Comment on the Uniqueness of the Electroweak Group from the Anomalies Viewpoint

Arguments for the uniqueness of the electroweak group and its representations are reexamined on the basis of freedom from the three known chiral gauge anomalies in four dimensions: the triangular chiral gauge anomaly, the global chiral gauge anomaly and the mixed chiral gauge-gravitational anomaly. While the standard chiral gauge group $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ gives rise to a unique minimal set of massless fermion representations of $SU(3)_C \otimes SU(2)_L$ and their $U(1)_Y$ charges that are in accord with experiment, it is shown that the unique minimal set of massless fermion representations of the gauge chiral group $SU(3)_C \otimes SU(3)_L \otimes U(1)_Y$—corresponding to the gauging of three light chiral quark flavors—does not yield a unique set of $U(1)_Y$ charges. We argue that this helps to explain nature's choice of quark and lepton chiral doublets rather than triplets.

Key Words: *electroweak theory, anomalies, left-right symmetry, Weyl fermions*

It is a striking fact that the generational grouping of quarks and leptons is in terms of (left-handed) chiral doublets and not triplets, as might be expected from the existence of three (and not two) light quarks. It will be recalled that the QCD symmetry group for n flavors of massless quarks is[1]:

$$SU(3)_C \otimes SU(n)_L \otimes SU(n)_R \otimes U(1)_L \otimes U(1)_R$$

where $SU(3)_C$ is the gauged color group and the rest is the global chiral quark flavor symmetry group. The confining $SU(3)_C$ group

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produces quark condensates $\langle 0|q\bar{q}|0\rangle \neq 0$ that leave $SU(3)_C$ and $U(1)_{L+R}$ (baryon charge) unbroken while breaking $SU(n)_L \otimes SU(n)_R$ down to the "diagonal sum" (vector) group $SU(n)_{L+R}$[2] [$SU(n)_{L+R}$ is the global quark flavor group $SU(n)_F$]; the color instantons break the axial $U(1)_{L-R}$ group down to the discrete group Z_{2n}.[3] Putting aside the instanton effects, QCD with n flavors tells us that $SU(3)_C$ is unbroken by the quark condensates whereas $SU(n)_L \otimes SU(n)_R$ is broken and gives rise to $(n^2 - 1)$ pseudoscalar Goldstone bosons ("pions").

Since, in nature, we have a triplet of light quarks ($m_q < \Lambda_{QCD}$ (Λ_{QCD} is the QCD scale)— all other quarks have $m_q >> \Lambda_{QCD}$), the largest global chiral quark flavor group that makes physical sense is $SU(3)_L \otimes SU(3)_R$. Indeed, the identification of the associated three chiral light quark flavor symmetry currents with the physical weak and electromagnetic currents explains the great success of current algebra. By paying attention to chiral quark–lepton flavor universality, one might then expect that the construction of a successful gauge theory of the electroweak interaction would be based on gauging $SU(3)_L$, rather than $SU(2)_L$, chiral flavor.

Nature has expressed a distinct preference for gauging two chiral quark flavors (the leptons must follow suit because of the anomaly-free constraints) and it is interesting to inquire whether making full use of freedom from anomalies can give us any insight into this choice at the electroweak scale. In this Comment, we shall show that the application of the three known chiral gauge anomaly-free conditions in four dimensions does distinguish between two and three chiral quark flavors and does lead to a unique minimal and empirically correct set of quantum numbers of the massless fermion (Weyl) representations for two chiral quark flavors but not for three. But first let us remind the reader of the three chiral gauge anomalies that are being considered: they are (1) the triangular (perturbative) chiral gauge anomaly,[4] which must be cancelled to avoid the breakdown of gauge invariance and, a fortiori, renormalizability of the theory; we call this *triangular* anomaly; (2) the global (non-perturbative) SU(2) chiral gauge anomaly,[5] which must be absent in order to define the fermion integral in a gauge invariant way; we call this the *global* anomaly; (3) the mixed (perturbative) chiral gauge-gravitational anomaly,[6] which must be cancelled in order to ensure general covariance of the theory; we call this the *mixed* anomaly.

The basic question is whether the requirements of minimality and freedom from all three chiral gauge anomalies for the candidate group yields a unique set of Weyl representations and their (hyper)charges that agree with experiment, i.e., whether the uniquely determined set of Weyl representations—after the breaking of the electroweak group to $U(1)_{EM}$—describes the observed quarks and leptons of one generation. We shall see that the standard group $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ comes through with flying colors whereas its generalization to three chiral quark flavors, i.e., $SU(3)_C \otimes SU(3)_L \otimes U(1)_Y$, does not.

The standard model has been worked out elsewhere[7] and here we simply summarize the main results. We begin by allowing an arbitrary[8] number of (left-handed) Weyl representations under the standard group, i.e.:

$$
\begin{array}{lll}
SU(3)_C \otimes & SU(2)_L \otimes & U(1)_Y \\
3 & 2 & Q_i \ (i = 1, 2, \ldots, j) \\
3 & 1 & Q'_i \ (i = 1, 2, \ldots, k) \\
\bar{3} & 1 & \overline{Q}_i \ (i = 1, 2, \ldots, l) \\
\bar{3} & 2 & \overline{Q}'_i \ (i = 1, 2, \ldots, m) \\
1 & 2 & q_i \ (i = 1, 2, \ldots, n) \\
1 & 1 & \overline{q}'_i \ (i = 1, 2, \ldots, p)
\end{array}
\qquad (1)
$$

where the integers j, k, l, m, n and p and the $U(1)_Y$ charges are all arbitrary. Freedom from the triangular anomalies then leads to the following equations:

$$[\mathrm{SU}(3)]^3: \quad \sum_{i=1}^{j} 2 + \sum_{i=1}^{k} 1 - \sum_{i=1}^{l} 1 - \sum_{i=1}^{m} 2 = 0, \tag{2a}$$

$$[\mathrm{SU}(3)]^2\mathrm{U}(1): \quad 2\sum_{i=1}^{j} Q_i + \sum_{i=1}^{k} Q'_i + \sum_{i=1}^{l} \overline{Q}'_i + 2\sum_{i=1}^{m} \overline{Q}'_i = 0, \tag{2b}$$

$$[\mathrm{SU}(2)]^2\mathrm{U}(1): \quad 3\sum_{i=1}^{j} Q_i + 3\sum_{i=1}^{m} \overline{Q}_i + \sum_{i=1}^{n} q_i = 0, \tag{2c}$$

$$\mathrm{U}(1)^3: \quad 6\sum_{i=1}^{j} Q_i^3 + 3\sum_{i=1}^{k} Q_i'^3 + 3\sum_{i=1}^{l} \overline{Q}_i^3 + 6\sum_{i=1}^{m} \overline{Q}_i'^3$$

$$+ 2\sum_{i=1}^{n} q_i^3 + \sum_{i=1}^{p} \overline{q}_i^3 = 0. \tag{2d}$$

The global anomaly-free condition is:

$$3j + 3m + n = N \tag{3}$$

where N is an even integer. Finally, the mixed anomaly-free condition is:

$$\mathrm{Tr}\, Y = 0. \tag{4}$$

The requirements of minimality and the three anomaly-free conditions (Eqs. (2)–(4)) lead to the values: $j = 1$, $k = 0$, $l = 2$, $m = 1$, $n = 1$, $p = 1$ and, in the obvious notation, to the four relations (note that there must be lepton as well as quark representations):

$$2Q_1 + \overline{Q}_1 + \overline{Q}_2 = 0 \tag{5a}$$

$$3Q_1 + q_1 = 0 \tag{5b}$$

$$6Q_1^3 + 3\overline{Q}_1^3 + 3\overline{Q}_2^2 + 2q_1^3 + \overline{q}_1^3 = 0 \tag{5c}$$

$$2q_1 + \overline{q}_1 = 0. \tag{6}$$

Equation (6) receives no contribution from SU(3) color triplets (quarks) because of the triangular anomaly-free condition (5a), so that, combining Eq. (6) with Eqs. (5), one gets:

$$Q_1 = \frac{1}{3}q_1, \quad \overline{Q}_1 = \frac{4}{3}q_1, \quad \overline{Q}_2 = -\frac{2}{3}q_1, \quad \overline{q}_1 = -2q_1 \tag{7}$$

It is seen from Eq. (7) that all the $U(1)_Y$ charges are uniquely determined in terms of a single $U(1)_Y$ charge, q_1; choosing the normalization $q_1 = -1$—consistent with zero electric charge for the neutrino—the resulting Weyl representations of $SU(3)_C$ and $SU(2)_L$ and their $U(1)_Y$ charges are those shown in Table I, in agreement with the standard model.

We thus find that the requirements of minimality and freedom from all three chiral gauge anomalies lead to a unique set of Weyl representations (and their $U(1)_Y$ charges) of the standard group that correspond to the observed quarks and leptons of one family. Furthermore, the $U(1)_Y$ charges of these quarks and leptons are quantized and correctly determined by adding the mixed anomaly-free condition. Clearly, if one lifts the minimality requirement, it

TABLE I

The quantum numbers of the (left-handed) Weyl representations under $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ when all three anomaly-free conditions are satisfied

Particles ($i = 1, 2, 3$)	$SU(3)_C$	$\otimes\ SU(2)_L$	$\otimes\ U(1)_Y$
$q_L^i = \begin{pmatrix} u \\ d \end{pmatrix}_L^i$	3	2	$\frac{1}{3}$
$\overline{u}_L^i$	$\overline{3}$	1	$-\frac{4}{3}$
$\overline{d}_L^i$	$\overline{3}$	1	$\frac{2}{3}$
$l_L^i = \begin{pmatrix} \nu \\ e \end{pmatrix}_L^i$	1	2	-1
$\overline{e}_L^i$	1	1	2

is possible to obtain as many copies of a quark–lepton family with the proper quantum numbers as one wishes.

It should be noted that our demonstration—that all three anomaly-free conditions are needed to determine the correct quantum numbers of one family of Weyl fermions—is based on the acceptance of the standard group as starting point. But the standard group only allows for left-handed neutrinos and the situation changes if, for example, the standard gauge group is enlarged to the left-right-symmetric (LRS) gauge group $SU(3)_C \otimes SU(2)_L \otimes SU(2)_R \otimes U(1)_{B-L}$.[9] With this group (and invoking minimality, as we did with the standard group), it is easily shown that the (left-handed) Weyl representations are those shown in Table II. In deriving Table II, it is only necessary to impose the first two anomaly-free conditions: the triangular and global anomaly-free conditions; the mixed anomaly-free condition is automatically satisfied in a manifestly left-right-symmetric theory such as the LRS model.

We now repeat the arguments outlined above when the $SU(3)_L$ chiral quark flavor gauge group replaces $SU(2)_L$. The simplest case—which illustrates the problem—is the analog of the standard group, i.e., $SU(3)_C \otimes SU(3)_L \otimes U(1)_Y$. Instead of beginning the

TABLE II

The quantum numbers of the (left-handed) Weyl representations under $SU(3)_C \otimes SU(2)_L \otimes SU(2)_R \otimes U(1)_{B-L}$ when the triangular and global SU(2) anomaly-free conditions are satisfied

Particles (i = 1, 2, 3)	$SU(3)_C$ ⊗	$SU(2)_L$ ⊗	$SU(2)_R$ ⊗	U(1)
$q_L^i = \begin{pmatrix} u \\ d \end{pmatrix}_L^i$	3	2	1	$\frac{1}{3}$
$\bar{q}_L^i = \begin{pmatrix} \bar{u} \\ \bar{d} \end{pmatrix}_L^i$	$\bar{3}$	1	2	$-\frac{1}{3}$
$l_L^i = \begin{pmatrix} \nu \\ e \end{pmatrix}_L^i$	1	2	1	−1
$\bar{l}_L^i = \begin{pmatrix} \bar{\nu} \\ \bar{e} \end{pmatrix}_L^i$	1	1	2	1

TABLE III

Minimal representations under $SU(3)_C \otimes SU(3)_L \otimes U(1)_Y$

$SU(3)_C$	$\otimes\ SU(3)_L$	$\otimes\ U(1)$
$\underline{3}$	3	Q
$\underline{3}$	1	Q_1
$\underline{3}$	1	Q_2
$\overline{3}$	$\underline{1}$	Q_3
1	$\underline{3}$	q_1
1	$\underline{3}$	q_2
1	$\overline{3}$	q_3

demonstration with an arbitrary set of Weyl representations (the result is the same), we accept minimality from the outset and ascertain whether the use of the three anomaly-free conditions enable us to fix uniquely the $U(1)_Y$ charges of the minimal representations. We write down these minimal Weyl representations under $SU(3)_C \otimes SU(3)_L \otimes U(1)_Y$ in Table III, but allow for arbitrary values of the $U(1)_Y$ charges.

The three triangular anomaly-free conditions become:

$$[SU(3)_C]^2U(1): \quad 3Q + \sum_{i=1}^{3} Q_i = 0 \tag{8a}$$

$$[SU(3)_L]^2U(1): \quad 3Q + \sum_{i=1}^{3} q_i = 0 \tag{8b}$$

$$[U(1)]^3: \quad 9Q^3 + 3\left(\sum_{i=1}^{3} Q_i^3\right) + 3\left(\sum_{i=1}^{3} q_i^3\right) = 0. \tag{8c}$$

The global anomaly-free condition is satisfied because the total number of zero modes of the 3 and $\overline{3}$ Weyl representations of $SU(3)_L$—containing $SU(2)_L$ as a subgroup—is *even*. The mixed anomaly-free condition becomes:

$$9Q + 3\sum_{i=1}^{3} Q_i + 3\sum_{i=1}^{3} q_i = 0. \tag{9}$$

Use of Eqs. (8) and Eq. (9) requires $Q = 0$ and the following three independent equations:

$$Q_1 + Q_2 + Q_3 = 0 \tag{10a}$$

$$q_1 + q_2 + q_3 = 0 \tag{10b}$$

$$Q_1^3 + Q_2^3 + Q_3^3 + q_1^3 + q_2^3 + q_3^3 = 0 \tag{10c}$$

It is easily shown that Eqs. (10) have no unique solution, even with the minimality condition, so that the $SU(3)_C \otimes SU(3)_L \otimes U(1)_Y$ model can not have its $U(1)_Y$ charges uniquely fixed even with the use of the mixed anomaly-free condition. It is interesting to note that one of the solutions: $Q_1 = +1$, $Q_2 = -1$, $Q_3 = 0$; $q_1 = -1$, $q_2 = 1$, $q_3 = 0$ comes from the 27 (fundamental) Weyl representation of E_6 when it breaks down to $SU(3)_C \otimes SU(3)_L \otimes U(1)_Y$ through the maximal intermediate subgroup $[SU(3)]^3$. However, neither this solution nor any other solution of Eqs. (10) yields the observed quark and lepton electric charges when $SU(2)_L \otimes U(1)_Y$ breaks down to $U(1)_{EM}$. This means that $E_6 \rightarrow SU(3)_C \otimes SU(3)_L \otimes U(1)_Y$ is not a possible symmetry-breaking path for a viable E_6 GUT theory.[10]

Our result for $SU(3)_C \otimes SU(3)_L \otimes U(1)_Y$ differs markedly from the case of the standard group $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$, where the application of the three anomaly-free conditions yields a unique set of $U(1)_Y$ charges that, moreover, agree with experiment. We consider this result an argument against the gauging of three chiral quark flavors instead of two. Thus, for the first time, the anomalies viewpoint has given us some inkling of why a gauge theory of the strong and electroweak interactions is natural within the framework of a replication of families of quark and lepton doublets. However, it must be admitted that, thus far, the anomalies viewpoint has given us no clue as to why the number of families is at least three and probably does not exceed four.

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