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REDUCTION OF THE FINITE-RANGE THREE-BODY PROBLEM TO TWO VARIABLES*

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ABSTRACT

It is shown explicitly that for finite-range two-body forces which contribute significant interactions in only $L + 1$ orbital angular momentum states, the Faddeev equations for the three-body T matrix with total angular momentum J can be reduced to well-defined integral equations for functions of two continuous variables with $3(L + 1) \times \min(2J + 1, 2L + 1)$ components. Hence numerical calculation for realistic interactions, and analytic investigation of the dependence on two-body dynamics (which is explicitly separated from the geometrical part of the problem), become possible.

Although the non-relativistic three-body problem has been given a well-defined mathematical structure by Faddeev[1] and reduced from six to three variables by Omnes,[2] the resulting equations are still so formidable that no one has yet attempted an exact solution for any specific problem using local two-particle interactions. We will show in what follows that for the case of interest for strong interactions, in which the finite range of the two-particle (pairwise) interactions insures the dominance of a finite number of two-particle angular momentum states, these equations can be reduced to coupled integral equations in two variables. These equations have a sufficiently simple structure to offer a reasonable prospect of numerical solution in physically interesting cases. Further, the reduction explicitly separates the geometrical (kinematical) part of the problem from that part which depends on two-body dynamics, and provides a useful starting point for discussions of the analytic structure of the dynamical part of the three-body problem.

The original Faddeev equations give the three-body transition matrix T for the transition from a state $\vec{p}_i$ to a state $\vec{p}_i'$ $(i=1,2,3)$ as the sum of three terms $T^{(i)}$ expressed in terms of integrals over two-body transition matrices $t^{(i)}$ in the same 9-dimensional Hilbert space. Omnes has shown that by changing variables to the three energies $\omega_i = \vec{p}_i^{\,2}/2m_i$, the total momentum $\vec{P} = \sum_i \vec{p}_i = 0 = \vec{P}'$, the total angular momentum J^2, its projection on a space-fixed axis, M_J, and its z-component along a body-fixed axis in the plane of the momentum triangle, M, the J component of these operators can be written in the four-dimensional space of $\vec{\omega} = (\omega_1, \omega_2, \omega_3)$, and M, $(-J \leq M \leq J)$. By taking matrix elements of these operators in this space, he finds that

$$T^{(i)}_{M'M}(\vec{\omega}', \vec{\omega}) = \frac{m_j m_k}{\sqrt{2m_i\omega_i}} \delta(\omega_i - \omega_i') \, t^{(i)}_{M'M}(\vec{\omega}', \vec{\omega}) +$$

$$\frac{m_j m_k}{\sqrt{2m_i\omega_i'}} \sum_{s=j,k} \int_0^\infty d\omega_i'' \int_0^\infty d\omega_j'' \int_{m_k^{-1}(\sqrt{m_i\omega_i''} - \sqrt{m_j\omega_j''})^2}^{m_k^{-1}(\sqrt{m_i\omega_i''} + \sqrt{m_j\omega_j''})^2} d\omega_k''$$

$$\sum_{M''=-J}^{M''=J} \frac{t^{(i)}_{M'M''}(\vec{\omega}', \vec{\omega}'') \, \delta(\omega_i'' - \omega_i')}{z - \omega_1'' - \omega_2'' - \omega_3''} \times T^{(s)}_{M''M}(\vec{\omega}'', \vec{\omega}) \, , \tag{1}$$

$i, j, k,$ cyclic on $1, 2, 3$.

The physical transition matrix is to be obtained by solving these equations and taking the limit $z \to E + i0$ with $E = \omega_1 + \omega_2 + \omega_3 = \omega_1' + \omega_2' + \omega_3' = E'$. Although the δ-functions in the kernels remove one of the integrations, these form a set of $3(2J+1)$ coupled integral equations in three continuous variables, as will become rapidly apparent to anyone who attempts to set them up for numerical computation; so far as we can see, this exceeds the capacity of any existing computer.

In order to reduce the problem further, we assume that the two-body (off-shell) transition matrix $t^{(i)}$ for the interaction between the jk pair contains significant interactions in only $L + 1$ orbital angular momentum states. We make use of the addition theorem for spherical harmonics to express the dependence on the angle between the initial center-of-mass momentum $\vec{q}_{jk}$ and the final momentum $\vec{q}_{jk}'$, in terms of the angle γ_i between $\vec{q}_{jk}$ and $\vec{p}_i$, the angle α_i between $\vec{p}_i$ and any arbitrarily chosen body-fixed axis in the plane of the triangle, and the similarly defined angles γ_i' and α_i' for the final state. The azimuthal integration over the angle u defined by Omnes can then be performed, and we find

$$t^{(i)}_{M'M}(\vec{\omega}', \vec{\omega}) = \sum_{\ell=o}^{L} \frac{m_j+m_k}{2\pi m_j m_k} (2\ell+1)\, t^{(i)}_{\ell}(\omega_1'+\omega_2'+\omega_3' - r_i\omega_i',\ \omega_1+\omega_2+\omega_3 - r_i\omega_i;\ z-\omega_i)$$

$$\times \sum_{\lambda=-\min(J,\ell)}^{\min(J,\ell)} \frac{(\ell-\lambda)!}{(\ell+\lambda)!}\, d^J_{M'\lambda}(-\alpha_i') P^{\lambda}_{\ell}(\cos\gamma_i')\, P^{\lambda}_{\ell}(\cos\gamma_i)\, d^J_{\lambda M}(\alpha_i) \quad ,$$

$$r_i = (m_1 + m_2 + m_3)/(m_j + m_k) \quad . \tag{2}$$

Here $t^{(i)}_{\ell}(q'^{\,2}_{jk}/2\mu_{jk},\ q^2_{jk}/2\mu_{jk};\ k^2/2\mu_{jk})$ are the usual partial wave transition amplitudes normalized to reduce to $e^{i\delta_\ell}\sin\delta_\ell/k$ on-shell. We write the arguments of $t^{(i)}_{\ell}$ in terms of the energy variables in order to emphasize the kinematic fact that $t^{(i)}_{M'M}(\vec{\omega}', \vec{\omega})$ depends on $\vec{\omega}'$ only through the combination $E' = \omega_1' + \omega_2' + \omega_3'$, and the energy of the noninteracting particle ω_i'. Aside from trival factors, $t^{(i)}_{M'M}(\vec{\omega}', \vec{\omega})$ is both the kernel and the inhomogeneous term of the Faddeev equations. As a result the solution, $T^{(i)}_{M'M}(\vec{\omega}', \vec{\omega})$, depends only on the pairs of variables E, ω_i and E', ω_i'. Furthermore the dependence on the magnetic quantum numbers M and M', occurs only through geometrically known separate factors. We exhibit this behavior explicitly by defining

$$T^{(i)}_{M'M}(\vec{\omega}', \vec{\omega}) = \sum_{\ell'=o}^{L} \sum_{\lambda'=-\min(J,\ell')}^{\min(J,\ell')} (2\ell'+1)\, \frac{(\ell'-\lambda')!}{(\ell'+\lambda')!}\, d^J_{M'\lambda'}(-\alpha_i')\, P^{\lambda'}_{\ell'}(\cos\gamma_i')\, F^{P(i)}_{\ell'\lambda'}(E', \omega_i') \tag{3}$$

The index P in the amplitude, $F^{P(i)}_{\ell'\lambda'}(E', \omega_i')$, is written to remind us that it will depend parametrically on z, $\vec{P} = 0$, J, M_J, M and $\vec{\omega}$ through the inhomogeneous term and (as we will see) on z only through the kernel. Our reduction to two variables will now be complete, provided only we can find an appropriate transformation of variables in the integrations.

The change of variables must be made with some care; the variable E' is common to each of the three equations, but the variable ω_i is different in each. However, this different variable on the left is precisely the variable ω_j or ω_k which must be used differently in the two integrations on the right if we are to preserve consistency. Again making use of the kinematics, we note that the variable orthogonal to E, ω_j is the angular function $\cos\gamma_j$ defined above, while for the k term we require E, ω_k, $\cos\gamma_k$. Explicitly, the transformations in the ith equation are

$$\omega_i'' = \frac{m_k}{m_k + m_i}\left\{E'' + \left(\frac{m_i m_j}{m_k(m_i + m_k)} - r_j\right) e_j'' + 2\sqrt{\frac{m_i m_j e_j''}{m_k(m_k + m_i)}}\sqrt{E'' - r_j e_j''}\cos\gamma_j''\right\},$$

$$\omega_j'' = e_j'' , \tag{4}$$

$$\omega_k'' = \frac{m_i}{m_i + m_k}\left\{E'' + \left(\frac{m_k m_j}{m_i(m_i + m_k)} - r_j\right) e_j'' - 2\sqrt{\frac{m_j m_k e_j''}{m_i(m_k + m_i)}}\sqrt{E'' - r_j e_j''}\cos\gamma_j''\right\},$$

for the term with $s = j$. The variable transformation for the term with $s = k$ can be obtained by letting $i \to j$, $j \to k$, and $k \to i$ on both sides of Eq. (4). The range of e_j'' is still from 0 to ∞ if $r_j e_j'' \leq E'' \leq \infty$, while $\cos\gamma_j''$ can vary from -1 to 1 independent of the energies. However, the argument of the δ-function does not necessarily lie in this range, which puts further obvious kinematic limits on the E'' integration. With this caution, the substitution of Eqs. (2) and (3) in Eq. (1) can now be carried through, and we find that

$$F^{P(i)}_{\ell'\lambda'}(E',e'_i) = \frac{(m_j+m_k)}{2\pi\sqrt{2m_i e'_i}} \Bigg\{ t^{(i)}_{\ell'}(E'-r_i\omega_i,\, E-r_i\omega_i,\, z-\omega_i)\, \delta(e'_i-\omega_i)\, d^J_{\lambda' M}(\alpha_i)\, P^{\lambda'}_{\ell'}(\cos\gamma_i)$$

$$+ \sum_{s=j,k} \int_o^\infty de'' \int_{r_s e''}^\infty dE'' \, \frac{t^{(i)}_{\ell'}(E'-r_i e'_i,\, E''-r_i e'_i,\, z-e'_i)}{z-E''}$$

$$\times \sum_{\ell'',\lambda''} K^{(i,s)}_{\ell'\lambda';\ell''\lambda''}(E',e'_i;E'',e'')\, F^{P(s)}_{\ell''\lambda''}(E'',e'') \Bigg\} \qquad (5)$$

where there are two θ-functions in the kernel which further restrict the E'' integration. We note that the amplitudes depend parametrically on $\vec{\omega}$ and M through known geometric functions in the inhomogeneous term and that the sum over M'' has disappeared. Explicitly

$$K^{(i,s)}_{\ell'\lambda';\ell''\lambda''}(E',e'_i;E'',e'') = \frac{(2\ell''+1)(\ell''-\lambda'')!}{(\ell''+\lambda'')!} \; \frac{\sqrt{2m_s e''}\sqrt{2\mu_{is'}(E''-r_s e'')}}{m_i+m_{s'}}$$

$$\times \int_{-1}^{1} d(\cos\gamma''_s)\, \delta\Big(e'_i - \omega''_i(E'',e'',\cos\gamma''_s)\Big)$$

$$\times P^{\lambda'}_{\ell'}\Big(\cos\gamma''_i(E'',e'',\cos\gamma''_s)\Big)\, P^{\lambda''}_{\ell''}(\cos\gamma''_s) \qquad (6)$$

$$\times \sum_{M''=-J}^{J} d^J_{\lambda' M''}\Big(\alpha''_i(E'',e'',\cos\gamma''_s)\Big)\, d^J_{M''\lambda''}\Big(-\alpha''_s(E'',e'',\cos\gamma''_s)\Big)$$

where $s' = k$ if $s = j$, or $s' = j$ if $s = k$. Doing the integration over the delta function and the sum over M'' gives

$$K^{(i,s)}_{\ell'\lambda';\ell''\lambda''}(E',e'_i;E'',e'') = \left\{ \Theta\left[E''-r_s e'' - \left(\sqrt{\frac{m_{s'}+m_i}{m_{s'}}}\, e'_i - \sqrt{\frac{m_i m_s e''}{m_{s'}(m_i+m_{s'})}}\right)^2\right] - \Theta\left[E''-r_s e'' - \left(\sqrt{\frac{m_{s'}+m_i}{m_{s'}}}\, e'_i + \sqrt{\frac{m_i m_s e''}{m_{s'}(m_i+m_{s'})}}\right)^2\right]\right\}$$

$$\times \frac{(2\ell''+1)(\ell''-\lambda'')!}{(\ell''+\lambda'')!}\, P^{\lambda'}_{\ell'}\left(\cos\gamma_i\,[E'',e'',\cos\Gamma_s(e'_i,E'',e'')]\right)$$

$$\times P^{\lambda''}_{\ell''}\left(\cos\Gamma_s(e'_i,E'',e'')\right)\; d^J_{\lambda'\lambda''}(\alpha''_i-\alpha''_s)\ , \tag{7}$$

where $\cos\Gamma_s$ is the value of $\cos\gamma''_s$ determined by the delta function in Eq. (6) and is given by

$$\cos\Gamma_s(e'_i,E'',e'') = \frac{\frac{m_{s'}+m_i}{m_{s'}}\, e'_i - E'' + r_s e'' + \frac{m_i m_s e''}{m_{s'}(m_i+m_{s'})}}{2\sqrt{\frac{m_i m_s e''}{m_{s'}(m_{s'}+m_i)}}\ \sqrt{E''-r_s e''}}\ , \tag{8}$$

for $s=j$ and the negative of this expression for $s=k$. Although we have carried through the algebra here only for the spinless case, it is obvious that the proof can be carried through immediately for arbitrary spin and isospin by introducing the appropriate spin-angular functions in the decomposition of the two-body t-matrices, the only effect being to complicate the parametric structure of the inhomogeneous term and the purely geometric kernel K. We believe this is better done for specific cases where the spin and isospin symmetries of the interactions can be directly utilized to simplify the geometric structure at an earlier stage, and do not attempt to give a general formula here.

We wish to emphasize that these are now well-defined integral equations in two continuous variables with a maximum of $3(L+1) \times \min(2J+1, 2L+1)$ components, and that the dynamical singularities of the two-body interactions have been explicitly separated, in so far as is physically allowable, from the purely geometrical coupling between the three interacting subsystems.

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