one simultaneously provides a necessary short-distance regularization of the theory and also truncates to a finite number of degrees of freedom so that the the theory can be simulated on a computer. This approach has been remarkably successful [see some of the many reviews for a summary, eg. 288–292], the result most relevant to us being the confirmation of the above intuition that the interaction potential between a quark and anti-quark is Coulombic at short distances where perturbation theory is applicable, but becomes linear at larger distances (see eg. [293–299]).

Despite its successes, lattice simulations have some limitations as theoretical tools. Firstly, to ensure proper convergence of the path-integral one is typically restricted to a Euclidean rather than Minkowskian description. This considerably limits the types of observables one can compute, because it obfuscates time dependence. Secondly, and more importantly, the simulations primarily function as a "black-box". One provides the YM action and coupling constant, and a lattice structure, and the computer code will compute observables. From simulations we do not necessarily get an intuitive physical description of the QCD string formation phenomenon, nor do we necessarily get clear insights on how to better predict the phenomenon analytically.

Recently though, there has been interest in the lattice community in studying the effects of a gluon condensate [300–304]. Notably, one finds that lattice measurements of the running coupling constant are not well fit by models unless the models include significant contributions from a dimension-2 gluon condensate, $\langle A^a_\mu A^a_\mu \rangle \neq 0$. [301, 302] Using operator product expansion (OPE) techniques, relationships have been made between this condensate and quark confinement. Additionally, using lattice simulations it has been observed that there are relationships between this condensate and a vacuum described by a topological instanton liquid [304]. This observation hints that if we more deeply understand the dimension-2 condensate, and the phenomenological consequences of its existence, we may come full circle back to the original topological/non-perturbative intuitions for the nature of quark confinement [278–281].

In this chapter we aim to provide a new analytical approach to the study of the quark confinement problem. We use a heavy quark approximation to study a quark-antiquark pair at fixed separation. We use the path-integral, and assume that the vacuum is described by a gluon condensate. We do not remark however on the mechanism which generates this condensate. The approach we use here has two primary features. Firstly, we use the first-order path-integral formalism so that the chromoelectric field is an explicit variable in the path-integral. Yang-Mills theory is a constrained theory, and in the first order formalism one sees explicitly that the chromoelectric field must satisfy the Yang-Mills analog of the Gauss law. Since it is a constraint, this equation holds exactly and does not rely on a semiclassical approximation to be valid in the quantum theory. This point is essential, as we do not expect a semi-classical description to be valid as the coupling flows to larger values. Secondly, we use a judicious gauge choice as well as a generalization of the transverse and longitudinal fields familiar from electrodynamics. These choices allow for a decomposition of the field variables such that the contribution to the quark-antiquark binding energy is determined solely by the solution to the constraint equation. We proceed to compute this binding energy as an asymptotic series with increasing powers of the particle separation r.

In section 6.2 we review the Yang-Mills path integral, introducing the: first-order formalism, gauge-covariant Coulomb gauge, and gauge-covariant transverse-longitudinal decomposition. We further demonstrate how with this choice of variables we can isolate the contributions to the quark-antiquark static potential as arising solely from the solution to the constraint equation.

In section 6.3 we discuss the constraint equation and introduce a path-integral description of the condensate following that of [305]. We then develop an asymptotic series expansion in powers of r for the interaction potential, with higher order terms predicted by a recursion relation.

In section 6.4 we relate our approach to the study of Wilson loops, and proceed to compute the first few terms in the interaction energy. We predict a Yang-Mills string tension in terms of the condensate strength, the coupling constant, and the quadratic Casimir eigenvalues of the adjoint representation and the chosen quark representation.

Finally, in section 6.5 we relate our prediction to predictions coming from OPE analyses and comment on various deficiencies of our model.

6.2 Non-abelian Yang Mills Theory Formalism

In Yang-Mills theory there is a gauge field A_μ which takes values in the Lie-algebra of the group G. From this one constructs the field strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu], \tag{6.1}$$

and then the action

$$S = \int d^4x \Big(-\frac{1}{2g^2} \text{tr}\, F_{\mu\nu}F^{\mu\nu} + \text{tr} A_\mu J^\mu \Big). \tag{6.2}$$

Here the trace is taken in the fundamental (lowest dimension) representation of the gauge group, and the generators T^a, $a = 1, ..., \dim(\text{G})$, in the fundamental representation have been normalized as usual

$$\text{tr}\, T^a T^b = \frac{1}{2}\delta^{ab}. \tag{6.3}$$

The source J^μ is a Lie-algebra valued current, and we'll specify to the case where it is the sum of contributions from particles on fixed worldlines

$$J^\mu(y) = \sum_n \int d\tau \frac{dx_n^\mu}{d\tau}\, q_n(\tau)\delta^4(y - x_n(\tau)). \tag{6.4}$$

In this we've introduced the Lie-algebra valued, time dependent, *color charge* $q_n(\tau)$.

Under infinitesimal gauge transformations $\Omega \approx 1 + i\omega$, with ω in the Lie-algebra, we have

$$\begin{aligned} \delta F_{\mu\nu} &= i[\omega, F_{\mu\nu}] \\ \delta A_\mu &= \partial_\mu \omega - i[A_\mu, \omega] \equiv D_\mu(A)\omega. \end{aligned} \tag{6.5}$$

The gauge-covariant derivative $D_\mu(A)$ defined here will appear frequently throughout the following calculations. From these transformation rules we can see that the Yang-Mills action is gauge invariant if the current satisfies

$$D_\mu(A)J^\mu = 0. \tag{6.6}$$

Since A is a dynamical quantum variable, this is not a requirement which could be imposed on a fixed external current, and so the internal *color* degree of freedom must also be quantum mechanical. It remains consistent though to fix the particle worldlines. This is effectively an approximation in which the masses of the charges have been taken to be arbitrarily large. For the time being, we'll treat J^μ as fixed and then later introduce the internal dynamics for the color variables.

The Yang-Mills generating functional is then just the path-integral

$$Z[J] = \int \mathcal{D}A_\mu \, e^{i\int d^4x\left(-\frac{1}{2g^2}\mathrm{tr}\, F_{\mu\nu}F^{\mu\nu} + \mathrm{tr} A_\mu J^\mu\right)}. \tag{6.7}$$

We can transition to a first-order form by introducing a Lie-algebra valued *chromoelectric* field variable E_j if we multiply eq. (6.7) by 1, represented as a particular gaussian integral. For convenience we also use the conventional redefinition $A_\mu \to gA_\mu$, $F^{\mu\nu} \to \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$ to make the coupling constant appear with the interaction terms. The result is

$$Z[J] = \int \mathcal{D}E_j \, e^{-i\mathrm{tr}\int d^4x\left(E_j + (\partial_0 A_j - \partial_j A_0)\right)^2} Z[J], \tag{6.8}$$

which can be better rewritten as

$$Z[J] = \int \mathcal{D}E_j \int \mathcal{D}A_\mu \, e^{iS}, \tag{6.9}$$

with first-order action

$$S = \mathrm{tr}\int d^4x\Big(- E^j E_j - 2E^j\partial_0 A_j - 2A_0 D_j(A)E^j - \frac{1}{2}F_{ij}F^{ij} + 2A_\mu J^\mu\Big). \tag{6.10}$$

We can now immediately integrate out A_0 since it appears as a Lagrange multiplier. Assuming sources with fixed locations in space, $J^j = 0, J^0 \equiv g\rho \neq 0$, then the sources will now only appear in the constraint equation

$$Z[J] = \int \mathcal{D}E_j \int \mathcal{D}A_j \;\delta(D_j(A)E^j - g\rho)\; e^{i\mathrm{tr}\int d^4x\left(-E^j E_j - 2E^j\partial_0 A_j - \frac{1}{2}F_{ij}F^{ij}\right)}. \tag{6.11}$$

It is now convenient to isolate the part of the chromoelectric field which is constrained from that which is not. To that end, we define the gauge-covariant-transverse and gauge-covariant-longitudinal chromoelectric fields via the relationship

$$E^j = E^j_T + E^j_L, \quad D_j(A)E^j_T = 0. \tag{6.12}$$

For brevity we'll hereafter refer to these as the GC-transverse and GC-longitudinal parts. This decomposition is similar to the typical transverse/longitudinal decomposition of vector fields which are defined by the relations

$$E^j = E^j_L + E^j_T, \quad \partial_j E^j_T = 0, \tag{6.13}$$

and are written in terms of the transverse projection

$$
\begin{aligned}
E_L^j &= \partial^j \left(\nabla^{-2}\partial_k\right) E^k \\
E_T^j &= \Big(\delta_k^j - \partial^j \left(\nabla^{-2}\partial_k\right)\Big) E^k,
\end{aligned} \tag{6.14}
$$

where ∇^{-2} is the Laplace Green's function. This decomposition of the field is particularly useful because transverse and longitudinal fields are orthogonal

$$
\int d^3x E_T^j E_L^j = 0. \tag{6.15}
$$

This orthogonality relation also holds for the GC-transverse decomposition, and is key to the decomposition being useful.

We can invert the defining relationship for these components to see that the GC-longitudinal part is a GC-gradient

$$
E_L^j = D^j(A)\Big(D_k(A)D^k(A)\Big)^{-1} D_i(A)E^i. \tag{6.16}
$$

We can hereafter use a scalar (Lie-algebra valued) variable V, where $E_L^j = -D^j(A)V$. Since the second order elliptic differential operator $D_j(A)D^j(A)$ is strictly positive, its inverse is well defined [306].

With this decomposition, the first-order form path-integral simplifies. Firstly, the Yang-Mills Gauss law constraint does not affect E_T^j, and all information about the sources is contained in the exact equation

$$
-D_j(A)D^j(A)V = g\rho, \tag{6.17}
$$

where A_j is a background and V is the variable to be solved for.

The next simplification is that the chromoelectric energy term separates because the two components are orthogonal

$$
\begin{aligned}
\mathrm{tr}\int d^3x E^j E_j &= \mathrm{tr}\int d^3x \big(E_L^j E_L^j + E_T^j E_T^j\big) \\
&= \mathrm{tr}\int d^3x \big(g\rho V[\rho, A_j] + E_T^j E_T^j\big).
\end{aligned} \tag{6.18}
$$

In the first term we used the constraint equation to rewrite E_L^j in terms of the Yang-Mills charge density and the solution $V[\rho, A_j]$ to the constraint equation eq. (6.17).

The final simplification comes in the "$p\dot{q}$" term in eq. (6.10). It can be rewritten as

$$
2\,\mathrm{tr}\int d^4x E^j \partial_0 A_j = 2\,\mathrm{tr}\int d^4x \Big(E_T^j \partial_0 A_j - V[\rho, A_j]\big(D^j(A)\partial_0 A_j\big)\Big), \tag{6.19}
$$

and we can eliminate this second term by a judicious choice of gauge. Indeed, it has been proven that the so-called generalized Coulomb gauge,

$$D^j(A)\partial_0 A_j = 0, \tag{6.20}$$

is a valid gauge condition which is free of Gribov ambiguities [306–308]. This gauge choice has a nice geometrical interpretation [306], the details of which we will not discuss here, which make it quite useful. There has been recent work trying to use this gauge choice in the canonical quantization using the constraint formalism of Dirac, however the authors' conclusions were that the canonical formulation was far more complicated than a path-integral formulation [309].

All together then, we have the first-order form for the generating functional

$$\begin{aligned} Z[J] = \int \mathcal{D}A_j \int \mathcal{D}E^T_j \,\Delta[A]\delta(D^j(A)\dot{A}_j)e^{-i\mathrm{tr}\int d^4x g\rho V[\rho,A_j]} \\ \times\, e^{i\mathrm{tr}\int d^4x\left(-E^j_T E^j_T - 2E^j_T\partial_0 A_j - \frac{1}{2}F_{ij}F^{ij}\right)}, \end{aligned} \tag{6.21}$$

where $\Delta[A]$ is the Faddeev-Popov determinant corresponding to the generalized Coulomb gauge. At this stage, we can conveniently integrate out the GC-transverse electric field to obtain a Lagrangian form

$$\begin{aligned} Z[J] = \int \mathcal{D}A_j \,\Delta[A]\delta(D^j(A)\dot{A}_j)e^{-i\mathrm{tr}\int d^4x g\rho V[\rho,A_j]} \\ \times\, e^{i\mathrm{tr}\int d^4x\left(\partial_0 A_j\partial_0 A^j - \frac{1}{2}F_{ij}F^{ij}\right)}. \end{aligned} \tag{6.22}$$

The most important difference between the non-abelian and abelian cases is that now the solution $V[\rho, A_j]$ depends on the "background" A_j. To compute the interaction energy between static sources, we must then solve eq. (6.17) for general background A_j and then evaluate the functional integral over A_j. This obviously cannot be done exactly, so what follows we will set-up approximate methods for doing so.

The interaction energy between the static sources is then given by the effective Hamiltonian $\mathcal{H}[\rho]$ defined by

$$e^{-i\int dt\mathcal{H}[\rho]} = \langle e^{-i\,\mathrm{tr}\int d^4x g\rho V[\rho,A_j]}\rangle, \tag{6.23}$$

where the angled brackets denote the vacuum expectation value for gauge fields in generalized Coulomb gauge,

$$\langle\mathcal{O}[A]\rangle = Z[0]^{-1}\int \mathcal{D}A_j\,\Delta[A]\delta(D^j(A)\dot{A}_j)\,\mathcal{O}[A]\,e^{i\mathrm{tr}\int d^4x\left(\partial_0 A_j\partial_0 A^j - \frac{1}{2}F_{ij}F^{ij}\right)}. \tag{6.24}$$

The interaction energy between static sources has now been isolated, and it remains then to solve the linear differential equation eq. (6.17) and to evaluate the functional integration over backgrounds.

6.3 Yang-Mills Gauss law

To understand the constraint equation we're going to expand out the Lie-Algebra valued fields in terms of the generators and work with components, $V = V^a T^a$, $\rho = \rho^a T^a$, with $a = 1, ..., \dim(\mathrm{G})$. The constraint equation (6.17) then reads

$$-\partial_j \partial^j V^a + 2g f^{abc} A^b_j \partial_j V^c - g^2 f^{abc} f^{cde} A^b_j A^d_j V^e = g\rho^a - g f^{abc} V^c \partial_j A^b_j, \tag{6.25}$$

where f^{abc} are the group's structure constants, $[T^a, T^b] = i f^{abc} T^c$. We won't attempt to solve this equation for a general background, rather we're going to try and understand the nature of the solution when there is a gluon condensate.

One can make a simple intuitive argument suggesting the instability of the empty gluon vacuum, a precursor for the formation of a condensate. The argument is actually borrowed from Fukuda [310]. Yang-Mills theory contains a three-point vertex, and thus "H" diagrams describing the interaction between gluons via the exchange of a gluon. It has been demonstrated that in the singlet channel the attractive force described by this gluon exchange is dominant over the repulsive force described by the four-gluon vertex. For very long-wavelength particles the binding energy will have larger magnitude than the kinetic energy of the particles and they will form a negative energy (bound) state. Moreover, since gluons are massless the bound states will appear as tachyonic poles in correlation functions and the entire field will be unstable to a pairing condensation quite analogous to the cooper pair condensation phenomenon in BCS superconductors. This argument is only suggestive[18], however one can substantiate this intuition quantitatively in Yang-Mills theory by using Bethe-Salpeter equations [310–312]. One can also argue for the instability of the empty Yang-Mills vacuum using background field methods [313, 314]. Additionally, recent numerical-lattice evidence has emerged suggesting that the Yang-Mills vacuum may

[18] One can ask why this argument doesn't also work for gravitation, and to this we do not have a concrete response but we do have a hypothesis. The gravitational interaction scales with the particle's kinetic energy, so perhaps the interaction energy between long wavelength gravitons is bounded from exceeding the kinetic energy.

be best described by an instanton liquid, the dimension-2 condensate being a consequence [304].

The gluon pairing intuition suggests that while $\langle A^a_j \rangle = 0$ one may still find non-zero vacuum expectation values for eg. the gauge-dependent dimension-2 operator $\langle A^a_\mu A^a_\mu \rangle$ or for the gauge-independent dimension-4 operator $\langle F^a_{\mu\nu} F^a_{\mu\nu} \rangle$. In the context of quark confinement, where one expects a dynamically generated QCD string tension σ to arise, it is the dimension-2 condensate which has the correct dimensions to generate a tension. It isn't obvious how this should occur though because $A^a_\mu A^a_\mu$ is not a gauge invariant observable.

Despite the dimension-2 condensate being gauge dependent, one can minimize its value over the gauge group. This minimal value is gauge invariant, and furthermore one finds that it is obtained in Landau gauge [315]. Since lattice QCD simulations have been performed in Landau gauge, with results demonstrating a non-zero value for the condensate [300, 301, 303], it is expected that regardless of gauge condition one should expect a positive definite value for the dimension-2 condensate.

To describe a gluon condensate we will use a pairing parameter ϕ_0, and assume vacuum expectation values,

$$\langle A^a_j \rangle = 0, \qquad \langle A^a_j A^a_j \rangle = \phi_0^2. \tag{6.26}$$

This approach has been discussed in the literature by [305, 316]. We will not aim to calculate ϕ_0, rather we will investigate the consequences of it being non-zero. It is natural to expect however that ϕ_0 would take a value of order the QCD scale, 0.2 GeV [317]. We will return to this point later.

To describe the gluon pairing condensate we'll follow [305] and decompose the gluon field in the path-integral as

$$A^a_j(x) = \mathscr{A}^a_j(x) + \phi_0 \eta^a_j, \tag{6.27}$$

where $\mathscr{A}^a_j(x)$ has no infinite wavelength mode, and ϕ_0, η^a_j are constants. Translation invariance will then ensure $\langle \mathscr{A}^a_j \rangle = 0$. The tensor η^a_j is constrained only to satisfy the normalization

$$\eta^a_j \eta^a_j = 1. \tag{6.28}$$

We'll then treat it as a β_G-dimensional unit vector with no preferred direction, where

$$\beta_G \equiv (d-1) \times \dim(\mathrm{G}), \tag{6.29}$$

and we'll always restrict to $d = 4$ spacetime dimensions throughout. The intuitive picture of this description of the vacuum is similar to that of spontaneous symmetry breaking. The effective potential, when computed in manner which accounts for instantons and other possible topological configurations, is assumed to have a non-trivial minimum which gives a vacuum expectation value of ϕ_0^2 to the pairing field $A_j^a A_j^a$. Every semiclassical solution then has $A_j^a = \phi_0 \eta_j^a$, however the vacuum is in a uniform superposition of equally likely *condensate angles* η_j^a, so that ultimately the vacuum expectation value of the gauge field is still vanishing, $\langle A_j^a \rangle = 0$.

Since we are ultimately interested in the static-long range force between sources, we will just formally evaluate the functional integration over the short-wavelength degrees of freedom $\mathscr{A}_j^a(x)$. We assume that this integration does two things, i) it generates the effective potential which allows ϕ_0 to take a non-zero and rigid value, and ii) that it leads to a running coupling g in our Gauss' law constraint equation eq. (6.25). We will not compute this scale dependence perturbatively as usual because we will not need the form of the function $g(p^2)$, rather we will just need certain assumptions about this function. We'll make our assumptions explicit at a later point when necessary.

With the above assumptions, the complicated Yang-Mills functional integral (6.24) is reduced to a simple integral over the condensate angle η_j^a. Vacuum expectation values are then computed as

$$\langle \mathcal{O}[A] \rangle = \frac{\int d\eta_j^a \delta(\eta_j^a \eta_j^a - 1)\, \mathcal{O}[\phi_0 \eta_j^a]}{\int d\eta_j^a \delta(\eta_j^a \eta_j^a - 1)}. \tag{6.30}$$

In this model we can exactly compute the generating function for correlation functions of the condensate angles η_j^a, the result being

$$z[b_j^a] = \sum_{m=0} \frac{1}{m!} \left(-\frac{1}{4} b_j^a b_j^a \right)^m \frac{\Gamma(\beta_G/2)}{\Gamma(\beta_G/2 + m)}, \tag{6.31}$$

where $\Gamma(x)$ is the Euler gamma function. A few examples of vacuum correlators are

$$\begin{aligned} \langle \eta_i^a \eta_j^b \rangle &= \frac{\delta^{ab}\delta_{ij}}{\beta_G} \\ \langle \eta_i^a \eta_j^b \eta_k^c \eta_l^d \rangle &= \frac{(\delta^{ab}\delta^{cd}\delta_{ij}\delta_{kl} + \text{all other contractions})}{\beta_G(\beta_G + 2)}, \end{aligned} \tag{6.32}$$

where "all other contractions" indicates that all pairings of indices into Kronecker delta symbols should be included in the sum. Note that this model is approximately gaussian in the sense that the higher order connected correlation functions are suppressed by factors of β_G. For example,

$$\begin{aligned}
&\langle \eta_i^a \eta_j^b \eta_k^c \eta_l^d \rangle_{conn.} \\
&\equiv \langle \eta_i^a \eta_j^b \eta_k^c \eta_l^d \rangle - \langle \eta_i^a \eta_j^b \rangle \langle \eta_k^c \eta_l^d \rangle - \langle \eta_i^a \eta_k^c \rangle \langle \eta_j^b \eta_l^d \rangle - \langle \eta_i^a \eta_l^d \rangle \langle \eta_j^b \eta_k^c \rangle \\
&= -\frac{1}{\beta_G} \frac{(\delta^{ab}\delta^{cd}\delta_{ij}\delta_{kl} + \text{all other contractions})}{\beta_G(\beta_G + 2)}.
\end{aligned} \tag{6.33}$$

In the large-N limit of $SU(N)$ theory, these higher order connected correlators would then vanish at least as fast as N^{-2}.

With the above description of the gluon condensate, and short-wavelength modes integrated out to give a running coupling, we arrive at the following effective Gauss law constraint equation which is local in Fourier space

$$p^2 V^a(p) + 2ig\phi_0 f^{abc} \eta_j^b p_j V^c(p) - (g\phi_0)^2 f^{abc} f^{cde} \eta_j^b \eta_j^d V^e(p) = g\rho^a(p). \tag{6.34}$$

This is now just a set of linear algebraic equations which could in principle be solved exactly. We will not attempt this here, rather we will set up a series expansion in powers of the condensate strength $g\phi_0$. Since ϕ_0 has dimensions of inverse length, the validity of this expansion will ultimately determined by the smallness of $rg\phi_0$, where r is a length scale characterizing the source charge density. We will soon see this explicitly.

To proceed we'll assume an expansion of the form

$$V^a(p) = \sum_n (g\phi_0)^n V^a_{(n)}(p). \tag{6.35}$$

The lowest order solutions are

$$\begin{aligned}
V^a_{(0)} &= \frac{g\rho^a}{p^2}, \\
V^a_{(1)} &= -2if^{abc} \frac{p_j \eta_j^b}{p^2} \left(\frac{g\rho^c}{p^2} \right),
\end{aligned} \tag{6.36}$$

and for $n \geq 2$ we have the recursion relation

$$V^a_{(n+2)} = -2if^{abc} \frac{p_j \eta_j^b}{p^2} V^c_{(n+1)} + f^{abc} f^{cde} \frac{\eta_j^b \eta_j^d}{p^2} V^e_{(n)}. \tag{6.37}$$

The interaction energy (before averaging over η_j^a) then has a series expansion

$$H = \frac{1}{2} \sum_n \int \frac{d^3p}{(2\pi)^3} (g\phi_0)^n g\rho^a(-p) V^a_{(n)}(p) \tag{6.38}$$

Some low order terms in the solution which are relevant to our upcoming discussion are

$$\rho^a V^a_{(2)} = \frac{g\rho^a f^{a\alpha b} f^{b\beta c}\rho^c}{p^4}\eta^\alpha_j \eta^\beta_k \left(\delta_{jk} - 4\frac{p_j p_k}{p^2}\right), \tag{6.39}$$

$$\begin{aligned}\rho^a V^a_{(4)} &= \frac{g\rho^a f^{a\alpha b} f^{b\beta c} f^{c\gamma d} f^{d\delta e}\rho^e}{p^6} \\ &\times \eta^\alpha_i \eta^\beta_j \eta^\gamma_k \eta^\delta_l \left(16\frac{p_i p_j p_k p_l}{p^4} + \delta_{ij}\delta_{kl} - 4\delta_{kl}\frac{p_i p_j}{p^2} - 4\delta_{ij}\frac{p_k p_l}{p^2} - 4\delta_{jk}\frac{p_i p_l}{p^2}\right).\end{aligned} \tag{6.40}$$

Note the factor of two in the recurrence relation eq. (6.37). This factor of two is responsible for the relative factor of four appearing in the tensor structure in eq. (6.39). Without this factor of four the tensor structure would be the standard transverse projector appearing often in QED calculations, but crucially this factor of four ensures that the longitudinal term dominates.

In the n^{th} order term in the solution, $V^a_{(n)}$, there is always a purely spatial (kinematic) tensor and a purely group theoretic tensor consisting of n structure constants contracted between the two source charges. These color and kinematic structures are contracted together by the product of n η^a_j's. Because of the rotational invariance of the vacuum state we know that, after the spatial tensor is contracted with the functionally integrated η^a_j's, it will be independent of p. We can then conclude that all of the p dependence in the solution $V^a_{(n)}$ will be in the charge densities ρ^a, the running coupling $g(p^2)$, and in the overall factors p^{-2-n}. Once we've specified the charge densities, we can then evaluate the fourier integral in eq. (6.38) without yet specifying details about the state of the internal colour degrees of freedom or the condensate angle η^a_j.

To set this up, we can implicitly define the p-independent part at each order,

$$G^{ab}_{(n)}\rho^b \equiv p^{-2-n}V^a_{(n)}. \tag{6.41}$$

We then have the interaction energy between the sources

$$H = \sum_n \frac{1}{2}\phi_0^n G^{ab}_{(n)} \int \frac{d^3p}{(2\pi)^3} g^{2+n}(p^2)\frac{\rho^a(-p)\rho^b(p)}{p^{2+n}}, \tag{6.42}$$

where $G^{ab}_{(0)} = \delta^{ab}$, $G^{ab}_{(2)}$ and $G^{ab}_{(4)}$ can be read off of eqs. (6.39) and (6.40), and higher order terms can be computed from the recurrence relation eq. (6.37).

To evaluate this integral we'll need to specify the spatial form the of charge density, and make some assumptions about the nature of the running coupling function.

Before proceeding to understand the sources, we can further simplify this expression. From the definition of the effective Hamiltonian, eq. (6.23), we need to exponentiate eq. (6.42) and evaluate the functional integral over the condensate angles η_j^a. We've observed however, eq. (6.33), that the higher-order connected correlation functions are suppressed by powers of β_G^{-1}, which is eg. for $SU(3)$ theory $\beta_{SU(3)}^{-1} = 1/24$. To leading order in a β_G^{-1} expansion we can then simply retain the mean field approximation for the effective Hamiltonian[19]

$$\langle e^{-i\,\mathrm{tr}\int d^4x g\rho V[\rho,A_j]}\rangle = e^{-i\,\mathrm{tr}\int d^4x g\rho \langle V[\rho,A_j]\rangle}, \tag{6.43}$$

where the angled brackets denote the condensate angle average eq. (6.30). We can then write the effective Hamiltonian for the Yang-Mills charges as

$$\mathcal{H} = \sum_n \frac{1}{2}\phi_0^n \langle G^{ab}_{(n)}\rangle \int \frac{d^3p}{(2\pi)^3} g^{2+n}(p^2)\frac{\rho^a(-p)\rho^b(p)}{p^{2+n}}. \tag{6.44}$$

For later use we list some of the expectations values for the $G^{ab}_{(n)}$. Clearly $\langle G^{ab}_{(0)}\rangle = \delta^{ab}$, and at lowest non-trivial order we have

$$\begin{aligned}\langle G^{ab}_{(2)}\rangle &= f^{a\alpha c} f^{c\beta b}\frac{\delta^{\alpha\beta}\delta_{jk}}{\beta_G}\left(\delta_{jk} - 4\frac{p_j p_k}{p^2}\right)\\ &= -\frac{f^{a\alpha c}f^{c\alpha b}}{\beta_G}\\ &= \delta^{ab}\frac{C_A}{\beta_G},\end{aligned} \tag{6.45}$$

where C_A is the quadratic Casimir eigenvalue in the adjoint representation. At the next order we find

$$\langle G^{ab}_{(4)}\rangle = \frac{1}{\beta_G(\beta_G+2)}\Big(- 4C_A^2\delta^{ab} + 7 f^{\alpha\beta\gamma}(f^{\alpha}_{\ ac}f^{\beta}_{\ cd}f^{\gamma}_{\ db})\Big). \tag{6.46}$$

[19] In $SU(N)$ theory we have $\beta_G^{-1} \sim N^{-2}$ at large N, which allows us to use the following expression exactly in the large N limit. We also have $\beta_G^{-1} \sim d^{-1}$ in a large number of dimensions d. We could then also use this tool to study Yang-Mills theory in many dimensions.

For $SU(N)$, there is a nice identity which allows us to see that the last term is proportional to the quadratic Casimir operator in the adjoint representation [318]. As a result, we find for $SU(N)$ Yang-Mills theory

$$\langle G^{ab}_{(4)} \rangle = -\delta^{ab} \frac{C_A^2}{2\beta_G(\beta_G + 2)}. \tag{6.47}$$

6.4 Sources

6.4.1 Charge Density

The formalism established thus far is applicable to any matter source which is i) described by a gauge invariant action and ii) static ($J^a_j = 0$, $J^a_0 \neq 0$). As a first application of this formalism we'll specify to the case of a heavy quark-antiquark pair separated by a fixed distance r. We use the term quark loosely to describe a particle in some representation R of the gauge group, its corresponding antiquark being in the conjugate representation. The Yang-Mills charge density for this quark-antiquark pair can be written as

$$\rho^a(x) = \rho^a_q \delta^{(3)}(x - r/2) + \rho^a_{\bar{q}} \delta^{(3)}(x + r/2). \tag{6.48}$$

We'll soon discuss the actual form of $\rho^a_q, \rho^a_{\bar{q}}$, but for now we'll first note that the numerator in the interaction energy is

$$\rho^a(-p)\rho^b(p) = \rho^a_q \rho^b_q + \rho^a_{\bar{q}} \rho^b_{\bar{q}} + \rho^a_q \rho^b_{\bar{q}} e^{i\vec{p}\cdot\vec{r}} + \rho^a_{\bar{q}} \rho^b_q e^{-i\vec{p}\cdot\vec{r}}. \tag{6.49}$$

Clearly the first two terms are self-energy terms which are independent of the mutual separation, and we'll simply ignore these. The interaction energy can then be written as

$$\mathcal{H} = (\rho^a_q \rho^b_{\bar{q}} + \rho^a_{\bar{q}} \rho^b_q) \sum_n \frac{1}{2} \phi_0^n \langle G^{ab}_{(n)} \rangle \int \frac{d^3p}{(2\pi)^3} g^{2+n}(p^2) \frac{e^{i\vec{p}\cdot\vec{r}}}{p^{2+n}}. \tag{6.50}$$

At the point we must make certain assumptions about the running coupling $g(p^2)$. The full non-perturbative form of this function, valid in both the UV and IR is obviously not known. As a result of asymptotic freedom one can approximate the functional perturbatively in the UV, but there is significant disagreement about the IR form. Indeed, it is not agreed upon whether as $p^2 \to 0$ the coupling diverges, vanishes, or "freezes" at a finite value (see [319] for a recent review). Even within these three camps there is quantitative disagreement [319]. We will not attempt to discuss all of the reasons for disagreement in the literature, but we note that a primary issue is that different calculation approaches lead to different definitions of the running coupling. For our purposes we want to understand whether our approach is more in-line with those suggesting g diverges in the IR or those suggesting the contrary.

It seems that generically, one finds an IR divergent running coupling if the interaction potential is written as

$$V(p) \sim \frac{g^2(p^2)}{p^2}, \tag{6.51}$$

and all non-Coulombic behaviour is folded into the running coupling [319]. In our expression for the interaction potential, eq. (6.42), the running coupling is separate from a series of terms of increasing inverse powers of p. We then expect that the divergent behaviour as $p^2 \to 0$ is already accounted for and will not arise in the running coupling. In what follows we will assume that $g(p^2)$ reaches a non-zero freezing as $p^2 \to 0$.

With the divergence set aside, we will assume that $g^2(p^2)$ is described well by perturbation theory at high-energies, and that its form for lower energies is just that of a smooth function which is not too rapidly varying. A number of calculations in the literature suggest that as energies are decreased $g^2(p^2)$ smoothly departs from the perturbative prediction, simply flattening out and approaching a constant value at zero energy [see the review 319, and the many refs. therein].

We can then start to understand the Fourier integral in eq. (6.50),

$$I_n(r) = \int \frac{d^3p}{(2\pi)^3} g^{2+n}(p^2) \frac{e^{i\vec{p}\cdot\vec{r}}}{p^{2+n}}. \tag{6.52}$$

For $n = 0$, this is just the Coulomb expression with running coupling and it follows from our above discussion that we do not expect the running coupling in this expression to drastically change the qualitative shape of the effective Coulomb potential.

To understand the higher order terms, We see two approaches. If the running coupling function was known, in particular if its analytic structure were known, then one might try to evaluate the integral using typical methods from complex analysis such as the residue theorem etc. Some properties of the analytic structure are understood, [320], so perhaps this could be a fruitful research direction.

In what follows however we will make the considerably more simple approach, and simply assume that we are interested only in short distance physics. In this approximation the coupling constant runs slowly and is essentially just equal to the value specified at the high-energy renormalization point. After a few standard manipulations, the integral for the higher order terms can be evaluated as a residue integral

$$I_n(r) = \frac{g_0^{2+n}}{4\pi} \mathrm{Re} \int_0^{\pi/2} d\theta \sin\theta \, i\mathrm{Res} \left(\frac{e^{ip|r\cos\theta|}}{p^n} \right) \Big|_{p=0}, \tag{6.53}$$

where it remains to evaluate the residue of the function within the parentheses. Here g_0 is the high-energy "constant" value of the running coupling, fixed at the renormalization point. For $n > 1$ we have a higher-order pole and the residue will involve various derivatives of this function. After integration we obtain the resulting Hamiltonian

$$\begin{aligned}\mathcal{H} =& (\rho_q^a \rho_{\bar{q}}^a) \int \frac{d^3p}{(2\pi)^3} g^2(p^2) \frac{e^{i\vec{p}\cdot\vec{r}}}{p^2} \\ &+ (\rho_q^a \rho_{\bar{q}}^b + \rho_{\bar{q}}^a \rho_q^b) \frac{g_0^2}{8\pi} \sum_{n=1}^{\infty} (-1)^n \langle G_{(2n)}^{ab} \rangle \frac{(\phi_0 g_0)^{2n} |r|^{2n-1}}{(2n)!}. \end{aligned} \tag{6.54}$$

6.4.2 Color Factors

At this point we must work to understand the color factor, $G_{(2n)}^{ab}(\rho_q^a \rho_{\bar{q}}^b + \rho_{\bar{q}}^a \rho_q^b)$, at least for the first few orders. The factor $G_{(2n)}^{ab}$ still depends on a product of $(2n)$ condensate angles η_j^a which will need to be functionally integrated over, but also, the internal color degrees of freedom contained in $\rho_q^a, \rho_{\bar{q}}^a$ are also quantum mechanical and this must be described as well. The effective Hamiltonian $\mathcal{H}[\rho]$ defined in eq. (6.23) is either an operator on the color Hilbert space, or equivalently, a function of color variables in a path-integral. To extract an energy which depends only on the quark-antiquark separation we'll need to average over the color degrees of freedom appropriately.

The most common discussion of the quark-antiquark pair involves the Wilson loop [321],

$$W[C] = \mathrm{tr}\, \mathcal{P} \exp \left(i \oint A \right), \tag{6.55}$$

where $\mathcal{P}$ denotes path ordering along the closed curve C, and the trace is performed in some representation R of the gauge group. If we specify to quarks in the fundamental representation of the gauge group $SU(N)$, it is straightforward to see that this Wilson loop comes from a path-integral over color degrees of freedom [322]. Explicitly it is

$$W[C] = \int \mathcal{D}\lambda \mathcal{D}w \mathcal{D}w^\dagger \, e^{iS_w[A]} w_j(\tau = \tau_f) w_j^\dagger(\tau = \tau_i), \tag{6.56}$$

where $w, w^\dagger$ are complex N-dimensional vectors and the action is

$$S_w[A] = \int_{\tau_i}^{\tau_f} d\tau \left[i w^\dagger \left(\frac{d}{d\tau} - iA(x(\tau)) \right) w + \lambda (w^\dagger w - 1) \right], \tag{6.57}$$

and the gauge-field 1-form is

$$A(x(\tau)) = \frac{dx^\mu(\tau)}{d\tau} T^a A^a_\mu(x(\tau)). \tag{6.58}$$

When the gauge field is turned off this is just a spin-coherent state path integral for the "spin"-vector of $SU(N)$.

Following common practice, we take the curve $x^\mu(\tau)$ to be a rectangle in spacetime composed of a straight spacelike segments of length $r = 2a$ at each of the far future and past $t \to \pm\infty$ which are connected by two straight timelike lines. With this configuration the contribution from the spacelike segments is irrelevant and we can rewrite the above colour path integral as the product of integrals for the quark and antiquark

$$\begin{aligned} W[C] = \int \mathcal{D}\lambda \mathcal{D}w \mathcal{D}w^\dagger \, e^{iS_w} \int \mathcal{D}\bar{\lambda} \mathcal{D}\bar{w} \mathcal{D}\bar{w}^\dagger \, e^{iS_{\bar{w}} + i\int d^4x J^\mu A_\mu} \\ \times w_j(t=\infty) \bar{w}_j(t=\infty) w^\dagger_k(t=-\infty) \bar{w}^\dagger_k(t=-\infty). \end{aligned} \tag{6.59}$$

Here we've pulled out the gauge interaction term, $S_w = S_w[A=0]$, and defined the current density

$$J^{a\,\mu}(x) = \delta^{0\mu}\delta^{(3)}(x - r/2) w^\dagger_j T^a_{jk} w_k - \delta^{0\mu}\delta^{(3)}(x + r/2) \bar{w}^\dagger_j (T^a_{jk})^* \bar{w}_k, \tag{6.60}$$

where the asterisk denotes complex conjugation. From this form, with the initial and final state operator insertions made explicit, we can see that this Wilson loop computes a transition amplitude between an initial singlet state for the quark-antiquark pair to a final singlet state.

It may not be immediately clear how the operator insertions at $t = \infty$ arose in this expression. Since the Wilson loop curve is a closed loop, when it is split into quark and anti-quark integrals there is an implicit delta function constraining their color variables in the future to be equal. This final state variable is integrated over in the path integral, but one can replace this with a sum over possible operator insertions. One can check that the constraint enforced by the Lagrange multiplier will set the amplitude to vanish unless there is one insertion of $\bar{w}^\dagger_j w^\dagger_k$ in the past and one insertion of $\bar{w}^\dagger_l w^\dagger_m$ in the future. The resolution of the identity in the future can then be expanded in a Fock basis and the constraint ensures that only the final singlet state leads to a non-zero result.

We can take the Wilson loop current density eq. (6.60) and extract from it the $SU(N)$ charges ρ_q^a, $\rho_{\bar{q}}^a$ defined in eq. (6.48),

$$\begin{aligned}\rho_q^a &= w_j^\dagger T_{jk}^a w_k, \\ \rho_{\bar{q}}^a &= -\bar{w}_j^\dagger (T_{jk}^a)^* \bar{w}_k. \end{aligned} \tag{6.61}$$

In principle, we could then compute the Yang-Mills vacuum expectation value of the Wilson-loop by evaluating the color path-integrals

$$\begin{aligned} W[C] = \int \mathcal{D}\lambda \mathcal{D}w \mathcal{D}w^\dagger \, e^{iS_w} \int \mathcal{D}\bar{\lambda} \mathcal{D}\bar{w} \mathcal{D}\bar{w}^\dagger \, e^{iS_{\bar{w}}} \, e^{-i\int dt \mathcal{H}} \\ \times\, w_j(t=\infty)\bar{w}_j(t=\infty) w_k^\dagger(t=-\infty)\bar{w}_k^\dagger(t=-\infty), \end{aligned} \tag{6.62}$$

with $\mathcal{H}$ given by eq. (6.54) and the charges written in terms of the w_j as in eq. (6.61). However as we will see shortly, the operator method is more straightforward.

Although we only have the explicit path-integral representation for the internal color variables of quarks in the fundamental representation of $SU(N)$, we can proceed more generally following the above statement that Wilson loops compute transition amplitudes for quark-antiquark pairs evolving from initial singlet states to final singlet states. In an operator representation, the Yang-Mills charges ρ_q^a, $\rho_{\bar{q}}^a$ for a general representation R of group G would then have the form

$$\begin{aligned}\rho_q^a &= \delta_{\bar{A}\bar{B}} T_{AB}^a(R), \\ \rho_{\bar{q}}^a &= -\delta_{AB} (T^a(R)^T)_{\bar{A}\bar{B}}, \end{aligned} \tag{6.63}$$

where T denotes the transpose, A, B are indicies in the quark-color Hilbert space and $\bar{A}, \bar{B}$ are indicies in the antiquark-color Hilbert space. In this basis the singlet state has wavefunction

$$\psi_{A\bar{A}}^{sing} = (\mathrm{dim(R)})^{-1/2} \delta_{A\bar{A}}. \tag{6.64}$$

In the Hamiltonian eq. (6.54), the color charge operators are contracted at each order with the factor $\langle G_{(2n)}^{ab} \rangle$. We've also demonstrated that the lowest order contributions, $\langle G_{(0)}^{ab} \rangle, \langle G_{(2)}^{ab} \rangle, \langle G_{(4)}^{ab} \rangle$, are all proportional to δ^{ab}. It then follows that up to $\mathcal{O}(\phi_0^4)$ the singlet state is an eigenstate of the Hamiltonian. We see this from the definition of the quadratic Casimir operator

$$\begin{aligned}(\rho_q^a \rho_{\bar{q}}^a)_{A\bar{A},B\bar{B}} \, \psi_{B\bar{B}}^{sing} &= -T_{AB}^a(R) T^a(R)_{B\bar{A}} (\mathrm{dim(R)})^{-1/2} \\ &= -C_R \, \psi_{A\bar{A}}^{sing}. \end{aligned} \tag{6.65}$$

When computing Wilson loops, since the color states start in an eigenstate of the Hamiltonian (up to $\mathcal{O}(\phi_0^4)$), they remain in this state and we can replace the charge operator $\rho_q^a \rho_{\bar{q}}^a$ by the eigenvalue in the singlet state, C_R. The energy eigenvalue for a singlet state is then

$$E_{sing} = - C_R \int \frac{d^3p}{(2\pi)^3} g^2(p^2) \frac{e^{i\vec{p}\cdot\vec{r}}}{p^2} + \frac{g_0^2}{4\pi} \frac{C_R C_A}{2\beta_G} (\phi_0 g_0)^2 |r| + \frac{g_0^2}{4\pi} \frac{C_R C_A^2}{4\beta_G(\beta_G + 2)} \frac{(\phi_0 g_0)^4 |r|^3}{4!} + \dots . \quad (6.66)$$

The first two terms are universal, whereas the r^3 term has been proven here only for the $SU(N)$ gauge group. The overall proportionality of the energy to the quadratic Casimir eigenvalue of the representation is a nice check of our result thusfar. It has been demonstrated quite convincingly in lattice simulations that this ought to occur [323–325].

Assuming $r(g_0\phi_0)$ is sufficiently small that we can neglect the non-linear terms in this series, we can read off an effective string tension from eq. (6.66),

$$\sigma = \frac{g_0^2}{4\pi} \frac{C_R C_A}{2\beta_G} (\phi_0 g_0)^2, \quad (6.67)$$

and for fundamental quarks in $SU(N)$ theory this is

$$\sigma = \frac{g_0^2}{48\pi} (g_0 \phi_0)^2. \quad (6.68)$$

The condensate strength $g_0\phi_0$ remains as an unfixed parameter.

6.5 Discussion

There are a number of ways that we could estimate the parameter $g_0\phi_0$ by comparing with reported values in the literature. Unfortunately, the dimension-2 condensate strength is gauge dependent. This precludes any direct comparison since we could not find previously reported results involving this quantities in the generalized Coulomb gauge. We'll instead make a rough estimates based on reported values in other gauges and also try to use phenomenological constraints.

Let's first return to the assumption that $r(g_0\phi_0)$ is small. We can look at the relative size of the cubic term to the linear term in the binding energy as a function of r. If the magnitude of cubic term is to be less than 5% of the magnitude of the linear term then we must have

$$r < \left(\frac{1}{g_0\phi_0}\right)\sqrt{\frac{12(\beta_G+2)}{5C_A}}. \tag{6.69}$$

For $SU(3)$ theory, with appropriate factors of $\hbar$ and c replaced, this is

$$r < \frac{0.9\,\text{fm GeV}}{g_0\phi_0}. \tag{6.70}$$

One expects that at sufficiently large separations, when it is energetically favourable, the QCD string will snap and a quark-antiquark pair will be produced. This leads to a flattened static potential above some critical separation r_c. Such an effect could not be seen in our heavy quark model, but it has been observed in lattice simulations with dynamical quarks. In these simulations a value $r_c \approx 1.2\,\text{fm}$ is found consistently [326, 327].

In the "quenched" simulations which do not include dynamical quarks, the linear rise of the potential has been shown to continue past the 1.2 fm mark and no cubic behavior has been conclusively demonstrated to arise [293, 296–299]. Furthermore, one can prove that invariance of Wilson loops under space/time interchange implies that the potential cannot grow faster that linearly with $|r|$ at large distances [328]. Unfortunately, these considerations then cast serious doubt on the validity of the model we've developed here.

Despite the inconsistencies between this model and the constraints from quenched lattice data, we can still ask whether the model may be rendered consistent by the inclusion of dynamical quarks, ie. string-breaking effects. Since the string-breaking phenomenon has been demonstrated to flatten the potential above $r = 1.2\,\text{fm}$ we can require only that our non-linear terms are negligible up to this distance. This then implies the bound

$$g_0\phi_0 \lesssim 0.75\,\text{GeV}. \tag{6.71}$$

Additionally, from Particle Data Group, ref. [317], we can borrow an approximate value for the strong coupling constant at, for an example, the scale of the Z-boson mass,

$$\frac{g^2(M_Z)}{4\pi} \approx 0.12. \tag{6.72}$$

Taking this together with the above constraint of the condensate strength, we can estimate a bound on our model's prediction of the QCD string tension

$$\sigma \lesssim (0.0056\,\text{GeV})^2. \tag{6.73}$$

This value is, however, completely inconsistent with the commonly understood value for the QCD string tension $\sigma \approx (0.18 - 0.22)\,\mathrm{GeV}^2$ [329].

Although the string tension predicted by our calculation is ruled out by meson phenomenology, it may simply indicate that we need to follow the alternative route which we've previously mentioned wherein one inserts a model function for the IR complete running coupling constant. We note that values have been quoted for the running coupling at zero energy ranging from $\alpha_s(0) = 2.97 - 4.74$, with the lower end coming from Landau gauge calculations and the higher end coming from Coulomb gauge calculations [319, 330, 331]. Inserting these values for the coupling into the bound eq. (6.71) we find

$$\sigma \lesssim (0.14 - 0.22)\,\mathrm{GeV}, \tag{6.74}$$

which is no longer inconsistent with the commonly understood value for the QCD string tension $\sigma \approx (0.18 - 0.22)\,\mathrm{GeV}$. To confidently rule out the model we've studied here, it seems that one would need to perform a more careful analysis using an (at least approximate) IR complete model of the running coupling function.

It also remains important to try and predict the model parameters from genuine calculations in the generalized Coulomb gauge, using tools which have been previously used in Landau or Coulomb gauge to understand the IR limit of Yang-Mills theory. For the IR behaviour of the coupling, one may try to use Schwinger-Dyson techniques as in, for example, [331] (see also the review [319] and refs. therein). For the condensate strength, one may try to use Bethe-Salpeter equation techniques [310–312, 316].

In addition to the above bound which we've used to assess the validity of this model, we can also try to compare with calculations of the QCD string tension coming from rather different approaches. Most studies involving a dimension-2 condensate are interested in the Lorentz invariant condensate $g^2\langle A^a_\mu A^\mu_a\rangle$, not just the spatial parts which we've isolated. We can try to find a comparison by using the Lorentz invariance of the vacuum, which implies

$$\langle A^a_j A^a_j\rangle = 3\langle A^a_0 A^a_0\rangle. \tag{6.75}$$

suggesting that in the commonly used Euclidean spacetime

$$\langle A^a_\mu A^\mu_a\rangle = (4/3)\phi_0^2. \tag{6.76}$$

One must be careful when comparing the RHS of this expression to our eq. (6.26) though; the dimension-2 condensate is in principle a gauge-dependent quantity and the above considerations may only apply in a covariant gauge.

Nonetheless we can still try make contact between our calculation on the QCD string tension and some previously reported calculations which used quite different approaches.

In the literature, investigations of the dimension-2 condensate and its studies have predominantly used operator product expansion (OPE) techniques [300–304, 332–336], we hope that the rather different approach we've provided here may prove to be complementary. For example, ref. [337] discusses the connection between the QCD string tension, the tachyonic gluon mass, and the dimension-2 condensate. Their analysis is an extension of refs. [338, 339] in which the physics of the condensate of modeled by assuming the existence of a term λ^2/Q^2 in the OPE of various QCD correlation functions $\Pi_J(Q^2)$.

We can actually make direct contact with some of this work. In ref. [337] the authors claim a short distance string tension for $SU(N)$ theory,

$$\sigma_0 = \frac{g^2}{72\pi}\frac{N^2}{N^2-1}g^2\langle A_\mu A^\mu\rangle, \tag{6.77}$$

and although it isn't explicitly stated, from their references it appears they are working in Euclidean spacetime. Using the 4/3 factor we then translate their expression into our notation

$$\sigma_0 = \frac{g_0^2}{54\pi}\frac{N^2}{N^2-1}g_0^2\phi_0^2. \tag{6.78}$$

Although this doesn't explicitly agree with our formula eq. (6.68) for general N, we do find agreement for the phenomenologically interesting case of $N = 3$. The disagreement for general N may just be the result of their relative rescaling of the condensate for general N, but it is not clear. Either way it is encouraging that using either OPE techniques, or our current formalism, one can arrive at apparently identical expressions for the QCD string tension. This provides optimism that despite the limitations of this approach, it may still prove to be a useful avenue for performing further Yang-Mills calculations.

6.6 Conclusions

In this chapter we have provided an analysis of the static quark-antiquark binding energy in the presence of a gluon condensate. We used the first-order path-integral formalism, wherein the chromoelectric field is an explicit variable. The benefit of this approach was that the Yang-Mills constraint equation was explicit and exact. We used a gauge-covariant generalization of the typical transverse/longitudinal decomposition from electrodynamics to separate out the constrained variables from the unconstrained variables. Doing all of this, and fixing the generalized Coulomb gauge condition, we demonstrated that the static quark-antiquark binding energy is determined solely by the solution to the Yang-Mills Gauss law equation.

We modeled the Yang-Mills vacuum as a dimension-2 gluon pairing condensate and used this to set-up a series expansion for the solution to the Yang-Mills Gauss law equation. As a consequence, we arrived at a series expansion for the static quark-antiquark binding energy in powers of the condensate strength, with a recursion relation to compute the higher-order terms. Our central result was a prediction of the coefficient of the term linear in quark separation, ie. the meson string tension. Our result matches a prediction coming from OPE analysis, suggesting that the techniques used here may indeed provide a useful complementary approach for more detailed calculations.

As mentioned previously, this chapter was quite tangential to the rest of the thesis. We ultimately found it to be an interesting investigation, ie. whether we could indeed exploit the Yang-Mills constraint equation to compute quark binding energy. We thought this may serve as a warm-up for calculation in non-linear quantum gravity, but it is quite unclear whether this will be the case. Regardless, we hope that in the hands of actual experts in particle physics some of the ideas of this chapter may eventually prove to be useful.

Part II

Foundations and applications of the Correlated Worldline Theory of quantum gravity

Chapter 7

The Correlated Worldline Theory of quantum gravity

7.1 Introduction

Classical gravitation is described by the classical field theory General Relativity (GR) and has been well tested for large scales and heavy objects with many constituents. This should be contrasted with conventional quantum theory which has been well tested at small scales for small mass objects composed of relatively few constituents[20]. In the quest for marrying gravitation with quantum theory it grows clear that there is an intermediate regime where both quantum and gravitational effects are expected to be relevant which is entirely untested. Taking the fundamental constants associated with quantum theory and GR $(\hbar, G, c)$ we can follow Planck and estimate the quantum gravity scale

$$\begin{aligned} \ell_P &= \sqrt{\frac{\hbar G}{c^3}} \sim 10^{-35}\,\mathrm{m}, \\ t_P &= \sqrt{\frac{\hbar G}{c^5}} \sim 10^{-44}\,\mathrm{s}, \\ E_P &= \sqrt{\frac{\hbar c^5}{G}} \sim 10^{19}\,\mathrm{GeV} \\ m_P &= \sqrt{\frac{\hbar c}{G}} \sim 10^{-8}\,\mathrm{kg} \end{aligned} \tag{7.1}$$

The Planck length and time are unfathomably small in standard units, and when written in units relevant for particle physics the Planck energy is unfathomably large. This is often taken as evidence that phenomena requiring both quantum theory and GR are not experimentally accessible. It must be remarked however, that the Planck mass is only large if one considers it

[20] Experiments involving "macroscopic" superpositions of currents in superconductors actually involve a relatively small number of superposed electrons [340].

in the context of elementary particle physics, a human eyelash has a mass $\sim 1\,m_P$. If quantum-gravitational phenomena are to be observed it is possible that they would be seen on the Planck mass scale, ie. for an object with mass $\sim 10^{-8}\,\mathrm{kg}$ composed of $\sim 10^{18}$ nucleons. To date no experiment has tested either GR or quantum theory at this scale, and so we do not necessarily have reason to assume that GR is valid below this scale or that quantum theory is valid above it.

7.2 Correlated Worldline Theory

The Correlated WorldLine (CWL) theory is an example of an "alternative" quantum gravity theory in which massive objects exhibit departures from the superposition principle. The primary mechanism in CWL theory is that different paths in a path integral for a single system now interact gravitationally [112, 113]. The intuition behind the CWL proposal is that the different paths describing the evolution of a spatially superposed mass will gravitate towards each other, which ultimately prevents large superpositions of macroscopic objects and thereby causes a quantum-to-classical transition.

A proper motivation for the principles of this theory and its mathematical definition were given by P.C.E. Stamp and collaborators in refs. [113, 114]. In what follows we will only have the space to just sketch some of these key arguments which determine the form of the theory. We do this to illustrate how one arrives at such a theory, however our perspective in this thesis will be more consequentialist than the perspective of Stamp in [113]. The research question taken here will be, "Assuming one is interested in modifying quantum theory such that various paths in a path integral for a single system are now coupled via gravity, is there a consistent mathematical theory to describe this, and what are the predictions of such a theory?".

To motivate the CWL theory one can start with the key insights of Kibble et al. on generalizations of QM [7, 74, 75]. Our mains takeaways from this are that: i) the non-linear nature of GR would require an alternative QG theory to be a non-linear generalization of quantum mechanics, and ii) that a non-linear quantum mechanics theory will be fraught with issues unless one abandons some of the formal structure of Hilbert space, projective measurement operators, etc. Motivated by this, Stamp has then argued [113] that attempts at generalizing QM should start from the path-integral rather from state vectors and operators.

In conventional quantum theory one describes the evolution of a particle via the propagator

$$K(B;A) \equiv K(x_B, t_B; x_A, t_A) = \int_{x_A}^{x_B} \mathcal{D}x \, e^{\frac{i}{\hbar}S[x]}. \tag{7.2}$$

As a consequence of the superposition principle each of the paths in the path integral is independent from any other path. If there are two "classical" paths, ie. two possible trajectories for which the action is stationary, then both paths contribute significantly to the propagator and interference phenomena will occur (eg. the two-slit system). One colloquially says that the particle took both paths simultaneously, or was in a superposition of trajectories during its evolution.

Starting from this path-integral, one must then determine what generalization to make which incorporates gravity. The goal here is, of course, to construct an alternative theory of quantum gravity, ie. one a theory in which QM breaks down for macroscopic systems. Here one must appeal to a physical principle to determine the mathematical structure, otherwise there is no constraint on the possible modifications that one can make. Stamp appeals to a principle central to classical general relativity [113], the equivalence principle.

Although there is a number of different statements of the equivalence principle, one could phrase it as Wesson has *"All test particles at the alike spacetime point, in a given gravitational field, will undergo the same acceleration, independent of their properties, including their rest mass"* [341]. Based on this, Stamp has argued that the gravitational field should not distinguish between: i) two paths of a single particle and ii) a path from each of two distinct but otherwise identical particles [113]. The consequence of this, is that one should allow for gravitational interactions between different paths <u>of a single system</u>. The same reasoning implies that gravitational interactions should be between all possible paths, not just pairs.

We will soon discuss how one formulates this mathematically, but for the moment we will follow this idea just heuristically to anticipate the general consequences. To do so, we'll start by considering only interactions between just one pair of paths for a non-relativistic particle. To do so we must consider two path-integrals for the particle, ie. an object of the form

$$K(B;A) = \int_{x_A}^{x_B} \mathcal{D}x_1 \int_{x_A}^{x_B} \mathcal{D}x_2 \, e^{\frac{i}{\hbar}(S[x_1]+S[x_2]+S_{cor}[x_1,x_2])}, \tag{7.3}$$

where the term S_{cor} couples two paths, and can be written simply as

$$S_{cor}[x_1, x_2] = \int_{t_A}^{t_B} dt \, \frac{Gm^2}{|x_1 - x_2|}. \tag{7.4}$$

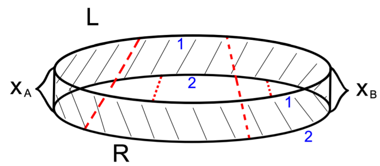


Figure 7.1: Two replicas (1 and 2) of a particle propagating from x_A to x_B. There are two dominate classical paths L and R. The particle paths are indicated by the solid black lines. The replica particles interact via S_{cor} (indicated by dashed red lines) when they are on different paths.

It is essential to understand that a new particle has not actually been introduced to the system, rather there is a new rule for describing the evolution of a single particle. We refer to the variables x_1, x_2 in the different path integrals as the different *replicas*[21]. The addition of S_{cor} has correlated the previously independent paths in the path integral.

A very simple model of this, which we can use to build intuition, is to assume

$$S_{cor}[x_1, x_2] = -\int_{t_A}^{t_B} dt\, \frac{1}{2} m \Omega_{corr}^2 (x_1 - x_2)^2. \tag{7.5}$$

The equivalence principle arguments imply that the only interaction between the paths are gravitational, however we can use this simple model to see what generally happens when paths are correlated by an attractive interaction.

If the particle action $S[x]$ is such that there are two classical paths L and R, then in the double path integral there are four "paths" for the coupled system. The two replicas may: both follow path L, both follow path R, replica 1 follows L while 2 follows R, or replica 1 follows R while 2 follows L (see fig. 7.1). When both replicas follow the same path, $S_{cor} = 0$, and there is no hint of the modification. When the replicas follow different paths, $S_{cor} > 0$ and there is a penalty. Thus by correlating the worldlines we have changed the rules for evolution so that the particle is less likely to be in a superposition, the paths tend to bunch together. Since the interaction (7.4) depends on the particle's mass, one expects microscopic systems to be unaffected while macroscopic systems strongly "path-bunch" and effectively collapse onto a classical trajectory.

Now of course Newton's law is not the full theory of gravitation, so to

[21]Note that the term "replica" here is not to be confused with replicas from the "replica trick" often used in spin-glass theory.

find a bunching mechanism which satisfies correspondence with classical GR in the appropriate limit we must introduce generally covariant interactions. We could then propose a first departure from conventional quantum theory

$$K(B;A) = \int \mathcal{D}g\, e^{\frac{i}{\hbar}S_G[g]} \int_{x_A}^{x_B} \mathcal{D}x_1 \int_{x_A}^{x_B} \mathcal{D}x_2\, e^{\frac{i}{\hbar}(S[x_1|g]+S[x_2|g])}. \tag{7.6}$$

where we have introduced the Einstein-Hilbert action S_G for general relativity and suppressed tensor indices on the metric.

The integral over the metric here is expected to have the same ambiguities with the measure, conformal instability, and gauge-redundancies as in conventional quantum gravity [27]. Here we will not address these issues, rather we will assume that the notation $\int \mathcal{D}g$ includes the appropriate measure factors. Furthermore, in this thesis we will assume a flat background on which metric fluctuations are treated quantum mechanically; we do not however restrict these fluctuations to be in the linear regime.

Again, the equivalence principle arguments imply that the theory must include gravitational interactions between not just two paths, but between all paths. One is then naturally led to the notion of *levels*. At level n there are n replicas interacting via gravity. Equation (7.6) above is then level 2, and we will introduce a product over all possible levels n, for $n = 1, 2, \ldots$. The CWL propagator is then written as[22]

$$K(B;A) = \lim_{N\to\infty} \left[\prod_{n=1}^{N} \int \mathcal{D}g^{(n)}\, e^{\frac{i}{\hbar}S_G[g^{(n)}]} \prod_{j=1}^{n} \int_{x_A}^{x_B} \mathcal{D}x_j^{(n)}\, e^{\frac{i}{\hbar}S[x_j^{(n)}|g^{(n)}]} \right]. \tag{7.7}$$

In the next few chapters we will investigate many consequences of this expression. We will find various small mathematical adjustments which need to be made to ensure the theory is consistent with known physics, but the core ideas will remain unchanged.

[22] It has previously been commented that the above logic does not necessarily imply the notion of levels, merely that one should consider a single level with infinitely many replicas. We will actually demonstrate later, that the two formulations are equivalent.

Chapter 8

First order perturbation theory in CWL theory

In this chapter we work to get a sense of how CWL theory describes departures from conventional quantum gravity. First we provide a technical introduction to CWL by reviewing the definition of the theory given in [114]. We then set up a perturbative expansion of the gravitational part of the generating functional for a general matter field theory, and identify the contributions up to $\mathcal{O}(G)$. The formal results of this chapter are later used to explicitly evaluate the CWL modifications to various quantities of physical interest, such as: correlation functions, time evolution of coherent states, and particle propagators.

In chapter 10 we will derive very general results for CWL theory which subsume most of the results in this chapter. We include this chapter to i) familiarize the reader with the language and notation we will be using elsewhere, and ii) because in this thesis we are aiming to provide a chronology of the author's involvement with the CWL theory.

8.1 Generating Functional

We will start our discussion from the definition of the "product version CWL theory", given in eqs. (85-91) of [114]. We will not concern ourselves with their alternative "summed version", because only the product version was properly consistent with General Relativity in the classical limit.

We will start with a conventional QFT for our "matter" (non-gravitational field). To avoid introducing additional labels we will restrict ourselves to a single scalar field ϕ. It is not necessary for the calculations which follow to restrict our matter to being a scalar field, or even just a single field. The generalization to multiple fields, scalars and vectors, is trivial and just requires carrying around extra indices and labels. The generalization to include fermions would require introducing the tetrad formalism, but we see no reason *a priori* that one could not perform calculations which parallel those in this chapter. We'll leave the matter action $S[\phi]$ unspecified, as it will be irrelevant for the following discussions. The formal calculations below will also not rely on a specific form of the gravitational action, in principle higher curvature terms could be included without difficulty, but we will consider only Einstein gravity for now.

We first recall that in conventional QFT, on a background spacetime with metric g one has the generating functional

$$Z[J,g] = \int \mathcal{D}\phi \, e^{iS[\phi,g]+i\int J\phi}. \tag{8.1}$$

The source is assumed to vanish asymptotically, and the integrals are over all of spacetime, with a slight rotation into the imaginary time direction. This expression then generates vacuum correlation functions. To simplify the writing, in what follows we will actually complete the "Wick rotation" to Euclidean time $t \to -i\tau$. The resulting expression is

$$Z[J,g] = \int \mathcal{D}\phi \, e^{-S[\phi,g]+\int J\phi}, \tag{8.2}$$

where we now use the "Euclidean action" for ϕ, and the integration is now over a 4-dimensional Riemannian manifold. This manouevre is standard in flat spacetime QFT, and commonly done in quantum gravity as well—though it is far from trivial in quantum gravity to see whether it is always valid [342–345]. From eq. (8.2) one obtains the connected parts of the correlation functions by functional differentiation

$$\langle \phi(x_1)\dots\phi(x_l)\rangle_c = \left.\frac{\delta^l \ln Z[J,g]}{\delta J(x_1)\dots\delta J(x_l)}\right|_{J=0}. \tag{8.3}$$

Now, we can state the definition of CWL theory given in [114]. We start with the conventional QFT of our matter ϕ, and replicate it into infinitely many copies ϕ_j with $j = 1, 2, ...$ We organize these copies into *levels* $n = 1, 2, ...$ such that at level n there are n *replicas* of the scalar field $\phi_k^{(n)}$ with $k = 1, ..., n$. The generating functional for such a theory is

$$\begin{aligned} \mathcal{Z}^U[J,g] &= \prod_{n=1}^{\infty}\prod_{k=1}^{n}\int \mathcal{D}\phi_k^{(n)} \, e^{-S[g|\phi_k^{(n)}]+\int \phi_k^{(n)} \frac{J}{f(n)}} \\ &= \prod_{n=1}^{\infty}\left(Z\left[\frac{J}{f(n)}, g\right]\right)^n, \end{aligned} \tag{8.4}$$

where $Z[J]$ is just the generating functional for a single copy of the field. We want to emphasize here that these are not the replicas of the "replica trick" used often in conventional quantum theory to study disorder or compute entropies. Here the replicas are a mathematical device introduced to allow for interactions between different realizations of the matter field's evolution, ie. different paths in a path integral. Moreover, in the "replica trick" one is typically interested in the behaviour of quantities near $n = 0$ or $n = 1$, and we will eventually see that in CWL theory one is much more interested in the large n behaviour.

The superscript U in eq. (8.4) denotes that the various replicas are *uncorrelated* with each other, in contrast with later discussed *correlated* worldline theory. Note that the same external source J couples to all of the replicas. This is done because the various fields are not meant to be physically distinct objects, just tools for describing novel correlations in a single field's path-integral. The constants $f(n)$ are unspecified as of yet, but they have been introduced to regulate possible divergences arising from the infinite number of levels.

To correct for the replication of all the fields, ref. [114] provides the following prescription for obtaining connected correlation functions from eq. (8.4),

$$\langle \phi(x_1) \dots \phi(x_l) \rangle_c = \left(\sum_{n=1}^{\infty} \frac{n}{f^l(n)} \right)^{-1} \frac{\delta^l}{\delta J(x_1) \dots \delta J(x_l)} \ln Z^U[J, g] \Bigg|_{J=0}, \quad (8.5)$$

and it is easy to see that this yields the same results as conventional QFT.

For each level one then introduces gravitation between each of the replicas, however each of the levels stay uncoupled. The result is the CWL generating functional

$$\begin{aligned} \mathcal{Z}[J] &= \prod_{n=1}^{\infty} \int \mathcal{D}g^{(n)} \, e^{-nS_G[g^{(n)}]} \prod_{k=1}^{n} \int \mathcal{D}\phi_k^{(n)} \, e^{-S[\phi_k^{(n)}, g^{(n)}] + \int \phi_k^{(n)} \frac{J}{f(n)}} \\ &\equiv \prod_{n=1}^{\infty} \mathcal{Z}_n[J] \end{aligned} \quad (8.6)$$

where $S_G[g]$ is the Einstein-Hilbert action[23],

$$S_G[g] = \frac{1}{16\pi G} \int d^4x \, \sqrt{g} R, \quad (8.7)$$

[23]Technically speaking we should also include the Gibbons-Hawking-York boundary term [161, 162], however it will be irrelevant for our purposes because we are ultimately interested in correlation functions of local operators and not the actual value of the partition function.

We've also written $\int \mathcal{D}g$, which schematically represents the functional integral over spacetime metrics. We will provide more meaning to this notation shortly.

Using the prescription (8.5) we can ensure that we obtain conventional QFT results as $G \to 0$, ie. when gravitational effects are irrelevant. This correspondence limit with QFT is the first of many checks which we will perform on CWL theory.

In eq. (8.6) a factor of n has been inserted in front of S_G. This is necessary so that we obtain the correct classical equations of motion. For example, at level n we have

$$\begin{aligned} 0 &= -\int \mathcal{D}g^{(n)} \prod_{k=1}^{n} \int \mathcal{D}\phi_k^{(n)} \frac{\delta}{\delta g^{(n)}} e^{-nS_G[g^{(n)}] - \sum_{k=1}^{n} S[\phi k^n, g^{(n)}] + \sum_{k=1}^{n} \int \phi_k^{(n)} \frac{J}{f(n)}} \\ &= \left\langle n \frac{\delta S_G}{\delta g^{(n)}} + \sum_{k=1}^{n} \frac{\delta S[\phi_k^{(n)}, g^{(n)}]}{\delta g^{(n)}} \right\rangle. \end{aligned} \tag{8.8}$$

This then implies the Einstein equation

$$\left\langle n G_{\mu\nu}(g^{(n)}) - 8\pi G \sum_{k=1}^{n} T_{\mu\nu}(\phi_k^{(n)}, g^{(n)}) \right\rangle = 0. \tag{8.9}$$

Since each of the replicas has identical boundary data and action, they will all yield the same stress tensor. The sum of stress tensors is then n times the result for a single scalar field. We can then see that the factor of n placed in front of the S_G was necessary to obtain the correct classical Einstein equation.

Note that this also implies that at level n we have the effective gravitational constant $G_n = G/n$, or equivalently, the effective Planck mass $M_{P\,n}^2 = nM_P^2$. This suppresses aspects of the "quantum" part of the quantum gravity theory at level n. In chapter 10 we will study the consequences of this in great detail. For now we observe that at each level n we just have conventional quantum gravity for n identical matter fields, and a compensating rescaling of the gravitational constant.

8.2 Perturbation theory in the gravitational constant: first order

Let us now set-up a perturbation series expansion to evaluate eq. (8.6) up to order G. One could certainly do this using Feynman diagrams; however we found it more convenient to write the terms explicitly, albeit in a very compact notation. We will later use diagrams to give intuition into the various terms in the series.

8.2.1 Notation

Before setting up the series expansion we will introduce the highly compact notation first introduced by DeWitt [40, 41]. This notation will allow us to proceed in a very general manner, without using explicit forms for the matter action nor the gravitational action. Furthermore, because of the proliferation of terms and indices in gravitational perturbation theory, this notation is almost essential to keep equations readable.

The first aspect of the DeWitt notation is to group together spacetime coordinates, discrete labels, and spacetime indices all together into a single label. For example, a the group of scalar fields at level n which are usually indexed by the distinct label $k = 1, .., n$ is now written as

$$\phi_k^{(n)}(x) \to \phi_i, \tag{8.10}$$

where i represents the tuple $(n, k, x^0, x^1, x^2, x^3)$. Another example is the metric, which can be condensed as

$$g_{\mu\nu}(x) \to g_a. \tag{8.11}$$

The second aspect of the DeWitt notation is to extend the Einstein summation convention to include integration over the continuous variables, eg.

$$T^a g_a = \int d^4x \, T^{\mu\nu}(x) g_{\mu\nu}(x). \tag{8.12}$$

This notation runs the risk of leaving the reader confused about what a certain index represents in large equations. We will attempt to be very consistent with the indexing. Latin letters from the start of the alphabet will be reserved for metric indices, latin letters from the middle of the alphabet will be reserved for matter field indices, and Greek letters will be reserved for indexing gauge (diffeomorphism) transformations.

8.2.2 Gauge fixing the diffeomorphism invariance

From eq. (8.6) it is clear that each level is independent. We will then discuss the perturbation theory only for a specific level, n, and combine the results only at the end. Since at each level we have conventional quantum gravity (up to n dependent factors), until we combine the results from various levels we are doing conventional quantum gravity calculations. The schematic generating functional at level n is

$$\mathcal{Z}_n[J] = \int \mathcal{D}g\mathcal{D}\phi_i \, e^{-S_G[g]-S[\phi,g]+J^i\phi_i}, \tag{8.13}$$

where the matter action is given by the sum of actions for each of the replicas,

$$S[\phi, g] = \sum_{i=1}^{n} S[\phi_i, g]. \tag{8.14}$$

Throughout this thesis we will omit the superscript (n) on the metric when we are considering only one level and there is no risk for confusion.

The gravitational path integral above has all of the typical issues: eg. measure ambiguities, unbounded conformal modes, and Gribov ambiguities (see eg. [27] for review). We will avoid these issues by using the path-integral to simply defines a perturbative series for an effective quantum field theory describing metric fluctuations about a classical background spacetime.

Even while working with metric perturbations, we need to deal with diffeomorphism invariance. When $J = 0$ the action is invariant under diffeomorphisms, ie. transformations of the field variables corresponding to general coordinate transformations. We treat this using the Faddeev-Popov (FP) trick to factor out redundant parts of the path-integral. For this we will need to consider only infinitesimal diffeomorphisms, $x^\mu \to x^\mu + \xi^\mu(x)$. Under this action the fields transform by a Lie derivative

$$\begin{aligned} \phi_i &\to \phi_i + \mathcal{L}_\xi \phi_i \\ g_a &\to g_a + \mathcal{L}_\xi g_a, \end{aligned} \tag{8.15}$$

and we write this using the generators R_μ,

$$\begin{aligned} \mathcal{L}_\xi \phi^i &= \xi^\mu R^i_\mu(\phi) \\ \mathcal{L}_\xi g^a &= \xi^\mu R^a_\mu(g). \end{aligned} \tag{8.16}$$

Using the metric as an example, we can unpack the notation and write the Lie derivative explicitly

$$\mathcal{L}_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu. \tag{8.17}$$

We can introduce an integral and a delta function, and integrate by parts to rewrite this as

$$\mathcal{L}_\xi g_{\mu\nu} = \int d^4y \, \xi^\alpha(y) R_{\alpha\,\mu\nu}(x, y|g), \tag{8.18}$$

with the generator written in the unpacked notation

$$R_{\alpha\,\mu\nu}(x,y|g) = \left(g_{\alpha\nu}(y)\frac{\partial}{\partial x^\mu} + g_{\alpha\mu}(y)\frac{\partial}{\partial x^\nu} \right) \delta^{(4)}(y-x). \tag{8.19}$$

To perform the FP gauge fixing, we introduce the gauge fixing function $\xi^\mu(g)$. As one often does in gauge theory, we assume this function to be linear in the quantum part of the metric, ie.

$$\chi^\mu(g) = \chi^\mu_a g^a - \chi^\mu_a g^a_0, \tag{8.20}$$

where we define g^a_0 as the solution to the classical vacuum Einstein equation.[24] Fixing this gauge requires introducing the FP determinant

$$\det|Q^\mu_\nu[g]| \equiv \det\left|\frac{\delta\chi^\mu[g^\xi]}{\delta\xi^\nu}\right|_{\xi=0}. \tag{8.21}$$

Our choice (8.20) simplifies our calculations, as we get the compact expression

$$Q^\mu_\nu = R^a_\nu \chi^\mu_a. \tag{8.22}$$

Finally, rather than inserting a strict gauge fixing factor $\delta(\chi^\mu)$ into our path-integral we can use the standard trick of smearing this delta function with a gaussian functional. In general this smearing functional has an invertible quadratic form $c_{\mu\nu}$.

The end result of all this is a more precisely defined generating functional for CWL at level n,

$$\mathcal{Z}_n[J] = \int \mathcal{D}g\mathcal{D}\phi_i \,\det|Q|\, e^{-\frac{n}{\ell_P^2}I[g]-S[\phi,g]+J^i\phi_i}, \tag{8.23}$$

where $\ell_P = (16\pi G)^{1/2}$, and we've defined

$$I[g] = \ell_P^2 S_G[g] + \frac{1}{2}\chi^\mu[g]c_{\mu\nu}\chi^\nu[g]. \tag{8.24}$$

For some applications one may choose to represent the FP determinant using fermionic "ghost" fields

$$\mathcal{Z}_n[J] = \int \mathcal{D}g\mathcal{D}\phi_i\mathcal{D}\bar{\omega}_\mu\mathcal{D}\omega^\nu \; e^{-\frac{n}{\ell_P^2}I[g]-\bar{\omega}_\mu Q^\mu_\nu[g]\omega^\nu - S[\phi,g]+J^i\phi_i}, \tag{8.25}$$

[24] At higher orders in perturbation theory we would actually need to solve the saddle point equation for the quantum effective action to self-consistently determine g_0.

but we'll find it more convenient to "integrate out" the ghosts and instead use the expression

$$\mathcal{Z}_n[J] = \int \mathcal{D}g\mathcal{D}\phi_i \, e^{-\frac{n}{\ell_P^2}I[g]+\mathrm{Tr}\ln Q[g]-S[\phi,g]+J^i\phi_i}. \tag{8.26}$$

This expression will be our starting point for perturbation theory.

8.2.3 Perturbative Expansion

We will now consider expanding the metric about a classical background g_0 which is a solution to the vacuum Einstein equation.[25] We will then expand the action in terms of the metric perturbation $h^a = g^a - g_0^a$.[26] We will refer to the n^{th} term in a Taylor series as

$$f_{(n)} = \frac{1}{n!}\frac{\delta^n f[g]}{\delta g_{a_1}\cdots\delta g_{a_n}}\bigg|_{g=g_0} h^{a_1}\cdots h^{a_n}, \tag{8.27}$$

The classical solution g_0 is defined by requiring

$$I_{(1)} = 0. \tag{8.28}$$

With all of the notation established we can finally start writing out the expansion. The lowest non-trivial order involves expanding the action to $\mathcal{O}(\ell_P^2)$,

$$\begin{aligned}\mathcal{Z}_n = \int \mathcal{D}\phi\mathcal{D}h \, \exp&\left[-n\frac{I_{(0)}}{\ell_P^2} + \mathrm{Tr}\ln Q_{(0)} - S[\phi,g_0] + J^i\phi_i - I_{(2)}\right] \\ \times \exp&\Bigg[-n^{-1/2}\ell_P I_{(3)} - n^{-1}\ell_P^2 I_{(4)} \\ &\quad + n^{-1/2}\ell_P(\mathrm{Tr}\ln Q)_{(1)} - n^{-1}\ell_P^2(\mathrm{Tr}\ln Q)_{(2)} \\ &\quad - n^{-1/2}\ell_P S_{(1)} - n^{-1}\ell_P^2 S_{(2)}.\Bigg]\end{aligned} \tag{8.29}$$

The notation we're using is sufficiently compact that one could straightforwardly go beyond lowest order, but we will reserve that discussion for chapter 10 .

[25] Although we did not write it explicitly, there is nothing here preventing us from including a cosmological constant, so that the vacuum may be AdS spacetime for example. Additionally, we could have included external classical matter sources to generate other non-trivial spacetimes, and the formal results to follow would be unchanged.

[26] To simplify the graviton propagator we will actually define the metric perturbation as $h^a = n^{-1/2}\ell_P(g^a - g_0^a)$.

The term $I_{(2)}$ is an $\mathcal{O}(\ell_P^0)$ quadratic form, and thus defines the graviton wave equation on the background g_0. Now we expand the exponential as a Taylor series in powers of ℓ_P. We also use the fact that the $\mathcal{O}(\ell_P^0)$ action is even in h to immediately drop terms in the expansion which are odd in h. The result is

$$\begin{aligned}
\mathcal{Z}_n[J] &= \exp\left[-n\frac{I_{(0)}}{\ell_P^2} + \mathrm{Tr}\ln Q_{(0)} - \frac{1}{2}\mathrm{Tr}\ln I_{ab}\right] \\
&\times \int \mathcal{D}\phi_i\, e^{-S[\phi,g_0]+J^i\phi_i}\left[1 + \frac{1}{n}\mathcal{E}[\phi, g_0]\right. \\
&\left. + \frac{\ell_P^2}{n}\left\langle \frac{1}{2}I_{(3)}^2 - I_{(4)} + (\mathrm{Tr}\ln Q)_{(2)} + \frac{1}{2}(\mathrm{Tr}\ln Q)_{(1)}^2 + I_{(3)}(\mathrm{Tr}\ln Q)_{(1)}\right\rangle_h + \mathcal{O}(\ell_P^4)\right],
\end{aligned} \tag{8.30}$$

where we've collected all of the terms involving the matter together as

$$\mathcal{E}[\phi, g_0] = \ell_P^2\left\langle \frac{1}{2}S_{(1)}^2 - S_{(2)} + I_{(3)}S_{(1)} + (\mathrm{Tr}\ln Q)_{(1)}S_{(1)}\right\rangle_h, \tag{8.31}$$

and where the angled brackets denote the expectation value in the graviton vacuum

$$\langle \mathcal{O}\rangle_h = e^{\frac{1}{2}\mathrm{Tr}\ln I_{ab}} \int \mathcal{D}h\, e^{-\frac{1}{2}I_{ab}h^a h^b}\,\mathcal{O}, \tag{8.32}$$

If we define the graviton Green's function via $I_{ab}D^{bc} = \delta_a^c$, then Wick's theorem ensures that

$$\langle h^a h^b\rangle_h = D^{ab} \tag{8.33}$$

and

$$\langle h^a h^b h^c h^d\rangle_h = D^{ab}D^{cd} + D^{ac}D^{bd} + D^{ad}D^{bc}. \tag{8.34}$$

We can use this to evaluate the various expectation values in eq. (8.31). For the functional derivatives of the terms in the action we will use the notation

$$\begin{aligned}
I_{a_1\ldots a_n} &\equiv \left.\frac{\delta^n I[g]}{\delta g_{a_1}\cdots\delta g_{a_n}}\right|_{g=g_0} \\
S_{a_1\ldots a_n} &\equiv \left.\frac{\delta^n S[\phi, g]}{\delta g_{a_1}\cdots\delta g_{a_n}}\right|_{g=g_0}.
\end{aligned} \tag{8.35}$$

The resulting expression is then

$$\mathcal{E}[\phi, g_0] = \ell_P^2\left[\frac{D^{ab}S_aS_b}{2} - \frac{D^{ab}S_{ab}}{2} - S_aD^{ab}\left(\frac{\delta R_\mu^c}{\delta g^b}\right)\chi_c^\mu(Q^{-1})_\mu^\nu + \frac{1}{2}S_aD^{ab}I_{bcd}D^{cd}\right]. \tag{8.36}$$

The terms in the last line of eq. (8.30) involve only gravitons and ghosts, and will not contribute to matter correlation functions.

It remains only to make sense of $\langle\langle \mathcal{E}[\phi, g_0]\rangle\rangle$, where we use the double angled brackets to indicate that this is an expectation value in the replicated field theory,

$$\langle\langle \mathcal{E}[\phi, g_0]\rangle\rangle_J = \frac{\int \mathcal{D}\phi_i \, e^{-S[\phi,g_0]+J^i\phi_i} \, \mathcal{E}[\phi, g_0]}{\int \mathcal{D}\phi_i \, e^{-S[\phi,g_0]+J^i\phi_i}}. \tag{8.37}$$

Recalling that the action is a sum over the actions for each of the replica fields, we can write $\langle\langle \mathcal{E}[\phi, g_0]\rangle\rangle_J$ explicitly as

$$\begin{aligned}\langle\langle \mathcal{E}[\phi, g_0]\rangle\rangle_J =& \ell_P^2 \sum_{j=1}^{n} \Bigg\langle\!\!\Bigg\langle \sum_{k=1}^{n} \frac{D^{ab} S_a[\phi_j] S_b[\phi_k]}{2} - \frac{D^{ab} S_{ab}[\phi_j]}{2} \\ & - S_a[\phi_j] D^{ab} \left(\frac{\delta R^c_\mu}{\delta g^b}\right) \chi^\mu_c (Q^{-1})^\nu_\mu + \frac{1}{2} S_a[\phi_j] D^{ab} I_{bcd} D^{cd} \Bigg\rangle\!\!\Bigg\rangle_J .\end{aligned} \tag{8.38}$$

We now see clearly that in the first term there are “diagonal” and “off-diagonal” contributions , $j = k$ and $j \neq k$ respectively. The diagonal contributions are just those of conventional quantum gravity for a single scalar field, but the off-diagonal terms are new CWL specific terms describing gravitation between replica fields. We now write this as

$$\langle\langle \mathcal{E}[\phi, g_0]\rangle\rangle_J = -nW_{CQG}[J, g_0] - n(n-1)W_{CWL}[J, g_0], \tag{8.39}$$

where

$$\begin{aligned}W_{CQG}[J, g_0] = -\ell_P^2 \Bigg[& \frac{1}{2} D^{ab} \langle S_a S_b\rangle_J - \frac{1}{2} D^{ab} \langle S_{ab}\rangle_J \\ & - \langle S_a\rangle_J D^{ab} \left(\frac{\delta R^c_\mu}{\delta g^b}\right) \chi^\mu_c (Q^{-1})^\nu_\mu + \frac{1}{2} \langle S_a\rangle_J D^{ab} I_{bcd} D^{cd} \Bigg],\end{aligned} \tag{8.40}$$

and where

$$W_{CWL}[J, g_0] = -\ell_P^2 \frac{1}{2} D^{ab} \langle S_a\rangle_J \langle S_b\rangle_J, \tag{8.41}$$

and angled brackets refer to a conventional QFT expectation value for a single scalar field.

Let us now pass to the generating functional for connected correlation functions, $\mathcal{W}_n = -\ln \mathcal{Z}_n$. Because of its lengthy name we will simply refer to $\mathcal{W}_n$ as the free energy. To compute this we evaluate the logarithm of eq. (8.30) to $\mathcal{O}(\ell_P^2)$. The result is

$$\mathcal{W}_n[J] = \frac{n}{\ell_P^2} W^G_{tree} + W^G_{1-loop} + \frac{\ell_P^2}{n} W^G_{2-loop} \\ + nW_0[J,g_0] + W_{CQG}[J,g_0] + (n-1)W_{CWL}[J,g_0]. \tag{8.42}$$

The pure gravitational terms here are

$$\begin{aligned} W^G_{tree} &= I_{(0)}, \\ W^G_{1-loop} &= \frac{1}{2}\mathrm{Tr}\ln I_{ab} - \mathrm{Tr}\ln Q_{(0)}, \\ W^G_{2-loop} &= -\left\langle \frac{1}{2} I^2_{(3)} - I_{(4)} + (\mathrm{Tr}\ln Q)_{(2)} + \frac{1}{2}(\mathrm{Tr}\ln Q)^2_{(1)} + I_{(3)}(\mathrm{Tr}\ln Q)_{(1)} \right\rangle_h, \end{aligned} \tag{8.43}$$

the conventional matter free energy for a single scalar field on a fixed background is

$$W_0[J,g_0] = -\ln \int \mathcal{D}\phi\, e^{-S[\phi,g_0]+J^i\phi_i}, \tag{8.44}$$

and the matter-gravity terms are defined in eqs. (8.40) and (8.41).

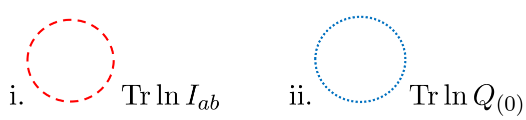


Figure 8.1: Feynman diagrams contributing to W^G_{1-loop}. Dashed red lines represent graviton propagators D^{ab}, and dotted blue lines represent ghost propagators $(Q^{-1})_\mu^\nu$.

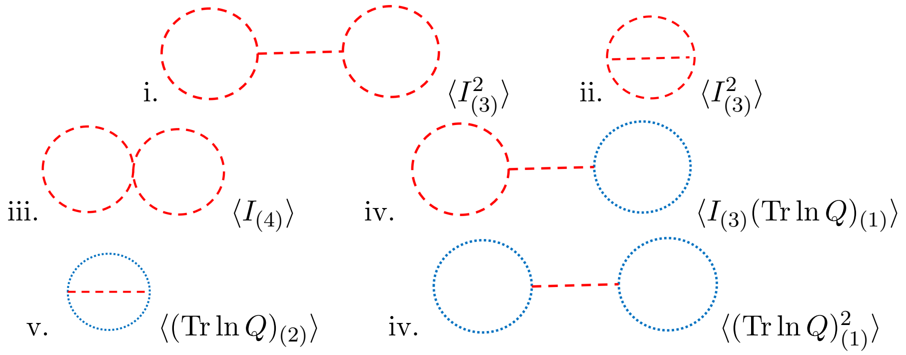


Figure 8.2: Feynman diagrams contributing to W^{G}_{2-loop}.

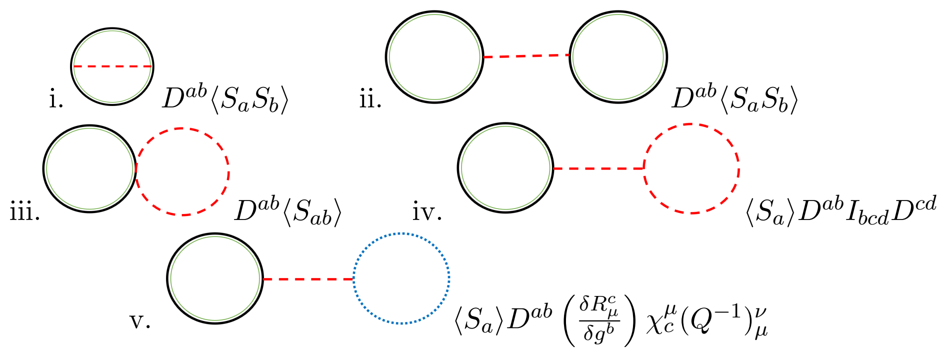


Figure 8.3: Feynman diagrams contributing to W_{CQG}. The solid black lines represent matter propagators. The inner color accent labels the replicas, and is useful for latter figures where replicas need be to distinguished.

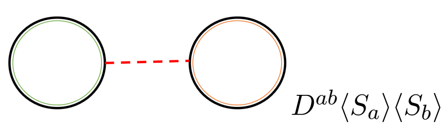


Figure 8.4: A Feynman diagram contributing to W_{CWL}. The inner color accent distinguishes the replicas.

Observe that eqs. (8.40), (8.43) and (8.44) are all conventional quantum gravity results, only eq. (8.41) is novel. Indeed, we can perform a sanity check and set that if we set $n = 1$ in eq. (8.42) we eliminate the CWL term and recover conventional quantum gravity at $\mathcal{O}(\ell_P^2)$.

Equation (8.42) is the primary formal result of this chapter. We will soon use this result, and the CWL "path-bunching" term (8.41) in particular, to study specific examples.

8.2.4 Gauge independence

The action we started with in eq. (8.13), is diffeomorphism invariant when $J = 0$. We used the Faddeev-Popov procedure to fix the gauge in the level-n generating functional (8.26), but because of the underlying diffeomorphism we have that $\mathcal{Z}_n[J = 0]$ is independent of the gauge fixing function χ_a^μ. One would like to verify however that this is indeed true order by order in the ℓ_P^2 expansion, to ensure that the result (8.42) can be trusted.

It turns out that one can explicitly demonstrate this at at $\mathcal{O}(\ell_P)^2$. We will not reproduce the proof here, we instead refer to our published work [126]. The computation is tedious, but essentially reduces to repeated applications of the fundamental Ward identities

$$\begin{aligned} &\langle R_\mu^a \frac{\delta S_G}{\delta g_a} \rangle = 0, \\ &\langle R_\mu^a \frac{\delta S}{\delta g^a} + R_\mu^i \frac{\delta S}{\delta \phi_i} \rangle = 0, \end{aligned} \tag{8.45}$$

and the descendant Ward identities obtained by functionally differentiating eq. (8.45) with respect to g_b, ϕ_i and J.

The main result we found is that the conventional quantum gravity term W_{CQG} is gauge independent, and separately the CWL term W_{CWL} is also gauge independent.

8.3 Correlation Functions

We can now proceed to set up the computation of connected correlation functions in the ℓ_P^2 approximation. At this stage, we just evaluate functional derivatives with respect to J.

Since the CWL generating functional is a product over levels, the CWL free energy is simply the sum over levels

$$\mathcal{W}[J] = \sum_{n=1}^{\infty} \mathcal{W}_n[J/f(n)]. \tag{8.46}$$

Using the prescription (8.5) for computing connected correlation functions we have

$$\langle\phi(x_1)\ldots\phi(x_l)\rangle_c = -\left(\sum_{m=1}^{\infty}\frac{m}{f^l(m)}\right)^{-1}\sum_{n=1}^{\infty}\frac{1}{f^l(n)}\frac{\delta^l}{\delta J(x_1)\ldots\delta J(x_l)}\mathcal{W}_n[J]\bigg|_{J=0}, \tag{8.47}$$

and if we insert the result (8.42) we arrive at the expression

$$\begin{aligned}\langle\phi(x_1)\ldots\phi(x_l)\rangle_c = &-\frac{\delta^l}{\delta J(x_1)\ldots\delta J(x_l)}\big(W_0[J,g_0]+W_{CWL}[J,g_0]\big)\bigg|_{J=0}\\ &-\mathcal{C}_l\frac{\delta^l}{\delta J(x_1)\ldots\delta J(x_l)}\big(W_{CQG}[J,g_0]-W_{CWL}[J,g_0]\big)\bigg|_{J=0},\end{aligned} \tag{8.48}$$

where the coefficient is

$$\mathcal{C}_l = \lim_{N\to\infty}\frac{\left(\sum_{n=1}^{N}\frac{1}{f^l(n)}\right)}{\left(\sum_{m=1}^{N}\frac{m}{f^l(m)}\right)} < 1. \tag{8.49}$$

Before we insert our results for the perturbative free energy functionals, we first note the simple rule for functionally differentiating expectation values,

$$\begin{aligned}\frac{\delta}{\delta J}\langle\mathcal{O}\rangle_J &= \frac{\delta}{\delta J}\frac{\int\mathcal{D}\phi\, e^{-S+\int J\phi}\mathcal{O}}{\int\mathcal{D}\phi\, e^{-S+\int J\phi}}\\ &= \langle\mathcal{O}\phi\rangle_J - \langle\mathcal{O}\rangle_J\langle\phi\rangle_J\end{aligned} \tag{8.50}$$

Scalar field

Let us start with a single scalar field in Minkowski spacetime. We'll assume it has $\phi\to-\phi$ symmetry, and that the vacuum energy density has been renormalized to zero. These two assumptions are actually sufficient to eliminate the CWL contribution to the two-point function. The result comes purely from conventional quantum gravity

$$\begin{aligned}\mathcal{G}(x_1,x_2) =&\langle\phi_1\phi_2\rangle - \mathcal{C}_2\ell_P^2\bigg[\frac{1}{2}D^{ab}\langle S_aS_b\phi_1\phi_2\rangle_c - \frac{1}{2}D^{ab}\langle S_{ab}\phi_1\phi_2\rangle_c\\ &-\langle S_a\phi_1\phi_2\rangle_c D^{ab}\left(\frac{\delta R^c_\mu}{\delta g^b}\right)\chi^\mu_c(Q^{-1})^\nu_\mu + \frac{1}{2}\langle S_a\phi_1\phi_2\rangle_c D^{ab}I_{bcd}D^{cd}\bigg],\end{aligned} \tag{8.51}$$

where again, the subscript c denotes the connected part, eg.

$$\langle S_{ab}\phi_1\phi_2\rangle_c = \langle S_{ab}\phi_1\phi_2\rangle - \langle S_{ab}\rangle\langle\phi_1\phi_2\rangle. \tag{8.52}$$

This result is identical to the conventional quantum gravity result, except for the coefficient $\mathcal{C}_2$ which serves to weaken the gravitational coupling.

We can also work out the CWL theory prediction for the four-point function. The conventional quantum gravity results are lengthy so we will not write them explicitly, but we note that they are ultimately suppressed by a factor of $\mathcal{C}_4$. The CWL specific term is however straightforward to compute, and we find the total result

$$\begin{aligned}&\mathcal{G}(x_1,x_2,x_3,x_4) = \langle\phi_1\phi_2\rangle\\ &+ \ell_P^2 G^{ab}\left[\langle S_a\phi_1\phi_2\rangle\langle S_a\phi_3\phi_4\rangle + \langle S_a\phi_1\phi_3\rangle\langle S_a\phi_2\phi_4\rangle + \langle S_a\phi_1\phi_4\rangle\langle S_a\phi_2\phi_3\rangle\right]\\ &+ \ell_P^2\mathcal{C}_4\big[\text{CQG ghost and graviton loop diagrams}\big]. \end{aligned} \tag{8.53}$$

Upon quick inspection one actually recognizes the three terms in the CWL contribution as the s, t, and u channels in $2 \to 2$ scattering. We then see that we get exactly the standard tree-level gravitational scattering term, while all of the loop diagrams are suppressed by a factor of $\mathcal{C}_4$.

Without a specification of the function $f(n)$ we cannot compute the $\mathcal{C}_l$, and thus we cannot work out precisely how strongly the loop diagrams are suppressed. In the following chapter we will work to understand the structure of CWL theory better, and ultimately we will use a correspondence limit with conventional QFT to fix the $f(n)$.

Non-relativistic systems

The idea for CWL theory was that gravitation would suppress superpositions of matter states by a "path-bunching" mechanism in the path integral. We then expect to see some signatures resembling this intuition when we look at the correlation functions predicted by the theory. So far, for the scalar field we have seen no such signature. There seems to be something peculiar going on, with a suppression of gravitational loop diagrams, but nothing yet describing the prevention of matter superpositions. To investigate this further, we will proceed in later chapters to simplify the discussion from a relativistic field theory to the non-relativistic quantum mechanics of particles. In that case we will better see the novelty of CWL theory.

For now, let us lay groundwork for a future calculation, and write the expression for a two-point function for a system which does not have a vanishing ground state stress-energy density (eg. a non-relativistic particle).

In this case, using ϕ as a placeholder degree-of-freedom we get the resulting two-point function

$$\begin{aligned}\mathcal{G}(x_1, x_2) = \langle\phi_1\phi_2\rangle \\ &+ \ell_P^2 G^{ab}\Big[\langle S_a\phi_1\rangle\langle S_b\phi_2\rangle + \langle S_a\rangle\langle S_b\phi_1\phi_2\rangle - \langle S_a\rangle\langle S_b\rangle\langle\phi_1\phi_2\rangle\Big] \\ &- \mathcal{C}_2\Big[\text{CQG ghost and graviton loops}\Big],\end{aligned} \tag{8.54}$$

. We will use this result in a later chapter.

8.4 Conclusions

In this chapter we started by reviewing the technical definition of CWL theory given in [114]. This included introducing the useful notion of matter *replicas* and *levels*, not to be confused with the replicas from the "replica trick" in conventional quantum theory. We then discussed the form of the CWL generating functional, and demonstrated that it describes a theory with the correct semi-classical Einstein equation. The prescription for computing correlation functions differs from conventional QFT, but we discussed how one gets conventional QFT results when gravity is negligible.

We then performed a systematic expansion of the CWL theory generating functional for connected correlation functions (free energy) to up to leading order, $\mathcal{O}(\ell_P^2)$, in an expansion in the gravitational coupling. The discussion was sufficiently general that it did not rely on the details of the matter system. Rather, we arrived at expressions which require us only to input the matter action and evaluate the resulting Feynman integrals. In a sense, what we've done is simply arrange the Feynman diagrammatic expansion, except we've done so symbolically rather than with diagrams.

We used the results to set-up calculations for the two and four-point functions in scalar field theory. We noted that the tree-level conventional gravity contributions were unchanged, and the only novel feature in CWL was that ghost and graviton loop diagrams were suppressed by factors $\mathcal{C}_{2,4}$ which remain unspecified.

In the following chapters we will work to better understand the structure of CWL theory in various ways including: looking at its Hilbert space formulation, discussing observables, embedding conventional QM within CWL, etc. The discussions there will allow us to determine the $\mathcal{C}_l$ for all l.

Chapter 9

CWL Hilbert space, states, and observables

In the previous chapter we reviewed the definition of CWL theory, as given in ref. [114], in terms of a generating functional. The generating functional allowed us to compute CWL corrections to correlation functions, however there are many other questions one would also like to see addressed. For example, one could ask how CWL theory would modify the interference pattern seen in a hypothetical large mass matter wave interferometer. This is precisely the type of thought experiment which should reveal the "path-bunching" mechanism which is expected to exist in CWL and preclude superpositions of large masses. Given that this basic example cannot be studied by computing n-point correlation functions, we seek a deeper formulation of CWL theory—one which gives rules for analyzing observables, computing probabilities etc.

To address these much more down to earth questions, we aim in this chapter to provide a formulation of CWL theory in terms of vectors in Hilbert space. We note, though, that much of the discussion in this chapter is tentative—we will raise a number of points which require further thought. The discussion in this chapter will allow us to understand CWL theory as an example of a particular type of non-linear quantum mechanics theory. In doing this reformulation we'll find that it takes some thought to determine how to properly ask and answer standard quantum mechanics questions within CWL theory.

We note here that this entire chapter goes against not only the spirit of the original formulation of CWL theory by P.C.E. Stamp [113], but also some of its motivating arguments. Indeed, Stamp argued that paths are primary and states/measurements are supposed to give way to a purely dynamical description of nature in terms of interactions and path-bunching. As mentioned previously, we believe that such a description may be in the future for CWL theory but it will require much more work. Here we take an entirely different perspective on the theory, where we'll actually try to retain as much of the conventional QM framework as possible. Despite going against the original intentions for the theory, we find that this approach has its benefits. In particular it allows us to use familiar language and intuitions from QM which ultimately guides us on how to embed QM questions into CWL.

9.1 States, Operators, and Correspondence with Conventional Quantum Theory

In CWL we have a great multiplicity of the fields (g, ϕ) which we use to describe nature. At each *level* n, we have a single metric g_n and n *replicas* $\phi_i^{(n)}$ of the matter. There are an infinite number of levels indexed by positive integers $n = 1, 2, ...\infty$. When performing calculations we truncate the number of levels at an arbitrary large integer N and then take the limit $N \to \infty$, assuming it exists. With this great multiplicity of degrees of freedom, we have a Hilbert space which is infinitely larger than that in QM[27]

$$\mathcal{H}_{CWL} = \bigotimes_{n=1}^{\infty} (\mathcal{H}_g)_n \bigotimes_{k=1}^{n} (\mathcal{H}_\phi)_k^{(n)}. \tag{9.1}$$

In general, a state in this enormous Hilbert space may have every matter replica in a different state and every metric in a different state

$$|\psi\rangle_{CWL} = |g_1\rangle_1 \otimes |\phi_1^{(1)}\rangle_1^{(1)} \otimes |g_2\rangle_2 \otimes |\phi_1^{(2)}\rangle_1^{(2)} \otimes |\phi_2^{(2)}\rangle_2^{(2)} \otimes \cdots . \tag{9.2}$$

The great enlargement of the Hilbert space presents a challenge in the proper formulation of CWL theory. Since the CWL Hilbert space is infinitely larger than the QM Hilbert space, there is clearly no one-to-one correspondence between states in $\mathcal{H}_g \otimes \mathcal{H}_\phi$ to states in $\mathcal{H}_{CWL}$. Suppose we have a state $|\psi\rangle = |g\rangle \otimes |\phi\rangle$ in QM and want to study the CWL corrections to its time evolution, clearly we first need a prescription telling us which state in $\mathcal{H}_{CWL}$ corresponds to $|\psi\rangle$. A naive guess would say that we should fix all replicas at all levels to be in the same state,

$$|\psi\rangle \to |\psi\rangle_{CWL} = |g\rangle_1 \otimes |\phi\rangle_1^{(1)} \otimes |g\rangle_2 \otimes |\phi\rangle_1^{(2)} \otimes |\phi\rangle_2^{(2)} \otimes \cdots , \tag{9.3}$$

[27] Here we are referring to the kinematical Hilbert space, where operators corresponding to the classical dynamical variables are defined. Einstein gravity is a constrained theory, and the subspace of the kinematical Hilbert space in which the constraints are obeyed as an eigenvalue equation is the physical Hilbert space. The physical Hilbert space does not factor into gravitation and matter Hilbert spaces (eg. Gauss' law tying static fields to particles), but the kinematical Hilbert space does indeed factor. Also, note that here we are being very casual with the term Hilbert space, as states in a QFT are not technically elements of a Hilbert space.

we will hereafter refer to such a state as a *replica equal* state. We will find that for some situations this is the appropriate choice, however not always.

For the remainder of this section, to avoid unnecessary complication, we'll assume a fixed background metric g_0 and simply turn off gravitational dynamics, (ie. set $G \to 0$). We'll continue to refer to the resulting replicated theory as the "uncorrelated worldline theory" (UWL) but of course we will always require UWL predictions to agree with conventional QM. In this case the generating functional is

$$\mathcal{Z}^U[g_0, J] = \prod_{n=1}^{\infty} \prod_{k=1}^{n} \int \mathcal{D}\phi_k^{(n)} \, e^{-S[g_0, \phi_k^{(n)}] + \int \frac{J}{f(n)} \phi_k^{(n)}} = \prod_{n=1}^{\infty} \left(Z\left[g_0, \frac{J}{f(n)}\right]\right)^n \tag{9.4}$$

and hereafter we will stop explicitly writing the g_0 argument.

The generating functional involves an integration over an infinite Euclidean time and hence computes vacuum correlation functions. Since every replica undergoes an integral over the same infinite time, a cut in the above path integral prepares the UWL vacuum state,

$$|0\rangle_{CWL} = |0\rangle_1^{(1)} \otimes |0\rangle_1^{(2)} \otimes |0\rangle_2^{(2)} \otimes \cdots , \tag{9.5}$$

in which every replica at every level is in the vacuum state. This is an example of a *replica equal* mapping from the vacuum $|0\rangle$ in QM to the vacuum $|0\rangle_{UWL}$ in UWL.

We can also ask about excited states such as $\phi(x)|0\rangle$ in QM, what is the appropriate corresponding state in UWL? There are two natural ways to build a state from the operators $\phi_k^{(n)}$ and the UWL vacuum; we can act on the vacuum with either the direct product of $\phi_k^{(n)}$ operators

$$\Phi_p = \bigotimes_{n=1}^{\infty} \bigotimes_{k=1}^{n} \phi_k^{(n)} \tag{9.6}$$

or the direct sum of $\phi_k^{(n)}$ operators

$$\Phi_s = \bigoplus_{n=1}^{\infty} c(n) \bigoplus_{k=1}^{n} \phi_k^{(n)}, \tag{9.7}$$

with the *level weights* $c(n)$ not yet determined. In the former case we'd create a *replica equal* state,

$$\Phi_p(x)|0\rangle_{UWL} = \left(\phi_1^{(1)}(x)|0\rangle_1^{(1)}\right) \otimes \left(\phi_1^{(2)}(x)|0\rangle_1^{(2)}\right) \otimes \left(\phi_2^{(2)}(x)|0\rangle_2^{(2)}\right) \otimes \cdots , \tag{9.8}$$

whereas in the latter case we'd create a *replica symmetric superposition* of excited states

$$\begin{aligned}\Phi_s(x)|0\rangle_{UWL} =& c(1)\left(\phi_1^{(1)}(x)|0\rangle_1^{(1)}\right)\otimes|0\rangle_1^{(2)}\otimes|0\rangle_2^{(2)}\otimes\cdots+\\ & c(2)\,|0\rangle_1^{(1)}\otimes\left(\phi_1^{(2)}(x)|0\rangle_1^{(2)}\right)\otimes|0\rangle_2^{(2)}\otimes\cdots+\\ & c(2)\,|0\rangle_1^{(1)}\otimes|0\rangle_1^{(2)}\otimes\left(\phi_2^{(2)}(x)|0\rangle_2^{(2)}\right)\otimes\cdots+\cdots\,,\end{aligned} \tag{9.9}$$

We see that in a replica symmetric superposition the excitation created by ϕ is "shared" over all of the replicas. The replicas are in an entangled state, in contrast with the product form of the replica equal state.

To determine which is the correct choice we use the fact that overlaps of states $\phi(x)|0\rangle$ in QM compute correlation functions. Since we already have a prescription for computing correlation functions from the generating functional (9.4) which match the correlation functions of QM, we can refer to that construction.

If we take a functional derivative with respect to J of the generating functional (9.4) we obtain

$$\begin{aligned}\frac{\delta}{\delta J(x)}\mathcal{Z}^U[J] &= \left(\prod_{n=1}^{\infty}\prod_{k=1}^{n}\int\mathcal{D}\phi_k^{(n)}\right)\frac{\delta}{\delta J(x)}\,e^{-S[\phi_k^{(n)}]+\sum_{n=1}^{\infty}\sum_{k=1}^{n}\int\frac{J}{f(n)}\phi_k^{(n)}}\\ &= \left(\prod_{n=1}^{\infty}\prod_{k=1}^{n}\int\mathcal{D}\phi_k^{(n)}\right)\left(\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{1}{f(n)}\phi_k^{(n)}(x)\right)\,e^{-S[\phi_k^{(n)}]+\sum_{n=1}^{\infty}\sum_{k=1}^{n}\int\frac{J}{f(n)}\phi_k^{(n)}}.\end{aligned} \tag{9.10}$$

Since the source couples to each replica linearly the functional derivative has brought down an operator which is the sum of replica fields. In operator language it is then clear that the source J couples to the direct sum,

$$\bigoplus_{n=1}^{\infty}\frac{1}{f(n)}\bigoplus_{k=1}^{n}\phi_k^{(n)} \tag{9.11}$$

which is just the operator Φ_s with the *level weights* $c(n) = f(n)^{-1}$. Thus when we compute correlation functions in UWL theory we are actually discussing overlaps of *replica symmetric superposition* states of the form

$$\begin{aligned}\Phi_s|0\rangle_{UWL} =& \frac{1}{f(1)}\left(\phi_1^{(1)}|0\rangle_1^{(1)}\right)\otimes|0\rangle_1^{(2)}\otimes|0\rangle_2^{(2)}\otimes\cdots+\\ & \frac{1}{f(2)}|0\rangle_1^{(1)}\otimes\left(\phi_1^{(2)}|0\rangle_1^{(2)}\right)\otimes|0\rangle_2^{(2)}\otimes\cdots+\\ & \frac{1}{f(2)}|0\rangle_1^{(1)}\otimes|0\rangle_1^{(2)}\otimes\left(\phi_2^{(2)}|0\rangle_2^{(2)}\right)\otimes\cdots+\cdots\,,\end{aligned} \tag{9.12}$$

and not *replica equal* states.

The next type of state we'd like to generalize from QM to CWL/UWL is the coherent state—are these replica equal or replica symmetric superposition states? Given that the free field vacuum is a particular coherent state centered on the classical field value $\phi = 0$ and that the CWL vacuum is a replica equal state, we might expect that more general coherent states will also be replica equal states. We will demonstrate below that this intuition is indeed correct.

To prepare a coherent state of a harmonic system (free fields or SHOs) in QM one couples the system to an external classical source and evolves it in time. A simple example of this is applying a static force to a harmonic oscillator. If we apply a static force f_0 to an oscillator with spring constant $k = m\omega^2$, the new ground state will be displaced from the origin by $\tilde{x} = f_0/k$. Thus if we consider time evolution from $t \in (-\infty, 0)$ (with a slight Euclidean rotation to prepare the ground state) under the influence of a force $f(t) = \tilde{x}_0 m\omega^2\,\theta(-t)$, we will obtain a coherent state at time $t = 0$ centered on $\tilde{x}_0$,

$$||x[f]\rangle\rangle = \mathcal{T}\left\{e^{i\int_{-\infty}^{0} dt f(t)x(t)}\right\}|0\rangle = e^{-i\tilde{x}_0\hat{p}}\,|0\rangle \equiv ||\tilde{x}_0\rangle\rangle. \tag{9.13}$$

Here and throughout this chapter we use double brackets on the kets to denote coherent states.

To understand coherent states in CWL we will then consider time evolution under the influence of an external source. Since all of the replicas in UWL theory are decoupled we can describe the time evolution of a CWL state using the direct sum of the Hamiltonians $H_i^{(n)}$ for each replica. Time evolution in the presence of a source is then

$$||\phi[J]\rangle\rangle_{UWL} = \mathcal{T}\left\{e^{i\int_{-\infty}^{0} J\Phi_s}\right\}|0\rangle_{UWL} = \mathcal{T}\left\{e^{i\sum_{n=1}^{\infty}\sum_{k=1}^{n}\int_{-\infty}^{0}\frac{J}{f(n)}\phi_k^{(n)}}\right\}|0\rangle_{UWL}. \tag{9.14}$$

Each of the replica fields commute with one another, so the state factorizes into the product of coherent states,

$$||\phi[J]\rangle\rangle_{UWL} = \bigotimes_{n=1}^{\infty}\bigotimes_{k=1}^{n} ||\phi[J_n]\rangle\rangle_k^{(n)} = ||\phi[J_1]\rangle\rangle_1^{(1)} \otimes ||\phi[J_2]\rangle\rangle_1^{(2)} \otimes ||\phi[J_2]\rangle\rangle_2^{(2)} \otimes \cdots \tag{9.15}$$

where $J_n = J/f(n)$.

Notice the effect of the *level weights* $f(n)$. These modify the coupling of each of the fields to the source, such that different levels see different source strengths. As a result, while the coherent states of each replica within a level are all the same, replicas in different levels are in different coherent states. The resulting UWL state is clearly not a *replica symmetric superposition*, and because of the function $f(n)$ it is also not a *replica equal* state. We've now seen a third type of state which naturally arises in UWL theory, a *level-weighted symmetric* state.

In principle one could imagine other types of states, but we don't have an interest at the moment in classifying all possible states in the UWL theory. Instead, we've been interested in finding the natural embedding of QM states in UWL theory. We've decided that the "natural" embedding is the one coming from a physical state preparation procedure.

9.2 Fixing the Source Coupling Function $f(n)$

As previously discussed (9.14) one can prepare a coherent state by evolving a system in the presence of a source. This time evolution can be described using a path integral. The wave functional of the coherent state $||\phi[\tilde{J}]\rangle\rangle$ is given by

$$\Psi_{\phi[\tilde{J}]}[\phi'(x)] = \langle\phi'(x)||\phi[\tilde{J}]\rangle\rangle = \int^{\phi'(x)} \mathcal{D}\phi\, e^{iS[\phi]+i\int \tilde{J}\phi}, \tag{9.16}$$

where the integral is taken over time $(-\infty, 0)$ (with a slight Euclidean rotation). Expectation values in this state are computed by considering operator insertions on a closed-time-path (CTP) or Schwinger-Keldysh contour which extends from past infinity to future infinity and wraps back to past infinity [114, 130, 131].The forward path represents the state ket and the return path represents the state bra,

$$\langle\langle\phi[\tilde{J}]||\,\mathcal{O}^{\sigma_1}\,||\phi[\tilde{J}]\rangle\rangle = \oint \mathcal{D}\phi'\, e^{iS[\phi^\sigma]+i\int \tilde{J}^\sigma\phi^\sigma}\,\mathcal{O}^{\sigma_1}. \tag{9.17}$$

Here $\sigma = \pm$ labels which path the operator is to be inserted on.

One can consider a CTP generating functional which generates correlation functions taken for the system prepared in the coherent state by introducing another (auxiliary) source J

$$Z_{\phi[\tilde{J}]}[J] = Z[\tilde{J} + J] = \oint \mathcal{D}\phi \, e^{iS[\phi] + i\int (\tilde{J}+J)\phi}, \tag{9.18}$$

where we've condensed the notation by not writing the path labels σ explicitly. By taking functional derivatives with respect to J and setting $J = 0$ (while leaving $\tilde{J} \neq 0$) we can compute correlation functions in the coherent state $||\phi[\tilde{J}]\rangle\rangle$.

To study coherent states in UWL theory we'll do the same procedure of translating the source and taking a CTP contour,

$$\begin{aligned}\mathcal{Z}_{UWL,\phi[\tilde{J}]}[J] =& \mathcal{Z}_{UWL}[\tilde{J} + J] = \prod_{n=1}^{\infty} \prod_{k=1}^{n} \oint \mathcal{D}\phi_k^{(n)} \, e^{iS[\phi_k^{(n)}] + i \frac{J+\tilde{J}}{f(n)} \phi_k^{(n)}} \\ &= \prod_{n=1}^{\infty} \left(Z \left[\frac{J + \tilde{J}}{f(n)} \right] \right)^n . \end{aligned} \tag{9.19}$$

If we now us the prescription (8.5) for computing connected correlation functions we obtain

$$\begin{aligned}\left(\sum_{n=1}^{\infty} \frac{n}{f^m(n)} \right)^{-1} & \frac{(-i)^m \delta^m}{\delta J(x_1) \ldots \delta J(x_m)} \log \mathcal{Z}_{UWL}[J + \tilde{J}] \bigg|_{J=0} = \\ &= \left(\sum_{n=1}^{\infty} \frac{n}{f^m(n)} \right)^{-1} \sum_{n=1}^{\infty} n \frac{(-i)^m \delta^m}{\delta J(x_1) \ldots \delta J(x_m)} \log Z \left[\frac{J + \tilde{J}}{f(n)} \right] \\ &= \left(\sum_{n=1}^{\infty} \frac{n}{f^m(n)} \right)^{-1} \sum_{n=1}^{\infty} \frac{n}{f^m(n)} \langle\langle \phi[\tilde{J}/f(n)] || \, \phi(x_1) \ldots \phi(x_m) \, || \phi[\tilde{J}/f(n)] \rangle\rangle \\ &\neq \langle\langle \phi[\tilde{J}] || \, \phi(x_1) \ldots \phi(x_m) \, || \phi[\tilde{J}] \rangle\rangle, \end{aligned} \tag{9.20}$$

and we do not agree with conventional QM.

Since the states themselves depend on $f(n)$ through the source coupling, the normalization procedure that works when $\tilde{J} = 0$ no longer works for general $f(n)$. The only resolution here is to set $f(n) = 1$, in which case the above expression does indeed yield the desired result

$$\left(\sum_{n=1}^{\infty} n\right)^{-1} \frac{(-i)^m \delta^m}{\delta J(x_1)...\delta J(x_m)} \log \mathcal{Z}_{UWL}[J+\tilde{J}]\Big|_{J=0} =$$
$$= \left(\sum_{n=1}^{\infty} n\right)^{-1} \left(\sum_{n=1}^{\infty} n\right) \langle\langle\phi[\tilde{J}]||\,\phi(x_1)...\phi(x_m)\,||\phi[\tilde{J}]\rangle\rangle$$
$$= \langle\langle\phi[\tilde{J}]||\,\phi(x_1)...\phi(x_m)\,||\phi[\tilde{J}]\rangle\rangle. \tag{9.21}$$

We then see that the free parameters $f(n)$ for $n = 1, 2, ..$ in CWL theory must all equal 1 if CWL theory is to reduce to conventional quantum theory in the limit $G = 0$. In hindsight this is all obvious, but it is nice to have an explicit demonstration here showing why we must indeed fix $f(n) = 1$ for all n.

One immediate consequence of setting the $f(n) = 1$ is that coherent states in CWL are proper *replica equal* states

$$||\phi[J]\rangle\rangle_{UWL} = \bigotimes_{n=1}^{\infty}\bigotimes_{k=1}^{n} ||\phi[J]\rangle\rangle_k^{(n)} = ||\phi[J]\rangle\rangle_1^{(1)} \otimes ||\phi[J]\rangle\rangle_1^{(2)} \otimes ||\phi[J]\rangle\rangle_2^{(2)} \otimes \cdots , \tag{9.22}$$

however we will shortly find that there are a number of other remarkable consequences.

Here we can get a glimpse into the consequences of setting $f(n) = 1$, but the detailed analysis will come in chapter 10. For now, let us refer back to the $\mathcal{C}_l$ coefficients from chapter 8. These coefficients (8.49) ultimately showed up suppressing conventional quantum gravity contributions to correlation functions (8.48). In particular, we saw in the scalar field two-point function (8.51) and four-point function (8.53) that $\mathcal{C}_2$ and $\mathcal{C}_4$ respectively suppressed contributions from ghost and graviton loop diagrams.

Upon setting $f(n) = 1$ we can now see quite simply that $\mathcal{C}_l = 0$ for all l. This is somewhat remarkable, since at least to lowest order in CWL perturbation theory we find that matter loops are preserved while graviton and ghost loops are completely suppressed. In eq. (8.48) we still have non-trivial CWL corrections to compute, and we will study these contributions further in upcoming chapters.

9.3 Redefining the CWL generating functional

The CWL generating functional (8.6) now has the form

$$\mathcal{Z}[J] = \prod_{n=1}^{\infty} \int \mathcal{D}g_n \, e^{-nS_G[g_n]} \prod_{k=1}^{n} \int \mathcal{D}\phi_k^{(n)} \, e^{-S[g_n, \phi_k^{(n)}] + \int J\phi_k^{(n)}}, \tag{9.23}$$

and the prescription for computing correlation functions is

$$\mathcal{G}(x_1, .., x_m) = \lim_{N\to\infty} \left(\sum_{N=1}^{\infty} n \right)^{-1} \frac{\delta^m}{\delta J(x_1)...\delta J(x_m)} \log \mathcal{Z}[J] \bigg|_{J=0}. \tag{9.24}$$

Note that this expression appears ill-defined; the pre-factor vanishes and the functional derivatives diverge because of the infinite number of replicas and levels. One can be reassured because the divergence does precisely cancel the zero; however it is clear that this expression for the correlation function is not the most convenient to work with. In this section we will reformulate CWL theory to avoid this issue.

Firstly, now that the normalization for the m-point function no longer depends on m, we can simply absorb the Green's function normalization factor into a redefinition of the generating functional. From now on, we will define the CWL generating functional as

$$\mathcal{Z}[J] = \lim_{N\to\infty} \left[\prod_{n=1}^{N} \int \mathcal{D}g^{(n)} \, e^{-nS_G[g^{(n)}]} \prod_{k=1}^{n} \int \mathcal{D}\phi_k^{(n)} \, e^{-S[\phi_k^{(n)}, g^{(n)}] + \int J\phi_k^{(n)}} \right]^{\alpha_N}, \tag{9.25}$$

with

$$\alpha_N \equiv \frac{1}{\left(\sum_{n=1}^{N} n\right)} = \frac{2}{N(N+1)}. \tag{9.26}$$

This is completely equivalent to the previous prescription, we've just absorbed a constant scaling into $\ln \mathcal{Z}$. With this definition of the CWL generating functional we compute the connected part of correlation functions by straightforwardly differentiating

$$\mathcal{G}(x_1, .., x_m) = \frac{\delta^m}{\delta J(x_1)...\delta J(x_m)} \log \mathcal{Z}[J] \bigg|_{J=0}. \tag{9.27}$$

We can see an immediate result of doing this if we return to the perturbative result (8.42) for the free energy. After rescaling we now have the expression for the generating functional at lowest order

$$\mathcal{Z}[J] = \lim_{N\to\infty} \exp\left[\alpha_N \sum_{n=1}^{N} \left(\frac{n}{\ell_P^2} W^G_{tree} + W^G_{1-loop} + \frac{\ell_P^2}{n} W^G_{2-loop} \right.\right.$$
$$\left.\left. + nW_0[J, g_0] + W_{CQG}[J, g_0] + (n-1)W_{CWL}[J, g_0] \right)\right]. \quad (9.28)$$

Evaluating the limit simplifies the expression considerably

$$\begin{aligned}\mathcal{Z}[J] &= \exp\left[\frac{1}{\ell_P^2} W^G_{tree} + W_0[J, g_0] + W_{CWL}[J, g_0]\right] \\ &= \exp\left[\frac{1}{\ell_P^2} W^G_{tree} + W_0[J, g_0] - \ell_P^2 \frac{1}{2} D^{ab} \langle S_a \rangle_J \langle S_b \rangle_J + \mathcal{O}(\ell_P^4)\right]. \quad (9.29)\end{aligned}$$

We then see explicitly that the graviton and ghost loop contributions vanish at this order. In chapter 10 we will explore this further, at higher orders in perturbation theory.

When written in this way it is obvious that we recover conventional quantum theory when gravitation is turned off. As $G \to 0$ we've seen already (8.43) that the gravitational path integral is dominated by the action of the classical vacuum solution. The value of the classical vacuum action is a constant and is typically renormalized to zero, ie. we have $I_{(0)} \equiv I[g_0] = 0$. The CWL generation functional is then,

$$\begin{aligned}\mathcal{Z}[J]\Big|_{G=0} &= \lim_{N\to\infty, G\to 0} \left[\prod_{n=1}^{N} \int \mathcal{D}g^{(n)}\, e^{-\frac{n}{16\pi G} I[g^{(n)}]} \prod_{k=1}^{n} \int \mathcal{D}\phi_k^{(n)}\, e^{-S[\phi_k^{(n)}, g^{(n)}] + \int J\phi_k^{(n)}}\right]^{\alpha_N} \\ &= \lim_{N\to\infty} \left[\prod_{n=1}^{N} \left(\int \mathcal{D}\phi\, e^{-S[\phi, g_0] + \int J\phi}\right)^n\right]^{\alpha_N} \\ &= \int \mathcal{D}\phi\, e^{-S[\phi, g_0] + \int J\phi}. \quad (9.30)\end{aligned}$$

We will see in upcoming sections, and also in chapter 10, that writing the generating functional in this form leads to a much simpler understanding of the nature of this theory.

9.4 Defining the propagator

To progress with our understanding of CWL theory, we'd like to understand its predictions for the dynamics of various systems. The place to start, given the theory's foundational formulation in terms of path integrals, is with is with the propagator, ie. transition amplitude between defined configurations. First, we must determine what the form of transition amplitudes in CWL theory should be. To do this we must take certain postulates. We will take the expression given for the generating functional eq. (9.25) as a foundational equation in CWL, and its form will determine for us the rules for computing various other quantities.

The generating functional (as defined as a path integral over all of spacetime with a Euclidean rotation) is by definition, a vacuum-to-vacuum transition amplitude in the presence of an external source. If we freeze gravity for a moment, and we turn off the external source, we know that the generating functional has the form in terms of conventional QM states

$$\mathcal{Z}[0] = \lim_{N\to\infty} \left[\prod_{n=1}^{N} \prod_{k=1}^{n} \langle 0|_k^{(n)} \, |0\rangle_k^{(n)} \right]^{\alpha_N}, \tag{9.31}$$

or completely equivalently

$$\mathcal{Z}[0] = \lim_{N\to\infty} \left[\left(\bigotimes_{n=1}^{N} \bigotimes_{k=1}^{n} \langle 0|_k^{(n)} \right) \left(\bigotimes_{n=1}^{N} \bigotimes_{k=1}^{n} |0\rangle_k^{(n)} \right) \right]^{\alpha_N}. \tag{9.32}$$

We then see that according to the prescription eq. (9.25) which we take as a postulate, to compute an amplitude in CWL theory we first take the inner product in the large replicated Hilbert space with N levels and then take the power α_N as $N \to \infty$.

With this prescription derived the last step before computing a particle propagator in CWL theory is to determine the natural embedding into CWL theory of the QM position eigenstate $|x\rangle$. We've seen previously that the natural embedding of vacuum states and coherent states are is the form of *replica equal* states, eqs. (9.5) and (9.22), where all replicas are identical. In comparison though, we've also seen that a single quantum excitation of a field is naturally described by a *replica symmetric superposition* state, (9.12), where the excitation is shared among the different replicas. It is not immediately obvious which, if either, of these two classes of states would naturally describe the embedding of a position eigenstate in CWL theory.

To proceed we can draw analogy with the coherent states. Both coherent states and position eigenstates are defined as eigenstates of certain fundamental operators in QM. For the coherent state, we can trivially confirm that eq. (9.22) is an eigenstate of the positive frequency part of the field operator

$$\hat{\Phi}^{(+)} = \bigotimes_{n=1}^{N} \bigotimes_{k=1}^{n} \hat{\phi}_k^{(n)(+)}, \tag{9.33}$$

with eigenvalue $\phi[J]^{\frac{N(N+1)}{2}}$, such that

$$\left[\langle\psi|\hat{\Phi}^{(+)}||\phi[J]\rangle\rangle_{CWL}\right]^{\alpha_N} = \phi[J]\left[\langle\psi||\phi[J]\rangle\rangle_{CWL}\right]^{\alpha_N}, \tag{9.34}$$

for arbitrary CWL state $|\psi\rangle$. If we then demand that the position eigenstate in CWL satisfies the eigenvalue equation above in the same way, except for the position operator, then we are led to conclude that the correct embedding of position eigenstates into CWL theory are is *replica equal* states,

$$|x\rangle_{CWL} = \bigotimes_{n=1}^{N} \bigotimes_{k=1}^{n} |x\rangle_k^{(n)}. \tag{9.35}$$

With the correct prescription for computing amplitudes determined, as well as the correct embedding for position eigenstates, we can now write down the definition of a particle propagator in CWL theory. The propagator for a particle propagating from position x_1 to x_2 is given by

$$\mathcal{K}(x_2, x_1) = \lim_{N\to\infty} \left[\prod_{n=1}^{N} \int \mathcal{D}g^{(n)}\, e^{inS_G[g^{(n)}]} \prod_{k=1}^{n} \int_{x_1}^{x_2} \mathcal{D}q_k^{(n)}\, e^{iS[q_k^{(n)}, g^{(n)}]}\right]^{\alpha_N}, \tag{9.36}$$

where $S[q|g]$ is the action for the particle on a background metric g. In following chapters we will study this propagator in some detail both perturbatively and non-perturbatively.

9.5 Observables and the Probability Interpretation in CWL Theory

The content of this section is the most recent of the research done by the author on the CWL theory. We stress that it is tentative, and has not been thoroughly scrutinized by the other researchers involved with CWL theory. Although this choice of thesis layout may be confusing, to organize the chapters and sections chronologically would likely be more confusing. In light of this, we ask the reader to be generous as they read the chapters.

9.5.1 Hilbert space formulation of CWL theory

We've already begun to discuss, in section 9.1, how one can formulate aspects of the CWL theory in terms of state vectors and operators. In this section we'd like to expand on that discussion more deeply to help to address the question of observables as well as to properly understand the probability interpretation of the theory.

As mentioned previously, the states in the CWL theory can also be understood as vectors in a Hilbert space. We have previously defined them in eq. (9.1) in terms of a state vector in the Hilbert space which has been replicated $\lim_{N\to\infty}\frac{1}{2}N(N+1)$ fold. One remarkable consequence of the rescaled version of the CWL theory generating functional eq. (9.25), which may not be apparent until we perform explicit calculations in the following chapter, is that the theory is equivalent to a similar but much more simple theory.

Recall that we have N levels, $n = 1, \dots, N$, each with n replicas of the matter fields. When the take the product of all the amplitudes and compute its fractional power α_N in the $N \to \infty$ limit, we'll see that the result is always completely dominated by the largest level, $n = N$. As a consequence we can dispense of the notion of levels entirely, and consider CWL theory as a theory with only one metric and $N \to \infty$ replicas. The generating functional can then be redefined once more as

$$\mathcal{Z}[J] = \lim_{N\to\infty}\left[\int \mathcal{D}g\, e^{-NS_G[g]} \prod_{k=1}^{N} \int \mathcal{D}\phi_k\, e^{-S[\phi_k,g]+J^k\phi_k}\right]^{1/N}. \tag{9.37}$$

Although the expression eq. (9.37) is equivalent to the previous eq. (9.25), the new definition allows for a much more simple discussion. Before imposing diffeomorphism invariance constraints, the states in CWL theory for a system consisting of gravity and a single matter field ϕ are now vectors in the kinematic Hilbert space

$$\mathcal{H}_{CWL} = \lim_{N\to\infty} \mathcal{H}_g \bigotimes_{k=1}^{N} \mathcal{H}_\phi. \tag{9.38}$$

Of course we have not strictly defined the limit here, rather the notation serves as a reminder that N will be taken to be arbitrarily large at the end of computations. In subsequent expressions we will not write this limit explicitly, but will instead keep in mind that N is an arbitrarily large integer.

Given orthonormal bases $\{|g_i\rangle\}$ of $\mathcal{H}_g$ and $\{|\phi_j\rangle\}$ of $\mathcal{H}_\phi$ from conventional quantum gravity, we can then write a generic state in CWL in the form

$$|\Psi\rangle = \sum_{i,j_1,\ldots,j_N} \Psi_{i,j_1,\ldots,j_N} |g_i\rangle|\phi_{j_1}\rangle \otimes \cdots \otimes |\phi_{j_N}\rangle. \tag{9.39}$$

In addition to the quantum gravitational constraints, there is a novel CWL constraint on the allowed wavefunctions $\Psi_{i,j_1,\ldots,j_N}$. Since all of the replicas are copies of the same matter system, the wavefunctions must be invariant under permutations of the replicas. That is, we impose the physical state condition

$$\Psi_{i,j_1,\ldots,j_N} = \Psi_{i,j_{\sigma(1)},\ldots,j_{\sigma(N)}}, \tag{9.40}$$

for $\sigma \in S_N$ a permutation of the set $\{1, 2, ..., N\}$ In this sense, the allowed wavefunctions are "bosonic". The previously discussed *replica identical* and *replica symmetric superposition* states are examples of such wavefunctions. It might be interesting to consider models wherein the allowed wavefunctions over the replica Hilbert space are fermionic but that seems to be quite different from our goals here, so we will not pursue it further.

To understand observables in CWL theory, we must determine the best prescription for embedding the operators of conventional quantum theory into CWL theory. The most natural way to do this (for operators on the matter Hilbert space) while maintaining the replica permutation invariance, is to use the replicated operator,

$$\mathbb{O} = \mathcal{O}^{\otimes N} = \bigotimes_{k=1}^{N} \mathcal{O}, \tag{9.41}$$

where $\mathcal{O}$ is the conventional operator.

Of course one could also consider operators which satisfy replica permutation invariance and cannot be written in this form, eg. using the direct sum

$$\begin{aligned} \mathscr{O} = \bigoplus_{k=1}^{N} \mathcal{O} = \mathcal{O} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \\ + \mathbb{1} \otimes \mathcal{O} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \\ + \cdots + \\ + \mathbb{1} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes \mathcal{O}, \end{aligned} \tag{9.42}$$

however operators of this type can often be derived from the direct product type operators. One way to see this is to use $\mathcal{O} = \exp i\epsilon A$ as a sort of generating function, and then compute various derivatives, eg.

$$-i\frac{d}{d\epsilon}\bigotimes_{k=1}^{N} e^{i\epsilon A}\bigg|_{\epsilon=0} = \bigoplus_{k=1}^{N} A. \tag{9.43}$$

9.5.2 Measurement and Probabilities

The above replicated system with its bosonic permutation symmetry is still a conventional quantum mechanics theory. The major departure of CWL theory from conventional quantum mechanics comes from the further rule for computing observables. When $G \to 0$, CWL theory must return the same results as conventional quantum mechanics. To enforce this, and to follow the same prescription as eq. (9.37), we would then naturally define the rule for computing observables to be

$$\langle \mathcal{O}_a \rangle_{CWL} = \Big[\langle \Psi | \mathbb{O}_a | \Psi \rangle\Big]^{1/N}, \tag{9.44}$$

where $\mathbb{O}_a$ is the appropriate operator on the replicated Hilbert space, and where it is understood that N will be taken arbitrarily large. We can show shortly how this cannot necessarily be correct.

Consider an observable $\mathcal{O}$ with possible outcomes $\{\lambda_j\}$. If we follow the prescription (9.44), we would conclude that the probability in CWL theory for obtaining outcome λ_j in state $|\Psi\rangle$ would be

$$p(\lambda_j) = \left[\langle \Psi | \Big(|\lambda_j\rangle\langle\lambda_j|\Big)^{\otimes N} |\Psi\rangle\right]^{1/N}. \tag{9.45}$$

The first check on whether this is a sensible definition is to check that it reduces to the conventional definition when $G \to 0$. It is trivial to check than in a replica identical state, $|\Psi\rangle = \bigotimes_{j=1}^{N} |\psi\rangle$, this prescription indeed reduces to the conventional definition

$$p(\lambda_j) = \langle\psi|\lambda_j\rangle\langle\lambda_j|\psi\rangle. \tag{9.46}$$

The second check is to see that the sum of probabilities for all outcomes is equal to 1, ie. that the set of probabilities is properly normalized. Let us first recall how this is seen in conventional quantum theory. Conventionally, we use the completeness relation for the eigenvectors of Hermitian operators to show

$$\begin{aligned}\sum_j p(\lambda_j) &= \langle\psi|\Big(\sum_j |\lambda_j\rangle\langle\lambda_j|\Big)|\psi\rangle \\ &= \langle\psi|\psi\rangle. \end{aligned}\tag{9.47}$$

Since unitary time evolution preserves the norm of states, if a state is normalized $\langle\psi|\psi\rangle = 1$, then the sum of probabilities will equal 1 for all time. In CWL theory, using the prescription (9.45), the sum of probabilities is

$$\sum_j p(\lambda_j) = \sum_j \left[\langle\Psi| \Big(|\lambda_j\rangle\langle\lambda_j|\Big)^{\otimes N} |\Psi\rangle\right]^{1/N} \neq \left[\langle\Psi| \Big(\sum_j |\lambda_j\rangle\langle\lambda_j|\Big)^{\otimes N} |\Psi\rangle\right]^{1/N}, \tag{9.48}$$

where the rightmost expression would equal 1 for normalized states.

We can see clearly that when the state $|\Psi\rangle$ is a replica equal state, then the sum of probabilities in CWL will indeed equal 1. If the state is not a replica equal state, then in general the sum of probabilities as given by the prescription eq. (9.45) will not equal 1. A natural question to ask is then, "If the matter system is initially in a replica equal state and evolves under the CWL time evolution, will it remain in a replica equal state?" To this the answer is simply no.

A replica equal state is a product state for the various replicas: there is no entanglement between replicas. The CWL evolution, before observables are measured and quantities are raised to powers of $1/N$, is just conventional unitary time evolution for a system consisting of gravity and N different copies of the matter system. When dynamical gravity is turned off, the replicas do not couple to each other, and thus remain in a product state. When gravity is turned on however, the replicas will interact with each other and generically they will evolve into an entangled state—and thus away from a replica equal state. It is interesting to observe that the failure of the probabilities to remain normalized is strongly linked to the development of entanglement between replicas—this relationship deserves more research at a later time.

Generally speaking then, it seems that we must "renormalize" the probabilities in CWL theory. Doing so, we arrive at the prescription for computing CWL probabilities

$$p(\lambda_j) = \frac{\left[\langle\Psi| \Big(|\lambda_j\rangle\langle\lambda_j|\Big)^{\otimes N} |\Psi\rangle\right]^{1/N}}{\sum_i \left[\langle\Psi| \Big(|\lambda_i\rangle\langle\lambda_i|\Big)^{\otimes N} |\Psi\rangle\right]^{1/N}}. \tag{9.49}$$

An immediate consequence of this is that the prescription for computing expectation values should be modified by this same renormalization,

$$\langle O\rangle_{CWL} = \frac{\left[\langle\Psi|\,\mathcal{O}^{\otimes N}|\Psi\rangle\right]^{1/N}}{\sum_i \left[\langle\Psi|\left(|\lambda_i\rangle\langle\lambda_i|\right)^{\otimes N}|\Psi\rangle\right]^{1/N}}. \tag{9.50}$$

9.5.3 CWL as a non-linear quantum mechanics theory

When written in this way (eqs. (9.49) and (9.50)), one sees that the CWL theory seems to be a particular example of a non-linear quantum mechanics theory, not unlike the class considered by Weinberg [99]. Authors including Weinberg, Czachor, Gisin, and Polchinski have argued that such theories have many potential issues including for example: issues with the sprobability interpretation [99], observables being basis dependent [99, 346], superluminal EPR communication [101–103], and ambiguities in generalizing from pure states to mixed states [103]. It is very important to check whether these issues also plague the CWL theory, or if it evades them in an interesting way.

We have not yet performed sufficient research to be conclusive on these issues, however we can share some preliminary observations. Our first observation is regarding the superluminal EPR communication proven by Gisin for non-linear quantum mechanics theories [101, 102]. In the models considered by Gisin, the proof of EPR communication relied on the projection postulate for quantum measurement. That is, the assumption that upon performing a measurement of the observable $\mathcal{O}$ on a state $|\psi\rangle$ and finding the outcome λ^j, the state instantaneously transforms

$$|\psi\rangle \to \frac{|\lambda^j\rangle\langle\lambda^j|\psi\rangle}{\sqrt{\langle\psi|\lambda^j\rangle\langle\lambda^j|\psi\rangle}}. \tag{9.51}$$

To this we note Polchinksi's observation that in such non-linear quantum mechanics theories, one may not need to impose the projection postulate, instead computing observables in a more operational sense [103]—relying on a relative-state (aka many-worlds) interpretation rather than the Copenhagen. In the CWL theory we will abandon the Copenhagen idea of the projection postulate. A genuine example of this is to view quantum theory merely as a "logic" for computing correlations between subsequent measurement outcomes— a perspective emphasized by Wigner [347] with a corresponding path-integral formalism developed by Caves [see 348, 349, and refs. therein].

We should add however, that in the original conception of the theory (ref. [113]) it was argued that in this theory one should ultimately abandon the unphysical mathematical crutch that is instantaneous "projective" measurements. The arguments of Stamp imply that one should instead describe the physical interaction between a system and a macroscopic apparatus, and that the CWL "path-bunching" would render the apparatus classical and force the system into a particular state as the result of *dynamics*. We believe that there has not been enough work done in CWL theory to prove that this mechanism indeed replicates the standard projective operator approach to measurements for microscopic systems, however it remains important to try and see whether it does indeed occur. In this thesis we will continue to use the conventional approach, where operator insertions describe idealized measurements.

The other comment we'd like to make on superluminal communication is regarding the arguments of Polchinski [103]. Polchinski makes no use of the projection postulate, and still demonstrates that a wide class of non-linear quantum mechanics theories have issues with EPR communication. The non-linear observables which Polchinski claims will not lead to superluminal communication are those which can be written as functions of the system's density matrix. If we simply rewrite $\rho = |\Psi\rangle\langle\Psi|$, then the CWL prescription eq. (9.50) can be written as

$$\langle \mathcal{O} \rangle_{CWL} = \frac{\Big[\mathrm{Tr}\big(\rho\, \mathbb{O}\big)\Big]^{1/N}}{\sum_i \Big[\mathrm{Tr}\Big(\rho\Big(|\lambda_i\rangle\langle\lambda_i|\Big)^{\otimes N}\Big)\Big]^{1/N}}. \tag{9.52}$$

We then see that the non-linear observables in CWL theory are precisely the type which Polchinski excludes from the superluminal signaling proof. It was also demonstrated in [103] however that such theories will necessarily involve communication between different branches of the wavefunction, but that is **exactly** what CWL theory is meant to model—gravitation between different paths in a path integral. More research is certainly required to be conclusive about whether CWL evades the various issues that other non-linear quantum theories suffer from, but as mentioned above, our early observations are promising.

Chapter 10

The intrinsic large-N limit of CWL theory

In this chapter we study aspects of the non-perturbative behaviour of a correlated worldline (CWL) theory of quantum gravity. We start with the generating functional to study how quantum field theory is modified in CWL and discover that because of the infinite number of "replica" paths in the path integral there is an intrinsic large-N limit to the theory. We study the novel CWL combinatorics for the Feynman diagrammatic expansion to develop an important intuition: that only tree diagrams will contribute in the limit $N \to \infty$.

We then use functional methods to prove that our intuition is correct and formally compute the exact CWL generating functional. There is an apparent inconsistency between the lack of loop diagrams and the necessity of loop diagrams for reproducing full non-linear General Relativity in the classical limit, however we demonstrate how it is resolved when the many "replicas" appear to coincide, thus enabling tree-level CWL diagrams to mimic conventional quantum field theory loop diagrams. This leads to rather different predictions for the CWL corrections to the correlations functions of a field theory in comparison with the CWL corrections for a particle propagator.

We close by identifying the relationship between CWL theory and large N approximations in conventional quantum gravity. As a consequence of all the simplifications, we conjecture that CWL theory is renormalizable with only a finite number of counter terms added to the Lagrangian.

10.1 CWL Generating Functional

In a later section we will return to further discuss propagators but for now we pivot the discussion back to field theory, where correlation functions are of more interest. Our starting point will be the generating functional $\mathcal{Z}[J]$, as written in eq. (9.25). From this we define the free-energy functional

$$\begin{aligned}\mathcal{W}[J] &= -\log \mathcal{Z}[J] \\ &= -\lim_{N\to\infty}\left[\alpha_N \sum_{n=1}^{N} \log \mathcal{Z}_n[J]\right]. \end{aligned} \tag{10.1}$$

Observe the utility of the normalizing factor α_N—in the limit $N \to \infty$, it will vanish as N^{-2}.

We will soon demonstrate that $\log \mathcal{Z}_n[J]$ can be expanded in powers of n, and that it grows no faster than linearly with n. This allows us to expand $\mathcal{W}_n[J] = -\log \mathcal{Z}_n[J]$ as

$$\mathcal{W}_n[J] = \mathcal{W}^{(1)}[J]\, n + \mathcal{W}^{(0)}[J]\, n^0 + \mathcal{O}(n^{-1}), \tag{10.2}$$

and when we substitute this into the full free energy functional (10.1) we obtain

$$\begin{aligned}\mathcal{W}[J] &= \lim_{N\to\infty}\left[\alpha_N \sum_{n=1}^{N}\Big(W^{(1)}[J]\, n + \mathcal{O}(n^0)\Big)\right] \\ &= W^{(1)}[J] \lim_{N\to\infty}\left[1 + \mathcal{O}(N^{-1})\right] \\ &= W^{(1)}[J]. \end{aligned} \tag{10.3}$$

Thus when computing the CWL free energy functional perturbatively we need only retain *connected* diagrams at each level n, which scale linearly with n. Any term scaling as n^P for $P < 1$ will be canceled out by the CWL normalization.

In this sense, CWL has a built-in "large-N limit" which is different in spirit from large-N limits considered in conventional QFT; here N refers not to the number of matter fields but to the number of *levels* (and thus to the number of matter replicas in the highest level, ie., the number of matter paths connected by CWL gravitational interactions).[28] The interpretation of the replicas in CWL is different from the interpretation of the multiple fields in a conventional QFT large-N limit. In conventional QFT each field has an independent existence and the permutations of the field labels is treated as a

[28] After this chapter was written, we became aware of some literature on the large-N approximation in quantum gravity, notably the works of Tomboulis, Hartle and Horowitz, and Kay [350–353]. We were originally unaware of this work, but in this chapter there is considerable overlap between our ideas in CWL and theirs in conventional quantum gravity. In particular, they discuss the connection between $1/N$ expansions and semiclassical gravity theories.

symmetry under which the states can be organized into representations. In CWL the replicas of a field are mathematical devices introduced to describe the evolution of that single object. In contrast with conventional QFT, replica permutation in CWL should be treated as a discrete gauge symmetry.

10.2 Diagrammatics

In this section we will return to the discussion of perturbation theory initiated in chapter 8. Now, equipped with the knowledge that we need only retain connected diagrams which are linear in n, we can take the diagrammatic expansion much further. We'll start by thinking in terms of conventional Feynman diagrams, then we will use a more general approach to classify diagrams at all orders.

10.2.1 Low orders - intuition

Let us first review the n dependence of various elements of Feynman diagrams. The level n generating functional is

$$\mathcal{Z}_n[J] = \int \mathcal{D}g\, e^{-nS_G[g]} \prod_{k=1}^{n} \int \mathcal{D}\phi_k\, e^{-S[\phi_k,g]+J^k\phi_k^{(n)}}. \tag{10.4}$$

We start by considering the metric perturbation about flat spacetime, $g_a = \eta_a + h_a$, and expand each action in powers of h. Since the matter action is independent of n each of the matter-graviton vertices will be the same as conventional quantum gravity. Since the Einstein-Hilbert action appears multiplied by n, each graviton-graviton vertex will come with a factor of n, and the graviton propagator (which is the inverse of the quadratic form in the action) will come with a factor of n^{-1}. Finally, factors of n will appear due to the combinatorics of diagrams due to the fact that there are n replicas. In Figure 10.1 we illustrate some diagrams contributing to the free energy functional in CWL scalar field theory to gain some intuition for which diagrams scale linearly with n.

From fig. 10.1 we can make a simple observation: only diagrams containing no graviton loops will scale linearly with n and thus contribute to the CWL free energy functional. It follows that there are no gravitational interactions coupling a replica to itself.

In conventional quantum gravity a diagram such as Figure 10.1-ii. contributes to $2 \to 2$ scattering. We see this by taking four functional derivatives of the diagram with respect to the source and then setting $J = 0$ to

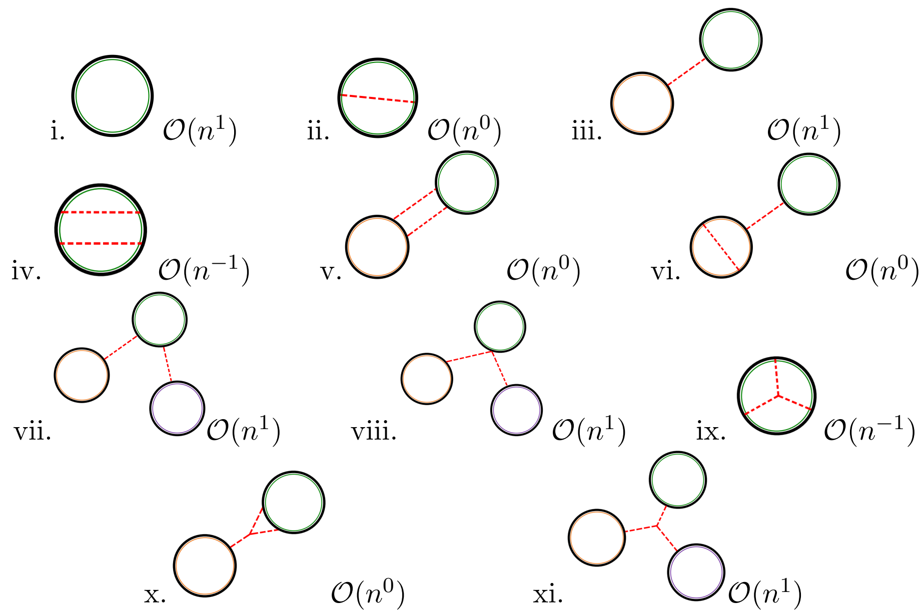


Figure 10.1: Some low order diagrams in perturbation theory contributing to $W_n[J]$ for scalar field theory. Thick solid black lines denote "sourced" matter propagators (2-point correlation functions computed while the source J is still on.). We use a thinner solid coloured line to distinguish an excitation of one replica field from an excitation in another replica field. Dashed red lines represent graviton propagators.

obtain the 4-pt correlation function. This "breaks" open the matter loop to give the tree-level scattering diagram (see Figure 10.2).

In CWL though, the diagram in Figure 10.1-ii. doesn't contribute to the free energy functional: instead the corresponding CWL diagram is Figure 10.1-iii. If we functionally differentiate this diagram with respect to the source to break open the matter lines we see that the lowest order diagrams describing $2 \to 2$ scattering in CWL between particles A and B involve particle A in one replica interacting with particle B in the other replicas.

The $2 \to 2$ scattering amplitude in CWL was computed perturbatively to lowest order in ref. [114] and it was shown to equal exactly the lowest order perturbative scattering amplitude from conventional quantum gravity. It was not emphasized in [114] how remarkable it was that one obtains the exact same result as conventional quantum gravity, despite the dominant interaction occurring between two excitations of different replica fields, rather than between excitations of the same field. This feature (that interactions

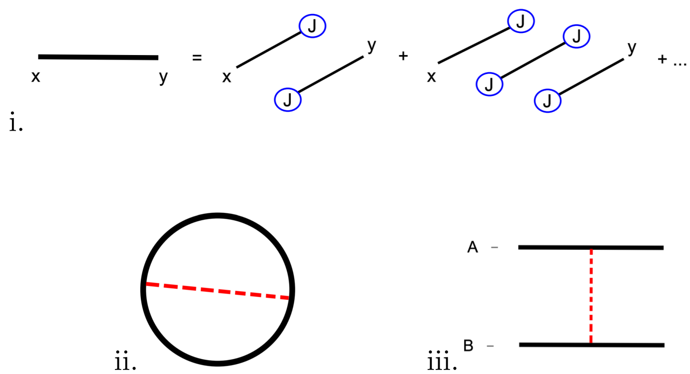


Figure 10.2: i. A perturbative expansion of the sourced propagator.
ii. Breaking open the sourced propagators in Figure 10.1-ii. to reveal the tree-level scattering diagram.

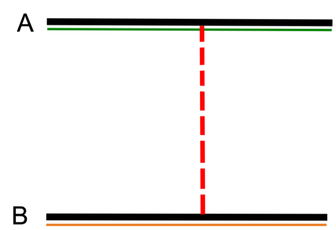


Figure 10.3: Gravitational scattering of particle A (an excitation in one replica field) with particle B (an excitation in a different replica field).

only occur between different replicas) will have significant implications for the higher-order diagrams.

Going beyond lowest order perturbation theory we expect new contributions to the $2 \to 2$ scattering amplitude. In conventional quantum gravity we would see diagrams such as those in Figure 10.4 where the matter lines exchange two gravitons, or where an exchanged graviton splits via a 3-graviton vertex, etc.

From eq. (10.3) we can see that the only CWL diagrams in which these higher order gravitational interactions arise are those involving three replicas (see Figures 10.1-vii, viii, xi.). If we take four functional derivatives with respect to J to break open the matter lines we find that in many of these diagrams the excitations either couple to a tadpole or there is a polarization

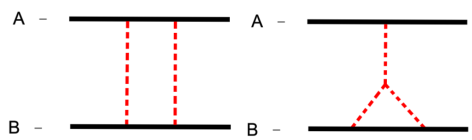


Figure 10.4: Some higher-order contributions to $2 \to 2$ scattering in conventional quantum gravity.

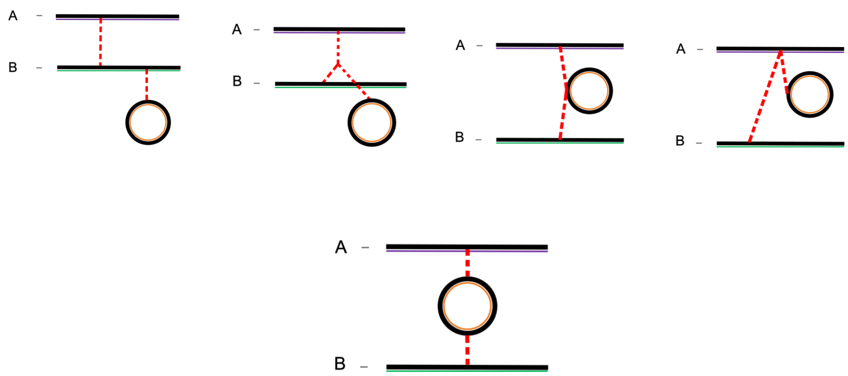


Figure 10.5: Breaking open higher order CWL free energy diagrams to study $2 \to 2$ scattering.

bubble affecting the graviton graviton propagator (see fig. 10.5). Upon setting $J = 0$, we then find that all of these second order diagrams except the polarization diagram will vanish! There are no diagrams involving graviton loops and as a consequence it seems that, up to a renormalization of the graviton propagator, the exact CWL 4-pt function is given by a single-graviton exchange diagram. In the following section we will prove this to be true.

We see from this example that these higher-order contributions involving multi-graviton exchange and graviton-graviton vertices do still arise in CWL. However they are seen only as interactions involving more than two replicas, and when we functionally differentiate to obtain the 4-pt correlation function all of the higher-order contributions vanish.

10.2.2 Higher orders

The following argument is somewhat standard in field theory texts for computing the powers of $\hbar$ in an arbitrary diagram, and we adapt the logic to compute the power of n. Although the argument is standard we will review it for completeness [55].

We can go further with our understanding if we formally integrate out the matter fields. Since each of the replicas has the same action we simply obtain

$$\mathcal{Z}_n[J] = \int \mathcal{D}g\, e^{-nS_G[g]+-nW_0[J|g]}, \tag{10.5}$$

where $W_0[J|g]$ is the free energy functional for the conventional QFT of a single replica on a fixed background metric g,

$$W_0[J|g] = -\log \int \mathcal{D}\phi\, e^{-S[\phi|g]+\int J\phi}. \tag{10.6}$$

At this stage only the metric is being integrated over so a diagrammatic expansion of this integral will contain only: graviton propagators, graviton-graviton vertices, and matter sources on which graviton lines can terminate. Furthermore, since n appears as a prefactor in front of both $S_G[g]$ and $W_0[J|g]$ we can see that every propagator will come with a factor of n^{-1} and every vertex will come with a factor of n. A diagram with I propagators and V vertices then scales as n^{V-I}.

We also know that for a connected graph i) every propagator comes with a 4-momentum integral and ii) every vertex comes with a momentum conserving delta function. One of these delta functions conserves total momentum and thus doesn't "cancel out" any of the integrals coming from the propagators. The number of remaining 4-momentum integrals (ie. number of loops) is then given by $L = I - (V - 1)$. It follows that a diagram with I propagators and V vertices will scale with n as n^{1-L}. Thus only those diagrams with zero graviton loops will scale linearly with n and therefore contribute to the CWL free energy functional $\mathcal{W}[J]$.

Although we did not explicitly include ghosts in this discussion, it is trivial to do so. The entire purpose of ghosts is to cancel out the unphysical "pure gauge" contributions running around graviton loops. Since we have determined that there are no such loops contributing to $\mathcal{W}[J]$ there will also be no ghost contributions.

Also, to be clear, this does not just eliminate closed loops containing only gravitons. Any diagram which has a loop involving a graviton is now eliminated.

This is a large departure from conventional quantum gravity which appears to spell disaster for CWL because of the well understood fact that graviton loop diagrams describe both quantum corrections and also classical GR non-linearities [354]. It appears that the $2 \to 2$ gravitational interaction in CWL is only that of linearized gravity and not of full general relativity. In the following sections we will elucidate this point and demonstrate that the appropriately defined semi-classical limit of CWL is indeed full non-linear general relativity.

10.3 Non-perturbative Results

10.3.1 Generating Functional

In the previous section we looked at some Feynman diagrams in CWL theory. From a simple power counting argument, we found that in CWL there is interplay between the combinatorics of the replicas and the rescaling of the gravitational coupling, and together they imply that the theory contains no diagrams with gravitons loops. In this section we will exploit this fact to explicitly evaluate the functional integral.

We can now formally evaluate the functional integral in eq. (10.5) using the stationary-phase method. Let's consider the expansion of the metric g about a stationary point $\bar{g}_J$ satisfying

$$\left(\frac{\delta S_G[g]}{\delta g} + \frac{\delta W_0[J|g]}{\delta g} \right) \bigg|_{g=\bar{g}_J} = 0. \tag{10.7}$$

Note that $\delta_g W_0[J|g]$ is proportional to the expectation value of the stress-energy tensor for the matter,

$$\begin{aligned} \frac{\delta W_0[J|g]}{\delta g^{\mu\nu}} &= \frac{-1}{Z[J|g]} \int \mathcal{D}\phi \, e^{-S[\phi|g] + \int J\phi} \left(\frac{-\delta S[\phi|g]}{\delta g^{\mu\nu}} \right) \\ &= -\frac{1}{2} \langle T_{\mu\nu}[g] \rangle_J \end{aligned} \tag{10.8}$$

and that $\delta_g S_G$ is proportional to the Einstein tensor, so $\bar{g}_J$ is the solution to the semi-classical Einstein equation

$$G_{\mu\nu}(\bar{g}_J) = 8\pi G \langle T_{\mu\nu}[\bar{g}_J]) \rangle_J. \tag{10.9}$$

We now write $g = \bar{g}_J + n^{-\frac{1}{2}} h$ and expand the effective action in powers of h,

$$\mathcal{Z}_n[J] = e^{-nS_G[\bar{g}_J]-nW_0[J|\bar{g}_J]}$$
$$\times \int \mathcal{D}h \, \exp\left[- \sum_{m=2}^{\infty} \frac{n^{1-m/2}}{m!} \frac{\delta^m}{\delta g^{a_1}...\delta g^{a_m}} \left(S_G[g] + W_0[J|g]\right)\Big|_{g=\bar{g}_J} \times h^{a_1}...h^{a_m} \right]. \tag{10.10}$$

In writing this we've included a "DeWitt" index on the metric. We've also omitted an overall factor of n raised to a power which came from the Jacobian of the integration variable change. After taking the logarithm of $\mathcal{Z}_n$ this factor will not be linear in n, and will not contribute to the free energy.

Now if we look at the remaining functional integral and consider evaluating it perturbatively in a Feynman diagram expansion, we see that the term quadratic in h^a in the effective action functional is proportional to n^0 and all vertices are proportional to n to a negative power. It follows then, that no diagram in the perturbative expansion of this functional integral will grow with n. We may then write the level-n generating functional as

$$\mathcal{Z}_n[J] = e^{-nS_G[\bar{g}_J]-nW_0[J|\bar{g}_J]+\mathcal{O}(n^0)}, \tag{10.11}$$

and if we refer back to our general statements from eqs. (10.1) and (10.3) we conclude that only the stationary-phase solution contributes to the final result

$$\mathcal{W}[J] = S_G[\bar{g}_J] + W_0[J|\bar{g}_J]. \tag{10.12}$$

The central result is then that the CWL generating functional can be written as

$$\mathcal{Z}[J] = e^{-S_G[\bar{g}_J]-W_0[J|\bar{g}_J]}, \tag{10.13}$$

where $W_0[J|g] = -\log Z[J|g]$ is the free energy functional for a conventional matter QFT on a background metric g, and $\bar{g}_J$ solves the semi-classical Einstein equation, eq. (10.9).

Despite the fact that the semi-classical Einstein equation appears here, it does not seem clear whether or not CWL is simply equal to semi-classical gravity. In semi-classical gravity one has a classical metric g_c which satisfies the same semi-classical differential equation. In CWL though the metric is not a classical variable, it is a a quantum variable and can ostensibly exist in a superposition state. The metric in CWL is not *equal* to the solution of the semi-classical Einstein equation but its evolution is however *determined* entirely by that solution. The situation is unclear though, because there does not seem to be a dynamical mechanism for preparing the metric in a superposition state.

In sections 10.5 and 11.2 when we discuss propagators it will become clear that CWL is certainly not the standard "in-in" semiclassical gravity, where the metric is sourced by the expectation value of the stress tensor [88, 89, 91]. Instead we will make clear that it is in the form of a "in-out" semi-classical gravity—the type discussed by Hartle and Horowitz and Kay [352, 353], where the metric is sourced by a certain matrix element of the stress tensor. In this case one can obtain a complex metric, and this metric also requires boundary data in the past and the future. These two facts each seem to preclude the typical interpretation of the saddle point solution $\bar{g}$ as "the" classical metric.[29] In-out models for semi-classical gravity were briefly considered in the literature but were quickly dismissed because they can produce a complex metric [355]; in CWL theory there is no *a priori* issue with a complex metric solution since it is not "the" classical metric.[30]

10.3.2 CWL as a Large N Theory

The result (10.13) is remarkably simple, and this is essentially a consequence of the normalizing power α_N. We saw in eqs. (10.1) and (10.3) that this allows us to retain only the connected diagrams at level n which scale linearly with n. We keep certain contributions from each level, but only those contributions which are common to every level. Said differently, the theory is entirely determined by the physics at the largest level, $n = N \to \infty$, where things simplify.

If we consider this largest level, level N, it is completely obvious that only diagrams which scale with N will contribute. From counting powers of n in the diagrammatic expansion we know that the free energy functional has a $1/N$ expansion,

$$\mathcal{Z}_N[J] = e^{-\sum_{i=0} N^{1-i}\mathcal{W}^{(1-i)}[J]}, \tag{10.14}$$

and for arbitrarily large N it is obvious that $\mathcal{W}^{(1)}$ completely dominates. In fact, eq. (10.11) tells us that this can be written as

$$\mathcal{Z}_N[J] = e^{-NS_G[\bar{g}]-NW_0[J|\bar{g}_J]+\mathcal{O}(N^0)}. \tag{10.15}$$

[29] In light of this strange semi-classical behaviour, we will still refer to CWL as a quantum gravity theory.

[30] If issues eventually arise because of the complex metric, then it is likely that the CWL generating functional would need to be redefined using an in-in framework. In that case one expects to recover standard semiclassical gravity.

If we strip off the factor of N we then exactly recover the result (10.13) of the previous section.

We then see that CWL theory, as defined by eq. (9.25), is completely equal to the theory with just one level containing an infinite number of replicas,

$$\mathcal{Z}[J] = \lim_{N\to\infty} \left[\int \mathcal{D}g\, e^{-NS_G[g]} \prod_{k=1}^{N} \int \mathcal{D}\phi_k\, e^{-S[\phi_k, g] + J^k \phi_k} \right]^{1/N} . \tag{10.16}$$

This result is just conventional quantum gravity with N matter fields with N taken arbitrarily large while keeping GN fixed. This is precisely how one sets up a $1/N$ expansion in field theory, and we see that the CWL generating function is precisely equal to the leading order result of this expansion.

10.3.3 Exact calculations of correlation functions

Let us now compute some exact correlation functions. We'll only concern ourselves with the connected part of correlation functions, so we can functionally differentiate the free-energy functional $\mathcal{W}[J]$ rather than the generating functional $\mathcal{Z}[J]$.

The free energy functional is a functional of the source J, but depends on J both explicitly and implicitly through the metric solution $\bar{g}_J$. Thus, functional derivatives are to be expanded using the chain rule. We illustrate this here using the 1-point function as an example,

$$\begin{aligned} \langle \phi^\alpha \rangle_{CWL} &= \frac{\delta}{\delta J^\alpha} \mathcal{W}[J] \Big|_{J=0} \\ &= \left(\frac{\partial}{\partial J^\alpha} + \frac{\delta \bar{g}^a_J}{\delta J^\alpha} \frac{\partial}{\partial g^a} \right) \big(S_G[g] + W_0[J|g] \big) \Big|_{g=\bar{g}_J,\, J=0} . \end{aligned} \tag{10.17}$$

Upon setting $g = \bar{g}_J$ and then $J = 0$, we effectively set $g^a = \eta^a$ (the Minkowski metric). If we were to compute higher correlation functions, we would have many functional derivatives which would have to commute past the $\delta \bar{g^a} / \delta J^\alpha$ term arising from the chain rule. This leads to a proliferation of terms which we will relegate to an appendix and here only discuss qualitative features of the calculation.

In eq. (10.17) above we have a few types of terms which arise that we should discuss before claiming the results more general correlation functions:

Firstly the direct derivative with respect to J, this will act only on the matter free-energy term and upon setting the metric to flat $g = \eta$ it generates a term $\langle\phi\rangle$ which is no different than that from standard flat space quantum field theory.

Secondly there is a factor $\delta\bar{g}/\delta J$. One does not need to solve the full Einstein equation to compute $\bar{g}_J$ since one is only interested in its derivative evaluated at $J = 0$. Said differently, if we had the full solution $\bar{g}_J$ and expanded it perturbatively in J, we would only be interested in the coefficient of the linear term, ie. the solution to the linearized Einstein equation.

Thirdly, in eq. (10.17) we are taking the functional derivative of $S_G[g] + W_0[J|g]$ with respect to g and then setting $g = \bar{g}_J$. Recall though that this is just the semi-classical equation of motion (eq. (10.7)) and $\bar{g}_J$ is precisely its solution. Thus the term with a functional derivative with respect to g will vanish.

All together, for the CWL one-point function we find

$$\langle\phi^\alpha\rangle_{CWL} = \frac{\partial}{\partial J^\alpha} W_0[J|g])\bigg|_{g=\bar{g}_J,\, J=0} = \langle\phi^\alpha\rangle, \tag{10.18}$$

There are no CWL corrections to the 1-point function, and this is a non-perturbative statement.

When computing higher order correlation functions one introduces higher-order vertices such as the three-graviton vertex, the "seagull" vertex, etc. The coefficients of these terms will involve factors such as $\delta\bar{g}/\delta J$, $\delta^2\bar{g}/\delta J^2$ etc., and these require knowledge of the solution to the Einstein equation.

In appendix A we perform explicit computations for the exact 1, 2, 3, and 4-point connected correlation functions for a theory with $\phi \to -\phi$ symmetry. The results are

$$\begin{aligned}\langle\phi_\alpha\phi_\beta\rangle_{CWL} &= \frac{\partial^2}{\partial J^\alpha \partial J^\beta} W_0[J|\eta]\bigg|_{J=0} \\ &= \langle\phi_\alpha\phi_\beta\rangle,\end{aligned} \tag{10.19}$$

$$\begin{aligned}\langle\phi_\alpha\phi_\beta\phi_\gamma\rangle_{CWL} &= \frac{\partial^3}{\partial J^\alpha \partial J^\beta \partial J^\gamma} W_0[J|\eta]\bigg|_{J=0} \\ &= 0.\end{aligned} \tag{10.20}$$

and

$$\begin{aligned}\langle\phi_\alpha\phi_\beta\phi_\gamma\phi_\delta\rangle_{CWL} &= \frac{\delta^4}{\delta J^\alpha \delta J^\beta \delta J^\gamma \delta J^\delta} W_0[J|\eta]\bigg|_{J=0} \\ &+ \langle\phi_\alpha\phi_\beta S_a\rangle\Big(S_{G,ab} + \langle S_a S_b\rangle\Big)^{-1}\langle\phi_\gamma\phi_\delta S_b\rangle.\end{aligned} \tag{10.21}$$

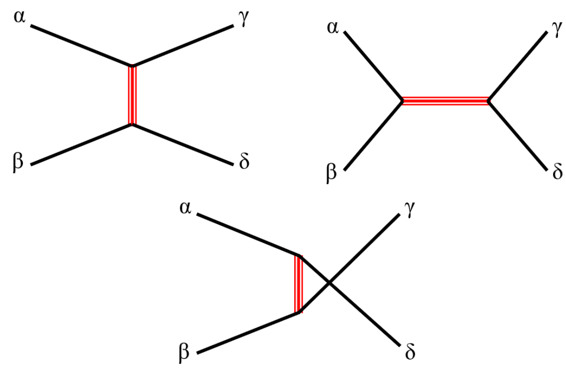


Figure 10.6: Diagram representation of the exact scalar 4-pt function in CWL theory, written in terms of the effective graviton propagator (thickened red line).

If we specialize to the case where the matter is non-interacting, except for gravitational interactions, then the stress tensor is quadratic in the fields,

$$S_a = \frac{1}{2}\hat{\tau}_a^{\alpha\beta}\phi_\alpha\phi_\beta. \tag{10.22}$$

Defining the matter Green's function as $\langle\phi_\alpha\phi_\beta\rangle = G_{\alpha\beta}$, we can write eq. (10.21) as

$$\begin{aligned}\langle\phi_\alpha\phi_\beta\phi_\gamma\phi_\delta\rangle_{CWL} &= G_{\alpha\sigma}G_{\beta\rho}\hat{\tau}_a^{\sigma\rho}\mathscr{D}^{ab}\hat{\tau}_b^{\lambda\eta}G_{\gamma\lambda}G_{\delta\eta}\\ &+ G_{\alpha\sigma}G_{\gamma\rho}\hat{\tau}_a^{\sigma\rho}\mathscr{D}^{ab}\hat{\tau}_b^{\lambda\eta}G_{\beta\lambda}G_{\delta\eta}\\ &+ G_{\alpha\sigma}G_{\delta\rho}\hat{\tau}_a^{\sigma\rho}\mathscr{D}^{ab}\hat{\tau}_b^{\lambda\eta}G_{\gamma\lambda}G_{\beta\eta}.\end{aligned} \tag{10.23}$$

This is just the sum of the three crossings of the tree diagram in fig. 10.2 from conventional QFT, but with a modified graviton propagator (see fig. 10.6)

$$\mathscr{D}^{ab} = \left(S_{G,ab} - \Pi_{ab}\right)^{-1}, \tag{10.24}$$

where the vacuum polarization is given by the matter bubble

$$\Pi_{ab} = \frac{1}{2}\hat{\tau}_a^{\alpha\beta}\hat{\tau}_b^{\gamma\delta}G_{\alpha\delta}G_{\beta\gamma}. \tag{10.25}$$

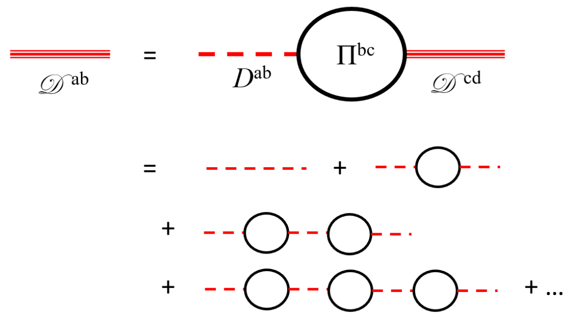


Figure 10.7: A diagrammatic representation of the effective graviton propagator (10.24) in terms of the "bare" graviton propagator and the polarization bubble Π.

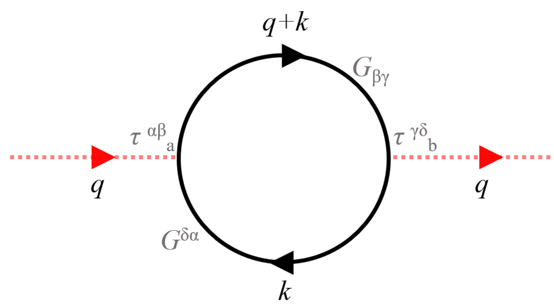


Figure 10.8: The Feynman diagram corresponding to the expression (10.25) for Π.

We see that the $1, 2,$ and 3 point correlation functions receive no corrections in CWL theory,and that the CWL correction to the 4-point correlation function is exactly equal to a single graviton exchange diagram with an effective graviton propagator, a small subset of the contributions which one finds in conventional quantum gravity. These are non-perturbative statements about the theory, including all possible diagrams.

10.3.4 Renormalizability

Before we move on, let us just comment on the effective graviton propagator (10.24). It has been long known that the divergences from this matter loop diagram cannot be renormalized by counterterms in the Einstein-Hilbert action, and that one must introduce additional higher curvature terms [44]. For completeness we'll review the heuristic argument for this.

If we fix harmonic gauge, then in momentum space we can write eq. (10.24),

$$\mathscr{D}^{\mu\nu\sigma\rho}(q) = \left(\ell_P^{-2} P^{\mu\nu\sigma\rho} q^2 - \Pi^{\mu\nu\sigma\rho}(q)\right)^{-1}, \tag{10.26}$$

where

$$P^{\mu\nu\sigma\rho} \equiv \frac{1}{2}\left(\eta^{\mu\sigma}\eta^{\nu\rho} + \eta^{\nu\sigma}\eta^{\mu\rho} - \eta^{\mu\nu}\eta^{\sigma\rho}\right). \tag{10.27}$$

The first term in eq. (10.26) highlights both that $S_G \propto \ell_P^{-2}$ and that the term quadratic in metric fluctuations is second order in derivatives, hence the q^2.

We can evaluate eq. (10.25) for a massless scalar matter field. We'll retain only the divergent parts to make our point. The 1-loop integral is

$$\Pi^{\mu\nu\sigma\rho}(q) = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4} \frac{\tau^{\mu\nu}(k, q+k)\tau^{\sigma\rho}(q+k, k)}{k^2(k+q)^2}, \tag{10.28}$$

where the vertex is

$$\tau^{\mu\nu}(p, k) = p^\mu k^\nu + k^\mu p^\nu - \eta^{\mu\nu} p\cdot k. \tag{10.29}$$

Evaluating this 1-loop integral using dimensional regularization, we obtain the result for the divergent parts

$$\Pi^{\mu\nu\sigma\rho}(q) = \frac{1}{16\pi^2} B^{\mu\nu\sigma\rho}_{\alpha\beta\gamma\delta}\, q^\alpha q^\beta q^\gamma q^\delta \left(\frac{2}{4-d} - \ln\left(\frac{q^2}{\mu^2}\right)\right), \tag{10.30}$$

where μ is an arbitrary renormalization scale, and where B is a tensor constructed from sums of products of the Minkowski metric with $\mathcal{O}(1)$ coefficients. The specific form of B is irrelevant for our heuristic argument.

The important feature of this result is that the divergence in $\Pi^{\mu\nu\sigma\rho}(q)$ goes as q^4. Since there is no term in S_G involving four derivatives of the metric, the 1-loop divergence cannot be absorbed into a renormalized coupling in S_G. One must then include higher-curvature terms in the action, R^2 and $R_{\mu\nu}R^{\mu\nu}$, in order to provide the appropriate counter terms.[31]

Since there are no higher-loop contributions to Π_{ab}, there is no need at this point to include cubic curvature scalars to cancel divergences proportional to q^6. It then seems possible that the inclusion of quadratic curvature invariants in the action renders CWL renormalizable.

[31] One need not consider $R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}$ since the Gauss-Bonnet identity ensures that it can be expressed in terms of the other two scalars.

If we look back at the derivation of eqs. (10.24) and (10.25) in appendix A, we can see that including higher curvature terms in the action would not change Π_{ab}, only $S_{G,ab}$, and thus would not seem to introduce any new divergences. All of this needs to be done rigorously, using eg. the background field method for example to maintain covariance throughout the calculation. We have not yet done this calculation, but it seems promising.

Concretely, we would like to determine whether or not the following generating functional generates finite correlation functions (after suitable redefinitions of the coupling constants etc.)

$$\mathbb{Q}[J] = \lim_{N\to\infty} \left[\int \mathcal{D}g \, e^{-N \int \sqrt{g}\left(\frac{1}{\ell_P^2} R + \alpha R^2 + \beta R_{\mu\nu} R^{\mu\nu}\right)} \prod_{k=1}^{N} \int \mathcal{D}\phi_k \, e^{-\int \sqrt{g} \frac{1}{2} g^{\mu\nu} \partial_\mu \phi_k \partial_\nu \phi_k + J^k \phi_k} \right]^{1/N} \tag{10.31}$$

N.B.: We note that after this chapter was originally written we discovered the papers of Tomboulis [350, 351], and of Stelle [356], in which they discussed the renormalizability of gravity with quadratic curvature invariants. Stelle proved the renormalizability of quadratic gravity with general matter fields using standard loop expansion methods; Tomboulis proved this in the context of a $1/N$ expansion. Tomboulis verifies our suspicion above, that the theory defined by eq. (10.31) is renormalizable and that the UV behaviour is dominated by the asymptotically free coupling of the quadratic terms.

It seems that quadratic gravity has been largely dismissed as an unphysical quantum gravity theory because the higher derivative terms imply of the existence of massive negative norm 'ghost' states, which are claimed to cause issues with unitarity [27, 357]. In the classical theory this manifests as Ostragradsky instabilities/acausalities in the theory. Lee and Wick actually considered precisely this type of problem well before quadratic gravity was proven renormalizable [358, 359]; their main point was that there is no issue with unitarity because these negative norm states are unstable and exist only for very short times. One loses the typical notion of causality at small length scales, but because this violation is deep within a quantum process and it cannot lead to deterministic inconsistencies at the classical level.

Recently quite a bit of attention has been given to reviving the ideas of Lee and Wick [eg. 360, 361], in particular by Donoghue and Menezes [362–364]. They further highlight that the classical instabilities/acausalities in quadratic gravity are utterly irrelevant because they occur at the Planck scale, where classical gravity is inapplicable and a quantum gravity description is necessary. One should then focus only on whether quadratic gravity is a sensible quantum field theory. Tomboulis actually explicitly demonstrated in the large$-N$ limit that quadratic gravity was unitary [350, 351]. Furthermore, arguments of Donoghue and Menezes imply that quadratic gravity is unitary, even away from the large-N limit. It then seems quite possible to us that quadratic gravity could actually be taken seriously as a quantum theory of gravity.

10.4 Where are the non-linearities?

The result for the CWL generating functional is remarkably simple to state but actually quite confusing. To evaluate the generating functional one must solve the full non-linear semi-classical Einstein equation to find the metric $\bar{g}_J$, so it seems like CWL is a non-perturbative quantum gravity theory. Additionally though, we've seen that only diagrams without graviton loops will contribute to free-energy functional, and as a consequence the connected 4-point function is described entirely by a diagram containing a single *renormalized* graviton exchange.

We thus have an apparent inconsistency. Single graviton exchange diagrams do not fully describe classical non-linearities; the tree-level $2 \to 2$ scattering amplitude only contains information on linearized gravity and not full GR. It is well established that the further non-linear classical effects are contained in the non-analytic parts of loop diagrams (see [52, 354] and refs. therein).

As an aside, we note that the non-analytic parts of loop diagrams in quantum gravity reveal genuine non-linear features of quantum gravity which are not just renormalized away by local counter terms. These contributions are easily separated into quantum and classical non-linear parts. In the early 70's it was recognized that one could use this to compute $\mathcal{O}(G^2)$ corrections for the two-body problem in classical GR [365, 366]. This idea gained considerable popularity in the 90's, when it was used to compute quantum corrections to the Newtonian potential and to black hole metrics [see eg. 52, 65, 67, 367–373]. Recently this idea has been central to a whole industry which uses using cutting edge techniques coming from S-matrix bootstrap/unitarity methods for computing quantum gravity scattering amplitudes, and uses them to compute state-of-the-art post-Minkowskian predictions in classical GR. [see eg. 374–387]

Returning to the question at hand here, since CWL theory seems to only have tree diagrams, we're lead to ask whether it is actually a quantum theory of full non-linear GR, or if it ultimately only describes linearized gravitational interactions.

To make sense of this, let us think about how we compute correlation functions from eq. (10.13). The details of the calculation can be found in appendix A, but for now we only need to focus on the part of the calculation involving functional derivatives with respect to $\bar{g}_J$.

Since we set $g = \bar{g}_J$ and $J = 0$ after taking functional derivatives we never need to explicitly solve the Einstein equation for $\bar{g}_J$, instead we'll only ever need to know a finite number of derivatives of $\bar{g}_J$ with respect to J. This implies that we can compute a CWL n-pt correlation function exactly by only computing $\bar{g}_J$ perturbatively in J up to a finite order determined by n. A remarkable conclusion from this is that when computing correlation functions in CWL perturbatively, the perturbative expansion will effectively truncate. For correlation functions with few operator insertions the perturbation series will truncate quite quickly, and for correlation functions with many operator insertions the perturbation series will truncate at a higher order. We then conclude that CWL is indeed a quantum theory of full non-linear gravity, but its non-linearities will only reveal themselves when the theory is probed by a large number of local operator insertions.

Supposing we were interested in the dynamics of a many particle state, perhaps a composite object formed from m particles for some number $m \gg 1$. In principle we could study its evolution by looking at the $2m$-point function in the appropriate channel. In doing so we would find gravitational non-linearities up to the $2m^{\text{th}}$ order in the graviton expansion. We then see that CWL treats objects differently depending on their relative numbers of constituents. Systems containing more constituents can more deeply probe the non-linear nature of gravity. For objects formed from a macroscopic number of constituents CWL has interactions very closely approximating full GR, but not exactly! It is fascinating to think about what implications this might have for the resolution of gravitational singularities.

10.5 Propagators

Let's now transition to studying propagators, ie. transition amplitudes between definite configurations. In conventional quantum field theory there is an equivalence between the 2-point correlator of fields and the one particle propagator. Indeed, for free scalar field theory we have that

$$\langle 0|\mathcal{T}\{\phi(x_1)\phi(x_2)\}|0\rangle = \int \frac{d^4p}{(2\pi)^4} e^{ip(x_1-x_2)} \frac{-i}{p^2+m^2-i\epsilon}, \tag{10.32}$$

and also that

$$\int \frac{d^4p}{(2\pi)^4} e^{ip(x_1-x_2)} \frac{-i}{p^2+m^2-i\epsilon} = \int_0^\infty ds \int_{x_1^\mu}^{x_2^\mu} \mathcal{D}x\, e^{\frac{-i}{2}\int_0^s d\tau \left(\dot{x}^\mu \dot{x}_\mu + m^2 - i\epsilon\right)}, \tag{10.33}$$

which is precisely the transition amplitude for a relativistic scalar particle. After incorporating CWL corrections we will find that this can no longer be true. We've already argued that the CWL correction to the 2pt function vanishes because any such correction would be a loop diagram, however we will shown here that the CWL correction to the propagator is actually non-trivial.

We'll start with our prescription for the CWL propagator, eq. (9.36), which we write again here for reference,

$$\mathcal{K}(x_2,x_1) = \lim_{N\to\infty} \left[\prod_{n=1}^{N} \int \mathcal{D}g^{(n)}\, e^{inS_G[g^{(n)}]} \prod_{k=1}^{n} \int_{x_1}^{x_2} \mathcal{D}q_k^{(n)}\, e^{iS[q_k^{(n)},g^{(n)}]} \right]^{\alpha_N}. \tag{10.34}$$

Before moving on to calculations, we should take a moment to address technical points regarding this propagator. Firstly, we've been cavalier with the functional integration measure. We acknowledge the subtleties in its definition, however we will continue to write it only symbolically. Secondly, in the above we've omitted the boundary data for the metric field. This omission is a subtle point which we'll expand on below.

Typically in flat spacetime quantum field theory one can prepare a vacuum state at a time $t=0$ by evolving an arbitrary state over an infinite amount of time T with a small rotation of the time contour into the imaginary direction,

$$|0\rangle \propto \lim_{\epsilon\to 0} \lim_{T\to\infty} e^{-i\hat{H}T(1-i\epsilon)}|\psi\rangle. \tag{10.35}$$

Since the state $|\Psi\rangle$ is arbitrary, one could equally well describe the vacuum by evolving from a state which is a superposition of all states $|\psi\rangle$ in some basis,

$$|0\rangle \propto \lim_{\epsilon\to 0} \lim_{T\to\infty} \int d\psi\, e^{-i\hat{H}T(1-i\epsilon)}|\psi\rangle. \tag{10.36}$$

Thus when computing vacuum expectation values of products of local operators, or vacuum persistence amplitudes, one often simply writes them as operator insertions in a path-integral which extends over all of spacetime with no fixed initial/final data, eg. in scalar field theory

$$\langle 0|\mathcal{T}\{\hat{\phi}(x_1)\hat{\phi}(x_2)\}|0\rangle \propto \int \mathcal{D}\phi \, e^{i\int d^4x\, \mathcal{L}(\phi,\partial_\mu\phi)}\phi(x_1)\phi(x_2), \tag{10.37}$$

with an appropriate $i\epsilon$ prescription in the Lagrangian density.

In quantum gravity one may try to do the same trick for the metric field, but the situation is far less clear. Firstly, because of the diffeomorphism invariance of General Relativity time evolution is not altogether distinct from an unphysical coordinate transformation—the Hamiltonian is a constraint, and thus $\hat{H}|\mathrm{phys}\rangle = 0$ for all physical states $|\mathrm{phys}\rangle$ [210, 245]. Without an obvious notion of energy, there is clearly no obvious notion of 'ground state' or 'vacuum state'.

One can get around this issue by breaking the diffeomorphism invariance in certain ways, by eg. imposing boundary conditions such as the spacetime being asymptotically flat or asymptotically Anti-de Sitter[32]. In each of these cases the asymptotic symmetry group of the spacetimes will include a time-like Killing vector which can be used to define the energy. One sees this explicitly by observing that Hamiltonian in gravity is the sum of 'bulk' and 'boundary' terms, and only the bulk term is required to vanish while the boundary term contains the physical information [390, 392]. Ostensibly this construction precludes any definition of energy/ground state for spacetimes with compact spatial slices, or for general spacetimes without these simple asymptotic forms. There is a very popular proposal for the 'natural' ground states in quantum gravity based on the Euclidean time rotation, ie. the Hartle-Hawking wave function of the universe [204], however recently this has received quite pointed criticism by Turok and collaborators [see eg. 393, 394], which bring doubt upon its validity.

Even for spacetimes with 'nice' asymptotics, one still must contend with fact that the Hamiltonian is not obviously bounded from below. There are classical positive energy theorems, such as the standard theorem proved by Schoen and Yau and by Witten [395–397], or the extension by Gibbons, Hawking, Horowitz, and Perry [398]. There is also the classical proof of the nonlinear stability of flat spacetime by Christodoulou and Klainerman [399]. According to Smolin though, it is quite unclear if the energy positivity and/or vacuum stability continue to hold in the non-perturbative quantum theory of gravitation [400].

[32] For just a few influential papers in this large literature, see some of the classic "ADM" papers [388, 389] and other influential works by Regge and Teitelboim and Henneaux and Teitelboim [390, 391].

It suffices to say that it is quite unclear what one means when they omit the boundary data for the metric path-integral as we've done in eq. (10.34). One is seemingly left with two options: i. prescribe boundary data (induced three-metrics on the bounding space-like hypersurfaces) and deal with the challenge of finding physically sensible data to prescribe, or ii. accept that the calculation is not truly background-independent and define the ground state perturbatively about a vacuum solution to Einstein's equations—hand-waving past possible risks with energy boundedness/vacuum stability. Here and in the remaining parts of the thesis, we accept the risks and choose option ii. The upshot of this discussion is that when field equations for the gravitational field arise, we will be imposing past-boundary conditions which correspond to, in the absence of matter, flat spacetime.

With this technical detour out of the way, we can proceed to study the propagator path integral in a similar manner to how we studied the generating functional. Let us first rewrite the standard propagator on a background g as

$$K_0(x_2, x_1|g) \equiv e^{i\psi_0(x_2,x_1|g)} = \int_{x_1}^{x_2} \mathcal{D}q\, e^{iS[q|g]}, \tag{10.38}$$

and also write the full CWL propagator as

$$\mathcal{K}(x_2, x_1) \equiv e^{i\Psi(x_2,x_1)}. \tag{10.39}$$

We then have the following expression for Ψ,

$$\Psi(x_2, x_1) = -i \lim_{N\to\infty} \left[\alpha_N \sum_{n=1}^{N} \log K_n(x_2, x_1)\right], \tag{10.40}$$

where the level-n propagator is

$$K_n(x_2, x_1) = \int \mathcal{D}g\, e^{inS_G[g]+in\psi_0(x_2,x_1|g)}. \tag{10.41}$$

To simplify this expression we can use essentially the same stationary phase arguments from the previous section that led us from eq. (10.5) to eq. (10.11). As a reminder, since α_N goes like N^{-2}, we need the log of K_n to return a quantity linear in n so that $\sum_n^N \log K_n$ will yield a factor proportional to $\sum_{n=1}^N n = \alpha_N^{-1} \sim N^2$. Thus when we evaluate the path integral for the level-n propagator eq. (10.41) we need only retain the part scaling as $e^{\mathcal{O}(n)}$. The arguments of the previous section demonstrate that this contribution to the path integral is precisely the stationary-phase part, ie.

$$K_n(x_2,x_1) = e^{inS_G[\bar{g}_{21}]+in\psi_0(x_2,x_1|\bar{g}_{21})+\mathcal{O}(n^0)}, \tag{10.42}$$

where the stationary-phase metric $\bar{g}_{21}$ satisfies the differential equation

$$\frac{\delta}{\delta g}\bigg(S_G[g]+\psi_0(x_2,x_1|g)\bigg)\bigg|_{g=\bar{g}_{21}} = 0. \tag{10.43}$$

Note that the metric solution here depends on x_1 and x_2, and thus on which amplitude is being computed.

Note that in contrast with $W_0[J=0|g]$ in eq. (10.7), the function $\psi_0(x_2,x_1|g)$ is not real valued in general. In some cases we must then solve for a complex metric! We have not yet fully investigated the consequences of this, but one possibility is that it could lead to a suppression of amplitudes for highly quantum states of states with significant gravitational interaction. In the following chapter we will study the propagator for systems with weak gravity, and in that context we can discuss when one expects the imaginary part of $\delta_g\psi_0$ to be significant.

If we substitute this expression into eq. (10.40) and take the limit $N\to\infty$ we obtain the following result for the CWL propagator,

$$\mathcal{K}(x_2,x_1) = e^{iS_G[\bar{g}_{21}]+i\psi_0(x_2,x_1|\bar{g}_{21})}, \tag{10.44}$$

up to an overall normalization.

We can understand this result by returning to the path integral definition of the propagator. A quick calculation reveals that the source here resembles a conditional matrix element of the stress tensor

$$\frac{\delta\psi_0(x_2,x_1|g)}{\delta g^{\mu\nu}(x)} = -\frac{1}{2}\frac{\int_{x_1}^{x_2}\mathcal{D}q\, e^{iS[q|g]}T_{\mu\nu}(x)}{\int_{x_1}^{x_2}\mathcal{D}q\, e^{iS[q|g]}}. \tag{10.45}$$

If we then define the notation

$$\langle\langle\hat{T}_{\mu\nu}(x)\rangle\rangle_{21} \equiv \frac{\int_{x_1}^{x_2}\mathcal{D}q\, e^{iS[q|g]}T_{\mu\nu}(x)}{\int_{x_1}^{x_2}\mathcal{D}q\, e^{iS[q|g]}}, \tag{10.46}$$

we see that the CWL propagator is given by eq. (10.44) where the stationary-phase metric $\bar{g}_{21}$ satisfies a type of semi-classical Einstein equation

$$G_{\mu\nu}(\bar{g}_{21}) = 8\pi G_N\langle\langle\hat{T}_{\mu\nu}\rangle\rangle_{21}. \tag{10.47}$$

We emphasize that (10.45) is a transition matrix element, not an expectation value. This is thus an "in-out" type semiclassical Einstein equation, not an "in-in" type [353].

From this we can conclude that gravitational interactions involved when computing the CWL propagator are those of full non-linear general relativity, albeit with a semi-classical matter source. We can now address the concern raised earlier, that is, how do we reconcile this result with the fact that there are no loop diagrams in CWL theory since we know that it is the non-analytic contributions from loop diagrams that generate classical non-linearities.

The resolution relies on the fact that there are replicas in CWL. When computing a propagator all replicas are evolving between the same end-points. In fig. 10.9 we sketch some diagrams describing the two-particle propagator in CWL. Each of the particles (A and B) has many replicas. For a given particle, each of its replicas will have the same dynamics and thus the many replicas appear to collapse down to a single matter object. The interactions occur between particle A and a replicas of particle B, but because all of the replicas approximately follow the same path, tree diagrams involving many replicas can mimic certain "classical" contributions from conventional loop diagrams. One can substantiate this properly using the Feynman tree theorem to decompose conventional loop diagrams into sums of tree diagrams, but we will not work through the details here as our result (eqs. (10.44) and (10.47)) already demonstrates the point.

Since propagators involve fully non-linear gravitational interactions and correlation functions do not, it is not true in CWL that 2-point correlation functions are equal to particle propagators. What then is the correct mathematical object to compute to describe the dynamics of particles? To this question we unfortunately do not have a definite answer. It seems that there is an issue in CWL with either of: i) our technique for computing correlators, ii) our interpretation of correlators, or iii) our definition of the propagator. We will need to spend more effort to understand this important point, but this is a work in progress.

One idea we have here pertains to objects composed from many constituents, such as a macroscopic mass. In this case, even if the propagator prescription is flawed and one should only look at correlators, then we would need to look at a $2m$-point correlation function to understand the dynamics of an object with m constituents. If m is a very large number, then we know that for all practical purposes we will involve full non-linear GR interactions. We then expect that the effective CWL description for the center of mass propagation of this macroscopic system should then be just as well described

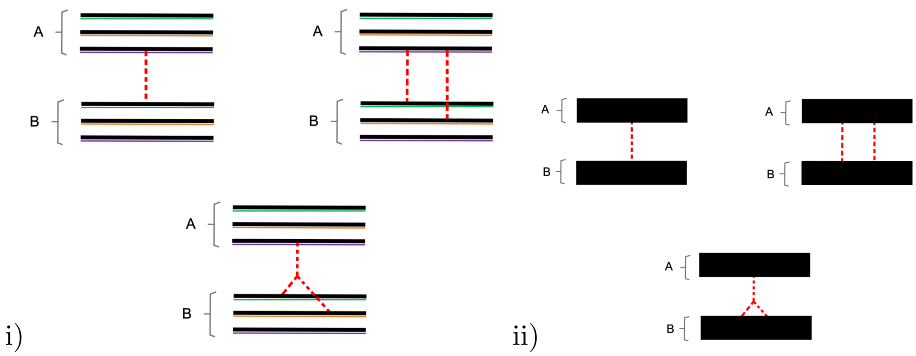


Figure 10.9: i) Some diagrams contributing to the two-particle propagator in CWL at level 3. Note that they are only tree diagrams, as CWL theory requires. ii) If there is a single dominant classical trajectory, all of the replicas follow roughly the same path and their lines appear to coincide. The resulting "collapsed replica" diagrams look like loop diagrams from field theory.

by the 1-particle propagator. We will focus more attention on precisely this idea in the upcoming chapter.

10.6 Conclusions

In this chapter we have studied CWL theory beyond the linearized limit both by looking at the combinatorics of higher-order Feynman diagrams and also by functional methods. We demonstrated that when computing field theory correlation functions perturbatively in CWL only tree diagrams contribute. This was a consequence of the intrinsic large-N limit in CWL coming from the infinite number of replicas. To see higher non-linear gravitational effects one needs to look at n-point correlation functions with sufficiently large n that higher-order vertices can arise without introducing loops.

We later provided a prescription for computing the exact CWL predictions for n-point correlation functions and in the appendix we do these computations explicitly up to $n = 4$. From the exact results for the 4-pt function we were able to see how CWL modifies the gravitational interaction between masses—the result being that the graviton propagator is modified by a polarization diagram which can be computed exactly. This led to a discussion of the possibility that CWL theory can be made renormalizable by including just quadratic curvature invariants into the gravitational action.

One corollary of our exact results is that correlation functions with many operator insertions are able to more deeply probe gravitational non-linearities, and as a consequence large composite systems interact gravitationally in a manner more resembling full GR than elementary particles do. As a consequence, large macroscopic objects (built from many operator insertions) will be almost exactly described by GR but the gravitational interactions involved will not involve arbitrarily high-order graviton vertices. One may then speculate that this could have implications for resolving space-time singularities. We will not speculate on these points further though until proper calculations have been done.

Finally, we looked at propagators within CWL theory. For these we showed that a similar semi-classical mechanism is at play, but the full non-linearity of the gravitational interaction is involved. Despite the fact that the theory does not allow for graviton loop diagrams (which we know contain classical non-linearities), the replicas in CWL can effectively coincide on one trajectory and tree diagrams will mimick loop diagrams—thereby generating the necessary classical non-linearities.

Chapter 11

Applications of the Correlated Worldline theory

In this chapter we will take all of the formal developments over the last few chapters and try to perform some concrete calculations. The ultimate goal here is to study physical systems which are experimentally relevant, and to try compute observable signatures of the CWL theory.

This project in ongoing, as we've only recently gotten to a stage in the formal development of CWL theory where we have an idea how to perform such calculations. As we've discussed previously there are a variety of experiments which may be sensitive to gravitationally induced departures from conventional QM, and we are only just starting to address these physical situations in the CWL theory.

Given the open questions remaining on the formal development side of things, there is a sense in which the upcoming calculations are premature. However we still believe it is important to proceed forward with physical calculations. If anything, by addressing concrete problems we may simply gain insight into fundamental flaws of the theory, and this too would be a worthwhile effort.

We start by considering the most simple system, a particle in a harmonic well. We'll look at the CWL predictions for its 2-point correlation function and also for the evolution of a coherent state. This study then covers both the *replica symmetric superposition* and *replica identical* states of the oscillator.

We then discuss the propagator for a non-relativistic particle in a general potential. We study two cases: when the particle has one dominant classical trajectory, and when there are two dominant classical trajectories (as in matter-wave interferometry experiments). In the former case we find sensible results, and for the later we discuss why the result is confusing. These results appear to have implications for the BMV gravity mediated entanglement experiment which we discussed at length for conventional quantum gravity in chapter 5.

Finally, we move beyond the idealized discussion of a point particle and consider an extended mass consisting of many bound particles. In this case we are able to generate estimates of the relevant time scales for experimental tests of CWL theory.

11.1 Quantum Simple Harmonic Oscillator

We'll consider an idealized situation here, where a massive particle has been trapped in an anisotropic potential such that it is effectively described by a one-dimensional oscillator with frequency ω. This effective theory has a high-energy cutoff scale determined by the energy levels of excitations in the perpendicular directions, ie. $\omega_\perp$. We will not explicitly write this cutoff dependence, but formulas which appear UV divergent should be understood as having this implicit regulator.

11.1.1 Ground state two-point function

We'll start our discussion here from the general perturbative result for the two-point function (8.54) adapted to the oscillator,

$$\mathcal{G}(t_1,t_2) = \langle x_1 x_2\rangle + \ell_P^2 D^{ab}\Big[\langle S_a x_1\rangle\langle S_b x_2\rangle + \langle S_a\rangle\langle S_b x_1 x_2\rangle - \langle S_a\rangle\langle S_b\rangle\langle x_1 x_2\rangle\Big]. \tag{11.1}$$

Here $x_{1,2} = x(t_{1,2})$, and we are considering the time-ordered correlation function.

We remind the reader that the DeWitt indices a, b encompass the spacetime indices of the metric tensor and also a spacetime coordinate. The source is a compact notation for the stress tensor

$$S_a = \frac{\delta S[x, g_{\mu\nu}]}{\delta g_{\mu\nu}(x)}\bigg|_{g_{\mu\nu}=\eta_{\mu\nu}} = -\frac{1}{2}T^{\mu\nu}(x). \tag{11.2}$$

For the oscillator only the mass-energy density is significant gravitationally, and in the non-relativistic limit the graviton Green's function D^{ab} simplifies to describe the Newtonian interaction. We then have

$$\begin{aligned}\mathcal{G}(t_1,t_2) - \langle x_1 x_2\rangle &= \\ = -\frac{\ell_P^2}{8\pi}\int dt \frac{d^3y d^3z}{|y-z|}\Big[&\langle S_{00}(y,t)x_1\rangle\langle S_{00}(z,t)x_2\rangle + \langle S_{00}(y,t)\rangle\langle S_{00}(z,t)x_1x_2\rangle \\ &- \langle S_{00}(y,t)\rangle\langle S_{00}(z,t)\rangle\langle x_1 x_2\rangle\Big],\end{aligned} \tag{11.3}$$

where the mass-energy density is just $S_{00}(y,t) = \frac{m}{2}\delta^3(\hat{x}(t)-y) = \frac{m}{2}|y(t)\rangle\langle y(t)|$.

The individual terms can be computed in terms of the oscillator's energy eigenfunctions $\psi_n(x)$:

$$\langle S_{00}(z,t)\rangle = \frac{m}{2}|\psi_0(z)|^2, \tag{11.4}$$

$$\langle S_{00}(z,t)x(t_2)\rangle = \frac{m}{2}\sqrt{m\omega}\,\psi_0(z)\psi_1(z)\,\langle x(t)x(t_2)\rangle, \tag{11.5}$$

and

$$\begin{aligned}
&\int dt\,\langle S_{00}(z,t)x(t_1)x(t_2)\rangle \\
&\quad = \langle x(t)x(t_1)\rangle\frac{m}{2}\left[|\psi_0(z)|^2\left(\int dt\right) + \frac{\sqrt{2}}{i\omega}\psi_2(z)\psi_0(z) + \frac{i}{\omega}|\psi_1(z)|^2\right] \\
&\quad + m^2\omega|\psi_1(z)|^2\int dt\,\langle x(t)x(t_1)\rangle\langle x(t)x(t_2)\rangle.
\end{aligned} \tag{11.6}$$

All together we find that the two point function in frequency space has the form

$$\begin{aligned}
\mathcal{G}(\Omega) &= G_0(\Omega) + \ell_P^2 A G_0(\Omega) + \ell_P^2 B G_0(\Omega)G_0(\Omega) \\
&= \frac{1+\ell_P^2 A}{G_0^{-1}(\Omega) - \ell_P^2 B},
\end{aligned} \tag{11.7}$$

where the contribution to the "field strength renormalization" is

$$\begin{aligned}
A &= \frac{-im^2}{32\pi\omega}\int\frac{dydz}{|y-z|}|\psi_0(z)|^2\Big(|\psi_1(y)|^2 - \sqrt{2}\psi_2(y)\psi_0(y)\Big) \\
&= \frac{-im^2}{64\pi\omega}\frac{1}{\sqrt{2\pi m\omega}}\left[3\left(\int\frac{dr}{|r|}e^{-\frac{1}{2}r^2}\right) - 1\right],
\end{aligned} \tag{11.8}$$

and the contribution to the "self energy" is

$$\begin{aligned}
B &= \frac{-m^2}{32\pi}(2m\omega)\int\frac{dydz}{|y-z|}\Big(|\psi_0(y)|^2|\psi_1(z)|^2 + \psi_0(y)\psi_1(y)\psi_0(z)\psi_1(z)\Big) \\
&= -\frac{m^2(m\omega)^{\frac{3}{2}}}{16\pi^2}\sqrt{\frac{\pi}{2}}\left(\int\frac{dr}{|r|}e^{-\frac{1}{2}r^2}\right).
\end{aligned} \tag{11.9}$$

Both of these quantities have logarithmic UV divergences, which are regulated by the cut-off scale of this effective theory, $\omega_\perp$.

The factor A doesn't contain much physical information and should be effectively eliminated by a redefinition (renormalization) of the "field strength" x, whereas the factor B has a very simple physical interpretation.

The term $\ell_P^2 B$ generates a shift in the pole of the two-point function, and we can compute the location of the new pole, $\mathcal{G}^{-1}(\Omega_*) = 0$. The pole is shifted from the bare frequency ω to a new frequency $\Omega_* = \omega + \Delta\omega$, where

$$\Delta\omega = \ell_P^2 \frac{B}{2m\omega} = -\frac{1}{\hbar}\frac{1}{2}\frac{G_N m^2}{\Delta x_0}\sqrt{\frac{1}{2\pi}}\left(\int \frac{dr}{|r|} e^{-\frac{1}{2}r^2}\right). \tag{11.10}$$

Here we've restored $\hbar$ where appropriate and introduced the width of the simple harmonic oscillator ground state, $\Delta x_0 = \left(\frac{\hbar}{m\omega}\right)^{\frac{1}{2}}$. The dimensional factor is exactly what one would guess, ie. the gravitational energy of two particles of mass m at a distance Δx_0 divided by $\hbar$ to convert the energy into a frequency. Indeed if we compute the gravitational energy of a mass density $\rho(y)$,

$$E_{SG} = -\frac{1}{2} G_N \int dydz \frac{\rho(y)\rho(z)}{|y-z|}, \tag{11.11}$$

and if we substitute the expectation value for the mass density of an oscillator in its ground state $\rho(y) = m\langle\delta(\hat{x} - y)\rangle = m|\psi_0(y)|$, then we obtain the gravitational energy of a "self-gravitating" wave-function

$$\begin{aligned} E_{SG} &= -\frac{1}{2} G_N m^2 \int \frac{dydz}{|y-z|} |\psi_0(y)|^2 |\psi_0(z)|^2 \\ &= -\frac{1}{2}\frac{G_N m^2}{\Delta x_0}\sqrt{\frac{1}{2\pi}}\left(\int \frac{dr}{|r|} e^{-\frac{1}{2}r^2}\right). \end{aligned} \tag{11.12}$$

We find that $\hbar\Delta\omega$ is precisely E_{SG}.

This shift is in principle unobservable: if we were to return to the action and write the frequency in terms of the physical value and a counter-term, $\omega = \omega_{phys} + \delta_\omega$, then we would fix δ_ω to cancel off the gravitational self-energy part. The oscillator two-point function would then have a pole at the physical frequency ω_{phys} and we would see no CWL correction to this function.

Although this is unobservable, it is interesting to see the CWL mechanics at play. There is a sense in which the wave-function is self-gravitating as it evolves, however the width of the wavefunction seems to be unchanging; if the wavefunction were narrowing with time we would see signatures of the instability of the ground state in the pole of this two-point function. We would likely need to go to higher orders in perturbation theory to see such an effect though, if it is even there.

11.1.2 1-pt function in a coherent state

To further probe the CWL oscillator we can compute correlation functions in states other than the ground state. We've actually discussed this already in section 9.2, where sources are used in a Schwinger-Keldysh closed-time-path (CTP) integral to prepare coherent states. The generating functional for this scenario is given by eq. (9.18).

In CWL the situation is the same except nominally one must include a trace over the final state of the gravitational field. As discussed in chapter 3 such a procedure is precisely how one computes an influence functional to describe the decoherence of the matter due to information leakage to gravitons.

For a non-relativistic matter source which evolves for a finite amount of time, gravitational radiation is completely negligible. In conventional quantum gravity one can then completely neglect the influence functional for such situations. Intuitively we should also be able to do this in CWL theory, and this is what we will do below.

To lowest order in gravitational coupling the CWL CTP generating functional has the same form as the CWL vacuum generating functional (9.29) except the expectation values in W_{WLC} are generalized from vacuum expectation values to CTP expectation values. The upshot of this is that we can compute CWL correlation functions in more general states without the machinery looking very different.

An interesting observable to check is the one-point function (ie. position expectation value) in a coherent state. In conventional QM one has the intuition that coherent states are the most classical type of states. They are minimal uncertainty states and their one-point functions oscillate exactly as classical oscillators. They are also classical states in a technical sense, because their Wigner functions are non-negative [401]. We'd like to see this "classical" property of coherent states preserved in the CWL theory because we live in a world where springs behave normally at macroscopic scales.

We'll consider an oscillator which has been subject to a static force for a sufficiently long time that it is in a displaced ground state at the location $\tilde{x}_0$. At time $t = 0$ we remove the force instantaneously, performing a rudimentary "quantum quench" on the system. We do so by including a classical source $j(t) = \theta(-t)j_0$, $j_0 = \tilde{x}_0/(m\omega^2)$, and this prepares a coherent state $||x[j]\rangle\rangle = ||\tilde{x}_0\rangle\rangle$ at time $t = 0$.

Computing

$$\mathcal{G}(t)_{\tilde{x}_0} = \frac{\delta}{\delta J(t)} \log \mathcal{Z}[J + j]\Bigg|_{J=0} \tag{11.13}$$

we then get

$$\mathcal{G}(t)_{\tilde{x}_0} = \langle x(t)\rangle_j + \ell_P^2 G^{ab}\langle S_a\rangle_j \Big[\langle S_b x(t)\rangle_j - \langle S_b\rangle_j\langle x(t)\rangle_j\Big]. \tag{11.14}$$

Here we use the subscript j on the angled brackets to denote the expectation value in the state $||\tilde{x}_0\rangle\rangle$. The conventional contribution is just $\langle x(t)\rangle_j = \tilde{x}_0\cos(\omega t)$. The CWL contribution can be computed and the result is

$$\begin{aligned}\Delta\mathcal{G}(t)_{\tilde{x}_0} &\equiv \mathcal{G}(t)_{\tilde{x}_0} - \langle x(t)\rangle_j \\ &= \frac{i\ell_P^2 m^2}{64\pi\omega}\int \frac{dydz}{|y-z|}|\psi_0(y)|^2\psi_0(z)\Big[\psi_0(z)\tilde{x}_0\cos(\omega t) + \psi_1(z)\Big].\end{aligned} \tag{11.15}$$

Both terms are UV divergent, but if we regulate the divergence we see that the last term vanishes by symmetry,

$$\begin{aligned}\int \frac{dydz}{|y-z|}|\psi_0(y)|^2\psi_0(z)\,\psi_1(z) &\propto \int \frac{dydz}{|y-z|}e^{-y^2-z^2}(y+z) \\ &= \left(\int_\epsilon^\infty dr\,\frac{e^{-\frac{r^2}{2}}}{|r|}\right)\left(\int_{-\infty}^\infty dR\,e^{-2R^2}R\right) \\ &= 0.\end{aligned} \tag{11.16}$$

The only remaining term involves only vacuum wavefunctions,

$$\Delta\mathcal{G}(t)_{\tilde{x}_0} = \frac{i\ell_P^2 m^2}{64\pi\omega}\left(\int \frac{dydz}{|y-z|}|\psi_0(y)|^2|\psi_0(z)|^2\right)\tilde{x}_0\cos(\omega t). \tag{11.17}$$

This is also UV divergent, but it oscillates as $\cos(\omega t)$.

If we return to the Lagrangian and write $j_0 = j_0(1+\delta_{j_0})$ where δ_{j_0} is a counter term for the coupling to the force, then the coherent state one-point function has the form

$$\mathcal{G}(t)_{\tilde{x}_0} = \left[\tilde{x}_0 + \frac{\delta_{j_0}\tilde{x}_0}{m\omega^2} + \frac{i\ell_P^2 m^2}{64\pi\omega}\left(\int \frac{dydz}{|y-z|}|\psi_0(y)|^2|\psi_0(z)|^2\right)\tilde{x}_0\right]\cos(\omega t). \tag{11.18}$$

We can clearly choose the counter term to cancel the UV divergence and we arrive at the result

$$\mathcal{G}(t)_{\tilde{x}_0} = \tilde{x}_0\cos(\omega t). \tag{11.19}$$

Thus to first lowest order in perturbation theory CWL predicts that oscillators in coherent states will still oscillate as classical oscillators.

We've seen that simple probes of a quantum harmonic oscillator do not provide evidence of any CWL corrections. One case we have not considered here is non-gaussian states. We hypothesize that CWL may not generate corrections for gaussian (pseudo-classical) states, but for non-gaussian states there may be novel CWL effects because they are "more quantum". In the next section, to further understand CWL theory, we will consider examples of non-gaussian states as we study particle propagators.

11.2 Propagators

11.2.1 Form of the weak-gravity propagator

Let us now evaluate the propagator for a non-relativistic particle in CWL theory. We'll make the assumption that only weak gravitational fields are relevant. While this is presumably appropriate for a non-relativistic particle we will discuss possible issues with this assumption at a later point.

We will start from our previously derived non-perturbative results eqs. (10.44) and (10.47),

$$\mathcal{K}(x_2, x_1) = e^{iS_G[\bar{g}_{21}]+i\psi_0(x_2,x_1|\bar{g}_{21})} \tag{11.20}$$

where $\psi_0(x_2, x_1|g_0)$ is determined by the matter propagator on a fixed background metric g_0

$$K_0(x_2, x_2|g_0) = e^{i\psi_0(x_2,x_1|g_0)} = \int_{x_1}^{x_2} \mathcal{D}q\, e^{iS[q|g_0]} \tag{11.21}$$

and the metric $\bar{g}_{21}$ satisfies a version of the semi-classical Einstein equation

$$G_{\mu\nu}(\bar{g}_{21}(x)) = 8\pi G \frac{\int_{x_1}^{x_2} \mathcal{D}q\, e^{iS[q|\bar{g}]} T_{\mu\nu}(x)}{\int_{x_1}^{x_2} \mathcal{D}q\, e^{iS[q|\bar{g}]}}. \tag{11.22}$$

This expression is obviously very non-linear, with the classical non-linearity already inherent in the Einstein tensor, and the further non-linearity introduced by the backreaction of the quantum matter. To proceed with it, we will make a weak field approximation.

We will perturb the metric about flat spacetime, $(\bar{g}_{21})_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. We then expand the phase $S_G[\bar{g}_{21}] + \psi_0(x_2, x_1|\bar{g}_{21})$ to leading order in $h_{\mu\nu}$ and insert the solution to just the linearized version of the semi-classical Einstein equation eq. (11.22).

To provide an example of a calculation in conventional notation, in this section we will expand the compact Dewitt notation used previously and be explicit about spacetime indices, coordinates, and integrations. Expanding the phase argument in powers of the metric perturbation, we find

$$\begin{aligned}
&\mathcal{K}(x_2, x_1) = e^{iS_G[\eta]+i\psi_0(x_2,x_1|\eta)} \\
&\times \exp\left[i\int d^4x \frac{\delta}{\delta g_{\mu\nu}(x)}\Big(S_G[g]+\psi_0[g]\Big)\Big|_{g=\eta} \times h_{\mu\nu}(x)\right] \\
&\times \exp\left[\frac{i}{2}\int d^4x \int d^4y \frac{\delta}{\delta g_{\mu\nu}(x)}\frac{\delta}{\delta g_{\sigma\rho}(y)}\Big(S_G[g]+\psi_0[g]\Big)\Big|_{g=\eta} \times h_{\mu\nu}(x)h_{\sigma\rho}(y)\right] \\
&\times \exp\Big[\mathcal{O}(h^3)\Big]. \qquad (11.23)
\end{aligned}$$

Note that here, and in the following spacetime integrals we omit the limits of integration. It should not be forgotten though that we are integrating over the spacetime region bounded by two constant time slices $x^0 = t_1$ and $x^0 = t_2$.

This expression can be simplified considerably. Firstly, since the flat spacetime metric is a solution to the vacuum Einstein equation and has vanishing action, we can immediately see that

$$S_G[\eta] = \frac{\delta}{\delta g_{\mu\nu}(x)} S_G[g]\Big|_{g=\eta} = 0. \qquad (11.24)$$

Secondly, from the linearized Einstein equation, it will be obvious that $\frac{\delta}{\delta g}\psi_0[g]\big|_{g=\eta} = \mathcal{O}(h)$. As a consequence, we will drop the matter term in the second line of eq. (11.23), as it is $\mathcal{O}(h^3)$. The resulting CWL propagator for a system with weak gravitational fields is

$$\begin{aligned}
\mathcal{K}(x_2, x_1) = e^{i\psi_0(x_2,x_1|\eta)} &\exp\left[i\int d^4x \frac{\delta\psi_0[g]}{\delta g_{\mu\nu}(x)}\Big|_{g=\eta} \times h_{\mu\nu}(x)\right] \\
\times &\exp\left[\frac{i}{2}\int d^4x \int d^4y \frac{\delta^2 S_G[g]}{\delta g_{\mu\nu}(x)\delta g_{\sigma\rho}(y)}\Big|_{g=\eta} \times h_{\mu\nu}(x)h_{\sigma\rho}(y)\right] \\
\times &\exp\Big[\mathcal{O}(h^3)\Big]. \qquad (11.25)
\end{aligned}$$

This expression simplifies one step further when we use the formal expression for the linearized semi-classical Einstein equation

$$\left(\int d^4y \frac{\delta^2 S_G[g]}{\delta g_{\mu\nu}(x)\delta g_{\sigma\rho}(y)} h_{\sigma\rho}(y) + \frac{\delta\psi_0[g]}{\delta g_{\mu\nu}(x)}\right)\Bigg|_{g=\eta} = 0, \qquad (11.26)$$

which gives the formal result for the propagator

$$\mathcal{K}(x_2, x_1) = K_0(x_2, x_1|\eta) \exp\left[\frac{i}{2}\int d^4x \frac{\delta\psi_0[g]}{\delta g_{\mu\nu}(x)}\bigg|_{g=\eta} \times h_{\mu\nu}(x) + \mathcal{O}(h^3)\right]. \tag{11.27}$$

All that remains is to explicitly solve the linearized semi-classical Einstein equation. The analogous calculation in classical gravity is standard [175], but we'll very briefly review here for completeness.

Linearizing the Einstein tensor we obtain

$$G^{(1)}_{\mu\nu}(\eta + h) = \frac{1}{2}(\partial^2 h_{\mu\nu} + \partial_\mu\partial_\nu h - \partial^\rho\partial_\nu h_{\rho\mu} - \partial^\rho\partial_\nu h_{\rho\mu} + \eta_{\mu\nu}\partial^\sigma\partial^\rho h_{\sigma\rho} - \eta_{\mu\nu}\partial^2 h), \tag{11.28}$$

where $h = \eta^{\mu\nu}h_{\mu\nu}$. Although in recent chapters we have not been explicit about how we are dealing with gauge fixing, all of our recent discussions have implicitly involved the Faddeev-Popov gauge fixing procedure discussed explicitly in eqs. (8.23) and (8.24). The upshot of this is that here we can choose to impose a gauge condition on $h_{\mu\nu}$.

In what follows we'll choose the familiar harmonic gauge. To do so, we define the trace reversed metric perturbation $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$, and impose the condition $\partial^\mu \bar{h}_{\mu\nu} = 0$. With harmonic gauge imposed, the linearized Einstein tensor is simply

$$G^{(1)}_{\mu\nu}(\eta + h) = \frac{1}{2}\partial^2 \bar{h}^{\mu\nu}. \tag{11.29}$$

Linearizing the matter side of the semi-classical Einstein equation fixes the source as equal to the flat-spacetime stress tensor, giving the result

$$\partial^2 \bar{h}_{\mu\nu}(x) = 16\pi G \frac{\int_{x_1}^{x_2} \mathcal{D}q\, e^{iS[q]} T_{\mu\nu}(x)}{\int_{x_1}^{x_2} \mathcal{D}q\, e^{iS[q]}}. \tag{11.30}$$

To save on writing in subsequent equations, we'll define the notation

$$\langle\langle \hat{T}_{\mu\nu}(x)\rangle\rangle_{21} = \frac{\int_{x_1}^{x_2} \mathcal{D}q\, e^{iS[q]} T_{\mu\nu}(x)}{\int_{x_1}^{x_2} \mathcal{D}q\, e^{iS[q]}}, \tag{11.31}$$

but we'd like to be clear: this iss **not** simply an *expectation value* of the stress-tensor operator.

At this stage we can simply invert the differential operator in eq. (11.30) using the retarded Green's function to obtain the solution

$$h_{\mu\nu}(x) = -4G \int d^4y \, \frac{\delta\big((x^0 - y^0) - |\vec{x} - \vec{y}|\big)}{|\vec{x} - \vec{y}|} \langle\langle \hat{\bar{T}}_{\mu\nu}(x) \rangle\rangle_{21}. \tag{11.32}$$

Inserting this solution into eq. (11.27), we obtain the final expression for the weak field CWL propagator

$$\begin{aligned} \mathcal{K}(x_2, x_1) &= K_0(x_2, x_1|\eta) \\ &\times \exp\left[iG \int d^4x \int d^4y \langle\langle \hat{T}_{\mu\nu}(x) \rangle\rangle_{21} \, \frac{\delta\big((x^0 - y^0) - |\vec{x} - \vec{y}|\big)}{|\vec{x} - \vec{y}|} \langle\langle \hat{\bar{T}}_{\mu\nu}(y) \rangle\rangle_{21} \right]. \end{aligned} \tag{11.33}$$

If we take the non-relativistic limit, then T_{00} is dominant and it is not changing on relativistic time scales, and we can simply the expression to

$$\begin{aligned} \mathcal{K}(x_2, x_1) &= K_0(x_2, x_1|\eta) \\ &\times \exp\left[\frac{i}{2} G \int_{t_1}^{t_2} dt \int d^3x d^3y \, \langle\langle \hat{T}_{00}(x) \rangle\rangle_{21} \, \frac{1}{|\vec{x} - \vec{y}|} \langle\langle \hat{\bar{T}}_{00}(y) \rangle\rangle_{21} \right]. \end{aligned} \tag{11.34}$$

This result looks somewhat natural, however we must recall the unusual form of this stress tensor eq. (11.31), which is not an expectation value but something like a "normalized" transition matrix element. Furthermore, one notices that to compute eq. (11.34), we need not actually study the dynamics of objects interacting via gravitation. Instead, we simply compute the two standard quantum mechanics quantities

$$\int_{x_1}^{x_2} \mathcal{D}q \, e^{iS[q]} \quad \text{and} \quad \int_{x_1}^{x_2} \mathcal{D}q \, e^{iS[q]} T_{00}(x), \tag{11.35}$$

and assemble the results into a quantity which computes $\mathcal{K}(x_2, x_1)$.

The result eq. (11.34) is valid for systems where only weak non-relativistic gravity is relevant. We could also consider the further approximation, wherein the CWL gravitational effect is small compared with the standard dynamics of the system, described by $\psi_0(x_2, x_1|\eta)$. In this case the CWL phase can be expanded perturbatively, giving the lowest order correction which can be written most transparently as

$$\mathcal{K}(x_2, x_1) = \frac{\int_{x_1}^{x_2} \mathcal{D}q \int_{x_1}^{x_2} \mathcal{D}q' \, e^{iSq] + iS[q']} \Bigg(1 + iS_{CWL}[q, q'] \Bigg)}{\int_{x_1}^{x_2} \mathcal{D}q \, e^{iS[q]}}, \tag{11.36}$$

where the CWL "path-bunching" action is given in this approximation by

$$S_{CWL}[q,q'] = \int_{t_1}^{t_2} dt \frac{Gm^2}{2}\frac{1}{|q-q'|}. \tag{11.37}$$

Written in the above form, we can develop a simple intuition for the perturbative CWL propagator. One considers just two copies of the system (sets of paths), interacting perturbatively via Newtonian gravity, and then divides the resulting two-body propagator by the original standard quantum mechanical result. In section 11.3 we will consider how this calculation is modified for a more realistic experimental mass, consisting of many constituent particles.

11.2.2 Evaluating the non-relativistic particle propagator

This subsection, and the following, are somewhat preliminary. These results have not yet been scrutinized by other researchers familiar with the CWL theory.

Let us now return to the expression eq. (11.34), and in particular, to the strange semi-classical stress tensor eq. (11.31). We'll actually find it useful to return to its definition eq. (10.45).

As we've previously shown using an eikonal calculation, eqs. (3.48) and (3.49), a particle propagating in a weak, slowly varying gravitational field will have a propagator its modified by just an additional phase factor,

$$K_0(x_2,x_1|h) \approx e^{-\frac{i}{2}\int d^4x h_{\mu\nu}(x)T^{\mu\nu}_{cl}(x)} K_0(x_2,x_1|\eta), \tag{11.38}$$

where $T^{\mu\nu}_{cl}(x)$ is the stress tensor for a particle following the "classical" path between the endpoints. Note that this above form holds when there is only one "classical" path between the endpoints, however we will soon consider the case where there are two dominant classical paths.

Written in this way it is straightforward to see that in this limit we effectively have

$$\frac{\delta\psi}{\delta h_{\mu\nu}(x)} = -\frac{1}{2}T^{\mu\nu}_{cl}(x), \tag{11.39}$$

and the CWL phase in the propagator eq. (11.34) would then just describe the gravitational self-energy of the classical trajectory for the particle,

$$\mathcal{K}(x_2,x_1) = K_0(x_2,x_1|\eta)\exp\left[\frac{i}{2}G_N \lim_{\epsilon\to 0}\int_{t_1}^{t_2} dt\, \frac{1}{|\vec{q}(t)-\vec{q}(t)|+\epsilon}\right]. \tag{11.40}$$

This result is divergent, because it effectively describes two interacting replicas which are infinitesimally close together. This then calls into question the assumptions of a weak, slowly-varying, gravitational fields which went into the use of eq. (11.38) for computing the semi-classical source. We will not take the result too literally, but will instead take it as a hint at what the structure of the propagator will look like for a single quantum particle with one dominant classical trajectory—it just accumulates a phase describing its gravitational self-energy. We also note that this result requires a proper regularization and renormalization procedure.

Let us now consider the propagator for a particle which has two dominant classical paths (labeled A and B) between its endpoints. One example of this would be a matter-wave interferometry experiment where the potential energy function is initially a single well, but is transformed into a two-well shape and then recombined into a single well. This is actually precisely what is proposed to be done for optically trapped nanoparticles [402]. We expect that the dynamics of each particle in the BMV experiment [13, 14], which we discussed at length in chapter 5, would also be well described by this model.

It is straightforward to discuss the propagator for this system in a WKB approximation. To do so we would perform a stationary phase approximation about each of the two classical paths and retain only the quadratic fluctuations about either path. Suppressing the arguments of functions, the WKB approximated propagator describing this scenario would be

$$K_0(x_2, x_1|h) = e^{-\frac{i}{2}\int h^{\mu\nu}T^{(A)}_{\mu\nu}} K_0^{(A)} + e^{-\frac{i}{2}\int h^{\mu\nu}T^{(B)}_{\mu\nu}} K_0^{(B)}, \tag{11.41}$$

where $T^{(A)}_{\mu\nu}$ ($T^{(B)}_{\mu\nu}$) denotes the classical stress tensor for the particle following path A (B), and where the contribution from each path is, up to an irrelevant overall constant,

$$K_0^{(A,B)} = \sqrt{\Delta_{(A,B)}}\, e^{iS[q_{cl}^{(A,B)}|\eta]}. \tag{11.42}$$

The factors $\Delta_{(A,B)}$ are the van Vleck determinants coming from integrating out the quantum fluctuations about the classical path to quadratic order, and $S[q^{(A,B)}]$ is the action evaluated on either classical solution, ie. Hamilton's principle function for each path.

If we evaluate the CWL semi-classical stress-tensor (11.31) for this two-path system we then have

$$\begin{aligned}\langle\langle T_{\mu\nu}\rangle\rangle_{21} &= -2\frac{\delta\psi_0}{\delta h_{\mu\nu}}\bigg|_{h=0} \\ &= \frac{T^{(A)}_{\mu\nu}K^{(A)}_0 + T^{(B)}_{\mu\nu}K^{(B)}_0}{K^{(A)}_0 + K^{(B)}_0}. \end{aligned} \tag{11.43}$$

Considering this two-well potential in which the particle is propagating, if the quadratic terms in the expansion of the potential about each classical path are the same, ie. have the same effective "oscillator frequency", then the van Vleck determinants will be the same for each path, $\Delta_{(A,B)} \equiv \Delta_0$. Let us assume this is true as we go forward. In this case the determinants will factor out of eq. (11.43) completely. The result can then be written nicely as

$$\langle\langle T_{\mu\nu}\rangle\rangle_{21} = \frac{T^{(A)}_{\mu\nu} + T^{(B)}_{\mu\nu}}{2} + i\frac{T^{(A)}_{\mu\nu} - T^{(B)}_{\mu\nu}}{2}\tan\left(\frac{S[q^{(A)}] - S[q^{(B)}]}{2}\right), \tag{11.44}$$

where we've omitted the tensor indices on the right side to make the resulting formulas more readable.

This expression is quite interesting; the effective stress tensor, and thus the saddle point solution for the metric, is complex! The real part of the CWL stress-tensor displays exactly the behaviour one might expect from a semi-classical stress tensor; it is the average of the stress tensors of the two classical paths. The imaginary of this expression is quite unusual though, and comes about because this is an "in-out" type semi-classical source, in contrast with an "in-in" expectation value, $\langle\psi|T_{\mu\nu}|\psi\rangle$, which would necessarily be real.

Let us first define the sum and difference of actions,

$$\begin{aligned} S_+ &\equiv \tfrac{1}{2}\Big(S[q^{(A)}] + S[q^{(B)}]\Big) \\ \Delta S &\equiv S[q^{(A)}] - S[q^{(B)}]. \end{aligned} \tag{11.45}$$

Note then that the imaginary part of this CWL stress tensor vanishes when $\Delta S = 2n\pi$ for integer n—precisely when the two paths interfere constructively. The imaginary part will diverge however when $\Delta S = (2n+1)\pi$, ie. when the paths interfere destructively.

We can now substitute this result, eq. (11.44) into the expression for the

CWL propagator eq. (11.34). We find,

$$\mathcal{K}(x_2,x_1) = 2\sqrt{\Delta_0}\, e^{iS_+} \cos\left(\frac{\Delta S}{2}\right)$$
$$\times \exp\left\{\frac{i}{2}G\int_{t_1}^{t_2} dt \int \frac{d^3x d^3y}{|\vec{x}-\vec{y}|}\left[\frac{1}{\cos^2(\Delta S/2)} T^{(A)}(x)T^{(B)}(y)\right.\right.$$
$$+\frac{1}{2}\Big(T^{(A)}(x)T^{(A)}(y) + T^{(B)}(x)T^{(B)}(y)\Big)(1-\tan^2(\Delta S/2))$$
$$\left.\left.+\, i\Big(T^{(A)}(x)T^{(A)}(y) - T^{(B)}(x)T^{(B)}(y)\Big)\tan(\Delta S/2)\right]\right\}, \qquad (11.46)$$

where we've written the mass density compactly as $T^{(j)} \equiv T^{(j)}_{00}$. In this expression we see the novel contributions from the imaginary part of the CWL stress tensor. The intuitive result is obtained by taking $\Delta S = 0$.

In regions of constructive interference, $\Delta S = 2n\pi$, we have conventional Newtonian gravity acting between the paths. This is precisely the "path-bunching" phenomenon that we anticipated in the introduction (chapter 7).

For general $\Delta S \neq 2n\pi$ though, we can notice a few changes. In the path bunching term the effective gravitational interaction is now strengthened as ΔS grows from zero and ultimately diverges for $\Delta S = \pm\pi$. The second term, which is the sum of self-energy terms for each path, is now modified by the $-\tan^2$ term. When $\Delta S = \pm\pi/2$ this term will cross over from positive to negative, ultimately diverging when $\Delta = \pm\pi$. The last term is imaginary, and corresponds to a difference in gravitational self-energies for the two paths. This term seems to have the possibility of suppressing of the two-path amplitude, however this speculation is questionable. Since the gravitational self-energy for each path should be equal, we actually expect this term to vanish in general.

Ultimately, we see that the CWL interaction between the two paths is a non-trivial function of ΔS, with wild departures from standard gravitational interactions in regions where the amplitudes for each path are destructively interfering. Unfortunately, we do not yet have a complete interpretation of this strange behaviour.

It seems that something strange is happening in regions where destructive interference would occur. If we ignore the self-energy terms and focus only on the interactions between path (A) and (B), we see that the gravitational interaction between the paths grows arbitrarily large as the replicas head towards a location where the paths would destructively interfere. In our calculation we used an eikonal approximation which assumed that the gravitational field is too weak to change the path of the particle, and this approximation would be valid if it were not for this $\sec^2(\Delta S)$ enhancement of the gravitational interaction. To properly understand this enhanced gravitation for destructively interfering trajectories we would then need to go beyond our eikonal approximations and self-consistently deal with the attraction between the paths.

This is a confusing result, but it is fascinating to see that something strange happens in the 'highly quantum' region where destructive interference should occur. More work will need to be done to see precisely what the effect is for interference fringes, and to determine whether CWL actually predicts a suppression of such fringes.

11.2.3 Applications to the BMV experiment

At this point we can see what CWL would predict for the BMV experiment [13, 14]. For this we consider two particles, each undergoing their own two-path evolution as above. If we label the classical paths for particle 2 as C and D, the conventional QM propagator describing this system is (in the WKB approximation)

$$K_0 = \left(\sqrt{\Delta_{(A)}}\, e^{iS[q^{(A)}]} + \sqrt{\Delta_{(B)}}\, e^{iS[q^{(B)}}\right)\left(\sqrt{\Delta_{(C)}}\, e^{iS[q^{(C)}]} + \sqrt{\Delta_{(D)}}\, e^{iS[q^{(D)}]}\right). \tag{11.47}$$

For simplicity we can assume the set-up is totally symmetric so that all the van Vleck determinants equal the same quantity, Δ_0. In this case, the four branches of the wavefunction are clear to see

$$\begin{aligned} K_0 = \Delta_0\big(&e^{iS[q^{(A)}]+iS[q^{(C)}]} + e^{iS[q^{(A)}]+iS[q^{(D)}]} \\ &+e^{iS[q^{(B)}]+iS[q^{(C)}]} + e^{iS[q^{(B)}]+iS[q^{(D)}]}\big). \end{aligned} \tag{11.48}$$

Let us recall our discussion in section 5.2. If we define the the gravitational phase

$$\Phi[T^{(1)}, T^{(2)}] = \frac{G}{2}\int_{t_i}^{t_f} dt \int d^3x d^3y \frac{T^{(1)}(x)T^{(2)}(y)}{|\vec{x}-\vec{y}|}, \tag{11.49}$$

then conventional quantum gravity predicts a different phase accumulating on each of these four branches

$$\begin{aligned} K_0 = \Delta_0\big(&e^{iS[q^{(A)}]+iS[q^{(C)}]}e^{i\Phi[T^{(A)},T^{(C)}]} + e^{iS[q^{(A)}]+iS[q^{(D)}]}e^{i\Phi[T^{(A)},T^{(D)}]} \\ &+e^{iS[q^{(B)}]+iS[q^{(C)}]}e^{i\Phi[T^{(B)},T^{(C)}]} + e^{iS[q^{(B)}]+iS[q^{(D)}]}e^{i\Phi[T^{(B)},T^{(D)}]}\big), \end{aligned} \tag{11.50}$$

and generically this is an entangled state of the two particles.

To see what CWL predicts, we first compute $\langle\langle T_{00}\rangle\rangle_{21}$ for this system. The result is just the sum for each particle

$$\begin{aligned}\langle\langle T_{00}\rangle\rangle_{21} = \frac{T^{(A)}+T^{(B)}}{2} + i\frac{T^{(A)}-T^{(B)}}{2}\tan\left(\frac{S[q^{(A)}]-S[q^{(B)}]}{2}\right)\\ + \frac{T^{(C)}+T^{(D)}}{2} + i\frac{T^{(C)}-T^{(D)}}{2}\tan\left(\frac{S[q^{(C)}]-S[q^{(D)}]}{2}\right). \quad (11.51)\end{aligned}$$

In the proposal of Bose et al. one recombines the paths symmetrically, such that $S[q^{(A)}] = S[q^{(B)}]$, and $S[q^{(A)}] = S[q^{(B)}]$. In this case the imaginary terms vanish and $\langle\langle T_{\mu\nu}\rangle\rangle_{21}$ is just the sum of the average stress tensors for each particle. We can then compute the CWL propagator for this system using eq. (11.34). The result is

$$\begin{aligned}\mathcal{K} = \Delta_0\Big(e^{iS[q^{(A)}]+iS[q^{(C)}]} + e^{iS[q^{(A)}]+iS[q^{(D)}]} + e^{iS[q^{(B)}]+iS[q^{(C)}]} + e^{iS[q^{(B)}]+iS[q^{(D)}]}\Big)\\ \times \exp\left[\frac{i}{4}\Phi\Bigg[\sum_{j=A,\ldots,D} T^{(j)}, \sum_{k=A,\ldots,D} T^{(k)}\Bigg]\right]. \quad (11.52)\end{aligned}$$

The gravitational interaction is between all 4 paths.

The essential point to notice here is that there is not a different gravitational phase sitting on each of the four branches of the wavefunction, rather there is a single overall phase describing the mutual interaction of all four branches. We could then trivially factorize this result

$$\begin{aligned}\mathcal{K} = \Delta_0\Big(e^{iS[q^{(A)}]} + e^{iS[q^{(B)}]}\Big)\Big(e^{iS[q^{(C)}]} + e^{iS[q^{(D)}]}\Big)\\ \times \exp\left[\frac{i}{4}\Phi_G\Bigg[\sum_{j=A,\ldots,D} T^{(j)}, \sum_{k=A,\ldots,D} T^{(k)}\Bigg]\right], \quad (11.53)\end{aligned}$$

and conclude that the CWL theory predicts no entanglement will develop between the two particles as a result of gravity.

If the BMV proposal [13, 14], and/or its relatives [15, 23, 115], are actually experimentally feasible, then they will be able to distinguish between conventional quantum gravity and CWL theory.

11.3 Many particle composite object

We'll conclude our study of CWL by considering the most experimentally relevant system, a "composite body" consisting of a large number of bound particles.

One can motivate this by estimating the scale at which CWL effects may be significant. This was done already by Stamp in [113]. Consider two quantum particles of mass m, (a proxy for the CWL replicas), interacting via Newtonian gravity. We can immediately port over results from atomic physics to this gravitational "atom" to find the binding energy for this system,

$$E_G = -\frac{E_P}{2}\left(\frac{m}{M_P}\right)^5, \tag{11.54}$$

where $E_P \sim 10^{19}\,\mathrm{GeV}$ is the Planck energy and $M_P \sim 10^{-5}\,\mathrm{g} \approx 10^{19}\,\mathrm{AMU}$ is the Planck mass. From this we can see that the CWL binding energy will only reach the eV scale for particles with mass $m \approx 2 \times 10^{15}\,\mathrm{AMU}$. CWL theory should then be irrelevant at atomic scales.

The above estimate only considered point particles, and a detailed calculation will need to be done for a composite body, but it is at least suggestive that experiments studying the quantum mechanics of large objects may be perfect for testing CWL theory.

The kind of systems we have in mind here range over many length scales. There is currently enormous experimental effort going towards studying increasing massive quantum systems. We currently have three specific scales in mind, which we very roughly classify as the: "highly quantum", "weakly quantum", and "potentially quantum".[33]

Examples of the highly quantum regime:

- Matter-wave interferometry of large molecules with ≈ 2000 atoms, a total mass $\sim 10^4\,\mathrm{AMU}$, and displacements over a few hundred nanometers. [12]
- Generation of entangled phonon states in spatially separated 3-mm-sized diamonds [403]. The masses here are $\sim 10^{17}\mathrm{AMU}$ but the relative displacement in the superposition is only $\sim 10^{-11}\,\mathrm{m}$.

An example of a weakly quantum system:

- Optically trapped/levitated 150-nm-diameter silica nanoparticles, with masses up to $\sim 10^{11}\,\mathrm{AMU}$. These have been cooled to their center-of-mass ground states [21], and there are detailed proposals for creating and verifying both non-gaussian squeezed states [404] and delocalizations of order 10^5 times the zero-point motion [402].

[33] We do not use the term "weakly" here to degrade the experimental achievements, it is just that their systems are much more massive and the demonstrated properties are still near the gaussian regime. Likewise, we use the term "potentially quantum" only because there is existing debate about whether the system is actually quantum mechanical.

For the potentially quantum system:

- The LIGO detector—Despite the LIGO mirrors having masses of 40 kg, the LIGO scientific collaboration has used squeezed-input techniques [originally concieved by W.G. Unruh 405] to perform measurements below the 'standard quantum limit' coming from quantum fluctuations in the apparatus [26].

  It is claimed in [26] that this experiment demonstrates non-classical correlations between photons and the center-of-mass degree of freedom of the mirror. However this claim is questioned by many [406], some of whom are even co-authors on ref. [26]. We also note that older work by some of the originators of the LIGO design argued [407] that non-classical photon correlations do not require a quantum mechanical mirror coordinate, so it is unclear to us whether or not LIGO serves as a test of macroscopic quantum mechanics.

With this wide array of potential applications, we will proceed quite generally. In this section we study N-particle systems in the non-relativistic perturbative limit of CWL theory. We use collective coordinate methods to separate the center of mass of the system from the internal relative motions of its constituent particles. The internal motions can be integrated out, and this generates an effective CWL "path-bunching" action for the center of mass coordinate.

A proper microscopic theory is set-up and it involves the general shape of the object and the spectrum of its internal phonons. We make no attempt to perform a sophisticated calculation though, as this would require detailed numerics and should only be done once a specific experiment is being analyzed. To proceed we make the somewhat crude approximation which treats the fluctuations of each particle as independent of the others. This allows us to write a general expression for the effective action in terms of a sum over all particles in both the body and its replica.

Our central result is that the $1/r$ singularity of the Newtonian potential is nicely smoothed out at short distances, but $1/r$ behavior is still found at large distances. We close by comparing the predicted CWL timescales with relevant experimental timescales.

11.3.1 Technical Introduction

In the non-relativistic effective CWL path-integral we have only pairwise Newtonian interactions. For a single 'elementary' particle of mass m, we have the double path integral (11.36)

$$K(\beta,\alpha) = K_0(\beta,\alpha)^{-1}\int_\alpha^\beta \mathcal{D}q \int_\alpha^\beta \mathcal{D}q'\, e^{iS[q]+iS[q']}\Big(1 + iS_{CWL}[q,q']\Big) \tag{11.55}$$

where the CWL-bunching action is given in this approximation by

$$S_{CWL}[q,q'] = \frac{Gm^2}{2}\int_{t_1}^{t_2} dt \frac{1}{|q(t)-q'(t)|}. \tag{11.56}$$

We stress again, that this is just the $\mathcal{O}(\ell_P^2)$ perturbative result. We also note that similar calculations to those which follow have been done independently by Stamp [408].

Although the particle position q is a three-dimensional vector, we refer to $\int \mathcal{D}q$ as a **single** path integral. This is just a choice of language to keep the counting more simple as we introduce more particles.

If we generalize to a system of N-particles mutually interacting by some non-gravitational forces, as in eg. a solid body, the CWL theory in the ℓ_P^2 approximation would have $2N$ path integrals.

Collective Coordinate Method

We will first work with the conventional QM propagator for the N-particle system and then later generalize it to CWL.

We'll start with some standard 'collective coordinate' manipulations to set up the path integral more conveniently. Each particle is indexed by $j = 1, 2, ..., N$, and we assume the particles interact via a pairwise potential in a translation invariant background. This gives us the propagator

$$K_0(\{\beta_j\},\{\alpha\}_j) = \prod_{j=1}^{N}\int_{\alpha_j}^{\beta_j} \mathcal{D}q_j\, e^{i\int_{t_1}^{t_2} dt\left(\sum_{j=1}^N \frac{1}{2}m_i\dot{q}_j^2 - \frac{1}{2}\sum_{i\neq j} V(|q_i-q_j|)\right)}. \tag{11.57}$$

Note that the action is invariant under shifts of all the coordinates $q_j = R + r_j$, for any constant displacement R.

To factor out the center of mass (CoM) motion of the whole system we can multiply the path integral by

$$1 = \int \mathcal{D}R(t)\,\delta\Big(R(t) - \frac{1}{M}\sum_{j=1}^N m_j q_j\Big), \tag{11.58}$$

where $M = \sum_j m_j$, and change variables from $q_j(t)$ to $r_j(t) = q_j(t) - R(t)$. Now the path integral for the system reads

$$K_0 = \prod_{j=1}^{N} \int_{a_j}^{b_j} \mathcal{D}r_j \int_A^B \mathcal{D}R \, e^{i \int_{t_1}^{t_2} dt \left(\frac{1}{2} M\dot{R}^2 + \dot{R} \sum_j m_j \dot{r}_j \right.} \times e^{\left. \frac{1}{2} \sum_j m_j \dot{r}^2 - \frac{1}{2} \sum_{i \neq j} V(|r_i - r_j|) \right)} \delta\Big(\frac{1}{M} \sum_j m_j r_j\Big). \tag{11.59}$$

Here we've taken the boundary data $\{\alpha_j\}, \{\beta_j\}$ for the q_j and used them to define boundary data $\{a_j\}, \{b_j\}$ and A, B for the relative coordinates $\{r_j\}$ and the CoM coordinate R respectively. The relationships are

$$\begin{aligned} a_j &= \alpha_j - A, \\ b_j &= \beta_j - B, \\ A &= \frac{1}{M} \sum_j m_j \alpha_j, \\ B &= \frac{1}{M} \sum_j m_j \beta_j. \end{aligned} \tag{11.60}$$

It may seem like we've turned the N degrees of freedom describing $\{q_j\}$ into $N+1$ degrees of freedom describing $\{r_j\}$ and R, but we haven't really: there is a constraint in the $\{r_j\}$ path integral which enforces these to be relative coordinates. This constraint actually sets the cross term ($\sim \dot{R}\dot{r}$) in the action to zero. The CoM is then explicitly decoupled from the internal degrees of freedom and the propagator can be written as

$$K_0 = \int_A^B \mathcal{D}R \, e^{i \int_{t_1}^{t_2} dt \frac{1}{2} M\dot{R}^2} \times \prod_{j=1}^{N} \int_{a_j}^{b_j} \mathcal{D}r_j \, e^{i \int_{t_1}^{t_2} dt \left(\frac{1}{2} \sum_j m_j \dot{r}^2 - \frac{1}{2} \sum_{i \neq j} V(|r_i - r_j|) \right)} \delta\Big(\frac{1}{M} \sum_j m_j r_j\Big). \tag{11.61}$$

Phonons

At this stage we can go through a standard phonon discussion. We assume that the object is a solid body, so that the inter-particle potential is minimized when each of the particles are at a certain fixed locations, $\bar{r}_j$.

We'll further assume that the system admits a sensible expansion about this minimum, $r_j = \bar{r}_j + u_j$ such that the potential need only be expanded to quadratic order in the fluctuations u_j.

$$\begin{aligned} V(|r_i - r_j|) &= V(|\bar{r}_i - \bar{r}_j|) + \frac{1}{2}V''(|\bar{r}_i - \bar{r}_j|)(u_i - u_j)^2 + ... \\ &\equiv V_{0,ij} + \frac{1}{2}k_{ij}(u_i - u_j)^2 + \dots . \end{aligned} \tag{11.62}$$

Note that we will neglect the terms above quadratic order. These terms correspond to anharmonic phonon-phonon interactions. The relevant experiments will most likely take place at temperatures much less than the Debye temperature of the system, and in this regime the effect should be negligible because the phonon occupation number will be low.

In this fluctuation expansion we now have the propagator for our system

$$\begin{aligned} K_0 = &\int_A^B \mathcal{D}R\, e^{i\int_{t_1}^{t_2} dt \frac{1}{2} M\dot{R}^2} \\ &\times \prod_{j=1}^{N} \int_{a_j - \bar{r}_j}^{b_j - \bar{r}_j} \mathcal{D}u_j\, e^{i\int_{t_1}^{t_2} dt \left(\frac{1}{2}\sum_j m_j \dot{u}^2 - \sum_{i<j} \frac{1}{2} k_{ij}(u_i - u_j)^2\right)} \delta\big(\frac{1}{M}\sum_j m_j(\bar{r}_j + u_j)\big), \end{aligned} \tag{11.63}$$

where we've discarded the ground state energy term.

In this above discussion there is no coupling between the CoM coordinate and the relative coordinates, and this is because there are no forces in consideration other than the translation invariant pairwise potential between the various particles. In eq. (11.57) we have boundary data fixed for each particle's position, and as a result eq. (11.63) has boundary data fixed for the CoM position and each of the relative fluctuation positions. Since the CoM is decoupled, a more natural quantity to compute would be the transition amplitude between product states with definite CoM positions and some states $|\psi_\alpha\rangle, |\psi_\beta\rangle$ describing the fluctuations,

$$K_0 = \langle B|\langle\psi_\beta|e^{-iHt}|\psi_\alpha\rangle|A\rangle. \tag{11.64}$$

For example we could consider choosing the phonon vacuum, $|\psi_\alpha\rangle = |\psi_\beta\rangle = |0\rangle$. A more experimentally relevant choice however would be to assume a thermal state for the phonons, but this would then force us into using density matrices and computing a density matrix propagator for the CoM coordinate.

If we were to introduce a potential coupling directly to some of the particle coordinates, eg. a $\tilde{V} = \sum_j V_j(q_j)$. Then, for non-linear V_j we would induce coupling between the CoM and relative coordinates. This is precisely what will happen in CWL theory. We will introduce a gravitational potential depending on the q_j, and this will induce a coupling between the relative coordinate fluctuations and the CoM displacement. In the spirit of the renormalization group, we will then integrate out the assumed 'fast' fluctuations to generate an effective gravitational potential for the 'slow' CoM coordinate. This effective gravitational potential is expected to have no short distance singularity because of a 'smoothing' effect from the fast fluctuations.

In the standard high-energy QFT renormalization group context the term "integrate out" comes loaded with an assumption that the 'fast' or high-energy modes start and end in their vacuum state: there are no incoming or outgoing high-energy excitations, rather they appear only as internal states in Feynman graphs. Because of the assumption that the initial state and final state of the high-energy modes is i) known and ii) unaffected by the evolution of low-energy degrees of freedom, one can "integrate out" the high-energy modes without "tracing" them out and forcing an influence functional description for the effective low-energy evolution. For our mundane purposes here, if we want an effective theory for the CoM which is describes pure state evolution rather than density matrix evolution via an influence functional, we will need to make an analogous assumption. We will need to assume that even if the phonon variables couple to the CoM variable, the initial and final state for the phonons is known and unaffected by the evolution of the CoM.

In terms of eq. (11.64) we could summarize the above by making the point that the following equation defining an effective Hamiltonian $\tilde{H}$ is sensible

$$\langle B|e^{-i\tilde{H}t}|A\rangle = \langle B|\langle 0|e^{-iHt}|0\rangle|A\rangle, \tag{11.65}$$

whereas

$$\langle B|e^{-i\tilde{H}t}|A\rangle = \langle B|\mathrm{tr}_{phon} e^{-iHt}\rho_{phon}|A\rangle \tag{11.66}$$

is not. In terms of path integrals, we will write the phonons evolving from vacuum to vacuum as

$$K_0(B,0;A,0) = \int_A^B \mathcal{D}R\, e^{i\int_{t_1}^{t_2} dt \frac{1}{2}M\dot{R}^2} \prod_{j=1}^{N} \int_{|0\rangle}^{|0\rangle} \mathcal{D}u_j\, e^{iS[\{u_j\}]} \delta\Big(\frac{1}{M}\sum_j m_j(\bar{r}_j + u_j)\Big), \tag{11.67}$$

where

$$S[\{u_j\}] = \int_{t_1}^{t_2} dt \left(\frac{1}{2}\sum_j m_j \dot{u}^2 - \sum_{i<j} \frac{1}{2} k_{ij}(u_i - u_j) \right) \tag{11.68}$$

From a condensed matter perspective, the phonon vacuum approximation is actually reasonable. There is discussion of this point in Stamp's original paper [113], which we recapitulate here. In the relevant experiments the temperatures will be much smaller than the Debye temperature of the solid. As a consequence there will be a very low occupation number for the phonons, except at very long wavelengths. One can then straightforwardly see that the motion of a single ion in a solid composed entirely of identical ions with mass m will be

$$\langle x^2 \rangle = \frac{\hbar^2}{2m} \int \frac{dE}{E} g(E)[1 + 2n(E)] \tag{11.69}$$

where $g(E)$ is the phonon density of states, and $n(E)$ is the Bose distribution. In any real solid $g(E) \sim E^2$ at low E, and is cut off for energies $\theta_D = k_B T_D$, where $T_D \sim 100 - 800\,\mathrm{K}$ depending of the material. It then follows that the mean distance of the ion from it's equilibrium position is

$$\bar{x} \sim \frac{3}{2} \hbar (1/m\theta_D)^{1/2} \tag{11.70}$$

to very high accuracy; for many solids this would be accurate to less than 1% even at a temperature $\sim 50\,\mathrm{K}$.

11.3.2 Composite object in CWL theory

Lets now take eq. (11.67) and generalize it to describe an N-particle object in CWL theory. Looking at eq. (11.55) we do this by making two modifications. First we must double the path-integral, ie. we introduce variables q'_j, or equivalently R' and u'_j, which have the same path integral and action as the original variables. Next, we introduce a new CWL bunching term $S_{CWL}[\{q_j\}, \{q'_j\}]$ which couples every particle via gravity. We now have $2N$ particles to consider.

We effectively have two identical universes which we can call replica universes. There are N particles in one universe, and N corresponding particles in the replica universe. For consistency, every particle must couple gravitationally to every other particle, whether in the same replica universe or not. Of course, for particles in the same replica universe mutual gravitational interactions are irrelevant. They correspond to extremely small corrections to the spring constants k_{ij}. Thus, we'll need only to consider interactions which couple a particle in one replica universe to a particle in the other replica universe.

We can then write the CWL propagator as

$$
\begin{aligned}
K(B,0;A,0) &= K_0^{-1}\int_A^B \mathcal{D}R \int_A^B \mathcal{D}R'\, e^{i\int_{t_1}^{t_2} dt \frac{1}{2}M\dot{R}^2+\frac{1}{2}M\dot{R}'^2} \\
&\times \prod_{j,k=1}^{N}\int_{|0\rangle}^{|0\rangle}\mathcal{D}u_j \int_{|0\rangle}^{|0\rangle}\mathcal{D}u_k'\, e^{iS[\{u_j\}]+iS[\{u_k'\}]}\,(1+iS_{CWL}) \\
&\times \delta\big(\frac{1}{M}\sum_j m_j(\bar{r}_j+u_j)\big)\delta\big(\frac{1}{M}\sum_k m_k(\bar{r}_k+u_k')\big),
\end{aligned}
\tag{11.71}
$$

where the CWL bunching action is

$$
S_{CWL} = \sum_{j,k=1}^{N} Gm_jm_k\int_{t_1}^{t_2} dt\frac{1}{|R(t)+\bar{r}_j+u_j(t)-R'(t)-\bar{r}_k-u_k'(t)|}.
\tag{11.72}
$$

Let's focus on the CWL correction term. We can actually write it a little more conveniently by using the ket representation of the phonon vacuum

$$
\begin{aligned}
K^{(1)}(B,0;A,0) &= K_0^{-1}\int_A^B \mathcal{D}R \int_A^B \mathcal{D}R'\, e^{i\int_{t_1}^{t_2} dt \frac{1}{2}M\dot{R}^2+\frac{1}{2}M\dot{R}'^2} \\
&\times \sum_{j,k=1}^{N} Gm_jm_k\int_{t_1}^{t_2} dt\langle 0|\langle 0|\frac{1}{|R(t)+\bar{r}_j+\hat{u}_j(t)-R'(t)-\bar{r}_k-\hat{u}_k'(t)|}|0\rangle|0\rangle.
\end{aligned}
\tag{11.73}
$$

The goal is to evaluate the vacuum expectation value here to compute the effective CWL action.

The difficulty in evaluating this effective action is two-fold. Firstly the coordinates u_j are not the normal-mode coordinates, so the ground state wavefunction will not be a simple Gaussian function of each u_j. Secondly, even if we could evaluate this expectation value we would still need to evaluate the sum over the N^2 terms desribing the interactions of each particle with every particle in the other replica universe.

Evaluating the effective action

Assuming the solid body has a lattice structure, a proper decomposition into normal-modes would involve writing

$$
\vec{u}_{n,a}(t) = \sum_{i,\vec{q}} Q_i(\vec{q},t)\vec{\epsilon}_a^{\,i}(\vec{q})\frac{e^{i\vec{q}\cdot\vec{R}_n}}{\sqrt{m_a\mathcal{N}}},
\tag{11.74}
$$

where the index j for each particle in the solid has been expanded to n, a, where n indexes the unit cell within the lattice and a indexes the atom within the unit cell. Here i labels the phonon polarization, $\vec{q}$ is the lattice momentum, $\vec{\epsilon}_a^{\,i}$ is a polarization vector, $\vec{R}_n$ is the vector pointing to unit cell n, $\mathcal{N}$ is the total number of unit cells in the solid, and $Q_i(\vec{q}, t)$ is the amplitude of the phonon with polarization i and lattice momentum $\vec{q}$. It is these Q which diagonalize the phonon Hamiltonian, and they have corresponding eigenfrequencies $\omega_i(\vec{q})$.

In this form we can see exactly what the constraint delta function is doing; it sets $\sum_j m_j u_j$ equal to a constant, ie. it sets $\sum_j m_j \dot{u}_j = 0$. This constraint (really three, one for each spatial direction) eliminates the zero frequency longitudinal phonon from the spectrum of fluctuations. This happens because we've already separated out this 'Nambu-Goldstone mode' with an explicit description in terms of the variable R.

In terms of these normal-mode coordinates the phonon vacuum wavefunction is a product of Gaussian functions for each mode,

$$\Psi_0[\{Q_i(\vec{q})\}] = \prod_{i=1}^{3} \prod_{\vec{q}\in B.Z.} \left(\frac{\omega_i(\vec{q})}{\pi}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\omega_i Q_i^2(\vec{q})}, \tag{11.75}$$

where $\vec{q}$ ranges over the Brillouin zone. To evaluate the expectation value, we'd then first need to insert eq. (11.74) in for each of u_j and u'_j into eq. (11.73). Then, we'd need to replace the vacuum kets by

$$\Psi_0[\{Q_i(\vec{q})\}]\Psi_0^*[\{Q_i(\vec{q})\}]\Psi_0[\{Q'_{i'}(\vec{q}')\}]\Psi_0^*[\{Q'_{i'}(\vec{q}')\}] \tag{11.76}$$

and integrate over all values of $Q_i(\vec{q})$ and $Q'_i(\vec{q})$ for each $i, i', \vec{q}$, and $\vec{q}'$. While any individual integral would be manageable, the fact that each the denominator in the Newtonian potential would contain a sum involving every $Q_i(\vec{q})$ makes the calculation completely intractable in practice.

Approximate evaluation of the effective action

At this stage we'll make a very crude approximation to the phonon spectrum, as this may be the only way to proceed analytically. We'll assume that each of the atomic fluctuations $u_{n,a}$ in the solid body is independent from one another, and that each of these can be described as an isotropic simple harmonic oscillator in its ground state, with zero point motion $\langle \vec{u}_j^{\,2} \rangle = 3\sigma_j^2$. This then defines each oscillator frequency to be $\omega_j = (\sigma_j^2 m_j)^{-1}$.

We'll parameterize the size of these fluctuations as $\sigma_j = \eta_j a_0$, where a_0 is the typical inter-atomic spacing in the solid, and $\eta_j \in (0, 1)$ are parameters characterizing the size of the displacement fluctuations for particles in the solid relative to the typical nearest neighbor distance. With the above assumption, the ground state of the lattice fluctuations is then modeled as

$$\Psi[\{\vec{u}_j\}] = \prod_{j=1}^{N} \left(\frac{1}{\pi\sigma_j^2}\right)^{\frac{3}{4}} e^{-\frac{1}{2}\sigma_j^{-2}\vec{u}_j^2}. \tag{11.77}$$

Using this model of the lattice fluctuation background we can write the effective action as

$$\begin{aligned}\tilde{S}_{CWL} = \sum_{j,k=1}^{N} Gm_j m_k \left(\frac{1}{\pi\sigma_j^2}\right)^{\frac{3}{2}} \left(\frac{1}{\pi\sigma_k^2}\right)^{\frac{3}{2}} \int_{t_1}^{t_2} dt \\ \times \int d^3u_j \int d^3u'_k \; e^{-\frac{1}{2}\sigma_j^{-2}\vec{u}_j^2} e^{-\frac{1}{2}\sigma_k^{-2}\vec{u'}_k^2} \frac{1}{|R(t) + \bar{r}_j + u_j(t) - R'(t) - \bar{r}_k - u'_k(t)|}.\end{aligned} \tag{11.78}$$

We can organize this as a sum over the effective gravitational interactions of every atom,

$$\tilde{S}_{CWL} = -\int_{t_1}^{t_2} dt \sum_{j,k=1}^{N} Gm_j m_k V(\vec{\Delta}_{jk}), \tag{11.79}$$

where we've defined the displacement between the equilibrium positions of atoms j and k, $\vec{\Delta}_{jk} \equiv \vec{R} - \vec{R'} + \vec{\bar{r}}_j - \vec{\bar{r}}_k$, and defined the effective gravitational potential for each pair of atoms as

$$\begin{aligned}V(\vec{\Delta}_{jk}) = \\ -\left(\frac{1}{\pi\sigma_j^2}\right)^{\frac{3}{2}} \left(\frac{1}{\pi\sigma_k^2}\right)^{\frac{3}{2}} \int d^3u_j \int d^3u'_k \; e^{-\frac{1}{2}\sigma_j^{-2}\vec{u}_j^2} e^{-\frac{1}{2}\sigma_k^{-2}\vec{u'}_k^2} \frac{1}{|\vec{\Delta}_{jk} + \vec{u}_j(t) - \vec{u'}_k(t)|}.\end{aligned} \tag{11.80}$$

This integral is obviously more manageable than we would have had if tried to perform the proper analysis in terms of the phonon spectrum.

To proceed we conveniently use sum and relative variables

$$\begin{aligned}\vec{l} &= \vec{\Delta}_{jk} + \vec{u}_j - \vec{u'}_k \\ \vec{s} &= \vec{u_j} + \vec{u'}_k,\end{aligned} \tag{11.81}$$

and we will assume for simplicity that all of the atoms are identical. The integral is then

$$V(\vec{\Delta}_{jk}) = -\left(\frac{1}{\pi\sigma^2}\right)^3 \int d^3D \, e^{-2\sigma^{-2}\vec{D}^2} \int d^3l \, e^{-\frac{1}{2}\sigma^{-2}(\vec{l}-\vec{\Delta}_{jk})^2} \frac{1}{|\vec{l}|}. \tag{11.82}$$

The integrations are elementary, and the result is the very simple effective gravitational potential

$$V(\vec{\Delta}_{jk}) = -\frac{1}{|\vec{\Delta}_{jk}|}\mathrm{Erf}\left(\frac{|\vec{\Delta}_{jk}|}{\sqrt{2}\sigma}\right). \tag{11.83}$$

We've found after integrating out atomic fluctuations, albeit in an crude model, that the effective gravitational potential between atom j in one replica universe and atom k in the other replica universe is just the standard potential but modified with an error-function form factor. At small $|\vec{\Delta_{jk}}|$ the effective potential between atoms has the expansion

$$V(\vec{\Delta}_{jk}) = -\sqrt{\frac{2}{\pi\sigma^2}} + \frac{1}{3\sqrt{2\pi}\sigma^3}|\vec{\Delta}_{jk}|^2 + \ldots. \tag{11.84}$$

Thus the CWL interaction between an atom and a replica which are very close together is just a harmonic potential! It is important to remember though that this is **not** the effective potential between the two replica N-particle bodies, ie. the effective potential felt by the CoM coordinates. To obtain the CoM effective gravitational potential we still need to sum over all particles in each body,

$$\tilde{S}_{CWL} = \int_{t_1}^{t_2} dt \sum_{j,k=1}^{N} Gm_j m_k \frac{1}{|\vec{R}(t) - \vec{R}'(t) + \vec{\bar{r}}_j - \vec{\bar{r}}_k|}\mathrm{Erf}\left(\frac{|\vec{R}(t) - \vec{R}'(t) + \vec{\bar{r}}_j - \vec{\bar{r}}_k|}{\sqrt{2}\sigma}\right). \tag{11.85}$$

Small body numerical approximation

There is a large number (N^2) of terms in the sum (11.85). We can get a sense for the result by considering a simple model solid. We'll consider a cubic object with cubic lattice and L atoms per side length. In fig. 11.1 we present numerical plots of the effective potential for the cases $L = 7, 13$, and 33. The computation time scales as $N^2 \sim \mathrm{length}^6$, so it is quite costly to compute for objects of significantly larger size. Note that in the plots we are considering CoM separation along one of the lattice vectors. We will see in the plots that this generated a periodic array of spikes in the effective CWL potential, which are just artifacts of this particular alignment. More generally, if the angle between the lattice vector differs from that of the displacement vector by $\sqrt{\langle \Delta x^2 \rangle}/a_0 \approx 0.01$, then one expects only the central spike to persist.

Moreover, in an amorphous solid one will certainly only retain the central spike.

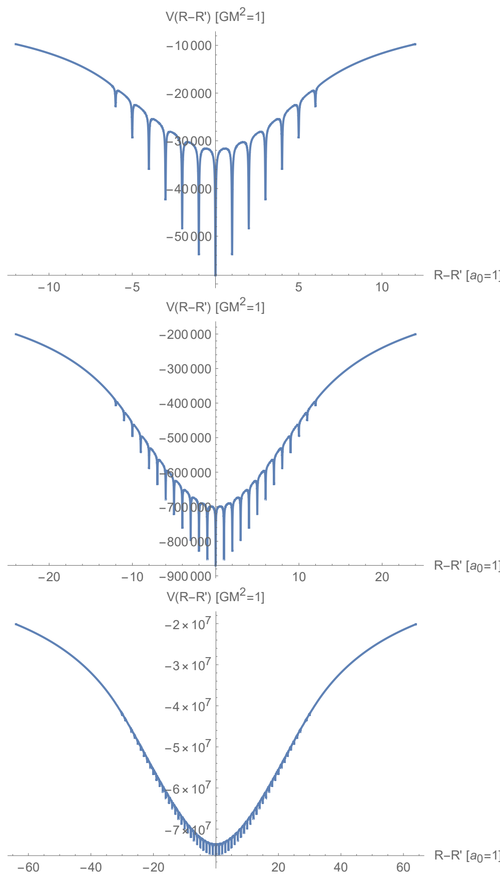


Figure 11.1: Plots of eq. (11.85), the effective CWL gravitational potential between cubic solids with cubic lattices with 7, 13, and 33 atoms per side length. The fluctuation parameter is $\eta = 0.01$, and the particle separation is presented in units of the nearest neighbor distance a_0. The units are such that $Gm^2 = 1$, where m is the atomic mass. The CoM separation is along one of the lattice vectors, so it is something of a special case and has a high degree of periodicity—hence the periodic spike structure in the plots.
.

Let's work to gain a qualitative understanding of these plots so that we can understand how they would scale for larger system size. There are really two important features to observe. All of the potentials take the form of a smooth well with a series of sharp spikes located at separations $|R-R'| = na_0$ for integer n, and $na_0 < 2L$. Additionally, these spikes have a size which decreases relative to the size of the smooth part of the potential as we increase the system size. This can be understood quite straightforwardly.

Lets look at eq. (11.85) at zero separation, $\vec{R}-\vec{R}'=0$. In the sum there are $N = L^3$ *diagonal* terms, ie. when $i = j$, and a particle is interacting with its own replica rather than a replica of another particle. For these diagonal contributions the Newtonian potential would diverge, but the error function smooths out the short distance behaviour and gives a $\frac{1}{\sigma}$ contribution. There are also $\mathcal{O}(N^2 = L^6)$ *off-diagonal* terms. For these terms the error function has large argument and is thus irrelevant. The contribution from the off-diagonal terms is the just that of classical gravitation between a particle at site i and all of the particles at sites $j \neq i$.

Quantitatively, at $\vec{R}-\vec{R}'=0$ we have the potential,

$$\tilde{V}(0) = -Gm^2 \sum_{i,j} \frac{1}{|\vec{r}_j - \vec{r}_k|} \mathrm{Erf}\left(\frac{|\vec{r}_j - \vec{r}_k|}{\sqrt{2}\sigma}\right). \tag{11.86}$$

We split it into diagonal and off-diagonal terms. If we use the Taylor series for the error function at small x, Erf(x) $\approx 2\pi^{-1/2}x$, and use the limiting value Erf(x) ≈ 1 at large x. This gives

$$\begin{aligned}\tilde{V}(0) &= -\frac{Gm^2}{\sigma}\sqrt{\frac{2}{\pi}}\left(\sum_i^N\right) - Gm^2 \sum_{i\neq j} \frac{1}{|\vec{r}_j - \vec{r}_k|} \\ &= -\frac{GMm}{\sigma}\sqrt{\frac{2}{\pi}} - \frac{Gm^2}{a_0}\sum_{\vec{n}\neq\vec{m}} \frac{1}{|\vec{n}-\vec{m}|},\end{aligned} \tag{11.87}$$

where we've defined the total object mass $M = mN$, and we've introduced $\vec{n}, \vec{m}$, which are vectors with integer valued components running over the lattice sites.

In the second term of eq. (11.87), we have L^6 terms and a length^{-1} contribution from each term. We then expect the sum to be of order $\mathcal{O}(L^5)$. We'll parameterize this contribution by defining the *geometric structure coefficient* γ as

$$\gamma = \frac{1}{L^5}\sum_{\vec{n}\neq\vec{m}} \frac{1}{|\vec{n}-\vec{m}|}. \tag{11.88}$$

As L, or equivalently as the volume V scales, γ should vary only weakly, and should reach a limiting value at large system size given by the integral limit

$$\gamma_{max} = V^{-5/3} \int_V d^3x \int_V d^3y \frac{1}{|\vec{x} - \vec{y}|}. \tag{11.89}$$

The geometrical structure coefficient primarily describes the shape of the object, not it's size. We've confirmed the weak scaling of γ with system size by directly evaluating the sum for a few examples and we found for $L = 7, 13, 19$ that $\gamma = 1.84, 1.87, 1.88$ respectively.

In terms of γ we can write the potential as

$$\begin{aligned}\tilde{V}(0) &= -\frac{GMm}{\sigma}\sqrt{\frac{2}{\pi}} - \frac{Gm^2}{a_0}L^5\gamma \\ &= -\frac{GMm}{\sigma}\sqrt{\frac{2}{\pi}} - \frac{GM^2}{La_0}\gamma. \end{aligned} \tag{11.90}$$

Since the second term comes from the off-diagonal contribution, ie. the gravitation from all other atoms, we expect it to generate the smooth part of the potential. Since the first term comes from the diagonal contribution, and would be absent if $\vec{R} - \vec{R}' \neq a_0\vec{n}$, it generates the spike.

The periodic array of spikes in the plots is then special, and comes about only because the displacement was assumed to be along a lattice vector. For general displacements we would only expect the central spike. Similarly for an amorphous solid, with no regular lattice structure, we would expect only a central spike for all displacements.

Note that the relative magnitude of the spike term to the smooth term is

$$\frac{V_{spike}}{V_{smooth}} = \frac{1}{N^{2/3}}\sqrt{\frac{2}{\pi}}\frac{1}{\eta\gamma}. \tag{11.91}$$

The fluctuation parameter η is typically small, $\eta = 0.01$, and the geometrical structure coefficient $\gamma \sim \mathcal{O}(1)$, so the prefactor in this ratio is numerically large. The scaling with system size however will suppress this ratio for large systems. For the cube $\gamma \approx 1.88$, we find that for $L = 7, 13, 33$ the ratio of spike size to smooth well size is $0.87, 0.25, 0.04$ respectively. At system size $L = 16$, the spikes are already small enough to be treated as perturbation of the smooth potential generated by the off-diagonal components.

Of course, the smooth part of the well is only relevant if the CoM separations are larger than $\sigma = 0.01a_0$. If only very small separations are relevant, then instead we may disregard the smooth part and consider only the the spike part of the potential.

We can verify this intuition in the following way. From the plots, the smooth part of the potential appears to have the shape

$$V_{smooth}(x) = -\frac{a}{x}\mathrm{Erf}\left(bx\right), \tag{11.92}$$

for constants a, b. We can determine these constants uniquely by matching the asymptotic behaviors and the minimum values of the functions. At large separation it is clear from eq. (11.85) that the true effective potential should be approximately

$$\tilde{V}(\vec{R}) \approx -\sum_{j,k=1}^{N} Gm_j m_k \frac{1}{|\vec{R}|}\mathrm{Erf}\left(\frac{|\vec{R}|}{\sqrt{2}\sigma}\right) = -\frac{GM^2}{|\vec{R}|}. \tag{11.93}$$

This determines the fit parameter $a = GM^2$. At zero separation we have

$$V_{smooth}(0) = -2\pi^{-1/2}ab, \tag{11.94}$$

and we can match this with the off-diagonal contribution in eq. (11.90),

$$-2\pi^{-1/2}ab = -\frac{GM^2}{La_0}\gamma. \tag{11.95}$$

This determines the remaining fit parameter b. We then have a fit to the smooth part of the function equal to

$$V_{smooth}(\vec{R}) = -\frac{GM^2}{|\vec{R}|}\mathrm{Erf}\left(\frac{\sqrt{\pi}}{2}\gamma\frac{|\vec{R}|}{a_0 L}\right). \tag{11.96}$$

This fit is quite good, as we see in fig. 11.2:

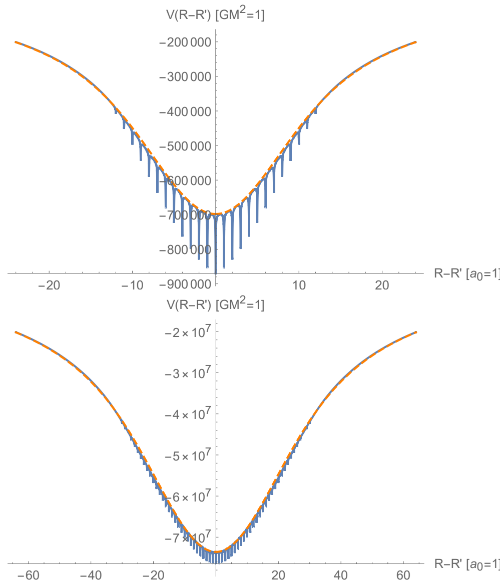


Figure 11.2: Plots of the effective CWL gravitational potential between cubic solids with cubic lattices with 13 and 33 atoms per side length. The solid blue curve is the true potential eq. (11.85), the dashed orange curve is a plot of the "smooth" or "off-diagonal" part eq. (11.96).
.

Approximate Harmonic Well of the Smooth Part

Now that we've approximated the effective potential by the smooth part eq. (11.96), we can extract an effective harmonic potential. From the Taylor series we find

$$V_{smooth}(\vec{R}) \approx -\frac{GM^2}{a_0 L}\gamma + \frac{\pi}{12}\gamma^3 G M \rho_{avg}|\vec{R}|^2, \tag{11.97}$$

where we've defined the average density of the object

$$\rho_{avg} = \frac{M}{(a_0 L)^3}. \tag{11.98}$$

From this we can then extract an effective oscillator frequency

$$\omega_{eff} = \left(\frac{\pi}{6}\gamma^3 G \rho_{avg}\right)^{1/2}. \tag{11.99}$$

The most important observation about this result, is that it is not determined by the total mass of the object, only its average density. This is somewhat discouraging if we hope to make experimental comparison.

Approximate Harmonic Well of the Spike

As mentioned previously, if experimental parameters are such that the separation between the CoM each of the replicas is always very small, ie. $|\vec{R} - \vec{R}'| \ll \sigma = 0.01 a_0$, then the effective CWL potential is just that of the central spike.

The spikes themselves are well modeled by a potential, $V_{spike} = -(a/x)\mathrm{Erf}(x/(\sqrt{2}\sigma))$ for constant a to be determined. To fit this, we only need to look at the value of the spike potential at the bottom of the well. Since we've already worked this out (eq. (11.87)) we just do some matching

$$V_{spike}(0) = \frac{2a}{\sqrt{2\pi}\sigma} - \frac{GMm}{\sigma}\sqrt{\frac{2}{\pi}}, \tag{11.100}$$

to determine $a = GMm$. For small separations the spike potential is then well approximated as

$$V_{spike}(|\vec{R}|) = -\frac{GMm}{|\vec{R}|}\mathrm{Erf}\left(\frac{|\vec{R}|}{\sqrt{2}\sigma}\right) = -\frac{GMm}{\sigma}\sqrt{\frac{2}{\pi}} + \frac{GMm}{3\sqrt{2\pi}\sigma^3}|\vec{R}|^2 + \dots . \tag{11.101}$$

We confirm this is a nice approximation by overlaying this fit with a zoom in on the exact $L = 33$ potential (fig. 11.3).

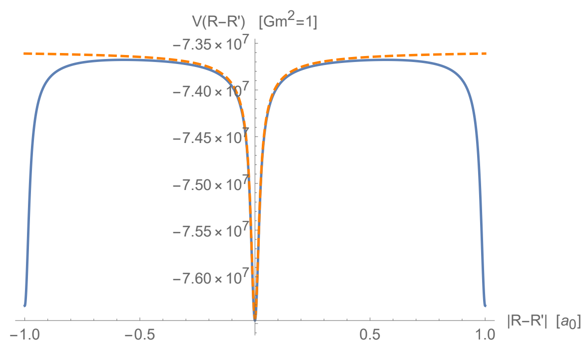


Figure 11.3: Plots of the effective CWL gravitational potential between cubic solids with cubic lattices with 33 atoms per side length. The solid blue curve is the true potential eq. (11.85), the dashed orange curve is a plot of the "spike" or "diagonal" part eq. (11.101). The fluctuation parameter is $\eta = 0.01$, and the particle separation is presented in units of the nearest neighbor distance a_0. The units are such that $Gm^2 = 1$, where m is the atomic mass.
.

From eq. (11.101) we can look at the quadratic part and extract an effective harmonic oscillator frequency for the CoM CWL potential

$$\omega_{eff} = \left(\sqrt{\frac{2}{\pi}}\frac{Gm}{3\sigma^3}\right)^{\frac{1}{2}}. \tag{11.102}$$

This too is independent of the total mass of the solid, however it has a significant larger effective density than the previous result valid for larger separations.[34]

11.3.3 Summary and Experimental Prospects

If we ignore the off-center spikes (since they are not generic), we then have the total effective CWL potential for the center of mass of the composite

[34]After preparing this section we came across the paper of Bassi et al. [409]. They consider the same system, except in the context of the Schrödinger-Newton equation for a self-gravitating wavefunction. Since we were unaware of this result, our calculation was independently done, however our result eq. (11.102) is exactly equal to the Schrödinger-Newton frequency they compute in their eq. (24).

bodies

$$\tilde{V}(|\vec{R}|) = -\frac{GMm}{|\vec{R}|}\mathrm{Erf}\left(\frac{|\vec{R}|}{\sqrt{2}\sigma}\right) - \frac{GM^2}{|\vec{R}|}\mathrm{Erf}\left(\frac{\sqrt{\pi}}{2}\gamma\frac{|\vec{R}|}{a_0 L}\right), \tag{11.103}$$

and this is characterized by two frequency scales: the slow

$$\omega_{eff} = \left(\frac{\pi}{6}\gamma^3 G\rho_{avg}\right)^{1/2}, \tag{11.104}$$

and the relatively fast

$$\omega_{eff} = \left(\sqrt{\frac{2}{\pi}}\frac{Gm}{3\sigma^3}\right)^{\frac{1}{2}}. \tag{11.105}$$

Let us close by comparing these frequencies with the experiments discussed in the introduction.

For both the levitated nanoparticles and the LIGO interferometer the masses are made primarily of silica. We can get an order of magnitude understanding of the CWL prediction by using $\sigma = 0.01*160\,\mathrm{pm}$ and $m = \frac{1}{3}*60\,\mathrm{g/mol}$, and the average density of $2.65\,\mathrm{g/cm}^3$. This gives a characteristic CWL frequency and timescale for the narrow spike part of the effective potential

$$f^{\mathrm{SiO_2}}_{spike} \approx 60.4\,\mathrm{mHz} \qquad \tau^{\mathrm{SiO_2}}_{spike} \approx 16\,\mathrm{s}, \tag{11.106}$$

and

$$f^{\mathrm{SiO_2}}_{smooth} \approx 0.78\,\mathrm{mHz} \qquad \tau^{\mathrm{SiO_2}}_{spike} \approx 21.2\,\mathrm{min}. \tag{11.107}$$

for the wide part of the potential.

These frequencies are many orders of magnitude smaller than the frequency of the harmonic well that the nanoparticle sits in, $(80 - 305)\,\mathrm{kHz}$ depending on the direction of displacement [21].

As for LIGO, we aren't quite sure how to compare, but we note that the interferometer was designed to detect strains in the frequency range $(10\,\mathrm{Hz} - 10\,\mathrm{kHz})$ [25]. This too is orders of magnitude faster than the CWL frequency. We note though, that the proposed space-based gravitational wave detector LISA is planned to be sensitive to frequencies in the range $(0.1\,\mathrm{mHz} - 1\,\mathrm{Hz})$ [24]. This would actually be the perfect frequency band to be sensitive to the predicted CWL frequency for the mirrors. What we've done thus far is very approximate, but, given that the LISA parameters are so near to our estimates, it is time for serious calculations to be performed which predict a CWL signature in the readout of the LISA experiment.

We note, however, that the LISA experiment will not use squeezed light to subvert the standard quantum limit. The utility of squeezed light in LIGO is central to the arguments that the experiment probes the quantum state of the 40 kg mirrors. Thus without squeezed light it is unclear whether LISA can actually probe CWL theory. In future research we plan to address these finer points more carefully.

Chapter 12

Conclusions

In this thesis we have discussed quantum gravity from many different angles. Ultimately we were interested in contributing towards an understanding of upcoming low-energy quantum gravity experiments, but we took a number of tangents to also address important technical problems which were related to our goal.

In Part I of this thesis we focused on conventional quantum gravity, as well as QED and non-abelian Yang-Mills theory. In chapter 3 we expanded on the discussion of decoherence via graviton emission presented in [117]. We developed a general framework for computing decoherence in systems where the central system and environment have a large separation of scales. We then applied the general framework to graviton emission in scattering processes. The primary results were that soft gravitons are completely correlated with the outgoing matter from scattering processes, and that tracing them out would completely decohere the system. These findings were in agreement with the findings of the Semenoff group, who performed a diagrammatic re-summation [121]. We then explained the results using density matrix Ward identities. Finally we commented on the relationships to the coherent state formalism and the black hole information problem.

In chapter 4 we developed methods for understanding manifestly gauge-invariant path integrals. The main utility here was to provide a path-integral first approach to understanding constraints in gauge theories. One application of this was to understand the nature of electric fields around quantum charged particles. We looked at a number of examples, including: non-relativistic scalars, Dirac fermions, and massless scalars. Our discussion naturally addressed the ideas of "large gauge transformations", and the relationship with the coherent state formalism.

In chapter 5 we reported on how some of the results from chapter 4 extend to linearized gravity. The results were not new, but we used them to discuss a controversy surrounding a proposed quantum gravity experiment [13, 14]. The experiment is meant to demonstrate the quantum nature of the gravitational field, however some claim that it does not actually probe the true gravitational degrees of freedom [123, 124]. We applied the gauge-invariance tools to substantiate these latter claims, and sharpen the arguments on each side. Finally, we agreed with those who proposed the experiment that it could indeed test conventional quantum gravity. We presented a set of arguments which imply that conventional quantum gravity is the only consistent theory which could yield a positive result in the experiment.

In chapter 6 we took a tangent to study the consequences of the constraint equation in Yang-Mills theory. We demonstrated that a vacuum gluon condensate will, according to the constraint equation, lead to quark confinement [125].

Part II of this thesis exclusively discussed the Correlated Worldline theory of quantum gravity.

We started with a very brief introduction to the theory in chapter 7. In chapter 8 we made a start on proper calculations by setting up a perturbative expansion in powers of the gravitational coupling ℓ_P^2. We evaluated the contributions up to leading order, and the main results were: i) conventional quantum gravity loop contributions are suppressed by undetermined parameters of the theory, ii) novel CWL contributions arise, but at this order they simply reproduce the contributions from conventional quantum gravity tree-diagrams, and iii) the theory retains the diffeomorphism symmetry of conventional quantum gravity.

In chapter 9 we tried to define the theory in terms of an infinitely replicated Hilbert space. In doing so, we were able to address how QM states and operators are embedded in CWL, fix the undetermined parameters, redefine the theory to simplify redundancies, define transition amplitudes. We later reported on some work in its infancy, as we tried to discuss measurements and the relation of CWL to a non-linear QM theory.

In chapter 10 we illustrated that CWL theory has an intrinsic large-N limit, and we used this fact to drastically simplify calculations. We demonstrated the equivalence of CWL to an "in-out" semiclassical gravity theory, and computed exact expressions for various correlations functions. From our results it became clear that CWL was likely renormalizable.

In chapter 11 we finally took the formal developments and applied them to calculations for simple physical systems. We showed that quantum mechanical oscillators are unaffected by CWL when in gaussian states. We discussed a general non-relativistic particle propagator and showed that there are no leading order CWL effects when there is a single dominant classical path. For a two-path system though, we demonstrated how the path bunching mechanism emerges. We then studied the gravitationally-mediated-entanglement experimental apparatus and determined that CWL will predict a null result for their experiment. Finally, we applied CWL theory to a many-particle object. For an object made of silica, as are the LIGO mirrors, we predicted relevant CWL corrections near the frequency band $(0.8 - 60)\,\mathrm{mHz}$. This is irrelevant to most current experiments, but right within the intended sensitivity band of LISA, giving some hope that the project may be able to test the CWL theory.

Bibliography

[1] A. Mari, G. De Palma, and V. Giovannetti, "Experiments testing macroscopic quantum superpositions must be slow," *Scientific Reports*, vol. 6, p. 22777, Mar 2016.

[2] A. Belenchia, R. M. Wald, F. Giacomini, E. Castro-Ruiz, Č. Brukner, and M. Aspelmeyer, "Quantum superposition of massive objects and the quantization of gravity," *Phys. Rev. D*, vol. 98, p. 126009, Dec 2018.

[3] A. Belenchia, R. M. Wald, F. Giacomini, E. Castro-Ruiz, Č. Brukner, and M. Aspelmeyer, "Information content of the gravitational field of a quantum superposition," *International Journal of Modern Physics D*, vol. 28, no. 14, p. 1943001, 2019.

[4] R. Feynman *et al.*, "Session viii sections 22,23," in *Report from Chapel Hill conference* (C. DeWitt and D. Rickles, eds.), 1957.

[5] F. Karolyhazy, "Gravitation and quantum mechanics of macroscopic objects," *Il Nuovo Cimento A (1965-1970)*, vol. 42, no. 2, pp. 390–402, 1966.

[6] F. Károlyházy, A. Frenkel, and B. Lukács, "On the possible role of gravity in the reduction of the wave function," in *Quantum Concepts in Space and Time* (R. Penrose and C. J. Isham, eds.), pp. 1–109, Oxford University Press, 1986.

[7] T. W. B. Kibble, "Is a Semiclassical Theory of Gravity Viable?," in *Quantum Gravity 2: A Second Oxford Symposium* (C. Isham, R. Penrose, and D. Sciama, eds.), Oxford Univeristy Press, 1980.

[8] T. W. B. Kibble and S. Randjbar-Daemi, "Nonlinear Coupling of Quantum Theory and Classical Gravity," *J. Phys. A*, vol. 13, p. 141, 1980.

[9] L. Diósi, "A universal master equation for the gravitational violation of quantum mechanics," *Physics Letters A*, vol. 120, no. 8, pp. 377–381, 1987.

[10] L. Diósi, "Models for universal reduction of macroscopic quantum fluctuations," *Phys. Rev. A*, vol. 40, pp. 1165–1174, 1989.

[11] R. Penrose, "On gravity's role in quantum state reduction," *Gen. Rel. Grav.*, vol. 28, pp. 581–600, 1996.

[12] Y. Y. Fein, P. Geyer, P. Zwick, F. Kiałka, S. Pedalino, M. Mayor, S. Gerlich, and M. Arndt, "Quantum superposition of molecules beyond 25 kDa," *Nature Physics*, vol. 15, no. 12, pp. 1242–1245, 2019.

[13] S. Bose, A. Mazumdar, G. W. Morley, H. Ulbricht, M. Toroš, M. Paternostro, A. A. Geraci, P. F. Barker, M. S. Kim, and G. Milburn, "Spin entanglement witness for quantum gravity," *Phys. Rev. Lett.*, vol. 119, p. 240401, Dec 2017.

[14] C. Marletto and V. Vedral, "Gravitationally induced entanglement between two massive particles is sufficient evidence of quantum effects in gravity," *Phys. Rev. Lett.*, vol. 119, p. 240402, Dec 2017.

[15] D. Carney, H. Müller, and J. M. Taylor, "Testing quantum gravity with interactive information sensing," 2021, arXiv:2101.11629 [quant-ph].

[16] D. Kleckner, I. Pikovski, E. Jeffrey, L. Ament, E. Eliel, J. van den Brink, and D. Bouwmeester, "Creating and verifying a quantum superposition in a micro-optomechanical system," *New Journal of Physics*, vol. 10, no. 9, p. 095020, 2008.

[17] M. J. Weaver, D. Newsom, F. Luna, W. Löffler, and D. Bouwmeester, "Phonon interferometry for measuring quantum decoherence," *Phys. Rev. A*, vol. 97, p. 063832, 2018.

[18] O. Romero-Isart, A. C. Pflanzer, F. Blaser, R. Kaltenbaek, N. Kiesel, M. Aspelmeyer, and J. I. Cirac, "Large quantum superpositions and interference of massive nanometer-sized objects," *Phys. Rev. Lett.*, vol. 107, p. 020405, 2011.

[19] M. Aspelmeyer, T. J. Kippenberg, and F. Marquardt, "Cavity optomechanics," *Rev. Mod. Phys.*, vol. 86, pp. 1391–1452, 2014.

[20] R. Kaltenbaek, M. Aspelmeyer, P. F. Barker, *et al.*, “Macroscopic quantum resonators (MAQRO): 2015 update,” *EPJ Quantum Technology*, vol. 3, p. 5, Mar 2016.

[21] U. Delić, M. Reisenbauer, K. Dare, D. Grass, V. Vuletić, N. Kiesel, and M. Aspelmeyer, “Cooling of a levitated nanoparticle to the motional quantum ground state,” *Science*, vol. 367, no. 6480, pp. 892–895, 2020.

[22] T. Westphal, H. Hepach, J. Pfaff, and M. Aspelmeyer, “Measurement of gravitational coupling between millimeter-sized masses,” 2021, arXiv:2009.09546 [gr-qc].

[23] R. Howl, V. Vedral, D. Naik, M. Christodoulou, C. Rovelli, and A. Iyer, “Non-gaussianity as a signature of a quantum theory of gravity,” *PRX Quantum*, vol. 2, p. 010325, Feb 2021.

[24] P. Amaro-Seoane et al., “Low-frequency gravitational-wave science with eLISA/NGO,” *Classical and Quantum Gravity*, vol. 29, no. 12, p. 124016, 2012.

[25] D. V. Martynov et al., “Sensitivity of the advanced ligo detectors at the beginning of gravitational wave astronomy,” *Phys. Rev. D*, vol. 93, p. 112004, Jun 2016.

[26] H. Yu et al., “Quantum correlations between light and the kilogram-mass mirrors of ligo,” *Nature*, vol. 583, pp. 43–47, Jul 2020.

[27] S. Carlip, “Quantum gravity: a progress report,” *Reports on Progress in Physics*, vol. 64, no. 8, pp. 885–942, 2001.

[28] M. Gell-Mann, M. L. Goldberger, and W. E. Thirring, “Use of causality conditions in quantum theory,” *Phys. Rev.*, vol. 95, pp. 1612–1627, Sep 1954.

[29] P. G. Bergmann, ““Gauge-invariant’’ variables in general relativity,” *Phys. Rev.*, vol. 124, pp. 274–278, Oct 1961.

[30] P. G. Bergmann, “Observables in general relativity,” *Rev. Mod. Phys.*, vol. 33, pp. 510–514, Oct 1961.

[31] C. G. Torre, “Gravitational observables and local symmetries,” *Phys. Rev. D*, vol. 48, pp. R2373–R2376, Sep 1993.

[32] W. Donnelly and S. B. Giddings, "Diffeomorphism-invariant observables and their nonlocal algebra," *Phys. Rev. D*, vol. 93, p. 024030, Jan 2016.

[33] W. Donnelly and S. B. Giddings, "Observables, gravitational dressing, and obstructions to locality and subsystems," *Phys. Rev. D*, vol. 94, p. 104038, Nov 2016.

[34] A. Ashtekar, *Asymptotic quantization: Based on 1984 Naples Lectures.* Bibliopolous P. (Monographs and Textbooks in Physical Science, 2), 1987.

[35] E. Witten, "Anti de sitter space and holography," *Adv. Theor. Math. Phys.*, vol. 2, pp. 253–291, 1998.

[36] S. Gubser, I. Klebanov, and A. Polyakov, "Gauge theory correlators from non-critical string theory," *Physics Letters B*, vol. 428, no. 1, pp. 105–114, 1998.

[37] A. Hamilton, D. Kabat, G. Lifschytz, and D. A. Lowe, "Local bulk operators in AdS/CFT correspondence: A holographic description of the black hole interior," *Phys. Rev. D*, vol. 75, p. 106001, May 2007.

[38] D. Harlow, "TASI Lectures on the Emergence of the Bulk in AdS/CFT," 2018, arXiv:1802.01040 [hep-th].

[39] B. S. DeWitt, "Quantization of fields with infinite-dimensional invariance groups. iii. generalized schwinger-feynman theory," *Journal of Mathematical Physics*, vol. 3, no. 6, pp. 1073–1093, 1962.

[40] B. S. DeWitt, *Dynamical theory of groups and fields.* Gordon and Breach, 1965.

[41] B. S. DeWitt, "Quantum theory of gravity. ii. the manifestly covariant theory," *Phys. Rev.*, vol. 162, pp. 1195–1239, Oct 1967.

[42] B. S. DeWitt, "Quantum theory of gravity. iii. applications of the covariant theory," *Phys. Rev.*, vol. 162, pp. 1239–1256, Oct 1967.

[43] S. Weinberg, *Ultraviolet divergences in quantum theories of gravitation.* Cambridge University Press.

[44] G. 't Hooft and M. J. G. Veltman, "One loop divergencies in the theory of gravitation," *Ann. Inst. H. Poincare Phys. Theor. A*, vol. 20, pp. 69–94, 1974.

[45] M. H. Goroff and A. Sagnotti, "The ultraviolet behavior of einstein gravity," *Nuclear Physics B*, vol. 266, no. 3, pp. 709–736, 1986.

[46] C. Rovelli, "Loop quantum gravity," *Living Reviews in Relativity*, vol. 1, p. 1, Jan 1998.

[47] A. Ashtekar, J. Olmedo, and P. Singh, "Quantum extension of the Kruskal spacetime," *Phys. Rev. D*, vol. 98, p. 126003, Dec 2018.

[48] A. Ashtekar, J. Olmedo, and P. Singh, "Quantum transfiguration of Kruskal black holes," *Phys. Rev. Lett.*, vol. 121, p. 241301, Dec 2018.

[49] J. Polchinski, *String theory. Vol. 1: An introduction to the bosonic string*. Cambridge Monographs on Mathematical Physics, Cambridge University Press, 2007.

[50] D. J. Gross and E. Witten, "Superstring modifications of Einstein's equations," *Nuclear Physics B*, vol. 277, pp. 1–10, 1986.

[51] A. Giveon, M. Porrati, and E. Rabinovici, "Target space duality in string theory," *Physics Reports*, vol. 244, no. 2, pp. 77–202, 1994.

[52] J. F. Donoghue, "General relativity as an effective field theory: The leading quantum corrections," *Phys. Rev. D*, vol. 50, pp. 3874–3888, Sep 1994.

[53] J. F. Donoghue, "Perturbative dynamics of quantum general relativity," in *Recent Developments in Theoretical and Experimental General Relativity, Gravitation, and Relativistic Field Theories (Proc. 8th Marcel Grossmann Meeting on General Relativity)* (T. Piran and R. Ruffini, eds.), 1999.

[54] C. P. Burgess, "Quantum gravity in everyday life: General relativity as an effective field theory," *Living Reviews in Relativity*, vol. 7, p. 5, Apr 2004.

[55] S. Weinberg, *The Quantum Theory of Fields*, vol. 1. Cambridge University Press, 1995.

[56] K. G. Wilson, "Renormalization group and critical phenomena. ii. phase-space cell analysis of critical behavior," *Phys. Rev. B*, vol. 4, pp. 3184–3205, Nov 1971.

[57] K. S. Stelle, "Classical gravity with higher derivatives," *General Relativity and Gravitation*, vol. 9, pp. 353–371, Apr 1978.

[58] M. Born and R. H. Fowler, "On the quantum theory of the electromagnetic field," *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character*, vol. 143, no. 849, pp. 410–437, 1934.

[59] M. Born, L. Infeld, and R. H. Fowler, "Foundations of the new field theory," *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character*, vol. 144, no. 852, pp. 425–451, 1934.

[60] A. A. Tseytlin, "Born-Infeld Action, Supersymmetry and String Theory," in *Many Faces of the Superworld: YURI GOLFAND MEMORIAL VOL*, pp. 417–452, 2000.

[61] A. O. Barvinsky and G. A. Vilkovisky, "The Generalized Schwinger-Dewitt Technique in Gauge Theories and Quantum Gravity," *Phys. Rept.*, vol. 119, pp. 1–74, 1985.

[62] N. E. J. Bjerrum-Bohr, J. F. Donoghue, and P. Vanhove, "On-shell techniques and universal results in quantum gravity," *Journal of High Energy Physics*, vol. 2014, p. 111, Feb 2014.

[63] J. F. Donoghue, "Low energy theorems of quantum gravity from effective field theory," *J. Phys. G: Nucl. Part. Phys.*, vol. 42, p. 103102, 2015.

[64] J. F. Donoghue, B. R. Holstein, B. Garbrecht, and T. Konstandin, "Quantum corrections to the ReissnerNordström and KerrNewman metrics," *Physics Letters B*, vol. 529, no. 1, pp. 132–142, 2002.

[65] N. E. J. Bjerrum-Bohr, J. F. Donoghue, and B. R. Holstein, "Quantum corrections to the Schwarzschild and Kerr metrics," *Phys. Rev. D*, vol. 68, p. 084005, Oct 2003.

[66] B. R. Holstein, "Metric modifications for a massive spin 1 particle," *Phys. Rev. D*, vol. 74, p. 084030, Oct 2006.

[67] N. E. J. Bjerrum-Bohr, J. F. Donoghue, and B. R. Holstein, "Quantum gravitational corrections to the nonrelativistic scattering potential of two masses," *Phys. Rev. D*, vol. 67, p. 084033, Apr 2003.

[68] N. E. J. Bjerrum-Bohr, J. F. Donoghue, B. R. Holstein, L. Planté, and P. Vanhove, "Bending of light in quantum gravity," *Phys. Rev. Lett.*, vol. 114, p. 061301, Feb 2015.

[69] N. E. J. Bjerrum-Bohr, J. F. Donoghue, B. R. Holstein, L. Plante, and P. Vanhove, "Light-like Scattering in Quantum Gravity," *JHEP*, vol. 11, p. 117, 2016.

[70] W. G. W. Richard P. Feynman, Fernando B. Morinigo, *Feynman Lectures on Gravitation.* Addison-Wesley, 1995.

[71] N. Rosen, "The relation between classical and quantum mechanics," *American Journal of Physics*, vol. 32, no. 8, pp. 597–600, 1964.

[72] C. Møller, "Colloques internationaux CNRS," *CNRS, Paris*, 1962.

[73] L. Rosenfeld, "On quantization of fields," *Nuclear Physics*, vol. 40, pp. 353–356, 1963.

[74] T. W. B. Kibble, "Relativistic models of nonlinear quantum mechanics," *Communications in Mathematical Physics*, vol. 64, no. 1, pp. 73 – 82, 1978.

[75] T. W. B. Kibble and S. Randjbar-Daemi, "Nonlinear Coupling of Quantum Theory and Classical Gravity," *J. Phys. A*, vol. 13, p. 141, 1980.

[76] S. Carlip, "Is quantum gravity necessary?," *Classical and Quantum Gravity*, vol. 25, no. 15, p. 154010, 2008.

[77] A. Bassi, K. Lochan, S. Satin, T. P. Singh, and H. Ulbricht, "Models of wave-function collapse, underlying theories, and experimental tests," *Rev. Mod. Phys.*, vol. 85, pp. 471–527, Apr 2013.

[78] A. Bassi, A. Großardt, and H. Ulbricht, "Gravitational decoherence," *Classical and Quantum Gravity*, vol. 34, no. 19, p. 193002, 2017.

[79] D. Carney, P. C. E. Stamp, and J. M. Taylor, "Tabletop experiments for quantum gravity: a user's manual," *Classical and Quantum Gravity*, vol. 36, no. 3, p. 034001, 2019.

[80] S. Donadi, K. Piscicchia, C. Curceanu, L. Diósi, M. Laubenstein, and A. Bassi, "Underground test of gravity-related wave function collapse," *Nature Physics*, vol. 17, pp. 74–78, Jan 2021.

[81] S. Adler in *Quatum Nonlocality and Reality: 50 Years of Bell's Theorem* (M. Bell and S. Gao, eds.), Cambridge: Cambridge University Press, 2016.

[82] G. Gasbarri, M. Toroš, S. Donadi, and A. Bassi, "Gravity induced wave function collapse," *Phys. Rev. D*, vol. 96, p. 104013, Nov 2017.

[83] N. V. Prokof'ev and P. C. E. Stamp, "Theory of the spin bath," *Reports on Progress in Physics*, vol. 63, no. 4, pp. 669–726, 2000.

[84] P. Stamp, "The decoherence puzzle," *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics*, vol. 37, no. 3, pp. 467–497, 2006. The arrows of time, 2006.

[85] W. G. Unruh, "Decoherence without dissipation," *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, vol. 370, no. 1975, pp. 4454–4459, 2012.

[86] S. L. Adler, "Lower and upper bounds on CSL parameters from latent image formation and IGM˜heating," *Journal of Physics A: Mathematical and Theoretical*, vol. 40, no. 44, pp. 13501–13501, 2007.

[87] S. Donadi, D.-A. Deckert, and A. Bassi, "On the spontaneous emission of electromagnetic radiation in the csl model," *Annals of Physics*, vol. 340, no. 1, pp. 70–86, 2014.

[88] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space*. Cambridge Monographs on Mathematical Physics, Cambridge, UK: Cambridge Univ. Press, 1984.

[89] T. Singh and T. Padmanabhan, "Notes on semiclassical gravity," *Annals of Physics*, vol. 196, no. 2, pp. 296–344, 1989.

[90] C.-I. Kuo and L. H. Ford, "Semiclassical gravity theory and quantum fluctuations," *Phys. Rev. D*, vol. 47, pp. 4510–4519, May 1993.

[91] C. Kiefer, "The semiclassical approximation to quantum gravity," in *Canonical Gravity: From Classical to Quantum* (J. Ehlers and H. Friedrich, eds.), (Berlin, Heidelberg), pp. 170–212, Springer Berlin Heidelberg, 1994.

[92] L. Diósi, "Gravitation and quantum-mechanical localization of macro-objects," *Physics Letters A*, vol. 105, no. 4, pp. 199–202, 1984.

[93] A. Ekert, R. Jozsa, R. Penrose, and R. Penrose, "Quantum computation, entanglement and state reduction," *Philosophical Transactions of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences*, vol. 356, no. 1743, pp. 1927–1939, 1998.

[94] K. Eppley and E. Hannah, "The necessity of quantizing the gravitational field," *Foundations of Physics*, vol. 7, pp. 51–68, Feb 1977.

[95] J. Mattingly, "Why Eppley and Hannah's thought experiment fails," *Phys. Rev. D*, vol. 73, p. 064025, Mar 2006.

[96] M. Albers, C. Kiefer, and M. Reginatto, "Measurement analysis and quantum gravity," *Phys. Rev. D*, vol. 78, p. 064051, Sep 2008.

[97] M. Bahrami, A. Großardt, S. Donadi, and A. Bassi, "The Schrödinger–Newton equation and its foundations," *New Journal of Physics*, vol. 16, no. 11, p. 115007, 2014.

[98] D. N. Page and C. D. Geilker, "Indirect evidence for quantum gravity," *Phys. Rev. Lett.*, vol. 47, pp. 979–982, Oct 1981.

[99] S. Weinberg, "Precision tests of quantum mechanics," *Phys. Rev. Lett.*, vol. 62, pp. 485–488, Jan 1989.

[100] S. Weinberg, "Testing quantum mechanics," *Annals of Physics*, vol. 194, no. 2, pp. 336–386, 1989.

[101] N. Gisin, "Stochastic quantum dynamics and relativity," *Helvetica Physica Acta*, vol. 62, no. 4, pp. 363–371, 1989.

[102] N. Gisin, "Weinberg's non-linear quantum mechanics and supraluminal communications," *Physics Letters A*, vol. 143, no. 1, pp. 1–2, 1990.

[103] J. Polchinski, "Weinberg's nonlinear quantum mechanics and the Einstein-Podolsky-Rosen paradox," *Phys. Rev. Lett.*, vol. 66, pp. 397–400, Jan 1991.

[104] W. G. Unruh in *Quantum Theory of Gravity: Essays in Honor of the 60th Birthday of Bryce S DeWitt* (S. Christensen, ed.), Bristol: Hilger, 1983.

[105] C. Rovelli, "Relational quantum mechanics," *International Journal of Theoretical Physics*, vol. 35, pp. 1637–1678, Aug 1996.

[106] A. Komar, "Construction of a complete set of independent observables in the general theory of relativity," *Phys. Rev.*, vol. 111, pp. 1182–1187, Aug 1958.

[107] S. B. Giddings, D. Marolf, and J. B. Hartle, "Observables in effective gravity," *Phys. Rev. D*, vol. 74, p. 064018, Sep 2006.

[108] B. L. Hu, "Stochastic gravity," *International Journal of Theoretical Physics*, vol. 38, pp. 2987–3037, Nov 1999.

[109] B. L. Hu and E. Verdaguer, "Stochastic gravity: Theory and applications," *Living Reviews in Relativity*, vol. 11, p. 3, Dec 2008.

[110] D. Kafri, J. M. Taylor, and G. J. Milburn, "A classical channel model for gravitational decoherence," *New Journal of Physics*, vol. 16, no. 6, p. 065020, 2014.

[111] B.-L. B. Hu and E. Verdaguer, *Semiclassical and Stochastic Gravity: Quantum Field Effects on Curved Spacetime.* Cambridge Monographs on Mathematical Physics, Cambridge University Press, 2020.

[112] P. C. E. Stamp, "Environmental decoherence versus intrinsic decoherence," *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, vol. 370, no. 1975, pp. 4429–4453, 2012.

[113] P. C. E. Stamp, "Rationale for a correlated worldline theory of quantum gravity," *New Journal of Physics*, vol. 17, no. 6, p. 065017, 2015.

[114] A. O. Barvinsky, D. Carney, and P. C. E. Stamp, "Structure of correlated worldline theories of quantum gravity," *Phys. Rev. D*, vol. 98, p. 084052, Oct 2018.

[115] H. Miao, D. Martynov, H. Yang, and A. Datta, "Quantum correlations of light mediated by gravity," *Phys. Rev. A*, vol. 101, p. 063804, Jun 2020.

[116] M. P. Blencowe, "Effective field theory approach to gravitationally induced decoherence," *Phys. Rev. Lett.*, vol. 111, p. 021302, Jul 2013.

[117] J. Wilson-Gerow, "Manifestly gauge invariant transition amplitudes and thermal influence functionals in qed and linearized gravity," Master's thesis, University of British Columbia, 2017.

[118] S. W. Hawking, M. J. Perry, and A. Strominger, "Soft hair on black holes," *Phys. Rev. Lett.*, vol. 116, p. 231301, 2016.

[119] J. Wilson-Gerow, C. DeLisle, and P. Stamp, "A functional approach to soft graviton scattering and BMS charges," *Class. Quant. Grav.*, vol. 35, no. 16, p. 164001, 2018.

[120] C. DeLisle, J. Wilson-Gerow, and P. Stamp, "Gravitational decoherence, asymptotic quantization, and entanglement measures," 2019, arXiv:1905.05333 [gr-qc].

[121] D. Carney, L. Chaurette, D. Neuenfeld, and G. W. Semenoff, "Infrared quantum information," *Phys. Rev. Lett.*, vol. 119, p. 180502, 2017.

[122] J. Wilson-Gerow and P. C. E. Stamp, "Gauge invariant propagators and states in quantum electrodynamics," 2020, arXiv:2011.05305 [hep-th].

[123] C. Anastopoulos and B.-L. Hu, "Comment on "a spin entanglement witness for quantum gravity" and on "gravitationally induced entanglement between two massive particles is sufficient evidence of quantum effects in gravity"," 2018, see alternative hosting, arXiv:1804.11315 [quant-ph].

[124] C. Anastopoulos and B. L. Hu, "Quantum superposition of two gravitational cat states," *Classical and Quantum Gravity*, vol. 37, no. 23, p. 235012, 2020.

[125] J. Wilson-Gerow, "Yang-Mills Gauss law and the heavy quark binding energy in the presence of a dimension-2 gluon condensate," 2020, arXiv:2011.05312 [hep-ph].

[126] A. O. Barvinsky, J. Wilson-Gerow, and P. C. E. Stamp, "Correlated worldline theory: Structure and consistency," *Phys. Rev. D*, vol. 103, p. 064028, Mar 2021.

[127] J. Wilson-Gerow and P. C. E. Stamp, "Paths and states in the correlated worldline theory of quantum gravity," 2020, arXiv:2011.14242 [gr-qc].

[128] H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems*. Oxford: Oxford University Press, 2007.

[129] R. P. Feynman and F. L. Vernon, Jr., "The Theory of a general quantum system interacting with a linear dissipative system," *Annals Phys.*, vol. 24, pp. 118–173, 1963.

[130] J. Schwinger, "Brownian motion of a quantum oscillator," *Journal of Mathematical Physics*, vol. 2, no. 3, pp. 407–432, 1961.

[131] L. Keldysh, "Diagram technique for nonequilibrium processes," *J. Exp. Theor. Phys.*, vol. 20, no. 4, p. 1018, 1964.

[132] A. Kamenev and A. Levchenko, "Keldysh technique and non-linear -model: basic principles and applications," *Advances in Physics*, vol. 58, p. 197319, May 2009.

[133] S. W. Hawking, "Breakdown of predictability in gravitational collapse," *Phys. Rev. D*, vol. 14, pp. 2460–2473, 1976.

[134] S. W. Hawking, "The unpredictability of quantum gravity," *Comm. Math. Phys.*, vol. 87, no. 3, pp. 395–415, 1982.

[135] W. G. Unruh and R. M. Wald, "Information loss," *Reports on Progress in Physics*, vol. 80, no. 9, p. 092002, 2017.

[136] D. Marolf, "The black hole information problem: past, present, and future," *Reports on Progress in Physics*, vol. 80, no. 9, p. 092001, 2017.

[137] A. Strominger, "On BMS invariance of gravitational scattering," *Journal of High Energy Physics*, vol. 2014, no. 7, p. 152, 2014.

[138] T. He, V. Lysov, P. Mitra, and A. Strominger, "BMS supertranslations and Weinberg's soft graviton theorem," *Journal of High Energy Physics*, vol. 2015, no. 5, p. 151, 2015.

[139] S. W. Hawking, M. J. Perry, and A. Strominger, "Superrotation charge and supertranslation hair on black holes," *Journal of High Energy Physics*, vol. 2017, no. 5, p. 161, 2017.

[140] S. Hyun, S.-A. Park, and S.-H. Yi, "Quasi-local charges and asymptotic symmetry generators," *Journal of High Energy Physics*, vol. 2014, no. 6, p. 151, 2014.

[141] T. Adamo, E. Casali, and D. Skinner, "Perturbative gravity at null infinity," *Classical and Quantum Gravity*, vol. 31, no. 22, p. 225008, 2014.

[142] T. He, P. Mitra, A. P. Porfyriadis, and A. Strominger, "New symmetries of massless QED," *Journal of High Energy Physics*, vol. 2014, no. 10, p. 112, 2014.

[143] M. Campiglia and A. Laddha, "New symmetries for the gravitational S-matrix," *Journal of High Energy Physics*, vol. 2015, p. 76, Apr 2015.

[144] M. Campiglia and A. Laddha, "Asymptotic symmetries of QED and weinberg's soft photon theorem," *Journal of High Energy Physics*, vol. 2015, no. 7, p. 115, 2015.

[145] A. Strominger, "Lectures on the infrared structure of gravity and gauge theory," 2018, 1703.05448.

[146] D. Kapec, M. Perry, A.-M. Raclariu, and A. Strominger, "Infrared divergences in QED revisited," *Phys. Rev. D*, vol. 96, p. 085002, Oct 2017.

[147] D. R. Yennie, S. C. Frautschi, and H. Suura, "The infrared divergence phenomena and high-energy processes," *Annals of Physics*, vol. 13, no. 3, pp. 379–452, 1961.

[148] S. Weinberg, "Infrared photons and gravitons," *Phys. Rev.*, vol. 140, pp. B516–B524, Oct 1965.

[149] B. Gabai and A. Sever, "Large gauge symmetries and asymptotic states in QED," *Journal of High Energy Physics*, vol. 2016, no. 12, p. 95, 2016.

[150] M. Mirbabayi and M. Porrati, "Dressed hard states and black hole soft hair," *Phys. Rev. Lett.*, vol. 117, p. 211301, Nov 2016.

[151] R. Bousso and M. Porrati, "Soft hair as a soft wig," *Classical and Quantum Gravity*, vol. 34, no. 20, p. 204001, 2017.

[152] R. Bousso, I. Halpern, and J. Koeller, "Information content of gravitational radiation and the vacuum," *Phys. Rev. D*, vol. 94, p. 064047, Sep 2016.

[153] V. Chung, "Infrared divergence in quantum electrodynamics," *Phys. Rev.*, vol. 140, pp. B1110–B1122, Nov 1965.

[154] T. W. B. Kibble, "Coherent soft-photon states and infrared divergences. iv. the scattering operator," *Phys. Rev.*, vol. 175, pp. 1624–1640, 1968.

[155] P. P. Kulish and L. D. Faddeev, "Asymptotic conditions and infrared divergences in quantum electrodynamics," *Theor. Math. Phys.*, vol. 4, p. 745, 1970.

[156] D. Carney, L. Chaurette, D. Neuenfeld, and G. W. Semenoff, "Dressed infrared quantum information," *Phys. Rev. D*, vol. 97, p. 025007, 2018.

[157] S. Choi and R. Akhoury, "BMS supertranslation symmetry implies faddeev-kulish amplitudes," *Journal of High Energy Physics*, vol. 2018, no. 2, p. 171, 2018.

[158] A. Stern, Y. Aharonov, and Y. Imry, "Phase uncertainty and loss of interference: A general picture," *Phys. Rev. A*, vol. 41, pp. 3436–3448, 1990.

[159] E. Fradkin, "Application of functional methods in quantum field theory and quantum statistics (ii)," *Nuclear Physics*, vol. 76, no. 3, pp. 588–624, 1966.

[160] G. Milekhin and E. Fradkin, "The doubly logarithmic approximation in quantum electrodynamics," *J. Exp. Theor. Phys.*, vol. 18, no. 5, p. 1323, 1964.

[161] J. W. York, "Role of conformal three-geometry in the dynamics of gravitation," *Phys. Rev. Lett.*, vol. 28, pp. 1082–1085, Apr 1972.

[162] G. W. Gibbons and S. W. Hawking, "Action integrals and partition functions in quantum gravity," *Phys. Rev. D*, vol. 15, pp. 2752–2756, May 1977.

[163] R. D. Jordan, "Effective field equations for expectation values," *Phys. Rev. D*, vol. 33, pp. 444–454, Jan 1986.

[164] E. Calzetta and B. L. Hu, "Nonequilibrium quantum fields: Closed-time-path effective action, Wigner function, and Boltzmann equation," *Phys. Rev. D*, vol. 37, pp. 2878–2900, May 1988.

[165] A. Schmid, "On a quasiclassical Langevin equation," *Journal of Low Temperature Physics*, vol. 49, pp. 609–626, Dec 1982.

[166] T. Oniga and C. H.-T. Wang, "Quantum gravitational decoherence of light and matter," *Phys. Rev. D*, vol. 93, p. 044027, Feb 2016.

[167] P. A. M. Dirac, "Generalized Hamiltonian dynamics," *Can. J. Math.*, vol. 2, pp. 129–148, 1950.

[168] P. A. M. Dirac, *Lectures on Quantum Mechanics.* Belfer Graduate School Sci. Mono., New York, NY: Belfer Graduate School of Science, 1964.

[169] H. Bondi, M. G. J. van der Burg, and A. W. K. Metzner, "Gravitational Waves in General Relativity. VII. Waves from Axi-Symmetric Isolated Systems," *Proceedings of the Royal Society of London Series A*, vol. 269, pp. 21–52, Aug. 1962.

[170] R. K. Sachs, "Gravitational waves in general relativity. 8. Waves in asymptotically flat space-times," *Proc. Roy. Soc. Lond. A*, vol. 270, pp. 103–126, 1962.

[171] R. Sachs, "Asymptotic symmetries in gravitational theory," *Phys. Rev.*, vol. 128, pp. 2851–2864, Dec 1962.

[172] S. Pasterski, "Asymptotic symmetries and electromagnetic memory," *Journal of High Energy Physics*, vol. 2017, p. 154, Sep 2017.

[173] A. Strominger and A. Zhiboedov, "Gravitational memory, BMS supertranslations and soft theorems," *Journal of High Energy Physics*, vol. 2016, p. 86, Jan 2016.

[174] M. Campiglia and A. Laddha, "Asymptotic symmetries of gravity and soft theorems for massive particles," *Journal of High Energy Physics*, vol. 2015, pp. 1–25, Dec 2015.

[175] C. Misner, K. Thorne, and J. Wheeler, *Gravitation.* San Francisco, CA: W.H. Freeman and Company, 1973.

[176] Y. B. Zel'dovich and A. G. Polnarev, "Radiation of gravitational waves by a cluster of superdense stars," *Soviet Astronomy*, vol. 18, p. 17, Aug. 1974.

[177] V. B. Braginskii and K. S. Thorne, "Gravitational-wave bursts with memory and experimental prospects," *Nature*, vol. 327, no. 6118, pp. 123–125, 1987.

[178] A. Strominger, "Black hole information revisited," 2017, arXiv:1706.07143 [hep-th].

[179] R. Utiyama, "Invariant theoretical interpretation of interaction," *Phys. Rev.*, vol. 101, pp. 1597–1607, Mar 1956.

[180] T. W. B. Kibble, "Lorentz invariance and the gravitational field," *Journal of Mathematical Physics*, vol. 2, no. 2, pp. 212–221, 1961.

[181] J. D. Norton, "General covariance and the foundations of general relativity: eight decades of dispute," *Reports on Progress in Physics*, vol. 56, no. 7, pp. 791–858, 1993.

[182] K. V. Kuchař, "Time and interpretations of quantum gravity," *International Journal of Modern Physics D*, vol. 20, no. supp01, pp. 3–86, 2011.

[183] N. Tsamis and R. Woodard, "Physical green's functions in quantum gravity," *Annals of Physics*, vol. 215, no. 1, pp. 96–155, 1992.

[184] J. B. Hartle, "Quantum kinematics of spacetime. i. nonrelativistic theory," *Phys. Rev. D*, vol. 37, pp. 2818–2832, May 1988.

[185] J. B. Hartle, "Quantum kinematics of spacetime. ii. a model quantum cosmology with real clocks," *Phys. Rev. D*, vol. 38, pp. 2985–2999, Nov 1988.

[186] J. B. Hartle, "Unitarity and causality in generalized quantum mechanics for nonchronal spacetimes," *Phys. Rev. D*, vol. 49, pp. 6543–6555, Jun 1994.

[187] R. Oeckl, "A general boundary formulation for quantum mechanics and quantum gravity," *Physics Letters B*, vol. 575, no. 3, pp. 318–324, 2003.

[188] R. Oeckl, "Schrödinger-feynman quantization and composition of observables in general boundary quantum field theory," *Advances in Theoretical and Mathematical Physics*, vol. 19, no. 2, pp. 451–506, 2015.

[189] F. Conrady, L. Doplicher, R. Oeckl, C. Rovelli, and M. Testa, "Minkowski vacuum in background independent quantum gravity," *Phys. Rev. D*, vol. 69, p. 064019, Mar 2004.

[190] R. Oeckl, "General boundary quantum field theory: Timelike hypersurfaces in the Klein-Gordon theory," *Phys. Rev. D*, vol. 73, p. 065017, Mar 2006.

[191] C. Kiefer, *Quantum Gravity*. Oxford University Press, 2012.

[192] M. G. G. Laidlaw and C. M. DeWitt, "Feynman functional integrals for systems of indistinguishable particles," *Phys. Rev. D*, vol. 3, pp. 1375–1378, Mar 1971.

[193] J. M. Leinaas and J. Myrheim, "On the theory of identical particles," *Nuovo Cim. B*, vol. 37, pp. 1–23, 1977.

[194] Y.-S. Wu, "General theory for quantum statistics in two dimensions," *Phys. Rev. Lett.*, vol. 52, pp. 2103–2106, Jun 1984.

[195] M. Stone, *Quantum Hall effect*. World Scientific, 1992.

[196] C. M. DeWitt, "Feynman's path integral: Definition without limiting procedure," *Communications in Mathematical Physics*, vol. 28, no. 1, pp. 47 – 67, 1972.

[197] D. S. Goldwirth, M. J. Perry, T. Piran, and K. S. Thorne, "Quantum propagator for a nonrelativistic particle in the vicinity of a time machine," *Phys. Rev. D*, vol. 49, pp. 3951–3957, Apr 1994.

[198] M. Visser, *Lorentzian Wormholes*. AIP - Press, 1996.

[199] G. Penington, S. H. Shenker, D. Stanford, and Z. Yang, "Replica wormholes and the black hole interior," 2020, arxiv:1911.11977 [hep-th].

[200] A. Almheiri, T. Hartman, J. Maldacena, E. Shaghoulian, and A. Tajdini, "Replica wormholes and the entropy of hawking radiation," *Journal of High Energy Physics*, vol. 2020, p. 13, May 2020.

[201] M. V. Raamsdonk, "Comments on wormholes, ensembles, and cosmology," 2020, arXiv:2008.02259 [hep-th].

[202] D. Marolf and H. Maxfield, "Transcending the ensemble: baby universes, spacetime wormholes, and the order and disorder of black hole information," *Journal of High Energy Physics*, vol. 2020, p. 44, Aug 2020.

[203] S. B. Giddings and S. Weinberg, "Gauge-invariant observables in gravity and electromagnetism: Black hole backgrounds and null dressings," *Phys. Rev. D*, vol. 102, p. 026010, Jul 2020.

[204] J. B. Hartle and S. W. Hawking, "Wave function of the universe," *Phys. Rev. D*, vol. 28, pp. 2960–2975, Dec 1983.

[205] P. Calabrese and J. Cardy, "Entanglement entropy and conformal field theory," *Journal of Physics A: Mathematical and Theoretical*, vol. 42, no. 50, p. 504005, 2009.

[206] A. Lewkowycz and J. Maldacena, "Generalized gravitational entropy," *Journal of High Energy Physics*, vol. 2013, p. 90, Aug 2013.

[207] C. Anastopoulos and B. L. Hu, "Probing a gravitational cat state," *Classical and Quantum Gravity*, vol. 32, no. 16, p. 165022, 2015.

[208] L. Faddeev and V. Popov, "Feynman diagrams for the yang-mills field," *Physics Letters B*, vol. 25, no. 1, pp. 29–30, 1967.

[209] P. A. M. Dirac, "Gauge invariant formulation of quantum electrodynamics," *Can. J. Phys.*, vol. 33, p. 650, 1955.

[210] B. S. DeWitt, "Quantum theory of gravity. i. the canonical theory," *Phys. Rev.*, vol. 160, pp. 1113–1148, Aug 1967.

[211] L. Prokhorov and S. Shabanov, "Invariant structures in gauge theories and confinement," *International Journal of Modern Physics A*, vol. 07, no. 31, pp. 7815–7840, 1992.

[212] S. Mandelstam, "Quantum electrodynamics without potentials," *Annals Phys.*, vol. 19, pp. 1–24, 1962.

[213] S. Mandelstam, "Quantization of the gravitational field," *Annals of Physics*, vol. 19, no. 1, pp. 25–66, 1962.

[214] S. Mandelstam, "Feynman rules for electromagnetic and yang-mills fields from the gauge-independent field-theoretic formalism," *Phys. Rev.*, vol. 175, pp. 1580–1603, Nov 1968.

[215] S. Mandelstam, "Feynman rules for the gravitational field from the coordinate-independent field-theoretic formalism," *Phys. Rev.*, vol. 175, pp. 1604–1623, Nov 1968.

[216] L. V. Prokhorov, D. V. Fursaev, and S. V. Shabanov, "String-like excitations in quantum electrodynamics," *Theoretical and Mathematical Physics*, vol. 97, pp. 1355–1363, Dec 1993.

[217] W. Donnelly and S. B. Giddings, "How is quantum information localized in gravity?," *Phys. Rev. D*, vol. 96, p. 086013, Oct 2017.

[218] W. Donnelly and S. B. Giddings, "Gravitational splitting at first order: Quantum information localization in gravity," *Phys. Rev. D*, vol. 98, p. 086006, Oct 2018.

[219] S. B. Giddings and A. Kinsella, "Gauge-invariant observables, gravitational dressings, and holography in ads," *Journal of High Energy Physics*, vol. 2018, p. 74, Nov 2018.

[220] S. B. Giddings, "Gravitational dressing, soft charges, and perturbative gravitational splitting," *Phys. Rev. D*, vol. 100, p. 126001, Dec 2019.

[221] M. Henneaux and C. Teitelboim, *Quantization of gauge systems*. Princeton University Press, 1992.

[222] N. Bao, A. Chatwin-Davies, J. Pollack, and G. N. Remmen, "Cosmological decoherence from thermal gravitons," *Journal of High Energy Physics*, vol. 2020, p. 65, Aug 2020.

[223] H. M. Fried, *Green's Functions and Ordered Exponentials*. Cambridge University Press, 2002.

[224] J.-L. Gervais and D. Zwanziger, "Derivation from first principles of the infrared structure of quantum electrodynamics," *Physics Letters B*, vol. 94, no. 3, pp. 389–393, 1980.

[225] D. Carney, L. Chaurette, D. Neuenfeld, and G. Semenoff, "On the need for soft dressing," *Journal of High Energy Physics*, vol. 2018, p. 121, Sep 2018.

[226] J. Jackson, *Classical Electrodynamics*. Wiley, 1962.

[227] G. Rossi and M. Testa, "The structure of yang-mills theories in the temporal gauge: (i). general formulation," *Nuclear Physics B*, vol. 163, pp. 109–132, 1980.

[228] G. Rossi and M. Testa, "The structure of yang-mills theories in the temporal gauge: (ii). perturbation theory," *Nuclear Physics B*, vol. 176, no. 2, pp. 477–499, 1980.

[229] F. Mattei, C. Rovelli, S. Speziale, and M. Testa, "From 3-geometry transition amplitudes to graviton states," *Nuclear Physics B*, vol. 739, no. 1, pp. 234–253, 2006.

[230] G. Klinkhammer and K. Thorne. unpublished.

[231] C. DeLisle. unpublished, we thank C. DeLisle for extensive discussions on this point.

[232] J. Lee and R. M. Wald, "Local symmetries and constraints," *Journal of Mathematical Physics*, vol. 31, no. 3, pp. 725–743, 1990.

[233] R. Ferrari and L. E. Picasso, "Spontaneous breakdown in quantum electrodynamics," *Nucl. Phys. B*, vol. 31, pp. 316–330, 1971.

[234] Y. Hamada, M.-S. Seo, and G. Shiu, "Large gauge transformations and little group for soft photons," *Phys. Rev. D*, vol. 96, p. 105013, Nov 2017.

[235] Y. Hamada and G. Shiu, "Infinite set of soft theorems in gauge-gravity theories as Ward-Takahashi identities," *Phys. Rev. Lett.*, vol. 120, p. 201601, May 2018.

[236] G. Compère, "Infinite towers of supertranslation and superrotation memories," *Phys. Rev. Lett.*, vol. 123, p. 021101, Jul 2019.

[237] P. Mao and W.-D. Tan, "Gravitational and electromagnetic memory," *Phys. Rev. D*, vol. 101, p. 124015, Jun 2020.

[238] T. W. B. Kibble, "Coherent softphoton states and infrared divergences. i. classical currents," *Journal of Mathematical Physics*, vol. 9, no. 2, pp. 315–324, 1968.

[239] T. W. B. Kibble, "Coherent soft-photon states and infrared divergences. ii. mass-shell singularities of green's functions," *Phys. Rev.*, vol. 173, pp. 1527–1535, 1968.

[240] T. W. B. Kibble, "Coherent soft-photon states and infrared divergences. iii. asymptotic states and reduction formulas," *Phys. Rev.*, vol. 174, pp. 1882–1901, 1968.

[241] C. DeLisle. unpublished manuscript.

[242] W. G. Unruh and R. M. Wald, "Time and the interpretation of canonical quantum gravity," *Phys. Rev. D*, vol. 40, pp. 2598–2614, Oct 1989.

[243] C. J. Isham, *Canonical Quantum Gravity and the Problem of Time*, pp. 157–287. Dordrecht: Springer Netherlands, 1993.

[244] P. A. M. Dirac, "Generalized hamiltonian dynamics," *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, vol. 246, no. 1246, pp. 326–332, 1958.

[245] P. A. M. Dirac, "The theory of gravitation in hamiltonian form," *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, vol. 246, no. 1246, pp. 333–343, 1958.

[246] P. A. M. Dirac, "Fixation of coordinates in the hamiltonian theory of gravitation," *Phys. Rev.*, vol. 114, pp. 924–930, May 1959.

[247] S. R. Moulick and P. K. Panigrahi, "Timelike curves can increase entanglement with LOCC," *Scientific Reports*, vol. 6, p. 37958, Nov 2016.

[248] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, "Quantum entanglement," *Rev. Mod. Phys.*, vol. 81, pp. 865–942, Jun 2009.

[249] Y. Chen and P. Stamp, "private communication," 2020.

[250] M. J. W. Hall and M. Reginatto, "On two recent proposals for witnessing nonclassical gravity," *Journal of Physics A: Mathematical and Theoretical*, vol. 51, no. 8, p. 085303, 2018.

[251] M. Reginatto and M. J. W. Hall, "Entanglement of quantum fields via classical gravity," 2018, 1809.04989.

[252] M. J. W. Hall and M. Reginatto, "Interacting classical and quantum ensembles," *Phys. Rev. A*, vol. 72, p. 062109, Dec 2005.

[253] C. Marletto and V. Vedral, "When can gravity path-entangle two spatially superposed masses?," *Phys. Rev. D*, vol. 98, p. 046001, Aug 2018.

[254] R. J. Marshman, A. Mazumdar, and S. Bose, "Locality and entanglement in table-top testing of the quantum nature of linearized gravity," *Phys. Rev. A*, vol. 101, p. 052110, May 2020.

[255] C. Marletto and V. Vedral, "Witnessing nonclassicality beyond quantum theory," *Phys. Rev. D*, vol. 102, p. 086012, Oct 2020.

[256] M. Christodoulou and C. Rovelli, "On the possibility of experimental detection of the discreteness of time," *Frontiers in Physics*, vol. 8, p. 207, 2020.

[257] M. Christodoulou and C. Rovelli, "On the possibility of laboratory evidence for quantum superposition of geometries," *Physics Letters B*, vol. 792, pp. 64–68, 2019.

[258] R. H. Kraichnan, "Special-relativistic derivation of generally covariant gravitation theory," *Phys. Rev.*, vol. 98, pp. 1118–1122, May 1955.

[259] S. Weinberg, "Derivation of gauge invariance and the equivalence principle from Lorentz invariance of the S-matrix," *Phys. Lett.*, vol. 9, no. 4, pp. 357–359, 1964.

[260] S. Weinberg, "Photons and Gravitons in S-Matrix Theory: Derivation of Charge Conservation and Equality of Gravitational and Inertial Mass," *Phys. Rev.*, vol. 135, pp. B1049–B1056, 1964.

[261] S. Deser, "Self-interaction and gauge invariance," *General Relativity and Gravitation*, vol. 1, pp. 9–18, Mar 1970.

[262] D. G. Boulware and S. Deser, "Classical general relativity derived from quantum gravity," *Annals of Physics*, vol. 89, no. 1, pp. 193–240, 1975.

[263] S. Deser, "Gravity from self-interaction in a curved background," *Classical and Quantum Gravity*, vol. 4, no. 4, pp. L99–L105, 1987.

[264] S. Deser, "Gravity from self-interaction redux," *General Relativity and Gravitation*, vol. 42, pp. 641–646, Mar 2010.

[265] W. Unruh, "False loss of coherence," in *Relativistic Quantum Measurement and Decoherence* (H.-P. Breuer and F. Petruccione, eds.), pp. 125–140, Springer, 2000.

[266] D. J. Gross and F. Wilczek, "Ultraviolet behavior of non-abelian gauge theories," *Phys. Rev. Lett.*, vol. 30, pp. 1343–1346, Jun 1973.

[267] H. D. Politzer, "Reliable perturbative results for strong interactions?," *Phys. Rev. Lett.*, vol. 30, pp. 1346–1349, Jun 1973.

[268] M. Gell-Mann, "A schematic model of baryons and mesons," *Physics Letters*, vol. 8, no. 3, pp. 214–215, 1964.

[269] T. Regge, "Introduction to complex orbital momenta," *Il Nuovo Cimento (1955-1965)*, vol. 14, pp. 951–976, Dec 1959.

[270] G. F. Chew and S. C. Frautschi, "Principle of equivalence for all strongly interacting particles within the S-matrix framework," *Phys. Rev. Lett.*, vol. 7, pp. 394–397, Nov 1961.

[271] G. F. Chew and S. C. Frautschi, "Regge trajectories and the principle of maximum strength for strong interactions," *Phys. Rev. Lett.*, vol. 8, pp. 41–44, Jan 1962.

[272] Y. Nambu, "Quark model and the factorization of the Veneziano amplitude," in *Symmetries and Quark Models* (R. Chand, ed.), p. 269, 1970.

[273] H. Nielsen in *15th Int. Conf. on high energy physics*, (Kiev), 1970.

[274] L. Susskind, "Dual-symmetric theory of hadrons.—I," *Nuovo Cimento A Serie*, vol. 69, no. 3, pp. 457–496, 1970.

[275] L. N. Chang and F. Mansouri, "Dynamics underlying duality and gauge invariance in the dual-resonance models," *Phys. Rev. D*, vol. 5, pp. 2535–2542, May 1972.

[276] P. Goddard, J. Goldstone, C. Rebbi, and C. Thorn, "Quantum dynamics of a massless relativistic string," *Nuclear Physics B*, vol. 56, no. 1, pp. 109–135, 1973.

[277] J. Kogut and L. Susskind, "Vacuum polarization and the absence of free quarks in four dimensions," *Phys. Rev. D*, vol. 9, pp. 3501–3512, Jun 1974.

[278] S. Mandelstam, "Vortices and quark confinement in non-abelian gauge theories," *Physics Reports*, vol. 23, no. 3, pp. 245–249, 1976.

[279] G. 't Hooft, "On the phase transition towards permanent quark confinement," *Nuclear Physics B*, vol. 138, no. 1, pp. 1–25, 1978.

[280] G. 't Hooft, "Topology of the Gauge Condition and New Confinement Phases in Nonabelian Gauge Theories," *Nucl. Phys. B*, vol. 190, pp. 455–478, 1981.

[281] H. Nielsen and P. Olesen, "A quantum liquid model for the QCD vacuum: Gauge and rotational invariance of domained and quantized homogeneous color fields," *Nuclear Physics B*, vol. 160, no. 2, pp. 380–396, 1979.

[282] J. M. Cornwall, "Quark confinement and vortices in massive gauge invariant qcd," *Nucl. Phys. B*, vol. 157, pp. 392–412, 1979.

[283] J. Ambjørn and P. Olesen, "On the formation of a random color magnetic quantum liquid in QCD," *Nuclear Physics B*, vol. 170, no. 1, pp. 60–78, 1980.

[284] G. Mack, "Predictions of a theory of quark confinement," *Phys. Rev. Lett.*, vol. 45, pp. 1378–1381, Oct 1980.

[285] G. Mack and E. Pietarinen, "Monopoles, vortices and confinement," *Nuclear Physics B*, vol. 205, no. 2, pp. 141–167, 1982.

[286] L. Del Debbio, M. Faber, J. Giedt, J. Greensite, and v. Olejník, "Detection of center vortices in the lattice yang-mills vacuum," *Phys. Rev. D*, vol. 58, p. 094501, Sep 1998.

[287] M. Chernodub, M. Polikarpov, A. Veselov, and M. Zubkov, "Aharonov-bohm effect, center monopoles and center vortices in su(2) lattice gluodynamics," *Nuclear Physics B - Proceedings Supplements*, vol. 73, no. 1, pp. 575–577, 1999.

[288] J. Drouffe and C. Itzykson, "Lattice gauge fields," *Physics Reports*, vol. 38, no. 3, pp. 133–175, 1978.

[289] J. B. Kogut, "The lattice gauge theory approach to quantum chromodynamics," *Rev. Mod. Phys.*, vol. 55, pp. 775–836, Jul 1983.

[290] A. Hasenfratz and P. Hasenfratz, "Lattice gauge theories," *Annual Review of Nuclear and Particle Science*, vol. 35, no. 1, pp. 559–604, 1985, https://doi.org/10.1146/annurev.ns.35.120185.003015.

[291] H. Wittig, "Lattice gauge theory," in *1999 International Europhysics Conference on High-Energy Physics*, pp. 181–200, 1999, hep-ph/9911400.

[292] M. C. Bañuls and K. Cichy, "Review on novel methods for lattice gauge theories," *Reports on Progress in Physics*, vol. 83, no. 2, p. 024401, 2020.

[293] G. S. Bali and K. Schilling, "Static quark-antiquark potential: Scaling behavior and finite-size effects in su(3) lattice gauge theory," *Phys. Rev. D*, vol. 46, pp. 2636–2646, Sep 1992.

[294] G. S. Bali and K. Schilling, "Running coupling and the Λ parameter from su(3) lattice simulations," *Phys. Rev. D*, vol. 47, pp. 661–672, Jan 1993.

[295] G. S. Bali, K. Schilling, and A. Wachter, "Complete $O(v^2)$ corrections to the static interquark potential from su(3) gauge theory," *Phys. Rev. D*, vol. 56, pp. 2566–2589, Sep 1997.

[296] G. S. Bali, B. Bolder, N. Eicker, T. Lippert, B. Orth, P. Ueberholz, K. Schilling, and T. Struckmann, "Static potentials and glueball masses from QCD simulations with Wilson sea quarks," *Phys. Rev. D*, vol. 62, p. 054503, Jul 2000.

[297] G. S. Bali, "Casimir scaling of SU(3) static potentials," *Phys. Rev. D*, vol. 62, p. 114503, Oct 2000.

[298] B. Bolder, T. Struckmann, G. S. Bali, N. Eicker, T. Lippert, B. Orth, K. Schilling, and P. Ueberholz, "High precision study of the $q\overline{Q}$ potential from wilson loops at large distances," *Phys. Rev. D*, vol. 63, p. 074504, Mar 2001.

[299] S. Necco and R. Sommer, "The nf=0 heavy quark potential from short to intermediate distances," *Nuclear Physics B*, vol. 622, no. 1, pp. 328–346, 2002.

[300] P. Boucaud, A. Le Yaouanc, J. Leroy, J. Micheli, O. Pène, and J. Rodríguez-Quintero, "Consistent ope description of gluon two- and three-point green functions?," *Physics Letters B*, vol. 493, no. 3, pp. 315–324, 2000.

[301] P. Boucaud, G. Burgio, F. D. Renzo, J. P. Leroy, J. Micheli, C. Parrinello, O. Pène, C. Pittori, J. Rodríguez-Quintero, C. Roiesnel, and K. Sharkey, "Lattice calculation of $1/p^2$ corrections to α_s and of $\lambda_{(QCD)}$ in the MOM-tilde scheme," *Journal of High Energy Physics*, vol. 2000, no. 04, pp. 006–006, 2000.

[302] F. D. Soto and J. Rodríguez-Quintero, "Remarks on the determination of the Landau gauge OPE for the asymmetric three gluon vertex," *Phys. Rev. D*, vol. 64, p. 114003, Oct 2001.

[303] P. Boucaud, A. Le Yaouanc, J. P. Leroy, J. Micheli, O. Pene, and J. Rodriguez-Quintero, "Testing Landau gauge OPE on the lattice with a $\langle A^2 \rangle$ condensate," *Phys. Rev. D*, vol. 63, p. 114003, 2001.

[304] P. Boucaud, J. P. Leroy, A. Le Yaouanc, J. Micheli, O. Pène, F. De Soto, A. Donini, H. Moutarde, and J. Rodríguez-Quintero, "Instantons and the $\langle A^2 \rangle$ condensate," *Phys. Rev. D*, vol. 66, p. 034504, Aug 2002.

[305] L. S. Celenza and C. M. Shakin, "Description of the gluon condensate," *Phys. Rev. D*, vol. 34, pp. 1591–1600, Sep 1986.

[306] J. L. Friedman and N. J. Papastamatiou, "On the canonical quantization of yang-mills theories," *Nuclear Physics B*, vol. 219, no. 1, pp. 125–142, 1983.

[307] C. Cronstrom, "The generalisation of the Coulomb gauge to yang-mills theory," 1998, arXiv: hep-th/9810002.

[308] C. Cronstrom, "Canonical structure of yang-mills theory," 1998, arXiv: hep-th/9810003v2.

[309] M. K. Gümüş and M. Boz, "Gauge fixing problem and the constrained quantization," 2017, arXiv:1701.09035 [hep-th].

[310] R. Fukuda and T. Kugo, "Dynamical Theory of the Yang-Mills Field. I: Instability of the Vacuum and Singlet Pair Condensation," *Progress of Theoretical Physics*, vol. 60, no. 2, pp. 565–583, 1978, https://academic.oup.com/ptp/article-pdf/60/2/565/5321393/60-2-565.pdf.

[311] R. Fukuda, "Tachyon bound state in yang-mills theory and instability of the vacuum," *Physics Letters B*, vol. 73, no. 1, pp. 33–38, 1978.

[312] V. Gusynin and V. Miransky, "On the vacuum rearrangement in massless chromodynamics," *Physics Letters B*, vol. 76, no. 5, pp. 585–588, 1978.

[313] G. Savvidy, "Infrared instability of the vacuum state of gauge theories and asymptotic freedom," *Physics Letters B*, vol. 71, no. 1, pp. 133–134, 1977.

[314] N. Nielsen and P. Olesen, "An unstable yang-mills field mode," *Nuclear Physics B*, vol. 144, no. 2, pp. 376–396, 1978.

[315] F. V. Gubarev, L. Stodolsky, and V. I. Zakharov, "On the significance of the vector potential squared," *Phys. Rev. Lett.*, vol. 86, pp. 2220–2222, 2001.

[316] R. Fukuda, "Gluon Condensation and the Properties of the Vacuum in Quantum Chromodynamics," *Progress of Theoretical Physics*, vol. 67, no. 2, pp. 648–654, 1982.

[317] P. Zyla *et al.*, "Review of particle physics," *PTEP*, vol. 2020, no. 8, p. 083C01, 2020.

[318] H. E. Haber, "Useful relations among the generators in the defining and adjoint representations of SU(N)," *SciPost Phys. Lect. Notes*, p. 21, 2021.

[319] A. Deur, S. J. Brodsky, and G. F. de Tramond, "The qcd running coupling," *Progress in Particle and Nuclear Physics*, vol. 90, pp. 1–74, 2016.

[320] E. Gardi, M. Karliner, and G. Grunberg, "Can the QCD running coupling have a causal analyticity structure?," *Journal of High Energy Physics*, vol. 1998, no. 07, pp. 007–007, 1998.

[321] K. G. Wilson, "Confinement of quarks," *Phys. Rev. D*, vol. 10, pp. 2445–2459, Oct 1974.

[322] D. Tong, "Lectures on gauge theory," 2018, http://www.damtp.cam.ac.uk/user/tong/gaugetheory.html.

[323] G. S. Bali, "Casimir scaling or flux counting?," *Nuclear Physics B - Proceedings Supplements*, vol. 83-84, pp. 422–424, 2000. Proceedings of the XVIIth International Symposium on Lattice Field Theory.

[324] G. S. Bali, "Casimir scaling of SU(3) static potentials," *Phys. Rev. D*, vol. 62, p. 114503, Oct 2000.

[325] V. Shevchenko and Y. Simonov, "Casimir scaling as a test of QCD vacuum models," *Phys. Rev. Lett.*, vol. 85, pp. 1811–1814, Aug 2000.

[326] G. Bali, T. Dssel, T. Lippert, H. Neff, Z. Prkain, and K. Schilling, "String breaking with dynamical wilson fermions," *Nuclear Physics B - Proceedings Supplements*, vol. 140, pp. 609–611, 2005. LATTICE 2004.

[327] G. S. Bali, H. Neff, T. Düssel, T. Lippert, and K. Schilling, "Observation of string breaking in QCD," *Phys. Rev. D*, vol. 71, p. 114513, Jun 2005.

[328] E. Seiler, "Upper bound on the color-confining potential," *Phys. Rev. D*, vol. 18, pp. 482–483, Jul 1978.

[329] G. S. Bali, "QCD forces and heavy quark bound states," *Physics Reports*, vol. 343, no. 1, pp. 1–136, 2001.

[330] C. Kellermann and C. S. Fischer, "Running coupling from the four-gluon vertex in landau gauge yang-mills theory," *Phys. Rev. D*, vol. 78, p. 025015, Jul 2008.

[331] W. Schleifenbaum, M. Leder, and H. Reinhardt, "Infrared analysis of propagators and vertices of yang-mills theory in landau and coulomb gauge," *Phys. Rev. D*, vol. 73, p. 125019, Jun 2006.

[332] M. Lavelle and M. Oleszczuk, "The operator product expansion of the QCD propagators," *Modern Physics Letters A*, vol. 07, no. 39, pp. 3617–3630, 1992.

[333] J. Ahlbach, M. Lavelle, M. Schaden, and A. Streibl, "Propagators and four dimensional condensates in pure QCD," *Physics Letters B*, vol. 275, no. 1, pp. 124–128, 1992.

[334] M. Lavelle and M. Schaden, "Propagators and condensates in QCD," *Physics Letters B*, vol. 208, no. 2, pp. 297–302, 1988.

[335] M. Lavelle and M. Schaden, "Propagators and condensates for QCD in the axial gauge," *Physics Letters B*, vol. 217, no. 4, pp. 551–559, 1989.

[336] M. Lavelle and M. Schaden, "The gluon condensate and the effective potential in QCD," *Physics Letters B*, vol. 246, no. 3, pp. 487–492, 1990.

[337] E. R. Arriola and W. Broniowski, "Dimension-two gluon condensate from large-N_c regge models," *Phys. Rev. D*, vol. 73, p. 097502, May 2006.

[338] K. Chetyrkin, S. Narison, and V. Zakharov, "Short-distance tachyonic gluon mass and $1/q^2$ corrections," *Nuclear Physics B*, vol. 550, no. 1, pp. 353–374, 1999.

[339] S. Narison and V. Zakharov, "Hints on the power corrections from current correlators in x-space," *Physics Letters B*, vol. 522, no. 3, pp. 266–272, 2001.

[340] J. I. Korsbakken, F. K. Wilhelm, and K. B. Whaley, "The size of macroscopic superposition states in flux qubits," *EPL (Europhysics Letters)*, vol. 89, no. 3, p. 30003, 2010.

[341] P. S. Wesson, *Five-Dimensional Physics.* World Scientific, 2006.

[342] G. Gibbons, S. Hawking, and M. Perry, "Path integrals and the indefiniteness of the gravitational action," *Nuclear Physics B*, vol. 138, no. 1, pp. 141–150, 1978.

[343] P. O. Mazur and E. Mottola, "The path integral measure, conformal factor problem and stability of the ground state of quantum gravity," *Nuclear Physics B*, vol. 341, no. 1, pp. 187–212, 1990.

[344] M. Visser, "How to Wick rotate generic curved spacetime," 2017, 1702.05572.

[345] A. Baldazzi, R. Percacci, and V. Skrinjar, "Wicked metrics," *Class. Quant. Grav.*, vol. 36, no. 10, p. 105008, 2019, 1811.03369.

[346] M. Czachor, "Elements of nonlinear quantum mechanics. 1: Nonlinear Schrodinger equation and two level atoms," *Phys. Rev. A*, vol. 53, p. 1310, 1996.

[347] E. P. Wigner, "The problem of measurement," *American Journal of Physics*, vol. 31, no. 1, pp. 6–15, 1963.

[348] C. M. Caves, "Quantum mechanics of measurements distributed in time. a path-integral formulation," *Phys. Rev. D*, vol. 33, pp. 1643–1665, Mar 1986.

[349] C. M. Caves, "Quantum mechanics of measurements distributed in time. ii. connections among formulations," *Phys. Rev. D*, vol. 35, pp. 1815–1830, Mar 1987.

[350] E. Tomboulis, "$1/N$ expansion and renormalization in quantum gravity," *Physics Letters B*, vol. 70, no. 3, pp. 361–364, 1977.

[351] E. Tomboulis, "Renormalizability and asymptotic freedom in quantum gravity," *Physics Letters B*, vol. 97, no. 1, pp. 77–80, 1980.

[352] J. B. Hartle and G. T. Horowitz, "Ground State Expectation Value of the Metric in the $1/N$ or Semiclassical Approximation to Quantum Gravity," *Phys. Rev. D*, vol. 24, pp. 257–274, 1981.

[353] B. S. Kay, "In-out semi-classical gravity and $1/N$ quantum gravity," *Physics Letters B*, vol. 101, no. 4, pp. 241–245, 1981.

[354] B. R. Holstein and J. F. Donoghue, "Classical physics and quantum loops," *Phys. Rev. Lett.*, vol. 93, p. 201602, Nov 2004.

[355] B. L. Hu, A. Roura, and E. Verdaguer, "Induced quantum metric fluctuations and the validity of semiclassical gravity," *Phys. Rev. D*, vol. 70, p. 044002, Aug 2004.

[356] K. S. Stelle, "Renormalization of higher-derivative quantum gravity," *Phys. Rev. D*, vol. 16, pp. 953–969, Aug 1977.

[357] D. Boulware, A. Strominger, and E. Tomboulis in *Quantum Theory of Gravity: Essays in Honor of the 60th Birthday of Bryce S DeWitt* (S. Christensen, ed.), Bristol: Hilger, 1983.

[358] T. Lee and G. Wick, "Negative metric and the unitarity of the S-matrix," *Nuclear Physics B*, vol. 9, no. 2, pp. 209–243, 1969.

[359] T. D. Lee and G. C. Wick, "Finite theory of quantum electrodynamics," *Phys. Rev. D*, vol. 2, pp. 1033–1048, Sep 1970.

[360] B. Grinstein, D. O'Connell, and M. B. Wise, "Causality as an emergent macroscopic phenomenon: The lee-wick $O(N)$ model," *Phys. Rev. D*, vol. 79, p. 105019, May 2009.

[361] A. Salvio, "Quadratic gravity," *Frontiers in Physics*, vol. 6, p. 77, 2018.

[362] J. F. Donoghue and G. Menezes, "Arrow of causality and quantum gravity," *Phys. Rev. Lett.*, vol. 123, p. 171601, Oct 2019.

[363] J. F. Donoghue and G. Menezes, "Unitarity, stability, and loops of unstable ghosts," *Phys. Rev. D*, vol. 100, p. 105006, Nov 2019.

[364] J. F. Donoghue and G. Menezes, "Quantum causality and the arrows of time and thermodynamics," *Progress in Particle and Nuclear Physics*, vol. 115, p. 103812, 2020.

[365] Y. Iwasaki, "Quantum theory of gravitation vs. classical theory*): Fourth-order potential," *Progress of Theoretical Physics*, vol. 46, pp. 1587–1609, Nov 1971.

[366] K. Hiida and H. Okamura, "Gauge transformation and gravitational potentials," *Progress of Theoretical Physics*, vol. 47, pp. 1743–1757, May 1972.

[367] J. F. Donoghue, "Leading quantum correction to the newtonian potential," *Phys. Rev. Lett.*, vol. 72, pp. 2996–2999, May 1994.

[368] H. Hamber and S. Liu, "On the quantum corrections to the newtonian potential," *Physics Letters B*, vol. 357, no. 1, pp. 51–56, 1995.

[369] I. J. Muzinich and S. Vokos, "Long range forces in quantum gravity," *Phys. Rev. D*, vol. 52, pp. 3472–3483, Sep 1995.

[370] A. Akhundov, S. Bellucci, and A. Shiekh, "Gravitational interaction to one loop in effective quantum gravity," *Physics Letters B*, vol. 395, no. 1, pp. 16–23, 1997.

[371] K. A. Kazakov, "Notion of potential in quantum gravity," *Phys. Rev. D*, vol. 63, p. 044004, Jan 2001.

[372] G. G. Kirilin and I. B. Khriplovich, "Quantum power correction to the newton law," *Journal of Experimental and Theoretical Physics*, vol. 95, pp. 981–986, Dec 2002.

[373] N. E. J. Bjerrum-Bohr, "Leading quantum gravitational corrections to scalar qed," *Phys. Rev. D*, vol. 66, p. 084023, Oct 2002.

[374] D. Neill and I. Z. Rothstein, "Classical spacetimes from the s-matrix," *Nuclear Physics B*, vol. 877, no. 2, pp. 177–189, 2013.

[375] N. Bjerrum-Bohr, J. F. Donoghue, and P. Vanhove, "On-shell techniques and universal results in quantum gravity," *Journal of High Energy Physics*, vol. 2014, no. 2, p. 111, 2014.

[376] V. Vaidya, "Gravitational spin hamiltonians from the S-matrix," *Phys. Rev. D*, vol. 91, p. 024017, Jan 2015.

[377] F. Cachazo and A. Guevara, "Leading singularities and classical gravitational scattering," 2017, 1705.10262.

[378] A. Guevara, "Holomorphic classical limit for spin effects in gravitational and electromagnetic scattering," 2017, arXiv:1706.02314 [hep-th].

[379] N. E. J. Bjerrum-Bohr, P. H. Damgaard, G. Festuccia, L. Planté, and P. Vanhove, "General relativity from scattering amplitudes," *Phys. Rev. Lett.*, vol. 121, p. 171601, Oct 2018.

[380] C. Cheung, I. Z. Rothstein, and M. P. Solon, "From scattering amplitudes to classical potentials in the post-minkowskian expansion," *Phys. Rev. Lett.*, vol. 121, p. 251101, Dec 2018.

[381] Z. Bern, C. Cheung, R. Roiban, C.-H. Shen, M. P. Solon, and M. Zeng, "Scattering amplitudes and the conservative hamiltonian for binary systems at third post-minkowskian order," *Phys. Rev. Lett.*, vol. 122, p. 201603, May 2019.

[382] Z. Bern, C. Cheung, R. Roiban, C.-H. Shen, M. P. Solon, and M. Zeng, "Black Hole Binary Dynamics from the Double Copy and Effective Theory," *JHEP*, vol. 10, p. 206, 2019, 1908.01493.

[383] C. Cheung and M. P. Solon, "Classical gravitational scattering at $\mathcal{O}(\mathrm{G}^3)$ from Feynman diagrams," *JHEP*, vol. 06, p. 144, 2020.

[384] C. Cheung and M. P. Solon, "Tidal effects in the post-minkowskian expansion," *Phys. Rev. Lett.*, vol. 125, p. 191601, Nov 2020.

[385] P. Di Vecchia, C. Heissenberg, R. Russo, and G. Veneziano, "The Eikonal Approach to Gravitational Scattering and Radiation at $\mathcal{O}(G^3)$," 2021, arXiv:2104.03256 [hep-th].

[386] E. Herrmann, J. Parra-Martinez, M. S. Ruf, and M. Zeng, "Radiative Classical Gravitational Observables at $\mathcal{O}(G^3)$ from Scattering Amplitudes," 2021, arXiv:2104.03957 [hep-th].

[387] N. E. J. Bjerrum-Bohr, P. H. Damgaard, L. Plant, and P. Vanhove, "Classical gravity from loop amplitudes," 2021, arXiv:2104.04510 [hep-th].

[388] R. Arnowitt, S. Deser, and C. W. Misner, "Dynamical structure and definition of energy in general relativity," *Phys. Rev.*, vol. 116, pp. 1322–1330, Dec 1959.

[389] R. Arnowitt, S. Deser, and C. W. Misner, "Coordinate invariance and energy expressions in general relativity," *Phys. Rev.*, vol. 122, pp. 997–1006, May 1961.

[390] T. Regge and C. Teitelboim, "Role of surface integrals in the hamiltonian formulation of general relativity," *Annals of Physics*, vol. 88, no. 1, pp. 286–318, 1974.

[391] M. Henneaux and C. Teitelboim, "Asymptotically anti-de sitter spaces," *Communications in Mathematical Physics*, vol. 98, pp. 391–424, Sep 1985.

[392] R. M. Wald and A. Zoupas, "General definition of "conserved quantities" in general relativity and other theories of gravity," *Phys. Rev. D*, vol. 61, p. 084027, Mar 2000.

[393] J. Feldbrugge, J.-L. Lehners, and N. Turok, "Lorentzian quantum cosmology," *Phys. Rev. D*, vol. 95, p. 103508, May 2017.

[394] J. Feldbrugge, J.-L. Lehners, and N. Turok, "Inconsistencies of the new no-boundary proposal," 2018, 1805.01609.

[395] R. Schoen and S.-T. Yau, "On the proof of the positive mass conjecture in general relativity," *Communications in Mathematical Physics*, vol. 65, pp. 45–76, Feb 1979.

[396] R. Schoen and S.-T. Yau, "Proof of the positive mass theorem. ii," *Communications in Mathematical Physics*, vol. 79, pp. 231–260, Mar 1981.

[397] E. Witten, "A new proof of the positive energy theorem," *Communications in Mathematical Physics*, vol. 80, pp. 381–402, Sep 1981.

[398] G. W. Gibbons, S. W. Hawking, G. T. Horowitz, and M. J. Perry, "Positive mass theorems for black holes," *Communications in Mathematical Physics*, vol. 88, pp. 295–308, Sep 1983.

[399] D. Christodoulou and S. Klainerman, "The global nonlinear stability of the minkowski space," *Séminaire Équations aux dérivées partielles (Polytechnique) dit aussi "Séminaire Goulaouic-Schwartz"*, 1989-1990. talk:13.

[400] L. Smolin, "Recent developments in non-perturbative quatum gravity," 1992, arXiv:9202022 [hep-th].

[401] R. Hudson, "When is the Wigner quasi-probability density non-negative?," *Reports on Mathematical Physics*, vol. 6, no. 2, pp. 249–252, 1974.

[402] T. Weiss, M. Roda-Llordes, E. Torrontegui, M. Aspelmeyer, and O. Romero-Isart, "Large quantum delocalization of a levitated nanoparticle using optimal control: Applications for force sensing and entangling via weak forces," 2020, arXiv:2012.12260 [quant-ph].

[403] K. C. Lee, M. R. Sprague, B. J. Sussman, J. Nunn, N. K. Langford, X.-M. Jin, T. Champion, P. Michelberger, K. F. Reim, D. England, *et al.*,

"Entangling macroscopic diamonds at room temperature," *Science*, vol. 334, no. 6060, pp. 1253–1256, 2011.

[404] A. A. Rakhubovsky, D. W. Moore, U. Delić, N. Kiesel, M. Aspelmeyer, and R. Filip, "Detecting nonclassical correlations in levitated cavity optomechanics," *Phys. Rev. Applied*, vol. 14, p. 054052, Nov 2020.

[405] W. Unruh, "Quantum noise in the interferometer detector," in *Quantum Optics, Experimental Gravity, and Measurement Theory* (P. Meystre and M. O. Scully, eds.), pp. 637–660, New York: Plenum Press, 1983.

[406] M. A. Yanbei Chen and J. McIver, "private communication," 2020.

[407] V. B. Braginsky, M. L. Gorodetsky, F. Y. Khalili, A. B. Matsko, K. S. Thorne, and S. P. Vyatchanin, "Noise in gravitational-wave detectors and other classical-force measurements is not influenced by test-mass quantization," *Phys. Rev. D*, vol. 67, p. 082001, Apr 2003.

[408] P. Stamp, "private communication and internal reports," 2015.

[409] A. Großardt, J. Bateman, H. Ulbricht, and A. Bassi, "Effects of newtonian gravitational self-interaction in harmonically trapped quantum systems," *Scientific Reports*, vol. 6, p. 30840, Aug 2016.

[410] M. J. Duff, "Quantum tree graphs and the schwarzschild solution," *Phys. Rev. D*, vol. 7, pp. 2317–2326, Apr 1973.

Appendix A

Computing Exact Correlation Functions in CWL Theory

We can start here from eq. (10.12) for the exact CWL free energy functional. We can compute connected correlation functions by functionally differentiating this with respect to the source J and setting $J = 0$. As a consequence of setting $J = 0$, the stationary phase solution $\bar{g}_J^{\mu\nu}$ simply becomes $\eta^{\mu\nu}$ because there is no matter source. In the body of the paper we do this for the 1-point function $\langle\phi^\alpha\rangle$ and conclude that CWL offers no correction for this. Here we will explicitly compute the $2, 3$, and 4-point functions in this way.

For the two-point function we have

$$\langle\phi_\alpha\phi_\beta\rangle_{CWL} = (-i)^2 \frac{\delta}{\delta J^\alpha}\frac{\delta}{\delta J^\beta}\mathcal{W}[J]\bigg|_{J=0}. \tag{A.1}$$

Since the free energy functional depends on the source J both explicitly and implicitly through the stationary phase solution $\bar{g}_J$, we must use the chain rule

$$\begin{aligned}\langle\phi_\alpha\phi_\beta\rangle_{CWL} =&(-i)^2\left(\frac{\partial}{\partial J^\alpha} + \frac{\delta \bar{g}_J^a}{\delta J^\alpha}\frac{\partial}{\partial g^a}\right)\left(\frac{\partial}{\partial J^\beta} + \frac{\delta \bar{g}_J^b}{\delta J^\beta}\frac{\partial}{\partial g^b}\right)\\ &\times (S_G[g] + W_0[J|g])\bigg|_{J=0,g=\eta}.\end{aligned} \tag{A.2}$$

Let us be clear about this notation, functional derivatives are typically written using the symbol δ rather than the d from standard calculus. Here though, we have a situation analogous to taking the convective time derivative

$$\frac{d}{dt}f(x^j(t),t) = \left(\frac{\partial}{\partial t} + \frac{dx^j}{dt}\frac{\partial}{\partial x^j}\right)f(x^j(t),t). \tag{A.3}$$

In standard calculus we have the symbols ∂ and d to distinguish partial from total derivatives and in our expressions above we are adopting the symbol

∂ to represent a "partial"-functional derivative (where we functionally differentiate the functional only with respect to a function where it appears explicitly) and we reserve the symbol δ for "total"-functional derivatives.

In evaluating the expression in eq. (A.2) we found it convenient to first expand out the derivatives before differentiating $\mathcal{W}[J]$. Doing this we find

$$\langle \phi_\alpha \phi_\beta \rangle_{CWL} = (-i)^2 \Bigg(\frac{\partial}{\partial J^\alpha} \frac{\partial}{\partial J^\beta} + \frac{\delta \bar{g}^a_J}{\delta J^\alpha} \frac{\partial}{\partial J^\beta} \frac{\partial}{\partial g^a} + \frac{\delta \bar{g}^b_J}{\delta J^\beta} \frac{\partial}{\partial J^\alpha} \frac{\partial}{\partial g^b} + \frac{\delta^2 \bar{g}^a_J}{\delta J^\alpha \delta J^\beta} \frac{\partial}{\partial g^a} + \frac{\delta \bar{g}^a_J}{\delta J^\alpha} \frac{\delta \bar{g}^b_J}{\delta J^\beta} \frac{\partial}{\partial g^a} \frac{\partial}{\partial g^b} \Bigg) (S_G[g] + W_0[J|g]) \Bigg|_{J=0, g=\eta} . \tag{A.4}$$

To simplify this we note that all terms linear in g derivatives will vanish because $\frac{\partial}{\partial g}(S_G[g] + W_0[J|g])\big|_{g=\bar{g}_J} = 0$ by the definition of $\bar{g}_J$. What remains is

$$\langle \phi_\alpha \phi_\beta \rangle_{CWL} = (-i)^2 \frac{\partial}{\partial J^\alpha} \frac{\partial}{\partial J^\beta} W_0[J|\eta] \Bigg|_{J=0} + (-i)^2 \frac{\delta \bar{g}^a_J}{\delta J^\alpha} \frac{\delta \bar{g}^b_J}{\delta J^\beta} \frac{\partial^2}{\partial g^a \partial g^b} (S_G[g] + W_0[J|g]) \Bigg|_{J=0, g=\eta} . \tag{A.5}$$

The first term is just the 2-point correlation function from conventional quantum field theory with no gravitational corrections and the second term is novel.

To compute the second term we must compute $\bar{g}_J$ at least to linear order in J. To to this we will set up an iterative procedure for solving the semi-classical Einstein equation in powers of J. We will do this mathematically and later interpret the result diagrammatically. We note that a similar procedure for understanding classical GR using quantum gravity was described in [410]. We start with the semi-classical Einstein equation

$$\frac{\delta}{\delta g^b} \Big(S_G[g] + W_0[J|g] \Big) \Bigg|_{\bar{g}_J} = 0. \tag{A.6}$$

Now expand the Einstein-Hilbert action about a flat background $g^a = \eta^a + h^a$ and separate the quadratic term from the cubic and higher terms.

$$\begin{aligned} S_G[g] &= \sum_{m=2}^{\infty} \frac{1}{m!} S_{G,a_1 \ldots a_m} h^{a_1} \ldots h^{a_m} \\ &= \frac{1}{2} S_{G,a_1 a_2} h^{a_1} h^{a_2} + S_G^{\text{int}}[h], \end{aligned} \tag{A.7}$$

where we use the comma notation for functional derivatives

$$S_{G,a_1...a_m} \equiv \left(\frac{\delta^m}{\delta g^{a_1} ... \delta g^{a_m}} S_G[g] \right) \Bigg|_{g=\eta}. \tag{A.8}$$

The expansion starts at quadratic order because Minkowski space is a solution to the vacuum Einstein equation with zero action. The semi-classical Einstein equation then reads

$$S_{G,ab} \bar{h}^a_J = -\frac{\delta}{\delta h^b} \left[S^{\text{int}}_G[\bar{h}_J] + W_0[J|\eta + \bar{h}_J] \right]. \tag{A.9}$$

We can formally solve this by inverting the quadratic form using the Green's function $D_{ab} \equiv (S_{G,ab})^{-1}$,

$$\bar{h}^a_J = -D^{ab} \left[S^{\text{int}}_{G,b}[\bar{h}_J] + W_{0,b}[J|\bar{g}_J] \right]. \tag{A.10}$$

When the source $J = 0$ the matter is in its vacuum state and $W_0[0|g]$ describes the vacuum energy. After appropriate renormalization of the vacuum energy we know that Minkowski spacetime is the solution to the vacuum Einstein equation with the appropriate boundary conditions. We can then conclude that $\bar{h}_{J=0} = 0$. To go beyond this to find derivatives of the solution we can implicitly differentiate eq. (A.10) with respect to J. Doing this we obtain

$$\begin{aligned} \frac{\delta \bar{h}^c_J}{\delta J^\alpha} = &- \left[\delta^c_a + D^{cb} S^{\text{int}}_{G,ba}[\bar{h}_J] + D^{cb} W_{0,ba}[J|\bar{g}_J] \right]^{-1} \\ &\times D^{ab} \frac{\partial}{\partial J^\alpha} W_{0,b}[J|\bar{g}_J]. \end{aligned} \tag{A.11}$$

We can now evaluate this when $J = 0$. Focus on the last factor, involving derivatives of the matter free energy functional. It is straightforward to show that it can be written

$$\frac{\partial}{\partial J^\alpha} W_{0,b}[J|\eta] \Bigg|_{J=0} = (i)^2 \langle \phi_\alpha S_b \rangle - i \langle \phi_\alpha \rangle \langle S_b \rangle, \tag{A.12}$$

where the correlation functions are to be evaluated in the conventional matter without gravity theory in Minkowski vacuum. If we assume a theory of matter which has vanishing vacuum n-point functions for odd n, then $\langle \phi_\alpha S_b \rangle = \langle \phi_\alpha \rangle \langle S_b \rangle = 0$. Thus the first derivative of the solution with respect to the source will vanish when evaluated at $J = 0$,

$$\frac{\delta \bar{h}^c_J}{\delta J^\alpha} \Bigg|_{J=0} = 0, \tag{A.13}$$

and then by eq. (A.5), the CWL corrections to the matter 2-point function will also vanish

$$\langle \phi_\alpha \phi_\beta \rangle_{CWL} = -i \frac{\partial}{\partial J^\alpha} \frac{\partial}{\partial J^\beta} W_0[J|\eta] \Big|_{J=0} . \tag{A.14}$$

This procedure can be extended to the 3-point function. After expanding out the derivatives using the chain rule we obtain

$$\begin{aligned} \langle \phi_\alpha \phi_\beta \phi_\gamma \rangle_{CWL} = (-i)^2 \Bigg(& \frac{\partial^3}{\partial J^\alpha \partial J^\beta \partial J^\gamma} \\ & + \frac{\delta \bar{g}^a_J}{\delta J_\alpha} \frac{\partial}{\partial g^a} \frac{\partial^2}{\partial J^\beta \partial J^\gamma} + \frac{\delta^2 \bar{g}^a_J}{\delta J_\alpha \delta J_\beta} \frac{\partial}{\partial g^a} \frac{\partial}{\partial J_\gamma} + \frac{\delta^3 \bar{g}^a_J}{\partial J^\alpha \partial J^\beta \partial J^\gamma} \frac{\partial}{\partial g^a} \\ & + \frac{\delta \bar{g}^a_J}{\delta J^\alpha} \frac{\delta \bar{g}^b_J}{\delta J^\beta} \frac{\partial^2}{\partial g^a \partial g^b} \frac{\partial}{\partial J^\gamma} + \frac{\delta \bar{g}^a_J}{\delta J^\alpha} \frac{\delta^2 \bar{g}^b_J}{\delta J^\beta \delta J^\gamma} \frac{\partial^2}{\partial g^a \partial g^b} \\ & + \frac{\delta \bar{g}^a_J}{\delta J^\alpha} \frac{\delta \bar{g}^b_J}{\delta J^\beta} \frac{\delta \bar{g}^c_J}{\delta J^\gamma} \frac{\partial^3}{\partial g^a \partial g^b \partial g^c} \\ & + \text{Permutations of} \, (\alpha, \beta, \gamma) \Bigg) (S_G[g] + W_0[J|g]) \Big|_{J=0, g=\eta} . \end{aligned} \tag{A.15}$$

If we use the previously discussed facts, ie. that terms linear in derivatives with respect to g^a as well as terms involving single derivatives of $\bar{g}_J$ will all vanish, then we see that all of the CWL corrections to the 3-point function vanish. The first term in eq. (A.15), involving only J derivatives, is the only surviving term and it is just the conventional QFT result

$$\langle \phi_\alpha \phi_\beta \phi_\gamma \rangle_{CWL} = (-i)^2 \frac{\partial^3}{\partial J^\alpha \partial J^\beta \partial J^\gamma} W_0[J|\eta] \Big|_{J=0} , \tag{A.16}$$

which itself vanishes in our case of interest.

Thusfar we've seen that CWL offers no corrections to the 1, 2 and 3-point functions for scalar field theory, but this is to be expected based on the conclusion from the text that no loop diagrams contribute in CWL theory.

Next, at 4-point we expect corrections. Expanding the derivatives we have

$$\begin{aligned}\langle\phi_\alpha\phi_\beta\phi_\gamma\phi_\delta\rangle_{CWL} = (-i)^3\bigg(&\frac{\delta^4}{\delta J^\alpha\delta J^\beta\delta J^\gamma\delta J^\delta}\\
&+\frac{\delta\bar{g}^a_J}{\delta J^\alpha}\frac{\partial}{\partial g^a}\frac{\partial^3}{\partial J^\beta\partial J^\gamma\partial J^\delta}+\frac{\delta\bar{g}^a_J}{\delta J^\alpha}\frac{\delta\bar{g}^b_J}{\delta J^\beta}\frac{\partial^2}{\partial g^a\partial g^b}\frac{\partial^2}{\partial J^\gamma\partial J^\delta}\\
&+\frac{\delta^2\bar{g}^a_J}{\delta J^\alpha\delta J^\beta}\frac{\partial}{\partial g^a}\frac{\partial^2}{\partial J^\gamma\partial J^\delta}+\frac{\delta\bar{g}^a_J}{\delta J^\alpha}\frac{\delta\bar{g}^b_J}{\delta J^\beta}\frac{\delta\bar{g}^c_J}{\delta J^\gamma}\frac{\partial^3}{\partial g^a\partial g^b\partial g^c}\frac{\partial}{\partial J^\delta}\\
&+\frac{\delta^2\bar{g}^a_J}{\delta J^\alpha\delta J^\beta}\frac{\delta\bar{g}^b_J}{\delta J^\gamma}\frac{\partial^2}{\partial g^a\partial g^b}\frac{\partial}{\partial J^\delta}+\frac{\delta^3\bar{g}^a_J}{\delta J^\alpha\delta J^\beta\delta J^\gamma}\frac{\partial}{\partial g^a}\frac{\partial}{\partial J^\delta}\\
&+\frac{\delta\bar{g}^a_J}{\delta J^\alpha}\frac{\delta\bar{g}^b_J}{\delta J^\beta}\frac{\delta\bar{g}^c_J}{\delta J^\gamma}\frac{\delta\bar{g}^d_J}{\delta J^\delta}\frac{\partial^4}{\partial J^\alpha\partial J^\beta\partial J^\gamma\partial J^\delta}\\
&+\frac{\delta^2\bar{g}^a_J}{\delta J^\alpha\delta J^\beta}\frac{\delta\bar{g}^b_J}{\delta J^\gamma}\frac{\delta\bar{g}^c_J}{\delta J^\delta}\frac{\partial^3}{\partial g^a\partial g^b\partial g^c}+\frac{\delta^2\bar{g}^a_J}{\delta J^\alpha\delta J^\beta}\frac{\delta^2\bar{g}^b_J}{\delta J^\gamma\delta J^\delta}\frac{\partial^2}{\partial g^a\partial g^b}\\
&+\frac{\delta^3\bar{g}^a_J}{\delta J^\alpha\delta J^\beta\delta J^\gamma}\frac{\delta\bar{g}^b_J}{\delta J^\delta}\frac{\partial^2}{\partial g^a\partial g^b}+\frac{\delta^4\bar{g}^a_J}{\delta J^\alpha\delta J^\beta\delta J^\gamma\delta J^\delta}\frac{\partial}{\partial g^a}\\
&+\text{Perms.}\bigg)(S_G[g]+W_0[J|g])\bigg|_{J=0,g=\eta},\end{aligned} \tag{A.17}$$

where "Perms." denotes all permutations of $\alpha,\beta,\gamma,\delta$ in the above which yield distinct terms. Applying the equation of motion and eq. (A.13) we find that in the above expression only the first term as well as the tenth term and its permutations are non-zero. We can then simplify the above to

$$\begin{aligned}\langle\phi_\alpha\phi_\beta\phi_\gamma\phi_\delta\rangle_{CWL} = (-i)^3&\frac{\delta^4}{\delta J^\alpha\delta J^\beta\delta J^\gamma\delta J^\delta}W_0[J|\eta]\bigg|_{J=0}\\
+\,i&\bigg(\frac{\delta^2\bar{g}^a_J}{\delta J^\alpha\delta J^\beta}\frac{\delta^2\bar{g}^b_J}{\delta J^\gamma\delta J^\delta}\frac{\partial^2}{\partial g^a\partial g^b}+\text{Perms.}\bigg)S_G[g]\bigg|_{J=0,g=\eta}\\
+\,i&\bigg(\frac{\delta^2\bar{g}^a_J}{\delta J^\alpha\delta J^\beta}\frac{\delta^2\bar{g}^b_J}{\delta J^\gamma\delta J^\delta}\frac{\partial^2}{\partial g^a\partial g^b}+\text{Perms.}\bigg)W_0[J|g]\bigg|_{J=0,g=\eta}\end{aligned} \tag{A.18}$$

The first term is of course just the connected 4-point correlation function from standard QFT. To understand the second term we need to compute the second derivative of the solution $\bar{h}_J$. To do so we implicitly differentiate eq. (A.10) twice with respect to J. We won't write the lengthy result here, but once we evaluate this at $J=0$ and use $\bar{h}_J\big|_{J=0}=\frac{\delta}{\delta J}\bar{h}_J\big|_{J=0}=0$, we obtain

$$\frac{\delta^2\bar{h}^a_J}{\delta J^\alpha\delta J^\beta}\bigg|_{J=0}=-\Big(S_{G,ac}+W_{0,ac}[0|\eta]\Big)^{-1}W_{0,c\alpha\beta}[0|\eta]. \tag{A.19}$$

Notice that here we have the inverse of the differential operator $S_{G,bc}+W_{0,bc}$, which itself shows up in eq. (A.18). This then leads to a rather drastic simplification when we substitute eq. (A.19) back into eq. (A.18),

$$\langle\phi_\alpha\phi_\beta\phi_\gamma\phi_\delta\rangle_{CWL} = (-i)^3\frac{\delta^4}{\delta J^\alpha\delta J^\beta\delta J^\gamma\delta J^\delta}W_0[J|\eta]\Big|_{J=0} + iW_{0,a\alpha\beta}[0|\eta]\left(S_{G,ab}+W_{0,ab}[0|\eta]\right)^{-1}W_{0,b\gamma\delta}[0|\eta] + \text{Perms.} \tag{A.20}$$

We can understand this expression by computing these derivatives of W_0, and then thinking in terms of diagrams. First we have

$$W_{0,a\alpha\beta}[0|\eta] = (i)^2\langle\phi_\alpha\phi_\beta S_a\rangle, \tag{A.21}$$

where all terms proportional to $\langle\phi_\alpha\rangle$ vanished because of the previously assumed internal symmetry of the Minkowski vacuum state. Also, all terms proportional to $\langle S_a\rangle$ vanish. This is because it is the vacuum expectation value of a local operator and since the Minkowski vacuum is Poincaré invariant it must equal a constant: and we renormalize this constant vacuum energy density to zero. We also have

$$W_{0,ab}[0|\eta] = i\langle S_a S_b\rangle + \langle S_{bc}\rangle - i\langle S_a\rangle\langle S_b\rangle. \tag{A.22}$$

Again, $\langle S_a\rangle$ vanishes here, but it is easy to see that S_{ab} is also a local operator and its Minkowski vacuum expectation value should also vanish after renormalization. Our final, exact expression for the 4-point function in CWL is then

$$\langle\phi_\alpha\phi_\beta\phi_\gamma\phi_\delta\rangle_{CWL} = (-i)^3\frac{\delta^4}{\delta J^\alpha\delta J^\beta\delta J^\gamma\delta J^\delta}W_0[J|\eta]\Big|_{J=0} + i\langle\phi_\alpha\phi_\beta S_a\rangle\left(S_{G,ab}+i\langle S_a S_b\rangle\right)^{-1}\langle\phi_\gamma\phi_\delta S_b\rangle. \tag{A.23}$$

This expression can be understood quite simply. Up to operator ordering, $\langle\phi_\alpha\phi_\beta S_a\rangle$ is proportional to the expectation value of the matter stress-energy tensor in the state created by perturbing the vacuum by sources J^α and J^β, ostensibly a two-particle state provided the source insertions are widely separated. We then have two stress-energy sources at a and b created by the source insertions. These stress-energy sources interact via the exchange of a single graviton with renormalized propagator. To see this we can use the Green's function $D^{ab} \equiv (S_{G,ab})^{-1}$,

$$\Big(S_{G,ab} + i\langle S_a S_b\rangle\Big)^{-1} = \Big(\delta^b_c + iD^{bd}\langle S_d S_c\rangle\Big)^{-1} D^{ca}$$
$$= \sum_{k=0}^{\infty}\Big(- iD^{bd}\langle S_d S_c\rangle\Big)^{k} D^{ca}, \qquad \text{(A.24)}$$

and see that this object is just the sum of diagrams contributing to the graviton 2-point function with all numbers of insertions of the vacuum polarization $\Pi_{dc} = \langle S_d S_c\rangle$. The exact 4-point function predicted by CWL is then described by the exchange of a single graviton which is renormalized by a vacuum polarization which can be computed in conventional flat space QFT.

If we consider a theory of matter which is free except for gravitational interactions, the stress-energy tensor will be quadratic in the fields $S_a = \frac{1}{2}\hat{\tau}_a^{\alpha\beta}\phi_\alpha\phi_\beta$ and the flat space correlation functions can be expressed in terms of the matter Green's function $\langle\phi_\alpha\phi_\beta\rangle = iG_{\alpha\beta} \equiv i(S^{\alpha\beta})^{-1}$ using Wick's theorem. The result for the exact 4-point function can then be written explicitly as

$$\begin{aligned}\langle\phi_\alpha\phi_\beta\phi_\gamma\phi_\delta\rangle_{CWL} &= iG_{\alpha\sigma}G_{\beta\rho}\hat{\tau}_a^{\sigma\rho}\mathscr{D}^{ab}\hat{\tau}_b^{\lambda\eta}G_{\gamma\lambda}G_{\delta\eta}\\ &+ iG_{\alpha\sigma}G_{\gamma\rho}\hat{\tau}_a^{\sigma\rho}\mathscr{D}^{ab}\hat{\tau}_b^{\lambda\eta}G_{\beta\lambda}G_{\delta\eta}\\ &+ iG_{\alpha\sigma}G_{\delta\rho}\hat{\tau}_a^{\sigma\rho}\mathscr{D}^{ab}\hat{\tau}_b^{\lambda\eta}G_{\gamma\lambda}G_{\beta\eta},\end{aligned} \qquad \text{(A.25)}$$

which is just the sum of the thee crossings of the tree diagram in fig. 10.2 from conventional QFT, **but** with a renormalized graviton propagator $\mathscr{D}^{ab} = \big((D^{ab})^{-1} + i\Pi_{ab}\big)^{-1}$ where the vacuum polarization is given by the matter bubble

$$\Pi_{ab} = -\frac{1}{2}\hat{\tau}_a^{\alpha\beta}\hat{\tau}_b^{\gamma\delta}G_{\alpha\delta}G_{\beta\gamma}. \qquad \text{(A.26)}$$