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Phenomenology of codimension-2 brane worlds: the importance of back-reaction

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PHENOMENOLOGY OF CODIMENSION-2 BRANE WORLDS

PHENOMENOLOGY OF CODIMENSION-2 BRANE WORLDS:
THE IMPORTANCE OF BACK-REACTION

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Abstract

In this thesis, we describe the properties of brane worlds embedded in a spacetime with two extra dimensions. We derive and describe the boundary conditions that branes impose on the bulk fields in the theory, and show that they reproduce known results for D7 branes in F-theory compactifications of type IIB supergravity. We show how brane-bulk couplings can stabilize moduli of a flux stabilized compactification of extra dimensions. An important new ingredient is that the branes can have a magnetic coupling to the flux that stabilizes the bulk. This coupling allows the system to relax the stringent constraints of flux quantization, which allows the bulk spacetime to respond to perturbations of the branes. We derive the dynamics of the lower-dimensional effective theory below the Kaluza-Klein scale, and show that the contributions of the magnetic coupling can be competitive with the tension of the brane.

We first describe the simplest flux compactification: an Einstein-scalar-Maxwell theory in 6 dimensions. We find that the effective potential in 4 dimensions gets minimized at the position one would naively expect – at the stationary point of the sum of all the brane Lagrangians – but its value at the minimum gets changed by the magnetic coupling to the brane.

Next we find that if the bulk is described by 6 dimensional gauged chiral supergravity, the effect of the magnetic coupling allows the curvature on the brane to be suppressed relative to the generic scale of the tension on the branes. We use this observation to construct an explicit brane-bulk system that has a technically natural cosmological constant of the correct size. The classical on-brane curvature vanishes in our construction, and the first order quantum corrections give a value to the cosmological constant of the right order of magnitude. We estimate higher loop corrections, and they are greatly suppressed.

Co-Authorship

Chapters (2) through (5) of this thesis all represent original papers written by myself, Leo van Nierop, and published in New Journal of Physics (chapter (3)) and Journal of High Energy Physics (chapters (2,4,5). The journal references are

Chapter (2): JHEP 0808:061, 2008 (arXiv:0802.4221)

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Chapter (4): JHEP 1102:094, 2011 (arXiv:1012.2638)

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Chapter (6) has been submitted to Journal of High Energy Physics as well, and is available as a preprint at (arXiv:1108:0345). The works are all collaborative, and my coauthors are:

Dr. C. P. Burgess on all papers

Dr. Claudia de Rham on chapter (2) (JHEP 0808:061, 2008)

Allan Bayntun, M.Sc. on chapter (3) (New Journal of Physics 12 (2010) 075015)

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> "Babble Babble Bitch Bitch Rebel Rebel Party Party Sex Sex Sex and don't forget the Violence."
>
> ---
>
> MARILYN MANSON (1969-)

“BRAAAAAAAAAAAAAAAAAAAAAANES”

Zombie Leo (2054-∞)

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Conventions and notation

In particle physics, there are various conventions on notation and choices of signs. Throughout this thesis, we use the following conventions unless specified otherwise:

- All equations assume natural units: $\hbar = c = 1$. The unit of choice for energy is GeV $\approx 1.6 \cdot 10^{-10}$J. The unit of choice for length is $(\text{ GeV})^{-1} \approx 2.0 \cdot 10^{-16}$m.
- The metric of spacetime has negative signature, $\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$ (and additional entries with $+1$ for any extra dimensions).
- The Riemann tensor, Ricci tensor and Ricci scalar are defined as

$$\begin{aligned} R^{\mu}{}_{\nu\rho\sigma} &= \partial_\sigma \Gamma^{\mu}_{\nu\rho} - \partial_\rho \Gamma^{\mu}_{\nu\sigma} + \Gamma^{\tau}_{\nu\rho}\Gamma^{\mu}_{\tau\sigma} - \Gamma^{\tau}_{\nu\sigma}\Gamma^{\mu}_{\tau\rho} \\ R_{\mu\nu} &= R^{\sigma}{}_{\mu\sigma\nu} \\ R &= g^{\mu\nu} R_{\mu\nu} \end{aligned} \tag{1}$$

  following Weinberg[1].

- When dealing with extra dimensions, Greek indices (μ, ν etc.) denote spacetime indices in the non-compact directions (usually, 4-dimensional spacetime). Capital latin indices (M, N etc.) denote indices in the full spacetime. Finally, lower case latin indices (m, n etc.) denote the extra dimensions only.

[1] Weinberg, *Gravitation and Cosmology*, Wiley 1973

Chapter 1

Introduction

In this thesis, we will discuss the issue of the naturalness of the cosmological constant. We explain the requirement of technical naturalness, and how the cosmological constant does not satisfy this within the standard model of particle physics[1]. We show an explicit construction of a theory that includes the standard model, but in which the cosmological constant *is* technically natural. Our model is based on the proposal of supersymmetric large extra dimensions (SLED, (1.1)), and we include sources that explain (and stabilize) the size of the extra dimensions.

1.1 Decoupling of scales in physics

The notion of naturalness of parameters in a theory of physics relies in part on the idea of decoupling. We give a quick outline of decoupling, and why it is relevant. The natural world that we observe can be described at many different scales. On astrophysical scales, it is well described by massive bodies

[1] The standard model contains all known forces except for gravity, and all known particles. It is of great experimental success, and the cosmological constant problem is one of the few conceptual difficulties within the standard model.

moving under gravity only. To understand the orbits of the planets, say, it is not necessary to understand any details about the planets. Modeling them as point masses gives a very good approximation. This is an example where details on short distances are irrelevant for the physics at large distances. It is this decoupling between wildly different scales that allows us to make any progress in physics, by allowing us to figure out small parts of the underlying laws, and use them to understand the more complicated physics.

With every length scale, there is an associated momentum scale through the de Broglie relation, $p = 1/2\pi\lambda$ (in natural units). There is an associated energy scale that comes from the relativistic relation $E^2 = p^2 + m^2$. In terms of energy, the decoupling of scales states that low energy processes can be understood well without knowing the details of any high energy processes that take place at the same time.

1.1.1 Effective field theories

We need to understand what the decoupling of scales entails for particle theory. A generic quantum field theory in particle physics consists of two parts: First, a Lagrangian that defines the particles and interactions in the theory. Second, a domain of validity. This usually takes the form of an energy scale at which this Lagrangian can no longer be trusted, because in the real world there are particles that can be created that are not described by this Lagrangian. As soon as the energies available in a process allow for the excitation of such a heavy state, a more complicated Lagrangian has to be used. Conversely, given a Lagrangian that is valid to some very high energy, the low energy behaviour of the theory can be studied in a reduced, effective theory. This is done by accounting for the heavy particles in how they change the interactions between light particles (1.2). The process of finding the low energy theory is referred to as integrating out the heavy states

Within the context of this thesis, any theory we work with will be an effective theory. The existence of a fundamental, UV complete theory is implicitly assumed, though it is not required: even if it is turtles all the way down, it is worth studying the next turtle!

1.1.2 Naturalness in particle physics

In particle physics, a generic model is described by a Lagrangian which is a functional of a set of fields and their derivatives. Typically there are various dimensionless coupling constants, as well as some number of dimensionful couplings (like masses of particles). All other things being equal, one would expect that the dimensionless couplings are all roughly of the same size. Similarly, the dimensionful couplings should be set by roughly the same scale.

Naturalness issues arise when there is a large ratio in physical quantities that are expected to be related to one another. In that case, it should be possible to explain why there is a large discrepancy. The need for such an explanation comes in two parts. The first question is why there is a large hierarchy when the ratio is calculated in the fundamental theory at very high energy. Since we do not know much about this theory, this question is not very pressing (and probably not answerable until much higher energies become experimentally available). The second part of the naturalness issue demands that the large ratio is understood in every effective theory below some scale, not just in one specific effective theory. This is called technical naturalness, and it is the question that we are addressing in this thesis.

1.1.3 Technical naturalness

Our view on technical naturalness is discussed in detail in chapter (6), here I will just give a broad overview. The point of technical naturalness is that we do not know the fundamental theory, so we cannot say if a large ratio emerges from the theory in a natural way. One way this can happen is through the running of couplings due to quantum effects, which can make some couplings grow while others shrink. What we do know, is an effective theory that describes the hierarchy. The question we can ask is how much this description changes when we integrate out heavy states to obtain a new effective theory which is only valid at lower energy. Suppose we start with an effective Lagrangian that is valid to some energy higher than the mass of a certain particle. Whenever we calculate an effective theory valid *below* the mass of that particle, this particle will no longer appear in the effective Lagrangian, and its effects need to be taken into account by changing the effective couplings in the rest of the Lagrangian.

In this process, a large ratio is considered natural if it is stable under integrating out heavy particles. On dimensional grounds, we can estimate for which type of couplings this may be problematic. A dimensionless coupling λ can only depend on ratios of scales, and the generic contributions due to loop effects are proportional to $\delta\lambda \propto \ln(E/M)$, with M the mass of the state that is integrated out, and E the energy scale of the process at hand (e.g. the center of mass energy of a scattering). Since the dependence on the high mass is only logarithmic, dimensionless couplings vary only mildly with scale, so they usually do not ruin an existing hierarchy.

On the other hand, couplings with positive mass dimension tend to get contributions proportional to the mass that is integrated out. Those contributions can be much larger than the final value of the coupling. In that case, there is an almost magical cancellation between the dimensionful coupling including the heavy state, and the contribution due to loops of the heavy state. As an example, the cosmological constant changes as $\Lambda_{\text{low}} = \Lambda_{\text{high}} + \#M^4$, with # a number that can be found from a vacuum loop graph. Here the cosmological constant problem becomes apparent: When all the contributions of all massive particles are added to the fundamental cosmological constant, we should end up with the physical value that sets the cosmic acceleration. This means there is an immense cancellation between all those contributions. Even if we assume that there are no new particles above the electroweak scale ($\propto$ 1 TeV), then the loop graphs have to cancel the bare cosmological constant to about 48 decimal places... but not beyond that! It is in this sense that the cosmological constant is not technically natural unless there is a mechanism at work that is unaccounted for in this description.

1.2 Extended objects in model building

The current best candidate for a complete theory of nature, which describes all fundamental interactions in a single framework, is string theory. It has been found (1.3) that the theory requires the existence of various extended objects, on which open strings can end. They have been dubbed branes[2], as a generalization of membrane.

[2]To distinguish between physics of the surface and physics of the spacetime in which they are embedded, the terms bulk and brane physics are used.

If we take the existence of branes seriously, then we need to know how their existence affects physics at low energy. In particular, when building models of particle physics beyond the standard model the inclusion of branes can have a large impact: all non-gravitational[3] particles are represented in string theory by open strings, which need to end on branes. This means that different particles can have their dynamics restricted to different numbers of dimensions.

The number of dimensions that particles feel has a large impact on their behaviour. As an example, the $1/r^2$ law for gravity and electromagnetism can be understood as a conservation law, because the area enclosed by the source grows as r^2. This means we can interpret the force law as the number of (virtual) force carriers per unit area. If the force carrier has access to extra dimensions, then the force law gets modified to $1/r^{2+n}$ when measured at distances smaller than the linear size of the n extra dimensions.

1.2.1 Brane world models

The realization that not all particles have to feel the same number of dimensions has opened up a new class of models to be studied. In those models, the universe as we know it is a brane, embedded in a higher dimensional spacetime. From the point of view of string theory (1.4) the total number of dimensions is expected to be 10 or 11. This means that 6 or 7 of those have to be compact, and of sufficiently small extent that their existence has evaded our detection so far.

The only particle that absolutely has to see all dimensions is the gravi-

[3] The closed string spectrum includes the gravity multiplet (metric, antisymmetric tensor, dilaton, and fermions related by supersymmetry), and some of the gauge potentials (their number and rank depend on the string theory).

ton (and its multiplet, if supersymmetry is present). The reason for this is that it is an excitation of the geometry, and all of the dimensions are part of the geometry of spacetime. This means that the bounds on the existence of extra dimensions in a brane world scenario are set by tests of gravity. Due to its weak coupling at energies that are accessible to accelerator experiments, there are no significant bounds on the extra dimensions from such experiments. Useful bounds come from tests of gravity at short distances, such as searches for deviations from the inverse square law (1.5) or the equivalence principle (1.6).

Collider physics *does* place very strong bounds on what extra dimensions can be seen by standard model particles. For any existing particle, there is a tower of excitations labeled by higher momentum wavefunctions in the extra dimensions. Pictorially, this is similar to the infinite square well in quantum mechanics. There is a tower of states, with the step in energy set by the Kaluza-Klein (1.7) scale, $M_{KK} \simeq 2\pi/r$ with r the linear size of the extra dimensions. If the extra dimensions are strongly curved, the new scale can be set by the inverse of the radius of curvature instead. Since we have not seen any such states in colliders, the KK mass scale for standard model particles must be at least on the order of the TeV scale from experiments at Tevatron. As we will see in the next section, such small dimensions are not particularly helpful for the hierarchy problems we try to address.

1.3 Extra dimensions and naturalness

The possible existence of extra dimensions introduces an additional ingredient in discussions of naturalness. The typical size of the extra dimensions $\mathcal{V}_n \propto r^n$

introduces a new energy scale $r^{-1} \simeq M_{KK}$. The question of naturalness of parameters now considers the fundamental, higher dimensional Lagrangian. At low energies, where the effective Lagrangian is 4 dimensional, parameters are set not just by the fundamental scale (the string tension (T_s), say), but also by the size of the extra dimensions. In particular, if the dimensionless combination $T_s r^2$ is much larger (or smaller) than 1, this can create an observed hierarchy that is natural.

At this point, the size of the extra dimensions takes the place of the unexplained hierarchy in the Lagrangian. This is not yet much progress, as it is just moving the problem around: why is the size of the extra dimensions not set by the inverse of the fundamental energy scale in the Lagrangian? Any model that attempts to address hierarchy problems with extra dimensions needs to answer this question.

As an example, there is a class of models called warped Randall-Sundrum (1.8) models, that have 1 extra dimension. In those models the extra dimension has a brane as its boundary (either on just one end, with the extra dimensions a half-line, or a boundary on both ends). The effect of the branes is to 'warp' the extra dimensions, in a way where distances along the brane directions depend exponentially on the position in the extra dimension. In this setup a large factor can appear by having a much smaller factor that becomes exponentiated. This means that ratios of the parameters that define the higher dimensional theory may be of order 10^2 or less, while still leading to measured hierarchies of order $\exp(10^2) \simeq 10^{43}$.

The reason that this is a desirable property, is that ratios of 100 or so happen all the time. Their technical naturalness is much less problematic, because the modifications to the scales from integrating out particles are not *just* set by the mass of the heavy state. Generally there are also factors of coupling constants involved, which can be fairly small as well. For example, the dimensionless electromagnetic coupling constant is $\alpha \approx 1/137$. If such a moderate hierarchy is natural, then exponentiating it automatically leads to a large hierarchy that is also natural.

1.3.1 Why 2 extra?

Most of this thesis deals with systems where the known 4 noncompact dimensions are supplemented by 2 compact, internal dimensions. There are various reasons to focus on two extra dimensions. One is a point of practicality: adding more and more extra dimensions adds to the complexity and intractability of the model. Systems with just one extra dimension have been studied extensively (1.8; 1.9), but even the qualitative results don't carry over to higher dimensions very well. The problem is that for more than 1 extra dimension, bulk fields often[4] diverge at the location of sources. On the contary, in codimension-1 systems all fields remain finite at the sources. This makes codimension-2 the simplest case that has a hope of being representative for generic brane worlds.

The other reason to focus on 2 extra dimensions has to do with the specific values of the hierarchies we wish to address. The relation between the 4 dimensional Planck mass and the higher dimensional gravity scale $M_\star$ is given by dimensional reduction:

$$M_p^2 \simeq M_\star^{2+n} r^n \,. \tag{1.1}$$

[4]For 2 extra dimensions, there are exceptions: a cosmic string, for example, has a conical singularity but no divergences

$M_\star$	n	$M_{KK} = r^{-1}$	r
100 GeV	1	10^{-32} GeV	$2 \cdot 10^{16}$ m
100 GeV	2	10^{-15} GeV	0.2 m
100 GeV	3	$5 \cdot 10^{-10}$ GeV	$4 \cdot 10^{-7}$ m
1 TeV	1	10^{-29} GeV	$2 \cdot 10^{13}$ m
1 TeV	2	10^{-13} GeV	$2 \cdot 10^{-3}$ m
1 TeV	3	$2 \cdot 10^{-8}$ GeV	10^{-8} m
10 TeV	1	10^{-26} GeV	$2 \cdot 10^{10}$ m
10 TeV	2	10^{-11} GeV	$2 \cdot 10^{-5}$ m
10 TeV	3	10^{-6} GeV	$2 \cdot 10^{-10}$ m

Table 1.1: Some examples of the radius of the extra dimensions for various numbers of dimensions and various gravity scales.

In order for the gravitational scale and electroweak scale to be natural, we wish to choose $M_\star$ near the TeV scale. Since the Planck scale is fixed, this means that we need to choose the radius of the extra dimensions as

$$r = \sqrt[n]{\frac{M_p^2}{M_\star^{2+n}}} \,. \tag{1.2}$$

From the sizes of the extra dimensions in table (1.1), it is obvious that only 1 extra dimension does not work in this context: The extra dimension has to be of macroscopic size to solve the hierarchy between the electroweak scale and the gravity scale. Of particular interest is the second to last line, highlighted with red, for two reasons. First, the size of the extra dimensions is comparable to the experimental limits on extra dimensions from tests of gravity (1.5). Second, the Kaluza-Klein (1.7) scale $M_{KK} = r^{-1} = 10^{-11}$ GeV is (more or less) of the right size for the cosmological constant: if the scale of the cosmological constant is set by the KK scale, this results in $\Lambda = M_{KK}^4 = 10^{-44}$ GeV. Comparing this to the cosmological constant, $\Lambda \simeq 10^{-38}$ GeV^4, the discrepancy is a factor of 10^6. However, in terms of the energy scale M_{KK} the discrepancy is only a factor of about 30.

The choices in the table (1.1) are only examples, and they depend (among other things) on the shape of the extra dimensions. This means that if the extra dimensions conspire to have an effective cosmological constant on the order of the KK scale, both the electroweak hierarchy problem and the cosmological constant problem can be solved with 2 extra dimensions with the size of tens of microns.

The effective cosmological constant is not generically set by the KK scale, there normally are larger contributions in the bulk. However, when the 2 large extra dimensions are described by a supergravity, the dominant contribution *is* set by the KK scale. This is the proposal of SLED (1.1) (Supersymmetric Large Extra Dimensions), and we summarize this result in chapter (6).

In addition to 2 large (micron) extra dimensions, a string theoretical UV completion requires another 4 extra dimensions. The 6 dimensional theory is an effective description with excitations in those 4 dimensions already integrated out. We will not be concerned with the other 4, and simply assume them to be stabilized at a sufficiently high scale.

In order to use this numerology as a genuine solution to the naturalness problems, it is not enough to just proclaim the size of the extra dimensions. They are required to be dynamically stabilized, and the parameters that are required to stabilize the two dimensions at this large size need to be natural in their own right. Chapter 6 deals with this stabilization, and discusses the naturalness of the construction.

1.4 Outline of the work

The previous sections describe the questions that guided the work in this thesis. Chapter (2) explores the interactions of a bulk scalar field with a codimension 2 brane, with applications to electroweak symmetry breaking. The extra dimensional scalar plays the role of the Higgs field by acquiring a v.e.v. through its interactions with the brane. The effects of gravity are neglected in this chapter. Chapter (3) sets up the framework for determining the back reaction of codimension 2 branes on a general bulk theory involving gravity, a set of scalar fields, and a bulk Maxwell field. We also describe the effective potential for the scalar fields below the Kaluza-Klein scale. In Chapter (4) we explore the consequences of those matching conditions for the simplest flux-stabilized extra dimensions. This chapter also describes the possibility of localizing part of the Maxwell flux on the branes due to a magnetic interaction. Chapter (5) develops the same construction for the Salam-Sezgin (1.10) solutions of gauged, chiral supergravity (1.11). Finally, chapter (6) takes the tools developed in the previous chapters to describe an explicit realization of the SLED (1.1) proposal, including the stabilization of the extra dimensions. The crucial new ingredient of this construction is the localization of Maxwell flux on the branes. This allows the construction to be robust against perturbations of the brane properties.

Bibliography

[1.1] Y. Aghababaie, C. P. Burgess, S. L. Parameswaran and F. Quevedo, "Towards a naturally small cosmological constant from branes in 6D supergravity," Nucl. Phys. B **680** (2004) 389 [arXiv:hep-th/0304256].

[1.2] TASI Lectures on Effective Field Theories, Ira Z. Rothstein, [arXiv:hep-ph/0308266v2]

[1.3] TASI Lectures on D-Branes, Joseph Polchinski, [arXiv:hep-th/9611050v2]

[1.4] Michael B. Green, John H. Schwarz, Edward Witten, *Superstring Theory: Volume I, Introduction* Cambridge Monographs on Mathematical Physics 1987

[1.5] E. G. Adelberger, B. R. Heckel and A. E. Nelson, "Tests of the gravitational inverse-square law," Ann. Rev. Nucl. Part. Sci. **53**, 77 (2003) [arXiv:hep-ph/0307284];

C. D. Hoyle, D. J. Kapner, B. R. Heckel, E. G. Adelberger, J. H. Gundlach, U. Schmidt and H. E. Swanson, "Sub-millimeter tests of the gravitational inverse-square law," Phys. Rev. D **70** (2004) 042004 [arXiv:hep-ph/0405262].

[1.6] C. M. Will, "The confrontation between general relativity and experiment," Living Rev. Rel. **4** (2001) 4 [arXiv:gr-qc/0103036]; Living Rev. Rel. **9** (2005) 3 [arXiv:gr-qc/0510072];
E. G. Adelberger, J. H. Gundlach, B. R. Heckel, S. Hoedl and S. Schlamminger, "Torsion balance experiments: A low-energy frontier of particle physics," Prog. Part. Nucl. Phys. **62** (2009) 102.

[1.7] T. Kaluza, "On The Unification Problem In Physics" HUPD-8401, Mar 1984.

O. Klein, "Quantentheorie und fnfdimensionale Relativittstheorie" Zeitschrift fr Physik A Hadrons and Nuclei Volume 37, Number 12, 895-906

[1.8] L. Randall, R. Sundrum, "A Large Mass Hierarchy from a Small Extra Dimension" Phys. Rev. Lett. **83** (1999) 3370 [hep-ph/9905221]; "An Alternative to Compactification" Phys. Rev. Lett. **83** (1999) 4690 [hep-th/9906064].

[1.9] TASI Lectures on Extra Dimensions and Branes, Csaba Csaki, [arXiv:hep-ph/0404096v1], and references therein

[1.10] A. Salam and E. Sezgin, "Chiral Compactification On Minkowski X S**2 Of N=2 Einstein-Maxwell Supergravity In Six-Dimensions," Phys. Lett. B **147** (1984) 47.

[1.11] H. Nishino and E. Sezgin, *Phys. Lett.* **144B** (1984) 187; "The Complete N=2, D = 6 Supergravity With Matter And Yang-Mills Couplings," Nucl. Phys. **B278** (1986) 353;

S. Randjbar-Daemi, A. Salam, E. Sezgin and J. Strathdee, "An Anomaly Free Model in Six-Dimensions" *Phys. Lett.* **B151** (1985) 351.

Chapter 2

The Hierarchy Problem and the Self-Localized Higgs

2.1 Preamble

This chapter is based on the work in (2.1). We describe the interactions of a 6 dimensional scalar field which couples to a 4 dimensional brane. The scalar field has a (large) mass M_B in the bulk. When the interaction with the brane is neglected all states in the effective theory have a mass larger than the bulk mass. The most important result of this work considering the theme of this thesis is that the coupling with the branes can drastically alter the energetics. In particular, the lowest energy state of the scalar field in the 4 dimensional effective theory has a mass that is *smaller* than the bulk mass.

This low mass state is separate from the Kaluza-Klein spectrum, which does start at M_B as usual. The reason that this state can have a smaller mass than the bulk mass is that it is divergent at the brane location. As a result the brane potential weighs in competitively with the bulk mass, despite being localized at a single point in the extra dimensions.

C.P. Burgess, Claudia de Rham and Leo van Nierop

2.2 Introduction

In the Standard Model (SM) the Higgs field is in many ways the odd man out. In the absence of the Higgs the only interactions that remain are gauge interactions, characterized by only a handful of coupling constants. But with the Higgs comes the deluge of parameters that parameterize our ignorance of the ultimate origins of the model's many masses and mixing angles. And among these parameters is the one dimensionful quantity, μ, that governs the size of the $\mu^2 H^* H$ term in the Higgs potential, and by fixing the size of the Higgs v.e.v. sets the scale for all masses. It is the sensitivity of this parameter to much heavier scales that is at the root of the hierarchy problem (2.2).

Historically, the hierarchy problem has been one of the main motivations for exploring brane-world scenarios for physics beyond the Standard Model (2.3; 2.4), for which all of the observed SM particles are trapped on a (3+1)-dimensional brane within an extra-dimensional bulk. Motivated by the observations that the Higgs is the lone SM particle yet to be observed, we here explore the idea that it is the only SM particle that is *not* confined to a brane: *i.e.* whereas all other SM particles live on a brane, the Higgs lives in the bulk. The hope is that this might account for its special role within the SM.

Brane-world models with the Higgs in the bulk have been examined in the literature, most often within the context of 5D Randall-Sundrum constructions (2.4). Yet these models differ from the present proposal in one of two ways: either by imagining the extra-dimensional Higgs to be related to other fields by supersymmetry (2.5; 2.6); or by taking the Higgs to be the 4D scalar component of what is ‘really’ an extra-dimensional gauge potential (2.7; 2.8). The motivation for doing so is the expectation that the extra-dimensional gauge symmetries can help alleviate the hierarchy problem, potentially allowing some of the properties of Higgs interactions to be unified with those of the gauge interactions. Implicit in this is the belief that a Higgs that is a bona-fide extra-dimensional scalar makes no progress towards alleviating the hierarchy problems of the usual 4D Higgs.

A model more similar to the one studied here was considered in ref. (2.9; 2.10), although from a different point of view. In ref. (2.9) the authors study the effects of codimension-2 brane couplings on a massless bulk scalar, with a focus on couplings close to the critical value for which the symmetry-breaking properties of the vacuum change. Ref. (2.10) generalizes to massive bulk fields, but without the focus of this paper on the hierarchy problem, and consequently without the study of couplings to fermions and gauge bosons described herein.

It is simple to see why extra dimensions in themselves are generally believed not to alleviate the hierarchy problem. This is because the Higgs potential,

$$U = -\frac{m_B^2}{2}\, H^* H + \frac{g}{4}\, (H^* H)^2 \,, \tag{2.1}$$

is always minimized by $H^*H = m_B^2/g$, where in n dimensions g has the (engineering) dimension of $(\text{mass})^{4-n}$ while m_B always simply has the dimension of mass. But the essence of the hierarchy problem is that because m_B is proportional to a positive power of mass, it generically receives contributions from heavy particles that grow with the mass, M, of the particles involved, and so is dominantly affected by the heaviest such particle that can contribute. Since m_B is a positive power of mass in any number of dimensions it is hard to see how the hierarchy problem can be ameliorated simply by placing the Higgs into the bulk.

In this paper we show why this simple argument is incorrect once the couplings between a bulk Higgs and the brane are properly taken into account. The brane-bulk interactions change the argument because the Higgs potential on the brane, U_b, and in the bulk, U_B, can disagree on which value of the Higgs v.e.v. has the least energy. In this case the system generically resolves this potential frustration by appropriately balancing these potential energies with the gradient energies which punish the field for attempting to interpolate between the two minima. But *if* the brane has codimension 2 (*i.e.* there are two dimensions transverse to the brane, such as for a (3+1)-dimensional brane situated in a 6D bulk), the Higgs likes to vary logarithmically near the branes, and the gradient energy associated with this variation is such that the resulting v.e.v. only depends *logarithmically* on the UV-sensitive term, m_B, of the bulk potential. Braneworld models can help with naturalness problems for a number of reasons; brane-bulk couplings provide a new way for them to do so. We show that the lunch is nevertheless not completely free, however, since the hierarchy problem gets partially recast as a requirement for the coefficients of the brane interactions $(H^*H)^2$ and $D_M H^* D^M H$ being required to be suppressed by very different scales. This kind of hierarchical suppression usually does not arise between two operators like these, that are not distinguished by low-energy symmetries or selection rules.

We also show how Higgs-brane interactions change another fundamental piece of widely-held intuition regarding the properties of a bulk Higgs. In the presence of a (positive) extra-dimensional mass term, $U_B = +\frac{1}{2}\, m_B^2 H^* H$, the spectrum of Kaluza-Klein (KK) states would usually be expected to consist of a multitude of levels (generically spaced by $M_c \sim 2\pi/L$ for a toroidal extra dimension of circumference L, say) that start at energies above a gap, $m_k \geq m_B$. We show here that brane-Higgs interactions can generically introduce a state which lives within this gap, $m < m_B$, that is 'bound' in the sense that its wave-function is localized at the position of the brane. We call this the 'self-localized' state inasmuch as its localization is a consequence only of the Higgs self-interactions and not on any geometric effects, such as those due to warping.

These arguments are presented in more detail in their simplest context in the next section, §2.3. §2.4 then tries to fashion an approach to the hierarchy problem by providing a preliminary discussion of the kinds of interactions that would be required for a realistic model, and the ways in which the low-energy Higgs couplings resemble and differ from those of the SM Higgs, as a function of the scales involved. §2.5 then follows with a discussion of some of the potential signatures and constraints such a scenario might have for Higgs physics. Our conclusions are briefly summarized in §2.6.

2.3 Vacuum Energetics of Extra-Dimensional Scalars

In this section we describe the interplay between brane and bulk energetics for the simplest toy model: a single real scalar, ϕ, in the presence of both brane and bulk potentials, U_b and U_B. We first review the more familiar situation of a codimension-1 brane in a 5D bulk, and then contrast this with the codimension-2 case with 6 bulk dimensions. (The situation for higher codimension is sketched in Appendix A.3.) Because they are peripheral to our main point we neglect gravitational effects in what follows, and so assume the mass scales involved are low enough for this to represent a good approximation.

2.3.1 Codimension-one

We first consider the codimension-one case, reproducing the results of ref. (2.11). Consider the following 5D scalar field theory, having both bulk- and brane-localized interactions,

$$S = -\int \mathrm{d}^4x \, \mathrm{d}y \, \left[\frac{1}{2} (\partial_M \phi \, \partial^M \phi) + U_B(\phi) + \delta(y) \, U_b(\phi) \right] , \tag{2.2}$$

with $\{x^M\} = \{x^\mu, y\}$. The field equation for this model is

$$\partial^M \partial_M \phi - U'_B(\phi) = \delta(y) U'_b(\phi) \, , \tag{2.3}$$

and the integration of this equation across the brane position (assuming continuity of ϕ) further implies the scalar jump condition

$$\left[\partial_y \phi \right]_0 = U'_b(\phi_0) \, , \tag{2.4}$$

where $\phi_0 = \phi(y = 0)$ and $[A]_0 = A(y = 0^+) - A(y = 0^-)$. The classical energy density per unit brane volume associated with a given field configuration in this model is then

$$\mathcal{H} = \int_{y_{\min}}^{y_{\max}} \mathrm{d}y \, \left[\frac{1}{2} \left(\dot{\phi}^2 + (\nabla \phi)^2 + (\partial_y \phi)^2 \right) + U_B(\phi) \right] + U_b(\phi_0) \, , \tag{2.5}$$

where $y_{\rm min} < 0 < y_{\rm max}$ and ∇ denotes differentiation in the in-brane spatial directions, $\{x^i\}$.

We now specialize to the case where the field has only a mass term in the bulk, while it has a quartic interaction on the brane. Keeping in mind that ϕ has dimension (mass)$^{3/2}$ in 5 dimensions,

$$U_B(\phi) = \frac{1}{2}\, m_B^2\, \phi^2 \quad \text{and} \quad U_b(\phi) = -\frac{1}{2}\, m_b\, \phi^2 + \frac{1}{4M_b^2}\, \phi^4 \,, \tag{2.6}$$

where $m_b > 0$ is chosen to ensure that the minimum of the brane potential occurs at the nonzero value $\phi^2 = M_b^2 m_b$, in contrast with the bulk potential which is minimized at $\phi = 0$.

Since U_b and U_B are not minimized by the same configuration, the vacuum solution need not correspond to a constant field configuration, $\partial_M \phi = 0$. Since the solutions to the field equations that only depend on y are exponentials, $\phi \propto e^{\pm m_B y}$, the general bulk solution is a linear combination of such terms. If the extra dimension is sufficiently large — $|m_B y_{\rm min}| \gg 1$ and $m_B y_{\rm max} \gg 1$ — then we can drop the solutions which grow exponentially far from the brane, just as if the extra dimension were noncompact. In this case the vacuum configuration should vanish at infinity, and the solution is therefore given by

$$\phi(y) = \bar\phi\, e^{-m_B |y|} \,, \tag{2.7}$$

where $\bar\phi$ is to be fixed using the boundary condition, eq. (2.4), at $y = 0$: *i.e.* $-2m_B \bar\phi = U_b'(\bar\phi)$, or

$$\left(2m_B - m_b + \frac{\bar\phi^2}{M_b^2} \right) \bar\phi = 0 \,. \tag{2.8}$$

When $m_b < 2\,m_B$ the only real solution allowed is $\bar{\phi} = 0$, but when $m_b > 2\,m_B$ there are three solutions for $\bar{\phi}$, corresponding to $\bar{\phi} = 0$ and $\bar{\phi} = \pm\phi_c$, with

$$\phi_c^2 = M_b^2\,(m_b - 2m_B)\;. \tag{2.9}$$

Since $\mathcal{H} = 0$ for $\bar{\phi} = 0$ and $\mathcal{H} = -\frac{1}{4}M_b^2\,(m_b - 2m_B)^2$ for $\bar{\phi}^2 = \phi_c^2$, we see that it is the nontrivial configuration which represents the classical ground state when $m_b > 2\,m_B$. This can also be seen more generally by writing the energy density as a function of $\bar{\phi}$,

$$\mathcal{H}(\bar{\phi}) = -\frac{1}{2}\,(m_b - 2m_B)\,\bar{\phi}^2 + \frac{1}{4M_b^2}\,\bar{\phi}^4\,, \tag{2.10}$$

which is indeed minimized, for $m_b > 2m_B$, by $\bar{\phi} = \pm M_b\sqrt{m_b - 2m_B}$, with the unstable stationary point, $\bar{\phi} = 0$, situated at a local maximum.

The resulting vacuum

$$\phi^2(y) = M_b^2\,(m_b - 2m_B)\,e^{-2m_B|y|}\,, \tag{2.11}$$

extrapolates from the bulk minimum ($\phi = 0$) for large y to the value $\phi_0 = \pm M_b\sqrt{m_b - 2m_B}$ at the brane. This represents a compromise between the bulk minimum, the value $\bar{\phi} = \pm M_b\sqrt{m_b}$, which minimizes U_b, and the gradient energy required to interpolate between the two. Notice that ϕ_0 approaches the brane minimum in the limit where the bulk potential is very flat, $m_B \ll m_b$.

2.3.2 Codimension-two

We now contrast the previous results with a similar analysis for the codimension-2 case of a real scalar field coupled to a 3-brane in 6 spacetime dimensions, where we show that the larger gradient energy more strongly favors the minimum of the bulk potential relative to that of the brane. Using the action

$$S = -\int \mathrm{d}^4x\, \mathrm{d}^2y \left[\frac{1}{2}(\partial_M\phi\, \partial^M\phi) + U_B(\phi) + \delta^2(y)\, U_b(\phi)\right], \tag{2.12}$$

we have the equation of motion

$$\partial^M\partial_M\phi - U_B'(\phi) = \delta^2(y) U_b'(\phi)\,. \tag{2.13}$$

Assuming a flat space-time metric

$$\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{d}r^2 + r^2\mathrm{d}\theta^2\,, \tag{2.14}$$

and integration of the equation of motion across a very small disc centered on the brane position at $r = 0$ (assuming continuity of ϕ) further implies the condition

$$\lim_{r\to 0}\,[2\pi r\partial_r\phi] = U_b'(\phi_0)\,, \tag{2.15}$$

where r measures the radial distance from the brane situated at $r = 0$. For configurations depending only on r, this corresponds to using the radial field equation

$$\frac{1}{r}\,\partial_r\Big(r\,\partial_r\phi\Big) - U_B'(\phi) = \frac{\delta_+(r)}{2\pi r}\,U_b'(\phi)\,, \tag{2.16}$$

where $\delta_+(r)$ is normalized so that $\int_0^a \mathrm{d}r \delta_+(r) f(r) = f(0)$, for any $a > 0$.

Since our interest is in how the system resolves the frustration of minimizing brane and bulk potentials having different minima, we specialize to the simple choices

$$U_B(\phi) = \frac{1}{2}\, m_B^2 \phi^2 \quad \text{and} \quad U_b(\phi) = -\frac{1}{2}\, \lambda_2\, \phi^2 + \frac{1}{4}\, \lambda_4\, \phi^4 \,, \tag{2.17}$$

with both λ_2 and λ_4 taken to be positive. Keeping in mind a 6D scalar field has dimension $(\text{mass})^2$, we see that the parameter λ_2 is dimensionless, while $\lambda_4 = 1/M_b^4$.

Provided the extra dimensional radius, L, satisfies $m_B L \gg 1$, it is a good approximation to demand the bulk vacuum configuration to vanish at large r, leading to the following solution

$$\phi(r) = \bar{\phi}\, K_0(m_B r) \,, \tag{2.18}$$

where the modified Bessel function, $K_0(z)$, falls exponentially with z for large z and diverges logarithmically as z approaches zero. Using $K_0(z) = -\ln(z/2) - \gamma + O(z)$ to evaluate $r\partial_r \phi \to -\bar{\phi}$ as $r \to 0$, allows the boundary condition, eq. (2.15), to be written

$$-2\pi\bar{\phi} = U_b'(\phi_0) \,, \tag{2.19}$$

and here we encounter the first difference from the codimension-2 case: $\phi(r)$ diverges logarithmically as $r \to 0$, making $\phi_0 = \phi(r = 0)$ ill defined. Regularizing[1] by evaluating at a small but nonzero radius, $r = \epsilon$, gives $\phi_\epsilon = \bar{\phi}\, z_\epsilon$,

[1] This regularization can be done more precisely by modelling the codimension-2 brane by a small codimension-1 circle at radius $r = \epsilon$, and using the codimension-1 jump conditions to relate the exterior bulk fields to the nonsingular fields in the circle's interior (2.12; 2.13).

where

$$z_\epsilon \equiv K_0(m_B\epsilon) = \ell + \ln 2 - \gamma + \mathcal{O}(\epsilon)\,, \tag{2.20}$$

with $\ell = -\ln(m_B\epsilon)$ diverging logarithmically when $\epsilon \to 0$ and $\gamma = 0.5772\ldots$ being the Euler-Mascheroni constant.

The trouble here lies in the fact that the classical solution for the bulk field coupled to a brane diverges when evaluated at the brane source. This is a completely generic feature for branes having codimension 3 or larger — *e.g.* the divergence of the Coulomb field at the position of the source charge. It is also generic for codimension 2, although exceptions in this case also arise, such as for the conical singularities arising in the static gravitational fields sourced by some (2.14; 2.15; 2.16) but not all (2.17; 2.12) codimension-2 branes. And the generic resolution to this problem lies in the need to renormalize the brane-bulk couplings *even at the classical level* (2.18; 2.19). As these references show (and is briefly summarized in Appendix A.2), the requirement that bulk ϕ propagators be finite implies the brane couplings also diverge logarithmically in the limit $\epsilon \to 0$, with the result

$$\lambda_2 = \frac{\bar\lambda_2}{1 + \bar\lambda_2 \hat\ell/2\pi} \quad \text{and} \quad \lambda_4 = \frac{\bar\lambda_4}{\left(1 + \bar\lambda_2 \hat\ell/2\pi\right)^4}\,,$$

where the $\bar\lambda_i$ are renormalized quantities that remain finite in the limit that $\epsilon \to 0$, and

$$\hat\ell = -\ln(\mu\epsilon) = \ell + \ln\left(\frac{m_B}{\mu}\right)\,, \tag{2.21}$$

for an arbitrary renormalization scale μ. For later purposes we remark that because the term in the action involving λ_2 is quadratic in ϕ, it is possible to evaluate the classical scalar propagator, including the brane-bulk mixing, without having to assume that λ_2 or $\bar{\lambda}_2$ are small (see Appendix A.2 for details). In particular the domain of validity of eq. (2.21) includes the regime of large $\bar{\lambda}_2$.

If we regularize by replacing ϕ_0 with ϕ_ϵ, the boundary condition which determines $\bar{\phi}$ becomes

$$-2\pi r\,\partial_r\phi + U_b'(\phi) = 2\pi\bar{\phi} + U_b'(\phi_\epsilon) = \left(2\pi - \lambda_2\, z_\epsilon + \lambda_4\, z_\epsilon^3\bar{\phi}^2\right)\bar{\phi} = 0\,. \qquad (2.22)$$

For $\lambda_2 < 2\pi/z_\epsilon$ this only admits the trivial solution, $\bar{\phi} = 0$, but for $\lambda_2 > 2\pi/z_\epsilon$ three solutions are possible: $\bar{\phi} = 0$ and $\bar{\phi} = \pm\phi_c$, with

$$\phi_c^2 = \frac{(\lambda_2 - 2\pi/z_\epsilon)}{\lambda_4\, z_\epsilon^2}\,. \qquad (2.23)$$

Notice that the criterion distinguishing the existence of one or three solutions depends only logarithmically on m_B (through its appearance in z_ϵ), and can be phrased in a regularization-independent manner by trading λ_2 for $\bar{\lambda}_2$. In particular, the condition $\lambda_2 < 2\pi/z_\epsilon$ ensuring only $\bar{\phi} = 0$ is a solution then becomes $\bar{\lambda}_2 < 2\pi/c$, where $c = \ln 2 - \gamma - \ln(m_B/\mu)$ defines the finite part of $z_\epsilon \equiv \hat{\ell} + c$.

The physical content of these expressions becomes clearer once the relative energy of these solutions is computed using the classical energy density, $\mathcal{H}(\bar{\phi})$, which is finite once it is expressed in terms of the renormalized quantities $\bar{\lambda}_i$. Explicitly, we have

$$\mathcal{H} = \lim_{\epsilon\to 0}\left\{2\pi\int_\epsilon^\infty r\mathrm{d}r\left[\frac{1}{2}(\partial_r\phi)^2 + \frac{1}{2}\,m_B^2\phi^2\right] + U_b(\phi_\epsilon)\right\} \qquad (2.24)$$

$$= \lim_{\epsilon\to 0}\left\{\pi\bar{\phi}^2\int_{m_B\epsilon}^\infty \mathrm{d}z\, z\left[(K_0')^2 + (K_0)^2\right] + U_b(\phi_\epsilon)\right\}\,. \qquad (2.25)$$

The integral may be evaluated in closed form (see Appendix A.1), to give

$$\begin{aligned} \mathcal{H} &= \lim_{\epsilon\to 0}\left\{-\frac{\pi}{2}\,\bar{\phi}^2 m_B^2\epsilon^2 K_0(m_B\epsilon)\Big[K_0(m_B\epsilon)-K_2(m_B\epsilon)\Big]-\frac{1}{2}\lambda_2\phi_\epsilon^2+\frac{1}{4}\lambda_4\phi_\epsilon^4\right\} \\ &= \lim_{\epsilon\to 0}\left\{\pi\bar{\phi}^2 z_\epsilon-\frac{1}{2}\lambda_2\bar{\phi}^2 z_\epsilon^2+\frac{1}{4}\lambda_4\bar{\phi}^4 z_\epsilon^4+\mathcal{O}(\epsilon)\right\}\,, \end{aligned} \tag{2.26}$$

which uses the asymptotic form $K_2(m_B\epsilon) \simeq 2/(m_B\epsilon)^2$ for small ϵ. Using the asymptotic limit of eq. (2.21) for $\bar{\lambda}_2\hat{\ell} \gg 2\pi$,

$$\lambda_2 \simeq \frac{2\pi}{\hat{\ell}}\left[1-\left(\frac{2\pi}{\bar{\lambda}_2\hat{\ell}}\right)+\cdots\right] \quad \text{and} \quad \lambda_4 \simeq \left(\frac{2\pi}{\bar{\lambda}_2\hat{\ell}}\right)^4\bar{\lambda}_4+\cdots\,,$$

we find the finite limit

$$\mathcal{H} = \frac{1}{2}\,g_2\,\bar{\phi}^2+\frac{1}{4}\,g_4\bar{\phi}^4 \quad \text{with} \quad g_2 = 2\pi\left(\frac{2\pi}{\bar{\lambda}_2}-c\right) \quad \text{and} \quad g_4 = \left(\frac{2\pi}{\bar{\lambda}_2}\right)^4\bar{\lambda}_4\,, \tag{2.27}$$

where $c = \ln 2 - \gamma - \ln(m_B/\mu)$, as above.

Notice the kinetic energy has combined with the bulk potential energy to partially cancel the quadratic term in the brane potential, with the solution $\bar{\phi} = 0$ being energetically preferred for $\bar{\lambda}_2 < 2\pi/c$ — the same criterion found earlier. Notice also that $c > 0$ if $\mu > \mu_\star = \frac{1}{2}\,e^\gamma m_B \simeq 0.89\,m_B$, and $c < 0$ if $\mu < \mu_\star$. c vanishes at the dividing case, $\mu = \mu_\star$, at which point the quadratic term is simply

$$g_2 = \frac{4\pi^2}{\lambda_{2\star}}\,, \tag{2.28}$$

with $\lambda_{2\star} \equiv \bar{\lambda}_2(\mu_\star)$. In terms of renormalized quantities the criterion for symmetry breaking becomes $\lambda_{2\star} < 0$, in which case the scalar v.e.v. is

$$\phi_c^2 = -\frac{g_2}{g_4} = -\frac{\lambda_{2\star}^3}{4\pi^2\bar{\lambda}_4}\,. \tag{2.29}$$

These calculations illustrate how the vacuum energetics of a bulk scalar depends crucially on the codimension of the brane to which it is coupled. In all cases the competition between gradient and potential energies in general allows the brane potential to drag the bulk scalar v.e.v. away from the value which minimizes U_B. But in the codimension-1 case the marginal strength of brane instability which distinguishes a nonzero from a vanishing v.e.v., $m_b = 2m_B$, depends strongly on the UV-sensitive scale m_B. By contrast, the corresponding criterion for codimension-2 branes, $\bar{\lambda}_2 = 2\pi/c$, is comparatively insensitive to m_B because it is the larger gradient energies which replace U_B in dominating the fight against U_b. (The situation for higher codimension is explored in Appendix A.3, below.)

2.3.3 The Self-Localized State

Since we expect the quadratic term in $\mathcal{H}$ to describe the mass of small fluctuations about the background configuration, there is a potential puzzle hidden in the weak dependence of g_2 on m_B. To see why, suppose the two extra dimensions are a square torus of volume $V_2 = L^2$, for which in the absence of the brane interactions we would normally expect a Kaluza Klein spectrum to be labelled by two integers, n_1 and n_2, with masses

$$M^2_{n_1 n_2} = m_B^2 + M_c^2(n_1^2 + n_2^2) \geq m_B^2 \,, \tag{2.30}$$

where $M_c = 2\pi/L$. The puzzle is that all of these states have masses larger than m_B, a result which seems hard to reconcile with a mass governed by the size of the quadratic term, $\frac{1}{2}\, g_2 \bar{\phi}^2$, of $\mathcal{H}$.

We next show that the resolution of this puzzle lies in the existence of a lower-mass 'bound' state whose mass lies in the gap, $m < m_B$, and whose presence relies on the influence of the interactions between ϕ and the brane. Furthermore, this state is localized near the brane by these interactions, in the sense that its wave-function falls exponentially away from the brane, with a characteristic size of order $a_B \sim 1/k$, where $k^2 = m_B^2 - m^2$. We call this the self-localized state, inasmuch as its localization is a direct consequence of the scalar-brane interactions (rather than due to a geometric effect, like warping, such as considered in ref. (2.20)).

The Fluctuation Spectrum

To this end consider small fluctuations in the bulk scalar field,

$$\phi(t, r, \theta) = \varphi(r) + \Phi_{n\omega}(r) e^{in\theta - i\omega t}\,, \tag{2.31}$$

labelled by their energy, ω, and angular momentum,[2] n. $\varphi(r)$ here denotes any of the vacuum configurations described above. The field equation obtained by linearizing eq. (2.13) in polar coordinates is

$$\frac{1}{r}\,\partial_r\Big(r\,\partial_r\Phi_{n\omega}\Big) - \frac{n^2}{r^2}\,\Phi_{n\omega} - k^2\Phi_{n\omega} = \frac{\delta_+(r)}{2\pi r}\,\left(-\lambda_2 + 3\lambda_4\varphi^2\right)\Phi_{n\omega}\,, \tag{2.32}$$

where $k^2 = m_B^2 - \omega^2$. For the purposes of identifying the bound state we further specialize to axially symmetric modes, and so set $n = 0$.

The steps for solving for Φ_ω closely parallel those taken above to find the background solution. Away from $r = 0$ the bulk solution is a linear combination of the modified Bessel functions, $K_0(kr)$ and $I_0(kr)$, although in

[2] We assume here an axially-symmetric bulk, such as might be generated (say) by two branes.

the limit $kL \gg 1$ the admixture of $I_0(kr)$ can be made negligibly small. In this case the background configuration is $\varphi = \bar{\phi}\, K_0(m_B r)$ and the fluctuation solutions are well approximated by[3]

$$\Phi_\omega(r) = N_\omega K_0(kr)\,, \tag{2.33}$$

with N_ω an appropriate normalization constant (*e.g.* $N_\omega^2 = k^2/\pi$ when $kL \gg 1$). (In this notation the tower of KK states having masses greater than m_B correspond to the ordinary Bessel functions obtained when k is pure imaginary.) The eigenvalue, k, is obtained by imposing the boundary condition at $r = 0$, which becomes

$$2\pi N_\omega + U_b''(\varphi)\Phi_\omega(r=0) = \left(2\pi - \lambda_2 \hat{z}_\epsilon + 3\lambda_4 z_\epsilon^2 \hat{z}_\epsilon \bar{\phi}^2\right) N_\omega = 0\,, \tag{2.34}$$

where $z_\epsilon = -\ln(m_B\epsilon/2) - \gamma = \hat{\ell} + c$ is as defined above, and $\hat{z}_\epsilon$ is the same quantity with $m_B \to k$: *i.e.* $\hat{z}_\epsilon = z_\epsilon + \ln(m_B/k)$. This equation is to be read as being solved for k, leading to the result $\hat{z}_\epsilon = 2\pi/(\lambda_2 - 3\lambda_4 z_\epsilon^2 \bar{\phi}^2)$, or

$$\begin{aligned} \ln\left(\frac{k}{m_B}\right) &= z_\epsilon - \frac{2\pi}{\lambda_2 - 3\lambda_4 z_\epsilon^2 \bar{\phi}^2} = \hat{\ell} + c - \frac{2\pi}{\lambda_2 - 3\lambda_4 (\hat{\ell}+c)^2 \bar{\phi}^2} \\ &= c - (2\pi/\bar{\lambda}_2) - (3\bar{\lambda}_4\bar{\phi}^2/\bar{\lambda}_2)(2\pi/\bar{\lambda}_2)^3 + \mathcal{O}(1/\hat{\ell}) \\ &\to -\left(\frac{2\pi}{\lambda_{2\star}}\right)\left[1 + \left(\frac{12\pi^2\bar{\lambda}_4\bar{\phi}^2}{\lambda_{2\star}^3}\right)\right] \quad \text{as } \epsilon \to 0\,. \end{aligned} \tag{2.35}$$

[3]Intriguingly, recasting the field equation to remove the single-derivative term, through the redefinition $\phi = \psi/r^{1/2}$, leads to the Schrödinger equation for motion of a point particle in a $1/r^2$ potential supplemented by a δ-function at the origin. This much-studied equation is known to exhibit the interesting phenomena of dimensional transmutation (2.21) and nontrivial limit cycles (2.22).

Consequently, $k = m_B \, e^{-2\pi/\lambda_{2\text{eff}}}$, or

$$\omega^2 = m_B^2 - k^2 = m_B^2 \left[1 - e^{-4\pi/\lambda_{2\text{eff}}}\right] , \tag{2.36}$$

where

$$\frac{1}{\lambda_{2\text{eff}}} = \frac{1}{\lambda_{2\star}} \left[1 + \left(\frac{12\pi^2 \bar{\lambda}_4 \bar{\phi}^2}{\lambda_{2\star}^3}\right)\right] . \tag{2.37}$$

Clearly this state lives in the gap, with $\omega < m_B$, provided only that $\lambda_{2\text{eff}} > 0$, and this mass can be hierarchically small if $\lambda_{2\text{eff}} \gg 4\pi$ (which lies within the domain of validity of the approximations used, as emphasized in Appendix A.2).

There are now two cases to consider. When $\lambda_{2\star} > 0$ we have $\bar{\phi} = 0$ and so $\lambda_{2\text{eff}} = \lambda_{2\star} > 0$, showing that the self-localized state exists. In the limit $\lambda_{2\star} \gg 4\pi$ we find $k \simeq m_B$ and $\omega^2 \simeq 4\pi m_B^2/\lambda_{2\star} = g_2 m_B^2/\pi \simeq g_2 N_\omega^2$, in agreement with the result computed from $\mathrm{d}^2\mathcal{H}/\mathrm{d}\bar{\phi}^2$ (once care is taken to canonically normalize the 4D scalar field). Alternatively, when $\lambda_{2\star} < 0$ we have $\bar{\phi} = \pm\phi_c$, with ϕ_c given by eq. (2.29), and so $\lambda_{2\text{eff}} = -\frac{1}{2}\lambda_{2\star} > 0$. Again a bound state exists whose mass agrees with the result, $-2\, g_2 N_\omega^2$, obtained by differentiating $\mathcal{H}(\bar{\phi})$.

2.4 A Self-Localized Bulk Higgs and the Hierarchy Problem

Because the above vacuum energetics show that the expectation value of a bulk scalar coupled to a codimension-2 (or higher codimension) brane is less sensitive to the details of the model's ultraviolet completion it can be used to provide a new approach to tackling the stability issue of the hierarchy problem. This section builds a simple illustrative example of this mechanism, in order to get a sense of its implications.

2.4.1 The Model

The mechanism's defining assumption is that the usual Standard Model Higgs doublet, $H(x,y)$, is located in an extra-dimensional bulk, while all of the other Standard Model particles — *i.e.* its gauge fields, $A^a_\mu(x)$, and fermions, $\psi_k(x)$ — reside on a brane whose codimension is at least two. (In practice we focus on the codimension-2 case in what follows, but generalizations to more general codimension are conceptually straightforward.) We take the brane potential to prefer an $SU_L(2) \times U_Y(1)$ breaking phase, while the bulk potential favors $SU_L(2) \times U_Y(1)$ invariance:

$$U_B = m_B^2 \, H^* H \quad \text{and} \quad U_b = -\lambda_2 \, H^* H + \lambda_4 \, (H^* H)^2 \,, \tag{2.38}$$

where m_B^2, λ_2 and λ_4 are all real and positive (evaluated at $m_B \epsilon \ll 1$).

We have seen that the classical vacuum of the higher-dimensional theory depends crucially on the sign of the renormalized coupling, $\lambda_{2\star}$, defined at the (large) scale $\mu_\star \simeq 0.89 \, m_B$. Notice in this regard that eq. (2.21) implies that both signs of $\lambda_{2\star}$ can be consistent with positive λ_2 when $\ell = -\ln(m_B\epsilon)$ is sufficiently large. We take $\lambda_{2\star} < 0$ in order to ensure that the total classical energy is minimized by an $SU_L(2) \times U_Y(1)$ breaking configuration.

If we had had $SU_L(2) \times U_Y(1)$ invariance throughout the bulk we would at this point be able to perform a gauge transformation to ensure that the Higgs doublet everywhere takes the unitary gauge form, $H = \frac{1}{\sqrt{2}} \, (0, \chi)^T$, with

χ real. However because we only have gauge invariance at the brane position this choice can only be made at $y^m = 0$: $H_0 = \frac{1}{\sqrt{2}}(0, \chi_0)^T$, where $H_0 \equiv H(x, 0)$. Away from the brane H in general contains 4 real fields, $H = \frac{1}{\sqrt{2}}(\zeta_1 + i\zeta_2, \chi + i\zeta_3)^T$, each of which must solve its appropriate field equations.

The arguments of the previous sections imply that the classical vacuum solutions may be constructed in terms of $K_0(m_B r)$ and $I_0(m_B r)$, with the coefficient of $I_0(m_B r)$ negligibly small when the extra dimensions are large compared with m_B^{-1} — *i.e.* $m_B L \gg 1$:

$$\zeta_i(r) = \bar{\zeta}_i \, K_0(m_B r) \quad \text{and} \quad \chi(r) = \bar{\chi} \, K_0(m_B r) \, . \tag{2.39}$$

As before the normalizations, $\bar{\zeta}_i$ and $\bar{\chi}$, are determined by the boundary conditions at $r = 0$, and so $\bar{\zeta}_i = 0$ due to the choice of unitary gauge at the brane, which implies $\zeta_i(0) = 0$ there. By contrast, the arguments of previous sections go through verbatim to imply $\bar{\chi} \equiv V^2$, with

$$V^4 = -\frac{g_2}{g_4} = -\frac{\lambda_{2\star}^3}{(2\pi)^2 \bar{\lambda}_4} = -\frac{\lambda_{2\star}^3}{(2\pi)^3} \, M_b^4 \, , \tag{2.40}$$

where we define $\bar{\lambda}_4 / 2\pi = 1/M_b^4$.

Similar arguments for the fluctuations, δH, show that in general all four components, $\delta\zeta_i$ and $\delta\chi$, are nonzero in the bulk. However the choice of unitary gauge at the brane endows $\delta\zeta_i$ with the boundary condition that it must vanish, and this in turn implies that none of these fields localizes at the branes in the same way that $\delta\chi$ does.

Since $SU(2) \times U(1)$ is only a global symmetry in the bulk, one might worry that its breaking by H implies that the $\delta\zeta_i$ contain KK towers of Goldstone modes that are systematically light compared with m_B. These could be phenomenologically dangerous, even if their couplings must be derivatively suppressed (2.23). However (as shown in appendix A.4 in more detail) the only Goldstone modes in the bulk-Higgs sector are the three self-localized states for the fields $\delta\zeta_i$ that are eaten by the brane gauge fields *via* the usual Higgs mechanism. All other states with energies smaller than m_B are typically removed by the boundary condition that requires $\delta\zeta_i$ to vanish at the brane, leaving the lightest remaining *bona fide* KK modes in $\delta\zeta_i$ with a mass of order m_B.

2.4.2 Scales and Naturalness

We now ask how V depends on the other scales in the problem, in order to identify whether the choices required to have sufficiently small masses for electroweak gauge bosons are technically natural – *i.e.* stable against integrating out very heavy degrees of freedom.

The model potentially involves several scales: among which are the compactification scale, M_c; the scale of extra-dimensional gravity, $M_* \gg M_c$, (or perhaps the string scale), which controls our neglect of gravitational physics; the scale of brane structure,[4] $\Lambda = 1/\epsilon$, used in earlier regularizations, and so on. In principle the UV scale, $M \gg M_c$, to which we imagine being potentially sensitive, can be any one of these, or some other scale associated with other types of heavy particles.

Our choices of scales are restricted by the domain of validity of approximations used in our calculations. For instance, use of codimension-2 branes

[4] For instance, such structure might ultimately arise if the 3-brane were really a higher-dimensional brane wrapped about further, smaller extra dimensions.

without resolving the brane structure when discussing the UV physics implies $\Lambda \gg M$. Ignoring (for simplicity) the influence of the second brane on the mode functions (*i.e.* dropping the admixture of $I_0(kr)$) assumes $k \gg M_c$, where $k^2 = m_B^2 - m^2$ for the self-localized mode. Neglect (for convenience) of gravitational effects requires both $M_* \gg M_c$, and the condition that the spacetime curvatures generated by the configurations of interest to be small compared with M_*^2. For instance if H takes values of order V^2 that change over distances of order ϵ, then the resulting gradient energies do not overly gravitate if $(\partial H)^2/M_*^6 \sim (V/M_*)^4(\Lambda/M_*)^2 \ll 1$. In what follows we assume all of these conditions to hold. The question we ask is *not* whether these hierarchies themselves are stable under renormalization (as this would require more information, such as specifying a stabilization mechanism for the size of the extra dimensions), but rather whether the choices required of the Higgs potential to obtain an acceptably small V are stable against renormalization, given the presence of these (and possibly other) scales.

This requires an estimate of the corrections to U_B and U_b that might arise as various kinds of heavy particles are integrated out. Although a precise statement of this requires specifying the theory's UV completion, some generic statements are possible on dimensional grounds for the corrections due to integrating out heavy particles that interact through small dimensionless couplings. This is because if such a particle has a large mass M, then its generic contribution to a coupling, λ_i, having dimension (mass)^n is $\delta\lambda_i \propto M^n$. According to this kind of estimate we expect

$$\delta m_B^2 \propto M^2 \,, \quad \delta\lambda_2 \propto \ln M \quad \text{and} \quad \delta\lambda_4 \propto M^{-4} \,. \tag{2.41}$$

As a result it is natural to expect the corrections to m_B to be dominated by the heaviest particles that can contribute, and so generically expect m_B to be comparable to the largest scales in the problem (and in particular to satisfy $m_B \gg M_c$ and $m_B \gg M_W$). It is the large size of these contributions to m_B that underlie the usual formulation of the hierarchy problem in 4 dimensions, because in this case the scale of the Higgs v.e.v. turns out to be proportional to $|m_B|$.

By contrast, in the 6D model of present interest we have seen that the size of the Higgs v.e.v. is largely independent of m_B, depending dominantly on the dimensionless coupling $\bar{\lambda}_2$ and the dimensionful coupling $\bar{\lambda}_4$. But $\bar{\lambda}_2$ is dimensionless, and so tends to depend only logarithmically on the large UV scale M. Potentially more dangerous is $\bar{\lambda}_4/2\pi = 1/M_b^4$ since this more directly sets the size of V. However this is also not UV sensitive because corrections to it vary inversely with the relevant particle mass on dimensional grounds, and so are dominated by the contributions of the *lightest* particles, rather than the heaviest.

As stated above, we emphasize that our goal here is not to provide an ultraviolet completion of the bulk-Higgs model, as would be required to understand in detail the conditions necessary to produce a large hierarchy in the first place, as this goes beyond the scope of this paper. Our goal is instead to point out how the introduction of Higgs bulk-brane couplings allows interestingly different mass-dependence in low-energy observables, and to study what this might imply for the low-energy sector.

2.4.3 Higgs-induced Mass Terms

The phenomenology of any such Higgs hinges on the form of its couplings to observed Standard Model particles, which are assumed in this framework to be localized on a brane.

Gauge Couplings

At first sight it is bizarre to restrict the SM gauge fields to a brane and yet allow a charged matter field (the Higgs doublet) live in the bulk. This is bizarre because the $SU_L(2) \times U_Y(1)$ symmetry transformations are global transformations in the bulk (since there is no spin-one field there to 'gauge' them), yet are local on the brane. Nonetheless, it must be possible because we could imagine the UV completion of the brane of interest being an ordinary gauge-Higgs theory containing vortex- or domain-wall-type defects. Since the Higgs field defining the defect typically vanishes at the interior of such a vortex/domain-wall, there generically should be spin-1 states which would be very massive given the nonzero Higgs in the bulk, but which can remain light by being localized on the brane. (D-branes also contain localized spin-1 fields.)

More precisely, it can be shown that gauge invariance of such a theory can always be ensured through an appropriate choice of effective interactions (or counter-terms) on the brane. Slightly generalizing the discussion of ref. (2.24) to codimension two, we may see this formally by taking the Higgs covariant derivatives to be

$$D_M H(x,y) = \partial_M H(x,y) - \delta^2(y)\delta^\mu_M i\kappa g A^a_\mu(x) T_a H(x,y)\,. \tag{2.42}$$

Here T_a are gauge generators, and as before the x^μ lie along the brane directions while the y^m are transverse. g here denotes the dimensionless gauge coupling on the brane and κ is a dimensionful constant, required in order to counter the dimensions of the delta function. One reason for the need for brane counterterms can be seen because the off-brane components of this covariant derivative are *not* actually covariant under the gauge transformation

$$\delta H(x,y) = \delta^2(y) i\kappa \Omega^a(x) T_a H(x,y)\,, \tag{2.43}$$

even when supplemented by the standard x^μ-dependent nonabelian transformations of $A^a_\mu(x)$. They are not because there is no gauge potential in $D_m H = \partial_m H$ to cancel the term arising when the derivative acts on the delta function. There is however a counterterm that can be added on the brane such that the entire combination *is* gauge invariant.

The implications of a bulk Higgs v.e.v. for gauge boson masses can be seen by writing out the bulk and brane kinetic terms

$$\begin{aligned} \mathcal{L}_{\rm kin} &= -\int \mathrm{d}^2 y\; D_{\scriptscriptstyle M} H^* D^{\scriptscriptstyle M} H - \kappa_b \mathcal{D}_\mu H_0^* \mathcal{D}^\mu H_0 \\ &= -\int \mathrm{d}^2 y\; \Big[\partial_{\scriptscriptstyle M} H^* \partial^{\scriptscriptstyle M} H\Big] - (\kappa+\kappa_b)\mathcal{D}_\mu H_0^* \mathcal{D}^\mu H_0 + \kappa \partial_\mu H_0^* \partial^\mu H_0 \\ &\qquad + \frac{\kappa g^2}{2}[1-\kappa\delta^2(0)]\left(H_0^*\left\{T_a,T_b\right\}H_0\right) A^a_\mu A^{b\mu}\,, \end{aligned} \tag{2.44}$$

where $\mathcal{D}_\mu H_0 \equiv \partial_\mu H_0 - igA^a_\mu T_a H_0$ is the standard covariant derivative on the brane. This shows that all of the gauge-boson mass terms appear in the brane kinetic term provided $\kappa\,\delta^2(0) = 1$ (and so $\kappa = O(\epsilon^2)$). Notice that this implies $\kappa + \kappa_b \sim \kappa_b$ for any scale $\kappa_b \gg O(\epsilon^2)$.

Superficially the gauge-boson mass obtained from these equations diverges as $\epsilon \to 0$, due to the divergence there of H_0. However, this divergence is countered by the renormalization of all Higgs-brane interactions due to the generic 'dressing' of these couplings (2.18; 2.19) by the Higgs-brane mixing, λ_2:

$$\kappa_b = \frac{\bar{\kappa}_b}{(1 + \bar{\lambda}_2 \hat{\ell}/2\pi)^2} \,. \tag{2.45}$$

Going to unitary gauge at the brane position, for which $H_0 = \frac{1}{\sqrt{2}}\,(0, \chi_0)^T$, with $\langle \chi \rangle = V^2 K_0(m_B r)$, we then have $\kappa_b \langle \chi_0 \rangle^2 = \bar{\kappa}_b V^4 (2\pi/\lambda_{2\star})^2$ as $\epsilon \to 0$.

The $SU_L(2) \times U_Y(1)$ doublet structure of the Higgs then leads in the standard way to the prediction $M_Z = M_W / \cos\theta_W$, where θ_W is the weak mixing angle, and the W-boson mass is, $M_W = \frac{1}{2}\, gv$, with

$$v^2 = \left(\frac{2\pi}{\lambda_{2\star}} \right)^2 \bar{\kappa}_b V^4 = (246 \text{ GeV})^2 \,. \tag{2.46}$$

Taking $\bar{\kappa}_b = 1/f^2$, this shows that successful phenomenology requires $V^2 = fv(|\lambda_{2\star}|/2\pi)$: *i.e.* V is the geometric mean between 246 GeV and the scale $|\lambda_{2\star}| f/2\pi$:

$$V \sim 10^9 \text{ GeV} \left(\frac{|\lambda_{2\star}| f/2\pi}{10^{15} \text{ GeV}} \right)^{1/2} \,. \tag{2.47}$$

Recall that within the present framework we have $V^4 = |\lambda_{2\star}/2\pi|^3 (2\pi/\bar{\lambda}_4)$ — *c.f.* eq. (2.40) — so defining M_b by $\bar{\lambda}_4/2\pi = 1/M_b^4$ as before we see that eq. (2.46) also implies that M_b must be of order

$$M_b \sim \sqrt{vf} \left(\frac{2\pi}{|\lambda_{2\star}|} \right)^{1/4} \,. \tag{2.48}$$

This requires either $M_b \sim f\sqrt{2\pi/|\lambda_{2\star}|} \sim v$, or a hierarchy $v \ll M_b \ll f\sqrt{2\pi/|\lambda_{2\star}|}$, if $f\sqrt{2\pi/|\lambda_{2\star}|} \gg v$. In the absence of a symmetry which forbids a Higgs kinetic term but allows a quartic $(H^*H)^2$ interaction on the brane, naturalness argues we should take f and M_b to be the same order of magnitude, in which case any hierarchy between M_b and v must be due to $|\lambda_{2\star}|/2\pi$ being very large or very small. Furthermore, having f and M_b both larger than v requires $|\lambda_{2\star}|/2\pi \lesssim O(1)$.

Fermion Couplings

Fermion masses in this picture are similarly given by Yukawa couplings between brane-based fermions, ψ_k, and the bulk Higgs doublet. In unitary gauge on the brane, $H_0 = \frac{1}{\sqrt{2}}\,(0, \chi_0)^T$, these have the form

$$\mathcal{L}_{\rm yuk} = \frac{y_{ij}}{F}\,(\overline{\psi}_i \psi_j)\chi_0\,, \tag{2.49}$$

for y_{ij} a collection of dimensionless Yukawa couplings, and F representing an appropriate ultraviolet scale. The resulting fermion masses are

$$m_{ij} = \frac{y_{ij}}{F}\,\langle \chi_0 \rangle = \frac{2\pi \bar{y}_{ij}\, V^2}{\lambda_{2\star} F} \tag{2.50}$$

with

$$y_{ij} = \frac{\bar{y}_{ij}}{1 + \bar{\lambda}_2 \hat{\ell}/(2\pi)}\,, \tag{2.51}$$

being the renormalized Yukawa coupling, as required to counter the divergence of H at the brane position, and the second equality in eq. (2.50) uses $y_{ij}\langle\chi_0\rangle = 2\pi\bar{y}_{ij}V^2/\lambda_{2\star}$ in the limit $\epsilon \to 0$, where $\langle \chi \rangle = V^2 K_0(m_B r)$.

Since $\mathcal{L}_{\rm yuk}$ breaks flavor symmetries — unlike the Higgs kinetic terms — the scale F need not be of the same order of magnitude as[5] f. In particular,

[5]This could arise, say, if the 3-brane is really a higher-dimensional brane wrapped in extra dimensions, and the flavor structure is associated with this wrapping, since this would suggest $F \simeq \Lambda \gg f$.

since the dominant contributions to couplings having dimensions of inverse mass come from the lightest scales to contribute, F is typically set by the smallest UV scale which involves flavor-violating physics while f can be much smaller than this. Because of this eqs. (2.46) and (2.50) may contain the seeds of an explanation of the observed smallness of most fermion masses relative to those of the electroweak gauge bosons, since

$$\frac{m_{ij}}{M_W} = \frac{\bar{y}_{ij}}{g} \left(\frac{2f}{F} \right) . \tag{2.52}$$

Even a mild hierarchy, $F \gg f$, removes some of the burden of having to require $\bar{y}_{ij}/g$ to be very small.

2.4.4 Couplings to the Higgs Fluctuations

We have seen that the spectrum of fluctuations in the Higgs field generically contains an assortment of KK modes, many of whose masses start above a large gap, $m_{KK} \gtrsim m_B$. For m_B sufficiently large these modes need not play an important role in low-energy observables. The two exceptions to the above statement are the bulk Goldstone modes, whose masses are generically of order M_c, and the self-localized state whose mass can lie within the gap below m_B, and be hierarchically smaller if $|\lambda_{2\star}| \gg 2\pi$. Furthermore, this latter state is present regardless of whether or not the Higgs v.e.v. is nonzero. These light states are likely to be the ones relevant to Higgs phenomenology in Bulk Higgs models, and so this section computes their couplings.

The bulk Goldstone modes

The simplest couplings to compute are those of the bulk Goldstone modes, $\delta\zeta_i$, because their vanishing at the brane position guarantees they completely drop out of any brane couplings that depend only on H_0 or $\partial_\mu H_0$, and not on off-brane derivatives like $\partial_m H_0$. In particular this ensures their removal (in unitary gauge) from the fermion Yukawa couplings and gauge couplings described above.

The self-localized state

Normalizing the wave-function of the self-localized state in the extra dimensions gives a canonically normalized 4D state h, where $\chi = h(x) N_\omega K_0(kr)$, so $y_{ij}\chi = (2\pi/\lambda_{2\star})(k/\sqrt{\pi})\bar{y}_{ij} h$, with $k^2 = m_B^2 - m_h^2$. The couplings of h to fermions are then given by interactions of the form

$$\mathcal{L}_{4D} = \frac{2\pi \bar{y}_{ij}}{\lambda_{2\star}} \left(\frac{k}{\sqrt{\pi} F} \right) (\overline{\psi}_i \psi_j) h \,, \tag{2.53}$$

leading to dimensionless 'physical' Yukawa couplings of order

$$\hat{y}_{ij} = \frac{2\pi \bar{y}_{ij}}{\lambda_{2\star}} \left(\frac{k}{\sqrt{\pi} F} \right) = y_{ij}^{\rm sm} \left(\frac{m_B}{\sqrt{\pi} f} \right) \left(\frac{2\pi}{\lambda_{2\star}} \right) e^{-4\pi/|\lambda_{2\star}|} \,, \tag{2.54}$$

where the argument of the exponential assumes $\lambda_{2\star} < 0$ (as required for a nonzero Higgs v.e.v.), and the last equality compares to what would be expected in the SM:

$$y_{ij}^{\rm sm} \equiv \frac{m_{ij}}{v} = \bar{y}_{ij} \left(\frac{f}{F} \right) . \tag{2.55}$$

Notice that the quantity $(2\pi/|\lambda_{2\star}|)\exp[-4\pi/|\lambda_{2\star}|]$ falls to zero for large and small $|\lambda_{2\star}|$, taking the maximum value of 0.18 when $|\lambda_{2\star}|/2\pi = 2$.

These expressions show that the self-localized Higgs couplings, $\hat{y}_{ij}$, can differ significantly from what would be expected in the SM, for two reasons. First, $\hat{y}_{ij}$ can be larger than $y^{\rm sm}_{ij}$ if $m_{\scriptscriptstyle B} \gg f$, and if sufficiently large the self-localized state becomes a strongly coupled broad resonance. Second, $\hat{y}_{ij}$ can differ from $y^{\rm sm}_{ij}$ because of its dependence on $\lambda_{2\star}/2\pi$, which acts to suppress $\hat{y}_{ij}/y^{\rm sm}_{ij}$ in the limit that $|\lambda_{2\star}|/2\pi$ is either very large or very small. This possibility of having $\hat{y}_{ij}$ differ from the SM expectation contrasts with 4D intuition based on the couplings of a single scalar whose v.e.v. generates mass, since such a scalar must have couplings given by the ratio m_{ij}/v. The reason this conclusion does not hold in the extra-dimensional case is that because the v.e.v., $\langle H(x,0)\rangle$, responsible for generating masses generically receives contributions from many KK modes and not just the v.e.v. of the single 4D self-localized state, h.

2.5 Possible Signatures of a Bulk Higgs Scenario

We next sketch some of the qualitative signatures and constraints that might be expected for the kind of Higgs scenario described above. What is to be expected depends somewhat on the choices made for the various scales in the problem, so we divide the discussion according to four simple options according to whether or not we take $|\lambda_{2\star}|$ to be large or small, and whether we take $M_c \sim 1$ TeV, or $M_c \sim 10^{-2}$ eV (as for large-extra-dimensional models).

2.5.1 Inclusive Processes

We first consider inclusive processes for which a specific Higgs state is not measured, and so which involve a summation over all possible KK modes. These are largely insensitive to the specifics of individual modes, such as the details of the self-localized state.

Fermion-fermion scattering

An important inclusive observable is the rate for fermion-fermion scattering mediated by a virtual Higgs. The amplitude for this process is of order

$$\mathcal{A}(\psi_i \psi_j \to H \to \psi_r \psi_s) \simeq \frac{y_{ij}\, y_{rs}}{F^2}\, iG_p(0;0)\, \delta^4(p_i + p_j - p_r - p_s)\,, \tag{2.56}$$

where $p^\mu \equiv (p_i + p_j)^\mu = (p_r + p_s)^\mu$. Here $G_p(y;y')$ is the bulk Higgs propagator, Fourier transformed in the brane directions, x^μ, but evaluated in position space in the off-brane directions, y^m. $G_p(0;0)$ denotes the same quantity evaluated at the brane position, and is given (see Appendix A.2 for details) in terms of the corresponding propagator in the absence of brane-Higgs couplings, $D_p(y;y')$, by

$$G_p(0;0) = \frac{D_p(0;0)}{1 - i\lambda_2 D_p(0;0)}\,. \tag{2.57}$$

Eliminating y_{ij}, y_{rs} and λ_2 in terms of the renormalized quantities, $\bar{y}_{ij}$, $\bar{y}_{rs}$ and $\bar{\lambda}_2$, and taking $\epsilon \to 0$, we find the finite result

$$\begin{aligned} \mathcal{A}(\psi_i \psi_j \to H \to \psi_r \psi_s) &\simeq \frac{\bar{y}_{ij}\, \bar{y}_{rs}}{\bar{\lambda}_2 F^2} \left[\frac{1}{1 - i\bar{\lambda}_2 D_p^\mu(0;0)} \right] \delta^4(p_i + p_j - p_r - p_s) \\ &\simeq \frac{y_{ij}^{\rm sm}\, y_{rs}^{\rm sm}}{\bar{\lambda}_2 f^2} \left[\frac{1}{1 - i\bar{\lambda}_2 D_p^\mu(0;0)} \right] \delta^4(p_i + p_j - p_r - p_s)\,, \end{aligned} \tag{2.58}$$

where $iD_p^\mu(0;0) = (1/2\pi)\ln(\mu/P)$, where $P^2 = p^2 + m_B^2$.

If this same process were computed using the exchange of a massive 4D SM Higgs scalar, we'd have instead obtained

$$\mathcal{A}^{\rm sm}(\psi_i\psi_j \to H \to \psi_r\psi_s) \simeq y_{ij}^{\rm sm}\, y_{rs}^{\rm sm} \left[\frac{1}{p^2 + m_H^2}\right]\, \delta^4(p_i + p_j - p_r - p_s)\,, \quad (2.59)$$

and so the leading effect is to replace the scale $p^2+m_H^2$ by $\bar\lambda_2 f^2[1-i\bar\lambda_2 D_p^\mu(0;0)]$. The absence of an observed signal therefore implies the order-of-magnitude bound

$$\bar\lambda_2 f^2 \left[1 + \frac{\bar\lambda_2}{2\pi}\ln\left(\frac{P}{\mu}\right)\right] \gtrsim O(100\ \text{GeV})^2\,, \quad (2.60)$$

where $P^2 = (p_i + p_j)^2 + m_B^2 = (p_r + p_s)^2 + m_B^2$.

If reactions of this type were to mediate flavor-changing neutral currents, the strong restrictions on these could potentially bound the scale F to be quite large. However, because the Yukawa couplings can have the same flavor structure as in the SM, there can be a GIM mechanism at work (2.25) that naturally suppresses the dangerous flavor-changing neutral current (FCNC) reactions produced by bulk-Higgs exchange. We henceforth assume this to be true, and therefore do not further worry about bounds on the fermion couplings due to FCNCs.

Vacuum Polarization

As is well known, the contributions to loops of the SM Higgs is well constrained by precision electroweak measurements. The main source of these contributions is through the Higgs contribution to the vacuum polarization of the electroweak gauge bosons. For an extra-dimensional bulk Higgs, this contribution is of order

$$\begin{aligned}
\Pi^{\mu\nu}_{ab}(p) &\simeq g^2\kappa_b^2\,\mathrm{tr}(T_aT_b)\int\frac{\mathrm{d}^4q}{(2\pi)^4}(2p-q)^\mu(2p-q)^\nu\,iG_q(0;0)\,iG_{p-q}(0;0)\\
&\simeq \frac{g^2\bar\kappa_b^2}{\bar\lambda_2^2}\,\mathrm{tr}(T_aT_b)\int\frac{\mathrm{d}^4q}{(2\pi)^4}\left[\frac{(2p-q)^\mu(2p-q)^\nu}{[1-i\bar\lambda_2D^\mu_p(0;0)][1-i\bar\lambda_2D^\mu_{p-q}(0;0)]}\right]\,,
\end{aligned} \tag{2.61}$$

plus a possible tadpole term. Since the remaining integration, d^4q, diverges in the ultraviolet it must be regularized, and this is most conveniently done using dimensional regularization.

Of most interest for phenomenological purposes is the contribution to the oblique parameters S, T and U (2.2; 2.26), which involve those terms in $\Pi^{\mu\nu}_{ab}$ having the tensor structure $(p^2\eta^{\mu\nu}-p^\mu p^\nu)$. Since the Higgs is an $SU_L(2)\times U_Y(1)$ doublet, it automatically preserves the accidental custodial $SU_c(2)$ symmetry (2.2; 2.27) that preserves the successful mass relation $M_W = M_Z\cos\theta_W$, thereby suppressing its contribution to T and making S of most interest. Because all mass dependence in eq. (2.61) is logarithmic, recalling the definition $\bar\kappa_b = 1/f^2$ and extracting the conventional factors of $g^2/4\pi$, we obtain the estimate

$$S\sim\frac{1}{4\pi\bar\lambda_2^2}\left(\frac{p^4}{f^4}\right)\,, \tag{2.62}$$

where p^2 represents the momentum transfer of interest. Applied to LEP experiments we may take $p^4 = M_Z^4$ and $|S| < 0.1$ to conclude $\bar\lambda_2 f^2 \gtrsim v^2$.

2.5.2 Higgs Decays to Fermions

Another class of observables involve specifying a specific Higgs KK mode. Perhaps the simplest of these is the decay rate for specific Higgs states into

SM particles (although this decay need not dominate the lifetime of a given KK mode because it must also compete with other channels, such as off-brane decays into the Goldstone modes $\delta\zeta_i$).

Generic KK states

For simplicity we start with the decay of a generic KK mode into brane fermions, assuming the KK wave-functions, $\Psi(y)$, extend throughout much of the extra-dimensional bulk so that $|\Psi(0)|^2 \simeq 1/V_2 \simeq M_c^2$. Once excited, such a heavy state can decay through the interaction (2.49), with the rate

$$\Gamma(\chi \to \bar{\psi}_i \psi_j) \simeq |\Psi(0)|^2 \, \frac{|y_{ij}|^2}{F^2} \, M_\chi \simeq |y_{ij}|^2 \left(\frac{M_c}{F} \right)^2 M_\chi \,, \tag{2.63}$$

where $M_\chi \geq m_B$ is the mass of the decaying mode. (Recall that the bulk Goldstone modes, $\delta\zeta_i$, do not decay in this way because of the requirement that they vanish at the brane.) We see that $\Gamma \ll M_\chi$ naturally follows from the smallness of the quantities y_{ij} and M_c/F (the latter of which is particularly small in the case of large extra dimensions). Whether these are the dominant decay channels depends on the availability of light states in the bulk (or on other branes) into which competing decays can proceed, and how efficiently these Higgs decays occur.

The self-localized state

Notice that y_{ij} vanishes, strictly speaking, when $\epsilon \to 0$ with $\bar{y}_{ij}$ and $\bar{\lambda}_2$ held fixed (making eq. (2.63) vanish logarithmically in this limit). The same is not true of the self-localized state, whose wave-function also diverges logarithmically at the position of the brane as $\epsilon \to 0$. In this case the decay rate can be computed using the interaction of eq. (2.53), leading (on neglect of final-state fermion masses) to the standard 4D expression

$$\Gamma(h \to \bar{\psi}_i \psi_j) = \frac{1}{8\pi} |\hat{y}_{ij}|^2 \, m_h = \Gamma^{\rm sm}(h \to \bar{\psi}_i \psi_j) \left(\frac{m_{\scriptscriptstyle B}^2}{\pi f^2} \right) \left(\frac{2\pi}{\lambda_{2\star}} \right)^2 e^{-16\pi/|\lambda_{2\star}|} \,, \tag{2.64}$$

which remains nonzero as $\epsilon \to 0$. This drops dramatically, as required in the unlocalized limit, as $|\lambda_{2\star}| \to 0$, and scales as $\Gamma^{\rm sm} \, m_h^4/(16\pi m_{\scriptscriptstyle B}^2 f^2)$ when $|\lambda_{2\star}| \gg 2\pi$.

2.5.3 TeV-Scale Compactifications

Suppose, first, the compactification scale, M_c, lies in the TeV range and so, in the absence of significant warping, the 4D Planck scale, $M_p \sim M_*^2/M_c$, comes out right if $M_* \sim 10^{10}$ GeV. This leaves lots of room to choose the other scales of interest to be much smaller than M_* in order to justify our neglect of gravitational interactions. We do not speculate as to how the extra-dimensional size is stabilized at this scale.

Choosing M_c this large also ensures that this is the mass of the lightest KK mode of the bulk Goldstone bosons, $\delta\zeta_i$, ensuring that these modes do not play much of a phenomenological role until energies are reached – at the LHC, God willing – that allow the direct production of KK excitations. The same is true of the generic KK modes of the field χ, provided we also choose $m_{\scriptscriptstyle B}$ to be large enough.

We have seen that the absence of Higgs detection in oblique or in 2-fermion to 2-fermion processes implies us to choose $f\sqrt{|\lambda_{2\star}|}$ to be at least several hundred GeV, whereas our use of a 6D calculational framework requires both f and $M_b \sim \sqrt{vf}(2\pi/|\lambda_{2\star}|)^{1/4}$ to be $\gtrsim M_c$. There are then two subcategories to consider, depending on the size of $|\lambda_{2\star}|/2\pi$.

Weak localization

Consider first the limit of small $|\lambda_{2\star}|$, for which $m_h \to m_B$ and $k \to 0$. Because k is small, the 'bound' state is not strongly localized relative to generic extra dimensional scales, and the breakdown of the approximation $k \gg M_c$ demands we go beyond the simple large-volume limits used above for the scalar v.e.v. and wave-function. Taking $|\lambda_{2\star}| \sim 0.01$ for illustrative purposes, we see that requiring $f > M_c \sim$ few TeV automatically ensures $f\sqrt{|\lambda_{2\star}|} \gtrsim$ several hundred GeV, and so is large enough to avoid the phenomenological bounds.

For weak localization, the exponential suppression of $\hat{y}_{ij}$ for small $|\lambda_{2\star}|$ allows us to choose m_B to be much larger than f without the Higgs-fermion couplings becoming strong. However we cannot have all χ states be too much higher than the TeV scale without there being a breakdown of the low-energy effective theory, such as through the development of unitarity problems in the scattering of longitudinal W bosons that the SM would suffer in the absence of a low-energy Higgs particle (2.2; 2.28; 2.29), and this puts an upper bound on how large m_B can be. In this case the χ spectrum resembles the usual intuition for bulk fields in the absence of brane couplings, consisting of a tower of Higgs KK modes starting above the gap at m_B. Furthermore, because these particles are likely to have a significant decay rate into the lighter bulk Goldstone states, any observed Higgs is likely to have a significant invisible width.

Because m_B cannot be made exceedingly large without running into troubles, and because M_c is typically smaller, it should be possible to observe some of the Higgs KK states at the LHC. Although the mass-M_c Goldstone states cost less energy, they are more difficult to produce because of the absence of direct couplings to the initial brane-based SM particles. The most likely channel for doing so is the virtual excitation of KK modes of the bulk state χ. Convincing evidence for these Goldstone states together with an absence for KK modes for the electroweak gauge bosons would provide the smoking gun for this scenario: with the Higgs in the bulk but gauge interactions localized to live only on the branes.

Strong localization

In the opposite limit, $|\lambda_{2\star}| \gg 2\pi$, the lowest energy state becomes localized to the brane with $k \simeq m_B$, and its mass drops to $m_h^2 \simeq 8\pi m_B^2/|\lambda_{2\star}| \ll m_B^2$. In this case m_B can be higher than it could for weak localization, provided that the self-localized state is lighter than a few TeV and so can unitarize the scattering of longitudinal gauge boson modes.

An upper limit to how large m_B can be is found from the condition that this light, localized Higgs state be weakly coupled

$$\left|\frac{\hat{y}_{ij}}{y_{ij}^{\rm sm}}\right|^2 \simeq \frac{8}{\pi}\left(\frac{m_B}{f}\right)^2\left|\frac{2\pi}{\lambda_{2\star}}\right|^3 \simeq \frac{1}{8\pi}\left(\frac{m_h}{f}\right)^2\left(\frac{m_h}{m_B}\right)^4 . \tag{2.65}$$

Large $|\lambda_{2\star}|$ also implies that the condition $f > M_c$ automatically ensures the validity of the phenomenological limits that require $f\sqrt{|\lambda_{2\star}|}$ to be larger than several hundred GeV, and makes the strongest constraint on f the theoretical condition that M_b be larger than M_c.

For instance for moderately large $|\lambda_{2\star}|/2\pi \sim 10^2$, then keeping m_h at the TeV scale requires $m_B \simeq 10$ TeV, and taking $M_b \sim 1$ TeV then implies $f \sim 10$ TeV. By contrast, if m_B should be the largest scale considered so far, $m_B \sim M_* \sim 10^{10}$ GeV, then $|\lambda_{2\star}|/2\pi \sim 10^{20}$, and so $M_b \sim 10^{-5} f > 1$ TeV implies a strong hierarchy between M_b and $f > 10^5$ GeV whose naturality would have to be explained. Notice that the physical couplings, $\hat{y}_{ij}$, are much smaller than for the SM given these scales.

In this case m_B could easily be large enough to preclude the direct detection of a Higgs KK spectrum, even at the LHC, leaving the burden of Higgs physics carried by the single self-localized Higgs state. In principle this can be distinguished from a SM Higgs in several ways. First, it could well have a large invisible width, if the mass of the self-localized state is sufficiently large compared with the mass, M_c, of the bulk Goldstone modes. Second, it can be distinguished by identifying the difference in the strength of its couplings to fermions from those expected in the SM.

2.5.4 Large Extra Dimensions

An alternative choice (2.3; 2.16; 2.30) would put the scale of extra-dimensional gravity at $M_* \sim 10$ TeV, which then requires $M_c \sim 10^{-2}$ eV. As a result, the upper bound $m_B < M_*$ automatically keeps the generic Higgs KK modes light enough to potentially be seen at the LHC, yet absence of the detection of Higgs KK modes also implies m_B cannot be much below the TeV scale.

An automatic consequence of having M_c so small is to make the bulk Goldstone states, $\delta\zeta_i$, essentially massless. This ensures that they are always kinematically available as final states for χ decays, making a significant invisible width for this state inevitable. In fact, the very lightest KK Goldstone modes in this scenario are light enough to mediate forces between macroscopic bodies, with generically near-gravitational strength, making them potentially relevant to precision tests of Newton's inverse-square law for gravity. Their presence is nonetheless unlikely to have been already ruled out due to the absence of direct couplings to brane matter, and the derivative nature of their Goldstone interactions.

In this scenario the conditions $f, M_b \gtrsim M_c$ pose no significant constraint, with more information coming from the phenomenological conditions that $f\sqrt{|\lambda_{2\star}|}$ be larger than a few hundred GeV. Notice that if we also require $f \lesssim M_*$ then we must have an upper bound $|\lambda_{2\star}| \lesssim 10^4$, and so the self-localized state cannot be more than a few orders of magnitude lighter than m_B.

Because the KK tower of modes is so narrowly spaced – by $O(M_c)$ – they provide almost a continuum of states. Although each of these modes couples with gravitational strength, their phase space makes their inclusive production cross section of order the weak-interaction size (2.3). Once the Higgs is produced, its phenomenology is likely to resemble that of extra-dimensional gravitons (2.31) or other bulk matter fields (2.32), including likely large invisible decay channels.

2.6 Conclusions

In this paper we examine a new way for brane-world scenarios to change how we think about low-energy naturalness problems. We do so by showing how oft-neglected couplings to branes can dramatically change the vacuum energetics and low-energy spectrum for bulk scalar fields. In particular, we show that when coupled to codimension-2 branes bulk scalar fields can have two unusual properties:

- They can acquire v.e.v.s that are only logarithmically related to the size of the UV-sensitive quadratic term, $\frac{1}{2}m_B^2\phi^2$, in the bulk Higgs potential;

- They can acquire low-energy KK modes that are localized to the branes (without the need for warping), and whose mass can lie inside the naive gap below the energy set by the mass scale m_B.

We further use these two observations to explore the possibility of building phenomenological brane-world models for which all Standard Model particles (save the Higgs) are trapped on a brane, but with the Higgs allowed to live in the bulk. We estimate the size of the effective couplings of such a Higgs to gauge bosons and fermions on the brane, and use these to estimate the sizes of masses and couplings to the Higgs KK modes.

We do not try to identify ultraviolet completions of the bulk-Higgs model, and so do not identify at a microscopic level why the electroweak hierarchy exists in the first place. Our focus is instead on whether such a hierarchy can be technically natural purely within the low-energy theory. We identify in eq. (2.4.3) the main obstacle to systematically raising the UV scale of this effective theory above the weak scale, since this equation generically requires the two dimensionful parameters f and M_b — governing the size of the brane potential term $(H^*H)^2/M_b^4$ and the brane kinetic term $(D_M H^* D^M H)/f^2$ — either to satisfy $M_b \sim f\sqrt{2\pi/\lambda_{2\star}}$ with both near the electroweak scale, or to satisfy the hierarchy $M_b \ll f\sqrt{2\pi/\lambda_{2\star}}$ if both are large compared with the electroweak scale. This latter hierarchy shows how the problem gets recast with a bulk Higgs, since both interactions are allowed by the same symmetries, making it unnatural for them to have coefficients suppressed by very different scales.

We provide a very preliminary discussion of possible signals and constraints on these models, including the observation that most realizations predict a significant invisible width for any observed 'Higgs', once detected. Simple estimates are made of Higgs decay rates into SM particles, the scattering rate for fermions due to virtual Higgs exchange, and the contribution of virtual Higgs loops to gauge boson vacuum polarization. These are used to outline the qualitative features of Higgs phenomenology within this class of models. In all cases we find that the phenomenology of these models is sufficiently interesting to bear further, more detailed study.

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Bibliography

[2.1] C.P. Burgess, Claudia de Rham, Leo van Nierop,
"The Hierarchy Problem and the Self-Localized Higgs"
JHEP0808:061,2008 (arXiv:0802.4221)

[2.2] For a recent summary of these issues see, C.P. Burgess and G.D. Moore, *The Standard Model: A Primer*, Cambridge University Press 2007.

[2.3] N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. **B429** (1998) 263 (hep-ph/9803315); Phys. Rev. **D59** (1999) 086004 (hep-ph/9807344);
I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, Phys. Lett. B **436**, 257 (1998) [arXiv:hep-ph/9804398].

[2.4] L. Randall, R. Sundrum, Phys. Rev. Lett. **83** (1999) 3370 [hep-ph/9905221], Phys. Rev. Lett. **83** (1999) 4690 [hep-th/9906064].

[2.5] A. Pomarol and M. Quiros, Phys. Lett. B **438** (1998) 255 [arXiv:hep-ph/9806263];
A. Delgado, A. Pomarol and M. Quiros, Phys. Rev. D **60** (1999) 095008 [hep-ph/9812489];
I. Antoniadis, Phys. Lett. B **246**, 377 (1990);

T. Gherghetta and A. Pomarol, Nucl. Phys. B **586** (2000) 141 [arXiv:hep-ph/0003129].

[2.6] L. J. Hall, Y. Nomura and D. R. Smith, Nucl. Phys. B **639** (2002) 307 [hep-ph/0107331];

I. Gogoladze, Y. Mimura and S. Nandi, Phys. Lett. B **562** (2003) 307 [hep-ph/0302176].

[2.7] G. Burdman and Y. Nomura, Nucl. Phys. B **656** (2003) 3 [hep-ph/0210257];

N. Haba and Y. Shimizu, Phys. Rev. D **67** (2003) 095001 [Erratum-ibid. D **69** (2004) 059902] [hep-ph/0212166];

N. Haba and T. Yamashita, JHEP **0404** (2004) 016 [hep-ph/0402157];

G. Martinelli, M. Salvatori, C. A. Scrucca and L. Silvestrini, JHEP **0510** (2005) 037 [hep-ph/0503179];

M. Sakamoto and K. Takenaga, Phys. Rev. D **75** (2007) 045015 [hep-th/0609067].

[2.8] C. A. Scrucca, M. Serone, L. Silvestrini and A. Wulzer, JHEP **0402** (2004) 049 [hep-th/0312267];

C. Biggio and M. Quiros, Nucl. Phys. B **703** (2004) 199 [hep-ph/0407348];

Y. Hosotani, S. Noda and K. Takenaga, Phys. Lett. B **607** (2005) 276 [hep-ph/0410193];

D. Hernandez, S. Rigolin and M. Salvatori, arXiv:0712.1980 [hep-ph].

[2.9] E. Dudas, C. Papineau and V. A. Rubakov, JHEP **0603**, 085 (2006) [arXiv:hep-th/0512276].

[2.10] E. Dudas and C. Papineau, JHEP **0611**, 010 (2006) [arXiv:hep-th/0608054].

[2.11] F. Coradeschi, S. De Curtis, D. Dominici and J. R. Pelaez, [arXiv:0712.0537 [hep-th]].

[2.12] J. Vinet and J. M. Cline, Phys. Rev. D **70** (2004) 083514 [hep-th/0406141].

[2.13] M. Peloso, L. Sorbo and G. Tasinato, Phys. Rev. D **73** (2006) 104025 [hep-th/0603026];

E. Papantonopoulos, A. Papazoglou and V. Zamarias, JHEP **0703** (2007) 002 [hep-th/0611311];

B. Himmetoglu and M. Peloso, Nucl. Phys. B **773** (2007) 84 [hep-th/0612140];

N. Kaloper and D. Kiley, JHEP **0705** (2007) 045 [hep-th/0703190];

C. P. Burgess, D. Hoover and G. Tasinato, JHEP **0709** (2007) 124 [arXiv:0705.3212 [hep-th]];

M. Minamitsuji and D. Langlois, Phys. Rev. D **76** (2007) 084031 [arXiv:0707.1426 [hep-th]];

F. Arroja, T. Kobayashi, K. Koyama and T. Shiromizu, JCAP **0712** (2007) 006 [arXiv:0710.2539 [hep-th]];

C. Bogdanos, A. Kehagias and K. Tamvakis, Phys. Lett. B **656** (2007) 112 [arXiv:0709.0873 [hep-th]].

[2.14] A. Vilenkin, Phys. Rev. **D23** (1981) 852;

R. Gregory and C. Santos, Phys. Rev. D **56**, 1194 (1997) [gr-qc/9701014].

[2.15] S. M. Carroll and M. M. Guica, [hep-th/0302067];

[2.16] Y. Aghababaie, C. P. Burgess, S. L. Parameswaran and F. Quevedo, Nucl. Phys. B **680** (2004) 389 [hep-th/0304256].

[2.17] G. W. Gibbons, R. Guven and C. N. Pope, Phys. Lett. B **595** (2004) 498 [hep-th/0307238]; Y. Aghababaie *et al.*, JHEP **0309** (2003) 037 [hep-th/0308064];

P. Bostock, R. Gregory, I. Navarro and J. Santiago, Phys. Rev. Lett. **92** (2004) 221601 [hep-th/0311074];

C. P. Burgess, F. Quevedo, G. Tasinato and I. Zavala, JHEP **0411** (2004) 069 [hep-th/0408109]; J. Vinet and J. M. Cline, Phys. Rev. D **71** (2005) 064011 [hep-th/0501098].

[2.18] W. D. Goldberger and M. B. Wise, Phys. Rev. D **65** (2002) 025011 [arXiv:hep-th/0104170].

[2.19] C. de Rham, JHEP **0801** (2008) 060 [arXiv:0707.0884 [hep-th]];

C. de Rham, AIP Conf. Proc. **957**, 309 (2007) [arXiv:0710.4598 [hep-th]].

[2.20] D. Langlois and M. Sasaki, Phys. Rev. D **68** (2003) 064012 [hep-th/0302069].

[2.21] W. Frank, D. J. Land and R. M. Spector, Rev. Mod. Phys. **43** (1971) 36;

G. Parisi and F. Zirilli, J. Math. Phys. **14** (1973) 243;

H. E. Camblong, L. N. Epele, H. Fanchiotti and C. A. Garcia Canal, Phys. Rev. Lett. **85** (2000) 1590 [hep-th/0003014];

S. R. Beane, P. F. Bedaque, L. Childress, A. Kryjevski, J. McGuire and U. v. Kolck, Phys. Rev. A **64** (2001) 042103 [quant-ph/0010073].

[2.22] E. Braaten and D. Phillips, [hep-th/0403168].

[2.23] See for instance, C. P. Burgess, Phys. Rept. **330** (2000) 193 [arXiv:hep-th/9808176].

[2.24] For a discussion of these issues in the codimension-1 case, see A. Kehagias and K. Tamvakis, Phys. Lett. B **628** (2005) 262 [hep-th/0507130].

[2.25] S. L. Glashow, J. Iliopoulos and L. Maiani, Phys. Rev. D **2** (1970) 1285.

[2.26] M. E. Peskin and T. Takeuchi, Phys. Rev. Lett. **65** (1990) 964; Phys. Rev. D **46** (1992) 381;

G. Altarelli and R. Barbieri, Phys. Lett. B **253** (1991) 161;

G. Altarelli, R. Barbieri and S. Jadach, Nucl. Phys. B **369** (1992) 3 [Erratum-ibid. B **376** (1992) 444];

C. P. Burgess, S. Godfrey, H. Konig, D. London and I. Maksymyk, Phys. Lett. B **326**, 276 (1994) [hep-ph/9307337]; Phys. Rev. D **49**, 6115 (1994) [hep-ph/9312291].

[2.27] S. Weinberg, Phys. Rev. D**19**, 1277 (1979);

L. Susskind, Phys. Rev. D**20**, 2619 (1979);

P. Sikivie, L. Susskind, M. Voloshin and V. Sakharov, Nucl. Phys. B**173** (1980) 189;

H. Georgi and D. B. Kaplan, Phys. Lett. B **145** (1984) 216.

[2.28] J.M. Cornwall, D.N. Levin and G. Tiktopoulos, Phys. Rev. D**10** 1145

(1974);

B.W. Lee, C. Quigg and H. Thacker, Phys. Rev. D**16** 1519 (1977);

M. Veltman, Acta. Phys. Pol. B**8** 475 (1977).

[2.29] C. P. Burgess and D. London, Phys. Rev. D **48** (1993) 4337 [hep-ph/9203216]; Phys. Rev. Lett. **69** (1992) 3428.

[2.30] C. P. Burgess, J. Matias and F. Quevedo, Nucl. Phys. B **706** (2005) 71 [hep-ph/0404135].

[2.31] G. F. Giudice, R. Rattazzi and J. D. Wells, Nucl. Phys. B **544** (1999) 3 [hep-ph/9811291];

T. Han, J. D. Lykken and R. J. Zhang, Phys. Rev. D **59** (1999) 105006 [hep-ph/9811350].

[2.32] D. Atwood, C. P. Burgess, E. Filotas, F. Leblond, D. London and I. Maksymyk, Phys. Rev. D **63** (2001) 025007 [hep-ph/0007178];

I. Antoniadis and K. Benakli, Int. J. Mod. Phys. A **15** (2000) 4237 [hep-ph/0007226];

J. L. Hewett and D. Sadri, Phys. Rev. D **69** (2004) 015001 [hep-ph/0204063];

P. H. Beauchemin, G. Azuelos and C. P. Burgess, J. Phys. G **30** (2004) N17 [hep-ph/0407196]; J. Phys. G **31** (2005) 1 [hep-ph/0401125].

Chapter 3

Codimension-2 Brane-Bulk Matching: Examples from Six and Ten Dimensions

3.1 Preamble

This chapter is based on the work in (3.1). We generalize the interactions between general codimension 2 branes and their surrounding bulk derived in (3.2; 3.3) to include interactions with a bulk gauge field. We describe the resulting boundary conditions for a class of bulk theories that consist of gravity, a Maxwell field and a set of scalars with an arbitrary potential and target space metric. This is sufficiently general to include the bosonic sector of many of the known higher dimensional supergravities.

In addition to the boundary conditions that the bulk has to satisfy, we derive the expression for the 4 dimensional effective potential, found by integrating out the bulk. We take the branes to be arbitrary functions of the bulk fields, but limit ourselves to the lowest term in a derivative expansion. We test our results in various examples at the end of this chapter, and find that the expressions give reasonable results. In particular the example of D-7 branes in F-theory agrees with known results from string theory, which justifies our approach to describing brane-bulk interactions.

This chapter sets up the general framework for brane-bulk interactions that we use heavily in the later chapters. Important for the naturalness of the cosmological constant is the observation that the low energy theory is insensitive to the value of the brane tensions. Rather, the leading brane contribution is set by the derivative of the tension with respect to the bulk scalar fields. The point of this is that it is the brane tension that changes when brane particles are integrated out, so the possibility that this quantity drops out of the 4 dimensional theory can give an explanation of the smallness of the cosmological constant that works on all scales.

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3.2 Introduction

Space-filling branes, situated around extra dimensions, provide a remarkable framework for approaching phenomenological problems. Besides being well-motivated — for instance arising very naturally within string theory — branes lead to novel kinds of low-energy physics that can cut to the core of many of the naturalness issues that currently plague particle physics and cosmology.

The realization that not all particles need 'see' the same number of dimensions (because brane-bound particles are trapped to move only along the branes) is the first type of brane-related insight to have made a major impact on physics, leading to the recognition that the scale of gravity can be much smaller than the Planck scale (3.4). A second major revelation came with the realization that the back-reaction of branes on their environment can strongly influence their low-energy properties, such as by providing deep gravitational potential wells within the extra dimensions that redshift the energy of those branes that live within them (3.5).

Although branes can in principle have a great variety of dimensions, almost all of the detailed exploration of brane-bulk back-reaction is specialized to the case of codimension-1 branes: *i.e.* those branes that span just one dimension less than the dimension of the full spacetime. This is partially because tools for describing how branes back-react on their surroundings are only well-developed for codimension-1 surfaces, since in this case the problem can be expressed in terms of the Israel junction conditions (3.6). This restriction to codimension-1 objects is potentially very limiting because the special nature of kinematics in one dimension makes it unlikely that back-reaction for codimension-1 branes is representative of back-reaction for branes with higher codimension.

The main obstacle to understanding how properties of higher-codimension branes are related to the bulk geometries they source is the fact that these bulk geometries typically diverge at the position of their sources. (The most familiar example of this for a codimension-3 object is the divergence of the Coulomb potential of a nucleus evaluated at the nuclear position.) It is one of the special features of codimension-1 objects that the bulk fields they source typically do not diverge at their positions. They instead cause discontinuities of derivatives across their surfaces, whose properties are captured by the Israel junction conditions.

The next-simplest case consists of codimension-2 objects, whose back-reaction is complicated enough to allow the possibility of bulk fields diverging at the positions of the sources. Although bulk fields *can* diverge for codimension-2 sources, they *needn't* do so in time-independent situations. (For instance, they can instead give rise to conical singularities, such as for cosmic strings in 4D spacetime (3.7). When bulk fields do not diverge the relation between bulk and brane properties is easier to formulate, and so better studied (3.8).) The potential for divergent bulk configurations makes codimension-2 branes more representative of systems with more generic codimension than are codimension-1 branes. But dynamics in two dimensions is still simple enough to allow explicit closed-form solutions to be known for the bulk configurations sourced by codimension-2 branes, allowing a detailed study of their properties.

Tools for describing how bulk fields respond to the properties of source branes were recently developed in the general case, including where the bulk fields diverge (3.2; 3.3; 3.8), opening up the properties of codimension-2 branes for phenomenological exploration. These tools — summarized (and slightly generalized) in §3.3 below for a fairly general class of scalar-tensor-Maxwell theories in n extra dimensions — boil down to a set of matching conditions that relate the near-brane limit of the radial derivatives of the bulk fields to the action for the brane in question.

In §3.4 we apply these tools to three kinds of examples: compact geometries sourced by D7 branes in F-theory compactifications of 10D Type IIB supergravity; 3-branes coupled to a bulk axion within unwarped, non-supersymmetric 6D scalar/Maxwell/Einstein theory; and 3-branes coupled to 6D chiral gauged supergravity. We draw the following lessons from these comparisons:

- F-theory compactifications (3.10) of 10D Type IIB supergravity sourced by D7-branes serve as a reality check, since string theory tells us the detailed form of both the brane and bulk actions (3.9), and explicit solutions are known for the transverse spacetimes that are sourced by these branes (3.21). We verify the codimension-2 brane/bulk matching conditions by checking that the asymptotic forms for the solutions are related to the known brane actions in the prescribed way.

- In 6D axion-Maxwell-Einstein theory, flux-compactified solutions are known for the bulk that interpolates between two 3-branes, and these are simple enough to allow the explicit calculation of how branes contribute to the low-energy axion potential (3.11). From the perspective of six dimensions the resulting axion stabilization arises through the requirement that both branes be consistent in their demands on the bulk. We show that the stabilized value agrees precisely with the result of minimizing the low-energy axion potential as seen by an observer who has integrated out the extra dimensions below the Kaluza-Klein (KK) scale. We also show how this potential gives the same value for the curvature of the maximally symmetric on-brane geometry as is calculated from the higher-dimensional field equations.

- Stable flux compactifications are also known for 6D chiral gauged supergravity (3.12), having up to two singularities that represent the positions of two source branes (3.13). These solutions are known in explicit closed form for the most general solutions having a flat on-brane geometry and axial symmetry in the bulk; and in a slightly more implicit form for solutions with de Sitter or anti-de Sitter on-brane geometry. In this case we use the matching conditions to show that the only bulk configurations that can be supported by positive-tension branes have flat induced on-brane geometries, with (possibly warped) bulk geometries with nonsingular limits as the source branes are approached. We also show how geometries that diverge at the brane positions can arise from specific kinds of negative-tension branes, while no maximally symmetric solutions exist at all for many kinds of brane sources (presumably corresponding to time-dependent runaway bulk geometries, such as those considered in (3.14)).

§3.5 briefly summarizes some of the implications of these results.

3.3 The Bulk-Brane system

We start by describing the brane-bulk framework within which we work. This starts with a statement of the scalar-metric-Maxwell system whose equations we use, followed by a statement of how the near-brane boundary conditions of the bulk fields are related to the action of the branes which are their source. Finally we describe the contribution of each brane to the low-energy scalar potential that is valid over distances much longer than the size of the extra dimensions, and identify a constraint which allows a simple description of this contribution given the properties of the brane tension.

3.3.1 The bulk

The starting point is the statement of the equations of motion that govern the bulk.

General formulation

We assume the following action for the n-dimensional bulk physics, describing a general scalar-tensor theory coupled to a Maxwell field,[1]

$$S = \int_{\mathcal{M}} \mathrm{d}^n x \; \mathcal{L}_B + \int_{\partial\mathcal{M}} \mathrm{d}^{n-1} x \; \mathcal{L}_{GH} \tag{3.1}$$

where

$$\mathcal{L}_B = -\sqrt{-g} \; \left\{ \frac{1}{2\kappa^2}\, g^{MN} \Big[\mathcal{R}_{MN} + \mathcal{G}_{AB}(\phi)\, \partial_M \phi^A \partial_N \phi^B \Big] + \frac{1}{4}\, f(\phi)\, F_{MN} F^{MN} + V(\phi) \right\} , \tag{3.2}$$

[1] Our metric is mostly plus, with Weinberg's curvature conventions (3.15), which differ from those of MTW (3.16) only by an overall sign in the definition of the Riemann tensor.

and the Gibbons-Hawking lagrangian (3.17) is

$$\mathcal{L}_{GH} = \frac{1}{\kappa^2}\sqrt{-\hat{\gamma}}\; K\,, \tag{3.3}$$

and is required in the presence of boundaries in order to make the Einstein action well posed. Here $F = \mathrm{d}A$ is the field strength of the Maxwell field, $\mathcal{R}$ is the Ricci scalar for the 6D spacetime metric, g_{MN}, and $\mathcal{G}_{AB}$ is the metric of the target space within which the scalar fields, ϕ^A, $A = 1, \dots, N$, take values. $\hat{\gamma}_{ij} = g_{MN}\,\partial_i x^M \partial_j x^N$ is the induced metric, and K is the trace, $\hat{\gamma}^{ij}K_{ij}$, of the extrinsic curvature, of the boundary surface, $\partial\mathcal{M}$.

This bulk action is chosen to be general enough to include the bosonic part of the supersymmetric theories of interest. Its field equations are

$$\frac{1}{2\kappa^2}\left(\mathcal{R}_{MN} + \mathcal{G}_{AB}\,\partial_M\phi^A\partial_N\phi^B\right) + \frac{f}{2}\;F_M{}^P F_{NP} + \frac{1}{n-2}\left[V - \frac{f}{4}\,F_{PQ}F^{PQ}\right]\,g_{MN} = 0, \tag{3.4}$$

$$\mathcal{G}_{AB}\Box\,\phi^B - \kappa^2\left[\frac{\partial V}{\partial\phi^A} + \frac{1}{4}\,\frac{\partial f}{\partial\phi^A}\,F_{MN}F^{MN}\right] = 0\,, \tag{3.5}$$

and

$$\nabla_M\left(fF^{MN}\right) = 0\,, \tag{3.6}$$

where

$$\Box\,\phi^A := g^{MN}\left[\nabla_M\partial_N\phi^A + \Gamma^A_{BC}(\phi)\partial_M\phi^B\partial_N\phi^C\right], \tag{3.7}$$

with $\Gamma^A_{BC}(\phi)$ being the Christoffel connection built from the metric $\mathcal{G}_{AB}$.

Metric *ansätze*

Our interest is in configurations whose geometries are maximally symmetric in the brane directions, for which it is convenient to specialize to the metric

$$\begin{aligned} \mathrm{d}s^2 = g_{MN}\,\mathrm{d}x^M\mathrm{d}x^N &= e^{2W}\hat{g}_{\mu\nu}\,\mathrm{d}x^\mu\mathrm{d}x^\nu + g_{mn}\,\mathrm{d}x^m\mathrm{d}x^n \\ &= e^{2W}\hat{g}_{\mu\nu}\,\mathrm{d}x^\mu\mathrm{d}x^\nu + e^{2C}\,\mathrm{d}z\,\mathrm{d}\overline{z}\,, \end{aligned} \tag{3.8}$$

where $\hat{g}_{\mu\nu}(x)$ denotes a maximally symmetric $(n-2)$-dimensional metric. The coordinates are $x^M = \{x^\mu, x^m\}$, with x^μ, $\mu = 0, \dots, n-3$ labelling the brane directions, and $m = n-2, n-1$ (or $z = x^{n-2} + ix^{n-1}$) being coordinates for the two dimensions transverse to the branes. The functions W and C are generally singular at the positions of any source branes. For instance, if $e^C = (\ell/r)^a$ for $r^2 = |z|^2$, then the proper distance becomes $\rho = [\ell/(1-a)](\ell/r)^{a-1}$ and $e^B = \ell(\ell/r)^{a-1} = (1-a)\rho$, showing that the metric in this case has a conical singularity at $r = \rho = 0$, with defect angle $\delta = 2\pi a$.

For some applications, particularly very near a brane, it is useful to further specialize to the most general *ansatz* consistent with cylindrical symmetry in the two transverse dimensions, $\{x^m, m = n-2, n-1\}$. This leads to the following metric:

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}\rho^2 + e^{2B}\mathrm{d}\theta^2 + e^{2W}\hat{g}_{\mu\nu}\,\mathrm{d}x^\mu\mathrm{d}x^\nu \\ &= e^{2C}\Big(\mathrm{d}r^2 + r^2\mathrm{d}\theta^2\Big) + e^{2W}\hat{g}_{\mu\nu}\,\mathrm{d}x^\mu\mathrm{d}x^\nu \end{aligned} \tag{3.9}$$

where θ labels the direction of cylindrical symmetry, and the functions $B = B(\rho)$ and $W = W(\rho)$ depend on the proper distance, ρ, only — or $C = C(r)$ is a function only of r.

The bulk scalars are similarly just functions of ρ, $\phi^A = \phi^A(\rho)$, and a gauge can be chosen to that the only nonzero component for the Maxwell field is $A_M = A_\theta(\rho)\,\delta^\theta_M$, and so

$$F_{\rho\theta} = -F_{\theta\rho} = A'_\theta\,, \tag{3.10}$$

where the prime denotes differentiation with respect to ρ.

The Einstein equations subject to this *ansatz* reduce to

$$\begin{aligned}
&\frac{1}{n-2}\,e^{-2W}\,\hat{R} + W'' + (n-2)(W')^2 + W'B' \\
&\qquad -\frac{1}{n-2}\kappa^2\,e^{-2B}\,f(A'_\theta)^2 + \frac{2\kappa^2 V}{n-2} = 0 \qquad (\mu\nu) && (3.11)\\
&B'' + (B')^2 + (n-2)W'B' \\
&\qquad +\frac{n-3}{n-2}\kappa^2\,e^{-2B}\,f(A'_\theta)^2 + \frac{2\kappa^2 V}{n-2} = 0 \qquad (\theta\theta) && (3.12)\\
&(n-2)\left[W'' + (W')^2\right] + B'' + (B')^2 + \mathcal{G}_{AB}\phi^{A\prime}\phi^{B\prime} \\
&\qquad +\frac{n-3}{n-2}\kappa^2\,e^{-2B}\,f(A'_\theta)^2 + \frac{2\kappa^2 V}{n-2} = 0 \qquad (\rho\rho)\,, && (3.13)
\end{aligned}$$

while the dilaton and Maxwell equations become

$$\begin{aligned}
&e^{-B-4W}\left(e^{B+4W}\,\mathcal{G}_{AB}\phi^{B\prime}\right)' + \mathcal{G}_{AB}\Gamma^B_{CD}\,\phi^{C\prime}\,\phi^{D\prime} \\
&\qquad -\kappa^2\left[\frac{\partial V}{\partial\phi^A} + \frac{1}{4}\,\frac{\partial f}{\partial\phi^A}\,e^{-2B}(A'_\theta)^2\right] = 0\,, && (3.14)
\end{aligned}$$

and

$$\left(e^{-B+4W} f\,A'_\theta\right)' = 0\,. \tag{3.15}$$

3.3.2 Boundary conditions for codimension-2 branes

General formulation

Suppose an $(n-2)$-dimensional, space-filling, codimension-2 brane is located at a position, $x^m = x_b^m$, within the 2 extra dimensions, with brane action

$$S_b = -\int_{x_b} \mathrm{d}^{n-2}x \, \sqrt{-\gamma} \, \Big[L_b \left(\phi^A, A_\theta, g_{\theta\theta} \right) + \cdots \Big] \,, \tag{3.16}$$

where L_b denotes the brane lagrangian, which is potentially a function of the bulk scalars, ϕ^A, and the tangential components of the bulk Maxwell field and metric, A_M and g_{MN}, but not their derivatives. (Ellipses denote the possible subdominant, higher-derivative effective interactions that can also be present.) We imagine the geometry surrounding the brane to be given by the axisymmetric *ansatz* of eq. (3.9), with the brane located at $\rho = 0$, so θ denotes the angular direction about its position. Because our interest is in maximally symmetric solutions along the brane directions we do not entertain a dependence of T_b on any components of A_M and g_{MN} apart from A_θ and $g_{\theta\theta}$.

The induced metric on the brane is $\gamma_{\mu\nu} = g_{MN} \, \partial_\mu x^M \, \partial_\nu x^N = e^{2W} \hat{g}_{\mu\nu}$. Because of the warp factor appearing in this metric, for later purposes it is convenient to define the 'warped' tension, T_b, by $T_b = e^{(n-2)W} L_b$, so that the brane action becomes

$$S_b = -\int_{x_b} \mathrm{d}^{n-2}x \, \sqrt{-\hat{g}} \, \Big[T_b(\phi, A_\theta, g_{\theta\theta}, W) + \cdots \Big] \,. \tag{3.17}$$

The back-reaction of such a brane onto the bulk geometry dictates the asymptotic near-brane behaviour of the bulk fields nearby,[2] through codimension-

[2]A familiar example of this from electrostatics is the $1/\rho$ dependence of the Coulomb

2 matching conditions that generalize (3.2; 3.3; 3.8) the more familiar ones that are encountered for codimension-1 branes. For the bulk scalars these state

$$\lim_{\rho \to 0} \oint_{x_b} \mathrm{d}\theta \, \left[\frac{1}{\kappa^2} \sqrt{-g} \, \mathcal{G}_{AB} \partial_\rho \phi^B \right] = -\frac{\delta S_b}{\delta \phi^A} \,, \tag{3.18}$$

where the integration is about a small circle of proper radius ρ encircling the brane position, x_b, which is taken to be situated at $\rho = 0$. Similarly, the Maxwell matching condition is

$$\lim_{\rho \to 0} \oint_{x_b} \mathrm{d}\theta \, \left[\sqrt{-g} \; f \, F^{\rho M} \right] = -\frac{\delta S_b}{\delta A_M} \,, \tag{3.19}$$

Finally, the metric matching condition is

$$\lim_{\rho \to 0} \oint_{x_b} \mathrm{d}\theta \, \left[\frac{1}{2\kappa^2} \sqrt{-g} \, \left(K^{ij} - K g^{ij} \right) - (\text{flat}) \right] = -\frac{\delta S_b}{\delta g_{ij}} \,, \tag{3.20}$$

where K_{ij} is the extrinsic curvature of the fixed-ρ surface, for which the local coordinates are those appropriate for surfaces of constant ρ: $\{x^i, i = 0, 1, \cdots, n-2\}$. Here 'flat' denotes the same result evaluated near the origin of a space for which the brane location $\rho = 0$ is nonsingular.

Axially symmetric ansatz

Specialized to the *ansatz* of eq. (3.9) the scalar-field matching condition becomes

$$\left[\frac{2\pi}{\kappa^2} \, e^{B+(n-2)W} \sqrt{-\hat{g}} \, \mathcal{G}_{AB} \, \phi^{B\prime} \right]_{x_b} = \frac{\partial}{\partial \phi^A} \left[\sqrt{-\hat{g}} \; T_b \right]. \tag{3.21}$$

potential that occurs in the immediate vicinity of a point charge situated at $\rho = 0$.

With the same *ansatz*, the corresponding result for the Maxwell field reduces to

$$\Big[2\pi\sqrt{-\hat{g}}\; e^{-B+(n-2)W}\; f\, A'_\theta\Big]_{x_b} = \frac{\partial}{\partial A_\theta}\Big[\sqrt{-\hat{g}}\;\, T_b\Big] := \sqrt{-\hat{g}}\; J_b(\phi)\,, \qquad (3.22)$$

where the last equality defines the quantity J_b.

Finally, for fixed-ρ surfaces in this *ansatz*, $K_{ij} = \frac{1}{2}\,\partial_\rho g_{ij}$, and the comparison 'flat' metric is $\mathrm{d}s^2_{\text{flat}} = \mathrm{d}\rho^2 + \rho^2\mathrm{d}\theta^2 + e^{2W_{\text{flat}}}\hat{g}_{\mu\nu}\,\mathrm{d}x^\mu\mathrm{d}x^\nu$, with $W'_{\text{flat}} \to 0$ as $\rho \to 0$. Since $K_{\theta\theta} = B'e^{2B}$ and $K_{\mu\nu} = W'\,e^{2W}\hat{g}_{\mu\nu}$, we have $K = g^{ij}K_{ij} = B' + (n-2)W'$, and so the $(\mu\nu)$ components of the metric matching conditions give

$$\left[-\frac{2\pi}{\kappa^2}\sqrt{-\hat{g}}\, e^{(n-2)W}[e^B\,((n-3)W' + B') - 1]\right]_{x_b} = \sqrt{-\hat{g}}\; T_b(\phi)\,, \qquad (3.23)$$

while the $(\theta\theta)$ components are,

$$\begin{aligned}\left[\frac{2\pi}{\kappa^2}\sqrt{-\hat{g}}\, e^{B+(n-2)W}\,((n-2)\,W')\right]_{x_b} &= -2\frac{\partial}{\partial g_{\theta\theta}}\Big[\sqrt{-\hat{g}}\,T_b\Big] \qquad (3.24)\\ &:= (n-2)\sqrt{-\hat{g}}\,U_b(\phi)\,,\end{aligned}$$

where the last equality defines U_b. Just as T_b physically represents the brane tension, J_b can be interpreted as describing microscopic axial currents within the brane, or equivalently any microscopic magnetic flux these currents enclose within the brane. Once the dimensions transverse to the brane are dimensionally reduced, U_b turns out (3.2; 3.3) to be related to the brane contribution to the scalar potential within the low-energy 4D effective theory defined below the KK scale (as is seen in more detail later).

3.3.3 The brane constraint

These matching conditions, when combined with the bulk equations of motion, imply an important constraint relating the quantities T_b, J_b and U_b (3.18; 3.2; 3.3). This constraint comes from eliminating second derivatives, ∂_ρ^2, of the fields from the field equations, and so can be regarded as the 'Hamiltonian' constraint on the initial data when integrating the field equations in the ρ direction. When written in the form given above, the relevant combination of Einstein equations is $(n-2)(\mu\nu) + (\theta\theta) - (\rho\rho)$, which imply

$$(n-3)(n-2)\,(W')^2 + 2(n-2)\,W'B' - \mathcal{G}_{AB}\,\phi^{A\prime}\phi^{B\prime} \\ -\kappa^2 e^{-2B} f\,(A'_\theta)^2 + e^{-2W}\hat{R} + 2\kappa^2 V = 0. \quad (3.25)$$

To turn this into a constraint on brane properties, multiply it through by $e^{2B+2(n-2)W}$ and take the limit $x \to x_b$, using the above matching conditions to eliminate the derivatives $\phi^{A\prime}$, B', W' and A'_θ in favour of the brane functions T_b, J_b and U_b. The required matching conditions are

$$\begin{aligned} \Big[e^B\phi^{A\prime}\Big]_{x_b} &= e^{-(n-2)W}\mathcal{G}^{AB}\frac{\partial\mathcal{T}_b}{\partial\phi^B} \quad \text{with} \quad \mathcal{T}_b := \frac{\kappa^2 T_b}{2\pi} \\ \Big[\kappa A'_\theta\Big]_{x_b} &= e^{-(n-2)W}\frac{\mathcal{J}_b}{f} \quad \text{with} \quad \mathcal{J}_b := \frac{\kappa\, e^B J_b}{2\pi} \\ \Big[e^B W'\Big]_{x_b} &= e^{-(n-2)W}\mathcal{U}_b \quad \text{with} \quad \mathcal{U}_b := \frac{\kappa^2\, U_b}{2\pi} \qquad (3.26) \\ \text{and} \quad \Big[e^B B' - 1\Big]_{x_b} &= -e^{-(n-2)W}\Big[\mathcal{T}_b + (n-3)\,\mathcal{U}_b\Big], \end{aligned}$$

where each of $\mathcal{U}_b$, $\mathcal{T}_b$ and $\mathcal{J}_b$ is dimensionless (keeping in mind e^B has dimensions of length). Using eqs. (3.26) in eq. (3.25) we find the desired constraint:

$$(n-3)(n-2)\,(\mathcal{U}_b)^2 + 2(n-2)\mathcal{U}_b\Big[e^{(n-2)W} - \mathcal{T}_b - (n-3)\,\mathcal{U}_b\Big] \qquad (3.27)$$
$$-\mathcal{G}^{AB}\,\frac{\partial \mathcal{T}_b}{\partial \phi^A}\,\frac{\partial \mathcal{T}_b}{\phi^B} - \frac{(\mathcal{J}_b)^2}{f} + e^{2B+2(n-2)W}\Big[e^{-2W}\hat{R} + 2\kappa^2 V\Big]_{x_b} = 0\,.$$

This crucially simplifies once we use the fact that near the brane $e^B \to 0$ as $\rho \to 0$. (This states that the circumference of small circles about the brane must vanish as the radius of the circles vanishes. If not true, the object at $\rho = 0$ would not be interpreted as a codimension-2 brane.) The key observation (3.2; 3.3) is that the quantities $\kappa\, e^{2B} J_b$, $e^{2B-2W}\hat{R}$ and $\kappa^2 e^{2B} V$ also tend to vanish in this limit (as would be true, for instance, if $e^{-2W}\hat{R}$, V and J_b were bounded at the brane positions), implying that the constraint becomes

$$(n-2)\mathcal{U}_b\Big[2e^{(n-2)W} - 2\mathcal{T}_b - (n-3)\,\mathcal{U}_b\Big] - (\mathcal{T}_b')^2 \simeq 0\,, \qquad (3.28)$$

where $(\mathcal{T}_b')^2 = \mathcal{G}^{AB}\,\partial_A \mathcal{T}_b\,\partial_B \mathcal{T}_b$.

What is important about this last form of the constraint is that the on-brane curvature drops out in this limit, meaning that eq. (3.28) cannot be read as being solved for $\hat{R}$. Instead, this constraint expresses a consistency condition for the brane action and junction conditions, imposed by the bulk equations of motion. In practice it provides a very simple method for computing the quantity $\mathcal{U}_b(\phi)$ once expressions for $\mathcal{T}_b(\phi)$ are given, since solving eq. (3.28) implies

$$\mathcal{U}_b = \frac{1}{n-3}\left[\left(e^{(n-2)W} - \mathcal{T}_b\right) \pm \sqrt{\left(e^{(n-2)W} - \mathcal{T}_b\right)^2 - \left(\frac{n-3}{n-2}\right)(\mathcal{T}_b')^2}\,\right]. \qquad (3.29)$$

Here the root is chosen for which $\mathcal{U}_b \to 0$ when $(\mathcal{T}_b')^2 \to 0$, and so is $\pm$ according to whether sign $\left(e^{(n-2)W} - \mathcal{T}_b\right)$ is $\mp$. This means that $\mathcal{U}_b$ has the same sign as does $\left(e^{(n-2)W} - \mathcal{T}_b\right)$. Notice also that requiring the square root never be complex requires

$$\frac{n-3}{n-2} \, (\mathcal{T}_b')^2 \leq \left(e^{(n-2)W} - \mathcal{T}_b\right)^2 \, . \tag{3.30}$$

This last condition can be nontrivial, even though control over the semiclassical approximation requires $|\mathcal{T}_b| \ll 1$ and $(\mathcal{T}_b')^2 \ll 1$. This is because it can happen that $e^W \to 0$ at the brane, in which case eq. (3.30) becomes a constraint on the size of $(\mathcal{T}_b')^2/\mathcal{T}_b^2$.

For $(\mathcal{T}_b')^2 \ll \left(e^{(n-2)W} - \mathcal{T}_b\right)^2$ eq. (3.29) becomes

$$\mathcal{U}_b \simeq \frac{(\mathcal{T}_b')^2}{2(n-2)\left(e^{(n-2)W} - \mathcal{T}_b\right)} + \frac{(n-3)(\mathcal{T}_b')^4}{8(n-2)^2\left(e^{(n-2)W} - \mathcal{T}_b\right)^3} + \cdots \, . \tag{3.31}$$

3.3.4 The classical low-energy on-brane effective action

Over distances much longer than the size of the two compact dimensions transverse to the brane the classical bulk dynamics is governed by the motion of the massless Kaluza-Klein states. The dynamics are effectively d-dimensional, with $d = n-2$. To understand the dynamics from this d-dimensional perspective, it is useful to integrate out the extra dimensions to obtain the low-energy lower-dimensional effective theory. At the classical level this amounts to eliminating all of the massive KK states as functions of their massless counterparts, using the bulk classical equations of motion.

In the present instance the massless KK states consist of the on-brane metric and Maxwell fields, $\hat{g}_{\mu\nu}$ and A_μ, as well as any d-dimensional scalars, φ^a, descending from ϕ^A and/or from moduli in the metric components, g_{mn}, in the extra dimensions. To obtain the low-energy potential, $V_{\rm eff}(\varphi)$, for the various d-dimensional scalars, φ^a, we eliminate the massive Kaluza-Klein modes in the action, as functions of $\hat{g}_{\mu\nu}$ and φ^a. The transverse metric, g_{mn}, is eliminated by using the trace reversed (mn) Einstein equations, which single out the kinetic terms for g_{mn}:

$$\frac{1}{2\kappa^2}\left(\mathcal{R}_{mn} + \mathcal{G}_{AB}\,\partial_m\phi^A\partial_n\phi^B\right) + \frac{f}{2}\,F_m{}^P F_{nP} + \frac{1}{n-2}\left[V - \frac{f}{4}\,F_{PQ}F^{PQ}\right] g_{mn} = 0, \tag{3.32}$$

These comprise two independent equations, which we take to be the sum and difference of the $(\rho\rho)$ and $(\theta\theta)$ components. The difference gives

$$(n-2)\Big(W'' + (W')^2 - W'B'\Big) + \mathcal{G}_{AB}\,\phi^{A\prime}\phi^{B\prime} = 0\,, \tag{3.33}$$

while the sum is equivalent to contracting eq. (3.32) with g^{mn}, to give

$$\frac{1}{2\kappa^2}\left(\mathcal{R}_{(2)} + \mathcal{G}_{AB}\partial_m\phi^A\partial^m\phi^B\right) = -\frac{n-3}{2(n-2)}\,f\,F_{mn}F^{mn} - \frac{2}{n-2}V\,, \tag{3.34}$$

where we write the higher-dimensional curvature scalar as

$$\begin{aligned}
\mathcal{R} = g^{MN}\mathcal{R}^P{}_{MPN} &= \mathcal{R}_{(n-2)} + \mathcal{R}_{(2)} \\
\text{where}\quad \mathcal{R}_{(2)} = g^{mn}\mathcal{R}^P{}_{mPn} &= R_{(2)} + (n-2)(\Box W + \nabla W\cdot\nabla W) \\
&= R_{(2)} + (n-2)\Big[W'' + (W')^2 + B'W'\Big] \qquad (3.35)\\
\text{and}\quad \mathcal{R}_{(n-2)} = g^{\mu\nu}\mathcal{R}^P{}_{\mu P\nu} &= e^{-2W}\hat{g}^{\mu\nu}\hat{R}_{\mu\nu} \\
&\quad + (n-2)[\Box W + (n-4)\nabla W\cdot\nabla W]
\end{aligned}$$

$$
\begin{aligned}
&= e^{-2W}\hat{g}^{\mu\nu}\hat{R}_{\mu\nu} \\
&\qquad +(n-2)\Big[W'' + (n-4)(W')^2 + B'W'\Big] .
\end{aligned}
$$

Here $R_{(2)} = g^{mn}R^p{}_{mpn}$ and $\hat{g}^{\mu\nu}\hat{R}_{\mu\nu}$ respectively denote the curvature scalars built from the 2D metric, g_{mn}, and the 4D metric, $\hat{g}_{\mu\nu}$.

Using eq. (3.34) to eliminate $\mathcal{R}_{(2)}$ from the bulk action then yields the bulk contribution to the lower-dimensional lagrangian density.[3] Using $\sqrt{-g} = \sqrt{-\hat{g}}\,\sqrt{g_2}\,e^{(n-2)W}$, we find

$$
\begin{aligned}
\mathcal{L}_{\text{eff}}(\varphi) &= -\int \mathrm{d}^2x\sqrt{g_2}\,e^{(n-2)W}\left[\frac{1}{2\kappa^2}\mathcal{R}_{(n-2)} + \frac{4-n}{4(n-2)}\,f\,F_{mn}F^{mn} + \frac{n-4}{n-2}V\right] \\
&= -\int \mathrm{d}^2x\sqrt{g_2}\,e^{(n-2)W}\left\{\frac{1}{2\kappa^2}\Big[e^{-2W}\hat{g}^{\mu\nu}\hat{R}_{\mu\nu}\right. \\
&\qquad\qquad +(n-2)\Big(W'' + (n-4)(W')^2 + B'W'\Big)\Big] \\
&\qquad\qquad \left. + \frac{4-n}{4(n-2)}\,f\,F_{mn}F^{mn} + \frac{n-4}{n-2}V\right\} \\
&= -\int \mathrm{d}^2x\sqrt{g_2}\,e^{(n-2)W}\left\{\frac{1}{2\kappa^2}\Big[e^{-2W}\hat{g}^{\mu\nu}\hat{R}_{\mu\nu}\right. \\
&\qquad\qquad +(n-2)\Big((n-5)(W')^2 + 2W'B'\Big) - \mathcal{G}_{AB}\phi^{A\prime}\phi^{B\prime}\Big] \\
&\qquad\qquad \left. + \frac{4-n}{4(n-2)}\,f\,F_{mn}F^{mn} + \frac{n-4}{n-2}V\right\} . \\
&= -\int \mathrm{d}^{n-2}x\sqrt{-\hat{g}}\,\left[\frac{1}{2\kappa_N^2}\,\hat{g}^{\mu\nu}\hat{R}_{\mu\nu} + V_B\right] ,
\end{aligned} \tag{3.36}
$$

where the second to last equality uses the second independent bulk field equation, eq. (3.33), the last equality defines the bulk potential, V_B, and the lower-

[3]Although in principle the extra-dimensional part of the trace reversed $(\mu\nu)$ Einstein equation, $ER_{\mu\nu}(x,y) = 0$ could also be used to eliminate massive KK modes, this *cannot* be used to eliminate $R_{(n-2)}$ from V_B because the integration in eq. (3.36) projects onto the zero-mode component of $E_{\mu\nu} = 0$.

dimensional Newton's constant, $\kappa_N^2 = 8\pi G_N$, is given by

$$\frac{1}{\kappa_N^2(\varphi)} := \frac{1}{\kappa^2} \int \mathrm{d}^2x \sqrt{g_2}\, e^{(n-4)W} \,. \tag{3.37}$$

In general this depends on the low-energy scalar fields, a dependence that can be removed by performing a Weyl rescaling to reach the lower-dimension Einstein frame.

To obtain the complete low-energy scalar potential, V_{eff}, the bulk contribution, V_B, must be combined with two other contributions, both associated with the source branes. The first of these comes from the boundary terms of the bulk action (3.2; 3.3), such as the Gibbons-Hawking term for the metric, evaluated at a small surface, Σ_b, situated a short proper distance, $\rho = \epsilon$, from the position of each of the source branes:

$$\begin{aligned} S_{GH} &= \sum_{b=0}^{1} \lim_{\epsilon\to 0} \oint_{\Sigma_b} \mathrm{d}\theta\, \mathrm{d}^{n-2}x\, \frac{1}{\kappa^2} \sqrt{-\hat{\gamma}}\; K \\ &= \frac{2\pi}{\kappa^2} \sum_{b=0}^{1} (-)^b \int_{\rho=\rho_b} \mathrm{d}^{n-2}x \sqrt{-\hat{g}}\; e^{B+(n-2)W} \Big[B' + (n-2)W' \Big] \\ &= -\sum_{b=0}^{1} \int_{\rho=\rho_b} \mathrm{d}^{n-2}x \sqrt{-\hat{g}}\; \Big\{ \Big[-T_b - (n-3)U_b \Big] + (n-2)U_b \Big\} \\ &= -\sum_{b=0}^{1} \int_{\rho=\rho_b} \mathrm{d}^{n-2}x \sqrt{-\hat{g}}\; \Big(U_b - T_b \Big) \,. \end{aligned} \tag{3.38}$$

Here we use the axisymmetric *ansatz*, as is appropriate very near the source branes. The relative sign, $(-)^b$, and the overall sign in the second line arise because primes denote $\mathrm{d}/\mathrm{d}\rho$ while the derivatives appearing in the Gibbons-Hawking action and matching conditions are outward directed, and this is in the $\mathrm{d}\rho$ direction for one brane and $-\mathrm{d}\rho$ for the other. The last line uses the matching conditions described earlier to exchange W' and B' for terms involving the brane action, using the fact that the contribution of $[e^B K]_{\rm flat}$ cancels between the two branes.

The second contribution to the 4D scalar potential comes from the contribution of the brane action itself, eq. (3.16). Combining these with $V_{4\rm B}$ above gives the full 4D scalar potential in the classical limit as in (3.3),

$$
\begin{aligned}
-\int \mathrm{d}^{n-2}x\sqrt{-\hat{g}}\; V_{\rm eff} &= -\int \mathrm{d}^{n-2}x\sqrt{-\hat{g}}\; V_B + \sum_{b=0}^{1}\Big[S_b + \lim_{\epsilon\to 0} S_{\rm GH}\Big] \qquad (3.39)\\
&= -\int \mathrm{d}^{n-2}x\sqrt{-\hat{g}}\; V_B - \sum_{b=0}^{1}\int \mathrm{d}^{n-2}x\,\sqrt{-\hat{g}}\,\Big[T_b+\Big(U_b - T_b\Big)\Big]\,,
\end{aligned}
$$

where the notation W_b is a reminder that W is evaluated at the brane position. This shows that (within the classical approximation) the effect of the Gibbons-Hawking terms is to ensure that the net contribution of each brane to the low-energy scalar potential is given by the quantity U_b, appropriately warped. The complete low-energy scalar potential is therefore,

$$
\begin{aligned}
V_{\rm eff} &= V_B + \sum_b U_b \qquad (3.40)\\
&= \sum_b U_b + \int \mathrm{d}^2x\sqrt{g_2}\; e^{(n-2)W}\left\{\frac{1}{2\kappa^2}\Big[(n-2)\left\{(n-5)(W')^2\right.\right.\\
&\qquad \left.\left. +2W'B' - \mathcal{G}_{AB}\phi^{A\prime}\phi^{B\prime}\right\}\Big]\\
&\qquad \left. +\frac{4-n}{4(n-2)}\,f\,F_{mn}F^{mn} + \frac{n-4}{n-2}\,V\right\}.
\end{aligned}
$$

Stationary points

For some purposes it is sufficient to obtain the value of the potential, $V_{\rm eff}(\phi_0)$, evaluated at its stationary point, where $V'_{\rm eff}(\varphi_0) = 0$. This can be obtained from the higher-dimensional action by eliminating fields using *all* of the equations of motion, and not just those of the massive KK modes. In this case we may directly use the equation of motion,

$$\frac{1}{2\kappa^2}\Big(\mathcal{R} + \mathcal{G}_{AB}\,\partial_M\phi^A\,\partial^M\phi^B\Big) = -\frac{(n-4)}{4(n-2)}\,f\,F_{MN}F^{MN} - \frac{nV}{n-2}\,, \tag{3.41}$$

rather than eq. (3.34) for $R_{(2)}$. Using this to eliminate $\mathcal{R}$ from the bulk action yields

$$\begin{aligned} S_{\rm ext} &= -\int {\rm d}^nx\sqrt{-g}\;\left[\frac{1}{2\kappa^2}\Big(\mathcal{R} + \mathcal{G}_{AB}\,\partial_M\phi^A\partial^M\phi^B\Big) + \frac{1}{4}\,f\,F_{MN}F^{MN} + V\right]_{\rm cl} \\ &= -\frac{2}{n-2}\int {\rm d}^nx\sqrt{-g}\;\left[\frac{1}{4}\,f\,F^{mn}F_{mn} - V\right]. \end{aligned} \tag{3.42}$$

When comparing with the low-energy theory we must also evaluate the low energy action at its stationary point. That is, we evaluate the action

$$S_{\rm eff} = -\int {\rm d}^{n-2}x\sqrt{-\hat g}\;\left[\frac{1}{2\kappa_N^2}\,\hat R_{(n-2)} + V_{\rm eff}\right], \tag{3.43}$$

at the solution to the low-energy field equations,

$$\frac{1}{2\kappa_N^2}\hat R_{(n-2)} = -\frac{(n-2)}{n-4}\,V_{\rm eff}\,, \tag{3.44}$$

leading to

$$S_{\rm ext} = \frac{2}{n-4}\int {\rm d}^{n-2}x\,\sqrt{-\hat g}\;V_{\rm eff}(\varphi_0)\,. \tag{3.45}$$

Using the previous results for $V_{\rm ext}$ and the brane contribution then gives

$$\frac{2}{n-4} V_{\rm eff}(\varphi_0) = -\sum_b e^{(n-2)W_b} U_b - \frac{2}{n-2} \int {\rm d}^2 x \sqrt{g_2}\, e^{(n-2)W} \left[\frac{1}{4} f\, F^{mn} F_{mn} - V \right] . \tag{3.46}$$

In many cases of interest the bulk contribution to this expression can itself also be written as a sum of contributions localized at the position of each brane. This is true, in particular, whenever the bulk action, $S_B = \int {\rm d}^n x\; \mathcal{L}_B$, enjoys a classical scaling symmetry, under which $\mathcal{L}_B[\lambda^{p_i} \psi_i] \equiv \lambda \mathcal{L}_B[\psi_i]$, for arbitrary real, constant λ. (This type of scale symmetry generically holds for higher-dimensional supergravity theories in particular.) When this is true the lagrange density satisfies the identity

$$\begin{aligned} \mathcal{L}_B &\equiv \sum_i p_i \left[\psi_i \frac{\partial \mathcal{L}_B}{\partial \psi_i} + \partial_\mu \psi_i \, \frac{\partial \mathcal{L}_B}{\partial(\partial_\mu \psi_i)} \right] \\ &= \sum_i \left\{ \partial_\mu \left[p_i \frac{\partial \mathcal{L}_B}{\partial \partial_\mu \psi_i} \right] + p_i \psi_i \left[\frac{\partial \mathcal{L}_B}{\partial \psi_i} - \partial_\mu \left(\frac{\partial \mathcal{L}_B}{\partial(\partial_\mu \psi_i)} \right) \right] \right\} , \end{aligned} \tag{3.47}$$

which shows (3.20) that the action becomes a total derivative whenever it is evaluated at an arbitrary classical solution. Whenever this is true the entire low-energy potential can be interpreted as the sum over brane contributions, much as was done for the Gibbons-Hawking term above.

3.4 Examples

It is instructive to test the above construction by applying it to situations for which explicit solutions are known for the higher-dimensional theory. We do so in this section using F-theory compactifications of 10D Type IIB supergravity to 8 dimensions in the presence of space-filling D7 branes, and using compactifications to 4 dimensions of supersymmetric and nonsupersymmetric six-dimensional theories.

3.4.1 D7 branes in F-Theory

We start with F-theory (3.10) compactifications of Type IIB supergravity to 8 dimensions, which serves as an example where explicit forms for the bulk and brane actions are known, as are closed-form expressions for the bulk sourced by various space-filling brane configurations (3.21). This provides a check on the validity of the matching conditions, and on the low-energy on-brane scalar potential.

The bulk fields to be followed in this case are the metric, g_{MN}, and the axio-dilaton,

$$\tau = C_0 + i\, e^{-\phi} \,, \tag{3.48}$$

where C_0 is the Ramond-Ramond scalar and ϕ is the 10D dilaton, for which the string coupling is $g_s = e^{\phi}$. The bulk action for these fields in the 10D Einstein frame is

$$S_B = -\frac{1}{2\kappa^2} \int \mathrm{d}^{10}x \sqrt{-g}\; g^{MN} \left[\mathcal{R}_{MN} + \frac{\partial_M \overline{\tau}\, \partial_N \tau}{2\,(\mathrm{Im}\,\tau)^2} \right] \,, \tag{3.49}$$

which is invariant under PSL(2,R) transformations

$$\tau \to \frac{a\tau + b}{c\tau + d} \,, \tag{3.50}$$

with the real parameters a through d satisfying $a\,d - b\,c = 1$. Quantum effects are expected to break this to PSL(2,Z), for which the parameters are restricted to be integers. Since $e^{\phi} \geq 0$ the field τ lives in the upper-half τ plane, but because of the symmetry it suffices to consider τ to live within the fundamental domain, $\mathcal{F}$, defined by modding out the upper half plane by a PSL(2,Z).

Bulk solutions

The scalar field equation for this action is

$$\partial\overline{\partial}\,\tau + \frac{2\,\partial\,\tau\,\overline{\partial}\,\tau}{\overline{\tau} - \tau} = 0\,, \tag{3.51}$$

which is satisfied by any holomorphic function, $\tau = \tau(z)$, for which $\overline{\partial}\,\tau = 0$.

Explicit solutions to the field equations to this model are known (3.21), for which two of the dimensions are compactified. Using complex coordinates, $z = x^8 + ix^9$, for the compact dimensions, the solutions are given by

$$j(\tau(z)) = P(z) \quad \text{and} \quad \mathrm{d}s^2 = \eta_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu + e^{2C(z,\overline{z})}\,\mathrm{d}\overline{z}\,\mathrm{d}z\,, \tag{3.52}$$

where the properties of the functions $j(\tau)$, $P(z)$ and $C(z,\overline{z})$ are now described.

The function $j(\tau)$, is the standard bijection from the fundamental domain, $\mathcal{F}$, to the complex sphere, given in terms of Jacobi ϑ-functions by

$$j(\tau) = \frac{1728\,\left[E_4(\tau)\right]^3}{\left[E_4(\tau)\right]^3 - \left[E_6(\tau)\right]^2}\,, \tag{3.53}$$

where $E_k(\tau)$ are the Eisenstein modular forms (3.22). For large $\mathrm{Im}\,\tau$, $j(\tau)$ diverges zero exponentially quickly, and the factor of 1728 is chosen so that it asymptotes to $j(\tau) \simeq e^{-2\pi i\tau} + \cdots$.

$P(z)$ is a holomorphic function, whose singularities occur at the locations of the source branes, $z = z_i$ for $i = 1, ..., N$. Since the singularities of the metric turn out to be conical when $P(z)$ has isolated poles as $z \to z_i$, it is convenient to choose $P(z)$ to be a ratio of polynomials. The simplest case could be taken as $P = 1/z$, describing a source at $z = 0$, but it turns out that the metric obtained from the Einstein equations is not compact in this case. The metric is compact when $P(z)$ has 24 zeroes, such as for the choice

$$P(z) = \frac{4(24f)^3}{27g^2 + 4f^3}\,, \tag{3.54}$$

with $f(z)$ a polynomial of degree 8 and $g(z)$ a polynomial of degree 12. This gives a compactification of Type IIB supergravity on CP^1, corresponding to an F-theory reduction on $K3$ (3.10).

Finally, the metric function $C(z, \overline{z})$ is chosen by solving the Einstein equation. Using $\mathcal{R}_{z\overline{z}} = 2\,\partial\overline{\partial}\,C$ and $\overline{\partial}\,\tau = 0$, this equation of motion is

$$2\,\partial\overline{\partial}\,C = \frac{\partial\,\tau\overline{\partial}\,\overline{\tau}}{(\tau - \overline{\tau})^2} = \partial\overline{\partial}\,\ln\Big(\mathrm{Im}\,\tau\Big)\,. \tag{3.55}$$

The required solution is

$$e^{2C(z,\overline{z})} = (\mathrm{Im}\,\tau)\,\left|\eta^2(\tau)\,\prod_{i=1}^{N}(z - z_i)^{-1/12}\right|^2\,, \tag{3.56}$$

where $\eta(\tau) = q^{1/24}\prod_k(1 - q^k)$, for $q = e^{2\pi i\tau}$, denotes the Dedekind η-function, and the product runs over the singularities of $P(z)$. The first factor of this expression is chosen to satisfy eq. (3.55), and the holomorphic factors are chosen to ensure invariance under PSL(2,Z), and by the requirement that the result does not vanish anywhere.

Brane sources

The presence of branes in these solutions is signaled by singularities where $P(z) \simeq c_i/(z - z_i)$, for which $q = e^{2\pi i\tau} \simeq (z - z_i)/c_i$, and so the above solution implies

$$\begin{aligned} \tau(z) &\simeq \frac{1}{2\pi i} \ln(z - z_i) + \cdots \\ \text{and} \quad e^{2C(z,\overline{z})} &\simeq k \, \mathrm{Im}\, \tau \,, \end{aligned} \tag{3.57}$$

for constant k. As $z \to \infty$, on the other hand, $P(z)$ remains bounded and so τ approaches some finite value. In this case the metric function becomes

$$e^{2C(z,\overline{z})} \propto (z\overline{z})^{-N/12} \,, \tag{3.58}$$

and so if we change coordinates to $z = 1/w$ we have

$$e^{2C} \, \mathrm{d}z \, \mathrm{d}\overline{z} \simeq |w|^{(N-24)/6} \, \mathrm{d}w \, \mathrm{d}\overline{w},$$

which is nonsingular because $N = 24$. But each individual brane contributed to this an amount $e^{2C} \simeq |w|^{1/6} \, \mathrm{d}w \, \mathrm{d}\overline{w} \propto r^{1/6} \, (\mathrm{d}r^2 + r^2 \mathrm{d}\theta^2)$, which we saw below eq. (3.9) corresponds to a deficit angle of $\delta = \pi/6$.

Matching conditions

We are now in a situation to use these solutions to test the matching conditions found in earlier sections. We can do so even though the geometry involved is not axisymmetric, because it becomes effectively axisymmetric in the near-brane limit.

To this end we assume a brane action of the form

$$S_b = -\int \mathrm{d}^8x\sqrt{-\gamma}\, T_b(\tau,\overline{\tau})\,, \tag{3.59}$$

where for a D7-brane in the Einstein frame we expect

$$T_b = T_*\, e^{\phi} = \frac{T_*}{\operatorname{Im}\tau} = \frac{2i\, T_*}{\tau-\overline{\tau}}\,, \tag{3.60}$$

for constant T_*.

Keeping in mind that $W = 0$ for the bulk solutions given above, the matching condition for the bulk scalar, eq. (3.21), becomes

$$\frac{2\pi}{\kappa^2}\left[\frac{e^B}{4\,(\operatorname{Im}\tau)^2}\,\partial_\rho\tau\right]_{x_b} = \frac{2\pi}{\kappa^2}\left[\frac{r}{4\,(\operatorname{Im}\tau)^2}\,\partial_r\tau\right]_{x_b} = \frac{\partial\, T_b}{\partial\,\overline{\tau}} = \frac{T_*}{2i\,(\operatorname{Im}\tau)^2}\,. \tag{3.61}$$

This uses the change of variables $\mathrm{d}\rho = e^C\,\mathrm{d}r$ and $e^B = r\,e^C$ to convert from proper distance to conformally-flat coordinates near the brane. Using the near-brane limit $\tau \simeq \ln r/2\pi i$ to evaluate $[r\,\partial\,\tau/\partial\,r]_{x_b} \simeq 1/(2\pi i)$, we find the matching condition becomes $T_* = 1/(2\kappa^2)$.

Notice that since e^ϕ is the string coupling constant, this semiclassical reasoning presupposes $\operatorname{Im}\tau = e^{-\phi}$ is large near the brane, so that $\kappa^2 T_b = \kappa^2 T_*/\operatorname{Im}\tau = 1/(2\operatorname{Im}\tau) \ll 1$. This is automatically satisfied as $r \to 0$ because $\operatorname{Im}\tau \simeq -(\ln r)/2\pi$.

The metric matching conditions can be understood in a similar way. First, matching the on-brane components of the metric gives, from eq. (3.23)

$$-\frac{2\pi}{\kappa^2}\Big[e^B\partial_\rho B - 1\Big]_{x_b} = -\frac{2\pi}{\kappa^2}\Big[r\,\partial_r B - 1\Big]_{x_b} = -\frac{2\pi}{\kappa^2}\Big[r\,\partial_r C\Big]_{x_b} = T_b(\tau,\overline{\tau}) = \frac{T_*}{\operatorname{Im}\tau}\,, \tag{3.62}$$

which again uses $e^B\partial_\rho = r\,\partial_r$ as well as $B = C + \ln r$. Using eq. (3.56) gives $e^{2C} \simeq \mathrm{Im}\,\tau$ near the brane, and so $r\,\partial_r C \simeq \frac{1}{2}(r\,\partial_r \mathrm{Im}\,\tau)/\mathrm{Im}\,\tau$ to get $[r\,\partial_r C]_{x_b} = -1/(4\pi\,\mathrm{Im}\,\tau)$. Once again the dependence on $\mathrm{Im}\,\tau$ is consistent on both sides and so the matching condition boils down to the statement $2\kappa^2 T_* = 1$, as above.

A further check comes from using the values for κ^2 and T_* for a D7-brane predicted in string theory (3.9). Using $T_* = 2\pi/\ell_s^8$ and $\kappa^2 = \ell_s^8/4\pi$, where $\ell_s = 2\pi\sqrt{\alpha'}$ is the string length, we have

$$2\kappa^2 T_* = 2\left(\frac{\ell_s^8}{4\pi}\right)\left(\frac{2\pi}{\ell_s^8}\right) = 1\,, \tag{3.63}$$

as required.

Finally, the absence of warping in the bulk solution — $W = 0$ — implies that the remaining metric matching condition, eq. (3.24), degenerates to $U_b = 0$. To compute U_b in the present instance we use the constraint, eq. (3.29), specialized to $n = 10$ dimensions

$$\mathcal{U}_b = \frac{1}{7}\left[(1-\mathcal{T}_b) - \sqrt{(1-\mathcal{T}_b)^2 - \frac{7}{8}\,(\mathcal{T}_b')^2}\,\right]\,, \tag{3.64}$$

where $\mathcal{T}_b = \kappa^2 T_b/2\pi = \kappa^2 T_*/(2\pi\,\mathrm{Im}\,\tau)$, and use

$$(\mathcal{T}_b')^2 = 2\,(\mathrm{Im}\,\tau)^2 \frac{\partial \mathcal{T}_b}{\partial \tau}\,\frac{\partial \mathcal{T}_b}{\partial \overline{\tau}} = \frac{1}{2\,(\mathrm{Im}\,\tau)^2}\left(\frac{\kappa^2 T_*}{2\pi}\right)^2 = \frac{1}{8\pi^2\,(\mathrm{Im}\,\tau)^2}\,. \tag{3.65}$$

Clearly $(\mathcal{T}_b')^2 = 0$ because $\mathrm{Im}\,\tau \to \infty$ as one approaches the brane, and this in turn ensures $U_b = 0$, as desired.

As a final check we compute the effective scalar potential, $V_{\rm eff}$, for the KK scalar zero mode in the 8D theory on the brane, after dimensional reduction. Because $U_b = 0$ this simply amounts to evaluating the action, eq. (3.49), at the classical solution to the extra-dimensional Einstein equations, which state

$$\mathcal{R}_{mn} + \frac{1}{4\,(\mathrm{Im}\,\tau)^2}\Big[\partial_m\,\tau\,\partial_n\,\overline{\tau} + \partial_n\,\tau\,\partial_m\,\overline{\tau}\Big] = 0\,. \qquad (3.66)$$

We see that $V_{\rm eff} = 0$ in the effective theory, which is consistent with the maximally symmetric on-brane geometry being flat.

3.4.2 Brane-axion couplings in 6D

We next apply the above matching conditions to the example of two branes coupled to a bulk Goldstone mode (axion), ϕ, in six dimensions. Since 6D examples with flat on-brane geometries are already discussed in some detail in refs. (3.3), we concentrate here on solutions to the higher-dimensional equations for which the on-brane geometry is known to be curved. Our purposes is to provide a nontrivial example for which the shape of the full low-energy potential, $V_{\rm eff}(\varphi)$, and its value at its stationary point, $V_{\rm eff}(\varphi_0)$, can be computed explicitly directly from the higher-dimensional theory. Because this allows a check on how $V_{\rm eff}$ varies from its minimum, it allows us to verify that the extremal point is actually a local minimum of the low-energy potential.

The simplest such a system starts with gravity coupled to a single bulk scalar and Maxwell field, with the bulk lagrangian density given by,

$$\mathcal{L}_B = -\sqrt{-g}\,\left\{\frac{1}{2\kappa^2}\,g^{MN}\Big[\mathcal{R}_{MN} + \partial_M\phi\,\partial_N\phi\Big] + \frac{1}{4}\,F_{MN}F^{MN} + \Lambda\right\}\,, \qquad (3.67)$$

where Λ is a bulk cosmological constant whose value can be chosen to obtain any desired curvature on the brane. Notice that the choices $f(\phi) = 1$ and

$V(\phi) = \Lambda$ ensure the action has a shift symmetry, $\phi \to \phi + \xi$, that guarantees the existence of a scalar KK zero mode having a constant profile across the bulk. This is the only such classically massless scalar KK mode, because the presence of the bulk cosmological term, Λ, breaks the rigid scaling symmetry that the Einstein action normally has. This breaking ensures that the presence of Λ removes the 'breathing' mode corresponding to rigid expansions of the extra dimensional geometry, that would have otherwise have been a low-energy scalar zero mode.

Bulk solutions

The field equations in this case admit explicit solutions for which the 4D on-brane geometry is maximally symmetric and the extra dimensions are axially symmetric (3.8; 3.11). Using the *ansatz* of eq. (3.9), a simple solution is

$$\mathrm{d}s^2 = \hat{g}_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{d}\rho^2 + \alpha^2 L^2 \sin^2\left(\frac{\rho}{L}\right)\mathrm{d}\theta^2 \tag{3.68}$$

$$F_{\rho\theta} = \alpha \mathcal{B}_0 L \sin\left(\frac{\rho}{L}\right), \tag{3.69}$$

with $\phi = \phi_0$ constant. The bulk field equations imply the following relation amongst the constants $\mathcal{B}_0$, L and Λ:

$$\mathcal{R}_{(2)} = -\frac{2}{L^2} = -\kappa^2\left(\frac{3\mathcal{B}_0^2}{2} + \Lambda\right), \tag{3.70}$$

and the curvature of the on-brane metric is given by

$$\hat{R} = 2\kappa^2\left(\frac{\mathcal{B}_0^2}{2} - \Lambda\right). \tag{3.71}$$

When $\alpha = 1$ the extra-dimensional metric describes a sphere of radius L. When $\alpha \neq 1$ the geometry would still look like a sphere if we redefine $\theta \to \alpha\vartheta$, although ϑ is then not periodic with period 2π. This indicates there are conical singularities at both $\rho = 0$ and $\rho = \pi L$, with defect angle given by $\delta = 2\pi(1 - \alpha)$.

Brane properties

We now ask for a pair of brane sources located at these two singularities that can support this geometry. We again take codimension-2 brane actions of the form

$$S_b = -\int \mathrm{d}^4x\sqrt{-\gamma}\, T_b(\phi)\,. \tag{3.72}$$

Because the bulk solution has constant scalar, $\phi = \phi_0$, its derivative, $\partial_\rho\phi$, vanishes at both branes. This is only consistent with the scalar matching condition if $T_b'(\phi)$ also vanishes for both branes when evaluated at the same place: $\phi = \phi_0$. The vanishing of $T_b'(\phi)$ at $\phi = \phi_0$ also ensures $U_b(\phi)$ vanishes there, and this is consistent with the $(\theta\theta)$ matching condition, eq. (3.24), because $W = 0$ throughout the bulk in the classical solution ensures $\partial_\rho W = 0$ at the brane positions.

Finally, the $(\mu\nu)$ matching condition, eq. (3.23), reads

$$-\frac{2\pi}{\kappa^2}\Big[e^B\, B' - 1\Big]_{x_b} = T_b(\phi_0)\,. \tag{3.73}$$

Using $e^B = \alpha L \sin(\rho/L)$ gives $e^B\, B' \to \alpha$ as $\rho \to 0$, and so this matching condition gives the usual expression for the defect angle in terms of the brane tension,

$$\delta = 2\pi(1 - \alpha) = \kappa^2\, T_b(\phi_0)\,, \tag{3.74}$$

and so $\mathcal{T}_b = \kappa^2 T_b/2\pi = 1 - \alpha$.

The 4D perspective

We now show how the above picture is reproduced in the low-energy 4D effective theory below the Kaluza-Klein scale. Although we cannot ask in the low-energy theory about the profiles of bulk fields within the extra dimensions, we can use it to understand the curvature, $\hat{R}$, of the 4D on-brane geometry and the value, ϕ_0, to which the low-energy scalar field is fixed.

To this end we explore the scalar potential, $V_{\rm eff}$, for the KK zero mode of the scalar, ϕ, as it is moved away from ϕ_0. To do so requires more information about the shape of $T_b(\phi)$, so we choose for simplicity,

$$T_b(\phi) = M_b^4 + \frac{\mu_b^4}{2}\,(\phi - \phi_0)^2\,, \tag{3.75}$$

although any choice for $T_b(\phi)$ would do, so long as both tensions share a common zero for $\partial T_b/\partial\phi$.

With this choice we have

$$\mathcal{T}_b = \frac{\kappa^2 M_b^4}{2\pi} + \frac{\kappa^2\mu_b^4}{4\pi}\,(\phi-\phi_0)^2\,, \qquad \mathcal{T}_b{}' = \frac{\kappa^2\mu_b^4}{2\pi}\,(\phi-\phi_0)\,, \tag{3.76}$$

and so to lowest nontrivial order in κ^2

$$\begin{aligned} \mathcal{U}_b &= \frac{1}{3}\left[(1-\mathcal{T}_b) - \sqrt{(1-\mathcal{T}_b)^2 - \frac{3}{4}\,(\mathcal{T}_b{}')^2}\right] \\ &\simeq \frac{(\mathcal{T}_b{}')^2}{8(1-\mathcal{T}_b)} + \frac{3(\mathcal{T}_b{}')^4}{128(1-\mathcal{T}_b)^3} + \cdots\,. \end{aligned} \tag{3.77}$$

Specialized to the above tension this becomes

$$U_b \simeq \frac{\kappa^2 \mu_b^8}{16\pi} \, (\phi - \phi_0)^2 + \cdots \, . \tag{3.78}$$

Notice (3.26) that because $\mathcal{U}_b$ is quadratic in $\mathcal{T}_b{}'$, both it and its derivative $\mathcal{U}_b'$ naturally vanish at zeroes of $\mathcal{T}_b{}'$. Furthermore, the coefficient of $(\phi - \phi_0)^2$ in U_b is suppressed relative to the same term in T_b by an additional power of the small dimensionless factor $\kappa^2 \mu_b^4 / 8\pi \ll 1$. The full expression for the effective potential (3.40) in this case reduces to

$$\begin{aligned} V_{\rm eff} &= \sum_b U_b + V_{\scriptscriptstyle B}(\phi_0) + \frac{1}{2} \, V_{\scriptscriptstyle B}''(\phi_0)(\phi - \phi_0)^2 + \cdots \\ &= \sum_b U_b + \int {\rm d}^2 x \sqrt{g_2} \; e^{4W} \left\{ -\frac{1}{8} \, F_{mn} F^{mn} + \frac{1}{2} \Lambda \right\} \\ &\qquad\qquad\qquad\qquad + \frac{1}{2} \, V_{\scriptscriptstyle B}''(\phi_0)(\phi - \phi_0)^2 + \cdots \\ &= \sum_b U_b + \frac{\pi}{2} \left(\Lambda - \frac{\mathcal{B}_0^2}{2} \right) \int_0^{\pi L} d\rho e^B + \frac{1}{2} \, V_{\scriptscriptstyle B}''(\phi_0)(\phi - \phi_0)^2 + \cdots \\ &= \left(\Lambda - \frac{\mathcal{B}_0^2}{2} \right) 2\pi\alpha L^2 + \frac{1}{2} \left[V_{\scriptscriptstyle B}''(\phi_0) + \sum_b \frac{\kappa^2 \mu_b^8}{8\pi} \right] (\phi - \phi_0)^2 + \cdots \, . \end{aligned}$$

using that both W' and ϕ' vanish when $\phi = \phi_0$. More explicit progress requires the calculation of $V_{\scriptscriptstyle B}''(\phi_0)$, although this can be expected to be non-negative due if the bulk solution is stable. This shows that $V_{\rm eff}(\phi)$ is minimized at $\phi = \phi_0$, and this is how the 4D theory understands the value at which ϕ is stabilized.

The value of the potential at this minimum has a direct physical interpretation, since it sets the value of the 4D curvature through the 4D Einstein equations. These read, as usual

$$\hat{R}_{\mu\nu} - \frac{1}{2}\,\hat{R}\,\hat{g}_{\mu\nu} - \kappa_N^2 V_{\rm eff}\,\hat{g}_{\mu\nu} = 0\,, \tag{3.79}$$

where the 4D Newton coupling is

$$\frac{1}{\kappa_N^2} = \frac{2\pi}{\kappa^2}\int_0^{\pi L} {\rm d}\rho\; e^B = \frac{4\pi\alpha L^2}{\kappa^2}\,, \tag{3.80}$$

and so

$$\hat{R} = -4\kappa_N^2 V_{\rm eff}(\phi_0) = 2\kappa^2\left(\frac{\mathcal{B}_0^2}{2} - \Lambda\right)\,, \tag{3.81}$$

in agreement with the higher-dimensional result, eq. (3.71). Notice that this agreement requires, in particular, that the brane tensions $T_b(\phi_0) = M_b^4$ drop out of the low-energy potential.

Finally, notice that evaluating the potential, eq. (3.79), at its minimum by evaluating the action at the classical solution gives a result that agrees with the general expression (3.46), which in the present instance evaluates to

$$\begin{aligned} V_{\rm eff}(\varphi_0) &= -\sum_b e^{4W_b} U_b - \frac{1}{2}\int {\rm d}^2x\sqrt{g_2}\; e^{4W}\left[\frac{1}{4}\,f\,F^{mn}F_{mn} - V\right] \\ &= \frac{1}{2}\left(4\pi\alpha L^2\right)\left(\Lambda - \frac{\mathcal{B}_0^2}{2}\right)\,. \end{aligned} \tag{3.82}$$

3.4.3 Warped and unwarped supersymmetric examples

A large class of examples of explicit flux compactifications with nontrivial warping and scalar profiles in the extra dimensions is provided by solutions (3.19; 3.20; 3.23; 3.24; 3.25; 3.13; 3.14) to chiral 6D supergravity (3.12). Our goal with this example is to identify the properties of the branes that are required to source the known solutions. In general the existence of solutions hinges on the consistency of these brane properties with the form of the intervening bulk, but these solutions are not known in closed form in the case where the on-brane dimensions are curved. In this situation it is much easier to investigate the existence of solutions using the equivalent formulation in terms of minima of the low-energy scalar potential, since it is much easier to determine when such solutions exist.

The solutions of interest take as their starting point the following bosonic part of the supersymmetric action

$$\mathcal{L}_B = -\sqrt{-g}\,\left\{\frac{1}{2\kappa^2}\, g^{MN}\Big[\mathcal{R}_{MN} + \partial_M\phi\,\partial_N\phi\Big] + \frac{1}{4}\, e^{-\phi}\, F_{MN}F^{MN} + \frac{2g^2}{\kappa^4}\, e^{\phi}\right\}, \tag{3.83}$$

where the constant g denotes the 6D gauge coupling for the Maxwell field. Because this lagrangian enjoys the property $\mathcal{L}_B \to \lambda^2\mathcal{L}_B$ when $e^\phi \to \lambda^{-1}e^\phi$ and $g_{MN} \to \lambda g_{MN}$, the arguments of section 3.3.4 imply it becomes a total derivative once evaluated at an arbitrary classical solution (3.20):

$$\mathcal{L}_B(g^c_{MN}, A^c_M, \phi^c) = \frac{1}{2\kappa^2}\,\sqrt{-g^c}\,\Box\,\phi^c\,. \tag{3.84}$$

Bulk solutions

For this system it is useful to choose a slightly different metric *ansatz* (3.23),

$$\mathrm{d}s^2 = \mathcal{W}^2\,\hat{g}_{\mu\nu}\,\mathrm{d}x^\mu\mathrm{d}x^\nu + a^2\Big(\mathcal{W}^8\mathrm{d}\eta^2 + \mathrm{d}\theta^2\Big)\,, \tag{3.85}$$

where $a = a(\eta)$, $\mathcal{W} = \mathcal{W}(\eta)$ and $\hat{g}_{\mu\nu}$ is, a maximally symmetric 4D de Sitter metric, with $\hat{R} = -12H^2$. With these choices the proper circumference of a circle along which θ varies from zero to 2π at fixed η is $2\pi a(\eta)$, and $\mathrm{d}\rho = a\mathcal{W}^4 \mathrm{d}\eta$. The dilaton is similarly taken to depend only on η, $\phi = \phi(\eta)$, and the Maxwell field is given by $A_\theta = A_\theta(\eta)$, so that

$$F_{\eta\theta} = Q\, a^2\, e^{\phi} \,. \tag{3.86}$$

In this case the content of Maxwell's equations is that Q must be a constant, while the dilaton and the trace-reversed Einstein equations become

$$\phi'' = \frac{2g^2}{\kappa^2}\, a^2 \mathcal{W}^8 e^{\phi} - \frac{\kappa^2 Q^2}{2}\, a^2 e^{\phi} \,, \tag{3.87}$$

and

$$(\mu\nu): \quad \frac{\mathcal{W}''}{\mathcal{W}} - \frac{(\mathcal{W}')^2}{\mathcal{W}^2} + \frac{1}{2}\,\phi'' = \left(\frac{\mathcal{W}'}{\mathcal{W}} + \frac{1}{2}\,\phi' \right)' = 3\, H^2 a^2 \mathcal{W}^6 \tag{3.88}$$

$$(\theta\theta): \quad \frac{a''}{a} - \frac{(a')^2}{a^2} + \frac{1}{2}\,\phi'' = \left(\frac{a'}{a} + \frac{1}{2}\,\phi' \right)' = -\kappa^2 Q^2\, a^2 e^{\phi} \,. \tag{3.89}$$

In all of these equations primes denote $\mathrm{d}/\mathrm{d}\eta$. The 'Hamiltonian constraint' — *i.e.* the $(\eta\eta)$ Einstein equation — in these variables is similarly

$$\frac{1}{2}\,(\phi')^2 - \frac{4\,a'\mathcal{W}'}{a\mathcal{W}} - \frac{6(\mathcal{W}')^2}{\mathcal{W}^2} = \frac{2g^2}{\kappa^2}\, a^2 \mathcal{W}^8 e^{\phi} - 6\, H^2 a^2 \mathcal{W}^6 - \frac{\kappa^2}{2} Q^2\, a^2 e^{\phi} \,. \tag{3.90}$$

The scale invariance of the full 6D field equations under $e^{\phi} \to e^{\phi}/\lambda$ and $g_{MN} \to \lambda g_{MN}$ can be seen from the invariance of the above equations under

$$\Big\{ \phi, a, \mathcal{W}, H \Big\} \to \Big\{ \phi + \phi_0, a\, e^{-\phi_0/2}, \mathcal{W}, H\, e^{\phi_0/2} \Big\} \,, \tag{3.91}$$

for ϕ_0 an arbitrary real constant. In the case $H = 0$ this symmetry implies the existence of a one-parameter family of classical solutions, and a corresponding flat direction (labelled by ϕ_0) that represents a classically massless KK zero mode coming from a combination of the metric and ϕ fields.

The above field equations are written so that their right-hand-sides tend to zero in the near-brane regions, for which $a \to 0$. For regions where these right-hand-sides are negligible the equations simplify to

$$\phi'' \simeq \left(\frac{\mathcal{W}'}{\mathcal{W}}\right)' \simeq \left(\frac{a'}{a}\right)' \simeq 0\,, \tag{3.92}$$

and so, letting $b = \{0, 1\}$ for the branes at $\eta = \{-\infty, +\infty\}$ respectively,

$$\phi \simeq (-)^b q_b \eta\,, \quad \mathcal{W} \simeq \mathcal{W}_b\, e^{(-)^b \omega_b \eta} \quad \text{and} \quad a \simeq a_b\, e^{(-)^b \alpha_b \eta}\,, \tag{3.93}$$

with different choices for the constants α_b, ω_b and q_b applying for the two limits, $\eta \to \pm\infty$. For both asymptotic regions these are related by the constraint, eq. (3.90), so that

$$q_b^2 = 4\omega_b(2\alpha_b + 3\omega_b)\,. \tag{3.94}$$

Notice that it is only consistent in the near-brane limit to ignore the quantities $a^2\mathcal{W}^6$, $a^2 e^{\phi}$ and $a^2\mathcal{W}^8 e^{\phi}$ on the right-hand sides of eqs. (3.88) through (3.90) if

$$2\alpha_b + 6\omega_b > 0\,, \quad 2\alpha_b + q_b > 0 \quad \text{and} \quad 2\alpha_b + 8\omega_b + q_b > 0\,. \tag{3.95}$$

The first of these also guarantees the convergence of the 4D gravitational constant, which is given by (*c.f.* eq. (3.37))

$$\frac{1}{\kappa_N^2} = \frac{2\pi}{\kappa^2} \int_{-\infty}^{\infty} \mathrm{d}\eta \, a^2 \mathcal{W}^6 \,. \tag{3.96}$$

Furthermore, since our interest is in solutions where $a \to 0$ at the positions of the brane sources, we demand $\alpha_b > 0$. This ensures that the circumference of small circles encircling the branes vanishes in the limit that the branes are approached. But if $\alpha_b > 0$, then ω_b must also be non-negative. To see this, suppose ω_b were negative. Then eq. (3.94) would imply $-2\alpha_b - 3\omega_b > 0$, and so adding this to the first of eqs. (3.95) would give $\omega_b > 0$, in contradiction with the assumption that it is negative. By contrast, the constant q_b can take either sign.

Solutions to these equations are known to exist for nonzero H (3.25), although not yet in an explicit closed form. Closed-form solutions are known, however, in the special case where H vanishes, given by (3.23; 3.20)

$$\begin{aligned} e^{\phi} &= \mathcal{W}^{-2} e^{\phi_0 - \lambda_3 \eta} \\ \mathcal{W}^4 &= \left(\frac{\kappa^2 Q \lambda_2}{2 g \lambda_1} \right) \frac{\cosh[\lambda_1(\eta - \eta_1)]}{\cosh[\lambda_2(\eta - \eta_2)]} \\ \text{and} \qquad a^{-4} &= \left(\frac{2 g \kappa^2 Q^3}{\lambda_1^3 \lambda_2} \right) e^{2(\phi_0 - \lambda_3 \eta)} \cosh^3[\lambda_1(\eta - \eta_1)] \cosh[\lambda_2(\eta - \eta_2)] \,. \end{aligned} \tag{3.97}$$

Here η_i and λ_j are integration constants, and there is no loss of generality in choosing, say, $\lambda_2 \geq 0$. The equations of motion require the constants to satisfy $\lambda_2^2 = \lambda_1^2 + \lambda_3^2$ — and so, in particular, $\lambda_2 \geq |\lambda_1|$ (with equality if and only if $\lambda_3 = 0$). ϕ_0 is an arbitrary constant corresponding to the scale invariance associated with the flat direction.

Because the terms involving H in the equations of motion become negligible in the near-brane limit, the $H = 0$ solutions also provide a more detailed picture of the asymptotic regions at $\eta \to \pm\infty$. The corresponding metric singularities are generically curvature singularities, except when $\lambda_3 = 0$, in which case they turn out to be conical (3.24). The $\lambda_3 = 0$ solutions include the unwarped, constant-dilaton 'rugby ball' configurations of ref. (3.19) as the special case where $\eta_1 = \eta_2$. Notice also that the limiting behaviour is as given in eq. (3.93), with

$$\alpha_b = \frac{1}{4}\left[3\lambda_1 + \lambda_2 + 2(-)^b\lambda_3\right] \geq 0\,, \quad \omega_b = \frac{1}{4}\left(\lambda_2 - \lambda_1\right) \geq 0\,, \tag{3.98}$$

and

$$q_b = (-)^{b+1}\lambda_3 - \frac{1}{2}\left(\lambda_2 - \lambda_1\right)\,. \tag{3.99}$$

Notice that the condition $\omega_b \geq 0$ follows from $\lambda_2 \geq |\lambda_1|$, while $\alpha_b \geq 0$ is a consequence of

$$3(\lambda_2 + \lambda_1) - 2\lambda_3 = \sqrt{\lambda_2 + \lambda_1}\left(3\sqrt{\lambda_2 + \lambda_1} - 2\sqrt{\lambda_2 - \lambda_1}\right) \geq 0\,. \tag{3.100}$$

A special role is played by the combination

$$\omega_b + \frac{q_b}{2} = (-)^{b+1}\frac{\lambda_3}{2}\,, \tag{3.101}$$

since this dictates the size of the Hubble constant, H. This can be seen by integrating eq. (3.88), and using eq. (3.96) to obtain (3.25),

$$3H^2\int_{-\infty}^{\infty} \mathrm{d}\eta\; a^2\,\mathcal{W}^6 = \frac{3\kappa^2 H^2}{2\pi\kappa_4^2} = \left[\left(\ln\mathcal{W} + \frac{\phi}{2}\right)'\right]_{\eta=-\infty}^{\eta=+\infty} = -\sum_b\left(\frac{q_b}{2} + \omega_b\right)\,. \tag{3.102}$$

When evaluated for the solutions of eq. (3.97), this reduces to the Friedmann equation

$$H^2 = -\frac{2\pi\kappa_4^2}{3\kappa^2}\sum_b \left[\frac{q_b}{2}+\omega_b\right] = \frac{\kappa_4^2}{3}\left[\frac{2\pi}{\kappa^2}\sum_b (-)^b\frac{\lambda_3}{2}\right] = 0 \tag{3.103}$$

as required. For more general solutions eqs. (3.97) hold only approximately in the near-brane region, so the constant λ_3 could differ for the asymptotic region near each brane.

Notice, in particular, that eq. (3.102) shows that $H^2 > 0$ (4D de Sitter space) requires at least one of the q_b to be negative. Furthermore, choosing $q_b < 0$ is sufficient to ensure that the contribution to H^2 of the corresponding brane is positive, because

$$-\left(\frac{q_b}{2}+\omega_b\right) = \frac{|q_b|}{2} - \omega_b = \sqrt{3\omega_b^2 + 2\alpha_b\omega_b} - \omega_b = \omega_b\left(\sqrt{3+\frac{\alpha_b}{\omega_b}} - 1\right) \geq 0\,. \tag{3.104}$$

This uses both eq. (3.94) and the property that α_b and ω_b are both non-negative.

Brane properties

As usual, the matching conditions relate the asymptotic bulk solutions to the properties of the source branes. Using $\mathcal{W} = e^W$, $a = e^B$ and $a\mathcal{W}^4 \mathrm{d}\eta = \mathrm{d}\rho$, and taking the brane action to be $S_b = -\int \mathrm{d}^4x\sqrt{-\gamma}\; L_b = -\int \mathrm{d}^4x\sqrt{-\hat{g}}\; T_b$, the scalar matching condition, eq. (3.21), becomes

$$\frac{2\pi}{\kappa^2}\left[e^{B+4W}\,\partial_\rho\,\phi\right]_{x_b} = \frac{\partial}{\partial\phi}\Big[\,e^{4W}\,L_b\Big] \quad\Longrightarrow\quad \Big[(-)^b\partial_\eta\,\phi\Big]_{x_b} = q_b = \frac{\kappa^2}{2\pi}\left(\frac{\partial\,T_b}{\partial\phi}\right), \tag{3.105}$$

where the sign arises because the direction away from the brane is $(-)^b \mathrm{d}\eta$ in the two asymptotic regions. The $(\theta\theta)$ metric matching condition, eq. (3.24), similarly becomes

$$\frac{2\pi}{\kappa^2}\left[e^{B+4W}\,\partial_\rho W\right]_{x_b} = U_b(\phi) \quad\Longrightarrow\quad \left[(-)^b\left(\frac{\partial_\eta \mathcal{W}}{\mathcal{W}}\right)\right]_{x_b} = \omega_b = \frac{\kappa^2 U_b}{2\pi}\,. \tag{3.106}$$

Finally, the $(\mu\nu)$ components of the metric matching conditions are

$$-\frac{2\pi}{\kappa^2}\left[e^{4W}[e^B\,(3\partial_\rho W + \partial_\rho B) - 1]\right]_{x_b} = T_b(\phi)\,, \tag{3.107}$$

and so

$$\left\{(-)^b\left[3\left(\frac{\partial_\eta \mathcal{W}}{\mathcal{W}}\right) + \left(\frac{\partial_\eta a}{a}\right)\right] - \mathcal{W}^4\right\}_{x_b} = 3\omega_b + \alpha_b - \mathcal{W}^4(x_b) = -\frac{\kappa^2 T_b}{2\pi}\,. \tag{3.108}$$

There are now two qualitatively different cases that are worth considering separately, depending on whether or not $\omega_b = 0$ or $\omega_b > 0$.

Solutions with only conical singularities:

If $\omega_b = 0$, then eq. (3.94) implies $q_b = 0$ as well, and so both ϕ and $\mathcal{W}$ asymptote to constants near the brane. Because $\omega_b = 0$ implies $\mathcal{W} \simeq \mathcal{W}_b$ is constant in the near-brane regime, the behaviour $a \sim e^{\alpha_b \eta}$ implies the extra-dimensional metric is proportional to

$$e^{2\alpha_b\eta}(\mathcal{W}_b^8 \mathrm{d}\eta^2 + \mathrm{d}\theta^2) = \mathrm{d}\rho^2 + \left(\frac{\alpha_b \rho}{\mathcal{W}_b^4}\right)^2 \mathrm{d}\theta^2\,, \tag{3.109}$$

showing that it has only a conical singularity at the brane position, with defect angle $\delta_b = 2\pi(1 - \alpha_b/\mathcal{W}_b^4)$.

When $\omega_b = q_b = 0$, the matching conditions boil down to

$$\frac{\kappa^2 T_b'}{2\pi} = \frac{\kappa^2 U_b}{2\pi} = 0 \quad \text{and} \quad \delta_b = \frac{\kappa^2 T_b}{\mathcal{W}_b^4} = \kappa^2 L_b \,. \tag{3.110}$$

The last of these relates the tension to the size of the conical defect angle in the usual way, while the first states that the value taken by ϕ near each brane must be at a stationary point of the tension on that brane. (Since this is also automatically a zero of U_b, the second condition is redundant.) In order for solutions to exist the two tensions must be related to one another by the known asymptotic limits of the given bulk solution. That is, if $\phi_b = \lim \phi(\eta)$ as $\eta \to -(-)^b\infty$, then T_b must satisfy $T_b'(\phi_b) = 0$ at both ends.

Since its right-hand-side is non-negative, eq. (3.88) shows that it is only possible to have $\omega_b = q_b = 0$ at *both* branes if $H = 0$. If $H = 0$ the solutions given in eqs. (3.97) have this property (for both branes) when $\lambda_3 = 0$ (and so also $\lambda_1 = \lambda_2 := \lambda$). Notice that $\mathcal{W}$ and $e^\phi = \mathcal{W}^{-2}$ need not be identically constant in this case unless $\eta_1 = \eta_2$.

From the point of view of the 4D theory the result $H = 0$ is understood for these solutions in terms of the vanishing of the classical low-energy 4D effective potential,

$$V_{\text{eff}} = V_B + \sum_b U_b = 0 \,. \tag{3.111}$$

This vanishes because eq. (3.84) (when $\phi' = 0$ near the branes) shows that the bulk contribution to the low energy potential vanishes, $V_B = 0$, and eq. (3.110) implies $U_b = 0$ for both branes.

If T_b' should vanish identically, then so must also U_b and V_{eff}. In this case the vanishing of V_{eff} shows that the flat direction, corresponding to the scaling $\phi \to \phi + \phi_0$ and $g_{MN} \to e^{-\phi_0} g_{MN}$, is not lifted by the classical couplings to the branes. But if T_b depends nontrivially on ϕ, then U_b becomes nonzero as soon as ϕ differs from its asymptotic value ϕ_b, implying that $V_{\rm eff}$ depends nontrivially on ϕ_0. Since $U_b(\phi_0)$ is given by

$$\mathcal{U}_b = \frac{1}{3}\left[(\mathcal{W}^4 - \mathcal{T}_b) - \sqrt{(\mathcal{W}^4 - \mathcal{T}_b)^2 - \frac{3}{4}\,(\mathcal{T}_b\,')^2}\,\right]\,, \tag{3.112}$$

where $\mathcal{T}_b = \mathcal{T}_b(\phi_b + \phi_0)$, it is non-negative (provided $\mathcal{T}_b < \mathcal{W}^4$). Because the bulk action is known to be stable against small fluctuations about the bulk solutions (3.27), it follows that $V_{\rm eff}(\phi_0)$ must be minimized by any configuration for which it vanishes, such as $\phi_0 = 0$ (which corresponds to $\lim \phi = \phi_b$). This shows how the 4D theory sees that the flat direction, ϕ_0, of the bulk equations becomes fixed at the same value as is chosen by the matching conditions when viewed from the higher-dimensional perspective.

Solutions with $\omega_b > 0$

On the other hand, if $\omega_b > 0$ then $e^W = \mathcal{W} \to 0$ as the brane is approached. In this case the scalar and $(\mu\nu)$ matching conditions are

$$q_b = \frac{\kappa^2 T_b'}{2\pi} = \mathcal{T}_b\,' \quad \text{and} \quad 3\,\omega_b + \alpha_b = -\frac{\kappa^2 T_b}{2\pi} = -\mathcal{T}_b\,. \tag{3.113}$$

Since α_b and ω_b are both positive, the last of these conditions implies $T_b < 0$. The third matching condition in this case is

$$\omega_b = \frac{\kappa^2 U_b}{2\pi} = \mathcal{U}_b = \frac{1}{3}\left[-\mathcal{T}_b - \sqrt{\mathcal{T}_b^2 - \frac{3}{4}\,(\mathcal{T}_b\,')^2}\,\right]\,, \tag{3.114}$$

which also requires $\mathcal{T}_b < 0$ if $\mathcal{U}_b$ and ω_b are to be positive.

Because we use coordinates for which the branes are situated at $\eta \to \pm\infty$, we demand that these matching conditions be satisfied as identities in η in the asymptotic regimes. Use of the asymptotic forms for the bulk solutions in this regime corresponds to expanding the brane tension about the value taken by ϕ at the brane.

This determines the functional form for the brane action, $T_b(\phi, a, W) = e^{4W} L_b(\phi, a)$, required to source the given bulk solution. Because e^ϕ and all metric functions behave as exponentials near the branes — *c.f.* eq. (3.93) — the brane action must have the form $L_b = -\Lambda_b \, e^{\xi_b \phi} \mathcal{F}\left(a \, e^{\zeta_b \phi}\right)$, where $\mathcal{F}(x)$ is an arbitrary function and the powers ξ_b and ζ_b are chosen to ensure the η-independence in the near-brane regime of

$$T_b = -\Lambda_b \, \mathcal{W}^4 e^{\xi_b \phi} \, \mathcal{F}\left(a \, e^{\zeta_b \phi}\right) \,, \tag{3.115}$$

for constant Λ_b. The parameters ξ_b and ζ_b therefore satisfy

$$4\,\omega_b + \xi_b \, q_b = \alpha_b + \zeta_b \, q_b = 0 \,. \tag{3.116}$$

In terms of $\mathcal{F}(x)$, the scalar matching condition becomes

$$q_b = \frac{\kappa^2}{2\pi} \left(\frac{\partial \, T_b}{\partial \phi} \right) = -\frac{\kappa^2 \Lambda_b}{2\pi} \, \mathcal{W}^4 \, e^{\xi_b \phi} \Big[\xi_b \, \mathcal{F}(x) + \zeta_b \, x \, \mathcal{F}'(x) \Big]_{x = a e^{\zeta_b \phi}} \,, \tag{3.117}$$

while the metric matching conditions similarly give

$$3\,\omega_b + \alpha_b = -\frac{\kappa^2 T_b}{2\pi} = \frac{\kappa^2 \Lambda_b}{2\pi} \, \mathcal{W}^4 \, e^{\xi_b \phi} \mathcal{F}\left(a \, e^{\zeta_b \phi}\right) \,, \tag{3.118}$$

and so on.

To go further requires making choices for the function $\mathcal{F}(x)$. We discuss for simplicity a power-law, $\mathcal{F}(x) = x^{\sigma_b}$, which to concretely illustrate the brane-bulk interaction.

Power-law tension: $\mathcal{F}(x) = x^{\sigma_b}$

Perhaps the simplest choice for the function $\mathcal{F}(x)$ appearing above is a power: $\mathcal{F}(x) = x^{\sigma_b}$, for σ_b a constant. In this case

$$T_b = -\Lambda_b \, \mathcal{W}^4 a^{\sigma_b} \, e^{\lambda_b \phi} \,, \tag{3.119}$$

where $\lambda_b = \xi_b + \zeta_b \sigma_b$, and so

$$4\omega_b + \sigma_b \alpha_b + \lambda_b q_b = 0 \,, \tag{3.120}$$

is required to ensure that the η-dependence cancels in T_b within the near-brane regime. This last equation is to be regarded as being solved for σ_b.

The scalar matching condition, eq. (3.105), then boils down to

$$q_b = -\lambda_b \, \mathcal{W}_b^4 a_b^{\sigma_b} \left(\frac{\kappa^2 \Lambda_b}{2\pi} \right) . \tag{3.121}$$

The $(\mu\nu)$ metric matching condition, eq. (3.108), similarly gives

$$3\,\omega_b + \alpha_b = \mathcal{W}_b^4 a_b^{\sigma_b} \left(\frac{\kappa^2 \Lambda_b}{2\pi} \right) . \tag{3.122}$$

Combining (3.121) and (3.122), gives the parameter λ_b as

$$\lambda_b = -\frac{q_b}{3\omega_b + \alpha_b} \,. \tag{3.123}$$

Clearly $q_b < 0$ implies $\lambda_b > 0$ and vice versa, because α_b and ω_b are both positive. Notice that $\lambda_b > 0$ implies $T_b \to 0$ in the 'weak-coupling' limit $e^\phi \to 0$.

Given α_b and ω_b, solving the above conditions gives $q_b = \pm 2\sqrt{\omega_b(2\alpha_b + 3\omega_b)}$ (from eq. (3.94)), λ_b (from eq. (3.123)), and the combination $\mathcal{W}_b^4 a_b^{\sigma_b}\,(\kappa^2\Lambda_b/2\pi)$ (from eq. (3.122)). The power of a appearing in T_b works out to be

$$\sigma_b = \frac{4\omega_b}{3\,\omega_b + \alpha_b} > 0\,. \tag{3.124}$$

One might think that the last matching condition, involving U_b, gives an independent equation that can be used to relate ω_b to α_b, but this turns out not to be independent due to the relation between U_b and T_b and the constraint, eq. (3.94).

The 4D perspective

In this section, we evaluate the full action at its classical solution to determine the value of V_{eff} at its minimum. For supergravity the full bulk action evaluates to a total derivative at any classical solution, giving

$$S_{B,\text{ext}} = \frac{1}{2\kappa^2}\int \mathrm{d}^6x\,\sqrt{-g}\;\Box\phi = \frac{\pi}{\kappa^2}\int \mathrm{d}^4x\,\sqrt{-\hat{g}}\,\Big[\partial_\eta\phi\Big]_{-\infty}^{\infty} = -\sum_b \frac{T_b'}{2}\,. \tag{3.125}$$

Adding to this the brane action and Gibbons-Hawking term, which combine to

$$\sum_b \Big(S_{GH} + S_b\Big) = -\int d^4x\sqrt{\hat{g}}\;U_b \tag{3.126}$$

gives the total action evaluated at the classical solution

$$S_{\text{ext}} = -\int \mathrm{d}^4x\sqrt{-\hat{g}}\ \sum_b \Big(U_b + \frac{T_b'}{2}\Big)\,. \tag{3.127}$$

Comparing this with eq. (3.45) (for $n = 6$) gives

$$V_{\text{eff}}(\phi_0) = -\sum_b \left(U_b + \frac{T_b'}{2}\right)\,. \tag{3.128}$$

Using this in the four-dimensional Einstein equations gives the 4D curvature

$$\hat{R} = -12H^2 = -4\kappa_N^2 V_{\text{eff}}(\phi_0)\,, \tag{3.129}$$

and so

$$H^2 = \frac{\kappa_N^2}{3} V_{\text{eff}} = -\frac{\kappa_N^2}{3}\sum_b \left(U_b + \frac{T_b'}{2}\right) = -\frac{2\pi\kappa_N^2}{3\kappa^2}\sum_b \left(\omega_b + \frac{q_b}{2}\right), \tag{3.130}$$

where the last equality uses the matching conditions to rewrite U_b and T_b' in terms of the bulk solution. This agrees with the bulk field equations, eq.(3.102), and so shows that the $4D$ and $6D$ pictures agree. In order to identify the value of ϕ_0 itself requires calculating V_{eff} away from its minimum, which requires a full dimensional reduction of the supergravity action.

3.5 Conclusions

This paper summarizes the bulk-brane matching conditions for codimension-2 objects (following the presentation given for scalar-tensor theories in (3.3), with generalizations to include a general coupling to the Maxwell field (3.2)), and describes several applications to higher-dimensional brane systems: F-theory compactifications involving space-filling codimension-2 D7-branes situated within 10 dimensions; unwarped 3-brane flux compactifications in 6 dimensional scalar-Maxwell-Einstein theory; and warped and unwarped 3-brane flux compactifications of 6D chiral gauged supergravity. The latter two cases involve geometries that are maximally symmetric — but possibly curved — in the directions parallel to the branes.

The comparison with the F-theory compactifications provides a sanity check on the junction conditions, since both the brane and bulk actions are explicitly known for Type IIB string vacua (3.9), as are explicit solutions for the surrounding bulk geometry (3.21). We show that the near-brane asymptotic form of the bulk configurations in this case precisely agrees with what the matching conditions would predict, given the explicit D7-brane action. Furthermore, this comparison lies within the weak-coupling regime since the bulk solution implies the string coupling becomes weak in the near-brane limit.

When applied to six-dimensional systems, the bulk-brane matching conditions can provide a stabilization mechanism for the bulk scalars (like a bulk axion, or the dilaton) provided the brane couplings break the appropriate symmetry that protects the scalar's mass. When this is so, the value to which the scalar stabilizes can be understood from the higher-dimensional point of view as being due to the consistency of the matching conditions at the two branes. Alternatively it can be regarded as the value which minimizes the effective potential in the low-energy, on-brane action below the KK scale, although this requires a calculation of the potential away from its minimum.

Although many of the bulk solutions considered in six dimensions (supersymmetric or not) have de Sitter curvature along the four brane directions

(3.8; 3.14), we show that for 6D gauged chiral supergravity only 4D-flat branes can be sourced by positive-tension branes. To establish this we first show that for any 6D theory a codimension-2 brane tension must be negative whenever the warp factor tends to zero near the brane. We then prove that the supergravity field equations imply the warping vanishes near the brane unless the near-brane geometry has a conical singularity. Finally, the desired result follows once the field equations are used to see that any geometry having only conical singularities necessarily is flat in the 4 brane directions.

This necessity for negative tension in order to obtain de Sitter and anti-de Sitter branes echoes the various no-go theorems for finding 4D-de Sitter solutions from extra-dimensional gravity (3.28), even though the curvatures of the bulk geometries considered make these theorems not directly apply. This suggests that the curvature assumptions made in these theorems may be somewhat stronger than is necessary.

The relation to 4D de Sitter geometries has potential applications to searches for cosmic inflation within an extra-dimensional context. This is because inflationary configurations often lay nearby pure de Sitter solutions. In particular, a broad class of time-dependent solutions are known (3.14) for the bulk field equations in 6D supergravity, and for some of these the on-brane 4D geometry is likely to undergo an accelerated expansion. The extension of the arguments of this paper to these time-dependent situations would be most worthwhile, since they could provide instances of explicit inflationary models for which there is both a higher- and lower-dimensional understanding of why the universe accelerates. (By contrast, current inflationary models typically rely on the low-energy 4D effective theory to conclude that the universe inflates.) Work along these lines is in progress (3.29).

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Bibliography

[3.1] Allan Bayntun, C.P. Burgess and Leo van Nierop
"Codimension-2 Brane-Bulk Matching: Examples from Six and Ten Dimensions "
New Journal of Physics 12 (2010) 075015 (arXiv:0912.3039)

[3.2] C. P. Burgess, D. Hoover and G. Tasinato, "UV Caps and Modulus Stabilization for 6D Gauged Chiral Supergravity," JHEP **0709** (2007) 124 [arXiv:0705.3212 [hep-th]].

[3.3] C. P. Burgess, D. Hoover, C. de Rham and G. Tasinato, "Effective Field Theories and Matching for Codimension-2 Branes," JHEP **0903** (2009) 124 [arXiv:0812.3820 [hep-th]].

[3.4] N. Arkani-Hamed, S. Dimopoulos and G. Dvali, *Phys. Lett.* **B429** (1998) 263 [hep-ph/9803315]; *Phys. Rev.* **D59** (1999) 086004 [hep-ph/9807344];
I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. Dvali, *Phys. Lett.* **B436** (1998) 257 [hep-ph/9804398];
K. Benakli, *Phys. Rev.* **D60**, 104002 (1999) [hep-ph/9809582];

C. P. Burgess, L. E. Ibáñez and F. Quevedo, *Phys. Lett.* **B447**, 257 (1999) [hep-ph/9810535];

P. Horava and E. Witten, *Nucl. Phys.* **B475** (1996) 94 [hep-th/9603142]; *Nucl. Phys.* **B460** (1996) 506 [hep-th/9510209];

E. Witten, *Nucl. Phys.* **B471** (1996) 135 [hep-th/9602070];

J. Lykken, *Phys. Rev.* **D54** (1996) 3693 [hep-th/9603133].

[3.5] L. Randall, R. Sundrum, Phys. Rev. Lett. **83** (1999) 3370 [hep-ph/9905221]; Phys. Rev. Lett. **83** (1999) 4690 [hep-th/9906064].

[3.6] K. Lanczos, Phys. Z. **23** (1922) 239–543; Ann. Phys. **74** (1924) 518–540;

C.W. Misner and D.H. Sharp, Phys. Rev. **136** (1964) 571–576;

W. Israel, Nuov. Cim. **44B** (1966) 1–14; *errata* Nuov. Cim. **48B** 463.

[3.7] A. Vilenkin, Phys. Rev. **D23** (1981) 852;

R. Gregory, Phys. Rev. Lett. **59** (1987) 740;

A. G. Cohen and D. B. Kaplan, "The Exact Metric About Global Cosmic Strings," *Phys. Lett.* **B215**, 67 (1988);

A. Vilenkin and P. Shellard, *Cosmic Strings and other Topological Defects*, Cambridge University Press (1994);

R. Gregory and C. Santos, "Cosmic strings in dilaton gravity," Phys. Rev. D **56**, 1194 (1997) [gr-qc/9701014].

[3.8] F. Leblond, R. C. Myers and D. J. Winters, "Consistency conditions for brane worlds in arbitrary dimensions," JHEP **0107** (2001) 031 [arXiv:hep-th/0106140];

S. M. Carroll and M. M. Guica, "Sidestepping the cosmological constant with football-shaped extra dimensions," [hep-th/0302067];

I. Navarro, "Codimension two compactifications and the cosmological constant problem," JCAP **0309** (2003) 004 [hep-th/0302129];

E. Papantonopoulos and A. Papazoglou, "Brane-bulk matter relation for a purely conical codimension-2 brane world," JCAP **0507** (2005) 004 [arXiv:hep-th/0501112].

[3.8] P. Bostock, R. Gregory, I. Navarro and J. Santiago, "Einstein gravity on the codimension 2 brane?," Phys. Rev. Lett. **92** (2004) 221601 [arXiv:hep-th/0311074];

J. Vinet and J. M. Cline, Phys. Rev. D **71** (2005) 064011 [hep-th/0501098];

M. Peloso, L. Sorbo and G. Tasinato, "Standard 4d gravity on a brane in six dimensional flux compactifications," Phys. Rev. D **73** (2006) 104025 [arXiv:hep-th/0603026];

B. Himmetoglu and M. Peloso, Nucl. Phys. B **773** (2007) 84 [hep-th/0612140];

B. Himmetoglu and M. Peloso, "Isolated Minkowski vacua, and stability analysis for an extended brane in the rugby ball," Nucl. Phys. B **773** (2007) 84 [hep-th/0612140];

C. de Rham, "The Effective Field Theory of Codimension-two Branes," JHEP **0801** (2008) 060 [arXiv:0707.0884 [hep-th]];

E. Papantonopoulos, A. Papazoglou and V. Zamarias, "Regularization of conical singularities in warped six-dimensional compactifications," JHEP **0703** (2007) 002 [arXiv:hep-th/0611311]; "Induced cosmology on a regularized brane in six-dimensional flux compactification," Nucl. Phys. B **797** (2008) 520 [arXiv:0707.1396 [hep-th]];

D. Yamauchi and M. Sasaki, "Brane World in Arbitrary Dimensions Without Z_2 Symmetry," Prog. Theor. Phys. **118** (2007) 245 [arXiv:0705.2443 [gr-qc]];

N. Kaloper and D. Kiley, "Charting the Landscape of Modified Gravity," JHEP **0705** (2007) 045 [hep-th/0703190];

M. Minamitsuji and D. Langlois, "Cosmological evolution of regularized branes in 6D warped flux compactifications," Phys. Rev. D **76** (2007) 084031 [arXiv:0707.1426 [hep-th]];

S. A. Appleby and R. A. Battye, "Regularized braneworlds of arbitrary codimension," Phys. Rev. D **76** (2007) 124009 [arXiv:0707.4238 [hep-ph]];

C. Bogdanos, A. Kehagias and K. Tamvakis, "Pseudo-3-Branes in a Curved 6D Bulk," Phys. Lett. B **656** (2007) 112 [arXiv:0709.0873 [hep-th]];

O. Corradini, K. Koyama and G. Tasinato, "Induced gravity on intersecting brane-worlds Part I: Maximally symmetric solutions," Phys. Rev. D **77** (2008) 084006 [arXiv:0712.0385 [hep-th]];

F. Arroja, T. Kobayashi, K. Koyama and T. Shiromizu, "Low energy effective theory on a regularized brane in 6D gauged chiral supergravity," JCAP **0712** (2007) 006 [arXiv:0710.2539 [hep-th]];

O. Corradini, K. Koyama and G. Tasinato, "Induced gravity on intersecting brane-worlds Part II: Cosmology," Phys. Rev. D **78** (2008) 124002 [arXiv:0803.1850 [hep-th]];

V. Dzhunushaliev, V. Folomeev and M. Minamitsuji, "Thick brane solutions," arXiv:0904.1775 [gr-qc].

[3.9] For a review with references see J. Polchinski, *TASI Lectures on D-Branes* [hep-th/9611050].

[3.10] C. Vafa, "Evidence for F-Theory," Nucl. Phys. B **469** (1996) 403 [arXiv:hep-th/9602022];
D. R. Morrison and C. Vafa, "Compactifications of F-Theory on Calabi–Yau Threefolds – I," Nucl. Phys. B **473** (1996) 74 [arXiv:hep-th/9602114]; "Compactifications of F-Theory on Calabi–Yau Threefolds – II," Nucl. Phys. B **476** (1996) 437 [arXiv:hep-th/9603161]. Sen, Ashoke, "F-theory and Orientifolds" Nucl. Phys. B **475** (1996) 562 [arXiv:hep-th/9605150].

[3.11] A. Kehagias, "A conical tear drop as a vacuum-energy drain for the solution of the cosmological constant problem," Phys. Lett. B **600** (2004) 133 [arXiv:hep-th/0406025];
T. Kobayashi and M. Minamitsuji, "Brane cosmological solutions in six-dimensional warped flux compactifications," JCAP **0707** (2007) 016 [arXiv:0705.3500 [hep-th]];
D. Kiley, "Rotating Black Holes on Codimension-2 Branes," Phys. Rev. D **76** (2007) 126002 [arXiv:0708.1016 [hep-th]];

[3.12] H. Nishino and E. Sezgin, *Phys. Lett.* **144B** (1984) 187; "The Complete N=2, D = 6 Supergravity With Matter And Yang-Mills Couplings," Nucl. Phys. **B278** (1986) 353; S. Randjbar-Daemi, A. Salam, E. Sezgin and J. Strathdee, *Phys. Lett.* **B151** (1985) 351.

[3.13] A. Salam and E. Sezgin, "Chiral Compactification On Minkowski X

S**2 Of N=2 Einstein-Maxwell Supergravity In Six-Dimensions," Phys. Lett. B **147** (1984) 47;

S. L. Parameswaran, G. Tasinato and I. Zavala, "The 6D SuperSwirl," Nucl. Phys. B **737** (2006) 49 [arXiv:hep-th/0509061];

H. M. Lee and C. Ludeling, "The general warped solution with conical branes in six-dimensional supergravity," JHEP **0601** (2006) 062 [arXiv:hep-th/0510026];

H. M. Lee and A. Papazoglou, "Supersymmetric codimension-two branes in six-dimensional gauged supergravity," JHEP **0801** (2008) 008 [arXiv:0710.4319 [hep-th]];

C. P. Burgess, S. L. Parameswaran and I. Zavala, "The Fate of Unstable Gauge Flux Compactifications," arXiv:0812.3902 [hep-th];

[3.14] A. J. Tolley, C. P. Burgess, C. de Rham and D. Hoover, "Scaling solutions to 6D gauged chiral supergravity," New J. Phys **8** (2006) 324 [arXiv:0608.083 [hep-th]];

A. J. Tolley, C. P. Burgess, C. de Rham and D. Hoover, "Exact Wave Solutions to 6D Gauged Chiral Supergravity," JHEP **0807** (2008) 075 [arXiv:0710.3769 [hep-th]];

M. Minamitsuji, "Instability of brane cosmological solutions with flux compactifications," Class. Quant. Grav. **25** (2008) 075019 [arXiv:0801.3080 [hep-th]].

H. M. Lee and A. Papazoglou, "Codimension-2 brane inflation," Phys. Rev. D **80** (2009) 043506 [arXiv:0901.4962 [hep-th]].

[3.15] S. Weinberg, *Gravitation and Cosmology*, Wiley 1973.

[3.16] C. W. Misner, J. A. Wheeler and K. S. Thorne, *Gravitation*, W. H. Freeman & Company 1973.

[3.17] G.W. Gibbons and S.W. Hawking, Phys. Rev. **D15** (1977) 2752.

[3.18] I. Navarro and J. Santiago, "Gravity on codimension 2 brane worlds," JHEP **0502** (2005) 007 [arXiv:hep-th/0411250].

[3.19] Y. Aghababaie, C.P. Burgess, S. Parameswaran and F. Quevedo, Nucl. Phys. **B680** (2004) 389–414, [hep-th/0304256];
C. P. Burgess, Annals Phys. **313** (2004) 283 [arXiv:hep-th/0402200]; AIP Conf. Proc. **743** (2005) 417 [arXiv:hep-th/0411140].

[3.20] Y. Aghababaie *et al.*, "Warped brane worlds in six dimensional supergravity," JHEP **0309** (2003) 037 [hep-th/0308064];

[3.21] B. R. Greene, A. D. Shapere, C. Vafa and S. T. Yau, "Stringy Cosmic Strings And Noncompact Calabi-Yau Manifolds," Nucl. Phys. B **337** (1990) 1.

[3.22] N. Koblitz, *Introduction to Elliptic Curves and Modular Forms*, Springer Verlag (1984).

[3.23] G. W. Gibbons, R. Guven and C. N. Pope, "3-branes and uniqueness of the Salam-Sezgin vacuum," Phys. Lett. B **595** (2004) 498 [hep-th/0307238];

[3.24] C. P. Burgess, F. Quevedo, G. Tasinato and I. Zavala, "General axisymmetric solutions and self-tuning in 6D chiral gauged supergravity," JHEP **0411** (2004) 069 [hep-th/0408109];

[3.25] A. J. Tolley, C. P. Burgess, D. Hoover and Y. Aghababaie, "Bulk singularities and the effective cosmological constant for higher co-dimension branes," JHEP **0603** (2006) 091 [arXiv:hep-th/0512218].

[3.26] C. P. Burgess, D. Hoover and G. Tasinato, "Technical Naturalness on a Codimension-2 Brane," JHEP **0906** (2009) 014 [arXiv:0903.0402 [hep-th]].

[3.27] H. M. Lee and A. Papazoglou, "Scalar mode analysis of the warped Salam-Sezgin model," [hep-th/0602208];
C. P. Burgess, C. de Rham, D. Hoover, D. Mason and A. J. Tolley, "Kicking the rugby ball: Perturbations of 6D gauged chiral supergravity," JCAP **0702** (2007) 009 [arXiv:hep-th/0610078].
S. L. Parameswaran, S. Randjbar-Daemi and A. Salvio, "Gauge fields, fermions and mass gaps in 6D brane worlds," Nucl. Phys. B **767** (2007) 54 [arXiv:hep-th/0608074]; "Stability and Negative Tensions in 6D Brane Worlds," JHEP **0801** (2008) 051 [arXiv:0706.1893 [hep-th]]; "General Perturbations for Braneworld Compactifications and the Six Dimensional Case," JHEP **0903** (2009) 136 [arXiv:0902.0375 [hep-th]].

[3.28] J. M. Maldacena and C. Nunez, "Supergravity description of field theories on curved manifolds and a no go theorem," Int. J. Mod. Phys. A **16** (2001) 822 [hep-th/0007018]; G. W. Gibbons, R. Kallosh and A. D. Linde, "Brane world sum rules," JHEP **0101** (2001) 022 [hep-th/0011225]; E. Teo, "A no-go theorem for accelerating cosmologies from M-theory compactifications," Phys. Lett. B **609** (2005) 181 [hep-th/0412164]; M. P. Hertzberg, S. Kachru, W. Taylor and M. Tegmark, "Inflationary Constraints on Type IIA String Theory," JHEP **0712**, 095

(2007) [arXiv:0711.2512 [hep-th]]; D. H. Wesley, "New no-go theorems for cosmic acceleration with extra dimensions," [arXiv:0802.2106 [hep-th]]; "Oxidised cosmic acceleration," [arXiv:0802.3214 [hep-th]];
P. J. Steinhardt and D. Wesley, "Dark Energy, Inflation and Extra Dimensions," Phys. Rev. D **79** (2009) 104026 [arXiv:0811.1614 [hep-th]].

[3.29] C.P. Burgess and L. van Nierop, "Sculpting the Extra Dimensions: Inflation from Codimension-2 Brane Back-reaction" [arXiv:1108.2553]

Chapter 4

Bulk Axions, Brane Back-reaction and Fluxes

4.1 Preamble

This chapter is based on the work in (4.1). This chapter describes the physics of branes in the simplest flux stabilization: 6 dimensional Einstein-Maxwell-scalar theory with extra dimensions in the shape of a rugby ball with branes on opposite ends. The rugby ball is sourced by two branes with equal and constant tension.

We explore the shape of the effective potential that is the result of an arbitrary small perturbation of the brane tensions. In addition, we find how the bulk fields respond to the perturbations, by solving the linearized equations of motion in the bulk in full generality, and relating the integration constants to the choices made on the branes. We find that the first subdominant term in the on-brane derivative expansion, a magnetic coupling to the stabilizing flux, is competitive with the lowest order contribution. The reason for this is that

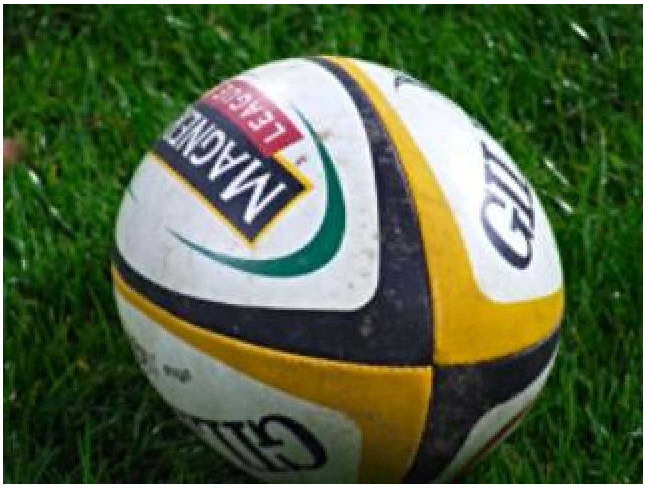

Figure 4.1: An example of a rugby ball (though the extra dimensions are less colorful).

this coupling allows some of the stabilizing flux to be localized at the branes, which reduces the energy cost of changing the volume.

The importance of this chapter is that we developed the description of the localization of bulk flux. In the supersymmetric case (see chapter (5)), this mechanism allows the bulk to relax when the source branes are perturbed.

C.P. Burgess and L. van Nierop

4.2 Introduction and Summary

Brane-world models — where known particles are localized on surfaces within extra dimensions — have proven to be fruitful places to seek novel kinds of low-energy physics. As studies over the past decade show, their low-energy physics can be novel (relative to 4D models, say) because of several different mechanisms:

Brane vs Bulk Kinematics: Because not all particles are trapped on the branes, different species experience the kinematics of different dimensions. This observation is what allows the existence of unusually large extra dimensions and low gravity scales (4.2; 4.3).

Brane Back-reaction: Because branes can be localized within the extra dimensions, which need not be homogeneous, physical properties can vary from place to place within the extra dimensions. Such position dependence is generic once the back-reaction of localized branes onto their geometry is included (4.4).

Dimensional stabilization: Because low-energy degrees of freedom can be lighter than the Kaluza-Klein (KK) scale, predictions for low-energy dynamics generically require an equally complete understanding of whatever physics stabilizes the extra dimensions (4.5).

These last two points in particular considerably complicate discussions of cosmology within a brane-world framework (4.6). What appear to be shallow directions for the scalar potential on a brane for fixed bulk geometry can turn into much steeper directions once the bulk geometry is allowed to move.

With a few exceptions, the interplay between brane back-reaction and the physics stabilizing the bulk remains relatively poorly explored. The main exception is the case of codimension-1 branes moving within one extra dimension, as for Randall-Sundrum (RS) models (4.4). Yet one wonders how representative codimension-1 systems are of the more generic situation having higher codimension, for which much less is known about brane back-reaction. Although some results exist for the back-reaction of branes on ten-dimensional geometries in string theory (4.7), it is often not possible to be as explicit about the form of the extra-dimensional geometry and its detailed interplay with brane back-reaction.

In this paper we compute the low-energy potential for a class of codimension-2 brane models for which explicit compactifications involving both bulk stabilization and brane back-reaction are known. We focus in particular on the simplest of brane/flux compactifications: (nonsupersymmetric) 6D Einstein-Maxwell theory with the extra dimensions stabilized through a competition between a background Maxwell flux and a bulk cosmological constant (4.8; 4.9; 4.10; 4.11; 4.12). To this system we couple a bulk Goldstone boson (axion), ϕ, whose shift symmetry, $\phi \to \phi+$(constant), is explicitly broken by its couplings to the two branes that source the bulk through interactions of the schematic form

$$S_b = \int_{\Sigma_b} \Big(\tau_b \, \omega + \Phi_b \, {}^\star\mathcal{F} \Big) \,, \tag{4.1}$$

where the integration is over the 4D brane world-sheet, Σ_b, whose volume form is denoted ω so $\tau_b(\phi)$ represents a ϕ-dependent brane tension. $\mathcal{F}_{MN}$ denotes the 6D Maxwell field strength — for which ${}^\star\mathcal{F}$ is the 6D Hodge dual — and the ϕ-dependent coupling $\Phi_b(\phi)$ can be interpreted as the amount of Maxwell flux carried by the brane in question.

By computing the back-reaction of the branes onto the bulk fields we obtain the low-energy potential for the resulting would-be Goldstone zero mode, φ, that becomes a pseudoGoldstone boson (pGB) (4.13) in the low-energy 4D effective theory. The branes back-react onto the bulk fields by changing their boundary conditions, through a codimension-2 generalization (4.14; 4.15; 4.16) of the more familiar codimension-1 Israel junction conditions (4.17).

The resulting bulk field equations subject to the brane boundary conditions can be solved explicitly in some generality if the axion-dependence of the brane tension is regarded as a small change to a background, axion-independent value. In this limit the would-be zero mode is stabilized to a fixed value, $\varphi = \varphi_\star$, which we compute in two separate ways: first by explicitly solving the linearized field equations of the full 6D theory; and second by minimizing the dimensionally reduced axion potential in the 4D low-energy effective theory. Both methods agree, and the generality of our result allows us to follow how the stabilized value and its energy density vary as a function of the axion couplings to the two branes. In particular we see what happens when the branes differ in the value for the axion that they prefer. As a by-product we also compute how the geometry of the extra dimensions changes due to the presence of the axion-brane couplings.

The calculation reveals the following generic features

1. In the absence of brane fluxes — *i.e.* $\Phi_b = 0$ — the low-energy 4D potential is very generally simply given by the sum of tensions, summed

over the branes present,

$$V_{\rm eff}(\varphi) = \sum_b \tau_b[\phi_b(\varphi)] \,, \tag{4.2}$$

where $\phi_b(\varphi)$ denotes the value taken by the (suitably renormalized) 6D scalar field at the corresponding brane position, regarded as a function of the zero mode φ. This agrees with the probe-brane approximation (which ignores brane back-reaction) since for brane tensions the contribution of the back-reaction first arises at second order. In particular the stabilized value, $\varphi = \varphi_\star$, satisfies

$$\sum_b \left(\frac{\partial \tau_b}{\partial \phi} \right)_{\varphi = \varphi_\star} = 0 \,. \tag{4.3}$$

Because the quantity $(\partial \tau_b / \partial \phi)_{\varphi_\star}$ governs the coupling of the lightest mode, φ, to matter localized on the brane, these couplings tend naturally to turn themselves off for small fluctuations of φ about its ground state. This could provide a phenomenologically useful mechanism for naturally decoupling light bulk scalars from brane matter, along the lines of similar earlier proposals (4.18).

2. By contrast, nonzero brane fluxes contribute at linear order in two ways, that are similar in size. The first arises because nonzero background fluxes are required to stabilize the bulk geometry, ${}^\star\mathcal{F} = -\mathcal{Q}\,\omega$ (with, as before, ω the volume form). Because of this the $\Phi_b {}^\star\mathcal{F}$ term modifies the value of the brane action, giving an 'effective tension'

$$T_b(\phi) = \tau_b(\phi) - \mathcal{Q}\,\Phi_b(\phi) \,. \tag{4.4}$$

The second contribution arises because quantization of total flux requires the amount of bulk flux to change in response to the presence of flux on the brane, leading to an additional energy cost over and above that measured by the difference $T_b - \tau_b$. Although the complete expression for $V_{\rm eff}(\varphi)$ that results involves an integration of Φ_b with respect to φ, the predictions for $\varphi_\star$ and $\varrho_{\rm eff} := V_{\rm eff}(\varphi_\star)$ turn out to be relatively simple. The prediction for $\varphi_\star$ is again given by eq. (4.3) — with no contribution from Φ_b — while the prediction for $\varrho_{\rm eff}$ becomes

$$\varrho_{\rm eff} = \sum_b \Big\{ T_b[\phi_b(\varphi_\star)] - \mathcal{Q}\,\Phi_b[\phi_b(\varphi_\star)] \Big\} = \sum_b \Big\{ \tau_b[\phi_b(\varphi_\star)] - 2\mathcal{Q}\,\Phi_b[\phi_b(\varphi_\star)] \Big\} . \tag{4.5}$$

We see in this way that the brane flux 'contributes twice' to the vacuum energy at low energies. In particular, $\varrho_{\rm eff} = 0$ for branes satisfying $\tau_b[\phi_b(\varphi_\star)] = 2\mathcal{Q}\,\Phi_b[\phi_b(\varphi_\star)]$, and so $T_b[\phi_b(\varphi_\star)] = \mathcal{Q}\,\Phi_b[\phi_b(\varphi_\star)]$.

3. Assuming a 6D kinetic energy of the form $F^4 \partial^M \phi\, \partial_M \phi$, in order of magnitude the mass of the light 4D would-be zero mode, φ, predicted by the low-energy potential is of order $m_\varphi \sim (\mu/F)^2 m_{KK} f(\varphi_\star)$, where m_{KK} denotes the KK mass scale and the ϕ-dependent part of the brane tension is assumed to be of order μ^4. In most cases of interest $f(\varphi_\star)$ is given by $\sum_b \partial^2 \tau_b / \partial \phi^2$ evaluated at $\varphi_\star$. When $\mu \ll F$ this mode satisfies $m_\varphi \ll m_{KK}$, allowing its properties to be described in the effective 4D theory. (Unlike for a purely 4D theory it makes sense to call ϕ a pseudo-Goldstone field even if $\mu > F$, since its shift symmetry is everywhere unbroken in the extra dimensions except at the positions of the branes. However, if $\mu > F$ then the mass of the would-be zero mode becomes

comparable with other KK modes, precluding calculating its properties within the 4D theory.[1])

The low-energy 4D theory obtained from models with bulk axions generically includes a light scalar field, whose small mass is technically natural because of the weakly broken shift symmetry. What the brane construction potentially provides is a UV completion that can explain why the masses and couplings are small in the first place. This could prove useful for a variety of low-energy applications, such as to extra-dimensional inflationary models, some of which we briefly describe below while examining specific examples of our general expressions.

We organize our detailed discussion as follows. Our main results are presented in the next section, which starts in §4.3.1 by setting out the field equations describing the system of interest. These are then solved by finding solutions that are perturbatively close to simple, well-known rugby-ball solutions involving two branes interacting with a spherical 2D bulk. §4.3.2 does so first for the simpler case where the branes do not couple to the bulk scalar, with the generalization to scalar-brane coupling following in §4.3.3. §4.4 then explores the features of these general solutions by examining in detail several simple illustrative special cases. These include situations where the two branes agree on the value at which the field φ stabilizes, and situations where they do not. §4.4 also considers several special cases of potential phenomenological interest for axion and inflationary applications.

[1]Exceptions to this can arise if the scalar is self-localized at the brane (4.19), but this usually requires a bulk scalar potential $U(\phi)$ that is forbidden in the current examples by the assumed shift symmetry, $\phi \to \phi + c$.

4.3 The bulk-brane system

This section defines the system of interest, which we take to be the simplest theory containing both codimension-two sources (with positive tension) and a flux mechanism for stabilizing the size of the extra dimensions. This suggests taking the bulk theory to be 6D Einstein-Maxwell gravity coupled to the Goldstone boson (axion) field.

4.3.1 Field equations and background solutions

Classical brane-bulk dynamics is defined by solving the bulk field equations, subject to the boundary conditions imposed by matching conditions at each brane.

Bulk field equations

The bulk action of interest is[2]

$$S_{\text{bulk}} = -\int \mathrm{d}^6x \,\sqrt{-g}\; \left\{ \frac{1}{2}\, g^{MN} \left(\frac{1}{\kappa^2}\, \mathcal{R}_{MN} + \frac{1}{\kappa_a^2}\, \partial_M\phi \,\partial_N\phi \right) + \frac{1}{4}\, \mathcal{F}_{MN}\mathcal{F}^{MN} + \Lambda \right\}, \tag{4.6}$$

where $\mathcal{R}_{MN}$ is the Ricci tensor constructed from the 6D metric g_{MN}, ϕ is the axion field and $\mathcal{F} = \mathrm{d}\mathcal{A}$ is the field strength for the Maxwell potential $\mathcal{A}_M$. The dimensionful parameters of the problem are the 6D gravitational coupling, $\kappa = 1/M_g^2$, the bulk axion decay constant, $\kappa_a = 1/F^2$, and the bulk cosmological constant, Λ (whose value is tuned to ensure the unperturbed solution is flat in the on-brane directions). Although κ_a can be absorbed into

[2]We use a 'mostly plus' metric and Weinberg's curvature conventions (4.20) (that differ from those of MTW (4.21) only by an overall sign in the definition of the Riemann tensor).

the normalization of ϕ, we do not do so because this changes the form of the brane couplings to ϕ.

The field equations obtained from this action are the (trace-reversed) Einstein equation

$$\mathcal{R}_{MN} + \lambda^2 \, \partial_M \phi \, \partial_N \phi + \kappa^2 \mathcal{F}_{MP} \mathcal{F}_N{}^P - \left[\frac{\kappa^2}{8} \, \mathcal{F}_{PQ} \mathcal{F}^{PQ} - \frac{\kappa^2 \Lambda}{2} \right] g_{MN} = 0 \,, \tag{4.7}$$

where $\lambda := \kappa/\kappa_a = F/M_g$. The Maxwell equation is

$$\sqrt{-g} \; \nabla_M \mathcal{F}^{MN} = \partial_M \Big(\sqrt{-g} \; \mathcal{F}^{MN} \Big) = 0 \,, \tag{4.8}$$

and

$$\sqrt{-g} \; \Box \phi = \partial_M \left(\sqrt{-g} \; \partial^M \phi \right) = 0 \,, \tag{4.9}$$

is the axion equation.

Rugby-ball solutions

We consider geometries that are maximally symmetric in the 4 on-brane directions and axially symmetric in the two extra dimensions. The corresponding *ansatz* for the metric, scalar and Maxwell fields is

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + e^{2B} \mathrm{d}\theta^2 + e^{2W} \, \hat{g}_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu \,, \tag{4.10}$$

and

$$\mathcal{F}_{\rho\theta} = \mathcal{A}_\theta' \,, \tag{4.11}$$

where $\hat{g}_{\mu\nu}$ is an x^μ-dependent maximally symmetric geometry and the functions B, W, ϕ and $\mathcal{A}_\theta$ depend only on ρ. Primes denote differentiation with respect to this coordinate.

Subject to this *ansatz* the bulk field equations reduce to

$$\begin{aligned}
\left(e^{B+4W}\,\phi'\right)' &= 0 & (\phi)\\
\left(e^{-B+4W}\mathcal{A}_\theta'\right)' &= 0 & (\mathcal{A}_\theta)\\
4\Big[W''+(W')^2\Big]+B''+(B')^2+\lambda^2(\phi')^2+\frac{3\kappa^2}{4}\,e^{-2B}\,(\mathcal{A}_\theta')^2+\frac{\kappa^2\Lambda}{2} &= 0 & (\rho\rho)\\
B''+(B')^2+4W'B'+\frac{3\kappa^2}{4}\,e^{-2B}\,(\mathcal{A}_\theta')^2+\frac{\kappa^2\Lambda}{2} &= 0 & (\theta\theta)\\
\frac{1}{4}\,e^{-2W}\hat{\mathcal{R}}+W''+4(W')^2+W'B'-\frac{\kappa^2}{4}\,e^{-2B}\,(\mathcal{A}_\theta')^2+\frac{\kappa^2\Lambda}{2} &= 0 & (\mu\nu)\,,
\end{aligned} \tag{4.12}$$

where $\hat{\mathcal{R}}$ is the curvature scalar built from the maximally symmetric metric $\hat{g}_{\mu\nu}$. The first two of these immediately integrate to give

$$e^{B+4W}\,\phi'=\varphi_1 \quad \text{and} \quad e^{-B+4W}\mathcal{A}_\theta'=\mathcal{Q}\,, \tag{4.13}$$

where φ_1 and $\mathcal{Q}$ are integration constants.

When $\varphi_1=0$ the full set of equations admits a particularly simple solution of the rugby-ball form (4.8)

$$\begin{aligned}
\mathrm{d}s^2 &= \mathrm{d}\rho^2+\alpha^2L^2\sin^2\left(\frac{\rho}{L}\right)\mathrm{d}\theta^2+\hat{g}_{\mu\nu}\,\mathrm{d}x^\mu\mathrm{d}x^\nu\\
\mathcal{F}_{\rho\theta} &= \mathcal{Q}\alpha L\,\sin\left(\frac{\rho}{L}\right)\,,
\end{aligned} \tag{4.14}$$

with constant $\phi=\varphi_0$ and $W=0$. The equations of motion imply the following relation amongst the integration constants:

$$\frac{2}{L^2}=\kappa^2\left(\frac{3\mathcal{Q}^2}{2}+\Lambda\right)\,, \tag{4.15}$$

as well as fixing the 4D curvature

$$\hat{\mathcal{R}} := \hat{g}^{\mu\nu}\hat{\mathcal{R}}_{\mu\nu} = \kappa^2 \left(\mathcal{Q}^2 - 2\Lambda\right) . \tag{4.16}$$

A final constraint relating parameters comes from flux quantization, due to the spherical topology of the extra dimensions. As usually framed, this implies

$$\frac{n}{g} = 2\alpha L^2 \mathcal{Q} \,, \tag{4.17}$$

where g is the gauge coupling of the Maxwell field and n is an arbitrary integer. However this expression assumes the absence of any flux localized on the source branes themselves (4.11). In the presence of brane-localized flux (more about this below and in Appendix B.1) the flux-quantization condition instead becomes

$$\frac{n}{g} = 2\alpha L^2 \mathcal{Q} + \sum_b \frac{\Phi_b}{2\pi} \,, \tag{4.18}$$

where the sum is over all of the branes present, each of which carries the localized flux, Φ_b.

Since our interest is in background solutions with flat geometries, $\hat{g}_{\mu\nu} = \eta_{\mu\nu}$, we further choose Λ so that $\hat{\mathcal{R}} = 0$:

$$\Lambda = \frac{\mathcal{Q}^2}{2} \quad \text{and so} \quad \kappa^2 L^2 \mathcal{Q}^2 = 1 \,. \tag{4.19}$$

With this choice all geometrical properties, like L and $\mathcal{Q}$, can be regarded as functions of the integration constant α together with the integer n and the lagrangian parameters κ and g (see Appendix B.2 for details).

The potential singularity in the geometry where $g_{\theta\theta} = e^B$ vanishes is just a coordinate artefact when $\alpha = 1$, in which case the two compact dimensions define a sphere. When $\alpha \neq 1$ the background has a conical singularity at $\rho = \rho_N := 0$ and $\rho = \rho_S := \pi L$. This is interpreted as describing the back-reaction of two codimension-two source branes located at these positions, having equal tensions, T (which includes the energy associated with the brane flux, see eqn. (4.21). Matching at the branes (see below) implies this tension is related to the deficit angle by

$$1 - \alpha = \frac{\kappa^2 T}{2\pi}\,, \tag{4.20}$$

similar to the relation between tension and deficit angle for a cosmic string (4.22).

Finally, notice that the value $\phi = \varphi_0$ is not determined by any of the equations of motion, due to the symmetry $\phi \to \phi + \text{constant}$. The parameter φ_0 labels a flat direction in the low-energy potential, that can be lifted if the coupling of ϕ to the branes breaks this symmetry (such as by allowing the tensions T to depend on ϕ). A primary goal of the next few sections is to identify the effective potential for this low-lying mode below the KK scale, to determine how the vacuum value after symmetry breaking, φ_*, is related to the couplings on the branes.

Brane matching conditions

As brane sources we use the most general form (involving the fewest derivatives) for a 4D brane action located at positions ρ_N and ρ_S (4.11)

$$S_{\text{branes}} \;=\; -\sum_{b=N,S} \int \mathrm{d}^4x\,\sqrt{-g_4}\,\left[\tau_b - \frac{\Phi_b}{2}\,\epsilon^{mn}\mathcal{F}_{mn}\right]$$

$$= \; -\sum_{b=N,S} \int \mathrm{d}^4x \, \sqrt{-\hat{g}} \, e^{4W} \left[\tau_b - \frac{\Phi_b}{2} \, \epsilon^{mn} \mathcal{F}_{mn} \right] , \qquad (4.21)$$

where $\epsilon^{\rho\theta} = 1/\sqrt{g_2} = e^{-B}$ transforms as a tensor, rather than a tensor density, in the two transverse dimensions. The parameter τ_b represents the tension of the brane, which can depend on all of ϕ, W and $g_{\theta\theta}$ without breaking the condition of maximal symmetry in the on-brane directions. As is shown below, the parameter Φ_b similarly denotes the magnetic charge (or flux) carried by the source branes (which could also depend on ϕ, W and $g_{\theta\theta}$).

The presence of such branes imposes a set of boundary conditions on the derivatives of the bulk fields in the near-brane limits, given by[3]

$$\begin{aligned} \Big[e^B \phi' \Big]_{\rho_b} &= \frac{\partial \mathcal{T}_b}{\partial \phi} \quad \text{with} \quad \mathcal{T}_b := \frac{\kappa^2 T_b}{2\pi} \\ \Big[e^B W' \Big]_{\rho_b} &= \mathcal{U}_b \quad \text{with} \quad \mathcal{U}_b := \frac{\kappa^2}{4\pi} \left(\frac{\partial T_b}{\partial g_{\theta\theta}} \right) \\ \text{and} \quad \Big[e^B B' - 1 \Big]_{\rho_b} &= -\Big[\mathcal{T}_b + 3\,\mathcal{U}_b \Big] , \end{aligned} \qquad (4.22)$$

where T_b is defined as the total lagrangian density of the source,

$$T_b = \tau_b - \; \Phi_b \, e^{-B} F_{\rho\theta} \, . \qquad (4.23)$$

The Bianchi identities ensure that only two of eqs. (4.22) are independent of one another, and as a consequence the quantities $\mathcal{U}_b$ and $\mathcal{T}_b$ are also not independent. They are subject to the constraint:

$$4\mathcal{U}_b \Big[2 - 2\mathcal{T}_b - 3\,\mathcal{U}_b \Big] - (\mathcal{T}_b')^2 = 0 \, , \qquad (4.24)$$

[3]Notice that we normalize the quantities $\mathcal{T}_b$ and $\mathcal{U}_b$ without including a factor of e^{4W} used in ref. (4.15).

where $\mathcal{T}_b' = \partial \mathcal{T}_b/\partial\phi$. Notice that for the rugby ball solutions $\mathcal{T}_b' = \mathcal{U}_b = 0$ and so eqs. (4.22) degenerate down to eq. (4.20).

As shown in Appendix B.1, the corresponding boundary condition for the Maxwell field implies that the integral of eq. (4.13) for $\mathcal{A}_\theta(\rho)$ for a patch containing each source brane is (4.11)

$$\begin{aligned} \mathcal{A}_\theta(\rho) &= \frac{\Phi_N}{2\pi} + \mathcal{Q}\int_{\rho_N}^{\rho} \mathrm{d}\hat\rho\; e^{B-4W} \qquad \text{Northern hemisphere} \\ &= -\frac{\Phi_S}{2\pi} + \mathcal{Q}\int_{\rho_S}^{\rho} \mathrm{d}\hat\rho\; e^{B-4W} \quad \text{Southern hemisphere}\,, \end{aligned} \tag{4.25}$$

and the signs are dictated by the observation that increasing ρ points away from (towards) the North (South) pole, together with the requirement that the two patches share the same orientation. Requiring these to differ by a gauge transformation, $g^{-1}\partial_\theta\Omega$, on regions of overlap implies the flux-quantization condition

$$\frac{n}{g} = \frac{\Phi_{\rm tot}}{2\pi} + \mathcal{Q}\int_{\rho_N}^{\rho_S} \mathrm{d}\rho\; e^{B-4W}\,, \tag{4.26}$$

where n is an integer, g is the gauge coupling and $\Phi_{\rm tot} = \Phi_N + \Phi_S$. It is this expression that identifies Φ_b as the fraction of the total Maxwell flux carried by each brane.

4.3.2 Perturbations I: the Einstein-Maxwell case

Next consider starting with a rugby-ball solution and independently perturbing each of the two brane tensions, $\tau_b = \tau + \delta\tau_b$, and brane-localized fluxes, $\Phi_b = \Phi + \delta\Phi_b$, implying a similar expansion for the total brane action, $T_b = \tau_b - e^B \mathcal{F}_{\rho\theta}\Phi_b = \tau_b - \mathcal{Q}\,\Phi_b$. This section starts simply and assumes both $\delta\tau_b$ and $\delta\Phi_b$ are independent of ϕ, with the resulting insights used to inform the next section's discussion of the more general case. The goal is to compute explicitly how the bulk fields respond to the perturbation, allowing a detailed examination of how the extra dimensions flex as their source branes change. In general, because the perturbed branes are different from one another, the scalar field acquires a nontrivial profile, $\phi = \phi(\rho)$, and the resulting geometry warps nontrivially, $W = W(\rho)$.

Linearized solutions

Because the brane perturbations are independent of ϕ, $\partial T_b / \partial \phi = 0$ and so there is no change to the ϕ boundary conditions. Consequently the unperturbed solution, $\phi = \varphi_0$, remains a solution. The scalar then drops out of the problem and the calculation involves only the Einstein-Maxwell system. Writing $e^B = e^{B_0}[1 + \delta B(\rho)]$ and $W = \delta W(\rho)$ — with B_0 given by the rugby-ball solution described by parameters $\mathcal{Q}$, L and α — we linearize the field equations in δB and δW.

The combination of the Einstein equations $(\rho\rho) - (\theta\theta)$ linearizes to

$$\delta W'' - B_0' \, \delta W' = \delta W'' - \frac{\delta W'}{L} \cot\left(\frac{\rho}{L}\right) = 0 \tag{4.27}$$

which has as its solution

$$\delta W = W_1 \cos\left(\frac{\rho}{L}\right) , \tag{4.28}$$

where we absorb an additive integration constant, W_0, into a re-scaling of the four on-brane coordinates, x^μ.

The perturbed gauge field again satisfies eq. (4.13), and so

$$\delta\mathcal{A}_\theta' = \Big[\delta\mathcal{Q} + \mathcal{Q}\,(\delta B - 4\delta W)\Big]\alpha L \sin\left(\frac{\rho}{L}\right) . \tag{4.29}$$

With this and eq. (4.28) the $\theta\theta$ Einstein equation linearizes to

$$\frac{[\delta B' \sin^2(\rho/L)]'}{\sin^2(\rho/L)} = \frac{10\,W_1}{L^2}\,\cos\left(\frac{\rho}{L}\right) - \frac{3}{2L^2}\left(\frac{\delta Q}{Q}\right) , \tag{4.30}$$

whose integral is

$$\delta B = \frac{3}{4}\left(\frac{\delta\mathcal{Q}}{\mathcal{Q}}\right)\frac{\rho}{L}\,\cot\left(\frac{\rho}{L}\right) - \frac{10W_1}{3}\,\cos\left(\frac{\rho}{L}\right) - B_1\cot\left(\frac{\rho}{L}\right) + \delta B_0\,. \tag{4.31}$$

Notice that the integration constant B_1 here is pure gauge, corresponding to an infinitesimal shift in the radial coordinate $\rho \to \rho + c$. We can fix this freedom by demanding that $\rho_N = 0$, and so $e^B \to 0$ as $\rho \to 0$. Since $e^B = \alpha L(1 + \delta B)\sin(\rho/L) \to -B_1$ at $\rho \to 0$, this implies $B_1 = 0$.

The linearized flux quantization condition, eq. (4.26) is

$$\frac{\delta\mathcal{Q}}{\mathcal{Q}} + \frac{1}{2}\int_0^{\pi L}\frac{\mathrm{d}\rho}{L}\sin\left(\frac{\rho}{L}\right)(\delta B - 4\delta W) + \frac{\kappa^2\mathcal{Q}}{4\pi\alpha}\,(\delta\Phi_N + \delta\Phi_S) = 0\,, \tag{4.32}$$

which uses the background relation $\kappa L\mathcal{Q} = 1$ to rewrite $\delta\Phi_b/L^2\mathcal{Q} = \kappa^2\mathcal{Q}\delta\Phi_b$. Solving this for $\delta\mathcal{Q}/\mathcal{Q}$ gives

$$\frac{\delta\mathcal{Q}}{\mathcal{Q}} = -4\delta B_0 - \frac{\kappa^2\mathcal{Q}}{\pi\alpha}\,(\delta\Phi_N + \delta\Phi_S)\ . \tag{4.33}$$

In summary, once coordinate conditions are used to eliminate W_0 and B_1, solutions to the bulk equations for δW, δB and $\delta\mathcal{A}_\theta$ involve three integration constants — W_1, δB_0 and $\delta\mathcal{Q}/\mathcal{Q}$ — among which flux quantization imposes one relation. The physical interpretation of the two remaining parameters is seen by connecting them to two physical quantities. One of these can be taken as the warping difference between the two branes,

$$\delta W_{_N} - \delta W_{_S} = 2W_1 \,, \tag{4.34}$$

which controls the relative redshift of energies on the two branes. The other can be chosen as the change in proper distance, $\rho_{_S} - \rho_{_N} = \pi(L + \delta L)$, between the two branes, where $\rho_{_N} = 0$ and $\rho_{_S}$ are defined as the places where $e^B \to 0$. Comparing

$$\lim_{\rho\to\pi L} e^B \simeq -\frac{3\pi\alpha L}{4}\left(\frac{\delta\mathcal{Q}}{\mathcal{Q}}\right) \,, \tag{4.35}$$

with the Taylor expansion $e^B(\pi L) \simeq (e^{B_0})'_{\rho_S}(-\pi\delta L) = +\pi\alpha\delta L$ gives

$$\frac{\delta L}{L} \simeq -\frac{3}{4}\left(\frac{\delta\mathcal{Q}}{\mathcal{Q}}\right) \,. \tag{4.36}$$

Matching to brane tensions

All that remains is to eliminate the integration constants δB_0 and W_1 in terms of the brane perturbations using the linearized brane matching conditions. In the present instance only the last of eqs. (4.22) is nontrivial. Besides imposing the background relation $\alpha = 1 - \kappa^2 T/(2\pi)$, for the linearized perturbations this condition implies

$$\delta\Big(e^B\Big)'_{\rho_b} = -\frac{\kappa^2\delta T_b}{2\pi} \,, \tag{4.37}$$

where $\delta T_b = \delta(\tau_b - \Phi_b \mathcal{Q} e^{-4W_b}) \simeq \delta\tau_b - \delta\Phi_b \mathcal{Q} - \Phi\mathcal{Q}(\delta\mathcal{Q}/\mathcal{Q} - 4\delta W_b)$. Evaluating this at $\rho = \rho_{\scriptscriptstyle N} = 0$ and[4] $\rho = \rho_S = \pi L + \delta\rho_S$, and keeping in mind that it is $-\rho$ that is the outward direction for the south brane, gives

$$\begin{aligned}
\alpha\left[\delta B_0 - \frac{10W_1}{3} + \frac{3}{4}\left(\frac{\delta\mathcal{Q}}{\mathcal{Q}}\right)\right] &= \alpha\left[-2\delta B_0 - \frac{10W_1}{3} - \frac{3\kappa^2\mathcal{Q}\delta\Phi_{\rm tot}}{4\pi\alpha}\right] = -\frac{\kappa^2\delta T_{\scriptscriptstyle N}}{2\pi} \\
\alpha\left[\delta B_0 + \frac{10W_1}{3} + \frac{3}{4}\left(\frac{\delta\mathcal{Q}}{\mathcal{Q}}\right)\right] &= \alpha\left[-2\delta B_0 + \frac{10W_1}{3} - \frac{3\kappa^2\mathcal{Q}\delta\Phi_{\rm tot}}{4\pi\alpha}\right] = -\frac{\kappa^2\delta T_{\scriptscriptstyle S}}{2\pi}\,.
\end{aligned} \tag{4.38}$$

When solving these we may approximate $\delta T_b \simeq \delta\tau_b - \mathcal{Q}\,\delta\Phi_b$, which involves dropping the back-reaction of those terms proportional to $\delta\mathcal{Q}/\mathcal{Q}$ and δW_b in $\kappa^2\delta T_b/2\pi$. This neglect is justified because their relative contribution is of order $\kappa^2\mathcal{Q}\Phi/2\pi$, which must be small to justify our classical treatment of gravity. The solution found within this approximation to eqs. (4.38) then is

$$\begin{aligned}
\delta B_0 &= \frac{\kappa^2}{8\pi\alpha}\Big[\delta T_{\scriptscriptstyle N} + \delta T_{\scriptscriptstyle S} - 3\mathcal{Q}(\delta\Phi_{\scriptscriptstyle N} + \delta\Phi_{\scriptscriptstyle S})\Big] \\
W_1 &= \frac{3\kappa^2}{40\pi\alpha}\Big(\delta T_{\scriptscriptstyle N} - \delta T_{\scriptscriptstyle S}\Big)\,,
\end{aligned} \tag{4.39}$$

Using the above,

$$\begin{aligned}
\frac{\delta L}{L} = -\frac{3}{4}\left(\frac{\delta\mathcal{Q}}{\mathcal{Q}}\right) &= \frac{3\kappa^2}{8\pi\alpha}\Big[(\delta T_{\scriptscriptstyle N} + \delta T_{\scriptscriptstyle S}) - \mathcal{Q}(\delta\Phi_{\scriptscriptstyle N} + \delta\Phi_{\scriptscriptstyle S})\Big] \\
&= \frac{3\kappa^2}{8\pi\alpha}\Big[(\delta\tau_{\scriptscriptstyle N} + \delta\tau_{\scriptscriptstyle S}) - 2\mathcal{Q}(\delta\Phi_{\scriptscriptstyle N} + \delta\Phi_{\scriptscriptstyle S})\Big]\,.
\end{aligned} \tag{4.40}$$

[4] To leading order $\delta\rho_S$ does not contribute, and is only mentioned for completeness.

On-brane geometry and the view from 4D

The curvature of the induced geometry on the branes comes from the linearized ($\mu\nu$) Einstein equation, which — using $e^{-2B}F^2_{\rho\theta} = \mathcal{Q}^2 e^{-8W}$ — gives

$$\begin{aligned}\hat{\mathcal{R}} &= -4\left[\delta W'' + \frac{\delta W'}{L}\,\cot\left(\frac{\rho}{L}\right)\right] - \frac{8\delta W}{L^2} + \frac{2}{L^2}\left(\frac{\delta\mathcal{Q}}{\mathcal{Q}}\right) = \frac{2}{L^2}\left(\frac{\delta\mathcal{Q}}{\mathcal{Q}}\right) \\ &= -\frac{\kappa^2}{\pi\alpha L^2}\Big[(\delta\tau_N + \delta\tau_S) - 2\mathcal{Q}(\delta\Phi_N + \delta\Phi_S)\Big]\,. \end{aligned} \tag{4.41}$$

From the point of view of a 4D observer localized on the brane this curvature would be interpreted as being due to a 4D energy density, $\varrho_{\rm eff}$. Since the 4D gravitational coupling, $\kappa_4^2 = 8\pi G_N$, is related to κ by

$$\frac{1}{\kappa_4^2} = \frac{2\pi\alpha L}{\kappa^2}\int_0^{\pi L} {\rm d}\rho\;\sin\left(\frac{\rho}{L}\right) = \frac{4\pi\alpha L^2}{\kappa^2}\,, \tag{4.42}$$

we have

$$\varrho_{\rm eff} = -\frac{\hat{\mathcal{R}}}{4\kappa_4^2} = \delta\tau_N + \delta\tau_S - 2\mathcal{Q}(\delta\Phi_N + \delta\Phi_S)\,. \tag{4.43}$$

Notice that this agrees with the naive expectation $\varrho_{\rm eff} = \delta\tau_N + \delta\tau_S$ in the absence of fluxes on the brane. The same is *not* true in the presence of brane fluxes, however, since the final result for $\varrho_{\rm eff}$ differs from $\delta T_N + \delta T_S = \delta\tau_N + \delta\tau_S - \mathcal{Q}(\delta\Phi_N + \delta\Phi_S)$. As the above calculation shows, $\varrho_{\rm eff} \propto -\delta\mathcal{Q}/\mathcal{Q}$ and so the energy cost of the perturbation arises from the change of flux required by the flux-quantization condition in response to the back-reaction of the branes on the bulk geometry. Since the flux is homogeneous across the extra dimensions, its energy cost is expensive since it scales with the volume. Localizing some of the flux into the branes reduces this extensive energy cost.[5]

[5]Of course, this possibility of back-reaction competing with brane tensions is already suggested by the complete absence of on-brane curvature in the initial rugby ball solution

The comparative importance of such back-reaction effects depends on the relative size of the two brane energy scales $\delta\tau_b$ and $\mathcal{Q}\delta\Phi_b = \delta\Phi_b/\kappa L$. If both $\delta\tau_b$ and $\delta\Phi_b$ are set by the same scale — *i.e.* $\tau_b \sim \Lambda^4$ and $\delta\Phi_b \sim \Lambda$ — then $\delta\tau_b \sim \mathcal{Q}\delta\Phi_b$ when $\Lambda \simeq \Lambda_\star := (\kappa L)^{-1/3}$. For Λ smaller than this the $\mathcal{Q}\delta\Phi_b$ term dominates, while $\delta\tau_b$ is the larger of the two when $\Lambda > \Lambda_\star$. (For a similar setup with n transverse dimensions this crossover would occur when $\Lambda \simeq (\kappa L)^{-2/(4+n)}$.) Although $\kappa\Lambda^2$ must be much smaller than one to justify semiclassical methods, for fundamental objects it is comparatively large (*e.g.* for D-branes $\kappa\Lambda^2$ is of order the string coupling, $g_s \simeq 0.01$ say), and so the tension contribution can therefore dominate. The flux contribution instead can dominate for lower-tension objects.

It is instructive to check this calculation by directly evaluating the low-energy potential through dimensional reduction of the 6D theory in the classical approximation. A general formula for this is computed (including brane back-reaction) in ref. (4.15), and when this is specialized to linear perturbations about a rugby ball it evaluates to

$$V_{\text{eff}} = 2\pi \int_0^{\pi L} \mathrm{d}\rho \; e^{B+4W} \left\{ \frac{1}{2\kappa^2} \left[\frac{8W'}{L} \cot\left(\frac{\rho}{L}\right) \right] - \frac{1}{4}(\mathcal{Q}+\delta\mathcal{Q})^2 e^{-8W} + \frac{\Lambda}{2} \right\} , \tag{4.44}$$

with the W' term arising from the extra-dimensional curvature and the $(\mathcal{Q} + \delta\mathcal{Q})^2$ term coming from the bulk Maxwell action. In the present instance all terms involving δW in this expression turn out to be proportional to $\sin(\rho/L)\cos(\rho/L)$ to linear order in the perturbations, and so integrate to zero and do not contribute to V_{eff}. Keeping in mind the background relations

despite the presence of the initial equal brane tensions, T.

$2\Lambda = \mathcal{Q}^2$ and $\kappa L \mathcal{Q} = 1$, the result therefore simplifies to

$$\begin{aligned} V_{\text{eff}} &\simeq 2\pi \int_0^{\pi L} \mathrm{d}\rho \, e^{B_0} \left\{ \left(-\frac{\mathcal{Q}^2}{4} + \frac{\Lambda}{2} \right) (1 + \delta B) - \frac{\mathcal{Q} \delta \mathcal{Q}}{2} \right\} \\ &= -\frac{2\pi\alpha}{\kappa^2} \left(\frac{\delta \mathcal{Q}}{\mathcal{Q}} \right) = -\frac{\pi \alpha L^2 \hat{\mathcal{R}}}{\kappa^2} = \varrho_{\text{eff}} \,, \end{aligned} \tag{4.45}$$

showing the equivalence between the 4D and 6D perspectives.

4.3.3 Perturbations II: the Einstein-Maxwell-axion case

In this section we generalize the previous discussion to consider branes and fluxes that can depend on ϕ. This allows us to follow how couplings to the brane lift the flat direction associated with the shift symmetry of the bulk theory, and so to see how the scalar zero mode, φ_0, becomes stabilized at a specific value, $\varphi_0 = \varphi_\star$.

It is instructive to ask how this stabilization happens from the point of view of the full six-dimensional theory. To this end imagine trying to integrate the field equations to obtain the bulk configuration that interpolates between the two branes. Specializing to solutions that are both axially symmetric in the transverse directions and maximally symmetric in the on-brane dimensions we seek bulk profiles as a function only of ρ, starting with initial conditions set by matching to the brane at $\rho = \rho_N$ (say). If this matching completely specified all of the fields and their first derivatives at this brane then the solution obtained by integration would completely determine the value of the fields and their radial derivatives at the second brane, and in general these need not be consistent with what would be obtained by matching to this second brane.

But matching to the first brane typically specifies only the derivatives of the fields at the first brane, and not separately the values of the fields themselves.[6] Consequently the values of the fields at the first brane can be adjusted to try to allow the solution to properly match to the properties of the second brane. It is in this way that the system can force $\varphi_0 = \varphi_\star$ if the brane actions do not preserve the bulk shift symmetry.

From the perspective of a low-energy 4D observer the energy cost responsible for this stabilization looks like a scalar potential for φ_0, and our goal in what follows is to compute its shape for configurations in the immediate neighborhood of $\varphi_0 = \varphi_\star$. As the above arguments show, a classical solution subject to our assumed ansatz should not exist as soon as $\varphi_0 \neq \varphi_\star$, and the part of the ansatz responsible is likely to be the condition of maximal symmetry in the on-brane directions. No maximally symmetric solution should exist for $\varphi_0 \neq \varphi_\star$ because this indicates the onset of time evolution in response to no longer sitting at the minimum of the 4D effective potential. (This development of time dependence in response to changes in the properties of mutually gravitating brane sources resembles what happens for a system of electric charges, which generically becomes time dependent when an equilibrium arrangement is disturbed).

Rather than trying to solve for the system's time-dependent response (see however refs. (4.23; 4.24)) when $\varphi_0 \neq \varphi_\star$ we instead focus on computing features of the low-energy potential that is responsible. We do so – in both the 4D and 6D theories – through the artifice of turning on a current that stops the time evolution, and so removes the obstruction to static solutions.

[6]Since in general the bulk fields can diverge at the brane positions, this argument should more precisely be made very near to, and not precisely at, the position of the first brane.

Since sufficiently small deviations from equilibrium should precipitate motion only along the low-energy flat directions, it suffices to couple the current only to these low-energy modes. By computing the amount of current required as a function of the low-energy scalar mode, we may Legendre transform in the usual way to determine the shape of the effective potential.

Linearized equations with currents

Since we work within a linearized approximation, we perturb the brane properties in a way that does not drive the low-energy scalar fields far from their initial values. This can be achieved if the potential energy of each brane has a minimum as a function of ϕ, and although the two branes need not agree on where this minimum is they should not disagree by too much. It suffices therefore to study the brane tensions in the vicinity of these minima, restricting to quadratic expansions in powers of ϕ. Writing $T_b = T + \delta T_b(\phi)$, we take

$$\delta T_b(\phi) = T_{b0} + \frac{T_{b2}}{2} \left(\phi - \hat{v}_b\right)^2 , \tag{4.46}$$

with $b = N$ and S and $T_b(\phi) = \tau_b(\phi) - \mathcal{Q}\, \Phi_b(\phi)$, as before. For technical reasons — see Appendix B.4 — we require that the minimum of the sum of the brane actions, $\sum_b T_b' = 0$, agrees with the minimum of the sum of the fluxes, $\sum_b \Phi_b' = 0$. When $\hat{v}_N \neq \hat{v}_S$ the two branes differ on which value for ϕ they prefer, and we assume that this difference is not so large as to invalidate a linearized integration of the field equations.

In the higher-dimensional theory the current used to stabilize the solutions against rolling is[7]

$$S_J := -\int \mathrm{d}^6x\,\sqrt{-g}\;J(\phi - \varphi_\star)\,, \tag{4.47}$$

and, to the extent that it suffices to stabilize just the KK zero mode, J can be taken to be independent of the extra-dimensional coordinates ρ and θ. In the presence of such a current the field equation for ϕ becomes

$$\partial_M\Big(\sqrt{-g}\;g^{MN}\partial_N\phi\Big) = \sqrt{-g}\;\kappa_a^2 J\,. \tag{4.48}$$

Perturbing around the rugby ball solution, our interest is in the lowest nontrivial order in J, corresponding to situations where the brane tensions only cause controllably small changes in ϕ. In this case the leading approximation to the axion fluctuation is obtained by solving eq. (4.48) with the metric evaluated at the rugby ball background,

$$\left[\sin\left(\frac{\rho}{L}\right)\delta\phi'\right]' = \frac{\epsilon_J}{L^2}\,\sin\left(\frac{\rho}{L}\right), \tag{4.49}$$

where the last equality defines the dimensionless current, assumed small: $\epsilon_J := \kappa_a^2 J L^2 \ll 1$.

The Maxwell equation is unchanged by the current, and integrates to give

$$F_{\rho\theta} = A'_\theta = (\mathcal{Q} + \delta\mathcal{Q})e^{-4W+B}\,. \tag{4.50}$$

[7]The additional coupling $J\,\varphi_\star$ is here inserted to ensure that J couples only to the light scalar mode, $\delta\varphi = \varphi_0 - \varphi_\star$, at the linearized level, and not also to the metric fluctuations. We keep this term even though, as discussed in Appendix B.3, for axions much lighter than the KK scale, $m \ll 1/L$, a misalignment that included metric modes only introduces subdominant contributions to the axion mass, of order $\delta m^2 \simeq m^4 L^2$.

Recall that both W and B in this expression include perturbations.

Including the stress energy from the current interaction, the linearized Einstein equations become

$$\begin{aligned}
\frac{\hat{\mathcal{R}}}{4} + \delta W'' + \frac{\delta W'}{L}\cot\left(\frac{\rho}{L}\right) - \frac{1}{2L^2}\left(\frac{\delta\mathcal{Q}}{\mathcal{Q}}\right) + \frac{2\delta W}{L^2} + \frac{\lambda^2\epsilon_J(\phi-\varphi_\star)}{2L^2} &= 0 \\
4\delta W'' + \delta B'' + \frac{2\delta B'}{L}\cot\left(\frac{\rho}{L}\right) + \lambda^2(\phi')^2 + \frac{3}{2L^2}\left(\frac{\delta\mathcal{Q}}{\mathcal{Q}}\right) & \\
-\frac{6\delta W}{L^2} + \frac{\lambda^2\epsilon_J(\phi-\varphi_\star)}{2L^2} &= 0 \\
\delta B'' + \frac{2\delta B'}{L}\cot\left(\frac{\rho}{L}\right) + 4\delta W'\cot\left(\frac{\rho}{L}\right) + \frac{3}{2L^2}\left(\frac{\delta\mathcal{Q}}{\mathcal{Q}}\right) & \\
-\frac{6\delta W}{L^2} + \frac{\lambda^2\epsilon_J(\phi-\varphi_\star)}{2L^2} &= 0\,,
\end{aligned} \tag{4.51}$$

where, as before, $\lambda = \kappa/\kappa_a = \kappa F$.

Linearized solutions

We now solve those equations to order ϵ_J. Because $(\phi')^2$ is order ϵ_J^2, to order ϵ_J the equation for the warping is unchanged from previous sections, giving the solution

$$\delta W = W_1 \cos\left(\frac{\rho}{L}\right)\,. \tag{4.52}$$

The current forces the axion to acquire a profile (which is desirable because this allows it to satisfy the new boundary conditions at the brane positions). The perturbed axion equation integrates to give

$$\delta\phi = \varphi_0 + \varphi_1 \ln\left|\frac{1-\cos(\rho/L)}{\sin(\rho/L)}\right| - \epsilon_J \ln\left|\sin\left(\frac{\rho}{L}\right)\right|\,, \tag{4.53}$$

with φ_1 and φ_0 integration constants. Since φ_0 parameterizes the (previously) flat direction we solve ϵ_J and all other integration constants in terms of it and brane properties.

Using these in the Einstein equations as before gives δB as the solution to

$$\frac{\left[\delta B' \sin^2(\rho/L)\right]'}{\sin^2(\rho/L)} = \frac{10W_1}{L^2}\cos\left(\frac{\rho}{L}\right) - \frac{3}{2L^2}\left(\frac{\delta\mathcal{Q}}{\mathcal{Q}}\right) - \frac{\lambda^2\epsilon_J(\varphi_0-\varphi_\star)}{2L^2}, \quad (4.54)$$

giving

$$\delta B = \left[\frac{3}{4}\left(\frac{\delta\mathcal{Q}}{\mathcal{Q}}\right) + \frac{\lambda^2\epsilon_J}{4}(\varphi_0-\varphi_\star)\right]\frac{\rho}{L}\cot\left(\frac{\rho}{L}\right) - \frac{10W_1}{3}\cos\left(\frac{\rho}{L}\right) + \delta B_0 . \quad (4.55)$$

Using this in the linearized flux-quantization condition finally gives a relation between $\delta\mathcal{Q}$ and δB_0,

$$\frac{\delta\mathcal{Q}}{\mathcal{Q}} = \lambda^2\epsilon_J(\varphi_0-\varphi_\star) - 4\delta B_0 - \frac{\kappa^2\mathcal{Q}}{\pi\alpha}\left(\delta\Phi_N + \delta\Phi_S\right) . \quad (4.56)$$

As before, the remaining integration constants — in this case φ_1, W_1 and δB_0 — are determined by solving the matching conditions at the brane positions. The fractional change in the proper distance between the source branes becomes

$$\frac{\delta L}{L} = -\frac{3}{4}\left(\frac{\delta\mathcal{Q}}{\mathcal{Q}}\right) - \frac{\lambda^2\epsilon_J}{4}(\varphi_0-\varphi_\star) . \quad (4.57)$$

Matching to branes

The matching condition for the axion at each brane is

$$\lim_{\rho\to\rho_b} \alpha\rho\,\phi' = \left.\frac{\kappa_a^2 T_b'(\phi)}{2\pi}\right|_{\rho\to\rho_b} , \quad (4.58)$$

but an additional complication arises because the right-hand side is ill defined due to the divergence in $\phi(\rho)$ at the brane positions. This requires a renormalization of the parameters defining the brane potentials (4.16; 4.25). To this end first regularize the matching condition by evaluating it at $\rho = \rho_N + \varepsilon_N$ and $\rho = \rho_S - \varepsilon_S$. Then define the renormalized parameters

$$\begin{aligned} v_N &:= \hat{v}_N - (\varphi_1 - \epsilon_J) \ln \left(\frac{\varepsilon_N}{L} \right) + \varphi_1 \ln 2 \\ v_S &:= \hat{v}_S + (\varphi_1 + \epsilon_J) \ln \left(\frac{\varepsilon_S}{L} \right) - \varphi_1 \ln 2 \,. \end{aligned} \tag{4.59}$$

Because the field profile satisfies

$$\phi(\varepsilon_N) = \varphi_0 + \varphi_1 \ln \left(\frac{\varepsilon_N}{2L} \right) - \epsilon_J \ln \left(\frac{\varepsilon_N}{L} \right) \,, \tag{4.60}$$

(and a similar result at $\rho = \pi L - \varepsilon_S$), these definitions ensure

$$\phi(\varepsilon_b) - \hat{v}_b = \varphi_0 - v_b \,, \tag{4.61}$$

and so remain finite in the limit $\varepsilon_b \to 0$. This makes the derivative of the tension (and the tension itself) finite when evaluated on the brane. Because the fluxes are also written in terms of $\phi - \hat{v}_b$, they do not need a separate renormalization.

In terms of renormalized quantities the matching conditions directly relate the integration constants,

$$\begin{aligned} \alpha(\varphi_1 - \epsilon_J) &= \frac{\kappa_a^2 \delta T'_N(\varphi_0)}{2\pi} = \left(\frac{\kappa_a^2 T_{N2}}{2\pi} \right) (\varphi_0 - v_N) \\ -\alpha(\varphi_1 + \epsilon_J) &= \frac{\kappa_a^2 \delta T'_S(\varphi_0)}{2\pi} = \left(\frac{\kappa_a^2 T_{S2}}{2\pi} \right) (\varphi_0 - v_S) \,, \end{aligned} \tag{4.62}$$

allowing the inference

$$\begin{aligned}\epsilon_J &= -\frac{\kappa_a^2}{4\pi\alpha}\Big[\delta T_N'(\varphi_0)+\delta T_S'(\varphi_0)\Big] = -\frac{\kappa_a^2}{4\pi\alpha}\Big[(T_{N2}+T_{S2})\,\varphi_0 - T_{N2}\,v_N - T_{S2}\,v_S\Big]\\ \varphi_1 &= \frac{\kappa_a^2}{4\pi\alpha}\Big[\delta T_N'(\varphi_0)-\delta T_S'(\varphi_0)\Big] = \frac{\kappa_a^2}{4\pi\alpha}\Big[(T_{N2}-T_{S2})\,\varphi_0 + T_{S2}\,v_S - T_{N2}\,v_N\Big]\,.\end{aligned} \tag{4.63}$$

The first of these identifies the field value where the flat direction gets stabilized, $\varphi_0 = \varphi_\star$, since this is the solution that corresponds to zero external current. The condition $\epsilon_J(\varphi_\star) = 0$ implies $\varphi_\star$ satisfies

$$\delta T_N'(\varphi_\star) + \delta T_S'(\varphi_\star) = 0\,, \tag{4.64}$$

and so when $\delta T_b(\varphi_0) = T_{b0} + \frac{1}{2}\,T_{b2}\,(\varphi_0 - v_b)^2$

$$\varphi_\star = \frac{T_{N2}v_N + T_{S2}v_S}{T_{N2}+T_{S2}}\,. \tag{4.65}$$

We again fix δB_0 and W_1 from the last of the matching conditions, eqs. (4.22),

$$\left(e^B\right)'_{\rho_b} = 1 - \frac{\kappa^2}{2\pi}\Big[T + \delta T_b(\varphi_0)\Big]\,, \tag{4.66}$$

which uses $\mathcal{U}_b(\varphi_0) \simeq 0$, as can be inferred either from the second of eqs. (4.22), or by solving eq. (4.24) to linear order in $\kappa^2 T_b$. As before this leads to the conditions

$$\begin{aligned}\alpha\left[-2\delta B_0-\frac{10W_1}{3}-\frac{3\kappa^2\mathcal{Q}\delta\Phi_{\rm tot}}{4\pi\alpha}+\lambda^2\epsilon_{\scriptscriptstyle J}(\varphi_0-\varphi_\star)\right] &= -\left(\frac{\kappa^2}{2\pi}\right)\delta T_{\scriptscriptstyle N}(\varphi_0)\\ \alpha\left[-2\delta B_0+\frac{10W_1}{3}-\frac{3\kappa^2\mathcal{Q}\delta\Phi_{\rm tot}}{4\pi\alpha}+\lambda^2\epsilon_{\scriptscriptstyle J}(\varphi_0-\varphi_\star)\right] &= -\left(\frac{\kappa^2}{2\pi}\right)\delta T_{\scriptscriptstyle S}(\varphi_0)\,.\end{aligned} \tag{4.67}$$

The result is

$$W_1=\frac{3\kappa^2}{40\pi\alpha}\Big[\delta T_{\scriptscriptstyle N}(\varphi_0)-\delta T_{\scriptscriptstyle S}(\varphi_0)\Big]\,, \tag{4.68}$$

and

$$\begin{aligned}\delta B_0 &= \frac{\lambda^2\epsilon_{\scriptscriptstyle J}}{2}(\varphi_0-\varphi_\star) \qquad (4.69)\\ &\quad+\frac{\kappa^2}{8\pi\alpha}\Big\{\delta T_{\scriptscriptstyle N}(\varphi_0)+\delta T_{\scriptscriptstyle S}(\varphi_0)-3\mathcal{Q}\Big[\delta\Phi_{\scriptscriptstyle N}(\varphi_0)+\delta\Phi_{\scriptscriptstyle S}(\varphi_0)\Big]\Big\}\\ &= \frac{\kappa^2}{8\pi\alpha}\left\{\sum_{b=N,S}\Big[\delta T_b(\varphi_0)-(\varphi_0-\varphi_\star)\delta T_b'(\varphi_0)\Big]-3\mathcal{Q}\delta\Phi_{\rm tot}(\varphi_0)\right\}\,,\end{aligned}$$

where the final line eliminates $\epsilon_{\scriptscriptstyle J}$ using eq. (4.63) and $\lambda^2\kappa_a^2=\kappa^2$. This implies

$$\begin{aligned}\frac{\delta\mathcal{Q}}{\mathcal{Q}} &= -\lambda^2\epsilon_{\scriptscriptstyle J}(\varphi_0-\varphi_\star)-\frac{\kappa^2}{2\pi\alpha}\Big\{\delta T_{\scriptscriptstyle N}(\varphi_0)+\delta T_{\scriptscriptstyle S}(\varphi_0)-\mathcal{Q}\delta\Phi_{\rm tot}(\varphi_0)\Big\} \qquad (4.70)\\ &= -\frac{\kappa^2}{2\pi\alpha}\left\{\sum_{b=N,S}\left[\delta T_b(\varphi_0)-\frac{1}{2}(\varphi_0-\varphi_\star)\,\delta T_b'(\varphi_0)\right]-\mathcal{Q}\delta\Phi_{\rm tot}(\varphi_0)\right\}\,.\end{aligned}$$

In terms of these the fractional change in the proper distance between branes becomes

$$\begin{aligned}\frac{\delta L}{L} &= -\frac{3}{4}\left(\frac{\delta\mathcal{Q}}{\mathcal{Q}}\right)-\frac{\lambda^2\epsilon_{\scriptscriptstyle J}}{4}(\varphi_0-\varphi_\star)\\ &= \frac{\lambda^2\epsilon_{\scriptscriptstyle J}}{2}(\varphi_0-\varphi_\star)+\frac{3\kappa^2}{8\pi\alpha}\Big\{\delta T_{\scriptscriptstyle N}(\varphi_0)+\delta T_{\scriptscriptstyle S}(\varphi_0)-\mathcal{Q}\delta\Phi_{\rm tot}(\varphi_0)\Big\}\end{aligned}$$

$$= \frac{3\kappa^2}{8\pi\alpha}\left\{\sum_{b=N,S}\left[\delta T_b(\varphi_0) - \frac{1}{3}(\varphi_0 - \varphi_\star)\,\delta T_b'(\varphi_0)\right] - \mathcal{Q}\delta\Phi_{\rm tot}(\varphi_0)\right\} \tag{4.71}$$

On-brane geometry and 4D effective potential

The linearized Einstein equation yields the following on-brane curvature

$$\begin{aligned}\hat{\mathcal{R}} &= -4\left[\delta W'' + \frac{\delta W'}{L}\,\cot\left(\frac{\rho}{L}\right)\right] - \frac{8\delta W}{L^2} + \frac{2}{L^2}\left(\frac{\delta\mathcal{Q}}{\mathcal{Q}}\right) - \frac{2\lambda^2\epsilon_J}{L^2}(\varphi_0 - \varphi_\star)\\ &= \frac{2}{L^2}\left[\frac{\delta\mathcal{Q}}{\mathcal{Q}} - \lambda^2\epsilon_J(\varphi_0 - \varphi_\star)\right] \qquad (4.72)\\ &= -\frac{\kappa^2}{\pi\alpha L^2}\left\{\sum_{b=N,S}\left[\delta T_b(\varphi_0) - (\varphi_0 - \varphi_\star)\,\delta T_b'(\varphi_0)\right] - \mathcal{Q}\delta\Phi_{\rm tot}(\varphi_0)\right\}.\end{aligned}$$

The presence of the current J complicates the determination of the effective potential, $V_{\rm eff}(\varphi)$, in the low-energy 4D theory. The appropriate matching calculation turns on a current in the low-energy theory as well, and asks what potential reproduces the previous results for $\hat{\mathcal{R}}$ and $\varphi_\star$.

The most general action for the 4D effective theory involving only the 4D metric, $\hat{g}_{\mu\nu}$, and the low-energy scalar, φ, is (up to two derivatives)

$$S_{\rm eff} = -\int {\rm d}^4x\,\sqrt{-\hat{g}}\,\left[\frac{1}{2\kappa_4^2}\,\hat{g}^{\mu\nu}\Big(\hat{\mathcal{R}}_{\mu\nu} + \lambda^2\,\partial_\mu\varphi\,\partial_\nu\varphi\Big) + V_{\rm eff}(\varphi) + j(\varphi - \varphi_\star)\right], \tag{4.73}$$

where j is the low-energy current, κ_4 is given by eq. (4.42) and the 4D axion decay constant is

$$f^2 = 4\pi\alpha L^2 F^4 = \frac{4\pi\alpha L^2}{\kappa_a^2} = \frac{\lambda^2}{\kappa_4^2}\,. \tag{4.74}$$

We couple the current j to the difference $\varphi - \varphi_\star$ purely as a matter of later convenience.

The equations of motion, specialized to constant scalar fields, $\varphi = \varphi_0$, and to maximally symmetric geometries, are

$$j = -V'_{\rm eff}(\varphi_0) \quad \text{and} \quad \frac{\hat{\mathcal{R}}}{4\kappa_4^2} = -j(\varphi_0 - \varphi_\star) - V_{\rm eff}(\varphi_0)\,, \tag{4.75}$$

from which j can be eliminated to give

$$(\varphi_0 - \varphi_\star)V'_{\rm eff}(\varphi_0) - V_{\rm eff}(\varphi_0) = \frac{\hat{\mathcal{R}}}{4\kappa_4^2} = \frac{\pi\alpha L^2\hat{\mathcal{R}}}{\kappa^2}\,. \tag{4.76}$$

The functional form for the potential $V_{\rm eff}$ is determined by requiring eq. (4.76) to reproduce the curvature, eq. (4.72), predicted by the 6D theory, regarded as a function of φ_0. This can be obtained by regarding eq. (4.76) as a differential equation for $V_{\rm eff}$, whose solution is

$$\begin{aligned} V_{\rm eff}(\varphi_0) &= (\varphi_0 - \varphi_\star)\int \frac{{\rm d}\varphi}{(\varphi-\varphi_\star)^2}\left[\frac{\pi\alpha L^2\hat{\mathcal{R}}(\varphi)}{\kappa^2}\right] \\ &= \delta T_N(\varphi_0) + \delta T_S(\varphi_0) \\ &\quad +(\varphi_0 - \varphi_\star)\left\{\int_{\varphi_\star}^{\varphi_0}{\rm d}\varphi\left[\frac{\mathcal{Q}\delta\Phi_{\rm tot}(\varphi)}{(\varphi-\varphi_\star)^2}\right] - \lim_{\varphi\to\varphi_\star}\left[\frac{\mathcal{Q}\delta\Phi_{\rm tot}(\varphi)}{\varphi-\varphi_\star}\right]\right\}\,. \end{aligned} \tag{4.77}$$

In general, the coefficient of the term linear in $(\varphi_0 - \varphi_\star)$ in $V_{\rm eff}$ is the integration constant, which is fixed in the second equality of eq. (4.77) by requiring $V'_{\rm eff}(\varphi_\star) = 0$, for $\varphi_\star$ as given in the 6D theory by eq. (4.64).

Two physical parameters of particular interest here are: (i) the effective on-brane cosmological constant, $\varrho_{\rm eff} := V_{\rm eff}(\varphi_\star)$, and the low-energy scalar mass, $m_\varphi^2 := V''_{\rm eff}(\varphi_\star)/f^2$. The first of these evaluates to

$$\begin{aligned} \varrho_{\rm eff} := V_{\rm eff}(\varphi_\star) &= \delta T_N(\varphi_\star) + \delta T_S(\varphi_\star) - \mathcal{Q}\Big[\delta\Phi_N(\varphi_\star) + \delta\Phi_S(\varphi_\star)\Big] \\ &= \delta\tau_N(\varphi_\star) + \delta\tau_S(\varphi_\star) - 2\mathcal{Q}\Big[\delta\Phi_N(\varphi_\star) + \delta\Phi_S(\varphi_\star)\Big]\,, \end{aligned} \tag{4.78}$$

whose value agrees with our earlier Einstein-Maxwell calculation when $\delta T_b' = 0$. Similarly[8]

$$m_\varphi^2 = \frac{V_{\rm eff}''(\varphi_\star)}{f^2} = \frac{1}{f^2}\left[\delta T_{\scriptscriptstyle N}''(\varphi_\star) + \delta T_{\scriptscriptstyle S}''(\varphi_\star) + \lim_{\varphi\to\varphi_\star}\left(\frac{\mathcal{Q}\delta\Phi_{\scriptscriptstyle N}'(\varphi) + \mathcal{Q}\delta\Phi_{\scriptscriptstyle S}'(\varphi)}{\varphi - \varphi_\star}\right)\right]. \tag{4.79}$$

In the absence of brane fluxes the effective potential is simply the sum of the brane potentials. But although the low-energy scalar always stabilizes at the stationary points of $\sum_b \delta T_b$, the scalar masses and 4D cosmological constant in general differ from what would be expected based just on $\sum_b \delta T_b$.

Comparison with dimensional reduction

As before, we can also evaluate $V_{\rm eff}$ at the classical level by direct dimensional reduction, which gives the integral

$$\begin{aligned} V_{\rm eff} &= 2\pi \int_0^{\pi L} {\rm d}\rho\; e^{B+4W}\left\{\frac{1}{2\kappa^2}\left[\frac{8W'}{L}\cot\left(\frac{\rho}{L}\right)\right] - \frac{1}{4}(\mathcal{Q}+\delta\mathcal{Q})^2 e^{-8W} \right.\\ &\qquad\qquad \left. +\frac{1}{2}\left(\Lambda + J\phi\right)\right\} \\ &= \frac{\pi\alpha L^2}{\kappa^2}\int_0^{\pi L}\frac{{\rm d}\rho}{L}\sin\left(\frac{\rho}{L}\right)\left[-\kappa^2\mathcal{Q}^2\left(\frac{\delta\mathcal{Q}}{\mathcal{Q}}\right) + \kappa^2 J\phi\right] \\ &= -\frac{2\pi\alpha}{\kappa^2}\left(\frac{\delta\mathcal{Q}}{\mathcal{Q}} - \lambda^2\epsilon_{\scriptscriptstyle J}\varphi_0\right) = -\frac{\pi\alpha L^2}{\kappa^2}\,\hat{\mathcal{R}}\,, \end{aligned} \tag{4.80}$$

in agreement with the 6D calculation above.

[8]Notice that m_φ^2 diverges if $\Phi_{\rm tot}$ vanishes linearly with $\varphi - \varphi_\star$. In this case the lowest energy KK mode is not properly captured by our ansatz — see Appendix B.4 — and so the low-energy potential misidentifies its size. It is for this reason that we require $\Phi_{\rm tot}$ and $T_{\rm tot}$ to agree on the value $\varphi_\star$ at which they are minimized.

4.4 Applications and special cases

This section seeks to illustrate the physical implications of the previous section's results by exploring several instructive examples.

4.4.1 Bulk response to stabilizing potentials

Consider first the response of the bulk geometry and the properties of the low-energy 4D scalar-tensor theory, distinguishing the cases where the two brane agree on, or compete for, the field value where the low-energy scalar is stabilized.

Shared minima

As an example where the fluxes and tensions on both branes are minimized at a common value of φ_0, consider the special case that all the fluxes and tensions have the following expansion $\tau_b = T + \delta\tau_b(\phi)$ and $\Phi_b = \Phi + \delta\Phi_b(\phi)$ with

$$\delta\tau_b(\varphi_0) = \delta\tau_{b0} + \frac{\delta\tau_{b2}}{2}(\varphi_0 - v)^2 \quad \text{and} \quad \delta\Phi_b(\varphi_0) = \delta\Phi_{b0} + \frac{\delta\Phi_{b2}}{2}(\varphi_0 - v)^2 \,, \tag{4.81}$$

and so

$$\delta T_b(\varphi_0) = \delta\tau_b(\varphi_0) - \mathcal{Q}\delta\Phi_b(\varphi_0) = \delta T_{b0} + \frac{\delta T_{b2}}{2}(\varphi_0 - v)^2 \,, \tag{4.82}$$

where $\delta T_{bk} = \delta\tau_{bk} - \mathcal{Q}\,\delta\Phi_{bk}$ are constants.

The condition fixing $\varphi_\star$ in this case is $\varphi_\star = v$, as one would expect. Inserting this into the formulae for the relative warping of the two branes and the fractional change in inter-brane distance gives

$$\begin{aligned} \delta W_{N} - \delta W_{S} &= \frac{3\kappa^2}{20\pi\alpha}\Big[\delta T_{N0} - \delta T_{S0}\Big] \\ \frac{\delta L}{L} &= \frac{3\kappa^2}{8\pi\alpha}\left[\Big(\delta T_{N0} + \delta T_{S0}\Big) - \mathcal{Q}\Big(\delta\Phi_{N0} + \delta\Phi_{S0}\Big)\right] . \end{aligned} \tag{4.83}$$

Similarly, the on-brane expressions for $\varrho_{\rm eff}$ and m_φ^2 yield

$$\begin{aligned} \varrho_{\rm eff} &= \delta T_{N0} + \delta\delta T_{S0} - \mathcal{Q}\Big(\delta\Phi_{N0} + \delta\Phi_{S0}\Big) \\ &= \delta\tau_{N0} + \delta\tau_{S0} - 2\mathcal{Q}\Big(\delta\Phi_{N0} + \delta\Phi_{S0}\Big) , \end{aligned} \tag{4.84}$$

and

$$\begin{aligned} m_\varphi^2 &= \frac{1}{f^2}\Big[\delta T_{N2} + \delta T_{S2} + \mathcal{Q}\Big(\delta\Phi_{N2} + \delta\Phi_{S2}\Big)\Big] \\ &= \frac{1}{f^2}\Big[\delta\tau_{N2} + \delta\tau_{S2}\Big] . \end{aligned} \tag{4.85}$$

Notice that only the second derivative of the tension, $\delta\tau_b''(\varphi_\star)$, contributes to the scalar mass, while both the tension, $\delta\tau_b(\varphi_\star)$, and the flux, $\delta\Phi_b(\varphi_\star)$, contribute to the on- and off-brane curvatures.

Brane competition

Consider next the case where the two branes each prefer φ_0 to stabilize at different values, causing them to compete in the value they ultimately determine. A representative example in this case is

$$\begin{aligned} \delta T_b(\varphi_0) &= \delta T_{b0} + \frac{\delta T_{b2}}{2}\,(\varphi_0 - v_b)^2 \\ \delta\Phi_b(\varphi_0) &= \delta\Phi_{b0} + \frac{\delta\Phi_{b2}}{2}\,(\varphi_0 - v_b)^2 . \end{aligned} \tag{4.86}$$

The stabilizing value for the scalar is now neither v_N nor v_S, but instead the intermediate value

$$\varphi_\star = \frac{\delta T_{N2} v_N + \delta T_{S2} v_S}{\delta T_{N2} + \delta T_{S2}}\,, \tag{4.87}$$

with the ratio $\delta T_{N2}/\delta T_{S2}$ controlling precisely where $\varphi_\star$ lies between v_N and v_S. Requiring $\delta\Phi_{\rm tot} = \delta\Phi_N + \delta\Phi_S$ also to have its minimum at the same value of $\varphi_\star$ then requires

$$\frac{\delta\Phi_{N2}}{\delta\Phi_{S2}} = \frac{\delta T_{N2}}{\delta T_{S2}}\,. \tag{4.88}$$

The extra-dimensional geometry satisfies

$$\delta W_N - \delta W_S = \frac{3\kappa^2}{20\pi\alpha}\left\{ \delta T_{N0} - \delta T_{S0} + \frac{\delta T_{N2}\delta T_{S2}}{2}\left[\frac{\delta T_{S2} - \delta T_{N2}}{(\delta T_{N2} + \delta T_{S2})^2}\right](v_N - v_S)^2 \right\}\,, \tag{4.89}$$

and $\delta L/L = (3\kappa^2 \varrho_{\rm eff}/8\pi\alpha)$, with the 4D vacuum energy given by

$$\begin{aligned} \varrho_{\rm eff} &= \delta T_{N0} + \delta T_{S0} - \mathcal{Q}\Big(\delta\Phi_{N0} + \delta\Phi_{S0}\Big) + \frac{1}{2}\left(\frac{\delta T_{N2}\delta T_{S2}}{\delta T_{N2} + \delta T_{S2}}\right)(v_N - v_S)^2 \\ &\qquad -\frac{\mathcal{Q}}{2}\left[\frac{\delta\Phi_{N2}\delta T_{S2}^2 + \delta\Phi_{S2}\delta T_{N2}^2}{(\delta T_{N2} + \delta T_{S2})^2}\right](v_N - v_S)^2 \\ &= \delta T_{N0} + \delta T_{S0} - \mathcal{Q}\Big(\delta\Phi_{N0} + \delta\Phi_{S0}\Big) \\ &\qquad + \frac{1}{2}\left[\frac{\delta T_{N2}(\delta T_{S2} - \mathcal{Q}\delta\Phi_{S2})}{\delta T_{N2} + \delta T_{S2}}\right](v_N - v_S)^2\,, \end{aligned} \tag{4.90}$$

where the last equality uses eq. (4.88). The result for m_φ^2 is again given by eq. (4.85). Because $\varphi_\star$ does not minimize the tension at either brane both the total tension and total flux get increased by positive amounts. These positive contributions then act oppositely in $\varrho_{\rm eff}$.

More complicated competitions can also occur if there is also symmetry-breaking in the bulk, in which case competition between the bulk and brane potentials can lead to self-localization (4.19).

Flux domination

A particular instance of the previous scenario corresponds to the case where $|\delta\tau_b| \ll |\mathcal{Q}\delta\Phi_b|$, since in this case $\delta T_b(\varphi) \simeq -\mathcal{Q}\delta\Phi_b(\varphi)$. Then the stabilizing value for the scalar becomes

$$\varphi_\star = \frac{\delta\Phi_{N2} v_N + \delta\Phi_{S2} v_S}{\delta\Phi_{N2} + \delta\Phi_{S2}} \,, \tag{4.91}$$

and

$$\delta W_N - \delta W_S = -\frac{3\kappa^2 \mathcal{Q}}{20\pi\alpha} \left\{ \delta\Phi_{N0} - \delta\Phi_{S0} + \frac{\delta\Phi_{N2}\delta\Phi_{S2}}{2} \left[\frac{\delta\Phi_{S2} - \delta\Phi_{N2}}{(\delta\Phi_{N2} + \delta\Phi_{S2})^2} \right] (v_N - v_S)^2 \right\} \,, \tag{4.92}$$

while $\delta L/L = (3\kappa^2 \varrho_{\rm eff}/8\pi\alpha)$, with

$$\varrho_{\rm eff} = -2\mathcal{Q}\Big(\delta\Phi_{N0} + \delta\Phi_{S0}\Big) - \frac{\mathcal{Q}}{2} \left(\frac{\delta\Phi_{N2}\delta\Phi_{S2}}{\delta\Phi_{N2} + \delta\Phi_{S2}} \right) (v_N - v_S)^2 \,. \tag{4.93}$$

Because in this case $m_\varphi^2 \simeq 0$ to leading order, the scalar mass — and so also the stability of the vacuum $\varphi_0 = \varphi_\star$ — is controlled by subdominant effects (like $\delta\tau_b$ or loops), even though the flux dominates the classical contribution to $\varrho_{\rm eff}$.

4.4.2 Axions

It is instructive to consider the relative sizes of the various scales that arise naturally when bulk axions receive masses through their couplings to branes, since these need not be related in the same way as when both axion and symmetry-breaking physics share the same number of dimensions. This section briefly examines several illustrative choices.

There are five scales that naturally arise in bulk-axion models. Three of these — the extra-dimensional Planck scale, $M_g = \kappa^{-1/2}$; the axion decay constant, $F = \kappa_a^{-1/2}$; and the KK scale, $m_{KK} = 1/L$ — characterize the bulk physics. The source branes are responsible for the other two: the scale Λ set by the ϕ-independent parts of the brane tensions and fluxes; and the scale μ set by the ϕ-dependent terms,

$$\tau_{b0} \simeq \Lambda^4 \,, \quad \Phi_{b0} \simeq \Lambda \,, \quad \tau_{b2} \simeq \mu^4 \quad \text{and} \quad \Phi_{b2} \simeq \mu \,. \tag{4.94}$$

These scales are not completely arbitrary. In general, control over the semiclassical approximation requires M_g to be much bigger than all of the others. Although the conditions $\kappa\Lambda^2 \ll 1$, $\kappa\mu^2 \ll 1$ and $\kappa/L^2 \ll 1$ follow fairly directly from standard arguments (4.26), the condition $F \ll M_g$ is a bit more indirect. Because F sets the scale of the bulk symmetry breaking for which ϕ is the would-be Goldstone boson, our upper bound on F assumes the UV completion describing this breaking intercedes below the Planck scale (before which the UV completion associated with gravity — such as string theory — should also intercede).

Furthermore, we generically expect $\mu \lesssim \Lambda$ for generic types of brane physics. This follows because it is difficult to have physics contribute to the φ mass without also contributing equivalently to the vacuum energy. Notice in this regard that it is technically natural to take $\mu \ll \Lambda$, because it is only μ that breaks the shift symmetry of the low-energy scalar: $\varphi \to \varphi +$ (constant).

Axion mass

In terms of these scales the mass of the light scalar in the effective 4D theory is of order

$$m_\varphi \simeq \frac{\mu^2}{f} \simeq \left(\frac{\mu}{F}\right)^2 \frac{1}{L} \simeq \left(\frac{\mu}{F}\right)^2 m_{KK}\,, \tag{4.95}$$

in all three of the scenarios considered above.[9] This result doesn't depend on which scenario is considered because for all three the scalar mass depends only on $\sqrt{\delta\tau_{N2}+\delta\tau_{S2}}/f$. Provided $\mu \ll F$, φ is much lighter than the KK scale as is appropriate for its description in the low-energy 4D effective theory.

For the higher-dimensional models of interest here, however, the regime $\mu \gg F$ can also make sense. Extra dimensions allow this regime even though the scale μ of explicit symmetry breaking is then much larger than the scale of the spontaneous breaking: F. Because all symmetry breaking is localized on the branes, even though $\mu > F$ the field ϕ behaves like a Goldstone boson for all energies lower than F in the bulk provided one stays away from the position of the branes. Although this regime is not amenable to a 4D description, the mass of all KK modes *can* be computed within the higher-dimensional theory. In this limit the 'zero mode' becomes lost among the generic massive KK states and is not singled out as being particularly light. In this regime it is clear that otherwise standard arguments, like cosmological bounds on axion properties, cannot be made purely within four dimensions without taking the full dynamics of the extra dimensions into account.

[9]In some circumstances additional suppression can be achieved, such as if the Goldstone symmetry is not completely broken by either brane separately (4.27).

Curvatures

A second robust prediction of all of the above scenarios is the relation between the change to the extra-dimensional size and the four-dimensional curvature:

$$\frac{\delta L}{L} = \frac{3\kappa^2 \varrho_{\rm eff}}{8\pi\alpha}\,, \tag{4.96}$$

although the size of $\varrho_{\rm eff}$ itself is not as model independent. This source of this model dependence is the competition between tension and flux contributions to $\varrho_{\rm eff}$, whose competing contributions are of order $\delta\varrho_{\rm eff} \simeq \sum_b \tau_{b0}$ or $\delta\varrho_{\rm eff} \simeq \sum_b \mathcal{Q}\Phi_{b0}$, with

$$\tau_{b0} \simeq \Lambda^4 \quad \text{and} \quad \mathcal{Q}\Phi_{b0} \simeq \mathcal{Q}\Lambda \simeq \frac{\Lambda}{\kappa L} \simeq \frac{M_g^2 \Lambda}{L} \simeq \sqrt{4\pi\alpha}\, \left(\frac{M_g^4 \Lambda}{M_p}\right)\,. \tag{4.97}$$

Special things happen for the BPS-like situation when the tension and charge are precisely related, $\tau_b(\varphi_\star) = 2\mathcal{Q}\Phi_b(\varphi_\star)$, since in this case the two contributions to $\varrho_{\rm eff}$ precisely cancel.

Whether the tension or the flux dominates in $\varrho_{\rm eff}$ depends on where Λ sits relative to the two geometrical scales $M_p \simeq 10^{18}$ GeV and $1/L$. Defining $\Lambda_\star^3 := \sqrt{4\pi\alpha}\, M_g^4/M_p$ we have $\varrho_{\rm eff} \simeq \Lambda^4$ if $\Lambda > \Lambda_\star$ and $\varrho_{\rm eff} \simeq \Lambda\Lambda_\star^3$ when $\Lambda < \Lambda_\star$. Some representative numerical values are given in Table 1. Intriguingly, $\Lambda_\star$ is of order the QCD scale in the extreme case of large extra dimensions ($M_g \lesssim 10$ TeV and $m_{KK} \lesssim 0.4$ eV (4.2)).

4.4.3 Gravitationally coupled scalars

The special case $F \simeq M_g$ is of particular interest because then $f \simeq M_p$ and the low-energy 4D scalar is gravitationally coupled. In this case the light scalar

M_g	$\Lambda_\star$	m_{KK}
10^{15}	4×10^{14}	4×10^{12}
10^{11}	2×10^{9}	4×10^{3}
10^{7}	8×10^{3}	4×10^{-4}
10^{4}	0.8	4×10^{-10}

Table 4.1: Values of m_{KK} and $\Lambda_\star$ as a function of M_g (in GeV).

mass is robustly of order $m_\varphi \simeq \mu^2/M_p$, and its small size is technically natural since it is protected by the underlying shift symmetry. There are two situations for which the existence of such light weakly-coupled scalars are of particular interest.

An inflationary mechanism

Inflationary models famously require light, weakly coupled scalars; something that is usually fairly difficult to achieve without fine-tuning in a real microscopic theory. The above estimates point to a fairly generic mechanism for achieving slow-roll inflation whenever a bulk axion acquires a potential through its interaction with codimension-two branes. This mechanism can be regarded as an ultraviolet completion of 4D 'natural inflation' models (4.28), that assume the inflaton to be a pseudo-Goldstone particle.

The mechanism rests on two assumptions: (i) the brane energy density, $\varrho_{\rm eff}$, must dominate any other contributions to the geometry in the on-brane directions; and (ii) the brane-axion couplings must have a local *maximum* rather than a minimum at $\varphi = \varphi_\star$, for which m^2_φ is of order μ^2/M_p (as above) but negative. In this case because the previous estimates apply near the potential's maximum, with an effective 4D scalar potential being of order

$$V_{\rm eff}(\varphi) \simeq A + BU(\varphi - \varphi_\star) \,, \tag{4.98}$$

with $B \simeq \mathcal{O}(\mu^4)$ and $A = \varrho_{\rm eff} \simeq \mathcal{O}(\Lambda^4)$ or $\varrho_{\rm eff} \simeq \mathcal{O}(\Lambda M_g^4/M_p)$ (whichever is larger). The technically natural choice $\mu \ll \Lambda$, $(M_g^4/M_p)^{1/3}$ therefore ensures $B \ll A$. Here $U(x)$ is a calculable, dimensionless, order-unity function, whose expansion for small arguments is (by assumption) $U(x) \simeq -\frac{1}{2}U_2\,x^2 + \cdots$ with $U_2 > 0$ and order unity.

Should this potential dominate the 4D geometry it produces a Hubble scale near this maximum that is of order $H \simeq \sqrt{\varrho_{\rm eff}}/M_p$ and so H is of order the larger of Λ^2/M_p or $(M_g^2/M_p)(\Lambda/M_p)^{1/2}$. Because of this, our choice $\mu \ll \Lambda$ automatically ensures $|m_\varphi^2| \ll H$. Provided that H is also small compared with the KK scale — as is easy to arrange — the resulting cosmology can be understood within the 4D effective theory, and describes an inflationary slow roll provided φ starts in an initially spatially homogeneous configuration near the potential's maximum. This slow roll is inflationary (despite having $f \simeq M_p$) because $B \ll A$, since the slow-roll parameters are of order $\epsilon \simeq (BU'/A)^2$ and $\eta \simeq BU''/A$. η is sufficiently small to inflate for ~ 60 e-foldings if $B/A \simeq 0.01$, in which case $\epsilon \simeq \eta^2$ is even smaller (and so the inflation typically does not produce an observable signal of primordial gravity-waves). If $A \simeq \Lambda^4$ then $B/A \simeq (\mu/\Lambda)^4$ and a sufficiently small ratio can be ensured for the comparably modest hierarchy $\mu/\Lambda \simeq 0.3$.

As an existence proof that all parameters can be chosen as required above consider the intriguing, but extreme, scenario where the QCD axion is a bulk scalar within large extra dimensions (the last line of Table 1). In this case taking $\mu \simeq \Lambda/3 \simeq \Lambda_{QCD} \simeq 0.2$ GeV both provides the right scale of axion-matter couplings, and ensures $\Lambda \simeq \Lambda_\star$ and so $\varrho_{\rm eff} \simeq \Lambda^4$ and $H \ll m_{KK}$. One might imagine that whatever solves the cosmological constant problem arranges the true ground state of the present epoch to be the unperturbed rugby-ball solution having $\hat{\mathcal{R}} \simeq 0$ and T of order the weak scale, with the perturbation $\delta\tau_b \simeq \Lambda_{QCD}^4$ arising in the early universe due to the vacuum energy associated with the QCD phase transition on the brane. Even if it were not to involve enough e-foldings to account for primordial fluctuations, such a very late inflationary period could be useful for removing unwanted relics — like moduli or KK modes — from the much earlier universe.

New long-range forces

Another potential application (or constraint) on the light bulk Goldstone mode described here comes from the long-range forces that it would mediate if its mass is sufficiently light. Indeed, one motivation to study the brane-bulk dynamics explored above is to find sensible UV completions which can have a technically light scalar whose presence could be sought when testing general relativity. Such tests provide strong constraints on the existence of any new forces competing with gravity in the solar system, with a precision that varies with the mass of the new scalar particle and the nature of its couplings to matter (4.29).

This section explores what brane-bulk dynamics might say about the couplings of the low-energy scalar to matter localized on the branes. We find these couplings can (but need not, depending on the brane properties) realize some earlier-proposed mechanisms (4.18) for dynamically vanishing when the scalar is in its ground state.

To see how φ couples to matter localized on the branes, we generalize the previous discussion to include brane-localized matter fields, generically denoted by ψ. All brane quantities like tension and flux are regarded as being functions of both brane and bulk fields,

$$\mathcal{T}_b = \mathcal{T}_b(\psi, \phi)\,, \quad \Phi_b = \Phi_b(\psi, \phi) \quad \text{and so on.} \tag{4.99}$$

The main point is that none of this affects the matching conditions and solutions described above, and so in a static (or adiabatic) configuration the ground-state value $\varphi = \varphi_\star(\psi)$ still adjusts to satisfy

$$\sum_b \left[\frac{\partial T_b(\psi, \phi)}{\partial \phi} \right]_{\varphi = \varphi_\star} = 0\,. \tag{4.100}$$

The new ingredient that appears in searches for new forces is the use of spatially inhomogeneous matter configurations as sources (*e.g.* planets, stars, *etc.*) of spatial variation, $\delta\varphi = \delta\varphi(x)$, for the fluctuation $\delta\varphi = \varphi - \varphi_\star$ along the on-brane directions. Regarded graphically, these constrain the amplitude for emitting a single φ particle from the source, with repeated emissions accumulating to give a coherent classical field. But the amplitude for φ-emission from matter localized on a specific brane, $b = b_0$, is controlled by the expansion of the brane action in powers of the fluctuation,

$$T_{b_0}(\psi, \varphi) = T_{b_0}(\psi, \varphi_\star) + \left[\frac{\partial T_{b_0}(\psi, \varphi)}{\partial \phi} \right]_{\varphi = \varphi_\star} \delta\varphi + \mathcal{O}(\delta\varphi^2)\,. \tag{4.101}$$

Of these interactions, it is only the term linear in $\delta\varphi$ that acts as an obstruction to solving the field equations with $\delta\varphi = 0$, and so it is this linear term that is subject to the strongest constraint from new-force searches. Unless T_{b_0} has special properties such a term could generate violations of the equivalence principle, which are strongly excluded once the range of the force becomes macroscopically large.

Comparing eqs. (4.100) and (4.101) reveals the mechanism for suppressing φ-matter couplings. If the action, T_{b_0}, for the specific brane on which we live should share the same extremum as does the sum of all branes, $\sum_b T_b$, then as $\varphi_\star$ adjusts to satisfy the condition (4.100), it would also automatically turn off the dangerous coupling of $\delta\varphi$ to matter localized at brane b_0. As the examples above show, the extremum of the sum of all brane actions need not agree with the extrema of each brane's action separately. But it automatically does so in two simple cases: (i) when none of the branes besides b_0 couple to ϕ at all; and (ii) when all of the branes couple to ϕ, but are all extremal for the same place.

Notice that the argument is not changed by the presence of ϕ-dependent brane fluxes, Φ_b. This is because they do not enter into eq. (4.100) independently from their contribution to T_b (even though they do contribute independently to the value of $\varrho_{\rm eff}$).

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Bibliography

[4.1] C.P. Burgess and L. van Nierop
"Bulk Axions, Brane Back-reaction and Fluxes"
JHEP 1102:094,2011 (arXiv:1012.2638)

[4.2] N. Arkani-Hamed, S. Dimopoulos and G. Dvali, *Phys. Lett.* **B429** (1998) 263 [hep-ph/9803315]; *Phys. Rev.* **D59** (1999) 086004 [hep-ph/9807344]; I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. Dvali, *Phys. Lett.* **B436** (1998) 257 [hep-ph/9804398].

[4.3] K. Benakli, *Phys. Rev.* **D60**, 104002 (1999) [hep-ph/9809582];
C. P. Burgess, L. E. Ibáñez and F. Quevedo, *Phys. Lett.* **B447**, 257 (1999) [hep-ph/9810535];
P. Horava and E. Witten, *Nucl. Phys.* **B475** (1996) 94 [hep-th/9603142];
Nucl. Phys. **B460** (1996) 506 [hep-th/9510209];
E. Witten, *Nucl. Phys.* **B471** (1996) 135 [hep-th/9602070];
J. Lykken, *Phys. Rev.* **D54** (1996) 3693 [hep-th/9603133].

[4.4] L. Randall, R. Sundrum, Phys. Rev. Lett. **83** (1999) 3370 [hep-ph/9905221]; Phys. Rev. Lett. **83** (1999) 4690 [hep-th/9906064].

[4.5] W. D. Goldberger and M. B. Wise, "Modulus stabilization with bulk fields", Phys. Rev. Lett. **83** (1999) 4922-4925 [arXiv:hep-ph/9907447]

[4.6] P. Binetruy, C. Deffayet, D. Langlois, "Non-conventional cosmology from a brane-universe" Nucl. Phys. B 565 (2000) 269-287 [arXiv:hep-th/9905012]

P. Kanti, I. I. Kogan, K. A. Olive, M. Pospelov, "Cosmological 3-brane solutions" Phys. Lett. B 468 (1999) 31-39 [arXiv:hep-ph/9909481]

J. M. Cline, C. Grojean, G. Servant, "Cosmological expansion in the presence of extra dimensions" Phys. Rev. Lett. 83 (1999) 4245 [arXiv:hep-ph/9906523]

[4.7] B. R. Greene, A. D. Shapere, C. Vafa and S. T. Yau, "Stringy Cosmic Strings And Noncompact Calabi-Yau Manifolds," Nucl. Phys. B **337** (1990) 1;

S. Kachru, R. Kallosh, A. D. Linde, J. M. Maldacena, L. P. McAllister and S. P. Trivedi, "Towards inflation in string theory," JCAP **0310**, 013 (2003) [arXiv:hep-th/0308055];

D. Baumann, A. Dymarsky, I. R. Klebanov, J. M. Maldacena, L. P. McAllister and A. Murugan, "On D3-brane potentials in compactifications with fluxes and wrapped D-branes," JHEP **0611** (2006) 031 [arXiv:hep-th/0607050];

D. Baumann, A. Dymarsky, I. R. Klebanov and L. McAllister, "Towards an Explicit Model of D-brane Inflation," JCAP **0801** (2008) 024 [arXiv:0706.0360 [hep-th]].

[4.8] A. Kehagias, "A conical tear drop as a vacuum-energy drain for the solution of the cosmological constant problem," Phys. Lett. B **600** (2004) 133 [arXiv:hep-th/0406025];

T. Kobayashi and M. Minamitsuji, "Brane cosmological solutions in six-dimensional warped flux compactifications," JCAP **0707** (2007) 016 [arXiv:0705.3500 [hep-th]];

D. Kiley, "Rotating Black Holes on Codimension-2 Branes," Phys. Rev. D **76** (2007) 126002 [arXiv:0708.1016 [hep-th]].

[4.9] F. Leblond, R. C. Myers and D. J. Winters, "Consistency conditions for brane worlds in arbitrary dimensions," JHEP **0107** (2001) 031 [arXiv:hep-th/0106140];

S. M. Carroll and M. M. Guica, "Sidestepping the cosmological constant with football-shaped extra dimensions," [hep-th/0302067];

I. Navarro, "Codimension two compactifications and the cosmological constant problem," JCAP **0309** (2003) 004 [hep-th/0302129];

E. Papantonopoulos and A. Papazoglou, "Brane-bulk matter relation for a purely conical codimension-2 brane world," JCAP **0507** (2005) 004 [arXiv:hep-th/0501112].

[4.10] A. Salam and E. Sezgin, "Chiral Compactification On Minkowski X S**2 Of N=2 Einstein-Maxwell Supergravity In Six-Dimensions," Phys. Lett. B **147** (1984) 47;

[4.11] Y. Aghababaie, C.P. Burgess, S. Parameswaran and F. Quevedo, Nucl. Phys. **B680** (2004) 389–414, [hep-th/0304256].

[4.12] S. L. Parameswaran, G. Tasinato and I. Zavala, "The 6D SuperSwirl," Nucl. Phys. B **737** (2006) 49 [arXiv:hep-th/0509061];

H. M. Lee and C. Ludeling, "The general warped solution with conical branes in six-dimensional supergravity," JHEP **0601** (2006) 062 [arXiv:hep-th/0510026];

H. M. Lee and A. Papazoglou, "Supersymmetric codimension-two branes in six-dimensional gauged supergravity," JHEP **0801** (2008) 008 [arXiv:0710.4319 [hep-th]];

C. P. Burgess, S. L. Parameswaran and I. Zavala, "The Fate of Unstable Gauge Flux Compactifications," arXiv:0812.3902 [hep-th];

[4.13] C. P. Burgess, "Goldstone and pseudo-Goldstone bosons in nuclear, particle and condensed-matter physics," Phys. Rept. **330** (2000) 193 [arXiv:hep-th/9808176].

[4.14] C. P. Burgess, D. Hoover and G. Tasinato, "UV Caps and Modulus Stabilization for 6D Gauged Chiral Supergravity," JHEP **0709** (2007) 124 [arXiv:0705.3212 [hep-th]];

C. P. Burgess, D. Hoover, C. de Rham and G. Tasinato, "Effective Field Theories and Matching for Codimension-2 Branes," JHEP **0903** (2009) 124 [arXiv:0812.3820 [hep-th]];

C. P. Burgess, D. Hoover and G. Tasinato, "Technical Naturalness on a Codimension-2 Brane," JHEP **0906** (2009) 014 [arXiv:0903.0402 [hep-th]];

[4.15] A. Bayntun, C. P. Burgess and L. van Nierop, "Codimension-2 Brane-Bulk Matching: Examples from Six and Ten Dimensions," New J. Phys. **12** (2010) 075015 [arXiv:0912.3039 [hep-th]].

[4.16] P. Bostock, R. Gregory, I. Navarro and J. Santiago, "Einstein gravity on the codimension 2 brane?," Phys. Rev. Lett. **92** (2004) 221601 [arXiv:hep-th/0311074];

I. Navarro and J. Santiago, "Gravity on codimension 2 brane worlds," JHEP **0502** (2005) 007 [arXiv:hep-th/0411250];

J. Vinet and J. M. Cline, "Codimension-two branes in six-dimensional supergravity and the cosmological constant problem," Phys. Rev. D **71** (2005) 064011 [hep-th/0501098];

M. Peloso, L. Sorbo and G. Tasinato, "Standard 4d gravity on a brane in six dimensional flux compactifications," Phys. Rev. D **73** (2006) 104025 [arXiv:hep-th/0603026];

B. Himmetoglu and M. Peloso, "Isolated Minkowski vacua, and stability analysis for an extended brane in the rugby ball," Nucl. Phys. B **773** (2007) 84 [hep-th/0612140];

B. Himmetoglu and M. Peloso, "Isolated Minkowski vacua, and stability analysis for an extended brane in the rugby ball," Nucl. Phys. B **773** (2007) 84 [hep-th/0612140];

E. Papantonopoulos, A. Papazoglou and V. Zamarias, "Regularization of conical singularities in warped six-dimensional compactifications," JHEP **0703** (2007) 002 [arXiv:hep-th/0611311];

"Induced cosmology on a regularized brane in six-dimensional flux compactification," Nucl. Phys. B **797** (2008) 520 [arXiv:0707.1396 [hep-th]];

D. Yamauchi and M. Sasaki, "Brane World in Arbitrary Dimensions Without Z_2 Symmetry," Prog. Theor. Phys. **118** (2007) 245 [arXiv:0705.2443 [gr-qc]];

N. Kaloper and D. Kiley, "Charting the Landscape of Modified Gravity," JHEP **0705** (2007) 045 [hep-th/0703190];

M. Minamitsuji and D. Langlois, "Cosmological evolution of regularized branes in 6D warped flux compactifications," Phys. Rev. D **76** (2007) 084031 [arXiv:0707.1426 [hep-th]];

S. A. Appleby and R. A. Battye, "Regularized braneworlds of arbitrary codimension," Phys. Rev. D **76** (2007) 124009 [arXiv:0707.4238 [hep-ph]];

C. Bogdanos, A. Kehagias and K. Tamvakis, "Pseudo-3-Branes in a Curved 6D Bulk," Phys. Lett. B **656** (2007) 112 [arXiv:0709.0873 [hep-th]];

O. Corradini, K. Koyama and G. Tasinato, "Induced gravity on intersecting brane-worlds Part I: Maximally symmetric solutions," Phys. Rev. D **77** (2008) 084006 [arXiv:0712.0385 [hep-th]];

F. Arroja, T. Kobayashi, K. Koyama and T. Shiromizu, "Low energy effective theory on a regularized brane in 6D gauged chiral supergravity," JCAP **0712** (2007) 006 [arXiv:0710.2539 [hep-th]];

O. Corradini, K. Koyama and G. Tasinato, "Induced gravity on intersecting brane-worlds Part II: Cosmology," Phys. Rev. D **78** (2008) 124002 [arXiv:0803.1850 [hep-th]];

V. Dzhunushaliev, V. Folomeev and M. Minamitsuji, "Thick brane solutions," arXiv:0904.1775 [gr-qc].

[4.17] K. Lanczos, Phys. Z. **23** (1922) 239–543; Ann. Phys. **74** (1924) 518–540;
C.W. Misner and D.H. Sharp, Phys. Rev. **136** (1964) 571–576;
W. Israel, Nuov. Cim. **44B** (1966) 1–14; *errata* Nuov. Cim. **48B** 463.

[4.18] T. Damour and A. M. Polyakov, "The String Dilaton And A Least Coupling Principle," Nucl. Phys. B **423** (1994) 532 [arXiv:hep-th/9401069];
P. Brax, C. van de Bruck, A. C. Davis and D. Shaw, "The Dilaton

and Modified Gravity," Phys. Rev. D **82** (2010) 063519 [arXiv:1005.3735 [astro-ph.CO]].

[4.19] E. Dudas, C. Papineau and V. A. Rubakov, "Flowing to four dimensions," JHEP **0603** (2006) 085 [arXiv:hep-th/0512276];
E. Dudas and C. Papineau, "Dual realizations of dynamical symmetry breaking," JHEP **0611** (2006) 010 [arXiv:hep-th/0608054];
C. P. Burgess, C. de Rham, L. van Nierop, "The Hierarchy Problem and the Self-Localized Higgs," JHEP **0808** (2008) 061. [arXiv:0802.4221 [hep-ph]].

[4.20] S. Weinberg, *Gravitation and Cosmology*, Wiley 1973.

[4.21] C. W. Misner, J. A. Wheeler and K. S. Thorne, *Gravitation*, W. H. Freeman & Company 1973.

[4.22] A. Vilenkin, "Gravitational field of vacuum domain walls and strings," Phys. Rev. **D 23**, (1981) 852-857
W. A. Hiscock, "Exact gravitational field of a string," Phys. Rev. **D 31**, (1985) 3288-3290

[4.23] A. J. Tolley, C. P. Burgess, C. de Rham and D. Hoover, "Scaling solutions to 6D gauged chiral supergravity," New J. Phys **8** (2006) 324 [arXiv:0608.083 [hep-th]];
A. J. Tolley, C. P. Burgess, C. de Rham and D. Hoover, "Exact Wave Solutions to 6D Gauged Chiral Supergravity," JHEP **0807** (2008) 075 [arXiv:0710.3769 [hep-th]];
M. Minamitsuji, "Instability of brane cosmological solutions with

flux compactifications," Class. Quant. Grav. **25** (2008) 075019 [arXiv:0801.3080 [hep-th]].

H. M. Lee and A. Papazoglou, "Codimension-2 brane inflation," Phys. Rev. D **80** (2009) 043506 [arXiv:0901.4962 [hep-th]].

[4.24] H. M. Lee and A. Papazoglou, "Scalar mode analysis of the warped Salam-Sezgin model," [hep-th/0602208];

C. P. Burgess, C. de Rham, D. Hoover, D. Mason and A. J. Tolley, "Kicking the rugby ball: Perturbations of 6D gauged chiral supergravity," JCAP **0702** (2007) 009 [arXiv:hep-th/0610078];

S. L. Parameswaran, S. Randjbar-Daemi and A. Salvio, "Gauge fields, fermions and mass gaps in 6D brane worlds," Nucl. Phys. B **767** (2007) 54 [arXiv:hep-th/0608074];

"Stability and Negative Tensions in 6D Brane Worlds," JHEP **0801** (2008) 051 [arXiv:0706.1893 [hep-th]];

"General Perturbations for Braneworld Compactifications and the Six Dimensional Case," JHEP **0903** (2009) 136 [arXiv:0902.0375 [hep-th]].

[4.25] W. D. Goldberger, M. B. Wise, "Renormalization group flows for brane couplings," Phys. Rev. **D65** (2002) 025011 [arXiv:hep-th/0104170];

E. Dudas, C. Papineau, V.A. Rubakov, "Flowing to four dimensions" JHEP **0603** (2006) 085 [arXiv:hep-th/0512276v1];

C. de Rham, "The Effective Field Theory of Codimension-two Branes," JHEP **0801** (2008) 060 [arXiv:0707.0884 [hep-th]].

F. del Aguila, M. Perez-Victoria, J. Santiago, " Effective description of brane terms in extra dimensions," JHEP **0610** (2006) 056 [arXiv:hep-ph/0601222]

K. A. Milton, S. D. Odintsov, S. Zerbini "Bulk versus brane running couplings" Phys. Rev. **D65** (2002)065012 [arXiv:hep-th/0110051]

[4.26] C. P. Burgess, "Quantum gravity in everyday life: General relativity as an effective field theory," Living Rev. Rel. **7** (2004) 5 [arXiv:gr-qc/0311082].

J. F. Donoghue, "Introduction to the Effective Field Theory Description of Gravity", [arXiv:gr-qc/9512024]

[4.27] C. P. Burgess, P. Grenier and D. Hoover, "Quintessentially flat scalar potentials," JCAP **0403** (2004) 008 [arXiv:hep-ph/0308252].

[4.28] Katherine Freese, Joshua A. Frieman and Angela V. Olinto, "Natural inflation with pseudo Nambu-Goldstone bosons" *Phys. Rev. Lett.***65** (1990) 3233—3236 .

[4.29] C. M. Will, "The confrontation between general relativity and experiment," Living Rev. Rel. **9** (2005) 3 [arXiv:gr-qc/0510072];

E. G. Adelberger, B. R. Heckel and A. E. Nelson, "Tests of the gravitational inverse-square law," Ann. Rev. Nucl. Part. Sci. **53** (2003) 77 [arXiv:hep-ph/0307284].

Chapter 5

Large Dimensions and Small Curvatures from Supersymmetric Brane Back-reaction

5.1 Preamble

This chapter is based on the work in (5.1). We follow the same steps as in chapter (4), but for a more complicated bulk theory: gauged chiral supergravity. The would-be zero mode in this case is a mixture between the scalar (dilaton) field and the extra dimensional metric. As a result, the stabilization of the zero mode by brane physics automatically also stabilizes the radius of the extra dimensions. This mechanism allows us to address the cosmological constant problem as a problem of stabilizing the extra dimensions at a sufficiently large size.

The other consequence of this is that dimensional reduction leads to a scalar-tensor theory in the Jordan frame. As a result of the Weyl rescaling to go to the Einstein frame, the effective classical cosmological constant is very generally set by the derivative of the brane tensions with respect to the dilaton. It is this fact that is exploited in chapter (6).

The localization of bulk fluxes on the brane is very important for the energetics of the bulk in this system. In the absence of the magnetic coupling, there is only one choice of constant tensions that leads to a rugby ball solution. When the tension is perturbed, the energy cost is large because the bulk flux has no way to relax: the strength of the magnetic field is set directly by parameters in the Lagrangian. The presence of the magnetic coupling allows the magnetic field to adjust to find a minimum of the effective potential.

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5.2 Introduction

Most of what is known about the physics of branes situated within extra dimensions either neglects their back-reaction onto their environment, or approximates the surrounding geometry as noncompact by ignoring the physics responsible for its stabilization at finite volume.[1] Although these are often good approximations, there are also very interesting situations where they are not.

A particularly interesting case where these effects cannot be neglected is when it is the back-reaction itself that stabilizes some of the extra-dimensional moduli. This case turns out to be important for compactifications whose extra-dimensional volume is very large, such as those arising within large-volume string vacua (5.4). In particular, the larger the extra dimensions the lower the string scale (5.5), and once the string scale gets as low as the TeV scale — such as in supersymmetric extensions (5.6) of ADD-type models (5.7) — supersymmetry becomes dominantly broken on the branes rather than by the fluxes in the bulk (5.8; 5.9). In this case it is known that brane-induced corrections can dominate the leading classical predictions for the potential governing the lightest moduli (5.10).

The need to include back-reaction when computing the shape of the low-energy scalar potential is both a potential asset and a liability. The downside is the additional complexity required to properly incorporate both the

[1] Randall-Sundrum models (5.2) are important exception to this statement, where back-reaction is incorporated through the Israel junction conditions (5.3), but these are restricted to the limiting special case of codimension-one branes.

extra-dimensional and brane dynamics within a controlled approximation. The upside is the potential for progress finding new mechanisms for understanding long-standing problems. Progress in particular on naturalness problems to do with the existence of light scalar masses and small vacuum energies, that hinge on understanding *all* contributions relevant to the low-energy scalar potential. And there are a variety of reasons for thinking that brane dynamics could be useful for understanding these problems (5.11; 5.6).

Six dimensional supergravities provide a fruitful place to explore these issues because they are complicated enough to exhibit many of the features of ten- or eleven-dimensional string vacua, yet they are simple enough often to allow explicit solutions and so more systematic exploration of the various configurations of physical interest. 6D gauged chiral supergravity (5.12) has proven particularly useful, providing early insights into chiral fermions and flux compactifications (5.13; 5.14). This has motivated finding a great many exact solutions to the classical field equations for this system, including a broad class of flux compactifications for which the two extra dimensions are a warped, squashed sphere with singularities at the positions of two positive-tension source 3-branes. These include solutions for which the on-brane geometry is flat (5.6; 5.15; 5.16) (also known as 'rugby-ball' solutions), de Sitter/anti de Sitter like (5.17), time dependent (5.18) or involves other bulk fields (5.19) or additional branes (5.20).

In this paper we explore brane back-reaction in this system by computing how the flat rugby-ball solutions respond to a general perturbation of the brane-bulk couplings. In particular, we assume the perturbed brane-bulk couplings to be given by the leading terms in a derivative expansion of the brane action,

$$S_b = \int_{\Sigma_b} \Big(\mathcal{T}_b \, \omega + \Phi_b \, {}^\star\mathcal{F} \Big) \,, \tag{5.1}$$

where ω is the volume form for the space-filling 3-brane, and ${}^\star\mathcal{F}$ denotes the 6D Hodge dual of the background Maxwell flux, $\mathcal{F}_{MN}$ (whose presence stabilizes some of what would otherwise be light bulk moduli, in the same way that 3-form fluxes stabilize some moduli in ten-dimensional flux compactifications (5.21)). The coefficient $\mathcal{T}_b$ denotes the tension of the brane in question, which can be an arbitrary function of the bulk scalar dilaton, ϕ, appearing in 6D gauged chiral supergravity. Φ_b has a similar interpretation (5.6; 5.22) as an on-brane flux, and can sometimes compete with $\mathcal{T}_b$ to play an important role in the low-energy energetics of the back-reaction.

The bulk geometry that interpolates between a generic pair of source branes is known to be time dependent (5.18), in much the same way that a random collection of mutually interacting electric charges is also not static. This is reflected by the generic absence of time-independent solutions once a brane-bulk system is perturbed. Unlike earlier stability analyses for these geometries (5.23), we do not deal with this by seeking the time-dependence of the solutions to the brane-perturbed bulk equations of motion. Rather, we instead couple an external current that stabilizes this time-dependence in order to study the energetics of the potential energy that drives it. In practice, at low energies this current need only couple to the massless Kaluza-Klein (KK) 'breathing' mode of the leading-order extra-dimensional geometry, since this is a flat direction in field space along which the time dependence dominantly lies.

In this way we find the response of the on- and off-brane geometries as a function of the perturbing brane couplings, as well as the shape of the scalar potential that stabilizes and gives a mass to the low-energy breathing mode, for general choices for the brane coupling functions τ_b and Φ_b. We find instances where the breathing mode is stabilized by the interaction of the branes on the bulk, as well as cases where it instead runs away to infinity (which, perhaps surprisingly, includes the simplest case where both τ_b and Φ_b are independent of the 6D bulk dilaton, ϕ).

When restricted to the special cases for which our results duplicate earlier calculations, we fully reproduce earlier expressions. But our systematic survey of perturbed solutions also reveals some new ones with surprising properties. These include (see §4 for a more detailed summary):

- Solutions whose extra-dimensional volumes stabilize at values that exponentiate any moderately large hierarchies among the brane-bulk couplings, naturally giving enormously large volumes;
- Solutions whose on-brane geometry can be parametrically small compared with the largest energy scales that appear in the brane-bulk couplings (though, alas, not yet small enough to describe the observed Dark Energy density);
- Solutions for which the value of the breathing mode along the low-energy flat direction defines the strength of both brane and bulk loop corrections, and for which this ensures that the above two properties can be stable against quantum effects;
- Models for which the brane-bulk couplings can have the form required to profit from a 'chameleon' mechanism (5.24).

Our presentation is organized as follows: The next section, §5.3, describes the linearized solutions to the bulk field equations, and how the integration constants in these solutions are determined by matching to the functions τ_b and Φ_b that define the codimension-2 bulk-brane interactions. These are then used to provide explicit expressions for the extra-dimensional and on-brane geometries as functions of these brane properties. The results of the full 6D calculation are compared with the effective 4D picture that captures the low-energy limit, since the scalar potential in this effective theory provides an efficient way to understand the implications of brane dynamics on low-energy properties. This section closely follows the logic of ref. (5.22), which performs a similar calculation in the non-supersymmetric case.

§5.4 then uses the general results of §5.3 to explore the implications of several simple illustrative choices for the coupling functions τ_b and Φ_b. A particularly simple toy model — for which $\sum_b \tau_b \propto \sum_b \Phi_b \propto \phi^\eta$, for small η — is also examined, that exhibits modulus stabilization at exponentially large volume and parametric suppression of the low-energy on-brane curvature (or vacuum energy). Finally, this section estimates the effects of brane and bulk loops for the toy model, and argues that the exponentially large volume, and the small on-brane vacuum energy (and scalar masses) can be technically natural.

Our conclusions are summarized in §5.5.

5.3 The bulk-brane system

This section defines the system of interest. The fields of interest are part of the bosonic sector of chiral gauged supergravity in six dimensions (5.12), for which we follow the implications of coupling to nonsupersymmetric branes. In particular we follow the metric, g_{MN}; a bulk Maxwell gauge potential, $\mathcal{A}_M$, whose presence helps stabilize the bulk geometry; and the 6D scalar dilaton, ϕ.

5.3.1 Field equations and background solutions

We first describe the bulk equations of motion and brane boundary conditions, followed by a simple class of rugby-ball solutions near which general solutions are sought.

Bulk equations

The bosonic action in the bulk is[2]

$$S_{\rm bulk} = -\int {\rm d}^6x\sqrt{-g}\;\left\{\frac{1}{2\kappa^2}\,g^{MN}\Big(\mathcal{R}_{MN}+\partial_M\phi\,\partial_N\phi\Big) + \frac{1}{4}\,e^{-\phi}\mathcal{F}_{MN}\mathcal{F}^{MN}+\frac{2\,g_R^2}{\kappa^4}\,e^{\phi}\right\}\,, \tag{5.2}$$

where the two dimensionful constants are the gauge coupling, g_R, for a specific $U_R(1)$ symmetry of the supersymmetry algebra, and the 6D gravitational constant, κ. One of these sets the overall scale of the bulk physics, leaving the dimensionless combination g_R^2/κ as a free parameter. Here $\mathcal{F} = {\rm d}\mathcal{A}$ denotes the gauge potential's field strength.

The equations of motion from this action are the (trace reversed) Ein-

[2]We use a 'mostly plus' metric and Weinberg's curvature conventions (5.25) (that differ from those of MTW (5.26) only by an overall sign in the definition of the Riemann tensor).

stein equations

$$\mathcal{R}_{MN} + \partial_M \phi \, \partial_N \phi + \kappa^2 e^{-\phi} \mathcal{F}_{MP} \mathcal{F}_N{}^P - \left(\frac{\kappa^2}{8} \, e^{-\phi} \mathcal{F}_{PQ} \mathcal{F}^{PQ} - \frac{g_R^2}{\kappa^2} \, e^{\phi} \right) g_{MN} = 0 \,, \quad (5.3)$$

the Maxwell equation

$$\nabla_M (e^{-\phi} \mathcal{F}^{MN}) = 0 \,, \quad (5.4)$$

and the dilation equation

$$\Box \phi - \frac{2 \, g_R^2}{\kappa^2} \, e^{\phi} + \frac{\kappa^2}{4} \, e^{-\phi} \mathcal{F}_{MN} \mathcal{F}^{MN} = 0 \,. \quad (5.5)$$

Since these field equations are invariant under the transformations

$$g_{MN} \to \zeta \, g_{MN} \quad \text{and} \quad e^{-\phi} \to \zeta \, e^{-\phi} \,, \quad (5.6)$$

with $\mathcal{A} \to \mathcal{A}$, any nonsingular solution is always part of a one-parameter family of solutions that are exactly degenerate (within the classical approximation).

Symmetry ansatz

In what follows we restrict attention to solutions that have maximal symmetry in the four on-brane directions, and axial symmetry in the two extra dimensions. This assumption restricts us to solutions involving at most two source branes. The corresponding *ansätze* for the metric and Maxwell field are

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + e^{2B} \mathrm{d}\theta^2 + e^{2W} \, \hat{g}_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu \quad \text{and} \quad \mathcal{A} = \mathcal{A}_\theta \, \mathrm{d}\theta \,, \quad (5.7)$$

where $\hat{g}_{\mu\nu}(x)$ is a maximally symmetric metric, and all of the functions W, B, ϕ and $\mathcal{A}_\theta$ depending only on ρ. The corresponding Maxwell field strength is

$\mathcal{F}_{\rho\theta} = \mathcal{A}'_\theta$, where primes denote differentiation with respect to the coordinate ρ.

Subject to this *ansatz* the bulk field equations reduce to

$$\begin{aligned}
\left(e^{-B+4W} e^{-\phi} \mathcal{A}'_\theta\right)' &= 0 \qquad (\mathcal{A}_\theta)\\
\left(e^{B+4W}\phi'\right)' - \left(\frac{2g_R^2}{\kappa^2}\, e^{\phi} - \frac{1}{2}\,\kappa^2 \mathcal{Q}^2\, e^{\phi} e^{-8W}\right) e^{B+4W} &= 0 \qquad (\phi)\\
4\Big[W'' + (W')^2\Big] + B'' + (B')^2 + (\phi')^2 + \frac{3}{4}\,\kappa^2\mathcal{Q}^2\, e^{\phi} e^{-8W} + \frac{g_R^2}{\kappa^2}\, e^{\phi} &= 0 \qquad (\rho\rho)\\
B'' + (B')^2 + 4W'B' + \frac{3}{4}\,\kappa^2\mathcal{Q}^2\, e^{\phi} e^{-8W} + \frac{g_R^2}{\kappa^2}\, e^{\phi} &= 0 \qquad (\theta\theta)\\
\frac{1}{4}\, e^{-2W}\hat{R} + W'' + 4(W')^2 + W'B' - \frac{1}{4}\,\kappa^2\mathcal{Q}^2\, e^{\phi} e^{-8W} + \frac{g_R^2}{\kappa^2}\, e^{\phi} &= 0 \qquad (\mu\nu)\,.
\end{aligned} \tag{5.8}$$

The first of these can be integrated once exactly, introducing an integration constant, $\mathcal{Q}$, labeling the bulk flux,

$$\mathcal{F}_{\rho\theta} = \mathcal{A}'_\theta = \mathcal{Q}\, e^{\phi} e^{B-4W}\,. \tag{5.9}$$

Rugby ball solutions

In the special case that the dilaton is constant, $\phi = \varphi_0$, these equations have a simple solution with extra dimensions having the shape of a rugby ball, sourced by two branes (5.6):

$$\begin{aligned}
\mathrm{d}s^2 &= e^{-\varphi_0}\left[\mathrm{d}\hat{\rho}^2 + \alpha^2 L^2 \sin^2\left(\frac{\hat{\rho}}{L}\right)\mathrm{d}\theta^2\right] + \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu\\
\mathcal{F}_{\rho\theta} &= \mathcal{F}_{\hat{\rho}\theta}\, e^{-\varphi_0/2} = \mathcal{Q} e^{\varphi_0/2}\alpha L \sin\left(\frac{\hat{\rho}}{L}\right),
\end{aligned} \tag{5.10}$$

where $\hat{g}_{\mu\nu} = \eta_{\mu\nu}$ denotes the usual flat metric of Minkowski space. The extra-dimensional metric becomes singular at the brane positions, $\hat{\rho}_{N} = 0$ and $\hat{\rho}_{S} = \pi L$, which are φ_0-independent because of the coordinate rescaling $\rho := e^{-\varphi_0/2}\hat{\rho}$.

The geometry generically has a conical singularity at these points, characterized by the defect angle $\delta = 2\pi(1-\alpha)$. In the special case $\alpha = 1$ the extra-dimensional geometry is a sphere, corresponding to the supersymmetric Salam-Sezgin solution (5.13). The deficit angle can be related to the common tension, T, of the two source branes by (5.27)

$$1 - \alpha = \frac{\kappa^2 T}{2\pi}\,. \tag{5.11}$$

The equations of motion impose two relations amongst the integration constants, requiring

$$\begin{aligned} \frac{2g_R^2}{\kappa^2} &= \frac{\kappa^2 \mathcal{Q}^2}{2} \qquad \text{(dilaton equation)} \\ \text{and} \quad \kappa^2 \mathcal{Q}^2 L^2 &= 1 \qquad\quad \text{(Einstein equation)}\,. \end{aligned} \tag{5.12}$$

Additionally, flux quantization due to the spherical topology of the extra dimensions implies

$$\frac{n}{g} = 2\alpha L^2 \mathcal{Q} = \frac{\alpha}{g_R}\,, \tag{5.13}$$

where g is the gauge coupling of the background Maxwell field and n is an integer. The couplings g and g_R are in general different because the background Maxwell field need not be the one that gauges the $U_R(1)$ symmetry. This last condition determines the deficit angle, α, and thereby constrains the tension of the source branes. As is elaborated in more detail below, a minor modification (5.6) of these solutions allows the source branes themselves to carry some of the total flux, $\Phi_{\rm branes}$, in which case eq. (5.13) generalizes to

$$\begin{aligned} \frac{n}{g} = \frac{\alpha}{g_R} + \frac{\Phi_{\rm branes}}{2\pi} &= \frac{1}{g_R}\left(1 - \frac{\kappa^2 T}{2\pi}\right) + \frac{\Phi_{\rm branes}}{2\pi} \\ &= \frac{1}{g_R}\left[1 - \frac{\kappa^2}{2\pi}\left(T - \frac{\mathcal{Q}\,\Phi_{branes}}{2}\right)\right], \end{aligned} \tag{5.14}$$

where the last equality uses eqs. (5.12), which imply $\mathcal{Q} = 2g_R/\kappa^2$. This can be regarded as allowing the tension in these solutions to be arbitrary, provided the on-brane flux is also dialed, $\Phi_{\rm branes}(T)$, to satisfy eq. (5.14).

For fixed brane flux the above construction describes only a one-parameter family of solutions, labeled by φ_0. This one-parameter degeneracy is the one required by the scale invariance, eq. (5.6), of the classical field equations. Because of the overall factor of $e^{-\varphi_0}$ in the extra-dimensional metric, eq. (5.10), the proper distance between the two branes is $\Delta\rho = e^{-\varphi_0/2}\pi L$ and the volume of the extra dimensions is

$$\mathcal{V}_2 = 4\pi\alpha L^2 e^{-\varphi_0}\,. \tag{5.15}$$

Our interest in what follows is in how this flat direction gets lifted by dilaton couplings to the branes. Its connection to the extra-dimensional volume makes this also a stabilization mechanism for the size of the extra dimensions; a codimension-2 generalization of the better-known Goldberger-Wise stabilization mechanism for codimension-1 branes (5.28) within RS models.

Brane matching conditions

We take the brane-bulk coupling to be defined by the following lowest-derivative action, including both a ϕ-dependent tension and a ϕ-dependent coupling to the Maxwell field (5.6):

$$\begin{aligned} S_{\text{branes}} &= -\sum_{b=N,S} \int d^4x \sqrt{-g_4}\, \left[\mathcal{T}_b - \frac{1}{2}\, \Phi_b\, e^{-\phi} \epsilon^{mn} \mathcal{F}_{mn} \right] \\ &= -\sum_{b=N,S} \int d^4x \sqrt{-\hat{g}}\, e^{4W} \left[\mathcal{T}_b - \frac{1}{2}\, \Phi_b\, e^{-\phi} \epsilon^{mn} \mathcal{F}_{mn} \right] , \qquad (5.16) \end{aligned}$$

where the coupling functions $\mathcal{T}_b$ and Φ_b can depend on all of ϕ, W and $g_{\theta\theta}$ without breaking the condition of maximal symmetry in the on-brane directions. Because of the explicit factor of $e^{-\phi}$ extracted from the Maxwell coupling, these interactions also do not break the bulk scaling symmetry, eq. (5.6), only when both $\mathcal{T}_b$ and Φ_b are ϕ-independent. Our conventions are such that $\epsilon^{\rho\theta} = 1/\sqrt{g_2} = e^{-B}$ transforms as a tensor, rather than a tensor density, in the two transverse dimensions. The parameter $\mathcal{T}_b$ has the physical interpretation of being the tension of the brane, and (as is shown below) the parameter Φ_b similarly denotes the magnetic charge (or flux) carried by the source branes.

The presence of such brane couplings imposes a set of boundary conditions on the derivatives of the bulk fields in the near-brane limits,[3] given by[4] (5.32):

$$\begin{aligned} \Big[e^B \phi' \Big]_{\rho_b} &= \frac{\partial \mathcal{T}_b}{\partial \phi} \quad \text{with} \quad \mathcal{T}_b := \frac{\kappa^2 T_b}{2\pi} \\ \Big[e^B W' \Big]_{\rho_b} &= \mathcal{U}_b \quad \text{with} \quad \mathcal{U}_b := \frac{\kappa^2}{4\pi} \left(\frac{\partial T_b}{\partial g_{\theta\theta}} \right) \qquad (5.17) \\ \text{and} \quad \Big[e^B B' - 1 \Big]_{\rho_b} &= -\Big[\mathcal{T}_b + 3\,\mathcal{U}_b \Big] , \end{aligned}$$

where, as before, primes denote differentiation with respect to ρ. T_b is defined

[3]These matching conditions can be derived (5.30) from codimension-1 microscopic models (5.31) for codimension-2 branes.

[4]Notice that we normalize the quantities $\mathcal{T}_b$ and $\mathcal{U}_b$ without including the factor of e^{4W} used in this reference.

in terms of τ_b and Φ_b as the total lagrangian density of the source,

$$T_b := \tau_b - \Phi_b \, e^{-\phi} e^{-B} \mathcal{F}_{\rho\theta} \,. \tag{5.18}$$

As shown in Appendix C.1, the corresponding boundary condition for the Maxwell field implies that the integral of $\mathcal{F}_{\rho\theta}$ to obtain $\mathcal{A}_\theta(\rho)$ in a coordinate patch containing each source brane gives

$$\begin{aligned} \mathcal{A}_\theta(\rho) &= \frac{\Phi_N}{2\pi} + \mathcal{Q} \int_{\rho_N}^{\rho} \mathrm{d}\rho \; e^{\phi+B-4W} \qquad \text{Northern hemisphere} \\ &= -\frac{\Phi_S}{2\pi} + \mathcal{Q} \int_{\rho_S}^{\rho} \mathrm{d}\rho \; e^{\phi+B-4W} \quad \text{Southern hemisphere} \,, \end{aligned} \tag{5.19}$$

where $\Phi_b := \lim_{\rho\to\rho_b} \Phi_b[\phi(\rho)]$ — appropriately renormalized (5.29) — and the signs are dictated by the observation that increasing ρ points away from (towards) the North (South) pole, together with the requirement that the two patches share the same orientation. Requiring these to differ by a gauge transformation, $g^{-1}\partial_\theta\Omega$, on regions of overlap implies the flux-quantization condition

$$\frac{n}{g} = \frac{\Phi_{\rm tot}}{2\pi} + \mathcal{Q} \int_{\rho_N}^{\rho_S} \mathrm{d}\rho \; e^{\phi+B-4W} \,, \tag{5.20}$$

which identifies $\Phi_{\rm tot} = \sum_b \Phi_b$ as the part of the total magnetic flux carried by the branes (5.6).

5.3.2 Perturbations

In this section we use the previous discussion to analyze how couplings to the brane lift the flat direction associated with the scaling symmetry of the bulk theory, and so to see how the scalar zero mode, φ_0, becomes stabilized at a specific value, $\varphi_0 = \varphi_\star$. Our discussion closely follows the discussion of the nonsupersymmetric system in ref. (5.22).

It is instructive to contrast how this stabilization differs from the non-supersymmetric system. To this end recall how the stabilization occurs in detail, from the point of view of six dimensions. Given two branes, we seek the bulk configuration satisfying the field equations that interpolates between the boundary conditions that each brane specifies. Specializing to solutions that are both axially symmetric in the transverse directions and maximally symmetric in the on-brane dimensions requires seeking bulk profiles that depend only on ρ.

What is important is that the brane boundary conditions only specify the derivatives of the fields near the branes, and not the values of the fields themselves there. Once the derivatives of the fields are specified at one brane, the values of the fields at the same brane can be adjusted to try to ensure that the derivatives take the values required by the other brane at the other brane's position. It is in this way that the stabilized value, $\varphi_0 = \varphi_\star$, is obtained if the brane actions break the classical bulk scaling symmetry.

This argument shows that a classical solution satisfying all of the boundary conditions is in general impossible given an arbitrary choice for φ_0. From the low-energy 4D perspective the absence of a solution when $\varphi_0 \neq \varphi_\star$ corresponds to the absence of a static solution for a value of φ_0 that is not an extremal of the low-energy effective potential, $V'_{\rm eff}(\varphi_0) \neq 0$. It can still be possible to map out the shape of the scalar potential for generic φ_0, however, provided we turn on an external current, J, coupled to φ_0 that is designed to ensure that φ_0 is a stationary point of the potential, including the current. The shape of the effective potential can be computed by seeing precisely how much current is required as a function of φ_0. In what follows we define the current coupling by adding the following term to the action[5]

$$S_J = -\int \mathrm{d}^6x\sqrt{-g}\; J\,, \tag{5.21}$$

where J is a constant (since our goal is only to couple a current to the would-be zero mode, φ_0).

In this kind of construction the stabilized value, $\varphi_\star$, corresponds to the choice for which no external current is necessary, $J(\varphi_\star) = 0$. An important difference between the supersymmetric system of interest here and the non-supersymmetric one studied in ref. (5.22) is that in the supersymmetric case it can (but need not) happen that there is no value of φ_0 for which $J(\varphi_0) = 0$. As we shall see, from the 4D point of view this corresponds to an effective potential that is a pure runaway, for which $V'_{\rm eff}(\varphi_0)$ only vanishes as $\varphi_0 \to \pm\infty$.

Linearized equations

Our goal is to solve the above field equations by linearizing them about a rugby-ball solution. This amounts to assuming that the ϕ-dependent contribution to τ_b is small relative to the tension that is responsible for the rugby-ball geometry itself:

$$\tau_b = \tau + \delta\tau_b(\phi) \quad \text{and} \quad \Phi_b = \Phi + \delta\Phi_b(\phi)\,, \tag{5.22}$$

with the background deficit angle sourced by $T = \tau - \mathcal{Q}\,\Phi$. The linearized equations of motion including the current term — derived in Appendix (C.3) — are given below. All background (rugby-ball) quantities are denoted by a

[5]As is shown in Appendix C.2, most of the low-energy physics of interest is insensitive to the detailed form of the current to which we couple, so long as it has a good overlap with the would-be zero mode.

subscript 0, and perturbations are universally denoted by δ: so $\mathcal{Q} = \mathcal{Q}_0 + \delta\mathcal{Q}$ *etc.* Since we ignore all second-order quantities we may write $\delta\mathcal{Q}/\mathcal{Q} \simeq \delta\mathcal{Q}/\mathcal{Q}_0$ and so can use either of these quantities interchangeably. Also, since $W_0 = 0$ for the rugby balls, $W = \delta W$.

To linear order the Maxwell field strength becomes

$$\mathcal{F}_{\rho\theta} = \mathcal{Q}\alpha L e^{\varphi_0/2} \sin\left(\frac{\hat{\rho}}{L}\right)\left(1 + \frac{\delta\mathcal{Q}}{\mathcal{Q}_0} + \delta B - 4\delta W\right), \tag{5.23}$$

the on-brane curvature is

$$\hat{R} = -4e^{\varphi_0}\left[\frac{2\delta W}{L^2} + \frac{1}{L}\cot\left(\frac{\hat{\rho}}{L}\right)\partial_{\hat{\rho}}\,\delta W + \partial^2_{\hat{\rho}}\,\delta W\right] + \frac{2e^{\varphi_0}}{L^2}\left(\frac{\delta\mathcal{Q}}{\mathcal{Q}}\right) - 2\kappa^2 J\,, \tag{5.24}$$

and the remaining linearized field equations become

$$\begin{aligned}
\partial_{\hat{\rho}}\left[\sin\left(\frac{\hat{\rho}}{L}\right)\partial_{\hat{\rho}}(\delta\phi)\right] &= \frac{1}{L^2}\left(4\delta W - \frac{\delta\mathcal{Q}}{\mathcal{Q}}\right)\sin\left(\frac{\hat{\rho}}{L}\right) \\
\frac{\partial_{\hat{\rho}}\left[\sin^2\left(\frac{\hat{\rho}}{L}\right)\partial_{\hat{\rho}}(\delta B)\right]}{\sin^2\left(\frac{\hat{\rho}}{L}\right)} &= -\frac{1}{L^2}\left[\delta\phi + \frac{3}{2}\left(\frac{\delta\mathcal{Q}}{\mathcal{Q}}\right) - 6\,\delta W + \kappa^2 J L^2 e^{-\varphi_0}\right] \\
&\quad -\frac{4}{L}\cot\left(\frac{\hat{\rho}}{L}\right)\partial_{\hat{\rho}}\,\delta W \\
\text{and}\quad \partial^2_{\hat{\rho}}\delta W &= \frac{1}{L}\cot\left(\frac{\hat{\rho}}{L}\right)\partial_{\hat{\rho}}\,\delta W\,.
\end{aligned} \tag{5.25}$$

Finally, the linearized flux quantization condition can be expressed as

$$\frac{\delta\mathcal{Q}}{\mathcal{Q}} = \frac{1}{2L}\int_0^{\pi L} \mathrm{d}\hat{\rho}\,\sin\left(\frac{\hat{\rho}}{L}\right)(4\delta W - \delta B - \delta\phi) - \frac{\kappa^2\mathcal{Q}}{4\pi\alpha}\Big(\delta\Phi_N + \delta\Phi_S\Big)\,. \tag{5.26}$$

5.3.3 Linearized solutions

The strategy is to construct the general solution to these linearized equations, and then to use the brane matching conditions to eliminate the resulting integration constants in terms of brane properties. To simplify expressions it is convenient to define the dimensionless coordinate $x := \hat{\rho}/L = (\rho/L)e^{-\varphi_0/2}$, keeping in mind that its implicit dependence on φ_0 brings this dependence to any bulk fields that depend on x. We have some freedom in how to group the perturbations; which we employ (without loss of generality) to simplify the linearized flux-quantization condition as much as possible.

First, we solve the equation for the warp factor, W, which has the general solution

$$\delta W(x) = W_0 + W_1 \cos x \,, \tag{5.27}$$

where W_0 and W_1 are integration constants, of which $W_0 = 0$ may be ensured by rescaling the on-brane coordinates, x^μ.

With this solution, the equation to be solved for the dilaton becomes

$$\partial_x \Big[\sin x \, \partial_x (\delta\phi) \Big] = \left(4W_1 \cos x - \frac{\delta\mathcal{Q}}{\mathcal{Q}} \right) \sin x \,, \tag{5.28}$$

which integrates to give

$$\delta\phi\,(x) = \delta\varphi_0 + \varphi_1 \ln \left| \frac{1 - \cos x}{\sin x} \right| - 2W_1 \cos x + \left(\frac{\delta\mathcal{Q}}{\mathcal{Q}} \right) \ln |\sin x| \,. \tag{5.29}$$

Here $\delta\varphi_0$ and φ_1 are integration constants, of which $\delta\varphi_0 = 0$ can be ensured without loss of generality by absorbing it into the otherwise arbitrary background value, φ_0.

Finally, the equation of motion for δB becomes

$$\frac{\partial_x\left[\sin^2 x\,\partial_x(\delta B)\right]}{\sin^2 x} = -\varphi_1\ln\left|\frac{1-\cos x}{\sin x}\right| - \frac{\delta\mathcal{Q}}{\mathcal{Q}}\left(\frac{3}{2}+\ln|\sin x|\right) + 12W_1\cos x - \kappa^2 JL^2 e^{-\varphi_0}\,. \tag{5.30}$$

This integrates to

$$\delta B = \delta\hat{B}_0 + B_1\cot x - 4W_1\cos x - \varphi_1\mathcal{M}_2(x) + \frac{\delta\mathcal{Q}}{\mathcal{Q}}\left[\frac{3}{4}\,x\cot x - \mathcal{H}_2(x)\right] + \frac{1}{2}\left(\kappa^2 JL^2 e^{-\varphi_0}\right)x\cot x\,, \tag{5.31}$$

where $\delta\hat{B}_0$ and B_1 are integration constants. Of these, B_1 is pure gauge in that it can be changed arbitrarily by reparameterizing the coordinate ρ. We fix this coordinate freedom by defining $\rho = 0$ to be the position of the 'north' brane, which requires $e^B \to 0$ as $\rho \to 0$; ensuring $B_1 = 0$. The functions $\mathcal{M}_2$ and $\mathcal{H}_2$ appearing here are defined by

$$\begin{aligned}\mathcal{M}_1(x) &:= \int_0^x \mathrm{d}y\,\sin^2 y\ln\left|\frac{1-\cos y}{\sin y}\right| \\ \mathcal{M}_2(x) &:= \int_0^x \mathrm{d}y\,\frac{\mathcal{M}_1(y)}{\sin^2 y}\,,\end{aligned} \tag{5.32}$$

and

$$\begin{aligned}\mathcal{H}_1(x) &:= \int_0^x \mathrm{d}y\,\sin^2 y\,\ln|\sin y| \\ \mathcal{H}_2(x) &:= \int_0^x \mathrm{d}y\,\frac{\mathcal{H}_1(y)}{\sin^2 y}\,.\end{aligned} \tag{5.33}$$

For later convenience when discussing flux quantization it is useful to absorb parts of these integrals into the definition of $\delta\hat{B}_0$, by writing

$$\delta B = \delta B_0 - 4W_1 \cos x + \varphi_1 \left[\frac{\overline{\mathcal{M}}}{2} - \mathcal{M}_2(x) \right] + \frac{\delta\mathcal{Q}}{\mathcal{Q}} \left[\frac{3}{4}\, x \cot x - \mathcal{H}_2(x) \right] + \frac{1}{2} \left(\kappa^2 J L^2 e^{-\varphi_0} \right) (x \cot x + 1) \,, \tag{5.34}$$

with the number $\overline{\mathcal{M}}$ defined by

$$\overline{\mathcal{M}} := \int_0^\pi \mathrm{d}x \, \sin x \, \mathcal{M}_2(x) \,. \tag{5.35}$$

Numerically this evaluates to the value[6] $\overline{\mathcal{M}} = -1$, which we use throughout what follows.

Flux quantization

Using the above expressions in the linearized flux quantization condition, eq. (5.26), gives

$$\begin{aligned} \frac{\delta\mathcal{Q}}{\mathcal{Q}} &= \frac{1}{2} \int_0^\pi \mathrm{d}x \, \sin x \Big(4\delta W - \delta B - \delta\phi \Big) - \frac{\kappa^2 \mathcal{Q}}{4\pi\alpha} \Big(\delta\Phi_{\scriptscriptstyle N} + \delta\Phi_{\scriptscriptstyle S} \Big) \\ &= -\delta B_0 + \frac{3}{4} \left(\frac{\delta\mathcal{Q}}{\mathcal{Q}} \right) - \frac{\kappa^2 \mathcal{Q}}{4\pi\alpha} \Big(\delta\Phi_{\scriptscriptstyle N} + \delta\Phi_{\scriptscriptstyle S} \Big) \,, \end{aligned} \tag{5.36}$$

which uses the integral

$$\overline{\mathcal{H}} := \int_0^\pi \mathrm{d}x \, \sin x \, \mathcal{H}_2(x) \simeq -0.613706 \simeq \ln 4 - 2 \,, \tag{5.37}$$

and the last approximate equality is a numerical inference.[7] The absence of φ_1 on the right-hand side of eq. (5.36) is a consequence of the definition of δB_0

[6] Maple 11, 10 digit precision, see Appendix (C.4)
[7] Mathematica 7, with thanks to Ben Jackel.

used in eq. (5.34). Solving this for δB_0 gives

$$\delta B_0 = -\frac{1}{4}\left(\frac{\delta \mathcal{Q}}{\mathcal{Q}}\right) - \frac{\kappa^2 \mathcal{Q}}{4\pi\alpha}\Big(\delta\Phi_{\scriptscriptstyle N} + \delta\Phi_{\scriptscriptstyle S}\Big)\,. \tag{5.38}$$

Finally, the linearized field equations return the following on-brane curvature,

$$\begin{aligned}\hat{R} &= -4e^{\varphi_0}\left[\frac{2\delta W}{L^2} + \frac{1}{L}\cot\left(\frac{\hat\rho}{L}\right)\partial_{\hat\rho}\,\delta W + \partial^2_{\hat\rho}\,\delta W\right] + \frac{2e^{\varphi_0}}{L^2}\left(\frac{\delta\mathcal{Q}}{\mathcal{Q}}\right) - 2\kappa^2 J \\ &= \frac{2e^{\varphi_0}}{L^2}\left(\frac{\delta\mathcal{Q}}{\mathcal{Q}}\right) - 2\kappa^2 J\,. \end{aligned} \tag{5.39}$$

Notice that all of the ρ-dependence cancels in this expression (as must happen given our assumption of maximal symmetry), leaving a result that is determined purely by the change of bulk Maxwell flux and the applied current.

5.3.4 Physical interpretation and renormalization

The above solutions are described by four physical integration constants, which we can take to be φ_0, φ_1, W_1 and $\delta\mathcal{Q}/\mathcal{Q}$. These can be traded for four physical properties of the bulk and on-brane geometries.

W_1 can be taken to be the difference between the value of the warping (which controls the gravitational redshift) between the two branes, which is given by

$$\delta W_{\scriptscriptstyle N} - \delta W_{\scriptscriptstyle S} = 2\,W_1\,. \tag{5.40}$$

Similarly, to linear order the near-brane geometry as $x = \hat\rho/L \to 0$ is governed by

$$e^B \simeq e^{-\varphi_0/2}\alpha L \sin x\Big[1+\delta B(x)\Big] \tag{5.41}$$
$$\simeq \alpha\rho\left[1+\delta B_0 - 4W_1 - \frac{\varphi_1}{2} + \frac{3}{4}\left(\frac{\delta\mathcal{Q}}{\mathcal{Q}}\right) + \kappa^2 JL^2 e^{-\varphi_0} + \mathcal{O}(\rho^2)\right],$$

which corresponds to a conical singularity (since the e^B vanishes linearly with ρ), having defect angle $\alpha_N = \alpha + \delta\,\alpha_N$ with

$$\frac{\delta\,\alpha_N}{\alpha} = \delta B_0 - 4W_1 - \frac{\varphi_1}{2} + \frac{3}{4}\left(\frac{\delta\mathcal{Q}}{\mathcal{Q}}\right) + \kappa^2 JL^2 e^{-\varphi_0}. \tag{5.42}$$

By contrast, as $x = \hat{\rho}/L \to \pi$ we have

$$e^B \to \pi\alpha L e^{-\varphi_0/2}\left[\left(1-\frac{1}{2}\ln 2\right)\left(\frac{\delta\mathcal{Q}}{\mathcal{Q}}\right) - \frac{1}{2}\left(\kappa^2 JL^2 e^{-\varphi_0}\right)\right], \tag{5.43}$$

which uses $\mathcal{H}_2(\pi-\varepsilon) = (1-\ln 4)(\pi/4\varepsilon) + \mathcal{O}(\varepsilon^0)$. In particular, this shows that e^B does *not* vanish at $\hat{\rho} = \pi L$. Instead e^B vanishes at $\hat{\rho} = \pi(L+\delta L)$, indicating a change in proper distance between the branes: $\rho_S - \rho_N = \pi(L+\delta L)e^{-\varphi_0/2}$. The amount of the change is obtained by comparing eq. (5.43) to the Taylor expansion of e^B about its new zero, giving

$$\frac{\delta L}{L} \simeq -\left[\frac{3}{4}\left(\frac{\delta\mathcal{Q}}{\mathcal{Q}}\right) + \frac{1}{2}\left(\kappa^2 JL^2 e^{-\varphi_0}\right)\right]. \tag{5.44}$$

The singularity at the 'south' brane is also conical (at linear order), with defect angle given by $\alpha_S = \alpha + \delta\,\alpha_S$ with

$$\frac{\delta\,\alpha_S}{\alpha} = \delta B_0 + 4W_1 + \frac{\varphi_1}{2} + \frac{3}{4}\left(\frac{\delta\mathcal{Q}}{\mathcal{Q}}\right) + \kappa^2 JL^2 e^{-\varphi_0}. \tag{5.45}$$

One might imagine that a further observable could be the difference between the value of the dilaton field at the two branes, $\phi_N - \phi_S$, since this governs the relative strength of some of the bulk couplings to each brane (such as the strength of the bulk Maxwell couplings to brane-localized charged particles). However a subtlety arises in this case because the profile $\phi(\rho)$ diverges in the limit that $\rho \to \rho_N$ and $\rho \to \rho_S$. For this reason we defer a discussion of this quantity to the next section, which deals with renormalizing these divergences.

Brane matching and renormalization

Ultimately the bulk integration constants should be related to physical properties of the branes that are the source of the bulk geometry; this is where the brane matching conditions play a role. In order to perform this matching we must specify a functional form for the brane tensions, τ_b, and fluxes, Φ_b. We take both of these to be smooth functions of ϕ, and in many (but not all) examples we imagine these functions to be extremized at $\phi = \hat{v}_b$: that is, $(\partial\tau_b/\partial\phi)_{\phi=\hat{v}_b} = 0$.

The problem in practice with matching is that the argument of τ_b and Φ_b is $\phi_b := \phi(\rho_b)$, but the profile $\phi(\rho)$ given in eq. (5.29) diverges as $\rho \to \rho_b$. For instance, for $x = \hat{\rho}/L = \varepsilon \ll 1$ and $x = \pi - \varepsilon$ we have

$$\begin{aligned} \phi\,(x=\varepsilon) \;&=\; \varphi_0 - 2W_1 + \varphi_1 \ln\left|\frac{\varepsilon}{2}\right| + \left(\frac{\delta\mathcal{Q}}{\mathcal{Q}}\right)\ln|\varepsilon| + \mathcal{O}(\varepsilon) \\ \text{and} \quad \phi\,(x=\pi-\varepsilon) \;&=\; \varphi_0 + 2W_1 + \varphi_1 \ln\left|\frac{2}{\varepsilon}\right| + \left(\frac{\delta\mathcal{Q}}{\mathcal{Q}}\right)\ln|\varepsilon| + \mathcal{O}(\varepsilon)\,. \end{aligned} \tag{5.46}$$

This divergence is dealt with by renormalizing the parameters that define the functional form of τ_b and Φ_b, and in particular those parameters that determine the values $\hat{v}_b$. It can be absorbed in the definitions of $\hat{v}_b$ by defining renormalized quantities, v_b:

$$\begin{aligned} v_N &= \hat{v}_N - \frac{\delta\mathcal{Q}}{\mathcal{Q}} \ln(\varepsilon) - \varphi_1 \ln(\varepsilon/2) \\ v_S &= \hat{v}_S - \frac{\delta\mathcal{Q}}{\mathcal{Q}} \ln(\varepsilon) + \varphi_1 \ln(\varepsilon/2)\,, \end{aligned} \tag{5.47}$$

where the first expression is relevant at $x = 0$ (the north brane positon), and the second one at $x = \pi$ (the south brane). With these definitions,

$$\begin{aligned} \lim_{\varepsilon\to 0}\Big[\phi(\varepsilon) - \hat{v}_N\Big] &= \varphi_0 - 2W_1 - v_N \\ \text{and} \quad \lim_{\varepsilon\to 0}\Big[\phi(\pi-\varepsilon) - \hat{v}_S\Big] &= \varphi_0 + 2W_1 - v_S\,, \end{aligned} \tag{5.48}$$

and so $\tau_b(\phi - \hat{v}_b) = \tau_b(\varphi_0 \pm 2W_1 - v_b)$ and so on. This is a useful redefinition because our interest really is in the value at which the zero mode, φ_0, gets stabilized, rather than on the value of ϕ itself at the brane position. And this is finite as $\varepsilon \to 0$ with renormalized quantities (like v_b) fixed.

With this construction in mind, there are four independent matching conditions:

$$\begin{aligned} \left[e^B\partial_\rho\phi\right]_{\rho=0} = \frac{\kappa^2}{2\pi}\left(\frac{\partial T_N}{\partial\phi}\right) \quad &\text{and} \quad \left[e^B\partial_\rho\phi\right]_{\rho=\pi L} = -\frac{\kappa^2}{2\pi}\left(\frac{\partial T_S}{\partial\phi}\right) \\ \left[e^B\partial_\rho B\right]_{\rho=0} = 1 - \frac{\kappa^2 T_N}{2\pi} \quad &\text{and} \quad \left[e^B\partial_\rho B\right]_{\rho=\pi L} = -1 + \frac{\kappa^2 T_S}{2\pi}\,, \end{aligned} \tag{5.49}$$

where, as before, $T_b = \tau_b - \mathcal{Q}\,\Phi_b\, e^{-4W(\rho_b)}$. The difference in signs between north and south brane arises because increasing ρ points away from the north brane but towards the south brane.

Specialized to the dilaton profile, eq. (5.29), the first two of the above conditions become

$$\begin{aligned} \alpha\left(\varphi_1 + \frac{\delta\mathcal{Q}}{\mathcal{Q}}\right) &= \frac{\kappa^2}{2\pi}\left(\frac{\partial\,\delta T_{\scriptscriptstyle N}}{\partial\phi}\right) \\ \alpha\left(\varphi_1 - \frac{\delta\mathcal{Q}}{\mathcal{Q}}\right) &= -\frac{\kappa^2}{2\pi}\left(\frac{\partial\,\delta T_{\scriptscriptstyle S}}{\partial\phi}\right), \end{aligned} \tag{5.50}$$

while the latter two evaluate to

$$\begin{aligned} \alpha\left[-4W_1 - \frac{\varphi_1}{2} + \frac{3}{4}\left(\frac{\delta\mathcal{Q}}{\mathcal{Q}}\right) + \delta B_0 + \kappa^2 J L^2 e^{-\varphi_0}\right] &= -\frac{\kappa^2}{2\pi}\,\delta T_{\scriptscriptstyle N} \\ \alpha\left[4W_1 - \frac{\varphi_1}{2} + \frac{3}{4}\left(\frac{\delta\mathcal{Q}}{\mathcal{Q}}\right) + \delta B_0 + \kappa^2 J L^2 e^{-\varphi_0}\right] &= -\frac{\kappa^2}{2\pi}\,\delta T_{\scriptscriptstyle S}, \end{aligned} \tag{5.51}$$

where the change in brane action from the background value, T, is $\delta T_b := T_b - T = \delta\tau_b - \mathcal{Q}\,\delta\Phi_b + \mathcal{Q}\,\Phi[4\delta W(\rho_b) - \delta\mathcal{Q}/\mathcal{Q}]$. However, the terms involving δW and $\delta\mathcal{Q}/\mathcal{Q}$ in $\kappa^2\delta T_b$ may be dropped in the matching conditions because their contributions are suppressed by an additional factor of $\kappa^2\mathcal{Q}\Phi/2\pi$ relative to the leading contributions. Hence, from here on we take $\delta T_b \simeq \delta\tau_b - \mathcal{Q}\,\delta\Phi_b$.

Eliminating δB_0 using eq. (5.38), and solving the above matching conditions gives

$$\begin{aligned} \frac{\delta\mathcal{Q}}{\mathcal{Q}} &= \frac{\kappa^2}{4\pi\alpha}\left[\delta T'_{\scriptscriptstyle N} + \delta T'_{\scriptscriptstyle S}\right] \\ \varphi_1 &= \frac{\kappa^2}{4\pi\alpha}\left[\delta T'_{\scriptscriptstyle N} - \delta T'_{\scriptscriptstyle S}\right] \\ W_1 &= \frac{\kappa^2}{16\pi\alpha}\left[\left(\delta T_{\scriptscriptstyle N} + \frac{1}{2}\,\delta T'_{\scriptscriptstyle N}\right) - \left(\delta T_{\scriptscriptstyle S} + \frac{1}{2}\,\delta T'_{\scriptscriptstyle S}\right)\right] \\ \kappa^2 J L^2 e^{-\varphi_0} &= -\frac{\kappa^2}{4\pi\alpha}\left[\left(\delta T_{\scriptscriptstyle N} + \frac{1}{2}\,\delta T'_{\scriptscriptstyle N} - \mathcal{Q}\,\delta\Phi_{\scriptscriptstyle N}\right) + \left(\delta T_{\scriptscriptstyle S} + \frac{1}{2}\,\delta T'_{\scriptscriptstyle S} - \mathcal{Q}\,\delta\Phi_{\scriptscriptstyle S}\right)\right], \end{aligned} \tag{5.52}$$

where $\delta T_b'$ denotes $\partial\, \delta T_b/\partial\phi$. These expressions allow the elimination of the three integration constants (φ_1, W_1 and $\delta \mathcal{Q}/\mathcal{Q}$) and the current, J, to be completely expressed in terms of brane properties and φ_0.

In particular, the condition $J = 0$ is satisfied when $\varphi_0 = \varphi_\star$, where

$$\left[\left(\delta T_{\scriptscriptstyle N} + \frac{1}{2}\,\delta T_{\scriptscriptstyle N}' - \mathcal{Q}\,\delta\Phi_{\scriptscriptstyle N}\right) + \left(\delta T_{\scriptscriptstyle S} + \frac{1}{2}\,\delta T_{\scriptscriptstyle S}' - \mathcal{Q}\,\delta\Phi_{\scriptscriptstyle S}\right)\right]_{\varphi_0=\varphi_\star} = 0\,. \quad (5.53)$$

This expression determines the stabilized value, $\varphi_0 = \varphi_\star$, as a function of the properties of the branes.

On-brane curvature

Finally, the curvature in the on-brane directions, regarded as a function of φ_0, becomes

$$\begin{aligned}\left(\frac{\pi\alpha L^2 e^{-\varphi_0}}{\kappa^2}\right)\hat{R}(\varphi_0) &= \frac{1}{2}\Big(\delta T_{\scriptscriptstyle N}' + \delta T_{\scriptscriptstyle S}'\Big) - \frac{2\pi\alpha}{\kappa^2}\left(\kappa^2 L^2 J e^{-\varphi_0}\right) \qquad (5.54)\\ &= \frac{1}{2}\left[\delta T_{\scriptscriptstyle N} + \delta T_{\scriptscriptstyle S} + \frac{3}{2}(\delta T_{\scriptscriptstyle N}' + \delta T_{\scriptscriptstyle S}') - \mathcal{Q}(\delta\Phi_{\scriptscriptstyle N} + \delta\Phi_{\scriptscriptstyle S})\right].\end{aligned}$$

Of particular interest is this result specialized to the value, $\varphi_0 = \varphi_\star$, that solves the field equations in the absence of the current J (if such a value exists – more about this below). The curvature evaluated at this value is the curvature predicted by the field equations for the brane geometry, and eq. (5.53) allows it to be written in two equivalent ways:

$$\left(\frac{\pi\alpha L^2 e^{-\varphi_\star}}{\kappa^2}\right)\hat{R} = \frac{1}{2}\Big(\delta T_{\scriptscriptstyle N}' + \delta T_{\scriptscriptstyle S}'\Big)_{\varphi_0=\varphi_\star} = -\Big[\delta T_{\scriptscriptstyle N} + \delta T_{\scriptscriptstyle S} - \mathcal{Q}(\delta\Phi_{\scriptscriptstyle N} + \delta\Phi_{\scriptscriptstyle S})\Big]_{\varphi_0=\varphi_\star}. \quad (5.55)$$

The second of these agrees precisely with the corresponding expression obtained in the nonsupersymmetric case studied in ref. (5.22). However this is *not* also equal to the first equality of eq. (5.55), because in the nonsupersymmetric case eq. (5.53) no longer holds, being instead replaced by $\delta T'_{\scriptscriptstyle N} + \delta T'_{\scriptscriptstyle S} = 0$.

From the point of view of a brane observer this must agree with the (maximally symmetric) curvature that is predicted by the 4D Einstein equations given a 4D vacuum energy, $\varrho_{\rm eff}$:

$$\hat{R} = -4\kappa_4^2 \, \varrho_{\rm eff} \,, \tag{5.56}$$

where κ_4 is the 4D gravitational coupling, given in terms of the 6D coupling, κ, by

$$\frac{1}{\kappa_4^2} = \frac{4\pi\alpha L^2 e^{-\varphi_\star}}{\kappa^2} \,. \tag{5.57}$$

Comparison gives

$$\begin{aligned} \varrho_{\rm eff} = -\frac{\hat{R}}{4\kappa_4^2} &= -\left(\frac{\pi\alpha L^2}{\kappa^2}\right)\hat{R} \\ &= -\frac{1}{2}\Big(\delta T'_{\scriptscriptstyle N} + \delta T'_{\scriptscriptstyle S}\Big)_{\varphi_0=\varphi_\star} \,. \end{aligned} \tag{5.58}$$

Notice that this agrees (to linear order) with the more general exact classical result obtained in eq. (3.81) of ref. (5.32),

$$\varrho_{\rm eff} = -\sum_b \left(U_b + \frac{1}{2}\,T'_b\right) \,, \tag{5.59}$$

given that U_b vanishes to linear order.

5.3.5 The low-energy 4D effective theory

This section constructs the effective 4D theory that reproduces the low-energy dynamics of φ_0 and the 4D metric predicted by the full 6D theory. We do so at the purely classical level, working perturbatively about a rugby ball solution, as above.

General form

In this section the two fields of interest in the low-energy theory are the 4D metric, $\hat{g}_{\mu\nu}$, describing the massless KK graviton, and a 4D scalar,[8] φ, describing the low-energy would-be zero mode, φ_0, associated with the scaling symmetry of the bulk field equations. (We ignore here any other low-energy fields, such as other 4D scalars or 4D gauge fields coming from $\mathcal{A}_M$ or the metric.)

The most general possible local 4D effective theory describing the interactions of φ and $\hat{g}_{\mu\nu}$, up to the two-derivative level, is

$$S_{\rm eff} = -\int {\rm d}^4x \sqrt{-\hat{g}} \; \Big\{ \hat{g}^{\mu\nu} \Big[f(\varphi) \hat{R}_{\mu\nu} + h(\varphi)\, \partial_\mu \varphi \, \partial_\nu \varphi \Big] + V_{\scriptscriptstyle JF}(\varphi) + j\, k(\varphi) \Big\} \,, \tag{5.60}$$

where f, h, $V_{\scriptscriptstyle JF}$ and k are all functions to be determined, and j denotes a low-energy current that is included to explore the shape of these functions (in precisely the same manner as J was included in the 6D theory). Our task is to identify these functions by matching the predictions of this theory with the low-energy predictions of the full 6D system.

The functions f, h and k differ from $V_{\scriptscriptstyle JF}$ in that they already receive

[8] We use φ to denote the 4D field in the effective theory, to distinguish it from the (closely related) parameter φ_0 appearing in the 6D solutions.

their leading contributions when the two source branes are described by their background tensions, T; without the symmetry-breaking, ϕ-dependent contributions $\delta T_b(\phi)$. These leading contributions can be obtained by simple dimensional reduction, which predicts

$$\begin{aligned} f(\varphi) = h(\varphi) &= \frac{4\pi\alpha L^2 e^{-\varphi}}{\kappa^2} = \frac{1}{2\kappa_4^2}\, e^{-(\varphi-\varphi_\star)} \\ \text{and} \quad k(\varphi) &= e^{-\varphi} \quad \text{if we define} \quad j \propto 4\pi\alpha L^2 J\,. \end{aligned} \tag{5.61}$$

To the same approximation the (Jordan frame) potential vanishes, $V_{JF}(\varphi) = 0$, since the background branes do not break the classical bulk scaling symmetry.

Low-energy matching conditions

The goal is to determine how these quantities are perturbed by the addition of ϕ-dependence to the brane action, $\delta T_b(\varphi)$. Our main focus is on the contribution to V_{JF}, since (unlike for the other functions) for V_{JF} this is the dominant contribution. We use the prediction for the low-energy scalar curvature, $\hat{R}$, as a function of φ — *i.e.* eq. (5.54) — as our means for doing so.

To make the comparison we compute $\hat{R}$ in the low-energy effective theory, assuming a maximally symmetric geometry. Defining for notational convenience $1/\hat{\kappa}_4^2 := e^{\varphi_\star}/\kappa_4^2 = 4\pi\alpha L^2/\kappa^2$, the metric and scalar equations of motion are

$$\begin{aligned} e^{-\varphi}\frac{\hat{R}}{4\hat{\kappa}_4^2} + je^{-\varphi} + V_{JF}(\varphi) = 0 \\ -e^{-\varphi}\frac{\hat{R}}{2\hat{\kappa}_4^2} - je^{-\varphi} + V_{JF}'(\varphi) = 0\,. \end{aligned} \tag{5.62}$$

Eliminating the current between these two equations gives the following expression for $\hat{R}$ as a function of φ,

$$e^{-\varphi}\frac{\hat{R}}{4\hat{\kappa}_4^2} = V_{JF} + V'_{JF} = e^{-\varphi}\frac{\mathrm{d}}{\mathrm{d}\varphi}\Big(e^{\varphi}\, V_{JF}\Big)\,. \tag{5.63}$$

To obtain V_{JF} we regard eq. (5.63) as a differential equation to be integrated with respect to φ, using eq. (5.54) to evaluate the left-hand side as an explicit function of φ. The integral yields

$$\begin{aligned} V_{JF}(\varphi) &= \frac{1}{2}\,e^{-\varphi}\int \mathrm{d}\varphi_0\, e^{\varphi_0}\left(\delta T_N + \delta T_S - \mathcal{Q}\,\delta\Phi_N - \mathcal{Q}\,\delta\Phi_S + \frac{3}{2}\,\delta T'_N + \frac{3}{2}\,\delta T'_S\right) \\ &= \frac{1}{2}\Big(\delta T_N + \delta T_S\Big) \\ &\quad +\frac{1}{2}\,e^{-\varphi}\int \mathrm{d}\varphi_0\, e^{\varphi_0}\left(-\mathcal{Q}\,\delta\Phi_N - \mathcal{Q}\,\delta\Phi_S + \frac{1}{2}\,\delta T'_N + \frac{1}{2}\,\delta T'_S\right)\,. \end{aligned} \tag{5.64}$$

The integration constant, C, implicit in this integration contributes an amount $C\,e^{-\varphi}$ to V_{JF}, with C fixed by matching to the 6D theory at a specific value of φ. A convenient place for doing so is the vacuum configuration (if this exists), $\varphi = \varphi_\star$, defined by $j(\varphi_\star) = 0$, for which a prediction — eq. (5.53) — is known in the 6D theory.

Specifically, solving eqs. (5.62) for $j(\varphi)$ gives

$$je^{-\varphi} = -\Big[V'_{JF}(\varphi) + 2V_{JF}(\varphi)\Big]\,, \tag{5.65}$$

and so $\varphi_\star$ satisfies

$$V'_{JF}(\varphi_\star) + 2V_{JF}(\varphi_\star) = 0\,. \tag{5.66}$$

This has a simple interpretation in the Einstein frame, which is defined by rescaling $\hat{g}_{\mu\nu} = e^{(\varphi-\varphi_\star)}g_{\mu\nu}$, so that the 4D action has a canonical Einstein-

Hilbert term

$$S_{\rm eff} = -\int {\rm d}^4x\sqrt{-g}\;\left\{\frac{1}{2\kappa_4^2}\,g^{\mu\nu}\left[R_{\mu\nu} + 5\,\partial_\mu\varphi\,\partial_\nu\varphi\right] + V_{\scriptscriptstyle EF}(\varphi) + j_{\scriptscriptstyle EF}\,e^{\varphi}\right\}\,, \quad (5.67)$$

with $j_{\scriptscriptstyle EF} := je^{-2\varphi_\star}$ and

$$V_{\scriptscriptstyle EF}(\varphi) := e^{2(\varphi-\varphi_\star)}\,V_{\scriptscriptstyle JF}(\varphi)\,. \quad (5.68)$$

Clearly $\varphi_\star$ therefore satisfies $V'_{\scriptscriptstyle EF}(\varphi_\star) = 0$, as might have been expected. Imposing $V'_{\scriptscriptstyle JF} + 2V_{\scriptscriptstyle JF} = 0$ when $\varphi = \varphi_\star$ satisfies eq. (5.53) then gives

$$\begin{aligned} V_{\scriptscriptstyle JF}(\varphi) \;&=\; \frac12\sum_b \delta T_b(\varphi) - \frac12\,e^{-(\varphi-\varphi_\star)}\sum_b\left[\frac12\,\delta T_b'(\varphi_\star) + \mathcal{Q}\,\delta\Phi_b(\varphi_\star)\right] \quad (5.69)\\ &\quad + \frac12\,e^{-\varphi}\int_{\varphi_\star}^{\varphi}{\rm d}\varphi_0\,e^{\varphi_0}\sum_b\left[\frac12\,\delta T_b'(\varphi_0) - \mathcal{Q}\,\delta\Phi_b(\varphi_0)\right]\,. \end{aligned}$$

Given the Einstein-frame potential, classical vacuum energy is

$$\begin{aligned} \varrho_{\rm eff} = V_{\scriptscriptstyle EF}(\varphi_\star) = V_{\scriptscriptstyle JF}(\varphi_\star) \;&=\; \sum_b\Big[\delta T_b(\varphi_\star) - \mathcal{Q}\,\delta\Phi_b(\varphi_\star)\Big]\\ &=\; -\frac12\sum_b \delta T_b'(\varphi_\star)\,, \quad (5.70) \end{aligned}$$

as found earlier (using eq. (5.53)). The scalar mass similarly is

$$\begin{aligned} m_\varphi^2 \;&=\; \frac{\kappa_4^2}{5}\,V''_{\scriptscriptstyle EF}(\varphi_\star) = \frac{\kappa_4^2}{5}\Big[V''_{\scriptscriptstyle JF}(\varphi_\star) - 4V_{\scriptscriptstyle JF}(\varphi_\star)\Big]\\ &=\; \frac{\kappa_4^2}{5}\sum_b\left[\frac34\,\delta T_b''(\varphi_\star) + \frac32\,\delta T_b'(\varphi_\star) - \frac12\,\mathcal{Q}\,\delta\Phi_b'(\varphi_\star)\right]\,. \quad (5.71) \end{aligned}$$

Similarly, chasing through the earlier expressions for the shape of the bulk geometry gives

$$
\begin{aligned}
\frac{\delta\alpha_b}{\alpha} &= -\frac{\kappa^2}{2\pi\alpha}\,\delta T_b(\varphi_\star)\\
\frac{\delta L}{L} &= -\frac{3\kappa^2}{16\pi\alpha}\sum_b \delta T_b'(\varphi_\star) = \frac{3\kappa^2\varrho_{\rm eff}}{8\pi\alpha}\\
\delta W_N - \delta W_S &= \frac{\kappa^2}{4\pi\alpha}\left[\left(\delta T_S + \frac{1}{2}\,\delta T_S'\right) - \left(\delta T_N + \frac{1}{2}\,\delta T_N'\right)\right]_{\varphi_\star}\\
&= \frac{\kappa^2}{8\pi\alpha}\Big[\mathcal{Q}\,\delta\Phi_N(\varphi_\star) - \mathcal{Q}\,\delta\Phi_S(\varphi_\star)\Big]\,. \qquad (5.72)
\end{aligned}
$$

Notice in particular that no warping arises unless the two branes carry different amounts of localized flux. This is by contrast with the nonsupersymmetric case (5.22), for which net warping always accompanies a tension difference for the two source branes. But in the supersymmetric case the flux quantization condition does not allow such a tension difference without some of the flux being forced onto the branes.

5.4 Illustrative examples

The previous formulae with which the previous section closed represent the main results of this paper. We now explore their consequences through a number of illustrative special choices for the φ-dependence of the tensions on each brane.

5.4.1 Dilaton-independent tensions and fluxes

Consider first the simplest example: where both quantities $\delta\tau_b$ and $\delta\Phi_b$ are independent of φ. In this case the condition, $J(\varphi_\star) = 0$, defining $\varphi_\star$ degenerates to

$$\sum_b \Big(\delta T_b - \mathcal{Q}\, \delta\Phi_b \Big) = \sum_b \Big(\delta\tau_b - 2\, \mathcal{Q}\, \delta\Phi_b \Big) = 0\,, \qquad (5.73)$$

so two situations need to be distinguished. Either a solution to the condition $J = 0$ exists — which requires $\sum_b \delta\tau_b = 2\mathcal{Q} \sum_b \delta\Phi_b$ — or it does not. Consider each of these in turn.

When $J = 0$ has solutions

If the constant quantities $\delta\tau_b$ and $\delta\Phi_b$ satisfy the condition $\sum_b \delta\tau_b = 2\mathcal{Q} \sum_b \delta\Phi_b$, then maximally symmetric solutions to the 6D field equations exist for any value of $\varphi_\star$. Because no particular value of φ_0 is selected, this shows that the flat direction that φ_0 parameterizes is not lifted. This is consistent with the observation that the brane action scales the same way as does the bulk action — and so does not break the bulk scaling symmetry — in the special case where $\delta\tau_b$ and $\delta\Phi_b$ are both φ-independent.

In this case formulae (5.70), (5.71) and (5.72) degenerate to $\varrho_{\rm eff} = m_\varphi^2 = \delta L/L = 0$, while eqs. (5.72) reveal $\delta\alpha_b = \kappa^2 \delta T_b/2\pi\alpha$, as usual, and $\delta W_{\scriptscriptstyle N} - \delta W_{\scriptscriptstyle S} = \kappa^2(\delta T_{\scriptscriptstyle N} - \delta T_{\scriptscriptstyle S})/8\pi\alpha$. The new perturbed solution in this case is a special instance of the general solution to the full nonlinear equations (5.15; 5.16; 5.17), all of which are known for the symmetries of interest to us. In particular, the assumption of constant brane action, $\delta T_b' = 0$, is known to be sufficient to ensure $\varrho_{\rm eff} = 0$, while $\delta T_{\scriptscriptstyle N} \neq \delta T_{\scriptscriptstyle S}$ induces warping. As initially argued in (5.6), it is the freedom to have nonzero on-brane flux, Φ_b, that prevents the flux quantization condition from being an obstruction to reaching these solutions as perturbations to the initial rugby ball (as one might naively have thought (5.33), if eq. (5.13) were read as forbidding the possibility of having perturbations to α_b, and hence also to T_b).

When $J \neq 0$ cannot be avoided

The perturbative solution found here also allows an exploration of what happens in the more general situation where the fluxes and tensions are *not* related to one another by $\sum_b \delta\tau_b = 2\mathcal{Q}\sum_b \delta\Phi_b$. In this case there is no choice for $\varphi_0 = \varphi_\star$ that can ensure $J(\varphi_\star) = 0$, implying that no solution exists at all to the linearized field equations, subject to the assumed axial symmetry and on-brane maximal symmetry. In this case studies of linearized stability (5.23) and exact time-dependent solutions (5.18) suggest that the relevant solutions are necessarily time-dependent.

We now show how this expectation for time-dependence can be made more precise in the present context, since $J \neq 0$ implies the absence of a stationary point to the (Einstein-frame) scalar potential, $V_{EF}(\varphi)$, for any finite value of φ. To show this we must reconsider the expression derived above for V_{JF}, but without using the condition $V'_{EF}(\varphi_\star) = 0$ to fix integration constants. For ϕ-independent $\delta\tau_b$ and $\delta\Phi_b$ expression (5.64) for V_{JF} becomes

$$\begin{aligned} V_{JF}(\varphi) &= \frac{1}{2}\, e^{-\varphi} \int \mathrm{d}\varphi_0\; e^{\varphi_0} \left(\delta T_N + \delta T_S - \mathcal{Q}\,\delta\Phi_N - \mathcal{Q}\,\delta\Phi_S + \frac{3}{2}\,\delta T'_N + \frac{3}{2}\,\delta T'_S \right) \\ &= \frac{1}{2} \sum_b \Big(\delta T_b - \mathcal{Q}\,\delta\Phi_b \Big) + C\, e^{-\varphi}\,, \end{aligned} \tag{5.74}$$

where C is the integration constant in question.

A natural choice for C is to demand that V_{JF} remain bounded as $e^{\varphi} \to 0$, since this corresponds to the weak-coupling limit for which both ϕ and $\mathcal{A}_M$ do not strongly self-interact in the bulk. More precisely, inspection of the 6D action, eq. (5.2), shows that the bulk scalar potential vanishes in this limit, allowing the constant part of ϕ to be absorbed into the definition $\tilde{\mathcal{A}}_M :=$ $e^{-\varphi_0/2}\mathcal{A}_M$. This argues that V_{JF} should not become unbounded in this limit, leading to the requirement $C = 0$.

With this choice the Einstein-frame scalar potential becomes

$$V_{EF}(\varphi) \propto e^{2\varphi} \sum_b \Big(\delta T_b - \mathcal{Q}\, \delta\Phi_b \Big) = e^{2\varphi} \sum_b \Big(\delta\tau_b - 2\mathcal{Q}\, \delta\Phi_b \Big) , \tag{5.75}$$

which describes a runaway to $\varphi \to \pm\infty$ — whose sign depends on the sign of $\sum_b(\delta T_b - \mathcal{Q}\,\delta\Phi_b)$. The absence of a solution here to $V'_{EF} = 0$ for any finite value of φ is what underlies the need for a time-dependent solution from the perspective of the low-energy 4D observer.

5.4.2 Dilaton-brane couplings, vacuum energy and volume stabilization

The next paragraphs explore some of the implications of nontrivial brane-dilaton couplings. Of particular interest is how the bulk and brane geometries depend on the choices made for these couplings. We start with the case where $\delta\tau_b$ and $\delta\Phi_b$ vary only weakly with φ, and move on to more strongly varying examples.

Linear dilaton-dependence

Consider therefore the simple situation where both brane tensions and fluxes are linear in φ, with

$$\tau_b = \tau_{b0} + \tau_{b1}\,\varphi \quad \text{and} \quad \Phi_b = \Phi_{b0} + \Phi_{b1}\,\varphi\,, \tag{5.76}$$

with τ_{bi} and Φ_{bi} constant. Since many — though not all — physical quantities depend only on the average brane action and flux, $T_{\rm eff} := \frac{1}{2}\sum_b T_b$ and $\Phi_{\rm eff} := \frac{1}{2}\sum_b \Phi_b$, it is useful to phrase our assumptions in terms of these, which have the form

$$T_{\rm eff}(\varphi) = T_0 + T_1\varphi \quad \text{and} \quad \Phi_{\rm eff}(\varphi) = \Phi_0 + \Phi_1\varphi\,, \tag{5.77}$$

where

$$T_i := \frac{1}{2}\sum_{b=N,S}(\tau_{bi} - \mathcal{Q}\,\Phi_{bi}) \quad \text{and} \quad \Phi_i := \frac{1}{2}\sum_{b=N,S}\Phi_{bi}\,. \tag{5.78}$$

We describe the resulting geometry as a perturbation about a rugby ball solution, characterized by a background tension, T, and brane flux, $\Phi(T)$, related by the background flux-quantization condition, eq. (5.14),

$$T - \mathcal{Q}\,\Phi = \frac{2\pi}{\kappa^2}\left[1 - \left(\frac{ng_R}{g}\right)\right]\,. \tag{5.79}$$

With this choice, the condition $J = 0$ defining $\varphi_\star$ becomes

$$\begin{aligned} 0 &= \delta T_{\rm eff}(\varphi_\star) - \mathcal{Q}\,\delta\Phi_{\rm eff}(\varphi_\star) + \frac{1}{2}\,\delta T'_{\rm eff}(\varphi_\star) \\ &= (T_0 - \mathcal{Q}\,\Phi_0) - (T - \mathcal{Q}\,\Phi) + (T_1 - \mathcal{Q}\,\Phi_1)\varphi_\star + \frac{T_1}{2}\,, \end{aligned} \tag{5.80}$$

whose solution,

$$\varphi_\star = \frac{1}{\mathcal{Q}\,\Phi_1 - T_1}\left[(T_0 - \mathcal{Q}\,\Phi_0) - (T - \mathcal{Q}\,\Phi) + \frac{T_1}{2}\right]\,, \tag{5.81}$$

in this case exists so long as $T_1 \neq \mathcal{Q}\,\Phi_1$.

We remark in passing that the assumed linear coupling does not preclude the existence of a vacuum configuration, $\varphi = \varphi_\star$, contrary to what happens for the nonsupersymmetric situation described in ref. (5.22). What is different in the nonsupersymmetric case is that $\varphi_\star$ satisfies $\sum_b \delta T_b'(\varphi_\star) = 0$ — rather than $\sum_b \left(\delta T_b + \frac{1}{2}\,\delta T_b' - \mathcal{Q}\,\delta\Phi_b\right)_{\varphi=\varphi_\star} = 0$ — which has no solutions if $\sum_b \delta T_b(\varphi)$ is a linear function of φ.

The Jordan-frame scalar potential, eq. (5.69), in the 4D effective theory then takes the simple form

$$V_{JF}(\varphi) = \mathcal{Q}\,\Phi_1 + (T_1 - \mathcal{Q}\,\Phi_1)(\varphi - \varphi_\star) - (T_1 + \mathcal{Q}\,\Phi_1)e^{-(\varphi-\varphi_\star)}\,, \tag{5.82}$$

and so the Einstein-frame potential becomes

$$V_{EF}(\varphi) = \Big[\mathcal{Q}\,\Phi_1 + (T_1 - \mathcal{Q}\,\Phi_1)(\varphi - \varphi_\star)\Big]e^{2(\varphi-\varphi_\star)} - (T_1 + \mathcal{Q}\,\Phi_1)e^{(\varphi-\varphi_\star)}\,. \tag{5.83}$$

Requiring the potential to be bounded from below implies $T_1 > \mathcal{Q}\,\Phi_1$. Notice that at $\varphi = \varphi_\star$ this satisfies $V_{EF}'(\varphi_\star) = 0$ automatically (by construction), and $V_{EF}(\varphi_\star) = -T_1$ there — which agrees with $-\frac{1}{2}\sum_b \delta T_b'(\varphi_\star) = -\delta T_{\rm eff}'(\varphi_\star)$, as it must. The physical parameters computed from V_{EF} using eqs. (5.70) and (5.71) in this case therefore are

$$\varrho_{\rm eff} = -T_1 \quad \text{and} \quad m_\phi^2 = \frac{\kappa_4^2}{5}\,(3T_1 - \mathcal{Q}\,\Phi_1)\,, \tag{5.84}$$

while the extra-dimensional response of eqs. (5.72) becomes

$$\begin{aligned}
\frac{\delta\alpha_b}{\alpha} &= -\left(\frac{\kappa^2}{2\pi\alpha}\right)\delta T_b(\varphi_\star) = -\frac{\kappa^2}{2\pi\alpha}\Big[\tau_b(\varphi_\star) - \mathcal{Q}\,\Phi_b(\varphi_\star)\Big] \\
\frac{\delta L}{L} &= \frac{3\kappa^2\varrho_{\rm eff}}{8\pi\alpha} = -\frac{3\kappa^2 T_1}{8\pi\alpha}\,,
\end{aligned} \tag{5.85}$$

and

$$W_{\scriptscriptstyle N} - W_{\scriptscriptstyle S} = \frac{\kappa^2}{8\pi\alpha}\Big[\mathcal{Q}\,\Phi_{\scriptscriptstyle N}(\varphi_\star) - \mathcal{Q}\,\Phi_{\scriptscriptstyle S}(\varphi_\star)\Big]\,. \tag{5.86}$$

For potentials that are bounded from below — *i.e.* those with $T_1 > \mathcal{Q}\,\Phi_1$ — the condition $T_1 > 0$ suffices to ensure $m_\varphi^2 > 0$ (and $\varrho_{\rm eff} < 0$).

Three important properties of these expressions bear special emphasis.

First, $\varrho_{\rm eff}$ quite generally depends on the background quantities T and Φ only through the combination $T - \mathcal{Q}\,\Phi$ whose value is constrained by flux quantization, eq. (5.79). Consequently $\varrho_{\rm eff}$ does not change at all as T is varied, because flux quantization demands Φ must also be adjusted in a way that precisely compensates. Any value of T is equally good, and what counts for physical predictions is only the extent to which the values $T_{\rm eff}(\varphi_\star) - \mathcal{Q}\,\Phi_{\rm eff}(\varphi_\star)$ differ from the flux-constrained background combination, $T - \mathcal{Q}\,\Phi$. This property also remains true for the more complicated examples discussed below.

Second, it is relatively easy to arrange $\varphi_\star \simeq -50$ using only a mild hierarchy of parameters on the branes. But eq. (5.15) then ensures that the volume of the extra dimensions, $\mathcal{V}_2 = 4\pi\alpha L^2 e^{-2\varphi_\star}$, is exponentially large compared with the intrinsic scales on the branes and in the bulk.

Third, what is most striking about this example is that the size of $\varrho_{\rm eff}$ and m_φ^2 is completely independent of T, $\mathcal{Q}\,\Phi$, T_0 and $\mathcal{Q}\,\Phi_0$. In this way this example captures part of the more general magic of codimension-2 constructions; they can admit classical solutions — like the rugby ball itself — for which large tensions coexist with flat (or weakly curved) on-brane geometries. Why is the result independent of the φ-independent part of $T_{\rm eff}$ and $\Phi_{\rm eff}$? Quite generally, we know from eq. (5.70) that $\varrho_{\rm eff} = \sum_b[\delta T_b(\varphi_\star) - \mathcal{Q}\,\delta\Phi_b(\varphi_\star)] = 2[\delta T_{\rm eff}(\varphi_\star) - \mathcal{Q}\,\Phi_{\rm eff}(\varphi_\star)]$, and so (apart for the special case where $\delta T_{\rm eff}$ cancels

$\mathcal{Q}\Phi_{\rm eff}$) the reason $\varrho_{\rm eff}$ can be small even when $T_0 - \mathcal{Q}\,\Phi_0$ is large is because the condition $J = 0$ drives $\varphi_\star$ out to such large values that the terms $T_0 - \mathcal{Q}\,\Phi_0$ and $(T_1 - \mathcal{Q}\,\Phi_1)\varphi_\star$ mostly cancel in $\varrho_{\rm eff}$.

One is drawn from this last observation to try to identify how robust this property is, both to the shape assumed for $\delta\tau_b(\varphi)$ and to the size of radiative corrections.

Power-law brane actions

In the previous example $|\varphi_\star|$ becomes very big if T_0 and $\mathcal{Q}\,\Phi_0$ are much larger in magnitude than are T_1 and $\mathcal{Q}\,\Phi_1$, and so the assumption that $\delta\tau_b$ is linear in φ typically cannot be justified simply as the first term in a Taylor expansion. It is useful therefore to examine slightly more complicated functional forms for $\delta\tau_b(\varphi)$ and $\delta\Phi_b(\varphi)$ in order to probe the robustness of the previous example.

Let us consider branes of the general form

$$\tau_b = \tau_{b0} + \tau_{b\eta}\,\varphi^\eta \quad \text{and} \quad \Phi_b = \Phi_{b0} + \Phi_{b\eta}\,\varphi^\eta\,, \tag{5.87}$$

again with constant τ_{bi} and Φ_{bi}. The effective brane action and flux, defined as before by $T_{\rm eff} := \frac12 \sum_b T_b$ and $\Phi_b := \frac12 \sum_b \Phi_b$, then give

$$T_{\rm eff}(\varphi) = T_0 + T_\eta\,\varphi^\eta \quad \text{and} \quad \Phi_{\rm eff}(\varphi) = \Phi_0 + \Phi_\eta\,\varphi^\eta\,, \tag{5.88}$$

where

$$T_i := \frac12 \sum_{b=N,S} (\tau_{bi} - \mathcal{Q}\,\Phi_{bi}) \quad \text{and} \quad \Phi_i := \frac12 \sum_{b=N,S} \Phi_{bi}\,. \tag{5.89}$$

As before we perturb about a rugby ball solution with background tension, T, and brane flux, Φ, related by the background flux-quantization condition, eq. (5.79), and so

$$\delta T_{\rm eff} = (T_0 - T) + T_\eta\, \varphi^\eta \quad \text{and} \quad \mathcal{Q}\, \delta \Phi_{\rm eff} = \mathcal{Q}(\Phi_0 - \Phi) + \mathcal{Q}\, \Phi_\eta\, \varphi^\eta \,. \qquad (5.90)$$

We find $\varphi_\star$ by using the condition $J(\varphi_\star) = 0$, or $\delta T_{\rm eff} - \mathcal{Q}\, \delta\Phi_{\rm eff} + \frac{1}{2}\delta T'_{\rm eff} = 0$, which in the present case gives

$$(T_0 - \mathcal{Q}\, \Phi_0) - (T - \mathcal{Q}\, \Phi) + \varphi_\star^{\eta-1} \Big[(T_\eta - \mathcal{Q}\, \Phi_\eta)\, \varphi_\star + \frac{\eta}{2}\, T_\eta \Big] = 0\,. \qquad (5.91)$$

Approximate solutions are possible when $|(T_\eta - \mathcal{Q}\, \Phi_\eta)\varphi_\star| \gg |\eta T_\eta/2|$, in which case the field stabilizes approximately at

$$\varphi_\star = \left(\frac{D}{T_\eta - \mathcal{Q}\, \Phi_\eta} \right)^{1/\eta} \,, \qquad (5.92)$$

where D is defined by

$$D = \mathcal{Q}(\Phi_0 - \Phi) - (T_0 - T) := -(\delta T_0 - \mathcal{Q}\, \delta \Phi_0)\,. \qquad (5.93)$$

This is a real solution if the signs of D and $T_\eta - \mathcal{Q}\, \Phi_\eta$ are the same. For $\eta > 0$ it is also large — and so justifies *a posteriori* making the large-$\varphi_\star$ approximation — if $|D| \gg |T_\eta - \mathcal{Q}\, \Phi_\eta|$. With this solution we find the low-energy cosmological constant is

$$\varrho_{\rm eff} = -\delta T'_{\rm eff}(\varphi_\star) = -\eta T_\eta \varphi_\star^{\eta-1} = -\frac{\eta T_\eta D}{T_\eta - \mathcal{Q}\, \Phi_\eta} \left(\frac{T_\eta - \mathcal{Q}\, \Phi_\eta}{D} \right)^{1/\eta} \,. \qquad (5.94)$$

This reduces to the cases previously considered in the special cases $\eta = 0$ and $\eta = 1$. Writing $\varrho_{\rm eff} = -(\eta D/\varphi_\star)/(1 - \mathcal{Q}\, \Phi_\eta/T_\eta)$ shows this result is generically suppressed relative to D within the approximations used, since these include $|\varphi_\star| \gg 1$. Because $\varphi \propto [D/(T_\eta - \mathcal{Q}\,\Phi_\eta)]^{1/\eta}$, with all other things equal this suppression becomes stronger for smaller $\eta > 0$.

Exponential branes

As our final example, consider several commonly occurring cases where the brane action depends exponentially on φ. A simple case of this type is when the entire tension and flux — *i.e.* both background and perturbation — involve a common exponential, $\tau_b(\varphi) = \tau_{b0} + \mathcal{A}_b\, e^{a\varphi}$ and $\mathcal{Q}\,\Phi_b(\varphi) = \mathcal{Q}\,\Phi_{b0} + \mathcal{B}_b\, e^{a\varphi}$.

In this case the average brane action and flux, $T_{\text{eff}} := \frac{1}{2}\sum_b T_b$ and $\Phi_{\text{eff}} := \frac{1}{2}\sum_b \Phi_b$, have the form

$$T_{\text{eff}}(\varphi) = T_0 + \mathcal{A}\, e^{a\varphi} \quad \text{and} \quad \mathcal{Q}\,\Phi_{\text{eff}}(\varphi) = \mathcal{Q}\,\Phi_0 + \mathcal{B}\, e^{a\varphi}\,, \tag{5.95}$$

where

$$\begin{aligned} T_0 &= \frac{1}{2}\sum_b (\tau_{b0} - \mathcal{Q}\,\Phi_{0b})\;, \quad \Phi_0 = \frac{1}{2}\sum_b \Phi_{0b}\,, \\ \mathcal{A} &= \frac{1}{2}\sum_b (\mathcal{A}_b - \mathcal{B}_b) \quad \text{and} \quad \mathcal{B} = \frac{1}{2}\sum_b \mathcal{B}_b\,. \end{aligned} \tag{5.96}$$

For instance, given the explicit factor of $e^{-\phi}$ in the definition of the brane-flux coupling, eq. (5.16), the special case $a = 1$ and $\mathcal{A}_b = \Phi_{b0} = 0$ (and so $\Phi_0 = \mathcal{A} + \mathcal{B} = 0$) corresponds to having brane actions that do not directly couple to the bulk scalar ϕ.

As before we perturb about a rugby ball solution with background tension, T, and brane flux, Φ, related by the background flux-quantization condition, eq. (5.14),

$$T - \mathcal{Q}\,\Phi = \frac{2\pi}{\kappa^2}\left(1 - \frac{n g_{_R}}{g}\right) . \tag{5.97}$$

The perturbations about this background become

$$\delta T_{\rm eff} = (T_0 - T) + \mathcal{A}\, e^{a\varphi} \quad \text{and} \quad \mathcal{Q}\,\delta\Phi_{\rm eff} = \mathcal{Q}(\Phi_0 - \Phi) + \mathcal{B}\, e^{a\varphi} . \tag{5.98}$$

The condition $J(\varphi_\star) = 0$ defining $\varphi_\star$ as usual is $\delta T_{\rm eff}(\varphi_\star) + \frac{1}{2}\,\delta T'_{\rm eff}(\varphi_\star) - \mathcal{Q}\,\delta\Phi_{\rm eff}(\varphi_\star) = 0$, which in this case becomes

$$\left[\mathcal{A}\left(1 + \frac{a}{2}\right) - \mathcal{B}\right] e^{a\varphi_\star} = D\,, \tag{5.99}$$

where $D := \mathcal{Q}(\Phi_0 - \Phi) - (T_0 - T)$. This has solutions if the sign of both sides is the same.

The low-energy Jordan-frame potential, eq. (5.69), then is

$$\begin{aligned} V_{_{JF}}(\varphi) &= C_1 e^{a\varphi} + C_2 e^{-\varphi} + C_3\,, \\ \text{with} \quad C_1 &= \frac{1}{a+1}\left[\left(1 + \frac{3a}{2}\right)\mathcal{A} - \mathcal{B}\right] \\ C_2 &= \left[\frac{2}{\left(\mathcal{A} + \frac{1}{2}a\mathcal{A} - \mathcal{B}\right)^{1/a}} - \frac{(2+a)C_1}{\left(\mathcal{A} + \frac{1}{2}a\mathcal{A} - \mathcal{B}\right)^{1+1/a}}\right] D^{1+1/a} \\ \text{and} \quad C_3 &= -D\,, \end{aligned} \tag{5.100}$$

leading to a similar expression for the Einstein-frame potential, $V_{_{EF}} = V_{_{JF}}\, e^{2(\varphi - \varphi_\star)}$. At $\varphi = \varphi_\star$ the cosmological constant becomes

$$\varrho_{\rm eff} = -\delta T'_{\rm eff}(\varphi_\star) = -a\mathcal{A}\, e^{a\varphi_\star} \;=\; a\left[\frac{(T_0 - T) - \mathcal{Q}(\Phi_0 - \Phi)}{1 + (a/2) - \mathcal{B}/\mathcal{A}}\right] \tag{5.101}$$

$$= \quad a \left[\frac{(T_0 - \mathcal{Q}\,\Phi_0) - (2\pi/\kappa^2)(1 - n g_R/g)}{1 + (a/2) - \mathcal{B}/\mathcal{A}} \right] ,$$

where the last equality uses the value of $T - \mathcal{Q}\,\Phi$ dictated by flux quantization. The scalar mass at the extremum is similarly

$$\begin{aligned} m_\varphi^2 &= \frac{a}{10} \left(\frac{6 + 3a - 2\mathcal{B}/\mathcal{A}}{2 + a - 2\mathcal{B}/\mathcal{A}} \right) \kappa_4^2 \Big[(T - T_0) - \mathcal{Q}(\Phi - \Phi_0) \Big] \qquad (5.102) \\ &= -\frac{a}{10} \left(\frac{6 + 3a - 2\mathcal{B}/\mathcal{A}}{2 + a - 2\mathcal{B}/\mathcal{A}} \right) \kappa_4^2 \left[(T_0 - \mathcal{Q}\,\Phi_0) - \frac{2\pi}{\kappa^2} \left(1 - \frac{n g_R}{g} \right) \right] . \end{aligned}$$

Eq. (5.101) identifies three potential mechanisms for suppressing $\varrho_{\rm eff}$.

1. The first is if $a \to 0$, in which case the brane actions become φ-independent and $\varphi_\star$ recedes to infinity. This is the suppression already encountered in the examples presented above.

2. The second is if the φ-independent parts, T_0 and Φ_0, are related to one another in the same way as flux quantization imposes on the background values, T and Φ. (For the special case where the bulk flux is chosen to lie in $U_R(1)$ direction (so $g = g_R$) and $n = 1$, the background lies in the same flux category as does the supersymmetric Salam-Sezgin solution (5.13), for which $T - \mathcal{Q}\,\Phi = 0$.)

3. Finally, the third potential suppression occurs even if $T - \mathcal{Q}\,\Phi \neq 0$, provided $|\mathcal{B}/\mathcal{A}| \gg 1$. That is, if $\delta\Phi_{\rm eff}$ dominates $\delta T_{\rm eff}$ then this only affects the value of $\varphi_\star$, leaving $\varrho_{\rm eff}$, as always, of order $\delta T'_{\rm eff}(\varphi_\star)$.

5.4.3 Quantum corrections and technical naturalness

All of the calculations of brane-bulk interactions provided in previous sections are performed purely within the classical approximation. As such they leave open the question of how robust their conclusions are to modification by quantum corrections. And since the streets are littered with classical examples having small vacuum energies, a proper treatment of quantum corrections is the crucial to any credible mechanism for understanding the small size of the observed vacuum energy.

In this section, we take a small step towards filling in this missing step, more in the spirit of indicating a promising line of inquiry than in providing a polished example. Our interest is in quantifying the stability of both the size of the low-energy cosmological constant, $\varrho_{\rm eff}$, and the size of the bulk volume, $\mathcal{V}_2$, (when this is large compared with more microscopic scales).

The starting point is an enunciation of the essence of the problem: once parameters are chosen to ensure a large value for $\varrho_{\rm eff}$ and/or $\mathcal{V}_2$, are these choices stable against the renormalization that results when heavy fields are integrated out? In extra-dimensional brane models this question necessarily has two parts, to do with integrating out heavy field on the brane and in the bulk.

We here use one of the previously discussed examples as a toy model for estimating the size of quantum corrections. We choose a model that has both has an exponentially large volume and a small 4D on-brane curvature — *i.e.* vacuum energy[9] — and estimate the size of quantum corrections. The main idea behind this model is that it is the bulk field ϕ itself that counts both

[9]In the model $\varrho_{\rm eff}$ is small inasmuch as it is parametrically suppressed relative to other scales, though not small enough numerically to describe the observed Dark Energy.

bulk and brane loops, with weak coupling corresponding to ϕ being large and negative. The influence of loop effects is then simply incorporated by tracking the ϕ-dependence of the quantum-corrected (1PI) action, for which the above arguments about brane-bulk back-reaction can be applied.

A toy model

The theory of interest is one of the 'power-law' models described earlier. For our starting point we take a background rugby-ball geometry whose background tension and flux satisfy the flux-quantization condition, eq. (5.79),

$$T - \mathcal{Q}\,\Phi = \frac{2\pi}{\kappa^2}\left[1 - \left(\frac{n g_R}{g}\right)\right], \tag{5.103}$$

with the right-hand-side being small enough to allow semiclassical reasoning, but not tuned to be inordinately small. Such geometries have flat on-brane directions, and as above we seek to see how brane-bulk interactions modify this, including loops.

For the perturbations to this geometry we choose the classical brane-bulk Lagrangian to have the power-law form,

$$\delta T_{\text{eff}} = T_\star \, (-\phi)^\eta \,, \tag{5.104}$$

with $0 < \eta < 1$ (and the smaller η is, the larger the suppression in ϱ_{eff}). Here $T_\star$ is a function of all of the on-brane degrees of freedom, ψ, such as

$$T_\star = \mu^4 + \hat{g}^{\mu\nu}\,\partial_\mu\psi\,\partial_\nu\psi + M^2\psi^2 + \lambda\,\psi^4 + \cdots \,, \tag{5.105}$$

which defines the scale μ. In the vacuum $\psi = 0$ and so $T_\star = \mu^4$.

A hierarchy is dialled in by choosing the scale μ in $T_\star$ to be small compared with the typical brane scale, M: *i.e.* $\mu^2 \ll M^2 \ll 1/\kappa$. Notice that taking $-\phi \gg 1$ does not affect the mass of the ψ particle (or other brane particles in general) at the classical level, because ϕ appears only as an overall factor in the brane action. The goal is to show that the energy scales set by $\varrho_{\rm eff}^{1/4}$ and $\mathcal{V}_2^{-1/2}$ can be hierarchically different from M, and that this can be protected from quantum effects. Since $\mathcal{V}_2$ turns out to depend exponentially on $T_\star$, a relatively small hierarchy between $T_\star$ and M^4 suffices to generate very large volumes.

The classical part of the story is worked out above, with (choosing $\Phi_{\rm eff} = 0$) eq. (5.92) implying

$$-\varphi_\star \simeq \left(\frac{T - \mathcal{Q}\,\Phi}{T_\star}\right)^{1/\eta} \simeq \left[\frac{2\pi}{\kappa^2\mu^4}\left(1 - \frac{ng_{\scriptscriptstyle R}}{g}\right)\right]^{1/\eta}, \tag{5.106}$$

from which eq. (5.15) gives the bulk volume,

$$\mathcal{V}_2 = 4\pi\alpha L^2 e^{-\varphi_\star} \simeq 4\pi\alpha L^2 \exp\left\{\left[\frac{2\pi}{\kappa^2\mu^4}\left(1 - \frac{ng_{\scriptscriptstyle R}}{g}\right)\right]^{1/\eta}\right\}. \tag{5.107}$$

Eq. (5.94) similarly gives the on-brane vacuum energy as

$$\varrho_{\rm eff} \simeq \eta T_\star(-\varphi_\star)^{\eta-1} \simeq \frac{\eta(T - \mathcal{Q}\,\Phi)}{(-\varphi_\star)} \simeq \frac{2\pi\eta}{\kappa^2}\left(1 - \frac{ng_{\scriptscriptstyle R}}{g}\right)\left[\frac{2\pi}{\kappa^2\mu^4}\left(1 - \frac{ng_{\scriptscriptstyle R}}{g}\right)\right]^{-1/\eta}, \tag{5.108}$$

revealing a power-law suppression of $\varrho_{\rm eff}$ relative to $1/\kappa^2$, whose strength improves the smaller η gets. (*e.g.* for $\eta = 1$ this gives $\varrho_{\rm eff} \sim \mu^4$ while $\eta = \frac{1}{2}$ implies $\varrho_{\rm eff} \propto \kappa^2\mu^8$, and so on.)

Loop corrections

We now argue that the choice $\mu \ll M$ underlying the classical hierarchy is technically natural. We do so using the observation that it is the expectation of the bulk zero-mode, φ, itself that controls the size of these loops, so loop corrections can be incorporated into the above argument by making a modified choice for the ϕ-dependence of $T_{\rm eff}$.

Brane loops:

To see why this is so, imagine first computing quantum corrections involving loops of the on-brane field, ψ. When computing these loops it is useful first to adopt a canonical normalization, $\psi \to \psi_c := (-\varphi)^{\eta/2}\,\psi$, after which the strength of the self-coupling becomes revealed to be $\lambda_c \psi_c^4$ with

$$\lambda_c = \frac{\lambda}{(-\varphi)^\eta}\,. \tag{5.109}$$

More generally, because $(-\varphi)^\eta$ pre-multiplies the entire brane action, for the purposes of power-counting brane perturbation theory it plays the role of $1/\hbar$. This ensures that each additional loop is parametrically suppressed by an additional factor of $(-\varphi)^{-\eta}$, with dimensions made up using the typical brane scale, M. In particular, integrating out a heavy field of mass M should give a Wilson action (or, alternatively a calculation of the 'quantum' 1PI brane action) of the form

$$\Gamma_{\rm eff} = T_\star(-\phi)^\eta + T_1 + \frac{T_2}{(-\phi)^\eta} + \cdots\,, \tag{5.110}$$

and so on.

Here the T_n generically depend on the brane fields,[10] much as did $T_\star$. The point of quantum hierarchy problems is that — on dimensional[11] grounds — each of the T_n is generically of order M^4, rather than the smaller μ^4. The question is whether this ruins the above conclusions about the size of $\varphi_\star$, and so also of $\mathcal{V}_2$ and $\varrho_{\rm eff}$.

Now comes the main point. Because each loop correction is suppressed by an additional factor of $(-\phi)^{-\eta}$, none of them has the same ϕ-dependence as does $T_\star$. In particular, none of them require the vacuum value of $T_\star$ also to be of order M^4 instead of μ^4. Better yet, having $T_1 \simeq M^4 \gg T_\star \simeq \mu^4$ keeps $\varphi_\star$ stabilized at large negative values, enforcing the dominance of the leading, classical, approximation.

To see this in detail we repeat the above classical calculation of the potential for the bulk modulus φ using the loop corrected action, eq. (5.110), rather than the classical expression, eqs. (5.104) and (5.105). For simplicity we take $\Phi_{\rm eff} = 0$ also at the loop level, though none of our conclusions would change if we were to assume a similar loop expansion for the brane flux,

$$\Phi_{\rm eff} \simeq \Phi_1 + \frac{\Phi_2}{(-\phi)^\eta} + \cdots \,, \tag{5.111}$$

with the dimensions of the Φ_n again set by the large mass, M, circulating in the loops.

To one-loop order, the new terms in $\Gamma_{\rm eff}$ are independent of ϕ, and so their modifications to the brane-bulk back-reaction are encompassed by the

[10] Although these would be local for the Wilson action, they need not be for the 1PI action (5.34).

[11] These are often stated to be of order the 'cutoff' scale, but we make the more conservative statement that they scale with the physical mass M because cutoffs generically cancel in all physical quantities (5.35).

analysis given in the previous section. Assuming $-\varphi_\star \gg 1$ this gives

$$-\varphi_\star \simeq \left(\frac{D}{T_\star}\right)^{1/\eta} \quad \text{and} \quad \varrho_{\text{eff}} \simeq \frac{\eta D}{(-\varphi_\star)}\,, \tag{5.112}$$

where $D = (T - T_1) - \mathcal{Q}(\Phi - \Phi_1) \simeq (T - \mathcal{Q}\,\Phi) - T_1$ is of order the larger of M^4 or $T - \mathcal{Q}\,\Phi \propto 2\pi/\kappa^2$. Since both of these scales are much larger than $T_\star \propto \mu^4$, the classical assumption that $-\varphi_\star$ is large (and all that comes with it) is not undermined by one-loop corrections. We assume here that D and $T_\star$ share the same sign.

The size of the two-loop correction can be similarly estimated. The equation that determines $\varphi_\star$ is, at two-loop order

$$-D + T_\star(-\varphi_\star)^\eta \left(1 + \frac{\eta}{2\varphi_\star}\right) + \frac{T_2}{(-\varphi_\star)^\eta}\left(1 - \frac{\eta}{2\varphi_\star}\right) \simeq 0\,. \tag{5.113}$$

Using $|D| \sim |T_2| \sim \mathcal{O}(M^4)$ and $(-\varphi_\star)^\eta \sim M^4/\mu^4 \gg 1$ shows the 2-loop term to represent a small correction to the value predicted for $(-\varphi_\star)$.

The two-loop contribution to ϱ_{eff} comes in two parts. The first of these is through the change in $\varphi_\star$, though because $|\delta\varphi_\star/\varphi_\star| \ll 1$ this contribution is subdominant to the value for ϱ_{eff} already computed at one loop. In particular, it doesn't ruin the suppression of ϱ_{eff} by the factor $1/\varphi_\star$.

The second type of 2-loop contribution to ϱ_{eff} comes from the fact that T'_{eff} now includes a new term of the form

$$-\delta T'_{\text{eff}} \simeq \eta\, T_2(-\varphi_\star)^{-\eta-1} = \left(\frac{\eta T_2}{-\varphi_\star}\right)\frac{1}{(-\varphi_\star)^\eta}\,, \tag{5.114}$$

which should be compared to the original contribution, $\sim \eta D/\varphi_\star$. Clearly the new term is subdominant if $|D| \sim |T_2| \gg T_\star$. And so it goes for higher loops, each of which is suppressed by an additional factor of $(-\varphi_\star)^{-\eta} \simeq \mu^4/M^4$, by virtue of the stability of the initial stabilization at large values of $-\varphi_\star$.

Notice that the brane loops also don't have a large relative impact on the light scalar mass, which eq. (5.71) gives to be

$$\begin{aligned} m_\varphi^2 &\simeq \frac{\kappa_4^2}{5}\left[\frac{3}{2}\,\delta T''_{\rm eff}(\varphi_\star) + 3\,\delta T'_{\rm eff}(\varphi_\star) - \mathcal{Q}\,\delta\Phi'_{\rm eff}(\varphi_\star)\right] && (5.115)\\ &\simeq \left(\frac{3\eta T_\star}{5M_p^2}\right)(-\varphi_\star)^{\eta-1} \simeq \frac{3\eta D}{5M_p^2\varphi_\star} \sim \frac{\eta M^4}{M_p^2}\left(\frac{\mu}{M}\right)^{4/\eta} && \text{(up to one loop)}\\ &\simeq \left(\frac{3\eta T_2}{5M_p^2}\right)(-\varphi_\star)^{-\eta-1} \simeq \frac{3\eta T_2 T_\star}{5M_p^2 D\varphi_\star} \sim \frac{\eta \mu^4}{M_p^2}\left(\frac{\mu}{M}\right)^{4/\eta} && \text{(two-loop term)}\,, \end{aligned}$$

with the final estimates using $T_2 \sim D \sim M^4$ and $T_\star \sim \mu^4$.

Bulk loops:

The previous loop estimates are restricted purely to brane loops because they rely on the assumed form of the brane-bulk coupling. But the back-reaction of the brane loops onto the bulk is computed classically (for the bulk theory) just as before. How big might be quantum corrections in the bulk sector?

An estimate for the size of bulk loops can be made in a manner very similar to the one just used for brane loops, because $e^{2\phi}$ is the loop-counting parameter for the bulk 6D supergravity. The simplest way to see this is to re-scale the 6D metric according to $g_{MN} \to \check g_{MN} := e^{-\phi} g_{MN}$, in terms of which the action of eq. (5.2) becomes

$$\begin{aligned} S_{\rm bulk} &= -\int {\rm d}^6x\sqrt{-\check g}\; e^{-2\phi}\left\{\frac{1}{2\kappa^2}\,\check g^{MN}\Big(\check{\mathcal{R}}_{MN} + \zeta\partial_M\phi\,\partial_N\phi\Big)\right.\\ &\qquad\left. + \frac{1}{4}\,\check g^{MP}\check g^{NQ}\mathcal{F}_{MN}\mathcal{F}_{PQ} + \frac{2\,g_R^2}{\kappa^4}\right\}, \qquad (5.116) \end{aligned}$$

where ζ is a constant. This shows that for bulk perturbation theory it is the constant value of $e^{2\phi}$ that plays the role of of $\hbar$. (The same also remains true once the action's fermion terms are included (5.12).) Each loop involving bulk fields therefore contributes an amount proportional to an additional power of $e^{2\phi}$, which is small when ϕ is large and negative (also the regime of weak brane coupling).

Now imagine integrating out fields in the bulk that are heavy relative to the KK scale. Loops of these fields potentially modify both the brane and bulk actions by new local interactions (5.36; 5.37). The loop-generated couplings arising in this way cannot depend on φ in the same way as does the classical action, (5.2), again indicating that these classical terms are not themselves renormalized. Loop-generated terms necessarily involve new interactions whose φ-dependence can be organized into a series in powers of $e^{2\varphi}$ (5.38).

This leads one to expect that each bulk loop is exponentially suppressed, by powers of $e^{2\varphi_\star} \propto 1/\mathcal{V}_2^2$ when $-\varphi_\star \gg 1$. In particular, these corrections to physical properties would therefore be expected to be sub-dominant to the brane loops considered above.

5.5 Conclusions

This paper computes the back-reaction of a pair of 4D codimension-two branes onto the 6D geometry that they source, within a framework of flux compactification that allows a complete calculation of modulus stabilization. Although performed with a particular (gauged, chiral (5.12)) 6D supergravity, the mechanisms exposed by our calculations rely only on broad features (like the presence of a dilaton and scale invariance of the classical equations) shared by a wide variety of higher-dimensional supergravities. This leads us to expect them to have a wider domain of validity than the particular 6D system studied here.

The main calculational assumptions are these: (*i*) we assume all energy densities and curvatures to be small enough to justify working within a semi-classical analysis; (*ii*) we compute brane-bulk couplings to leading order in a derivative expansion (making the dominant players the brane tensions and brane-localized fluxes); (*iii*) we seek solutions that are axially symmetric in the two dimensions transverse to the branes and whose on-brane geometries are maximally symmetric; and (*iv*) we linearize the brane properties about the choices that source simple rugby-ball geometries (which have flat on-brane geometries despite having nonzero tensions).

The last two of these assumptions deserve some motivation. The linearization about rugby ball geometries (5.6; 5.13; 5.15; 5.16) is made in order to allow the search of their immediate neighborhoods in field space to be systematic; the linearity of the equations allows the construction of their most general solutions. We do not believe that the qualitative features of our results (like the existence of very large volume solutions, and the suppression of on-brane curvatures) depend strongly on this assumption.

By contrast, at first blush the assumption of maximal symmetry might seem more restrictive, since maximal symmetry is known *not* to be possible for a majority of brane configurations and the general situation is expected to be time-dependent (5.18). We employ a trick to explore such configurations: we stabilize the time-dependent runaway by turning on an external current that couples to the system's low-energy moduli. In this way we can explore the potential energy cost that drives these runaway solutions, at least at the low energies of main interest.

The supergravity of interest has a one-parameter flat direction, labeled by a particular combination of the 6D dilaton and the breathing mode of the extra-dimensional metric. Our main calculation interest for this theory is in the potential energy generated for this flat direction by the back-reaction of the bulk-brane couplings; and in the related change of shape of the extra-dimensional geometry. We find that these display the following noteworthy features:

- *Volume stabilization:* Any non-derivative coupling of the branes to the bulk dilaton, ϕ, breaks the classical scaling symmetry of the bulk field equations, and so lifts the degeneracy of the classical zero mode. Because the bulk volume depends exponentially on the canonically normalized dilaton, ϕ, we find that a mild hierarchy in the brane-bulk coupling parameters can easily generate an exponentially large extra-dimensional volume. The exception to this is if the branes also couple only to exponentials of ϕ, as is in particular often true for D-branes.

- *Suppressed on-brane curvature:* A remarkable feature of rugby ball geometries is that their on-brane directions are flat despite the presence of large brane tensions. We find that perturbations about these geometries can – but need not – share this feature, having on-brane curvatures that are parametrically small compared with the generic size of the on-brane tensions. In particular, a mechanism for achieving such solutions arises for some types of dilaton-brane couplings since the dilaton can be driven to roll out to large fields along the flat direction to find places where the on-brane tension and curvature are the smallest.

- *Relevance of on-brane fluxes:* Flux quantization within the bulk provides a strong constraint on flux-stabilized rugby-ball geometries, and in the simplest examples gives rugby-ball perturbations whose on-brane curvatures are *not* suppressed. An important part of our ability to find other solutions with lower curvature is our inclusion of brane-localized flux, corresponding to a magnetic coupling of the branes to the geometry-stabilizing fluxes. In this way our calculations bear out the earlier expectations of ref. (5.6).

- *Quantum corrections:* In section 5.4.3 we provide a preliminary estimate of the size of quantum corrections for a particularly promising toy model, with both exponentially large volumes and a suppressed on-brane curvature. Our estimates indicate these properties need not be destabilized by quantum effects on the brane or in the bulk. They do not do so because it is the value taken by the dilaton along the classical flat direction itself that plays the role of the loop-counting parameter (similar to what happens for string vacua), and this constrains how quantum effects can alter the dynamics that determines what this value is. In particular, we find that provided the brane couplings are arranged to lie within the regime of weak coupling, the conclusions of the classical analysis are protected from loop effects.

- *Bulk distortion:* Although the rugby ball solutions themselves are simple we find that the nearby geometries are more generic, including warping and nontrivial dilaton profiles across the extra dimensions.

- *Modulus-matter couplings:* Having the dilaton couple to branes only through an overall prefactor implies a universal coupling to ordinary matter, if this resides on a brane. This is likely to have interesting phenomenological implications if the moduli can be arranged to be light enough to mediate macroscopic forces. It is noteworthy that these couplings can have the form required to profit from a 'chameleon' mechanism (5.24).

We regard these properties to be new examples of how low-energy brane dynamics can change how one thinks about technical naturalness and hierarchies of scale. In particular, the natural generation of exponentially large volumes in the 6D model explored here fills in a key missing step in efforts to use large volumes to solve the gauge hierarchy problem.

Although none of the solutions explored here have on-brane curvatures that are low enough to describe the Dark Energy density, the existence in some models of a mechanism for robust parametric suppression of the on-brane curvature is very suggestive. We regard this as encouragement to continue to explore this direction for new approaches to the cosmological constant problem.

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Bibliography

[5.1] C.P. Burgess and L. van Nierop, "Large Dimensions and Small Curvatures from Supersymmetric Brane Back-reaction" JHEP 1104:078,2011 [arXiv:1101.0152].

[5.2] L. Randall, R. Sundrum, "A Large Mass Hierarchy from a Small Extra Dimension" Phys. Rev. Lett. **83** (1999) 3370 [hep-ph/9905221]; "An Alternative to Compactification" Phys. Rev. Lett. **83** (1999) 4690 [hep-th/9906064].

[5.3] K. Lanczos, Phys. Z. **23** (1922) 239–543; Ann. Phys. **74** (1924) 518–540;
C.W. Misner and D.H. Sharp, "Relativistic Equations for Adiabatic, Spherically Symmetric Gravitational Collapse" Phys. Rev. **136** (1964) 571–576;
W. Israel, "Singular hypersurfaces and thin shells in general relativity" Nuov. Cim. **44B** (1966) 1–14; *errata* Nuov. Cim. **48B** 463.

[5.4] V. Balasubramanian, P. Berglund, J. P. Conlon and F. Quevedo, "Systematics of moduli stabilisation in Calabi-Yau flux compactifications," JHEP **0503**, 007 (2005) [arXiv:hep-th/0502058];

J. P. Conlon, F. Quevedo and K. Suruliz, “Large-volume flux compactifications: Moduli spectrum and D3/D7 soft supersymmetry breaking,” JHEP **0508** (2005) 007 [hep-th/0505076];

[5.5] P. Horava and E. Witten, “Eleven-Dimensional Supergravity on a Manifold with Boundary” *Nucl. Phys.* **B475** (1996) 94 [hep-th/9603142]; “Heterotic and Type I String Dynamics from Eleven Dimensions” *Nucl. Phys.* **B460** (1996) 506 [hep-th/9510209];

E. Witten, “Strong Coupling Expansion Of Calabi-Yau Compactification” *Nucl. Phys.* **B471** (1996) 135 [hep-th/9602070];

J. Lykken, “Weak Scale Superstrings” *Phys. Rev.* **D54** (1996) 3693 [hep-th/9603133];

K. Benakli, “Phenomenology of low quantum gravity scale models,” *Phys. Rev.* **D60**, 104002 (1999) [hep-ph/9809582];

C. P. Burgess, L. E. Ibáñez and F. Quevedo, “Strings at the intermediate scale or is the Fermi scale dual to the Planck scale?” *Phys. Lett.* **B447**, 257 (1999) [hep-ph/9810535];

I. Antoniadis and K. Benakli, “Large dimensions and string physics in future colliders,” Int. J. Mod. Phys. A **15** (2000) 4237 [arXiv:hep-ph/0007226].

[5.6] Y. Aghababaie, C.P. Burgess, S. Parameswaran and F. Quevedo, “Towards a Naturally Small Cosmological Constant from Branes in 6D Supergravity” Nucl. Phys. **B680** (2004) 389–414, [hep-th/0304256].

[5.7] N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, “The hierarchy problem and new dimensions at a millimeter,” Phys. Lett. B **429** (1998) 263 [arXiv:hep-ph/9803315];

I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, "New dimensions at a millimeter to a Fermi and superstrings at a TeV," Phys. Lett. B **436** (1998) 257 [arXiv:hep-ph/9804398].

[5.8] D. Atwood, C. P. Burgess, E. Filotas, F. Leblond, D. London and I. Maksymyk, "Supersymmetric large extra dimensions are small and/or numerous," Phys. Rev. D **63** (2001) 025007 [arXiv:hep-ph/0007178];
J. L. Hewett and D. Sadri, "Supersymmetric extra dimensions: Gravitino effects in selectron pair production," Phys. Rev. D **69** (2004) 015001 [arXiv:hep-ph/0204063];

[5.9] C. P. Burgess, "Supersymmetric large extra dimensions and the cosmological constant: An update," Annals Phys. **313** (2004) 283 [arXiv:hep-th/0402200];
"Towards a natural theory of dark energy: Supersymmetric large extra dimensions," AIP Conf. Proc. **743** (2005) 417 [arXiv:hep-th/0411140];
C. P. Burgess, J. Matias and F. Quevedo, "MSLED: A Minimal supersymmetric large extra dimensions scenario," Nucl. Phys. B **706** (2005) 71 [arXiv:hep-ph/0404135];
J. Matias and C. P. Burgess, "MSLED, neutrino oscillations and the cosmological constant," JHEP **0509** (2005) 052 [arXiv:hep-ph/0508156];
P. Callin and C. P. Burgess, "Deviations from Newton's law in supersymmetric large extra dimensions," Nucl. Phys. B **752** (2006) 60 [arXiv:hep-ph/0511216].

[5.10] C. P. Burgess, A. Maharana and F. Quevedo, "Uber-naturalness: unexpectedly light scalars from supersymmetric extra dimensions," arXiv:1005.1199 [hep-th].

[5.11] J. W. Chen, M. A. Luty and E. Ponton, "A Critical cosmological constant from millimeter extra dimensions" JHEP **0009**(2000)012 [arXiv:hep-th/0003067]

C. P. Burgess, R. C. Myers and F. Quevedo, "A naturally small cosmological constant on the brane?," Phys. Lett. B **495** (2000) 384 [arXiv:hep-th/9911164];

N. Arkani-Hamed, S. Dimopoulos, N. Kaloper and R. Sundrum, "A small cosmological constant from a large extra dimension," Phys. Lett. B **480** (2000) 193 [arXiv:hep-th/0001197];

S. Kachru, M. B. Schulz and E. Silverstein, "Self-tuning flat domain walls in 5d gravity and string theory," Phys. Rev. D **62** (2000) 045021 [arXiv:hep-th/0001206].

S. Dimopoulos, S. Kachru, N. Kaloper, A. E. Lawrence and E. Silverstein, "Small numbers from tunneling between brane throats," Phys. Rev. D **64** (2001) 121702 [arXiv:hep-th/0104239];

S. M. Carroll and M. M. Guica, "Sidestepping the cosmological constant with football-shaped extra dimensions," arXiv:hep-th/0302067;

I. Navarro, "Codimension two compactifications and the cosmological constant problem," JCAP **0309** (2003) 004 [hep-th/0302129];

C. P. Burgess, P. Grenier and D. Hoover, "Quintessentially flat scalar potentials," JCAP **0403** (2004) 008 [arXiv:hep-ph/0308252];

[5.12] H. Nishino and E. Sezgin, *Phys. Lett.* **144B** (1984) 187; "The Complete N=2, D = 6 Supergravity With Matter And Yang-Mills Couplings," Nucl. Phys. **B278** (1986) 353;

S. Randjbar-Daemi, A. Salam, E. Sezgin and J. Strathdee, "An Anomaly

Free Model in Six-Dimensions" *Phys. Lett.* **B151** (1985) 351.

[5.13] A. Salam and E. Sezgin, "Chiral Compactification On Minkowski X S**2 Of N=2 Einstein-Maxwell Supergravity In Six-Dimensions," Phys. Lett. B **147** (1984) 47.

[5.14] Y. Aghababaie, C. P. Burgess, S. L. Parameswaran and F. Quevedo, "SUSY breaking and moduli stabilization from fluxes in gauged 6D supergravity," JHEP **0303** (2003) 032 [arXiv:hep-th/0212091];
C. P. Burgess, S. L. Parameswaran and I. Zavala, "The Fate of Unstable Gauge Flux Compactifications," JHEP **0905** (2009) 008 [arXiv:0812.3902 [hep-th]].

[5.15] G. W. Gibbons, R. Guven and C. N. Pope, "3-branes and uniqueness of the Salam-Sezgin vacuum," Phys. Lett. B **595** (2004) 498 [hep-th/0307238];
Y. Aghababaie *et al.*, "Warped brane worlds in six dimensional supergravity," JHEP **0309** (2003) 037 [hep-th/0308064].

[5.16] C. P. Burgess, F. Quevedo, G. Tasinato and I. Zavala, "General axisymmetric solutions and self-tuning in 6D chiral gauged supergravity," JHEP **0411** (2004) 069 [hep-th/0408109];

[5.17] A. J. Tolley, C. P. Burgess, D. Hoover and Y. Aghababaie, "Bulk singularities and the effective cosmological constant for higher co-dimension branes," JHEP **0603** (2006) 091 [arXiv:hep-th/0512218].

[5.18] A. J. Tolley, C. P. Burgess, C. de Rham and D. Hoover, "Scaling solutions to 6D gauged chiral supergravity," New J. Phys **8** (2006) 324 [arXiv:0608.083 [hep-th]];

A. J. Tolley, C. P. Burgess, C. de Rham and D. Hoover, "Exact Wave Solutions to 6D Gauged Chiral Supergravity," JHEP **0807** (2008) 075 [arXiv:0710.3769 [hep-th]];

M. Minamitsuji, "Instability of brane cosmological solutions with flux compactifications," Class. Quant. Grav. **25** (2008) 075019 [arXiv:0801.3080 [hep-th]].

H. M. Lee and A. Papazoglou, "Codimension-2 brane inflation," Phys. Rev. D **80** (2009) 043506 [arXiv:0901.4962 [hep-th]].

[5.19] S. L. Parameswaran, G. Tasinato and I. Zavala, "The 6D SuperSwirl," Nucl. Phys. B **737** (2006) 49 [arXiv:hep-th/0509061];

[5.20] H. M. Lee and C. Ludeling, "The general warped solution with conical branes in six-dimensional supergravity," JHEP **0601** (2006) 062 [arXiv:hep-th/0510026];

[5.21] S. B. Giddings, S. Kachru and J. Polchinski, "Hierarchies from fluxes in string compactifications," Phys. Rev. D **66**, 106006 (2002) [arXiv:hep-th/0105097];

K. Dasgupta, G. Rajesh and S. Sethi, "M theory, orientifolds and G-flux," JHEP **9908** (1999) 023 [arXiv:hep-th/9908088].

[5.22] C. P. Burgess and L. van Nierop, "Bulk Axions, Brane Back-reaction and Fluxes," arXiv:1012.2638 [hep-th].

[5.23] H. M. Lee and A. Papazoglou, "Scalar mode analysis of the warped Salam-Sezgin model," [hep-th/0602208];

C. P. Burgess, C. de Rham, D. Hoover, D. Mason and A. J. Tolley, "Kicking the rugby ball: Perturbations of 6D gauged chiral supergravity," JCAP **0702** (2007) 009 [arXiv:hep-th/0610078].

S. L. Parameswaran, S. Randjbar-Daemi and A. Salvio, "Gauge fields, fermions and mass gaps in 6D brane worlds," Nucl. Phys. B **767** (2007) 54 [arXiv:hep-th/0608074]; "Stability and Negative Tensions in 6D Brane Worlds," JHEP **0801** (2008) 051 [arXiv:0706.1893 [hep-th]]; "General Perturbations for Braneworld Compactifications and the Six Dimensional Case," JHEP **0903** (2009) 136 [arXiv:0902.0375 [hep-th]].

[5.24] J. Khoury and A. Weltman "Chameleon cosmology" phys. rev. D **69** (2004) 044026 [arXiv:0309411 [astro-ph]]

[5.25] S. Weinberg, *Gravitation and Cosmology*, Wiley 1973.

[5.26] C. W. Misner, J. A. Wheeler and K. S. Thorne, *Gravitation*, W. H. Freeman & Company 1973.

[5.27] A. Vilenkin, Phys. Rev. "Gravitational field of vacuum domain walls and strings" **D23** (1981) 852;

R. Gregory, "Gravitational stability of local strings" Phys. Rev. Lett. **59** (1987) 740;

A. G. Cohen and D. B. Kaplan, "The Exact Metric About Global Cosmic Strings," *Phys. Lett.* **B215**, 67 (1988);

A. Vilenkin and P. Shellard, *Cosmic Strings and other Topological Defects*, Cambridge University Press (1994);

R. Gregory and C. Santos, "Cosmic strings in dilaton gravity," Phys. Rev. D **56**, 1194 (1997) [gr-qc/9701014].

[5.28] W. D. Goldberger and M. B. Wise, "Modulus stabilization with bulk fields", Phys. Rev. Lett. **83** (1999) 4922-4925 [arXiv:hep-ph/9907447]

[5.29] W. D. Goldberger, M. B. Wise, "Renormalization group flows for brane couplings," Phys. Rev. **D65** (2002) 025011 [hep-th/0104170];
E. Dudas, C. Papineau, V.A. Rubakov, "Flowing to four dimensions" JHEP0603:085,2006 arXiv:hep-th/0512276v1;
C. de Rham, "The Effective Field Theory of Codimension-two Branes," JHEP **0801** (2008) 060 [arXiv:0707.0884 [hep-th]].

[5.30] C. P. Burgess, D. Hoover and G. Tasinato, "UV Caps and Modulus Stabilization for 6D Gauged Chiral Supergravity," JHEP **0709** (2007) 124 [arXiv:0705.3212 [hep-th]];
C. P. Burgess, D. Hoover, C. de Rham and G. Tasinato, "Effective Field Theories and Matching for Codimension-2 Branes," JHEP **0903** (2009) 124 [arXiv:0812.3820 [hep-th]].

[5.31] P. Bostock, R. Gregory, I. Navarro and J. Santiago, "Einstein gravity on the codimension 2 brane?," Phys. Rev. Lett. **92** (2004) 221601 [arXiv:hep-th/0311074];
I. Navarro and J. Santiago, "Gravity on codimension 2 brane worlds," JHEP **0502** (2005) 007 [arXiv:hep-th/0411250];
J. Vinet and J. M. Cline, "Codimension-two branes in six-dimensional supergravity and the cosmological constant problem," Phys. Rev. D **71** (2005) 064011 [hep-th/0501098];
M. Peloso, L. Sorbo and G. Tasinato, "Standard 4d gravity on a brane in six dimensional flux compactifications," Phys. Rev. D **73** (2006) 104025 [arXiv:hep-th/0603026];

B. Himmetoglu and M. Peloso, "Isolated Minkowski vacua, and stability analysis for an extended brane in the rugby ball," Nucl. Phys. B **773** (2007) 84 [hep-th/0612140];

B. Himmetoglu and M. Peloso, "Isolated Minkowski vacua, and stability analysis for an extended brane in the rugby ball," Nucl. Phys. B **773** (2007) 84 [hep-th/0612140];

C. de Rham, "The Effective Field Theory of Codimension-two Branes," JHEP **0801** (2008) 060 [arXiv:0707.0884 [hep-th]];

E. Papantonopoulos, A. Papazoglou and V. Zamarias, "Regularization of conical singularities in warped six-dimensional compactifications," JHEP **0703** (2007) 002 [arXiv:hep-th/0611311]; "Induced cosmology on a regularized brane in six-dimensional flux compactification," Nucl. Phys. B **797** (2008) 520 [arXiv:0707.1396 [hep-th]];

D. Yamauchi and M. Sasaki, "Brane World in Arbitrary Dimensions Without Z_2 Symmetry," Prog. Theor. Phys. **118** (2007) 245 [arXiv:0705.2443 [gr-qc]];

N. Kaloper and D. Kiley, "Charting the Landscape of Modified Gravity," JHEP **0705** (2007) 045 [hep-th/0703190];

M. Minamitsuji and D. Langlois, "Cosmological evolution of regularized branes in 6D warped flux compactifications," Phys. Rev. D **76** (2007) 084031 [arXiv:0707.1426 [hep-th]];

S. A. Appleby and R. A. Battye, "Regularized braneworlds of arbitrary codimension," Phys. Rev. D **76** (2007) 124009 [arXiv:0707.4238 [hep-ph]];

C. Bogdanos, A. Kehagias and K. Tamvakis, "Pseudo-3-Branes in a Curved 6D Bulk," Phys. Lett. B **656** (2007) 112 [arXiv:0709.0873 [hep-th]];

O. Corradini, K. Koyama and G. Tasinato, "Induced gravity on intersecting brane-worlds Part I: Maximally symmetric solutions," Phys. Rev. D **77** (2008) 084006 [arXiv:0712.0385 [hep-th]]; "Induced gravity on intersecting brane-worlds Part II: Cosmology," Phys. Rev. D **78** (2008) 124002 [arXiv:0803.1850 [hep-th]];

F. Arroja, T. Kobayashi, K. Koyama and T. Shiromizu, "Low energy effective theory on a regularized brane in 6D gauged chiral supergravity," JCAP **0712** (2007) 006 [arXiv:0710.2539 [hep-th]];

V. Dzhunushaliev, V. Folomeev and M. Minamitsuji, "Thick brane solutions," arXiv:0904.1775 [gr-qc].

[5.32] A. Bayntun, C.P. Burgess and L. van Nierop, "Codimension-2 Brane-Bulk Matching: Examples from Six and Ten Dimensions," New J. Phys. **12** (2010) 075015 [arXiv:0912.3039 [hep-th]].

[5.33] G. W. Gibbons and C. N. Pope, "Consistent S**2 Pauli reduction of six-dimensional chiral gauged Einstein-Maxwell supergravity," Nucl. Phys. B **697** (2004) 225 [arXiv:hep-th/0307052];

[5.34] C. P. Burgess, "Introduction to effective field theory," Ann. Rev. Nucl. Part. Sci. **57** (2007) 329 [arXiv:hep-th/0701053];

"Quantum gravity in everyday life: General relativity as an effective field theory," Living Rev. Rel. **7** (2004) 5 [arXiv:gr-qc/0311082].

[5.35] C. P. Burgess and D. London, "Uses and abuses of effective Lagrangians," Phys. Rev. D **48** (1993) 4337 [arXiv:hep-ph/9203216].

[5.36] J. Schwinger, "On Gauge Invariance and Vacuum Polarization" Phys. Rev. 82 (1951) 664;

B.S. De Witt, "Dynamical Theory of Groups and Fields," in *Relativity, Groups and Topology*, B.S. De Witt and C. De Witt, (New York, Gordon and Breach, 1965);

P. B. Gilkey, "The Spectral Geometry Of A Riemannian Manifold," J. Diff. Geom. **10**, 601 (1975);

D.M. McAvity and H. Osborn, "A DeWitt expansion of the heat kernel for manifolds with a boundary" *Class. Quant. Grav.* **8** (1991) 603–638.

D. Hoover and C. P. Burgess, "Ultraviolet Sensitivity in Higher Dimensions," JHEP **0601** (2006) 058 [arXiv:hep-th/0507293];

"UV Sensitivity in Supersymmetric Large Extra Dimensions: The Ricci-flat Case," Nucl. Phys. B **772** (2007) 175 [arXiv:hep-th/0504004].

[5.37] A. O. Barvinsky and G. A. Vilkovisky, "The Generalized Schwinger-Dewitt Technique In Gauge Theories And Quantum Gravity," Phys. Rept. **119** (1985) 1;

D. V. Vassilevich, "Heat kernel expansion: User's manual," Phys. Rept. **388**, 279 (2003) [hep-th/0306138].

[5.38] E. Witten, "New *gauge* theories in six dimensions," JHEP **9801** (1998) 001 [Adv. Theor. Math. Phys. **2** (1998) 61] [arXiv:hep-th/9710065];

A. Hanany and A. Zaffaroni, "Branes and six dimensional supersymmetric theories," Nucl. Phys. B **529** (1998) 180 [arXiv:hep-th/9712145];

M. Alishahiha and Y. Oz, "Supergravity and 'new' six-dimensional gauge theories," Phys. Lett. B **495** (2000) 418 [arXiv:hep-th/0008172].

Chapter 6

Technically Natural Cosmological Constant From Supersymmetric 6D Brane Backreaction

6.1 Preamble

This chapter is based on the work in (6.1). We use the techniques of the previous chapters to construct a 6 dimensional extension to the standard model. This model is an explicit realization of the proposal (6.2) of supersymmetric large extra dimensions (SLED) as an approach to tackle the cosmological constant problem. The possibility to have part of the stabilizing flux localized on the branes addresses a previous obstruction: the flux quantization condition in our construction can be satisfied for any value of the brane tensions by adjusting the value of the dilaton.

The least attractive feature of this construction is that the brane magnetic coupling is huge — much larger than any other scale in the problem.

However, the coupling never appears without an accompanying factor of the exponent of the dilaton. The dilaton is stabilized at a large negative value, which causes the combination of dilaton and magnetic coupling that can appear in loops to be sufficiently small. This means that in the end the construction is stable under quantum corrections, including the large magnetic coupling. This means that this is a technically natural construction that has a sufficiently small cosmological constant.

C.P. Burgess and Leo van Nierop

6.2 Introduction

The cosmological constant problem (6.3) remains an important conceptual obstacle to our understanding of the hierarchies of the physical world. The puzzle of why the electroweak scale is much smaller than the Planck (and possibly GUT) scale has motivated many proposals for what kinds of physics might lie at TeV energies — supersymmetry, compositeness, extra dimensions and so on — that have been famously used to motivate many choices made when designing the now-operational Large Hadron Collider (LHC). But the same reasoning applied at the much lower, sub-eV energies relevant to the scale of Dark Energy seems to fail to explain how the vacuum energy can gravitate as weakly as it appears to do.

Many have remarked that extra dimensions (and large ones) can help with the cosmological constant problem, because they break the connection between the energy density of a 4-dimensional vacuum (which we believe should be large), and the curvature of the visible universe (which we observe to be small) (6.4; 6.5; 6.6; 6.2; 6.7). The problem in four dimensions is that the Einstein equations force these to be the same, since $\langle T_{\mu\nu}\rangle = \frac{1}{4}\,T\,g_{\mu\nu}$ for any Lorentz-invariant state, implying $R_{\mu\nu} = -2\pi G\,T\,g_{\mu\nu}$. But once there are more than four dimensions then we need not demand the vacuum be Lorentz-invariant in the higher dimensions. And a large energy density that is Lorentz invariant in the 4D sense (such as a brane tension), can curve the extra dimensions rather than the four dimensions that are Lorentz-invariant. In particular, explicit solutions to the higher-dimensional field equations with brane sources are known that have this property, at least when there are not too many extra dimensions (6.4; 6.6; 6.2).

But the existence of some choices for brane sources for which bulk solutions can be flat is not in itself a solution to the cosmological constant problem. What must be shown is that these choices are sufficiently stable against integrating out heavy fields, including the electron.

In this paper we provide an explicit example of an extra-dimensional model which we believe predicts a 4D curvature whose size is controlled by the Casimir energy of the extra dimensions, $R \simeq m_{KK}^4/M_p^2$, where m_{KK} is the Kaluza-Klein (KK) scale and M_p is the Planck scale. In particular, it can be much smaller than what would be expected from the scales $M \gg m_{KK}$ of particle physics. We regard it as a realization of an earlier general proposal — supersymmetric large extra dimensions (SLED) (6.2) — wherein ordinary particles are localized on a space-filling (3+1)-dimensional codimension-two brane that sits within a (5+1)-dimensional bulk spacetime with two compact dimensions transverse to the brane. We take the bulk to be described by a particular 6D supergravity (chiral, gauged supergravity (6.8)), but we believe the underlying mechanism applies equally well to other 6D supergravities, and more generally to other low-codimension brane systems interacting through bulk supergravities, once their back-reaction onto the bulk is accurately included.

A proper description of the vacuum energy must include in particular the energetics that stabilize the extra dimensions, and an advantage of the particular model we study is that many features of the extra-dimensions are stabilized within the bulk (without reference to the branes) at the classical level by a simple flux compactification (6.9), leaving only the single flat direction that is guaranteed by the classical scale invariance of the bulk supergravity.

It has been known for some time that brane back-reaction can lift this last flat direction (6.10; 6.11; 6.12; 6.13), and what is new about our contribution here is to show that there is a simple choice for the brane-bulk couplings that can fix this last flat direction without generating an on-brane curvature, assuming we work only to within the classical approximation in the bulk (more about quantum corrections below). Two brane properties are required: (*i*) the absence of a direct brane coupling to the bulk dilaton (a scalar superpartner of the graviton in six dimensions); and (*ii*) a Maxwell-brane coupling that allows one of the branes to carry a localized amount of the bulk-stabilizing flux.

What is remarkable is that these properties are unchanged under arbitrary loops of the on-brane fields, including in particular loops of all ordinary particles of everyday experience (which we assume to be localized on one of the branes). This is possible because property (*i*) — the *absence* of a coupling to a bulk field — is automatically preserved by brane loops if it is true at the classical level. Loops of ordinary particles also cannot alter the brane-localized flux coupling required by property (*ii*) if the brane-bound particles do not couple to the bulk flux field. In the model explored below we assume the brane-flux coupling occurs on a different brane from that on which all brane particles are localized.

The requirement for brane-localized flux — property (*ii*) above — turns out to be the new crucial ingredient, since it is the possibility of being able to localize some bulk fluxes onto the brane that allows the system to respond with little energy to changes of brane tension (6.13). It is also this brane-localized flux that is responsible for stabilizing the remaining flat direction in the bulk, which it can do because the bulk flux field couples to the 6D scalar dilaton that parameterizes this flat direction.

We lay out our arguments in the following way. First, the remainder of this section carefully defines the notion of 'technical naturalness,' whose absence is the essence of the cosmological constant problem. We do so because we believe that a resolution of this problem ultimately points towards a world with two supersymmetric extra dimensions at sub-eV scales. Although this seems an extreme possibility, there seem to be no alternatives short of abandoning technical naturalness altogether. In the words of Sherlock Holmes (6.14) "...when you have eliminated the impossible, whatever remains, however improbable, must be the truth."

§6.3 then describes in detail the simple bulk and brane systems on whose properties our proposal rests. In particular this section describes the exact classical solutions that capture the back-reaction of the branes to the bulk and which govern the geometry of both the on-brane and off-brane dimensions. The size of quantum corrections is the topic of §6.4, which studies both the implications of loops of on-brane and bulk modes. This section argues why loops of brane-localized fields do not change the conclusions of §6.3 at all, and why the leading contributions come from bulk loops only. The contributions of massless and massive fields in the bulk are contrasted, and both are argued only to generate contributions to the low-energy 4D scalar potential that are of order m_{KK}^4. Our conclusions are summarized in §6.5, including a qualitative discussion of why both extra dimensions and supersymmetry are required. This section closes with a brief summary of what is known about the potentially rich observational signatures that are implied by the present framework, together with a summary of issues needing further study. Three appendices deal with technical issues about localizing flux on branes; calculate the low-energy 4D potential for arbitrary small perturbations to the brane actions; and examine two common objections to the possibility of using extra dimensions to help with the cosmological constant problem.

6.2.1 Technical naturalness (without cutoffs)

Notions of 'technical naturalness' are central to our motivation, so we pause here to state these carefully. We believe our discussion largely keeps to the party line, though we make an effort not to cast the issues in terms of cutoffs in divergent integrals. Those familiar with the issues should feel free to skip this section entirely.

An understanding of hierarchies of scale, like the electroweak hierarchy or the cosmological constant problems, comes in two parts. The first part asks why the hierarchy of scales exists in the first place in the fundamental theory at very small distances. Because this question is sensitive to physics at the fundamental scale — possibly the string scale or some other quantum gravity scale — it might not be answered until we ultimately understand this fundamental theory in detail. The second part of the understanding asks why the hierarchy is stable when various massive states are integrated out to produce one of the effective theories that describes the implications of the fundamental theory at the low energies we can observe.

Technical naturalness is addressed at this second part of the problem, since the low-energy effective theory is not unique (depending as it does on the energy range that is of interest for a particular low-energy observer). Yet it is implicit in our understanding of physics that a large hierarchy can be understood equally well in *any* of the effective theories for which we choose to ask the question.

For example, a large hierarchy that is well-understood is the small size of the nucleus, ℓ_n, relative to the size of an atom, a_0. Within the standard model the small ratio $\ell_n/a_0 \simeq 10^{-5}$ is understood as being a consequence of two other experimental facts: the electromagnetic coupling constant is weak: $\alpha = e^2/4\pi \simeq 10^{-2}$; and the electron is light compared with the QCD scale, $m_e/\Lambda_{QCD} \simeq 10^{-3}$. The small size of the nucleus is then a consequence of the relations $a_0^{-1} \simeq \alpha m_e$ and $\ell_n^{-1} \simeq \Lambda_{QCD}$.

But the same question might again be asked within a lower-energy effective theory below the QCD scale, say within the quantum electrodynamics of electrons, protons and neutrons. In this theory the small ratio of observables, ℓ_n/a_0, is instead understood as a consequence of the small size of α in this theory, together with the electron being much lighter than the proton: $m_e \ll m_p$. The process of integrating out the quarks and gluons to give the proton and neutron (or integrating out the muon or other particles) does not fundamentally change the way we think about nuclei being small, and this is what it means[1] for this hierarchy to be 'technically natural.'

Contrast this with our understanding of the small ratio between the observed Dark Energy density, $\rho_{\rm vac}$, and the electron mass (say): $\rho_{\rm vac}/m_e^4 \simeq 10^{-36}$. Consider the low-energy theory well below the electron mass, for which the fundamental particles might be taken to be photons, neutrinos and gravitons. For this theory ρ is given by the cosmological constant that appears in the low-energy effective action, plus the loop contributions of these very

[1] As originally formulated (6.15), a small ratio is said to be technically natural if a new symmetry emerges when the ratio goes to zero. This is a particularly important way of ensuring technical naturalness in the way we define it here.

low-mass particles:

$$\rho_{\rm vac} \simeq \lambda_{\rm le} + \text{low-energy loops}\,, \quad \text{where} \quad \mathcal{L}_{\rm eff} = -\sqrt{-g}\,\Big(\lambda_{\rm le} + \cdots\Big)\,. \qquad (6.1)$$

Compare this with the same calculation, performed in the effective theory defined above the electron mass, containing electrons in addition to the previously considered low-energy particles. There is an effective cosmological constant, $\lambda_{\rm he}$, also in this effective theory, whose value is related to $\lambda_{\rm le}$ by a matching condition that states that the physical quantity, ρ, should be the same in this theory as in the lower-energy effective theory. This implies that the renormalized value of $\lambda_{\rm he}$ in the effective theory above the electron mass is related to that below, $\lambda_{\rm le}$, by

$$\lambda_{\rm le} \simeq \lambda_{\rm he} + \frac{k\, m_e^4}{16\pi^2}\,, \qquad (6.2)$$

where k is an order-unity number whose value is computed by evaluating an electron loop graph. This kind of shift, $\lambda_{\rm le} \to \lambda_{\rm he}$ occurs as we match across the electron threshold because the low-energy theory is obtained by integrating out the electron, meaning electrons are not present there to contribute to ρ through loops. Eq. (6.2) expresses how λ must change between the two theories to ensure that the low-energy theory 'knows' about the contributions of virtual electrons to the vacuum energy.

Now comes the main point. Since all of the masses of particles in the very low-energy theory below the electron threshold are small, $\lambda_{\rm le}$ is of the same order of magnitude as is $\rho_{\rm vac}$. Consequently eq. (6.2) implies $\lambda_{\rm he}$ must be much larger than ρ in the effective theory above the electron threshold. This nevertheless produces a small value for ρ in this higher-energy theory because of a cancelation of roughly 36 decimal places between $\lambda_{\rm he}$ and an equally large electron loop, with the much smaller value, $\lambda_{\rm le}$, of the lower-energy theory emerging as the residue.

Instead of there being an understanding in *all* effective theories why λ is smaller than m_e^4, the small size of $\lambda_{\rm le}$ in the very low-energy theory is understood as arising as an incredibly detailed cancelation between much larger quantities like $\lambda_{\rm he}$ and loops of the many much heavier particles the higher-energy theories contain. Of course, it is logically possible that this is the way nature works. But although we know about very many other hierarchies in nature (besides that between atoms and nuclei), so far as we know *none* of these are understood in this way.

It is a radical proposal that advocates that new hierarchies should be understood so very differently than those we've understood well in the past. A more scientifically conservative approach is instead to seek a technically natural understanding of poorly understood hierarchies like the electroweak hierarchy and the Dark Energy density. Of course this is a very tall order for the Dark Energy, since its very small size means that any successful approach must modify the properties of comparatively low-energy particles (like the electron).

Purging cutoffs

Notice that the previous paragraphs are all formulated in terms of physical, or of renormalized, masses. Technical naturalness is often stated in a cutoff-dependent way, in terms of the absence of quadratic or quartic divergences when loop contributions are computed within a low-energy effective theory.

We deliberately do not phrase things here in terms of cutoffs because we believe this can be a confusing way to express the physical issues at stake.

At face value quadratic dependence on a cutoff sounds like the same thing as sensitivity to heavy particle masses, because within a cutoff regularization the value of the cutoff very concretely specifies where the high-energy theory starts and a low-energy effective theory breaks down. Furthermore, quadratic or higher dependence on a cutoff indicates a strong sensitivity of a loop integral to the details of the unknown high-energy physics. However, from the point of view of a Wilsonian effective field theory, cutoffs are one of the few things we can be sure never enter into physical quantities, because they are an artefact of how a theorist decides to organize a calculation into a low- and high-energy contribution (6.16; 6.17). In particular, damage done by using a silly or inconvenient regularization, can be undone by appropriately renormalizing the resulting theory.

From this point of view the scale of the cutoff in the low-energy theory is really only a proxy for a bona fide mass of a state in the UV completion, and the presence of quadratic divergences really only provide a qualitative indication of when heavy masses can appear as an enhancement when integrating out a heavy particle. But in the end, the relation between cutoffs and heavy masses is not quantitative, and to properly decide whether heavy masses contribute significantly to an observable really requires knowledge of the UV completion that describes its properties, and cannot be decided purely within the low-energy theory. In general, the coefficients of quadratic divergences do not track those of heavy masses, and one can get burned by taking the correspondence between heavy masses and cutoffs too seriously (6.18).

6.3 Classical brane-bulk dynamics

We start by summarizing the bulk-brane system of interest, which we choose closely following ref. (6.13). Since our results depend only on the dynamics of codimension-2 branes within higher-dimensional supergravity (together with the classical scale invariance these supergravities naturally enjoy), we believe our results not to be limited to the specific six-dimensional supergravity we examine here in detail.

6.3.1 The bulk system

We take chiral gauged supergravity in six dimensions (6.8) to govern the bulk physics, to which we couple two space-filling, positive-tension branes that strongly break supersymmetry. The bulk fields whose dynamics we follow in detail are the metric, g_{MN}; a flux-stabilizing bulk Maxwell gauge potential, $\mathcal{A}_M$; and the 6D scalar dilaton, ϕ.

Field equations

The bulk bosonic action restricted to these fields is[2]

$$S_{\rm bulk} = -\int {\rm d}^6x\sqrt{-g}\;\left\{\frac{1}{2\kappa^2}\,g^{MN}\Big(\mathcal{R}_{MN}+\partial_M\phi\,\partial_N\phi\Big) + \frac{1}{4}\,e^{-\phi}\mathcal{F}_{MN}\mathcal{F}^{MN}+\frac{2\,g_R^2}{\kappa^4}\,e^{\phi}\right\}, \quad (6.3)$$

where $\mathcal{F} = {\rm d}\mathcal{A}$ denotes the gauge potential's field strength, and κ and g_R are, respectively, the dimensionful coupling constants for gravity and for a specific

[2]We use a 'mostly plus' metric and Weinberg's curvature conventions (6.19) (that differ from those of MTW (6.20) only by an overall sign in the definition of the Riemann tensor).

$U_R(1)$ symmetry of the supersymmetry algebra. The full gauged supergravity has more bosonic fields than this, but the rest can be set to zero consistent with their equations of motion. The background gauge field, $\mathcal{A}_M$, need not be the one that gauges the $U_R(1)$ symmetry so its gauge coupling, g, need not equal g_R.

The equations of motion from this action are the (trace reversed) Einstein equations

$$\mathcal{R}_{MN} + \partial_M\phi\,\partial_N\phi + \kappa^2 e^{-\phi}\mathcal{F}_{MP}\mathcal{F}_N{}^P - \left(\frac{\kappa^2}{8}\,e^{-\phi}\mathcal{F}_{PQ}\mathcal{F}^{PQ} - \frac{g_R^2}{\kappa^2}\,e^{\phi}\right) g_{MN} = 0\,, \quad (6.4)$$

the Maxwell equation

$$\nabla_M(e^{-\phi}\mathcal{F}^{MN}) = 0\,, \quad (6.5)$$

and the dilaton equation

$$\Box\phi - \frac{2\,g_R^2}{\kappa^2}\,e^{\phi} + \frac{\kappa^2}{4}\,e^{-\phi}\mathcal{F}_{MN}\mathcal{F}^{MN} = 0\,. \quad (6.6)$$

These field equations enjoy the exact classical symmetry

$$g_{MN} \to \zeta\, g_{MN} \quad \text{and} \quad e^{-\phi} \to \zeta\, e^{-\phi}\,, \quad (6.7)$$

with $\mathcal{A}_M \to \mathcal{A}_M$. This ensures the theory has three important properties:

- It ensures any nonsingular solution is always part of a one-parameter family of solutions that are exactly degenerate (within the classical approximation);
- It ensures that there exists a Weyl rescaling, $\check{g}_{MN} = e^{\phi}\, g_{MN}$, for which ϕ appears undifferentiated in the bulk action only as an overall factor.

That is,

$$S_{\rm bulk} = -\int {\rm d}^6x\sqrt{-\check{g}}\; e^{-2\phi}\mathcal{L}(\check{g}_{MN}, \partial_M\phi, F_{MN})\,, \tag{6.8}$$

where $\mathcal{L}$ only depends on derivatives of ϕ. Eq. (6.8) is significant because it shows that the quantity $e^{2\phi}$ plays the role of $1/\hbar$, and so is the loop-counting parameter for the bulk part of the theory.

- It ensures that once evaluated at *any* solution of the field equations — *i.e.* eqs. (6.4) through (6.6) — the action, eq. (6.3), evaluates to a total derivative (6.21),

$$S_{\rm bulk}\Big|_{\rm soln} = \frac{1}{2\kappa^2}\int {\rm d}^6x\sqrt{-g}\;\Box\phi\,. \tag{6.9}$$

6.3.2 Brane properties

We focus on configurations involving two space-filling (3+1)-dimensional branes, whose coupling to the bulk fields we take to be given by the leading terms in a derivative expansion:

$$S_{\rm branes} = -\sum_b \int d^4x\sqrt{-g_4}\;\left[T_b(\phi) - \frac{1}{2}\,\Phi_b(\phi)\,\epsilon^{mn}\mathcal{F}_{mn} + \cdots\right]\,, \tag{6.10}$$

where the ellipses indicate terms involving two derivatives or more.[3] In general the coupling functions T_b and Φ_b can depend on ϕ, as well as any fields localized on the branes (which we denote collectively by ψ). If T_b is independent of ϕ and $\Phi_b \propto e^{-\phi}$, then the brane action transforms under the classical scaling symmetry, eq. (6.7), in the same way as does the bulk action, ensuring that the brane couplings do not break the classical bulk scale invariance.

[3]Notice that we normalize the quantity Φ_b slightly differently than in ref. (6.13).

The parameter T_b represents the tension of the brane, and our conventions are such that $T_b > 0$ corresponds to positive tension. The parameter Φ_b corresponds physically (6.13) to the amount of magnetic flux that is localized on the source branes (see eq. (6.23) below). When T_b drops out of the low-energy energetics – as is the case below – keeping nominally subdominant terms in the derivative expansion like the magnetic coupling to the brane becomes important (6.2).

Let us now specify the properties of the two source branes in more detail. First is the observer's brane, S_o, on which all ordinary particles are imagined to reside,

$$T_o = \tau_o + g^{\mu\nu}\partial_\mu\psi^*\partial_\nu\psi + M^2\psi^*\psi + \cdots \,, \tag{6.11}$$

and on which there is no flux,[4] $\Phi_o = 0$. ψ here could represent the Higgs boson, but more broadly is meant as a proxy for all of the fields of the Standard Model. The goal of later sections is to show that the on-brane curvature can be made systematically small compared with M^4/M_p^2 in a technically natural way.

Second is what we call the 'flux' brane, on which no fields are localized and for which

$$T_f = \tau_f \quad \text{and} \quad \Phi_f = \mu \,. \tag{6.12}$$

Here τ_o, τ_f, μ and M are dimensionful parameters that define the energy scales of the system. Although the validity of semiclassical reasoning requires quantities like $\kappa^2\tau_b$ and $\kappa^2 M^4$ to be smaller than order unity, we do not assume τ_f, τ_o or M^4 to be particularly small relative to one another, and we wish to

[4] Since our conclusions depend only on the total brane flux, $\Phi_o + \Phi_f$, the vanishing of Φ_o is not necessary.

identify when the low-energy 4D curvature is set by scales that are much smaller than these.

What is important for what follows is that the choices (6.11) and (6.12) ensure that the classical brane actions are both independent of the bulk field ϕ. Our choices also ensure there is no direct coupling between the brane-localized fields ψ and the bulk Maxwell field, $\mathcal{F}_{MN}$.

6.3.3 Bulk-brane interactions

We next turn to the bulk configurations to which these two branes give rise. In what follows it suffices to focus on solutions that are maximally symmetric in the four on-brane directions, and are symmetric under rotations in the extra dimensions about the two brane positions. This leads to the following *ansätze* for the metric and Maxwell field:

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + e^{2B}\mathrm{d}\theta^2 + e^{2W}\,\hat{g}_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu \quad \text{and} \quad \mathcal{A} = \mathcal{A}_\theta\,\mathrm{d}\theta\,, \tag{6.13}$$

where $\hat{g}_{\mu\nu}(x)$ is a maximally symmetric metric, and the functions W, B, ϕ and $\mathcal{A}_\theta$ depend only on ρ.

In this case the bulk field equations reduce to

$$\begin{aligned}
\left(e^{-B+4W}e^{-\phi}\mathcal{A}_\theta'\right)' &= 0 && (\mathcal{A}_\theta)\\
\left(e^{B+4W}\phi'\right)' - \left(\frac{2g_R^2}{\kappa^2}\,e^{\phi} - \frac{1}{2}\,\kappa^2\mathcal{Q}^2\,e^{\phi}e^{-8W}\right)e^{B+4W} &= 0 && (\phi)\\
4\Big[W'' + (W')^2\Big] + B'' + (B')^2 + (\phi')^2 + \frac{3}{4}\,\kappa^2\mathcal{Q}^2\,e^{\phi}e^{-8W} + \frac{g_R^2}{\kappa^2}\,e^{\phi} &= 0 && (\rho\rho)\\
B'' + (B')^2 + 4W'B' + \frac{3}{4}\,\kappa^2\mathcal{Q}^2\,e^{\phi}e^{-8W} + \frac{g_R^2}{\kappa^2}\,e^{\phi} &= 0 && (\theta\theta)\\
\frac{1}{4}\,e^{-2W}\hat{R} + W'' + 4(W')^2 + W'B' - \frac{1}{4}\,\kappa^2\mathcal{Q}^2\,e^{\phi}e^{-8W} + \frac{g_R^2}{\kappa^2}\,e^{\phi} &= 0\,, && (\mu\nu)
\end{aligned}$$

(6.14)

where primes denote differentiation with respect to the coordinate ρ. The first of these can be integrated once exactly, introducing an integration constant, $\mathcal{Q}$:

$$\mathcal{F}_{\rho\theta} = \mathcal{A}'_\theta = \mathcal{Q}\, e^{\phi} e^{B-4W} \,. \tag{6.15}$$

Evaluated with this *ansatz*, the brane action, eq. (6.10), becomes

$$S_{\text{branes}} = -\sum_{b=o,f} \int d^4x \sqrt{-\hat{g}_4}\; e^{4W} L_b \,, \tag{6.16}$$

where L_b is given for each brane in terms of T_b and Φ_b by

$$L_b := T_b - \Phi_b\, e^{-B} \mathcal{F}_{\rho\theta} + \cdots = T_b - \mathcal{Q}\Phi_b\, e^{\phi}\, e^{-4W} + \cdots \,. \tag{6.17}$$

Brane matching conditions

The brane-bulk couplings impose a set of boundary conditions on the derivatives of the bulk fields in the near-brane limits, that are the generalization to codimension-2 of the more familiar Israel junction conditions (6.22) of codimension-1. The precise conditions were recently worked out for codimension-2 branes (6.10; 6.11; 6.12) (see also (6.23; 6.24)), and state:

$$\Big[e^B \phi'\Big]_{\rho_b} = \frac{\partial \mathcal{L}_b}{\partial \phi} \,, \quad \Big[e^B W'\Big]_{\rho_b} = \mathcal{U}_b \quad \text{and} \quad \Big[e^B B' - 1\Big]_{\rho_b} = -\Big[\mathcal{L}_b + 3\,\mathcal{U}_b\Big] \,, \tag{6.18}$$

where both sides are evaluated in the near-brane limit,[5] $\rho \to \rho_b$, and as before primes denote differentiation with respect to ρ. The quantities $\mathcal{L}_b$ and $\mathcal{U}_b$ appearing here are

$$\mathcal{L}_b := \frac{\kappa^2 L_b}{2\pi} \quad \text{and} \quad \mathcal{U}_b := \frac{\kappa^2}{4\pi} \left(\frac{\partial L_b}{\partial g_{\theta\theta}} \right) . \tag{6.19}$$

Notice that it is not necessary to know how L_b depends on $g_{\theta\theta}$ in order to evaluate $\mathcal{U}_b$, because the bulk field equations require $\mathcal{U}_b$ must satisfy the constraint (6.10; 6.11; 6.12)

$$4\mathcal{U}_b \Big[2 - 2\mathcal{L}_b - 3\mathcal{U}_b \Big] - (\mathcal{L}_b')^2 \simeq 0 \,, \tag{6.20}$$

where $\mathcal{L}_b' = \partial \mathcal{L}_b / \phi$, and so

$$\mathcal{U}_b = \frac{1}{3} \left[(1 - \mathcal{L}_b) - \sqrt{(1-\mathcal{L}_b)^2 - \frac{3}{4} (\mathcal{L}_b')^2} \right] \simeq \frac{(\mathcal{L}_b')^2}{8(1-\mathcal{L}_b)^2} + \cdots \,, \tag{6.21}$$

where the root is chosen so $\mathcal{U}_b \to 0$ when $(\mathcal{L}_b')^2 \to 0$.

The corresponding boundary condition for the Maxwell field implies that in a coordinate patch containing each source brane, eq. (6.15) integrates to (see also Appendix D.1) (6.13)

$$\begin{aligned} \mathcal{A}_\theta(\rho) &= \left(\frac{\Phi_o}{2\pi} \right) e^{\phi_o} + \mathcal{Q} \int_{\rho_o}^{\rho} \mathrm{d}\tilde{\rho} \; e^{\phi + B - 4W} \qquad \text{observer brane} \\ &= -\left(\frac{\Phi_f}{2\pi} \right) e^{\phi_f} + \mathcal{Q} \int_{\rho_f}^{\rho} \mathrm{d}\tilde{\rho} \; e^{\phi + B - 4W} \quad \text{flux brane} \,, \end{aligned} \tag{6.22}$$

where $\Phi_o := \lim_{\rho \to \rho_o} \Phi_o[\phi(\rho)]$ — appropriately renormalized (6.25) — and so

[5] An important complication for codimension-2 branes over codimension-1 is that both sides of eqs. (6.18) generically diverge in the near-brane limit; requiring a renormalization of the brane action (6.25). (This renormalization turns out to be unnecessary in the special case of D7-branes in Type IIB supergravity (6.12).)

on.

Requiring these two solutions to differ by a gauge transformation, $g^{-1}\partial_\theta\Omega$, on regions of overlap between the two patches implies the flux-quantization condition

$$\frac{n}{g} = \frac{1}{2\pi}\left(\Phi_o\, e^{\phi_o} + \Phi_f\, e^{\phi_f}\right) + \mathcal{Q}\int_{\rho_o}^{\rho_f} \mathrm{d}\rho\; e^{\phi+B-4W}\,. \tag{6.23}$$

This identifies $\Phi_{\rm tot} = \sum_b \Phi_b\, e^{\phi_b}$ as the part of the total magnetic flux carried by the branes (6.2).

6.3.4 Explicit solutions

A great many explicit solutions to the above field equations and boundary conditions have been found, starting almost 30 years ago (6.9). Some of these are known at the linearized level (6.26; 6.13), while others are exact solutions (6.2; 6.27; 6.21; 6.28; 6.29; 6.30; 6.31; 6.32). Many of these solutions provide explicit compactifications from six to four dimensions, and provide among the earliest examples of flux-stabilized compactifications.

For the present purposes, what is most interesting about the exact solutions is that the most general solutions are known (6.2; 6.27; 6.21; 6.28) for the special case where the dilaton's radial derivative, ϕ', tends to zero at both brane positions. As is clear from the boundary conditions, eqs. (6.18), these solutions are the ones appropriate for the case where the brane actions do not depend on ϕ: $\partial L_b/\partial\phi = 0$. What is remarkable about these solutions is that for *all* of them the on-brane geometries are flat: $\hat{g}_{\mu\nu} = \eta_{\mu\nu}$. We now briefly summarize these solutions in more detail.

Rugby balls

A particularly simple situation is the special case where the dilaton is constant, $\phi = \varphi_0$, since then the solution is very easy to visualize: a rugby ball, sourced by two branes (6.2):

$$\begin{aligned} \mathrm{d}s^2 &= e^{-\varphi_0}\left[\mathrm{d}\hat{\rho}^2 + \alpha^2\ell^2\sin^2\left(\frac{\hat{\rho}}{\ell}\right)\mathrm{d}\theta^2\right] + \hat{g}_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu \\ \mathcal{F}_{\rho\theta} &= \mathcal{F}_{\hat{\rho}\theta}\, e^{-\varphi_0/2} = \mathcal{Q}e^{\varphi_0/2}\alpha\ell\sin\left(\frac{\hat{\rho}}{\ell}\right) . \end{aligned} \tag{6.24}$$

With this metric the volume of the extra dimensions is

$$\mathcal{V}_2 = 4\pi\alpha\ell^2 e^{-\varphi_0}\,, \tag{6.25}$$

showing that the flux-stabilization fixes the extra-dimensional volume in terms of the scalar-field value, φ_0.

In these coordinates[6] the two source branes for this geometry are situated at $\hat{\rho}_o = 0$ and $\hat{\rho}_f = \pi\ell$. This geometry has a conical singularity at these points, characterized by the defect angle $\delta = 2\pi(1-\alpha)$. In the special case $\alpha = 1$ the extra-dimensional geometry is a sphere (6.9). For these rugby-ball solutions the matching conditions, eqs. (6.18), degenerate to a relationship between the deficit angle and the lagrangian density, $L_o = L_f = L$, of the two source branes,

$$1 - \alpha = \frac{\kappa^2 L}{2\pi}\,, \tag{6.26}$$

as expected from other approaches (6.33).

The equations of motion determine the values of $\mathcal{Q}$ and ℓ as well as the

[6]Notice the coordinate rescaling $\rho := e^{-\varphi_0/2}\hat{\rho}$ between this solution and the *ansatz* of eq. (6.13).

curvature of the on-brane directions:

$$\mathcal{Q} = \pm\frac{2g_R}{\kappa^2} \qquad \ell = \frac{\kappa}{2g_R} \quad \text{and} \quad \hat{g}_{\mu\nu} = \eta_{\mu\nu}\,, \tag{6.27}$$

while the flux quantization condition implies

$$\frac{n}{g} = \frac{\alpha}{g_R} + \left(\frac{\Phi_o + \Phi_f}{2\pi}\right) e^{\varphi_0}\,, \tag{6.28}$$

where the last equality uses eqs. (6.27).

The interpretation of this last equation differs according to whether or not $\sum_b \Phi_b\, e^{\phi_b}$ depends on φ_0. If not — such as in the scale-invariant case where $\Phi_b \propto e^{-\phi_b}$ — then eq. (6.28) must be regarded as a constraint on the parameters of the brane action which, if not satisfied, is an obstruction to the existence of solutions satisfying our assumed symmetry *ansatz*. But if $\sum_b \Phi_b\, e^{\phi_b}$ depends on φ_0 then this equation can be read as determining the value of φ_0, which is not fixed by any of the other field equations.

These properties have a simple interpretation from the point of view of the low-energy 4D effective theory (6.13; 6.34). The reason φ_0 is not fixed by the other equations is because it is the parameter that labels the one-parameter family of solutions whose existence is guaranteed by the scale invariance, eq. (6.7), of the classical bulk field equations. If both T_b and $\Phi_b\, e^{\phi}$ do not depend on ϕ the brane couplings do not break this scale invariance, implying the classical low-energy potential for φ_0 must have an exponential form, $V_{\text{eff}} = Ae^{2\varphi_0}$. In this case there are two situations: (*i*) φ_0 labels a flat direction (and so is undetermined by the field equations) if $A = 0$, or (*ii*) φ_0 necessarily runs away to $\pm\infty$ if $A \neq 0$ (and so for finite φ_0 there are no solutions to the equations that are maximally symmetric in the on-brane directions). As is shown in (6.13) (see also Appendix D.2), eq. (6.28) corresponds in the low-energy theory to the condition $A = 0$.

However, in the case of interest – *c.f.* eqs. (6.11) and (6.12) – $\sum_b \Phi_b \, e^{\phi}$ *does* depend on ϕ, and so the brane-bulk couplings break the bulk scaling symmetry. In this case φ_0 appears only in eq. (6.28), which should be read as fixing its value. From the point of view of the low-energy 4D theory (details in Appendix D.2), φ_0 gets fixed because the breaking of scale invariance lifts the flat direction, through a Goldberger-Wise-like (6.35) stabilization mechanism for codimension-2 branes. In this case all of the field equations are not satisfied unless φ_0 is chosen to minimize this potential.

Unequal tensions

Because the above rugby ball solutions have equal defect angles at both brane positions, they only describe situations where the two branes have precisely equal tensions. But the general solutions that apply when $\phi' \to 0$ at both branes are known (6.21; 6.27; 6.28), including those having two unequal brane tensions, which we now describe.

In this case the metric can be written

$$\mathrm{d}s^2 = \mathcal{W}^2 \, \hat{g}_{\mu\nu} \, \mathrm{d}x^\mu \mathrm{d}x^\nu + a^2 e^{-\varphi_0} \Big(\mathcal{W}^8 \mathrm{d}\eta^2 + \mathrm{d}\theta^2 \Big) \,, \tag{6.29}$$

where $a = a(\eta)$, $\mathcal{W} = \mathcal{W}(\eta)$ and (as before) $\hat{g}_{\mu\nu}$ is a maximally symmetric on-brane geometry. The dilaton is similarly taken to depend only on η, $\phi = \phi(\eta)$, and the Maxwell field is given by $\mathcal{A}_\theta = \mathcal{A}_\theta(\eta)$, so that

$$\mathcal{F}_{\eta\theta} = Q\, a^2\, e^{\phi} e^{-\varphi_0}\,, \tag{6.30}$$

with Q an integration constant. With these choices proper distance along the direction between the branes is $\mathrm{d}\rho = a\mathcal{W}^4 \mathrm{d}\eta\, e^{-\varphi_0/2}$, the proper circumference of a circle along which θ varies from zero to 2π at fixed η is $\mathcal{C} = 2\pi a(\eta)\, e^{-\varphi_0/2}$, and the extra-dimensional volume is $\mathcal{V}_2 = 2\pi e^{-\varphi_0} \int \mathrm{d}\eta\, a^2 \mathcal{W}^4$. In particular, when φ_0 is moderately large and negative — so the bulk coupling satisfies $e^{\varphi_0} \ll 1$ — then the 'radius' defined by $\mathcal{V}_2 = r^2$ can become exponentially large: $r^2 \propto e^{-\varphi_0}$.

The general solution to the bulk field equations having only conical defects is known explicitly for these variables, given by $\hat{g}_{\mu\nu} = \eta_{\mu\nu}$ together with

$$\begin{aligned} e^{\phi} &= \mathcal{W}^{-2} e^{\varphi_0} \\ \mathcal{W}^4 &= \left(\frac{\kappa^2 Q}{2g_R}\right) \frac{\cosh[\lambda(\eta-\eta_1)]}{\cosh[\lambda(\eta-\eta_2)]} \\ \text{and} \qquad a^{-4} &= \left(\frac{2g_R\kappa^2 Q^3}{\lambda^4}\right) \cosh^3[\lambda(\eta-\eta_1)]\cosh[\lambda(\eta-\eta_2)]\,, \end{aligned} \tag{6.31}$$

showing that $\eta_2 - \eta_1$, λ, φ_0 and Q are the independent integration constants.

The position of the two source branes in these coordinates is $\eta \to \pm\infty$. Since the near-brane limit of the proper distance is

$$\mathrm{d}\rho = \mp e^{-\varphi_0/2} a\mathcal{W}^4\, \mathrm{d}\eta \to \mp C\, e^{\mp\lambda\eta}\, \mathrm{d}\eta\,, \tag{6.32}$$

the defect angle in the geometry as $\eta \to \pm\infty$ turns out to be

$$\alpha_\pm := \left(\frac{2\lambda g_R}{\kappa^2 Q}\right) e^{\mp\lambda(\eta_2-\eta_1)} \,. \tag{6.33}$$

The product of these last two expressions show how the integration constant Q is fixed in terms of the tensions of the two branes:

$$\alpha_+\alpha_- = \left(\frac{2\lambda g_R}{\kappa^2 Q}\right)^2 \,. \tag{6.34}$$

It is fixed in this way because it must be adjusted to ensure that the solution to the dilaton field equation is consistent with the boundary condition that $\phi' \to 0$ at both branes. Once this is done the solutions have three independent parameters that may be dialed: the two tensions (or defect angles) and the parameter φ_0 that labels the orbit of the classical scale symmetry.

The flux-quantization condition is found by computing $\mathcal{A}_\theta(\eta)$ near the brane at $\eta \to \pm\infty$, giving

$$\begin{aligned} \mathcal{A}^\pm_\theta &= \left(\frac{\Phi_o}{2\pi}\right) e^{\phi_o} + \frac{\lambda}{\kappa^2 Q}\Big\{\tanh[\lambda(\eta-\eta_1)] + 1\Big\} \quad \text{observer brane} \\ &= -\left(\frac{\Phi_f}{2\pi}\right) e^{\phi_f} + \frac{\lambda}{\kappa^2 Q}\Big\{\tanh[\lambda(\eta-\eta_1)] - 1\Big\} \quad \text{flux brane}\,, \end{aligned} \tag{6.35}$$

and so flux quantization becomes

$$\begin{aligned} \frac{n}{g} &= \frac{2\lambda}{\kappa^2 Q} + \frac{1}{2\pi}\left(\Phi_o\, e^{\phi_o} + \Phi_f\, e^{\phi_f}\right) \\ &= \frac{(\alpha_+\alpha_-)^{1/2}}{g_R} + \frac{1}{2\pi}\left(\frac{\Phi_o}{\mathcal{W}_o^2} + \frac{\Phi_f}{\mathcal{W}_f^2}\right) e^{\varphi_0}\,, \end{aligned} \tag{6.36}$$

where $\mathcal{W}_0^4 = \lambda/\alpha_+$ and $\mathcal{W}_f^4 = \lambda/\alpha_-$. Notice this reduces to the rugby-ball quantization condition in the limit that[7] $\alpha_+ = \alpha_- = \alpha$.

Response to brane perturbations

Crucial to what follows is what happens to these solutions when properties of the source branes are varied. Most importantly, the above solutions require their source branes to satisfy two separate conditions:

$$(i) \quad \frac{\partial L_b}{\partial \phi} = 0 \qquad (6.37)$$
$$(ii) \quad \text{flux quantization } (\textit{i.e.} \text{ eq. } (6.36)) \,.$$

Notice in particular that it is *not* necessary to require $L_o = L_f$, which just corresponds to the special case of rugby-ball solutions.

These conditions provide the motivation for the choices made for the brane lagrangians given above — eqs. (6.11) and (6.12). The ϕ-independence of both T_b and Φ_b is designed so that both branes do not couple to ϕ, ensuring $\partial L_o/\partial\phi = \partial L_f/\partial\phi = 0$ as required by condition (i). But because these choices imply that the flux-quantization condition depends on φ_0, condition (ii) is automatically satisfied for an appropriate choice $\varphi_0 = \varphi_\star$. Using $\Phi_o = 0$ and $\Phi_f = \mu$:

$$e^{\varphi_\star} = \frac{2\pi \mathcal{W}_f^2}{\mu} \left[\frac{n}{g} - \frac{(\alpha_+\alpha_-)^{1/2}}{g_R} \right] . \qquad (6.38)$$

This adjustment of φ_0 also has an energetic interpretation. This can be shown explicitly for small perturbations about rugby-ball geometries (see Appendix D.2 for details), for which condition (ii) can be seen to be the condition for minimizing the brane-generated scalar potential that lifts the

[7]Notice that in the equal-tension limit the warp factor at the brane position is $\mathcal{W}_o^4 = \mathcal{W}_f^4 = \lambda/\alpha$, which was set to one in the rugby-ball solution by rescaling the coordinates x^μ.

flat direction for φ_0 in the low-energy 4D effective theory.

The same thing can also be shown beyond the linearized approximation. On general grounds, for the system studied here the effective 4D scalar potential responsible for the on-brane curvature is given by (6.10; 6.12)

$$V_{\rm eff}(\varphi_0) = V_{\rm brane}(\varphi_0) + V_{\rm bulk}(\varphi_0)\,, \tag{6.39}$$

where the 'bulk' contribution is given by evaluating the bulk action at the bulk solution generated by the source branes,

$$\begin{aligned}\sqrt{-g_4}\; V_{\rm bulk} &= -\int {\rm d}^2x \mathcal{L}_{\rm bulk} = -\frac{1}{2\kappa^2}\int {\rm d}^2x\sqrt{-g}\;\Box\phi \\ &= \frac{2\pi}{2\kappa^2}\sum_b \sqrt{-g}\; n_{\scriptscriptstyle M}\partial^{\scriptscriptstyle M}\phi = \frac{1}{2}\sqrt{-g_4}\; L_b'\,.\end{aligned} \tag{6.40}$$

where we use the general result, eq. (6.9), $n_{\scriptscriptstyle M}$ is the normal vector directed *into* the bulk, evaluated at the position of each brane, and we use the dilaton matching condition, eq. (6.18), to trade $n_{\scriptscriptstyle M}\partial^{\scriptscriptstyle M}\phi$ for $L_b' = {\rm d}L_b/{\rm d}\phi$ evaluated at the brane.

The 'brane' contribution to eq. (6.39) is similarly given by the sum of the brane action, L_b, and a 'Gibbons-Hawking' contribution, for each brane. This leads to

$$V_{\rm brane} = \sum_b U_b\,, \tag{6.41}$$

with U_b given by eqs. (6.19) and (6.21). Notice in particular that near a zero of L_b' the function U_b vanishes quadratically:

$$\mathcal{U}_b = \frac{\kappa^2 U_b}{2\pi} \simeq \frac{(\mathcal{L}_b')^2}{8(1-\mathcal{L}_b)^2} + \cdots\,, \tag{6.42}$$

where $\mathcal{L}_b = \kappa^2 L_b/2\pi$.

Combining terms, the total brane-generated effective potential becomes

$$V_{\text{eff}} = \sum_b \left(U_b + \frac{1}{2} \frac{\partial L_b}{\partial \phi} \right) . \tag{6.43}$$

Notice in particular how V_{eff} vanishes whenever L_b' does. This is required by consistency since all exact solutions are known with $\phi' \to 0$ at the branes (as is the case whenever $L_b' = 0$), and the on-brane geometry of all of these solutions is flat. Furthermore, since U_b is quadratic in $\kappa^2 L_b'$ when L_b' is small, it is only the second term in the sum in eq. (6.43) that contributes in a linearized deviation away from these flat solutions, consistent with the explicit linearized analysis of Appendix D.2 and refs. (6.13; 6.34). The U_b term provides the generalization of the brane-generated potential beyond linear order, that is exact (up to classical level in the bulk physics) (6.10; 6.11; 6.12).

6.4 Loop effects

In order to have a technically natural cosmological constant, it is not enough just to have a vanishing classical contribution. Since the cosmological constant problem is in essence a quantum problem, the problem hasn't become hard until loop effects are included. Generically, because the vacuum energy has low (mass) dimension, it is the largest mass scale that can appear in the loop that is the most dangerous. In the present instance there are two separate kinds of loop effects to distinguish: those involving only particles localized on the brane (which we imagine also includes all the known standard-model loops); and those that also involve virtual contributions from the bulk supergravity.

We briefly discuss each in turn.

6.4.1 Brane loops

Consider first those loops involving only brane-bound states. For realistic brane-world models these include loops of all ordinary Standard Model particles. Neglecting (for the moment) bulk loops amounts to asking how the bulk and on-brane geometry classically respond to brane-loop-generated changes to the brane action.

Now comes the main point. What is important for these purposes is the observation that brane loops cannot in themselves invalidate the two conditions, (6.37), given that these are satisfied by the classical brane action (*i.e.* such as by eq. (6.10) with eqs. (6.11) and (6.12)). That is, a sufficient condition for obtaining zero on-brane curvature (at the bulk classical level) is the *absence* of a coupling between the bulk dilaton, ϕ, and the branes, since this ensures the validity of both conditions (*i*) and (*ii*) (6.10; 6.11).

From this point of view the effects of brane loops can be regarded as generic $\mathcal{O}(M^4)$ perturbations to the initial brane function T_o. For the model considered brane loops alone also cannot modify Φ_o because brane fields do not initially couple to the bulk gauge potential, $\mathcal{A}_M$. The assumed absence of heavy brane-localized fields on the flux brane, together with the physical separation between the observer and flux branes, similarly ensures that brane loops cannot modify[8] flux-brane properties like T_f or Φ_f.

The upshot is that brane loops can only renormalize the brane actions (and in the model considered here, only for the observer brane) in a

[8]That is, the only influence at this order between the two branes is due to the classical response of the bulk fields, which are computed exactly in the above solutions, and do not correspond to changes to the flux-brane action.

ϕ-independent way. But this does not change the bulk response since we in any case did not assume anything special about the typical energy scale for L_o when inferring the flatness of the on-brane geometry.

6.4.2 Bulk loops

Since brane loops cannot lift the flatness of the on-brane directions, the dominant corrections come from bulk loops. And these can come in a number of varieties, depending on whether or not the bulk states in the loop are short- or long-wavelength. The purpose of this section is to recap earlier arguments (6.10; 6.11) that the contribution of bulk loops to the low-energy scalar potential can be naturally of order m_{KK}^4 in supersymmetric theories.

We first estimate the generic size of bulk loops in non-supersymmetric theories, and then how bulk supersymmetry changes these estimates.

Loops involving massless 6D fields

On dimensional grounds the contributions of massless 6D fields to the low-energy 4D scalar potential is of order $\delta V_{\text{eff}} \simeq m_{KK}^4 \propto 1/r^4$, and various contributions of this type have been explicitly calculated for specific extra-dimensional geometries as Casimir energy calculations (6.36; 6.37; 6.38; 6.39). Because the bulk states that dominate in the loop have wavelengths comparable to the size of the extra dimensions, this contribution to V_{eff} need not have a local interpretation from the point of view of the extra dimensions.

We now argue that order m_{KK}^4 contributions are the generic size when the bulk is supersymmetric, since (unusually) the contribution of heavier fields is not larger than this.

Massive 6D states

The Casimir energy contributed by 6D states of mass m has also been computed (6.37; 6.38) for simple extra-dimensional geometries. In general this depends in a complicated way on the dimensionless ratio m/m_{KK}, but the simplifies considerably when $m \gg m_{KK}$. The simplification arises because in this limit the wavelength that dominates the loop is much shorter than the size of the extra dimensions, leading to a result that can be described by a local contribution to the higher-dimensional effective action. This simplification allows a very general calculation (6.40) of the contributions of heavy fields to the low-energy theory to be performed, using general tools (6.41) for studying the small-distance singularities in correlation functions on curved space.

There are two kinds of such local contributions that massive loops can generate. Quantum fluctuations that take place further than $\mathcal{O}(m^{-1})$ from the branes are described by local contributions to the bulk action, integrated over the full 6D spacetime. Those that occur nearer to the branes themselves can also contribute local 4D corrections to the brane action. We consider each of these in turn.

Far from the brane

The contributions in the bulk can be organized in a derivative expansion, leading to the schematic terms

$$\begin{aligned} -\frac{\delta \mathcal{L}_{\rm eff}}{\sqrt{-g_6}} &= a_1 m^6 + m^4 \Big[b_1 R + b_2 (\partial_M \phi \, \partial^M \phi) + \cdots \Big] \\ &\quad + m^2 \Big[c_1 R^2 + c_2 (\partial_M \phi \, \partial^M \phi)^2 + \cdots \Big] \\ &\quad + \log \left(\frac{m^2}{\mu^2} \right) \Big[d_1 R^3 + d_2 R (\partial_M \phi \, \partial^M \phi)^2 + \cdots \Big] + \cdots , \end{aligned} \tag{6.44}$$

where all possible terms containing a fixed number of derivatives are included, for each of which the coefficients $a_i(\phi)$, $b_i(\phi)$, $c_i(\phi)$ and $d_i(\phi)$ are calculable (and generically nonzero) for any given choice for the heavy fields circulating in the loops (6.40).[9] As indicated, in general these coefficients can be functions of the background scalar field, ϕ.

The contribution of this kind of loop to the low-energy 4D potential for the zero-mode φ_0 may be estimated by replacing all derivatives by $1/r$, where r is a measure of the extra-dimensional size, with the result integrated over the extra dimensions:

$$\begin{aligned}\delta V_{\text{eff}}(\varphi_0) \;\simeq\; & \tilde{a}_1 m^6 r^2 + m^4\Big[\tilde{b}_1 + \tilde{b}_2 + \cdots\Big] + \frac{m^2}{r^2}\Big[\tilde{c}_1 + \tilde{c}_2 + \cdots\Big] \\ & + \frac{1}{r^4}\,\log\left(\frac{m^2}{\mu^2}\right)\Big[\tilde{d}_1 + \tilde{d}_2 + \cdots\Big] + \cdots\,, \qquad (6.45)\end{aligned}$$

where the coefficients $\tilde{a}_i$ through $\tilde{d}_i$ are proportional to a_i through d_i, with numerical factors (and possibly logs of m and r) arising from the details of the evaluation of the derivatives and the extra-dimensional integration. This shows that when $m \gg m_{KK} \simeq 1/r$, it is the contributions involving a_i, b_i and c_i that are much greater than $\mathcal{O}(m_{KK}^4)$.

Here is where supersymmetry in the bulk plays its part. If we specialize to the classical level in the bulk, then there is no Casimir energy, $\delta\mathcal{L}_{\text{eff}} = 0$, but the classical bulk lagrangian, $\mathcal{L}_{\text{bulk}}$ given by eq. (6.3), itself has the form of eq. (6.44). And working to classical level in the bulk we know that in this case the work of the previous sections shows that the contributions to the

[9]For simple toroidal examples it can happen that the vanishing of $R^M{}_{NPQ}$ and $\partial_M\phi$ in the background can imply that only the first of these survives, making the $\mathcal{O}(m^6)$ contribution the only one that grows in the large-m limit (6.38).

low-energy potential cancel[10] to give $V^c_{\rm eff} = 0$.

Bulk loops change this, but their ϕ-dependence is easy to establish because for supersymmetric theories the classical scale invariance implies $e^{2\phi}$ is the loop-counting parameter. In the frame where the classical lagrange density has the form $\mathcal{L}_{\rm bulk} \propto \sqrt{-\breve{g}}\; e^{-2\phi}$, the ℓ-loop corrections obtained after integrating out heavy 6D states of mass m are therefore proportional to $\delta\mathcal{L}_\ell \propto \sqrt{-\breve{g}}\; e^{2(\ell-1)\phi}$, which implies in Einstein frame $\mathcal{L}_{\rm bulk} + \delta\mathcal{L}_{\rm eff}$ is given by

$$
\begin{aligned}
-\frac{\mathcal{L}_{\rm bulk} + \delta\mathcal{L}_{\rm eff}}{\sqrt{-g_6}} \;=\;& \left[\frac{2g_{\scriptscriptstyle R}^2}{\kappa^4}\, e^{\phi} + a_1\, m^6\, e^{3\phi} + \mathcal{O}(e^{5\phi})\right] \\
&+ \left[\frac{1}{2\kappa^2} + b_1\, m^4\, e^{2\phi} + \mathcal{O}(e^{4\phi})\right] R + \cdots \\
&+ \left[c_1\, e^{\phi} + \mathcal{O}(e^{3\phi})\right] m^2\, R^2 + \cdots \\
&+ \left[d_1 + \mathcal{O}(e^{2\phi})\right] R^3 \log\left(\frac{m^2}{\mu^2}\right) + \cdots\,.
\end{aligned}
\tag{6.46}
$$

Notice in particular that all of the corrections beyond the classical terms are at least $\mathcal{O}(1/r^6)$ once evaluated with derivatives of order $1/r$ and using the classical flux-stabilization condition, $e^\phi \propto 1/r^2$. This ensures all such contributions to the 4D potential are at most of order $\delta V_{\rm eff} \propto 1/r^4 \simeq m_{\scriptscriptstyle KK}^4$ for large r, as claimed. Evidence from explicit calculations for this supersymmetric suppression is also available for some kinds of compactifications (6.40).

Furthermore, validity of the semiclassical tools used here ensure the coefficient of proportionality of $1/r^4$ also cannot be large. For example, for the rugby ball if we define r using the extra-dimensional volume, so $\mathcal{V}_2 := r^2$,

[10] Explicitly, for the rugby-ball solutions it is the coefficients of the R, $e^{-\phi}F^2$ and e^ϕ terms that cancel amongst themselves, which is possible because $e^\phi \propto 1/r^2$ for the classical solution.

then eqs. (6.25) and (6.27) imply

$$r^2 e^{\varphi_0} = 4\pi\alpha \left(\frac{\kappa}{2g_{_R}}\right)^2 , \tag{6.47}$$

and so, for example, $\mathcal{V}_2\, m^6 e^{3\varphi_0} = (\pi\alpha\,\kappa^2 m^2/g_{_R}^2)^3 (1/r)^4 \propto (m/M_g)^6 (1/r)^4$, where we take $g_{_R}^2 \simeq \kappa \simeq 1/M_g^2$ with M_g the 6D gravity scale. The validity of the semiclassical approximation in the low-energy theory requires $m \ll M_g$, which keeps the coefficient of $1/r^4$ small.

More precisely, loops involving states with mass $\kappa^2 m^2 > 1$ would have to be computed in the UV completion of the low-energy supergravity. Although string theory provides a natural choice for this, we cannot yet compute these loops for the 6D supergravity studied here since its string-theoretic provenance is not yet known (see however (6.42; 6.43)). We do know, however, that at such high energies there are a variety of mechanisms (6.44), including supersymmetry and the general softening of UV dependence that string theory brings, that can suppress these contributions from the extreme ultraviolet.

Near the brane

A similar discussion applies to quantum fluctuations of heavy bulk fields located near the branes. These also have a local interpretation if the bulk fields involved have masses $m \gg m_{_{KK}}$. Proximity to the brane allows such loops to modify the brane lagrangian as well as the bulk one. Since each bulk loop comes with a factor of $e^{2\phi}$, near-brane loop effects are counted by also writing the brane lagrangians as a series in this variable,

$$T_b = T_b^{(0)} + T_b^{(1)}\, e^{2\phi} + \cdots , \quad \text{and} \quad \Phi_b = \Phi_b^{(0)} + \Phi_b^{(1)}\, e^{2\phi} + \cdots , \tag{6.48}$$

and so on. Such corrections are potentially dangerous because they clearly introduce a ϕ-dependence to the brane action, and so violate condition (i), above, that ensured the flatness of the on-brane directions.

The effect of integrating out very massive bulk states therefore corresponds to modifying both the brane and bulk actions as a series in $e^{2\phi}$. And the implications of the corrections to the brane action can then be estimated by following how these changes modify the bulk solutions, through the changes they induce in the bulk boundary conditions. This is evaluated in detail in Appendix D.2, showing that the result is a contribution to the effective potential that is of order $e^{2\varphi_\star}$, where $\varphi_\star$ is the lowest-order value of the localized dilaton. The rest of the story is by now familiar: because $e^{2\varphi_\star} \propto 1/r^4$, the resulting contribution to the 4D potential is again $\delta V_{\text{eff}} \simeq 1/r^4 \simeq m_{KK}^4$, as claimed.

The upshot is this: brane loops in themselves cannot cause on-brane curvature because they cannot introduce a ϕ-dependence of the brane action if this was absent at lowest order. Bulk loops can cause on-brane curvature, but the result corresponds to an effective 4D potential that is of order $\delta V_{\text{eff}} \simeq m^4 e^{2\phi} \simeq 1/r^4$, and so is very small for large r because the bulk coupling is so very weak in this limit. Although the calculation of contributions from loops arising from above the gravity scale remain beyond our present calculation reach, similar kinds of volume suppression are known to arise in other explicit large-volume string compactifications (6.43; 6.44).

6.5 Conclusions

We provide here an explicit model of brane-localized matter for which both brane-backreaction and fluxes play a role in stabilizing the size of the extra dimensions. Remarkably, the stabilization mechanism produces an on-brane curvature that is parametrically suppressed relative to the generic scales of masses that define the brane-localized tensions (including loops).

The model of interest involves a generic field theory (a proxy for the Standard Model, say) localized on a nonsupersymmetric codimension-two brane within a six-dimensional spacetime whose bulk dynamics is supersymmetric. The on-brane curvature is found to be of order $R \sim V/M_p^2$, where $V \sim m_{KK}^4$. This is true even if the Kaluza-Klein scale, m_{KK}, is much smaller than the generic particle mass, M, on the brane.

The small size of the low-energy effective potential is a consequence of a cancelation between the direct contributions of the brane and the contributions of the bulk to which the branes give rise. What is new in this paper is an explicit calculation of how the system responds to arbitrary small perturbations in brane properties, which confirms in detail the more general arguments (6.21; 6.10; 6.11; 6.12; 6.28; 6.40; 6.44) that have emerged over the years from the SLED proposal (6.2; 6.7).

We believe our example provides a useful explicit particular realization of the general SLED proposal, but expect the result to apply more generally than just for the specific 6D supergravity considered here. The ingredients we believe to be necessary to suppress the brane curvature are:

- Codimension-two branes, for which the back-reaction on the surrounding bulk only varies logarithmically with distance, and so cannot be neglected even when comparatively far from the brane;

- A higher-dimensional bulk, described by a supergravity that enjoys a classical scale invariance under which the metric scales by a constant factor while a bulk dilaton shifts. Such a scale invariance appears to be generic for many supergravities in six and higher dimensions. The bulk itself need not be required to be invariant under any of the supersymmetries.

- A bulk-stabilizing flux that can be localized on at least one of the branes present.

In particular, we expect the mechanism to generalize to 3-branes localized within a bulk described by more generic 6D supergravities than the particular Nishino-Sezgin gauged supergravity studied here. But supersymmetry is crucial, since this is what allows the suppression of bulk loop effects.

We believe the models described here provide a context for understanding why the observed Dark Energy density is so much smaller than all of the other scales we know about in particle physics. Because the stabilization mechanism for the extra dimensions both explains the dark energy density and the electroweak hierarchy, these problems are related in this framework.

6.5.1 Some observational implications

Realistic applications to an effectively 6D world with large supersymmetric dimensions require $m_{KK} \simeq 10^{-2}$ eV, which corresponds to $\mathcal{V}_2/\kappa \simeq (M_g r)^2 \simeq e^{-\varphi_\star} \simeq 10^{30}$. Notice flux-quantization gives the value of the stabilized dilaton by $e^{-\varphi_\star} \simeq g_R \mu/\kappa^2 T \simeq 10^{30}$, where T is a generic brane tension and $\Phi = \mu$ is the flux parameter on the flux-brane. The extra dimensions would be expected to be of order $m_{KK} \simeq 10^{-2}$ eV, or $r \simeq 10$ microns, and the 6D gravity scale could be as low as[11] $M_g \gtrsim 10$ TeV.

If this is how nature works we will soon know, since the SLED picture necessarily has many striking observational consequences, some of which are shared by the non-supersymmetric proposal for sub-eV dimensions (6.45). Since some of these are described in more detail elsewhere, we only briefly list some of the main ones here.

- *Deviations from Newton's Law:* Since the size of the Dark Energy density is set by the KK scale, the extra dimensions must generically be of order micron scales. Deviations from Newton's inverse-square law must arise once distances of order this size are probed. This is the smoking gun for the SLED scenario, since it cannot be avoided. Since only two dimensions can possibly be this large, the predicted change is a crossover to an inverse fourth power, although the precise shape depends somewhat on the details of the extra-dimensional shape (6.49). Present bounds probe down to about 45 microns (6.50; 6.51) and so are getting close.

- *String and gravity physics at the LHC:* Given the size of the extra dimensions, the measured strength of gravity dictates the gravity scale in the extra dimensions. The 6D gravity scale to which this points is of order tens of TeV (though astrophysics requires it to be no smaller than 10 TeV). The string scale and the KK scale for any other extra dimen-

[11] A gravity scale lower than this produces too much energy loss from supernovae (6.45; 6.46). Much stronger, but more model-dependent, bounds are also possible if extra-dimensional states can decay visibly (such as into photons) (6.47; 6.48), but these bounds can be avoided if visible channels are swamped by invisible higher-dimensional ones (6.45; 6.46) (some potential examples of which are discussed in (6.43)).

sions is then generically found to be lower than this (6.46; 6.43). This means that quantum gravity is becoming strong at LHC energies. This implies a variety of signals for the LHC, including excited string states for all Standard Model particles (6.52), new neutral gauge bosons (6.53), energy loss into gravitons (6.54) and other particles (6.46; 6.55; 6.56) in the extra dimensions, and possibly black holes (6.57) or other aspects of high-energy gravity (6.58). Even though supersymmetry is broken only at very low scales in the bulk, supersymmetry must be nonlinearly realized on any brane and so superpartners for ordinary particles (and so also the MSSM) are *not* predicted (6.46). Results for new experimental searches at the LHC are even now starting to come out (6.59; 6.60).

- *Dark Energy quintessence cosmology:* The same physics that makes the value of the potential, $\rho = V_\star$, small at its minimum (and thereby gives a small Dark Energy density) also makes the mass of the would-be zero mode very light: $m^2 \simeq \sqrt{V_\star}/M_p$ (and in an equally technically natural way). Since this is of order the present-day Hubble scale, Dark Energy phenomenology is that of a quintessence model rather than of a cosmological constant (6.61). The same requirement that makes the on-brane curvature small — the absence of a direct brane-dilaton coupling — also ensures that the light scalar field naturally has quasi-Brans-Dicke couplings to brane matter. This means they can naturally evade tests of the equivalence principle (6.50), but the couplings need not be small and so are potentially constrained by a variety of long-distance tests of General Relativity that bound scalar-tensor models (6.62; 6.63), as well as laboratory bounds on light scalars with an effective 2-photon coupling

(6.64). Present-day bounds on deviation from GR in the solar system provide nontrivial constraints, but need not be fatal (6.61). One reason for this is because the Brans-Dicke couplings of the light scalar turn out to be field dependent, and so can evolve cosmologically. For parts of parameter space (6.61) these couplings can be acceptably small in the solar system during the present cosmological epoch.

- *Exotic sterile neutrino physics:* Although not absolutely required, the SLED scenario predicts there to be a variety of massless fermions in the extra dimensions, whose mass is protected to be small because they are related by supersymmetry to the graviton or bulk gauge fields. These fermions can mix with Standard Model neutrinos, leading to a potentially rich spectrum of sterile neutrino mixing (6.65) whose masses are naturally in the sub-eV range due to the large size of the extra dimensions (6.66; 6.67).

There are likely even more consequences, since only the surface of what might be seen has yet been scratched. Should all of these be seen together, there could be little doubt about what is going on.

6.5.2 Outstanding issues

We now summarize potential challenges that these models remain to face.

First, it is an unpleasant — though technically natural — feature of the model that a large number must be inserted for μ as a parameter in the lagrangian in order to obtain a sufficiently low KK scale. This does not cause a problem with the approximations made, however, since it is only the combination $g_R \mu e^{\varphi_\star} \simeq \kappa^2 T \lesssim 0.1$ that appears in the brane lagrangian. $\varphi_\star$ also appears on its own in the bulk lagrangian, but the loop approximation in the bulk is under good control because the loop counting parameter there is $e^{2\varphi_\star} \simeq 10^{-60}$. We expect this feature is likely something that can be improved in more complicated examples, preferably with more explicit contact with a UV string construction, since most of the known 10-dimensional string compactifications having very large volumes (6.68) generically obtain equally large volumes as are required here without having to dial in such small parameters. They do so because they predict the volume to arise as the exponential of another, much smaller, modulus, for which parameters of order 10 need be used.

Second, much could be gained if this picture could be properly embedded into a controlled UV completion, such as if it were obtained from an explicit string vacuum. Until this is done the contributions to ρ from states in the far UV cannot be properly computed.

Third, SLED models face a variety of phenomenological challenges as well as opportunities. In particular, as mentioned above, strong bounds on light gravitationally coupled fields must be evaded in order not to conflict with known physics in the solar system. The cosmology of the universe before nucleosynthesis is also challenging, due to constraints from energy loss into the extra dimensions (together with stronger, but more model-dependent bounds that arise if extra dimensional fields can decay too frequently to visible states). The nature of inflationary cosmology is also unknown (see however (6.69) for first steps towards an inflationary theory where extra dimensions evolve during inflation, allowing the gravity scale to be much higher during inflation than it is at present). We regard these to be model building challenges, but much easier to solve than is the cosmological constant problem itself.

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Bibliography

[6.1] C.P. Burgess and Leo van Nierop "Technically Natural Cosmological Constant From Supersymmetric 6D Brane Backreaction" [arXiv:1108.0345]

[6.2] Y. Aghababaie, C. P. Burgess, S. L. Parameswaran and F. Quevedo, "Towards a naturally small cosmological constant from branes in 6D supergravity," Nucl. Phys. B **680** (2004) 389 [arXiv:hep-th/0304256].

[6.3] S. Weinberg, "The Cosmological Constant Problem," Rev. Mod. Phys. **61** (1989) 1-23.

[6.4] J. W. Chen, M. A. Luty and E. Ponton, "A Critical cosmological constant from millimeter extra dimensions" JHEP **0009**(2000)012 [arXiv:hep-th/0003067] ; S. M. Carroll and M. M. Guica, "Sidestepping the cosmological constant with football-shaped extra dimensions," arXiv:hep-th/0302067;
I. Navarro, "Codimension two compactifications and the cosmological constant problem," JCAP **0309** (2003) 004 [hep-th/0302129].

[6.5] C. P. Burgess, R. C. Myers and F. Quevedo, "A naturally small cosmological constant on the brane?," Phys. Lett. B **495** (2000) 384 [arXiv:hep-th/9911164];

S. Dimopoulos, S. Kachru, N. Kaloper, A. E. Lawrence and E. Silverstein, "Small numbers from tunneling between brane throats," Phys. Rev. D **64** (2001) 121702 [arXiv:hep-th/0104239];

C. P. Burgess, P. Grenier and D. Hoover, "Quintessentially flat scalar potentials," JCAP **0403** (2004) 008 [arXiv:hep-ph/0308252];

[6.6] N. Arkani-Hamed, S. Dimopoulos, N. Kaloper and R. Sundrum, "A small cosmological constant from a large extra dimension," Phys. Lett. B **480** (2000) 193 [arXiv:hep-th/0001197];

S. Kachru, M. B. Schulz and E. Silverstein, "Self-tuning flat domain walls in 5d gravity and string theory," Phys. Rev. D **62** (2000) 045021 [arXiv:hep-th/0001206].

[6.7] C. P. Burgess, "Supersymmetric large extra dimensions and the cosmological constant: An update," Annals Phys. **313** (2004) 283 [arXiv:hep-th/0402200];

"Towards a natural theory of dark energy: Supersymmetric large extra dimensions," AIP Conf. Proc. **743** (2005) 417 [arXiv:hep-th/0411140];

[6.8] H. Nishino and E. Sezgin, *Phys. Lett.* **144B** (1984) 187; "The Complete N=2, D = 6 Supergravity With Matter And Yang-Mills Couplings," Nucl. Phys. **B278** (1986) 353;

S. Randjbar-Daemi, A. Salam, E. Sezgin and J. Strathdee, "An Anomaly Free Model in Six-Dimensions" *Phys. Lett.* **B151** (1985) 351.

[6.9] A. Salam and E. Sezgin, "Chiral Compactification On Minkowski X S**2

Of N=2 Einstein-Maxwell Supergravity In Six-Dimensions," Phys. Lett. B **147** (1984) 47.

[6.10] C. P. Burgess, D. Hoover and G. Tasinato, "UV Caps and Modulus Stabilization for 6D Gauged Chiral Supergravity," JHEP **0709** (2007) 124 [arXiv:0705.3212 [hep-th]];
C. P. Burgess, D. Hoover, C. de Rham and G. Tasinato, "Effective Field Theories and Matching for Codimension-2 Branes," JHEP **0903** (2009) 124 [arXiv:0812.3820 [hep-th]].

[6.11] C. P. Burgess, D. Hoover, G. Tasinato, "Technical Naturalness on a Codimension-2 Brane," JHEP **0906** (2009) 014. [arXiv:0903.0402 [hep-th]].

[6.12] A. Bayntun, C.P. Burgess and L. van Nierop, "Codimension-2 Brane-Bulk Matching: Examples from Six and Ten Dimensions," New J. Phys. **12** (2010) 075015 [arXiv:0912.3039 [hep-th]].

[6.13] C.P.Burgess and L. van Nierop, "Large Dimensions and Small Curvatures from Supersymmetric Brane Back-reaction," [arXiv:1101.0152 (hep-th)].

[6.14] A. Conan Doyle, "The Sign of the Four," ch. 6 (1890).

[6.15] G. 't Hooft, in *Cargese Summer Inst.* 1979:135 (QCD161:S77:1979) (reprinted in *'t Hooft, G. (ed.): Under the spell of the gauge principle* 352-374, and in *Farhi, E. (ed.), Jackiw, R. (ed.): Dynamical gauge symmetry breaking* 345-367).

[6.16] K.G. Wilson and J.B. Kogut, *Phys. Rept.* **12** (1974) 75;

J. Polchinski, "Renormalization and Effective Lagrangians," Nucl. Phys. **B231** (1984) 269-295.

[6.17] For a review of some of these issues, see for example
C. P. Burgess, "Quantum gravity in everyday life: General relativity as an effective field theory," Living Rev. Rel. **7** (2004) 5. [gr-qc/0311082]; "Introduction to Effective Field Theory," Ann. Rev. Nucl. Part. Sci. **57** (2007) 329-362. [hep-th/0701053].

[6.18] For a discussion of the pitfalls of using cutoffs within a non-gravitational context, see
C.P. Burgess and David London, "Uses and abuses of effective Lagrangians," Phys. Rev. **D48** (1993) 4337-4351 [hep-ph/9203216]; " On anomalous gauge boson couplings and loop calculations," Phys. Rev. Lett. **69** (1992) 3428-3431.

[6.19] S. Weinberg, *Gravitation and Cosmology*, Wiley 1973.

[6.20] C. W. Misner, J. A. Wheeler and K. S. Thorne, *Gravitation*, W. H. Freeman & Company 1973.

[6.21] Y. Aghababaie, C. P. Burgess, J. M. Cline, H. Firouzjahi, S. L. Parameswaran, F. Quevedo, G. Tasinato, I. Zavala, "Warped brane worlds in six-dimensional supergravity," JHEP **0309** (2003) 037. [hep-th/0308064].

[6.22] K. Lanczos, Phys. Z. **23** (1922) 239–543; Ann. Phys. **74** (1924) 518–540; C.W. Misner and D.H. Sharp, "Relativistic Equations for Adiabatic, Spherically Symmetric Gravitational Collapse" Phys. Rev. **136** (1964) 571–576;

W. Israel, "Singular hypersurfaces and thin shells in general relativity" Nuov. Cim. **44B** (1966) 1–14; *errata* Nuov. Cim. **48B** 463.

[6.23] P. Bostock, R. Gregory, I. Navarro and J. Santiago, "Einstein gravity on the codimension 2 brane?," Phys. Rev. Lett. **92** (2004) 221601 [arXiv:hep-th/0311074];
I. Navarro and J. Santiago, "Gravity on codimension 2 brane worlds," JHEP **0502** (2005) 007 [arXiv:hep-th/0411250];
J. Vinet and J. M. Cline, "Codimension-two branes in six-dimensional supergravity and the cosmological constant problem," Phys. Rev. D **71** (2005) 064011 [hep-th/0501098];
C. de Rham, "The Effective Field Theory of Codimension-two Branes," JHEP **0801** (2008) 060 [arXiv:0707.0884 [hep-th]];
E. Papantonopoulos, A. Papazoglou and V. Zamarias, "Regularization of conical singularities in warped six-dimensional compactifications," JHEP **0703** (2007) 002 [arXiv:hep-th/0611311]; "Induced cosmology on a regularized brane in six-dimensional flux compactification," Nucl. Phys. B **797** (2008) 520 [arXiv:0707.1396 [hep-th]];
D. Yamauchi and M. Sasaki, "Brane World in Arbitrary Dimensions Without Z_2 Symmetry," Prog. Theor. Phys. **118** (2007) 245 [arXiv:0705.2443 [gr-qc]];
N. Kaloper and D. Kiley, "Charting the Landscape of Modified Gravity," JHEP **0705** (2007) 045 [hep-th/0703190];
M. Minamitsuji and D. Langlois, "Cosmological evolution of regularized branes in 6D warped flux compactifications," Phys. Rev. D **76** (2007) 084031 [arXiv:0707.1426 [hep-th]];

S. A. Appleby and R. A. Battye, "Regularized braneworlds of arbitrary codimension," Phys. Rev. D **76** (2007) 124009 [arXiv:0707.4238 [hep-ph]];

C. Bogdanos, A. Kehagias and K. Tamvakis, "Pseudo-3-Branes in a Curved 6D Bulk," Phys. Lett. B **656** (2007) 112 [arXiv:0709.0873 [hep-th]];

O. Corradini, K. Koyama and G. Tasinato, "Induced gravity on intersecting brane-worlds Part I: Maximally symmetric solutions," Phys. Rev. D **77** (2008) 084006 [arXiv:0712.0385 [hep-th]]; "Induced gravity on intersecting brane-worlds Part II: Cosmology," Phys. Rev. D **78** (2008) 124002 [arXiv:0803.1850 [hep-th]];

F. Arroja, T. Kobayashi, K. Koyama and T. Shiromizu, "Low energy effective theory on a regularized brane in 6D gauged chiral supergravity," JCAP **0712** (2007) 006 [arXiv:0710.2539 [hep-th]];

V. Dzhunushaliev, V. Folomeev and M. Minamitsuji, "Thick brane solutions," arXiv:0904.1775 [gr-qc].

[6.24] M. Peloso, L. Sorbo and G. Tasinato, "Standard 4d gravity on a brane in six dimensional flux compactifications," Phys. Rev. D **73** (2006) 104025 [arXiv:hep-th/0603026];

B. Himmetoglu and M. Peloso, "Isolated Minkowski vacua, and stability analysis for an extended brane in the rugby ball," Nucl. Phys. B **773** (2007) 84 [hep-th/0612140];

[6.25] W. D. Goldberger, M. B. Wise, "Renormalization group flows for brane couplings," Phys. Rev. **D65** (2002) 025011 [hep-th/0104170];

E. Dudas, C. Papineau, V.A. Rubakov, "Flowing to four dimensions" JHEP0603:085,2006 arXiv:hep-th/0512276v1;
C. de Rham, "The Effective Field Theory of Codimension-two Branes," JHEP **0801** (2008) 060 [arXiv:0707.0884 [hep-th]];
C. P. Burgess, C. de Rham, L. van Nierop, "The Hierarchy Problem and the Self-Localized Higgs," JHEP **0808** (2008) 061. [arXiv:0802.4221 [hep-ph]].

[6.26] H. M. Lee and A. Papazoglou, "Scalar mode analysis of the warped Salam-Sezgin model," [hep-th/0602208];
C. P. Burgess, C. de Rham, D. Hoover, D. Mason and A. J. Tolley, "Kicking the rugby ball: Perturbations of 6D gauged chiral supergravity," JCAP **0702** (2007) 009 [arXiv:hep-th/0610078].
S. L. Parameswaran, S. Randjbar-Daemi and A. Salvio, "Gauge fields, fermions and mass gaps in 6D brane worlds," Nucl. Phys. B **767** (2007) 54 [arXiv:hep-th/0608074]; "Stability and Negative Tensions in 6D Brane Worlds," JHEP **0801** (2008) 051 [arXiv:0706.1893 [hep-th]]; "General Perturbations for Braneworld Compactifications and the Six Dimensional Case," JHEP **0903** (2009) 136 [arXiv:0902.0375 [hep-th]].

[6.27] G. W. Gibbons, R. Guven and C. N. Pope, "3-branes and uniqueness of the Salam-Sezgin vacuum," Phys. Lett. B **595** (2004) 498 [hep-th/0307238];

[6.28] C. P. Burgess, F. Quevedo, G. Tasinato and I. Zavala, "General axisymmetric solutions and self-tuning in 6D chiral gauged supergravity," JHEP **0411** (2004) 069 [hep-th/0408109];

[6.29] A. J. Tolley, C. P. Burgess, D. Hoover and Y. Aghababaie, "Bulk singularities and the effective cosmological constant for higher co-dimension branes," JHEP **0603** (2006) 091 [arXiv:hep-th/0512218].

[6.30] A. J. Tolley, C. P. Burgess, C. de Rham and D. Hoover, "Scaling solutions to 6D gauged chiral supergravity," New J. Phys **8** (2006) 324 [arXiv:0608.083 [hep-th]];

A. J. Tolley, C. P. Burgess, C. de Rham and D. Hoover, "Exact Wave Solutions to 6D Gauged Chiral Supergravity," JHEP **0807** (2008) 075 [arXiv:0710.3769 [hep-th]];

M. Minamitsuji, "Instability of brane cosmological solutions with flux compactifications," Class. Quant. Grav. **25** (2008) 075019 [arXiv:0801.3080 [hep-th]].

H. M. Lee and A. Papazoglou, "Codimension-2 brane inflation," Phys. Rev. D **80** (2009) 043506 [arXiv:0901.4962 [hep-th]].

[6.31] S. L. Parameswaran, G. Tasinato and I. Zavala, "The 6D SuperSwirl," Nucl. Phys. B **737** (2006) 49 [arXiv:hep-th/0509061];

[6.32] H. M. Lee and C. Ludeling, "The general warped solution with conical branes in six-dimensional supergravity," JHEP **0601** (2006) 062 [arXiv:hep-th/0510026];

[6.33] A. Vilenkin, Phys. Rev. "Gravitational field of vacuum domain walls and strings" **D23** (1981) 852;

R. Gregory, "Gravitational stability of local strings" Phys. Rev. Lett. **59** (1987) 740;

A. G. Cohen and D. B. Kaplan, "The Exact Metric About Global Cosmic Strings," *Phys. Lett.* **B215**, 67 (1988);
A. Vilenkin and P. Shellard, *Cosmic Strings and other Topological Defects*, Cambridge University Press (1994);
R. Gregory and C. Santos, "Cosmic strings in dilaton gravity," Phys. Rev. D **56**, 1194 (1997) [gr-qc/9701014].

[6.34] C. P. Burgess and L. van Nierop, "Bulk Axions, Brane Back-reaction and Fluxes," arXiv:1012.2638 [hep-th].

[6.35] W. D. Goldberger and M. B. Wise, "Modulus stabilization with bulk fields", Phys. Rev. Lett. **83** (1999) 4922-4925 [arXiv:hep-ph/9907447]

[6.36] H.B.G. Casimir, Proc. K. Ned. Akad. Wet. **51** (1948) 793;
for a review with references see: M. Bordag, U. Mohideen and V.M. Mostepanenko, *New Developments in the Casimir Effect*, Phys. Rept. **353**, (2001) 1 (arXiv:quant-ph/0106045).

[6.37] P. Candelas and S. Weinberg, "Calculation of Gauge Couplings and Compact Circumferences from Self-consistent Dimensional Reduction," Nucl. Phys. **B 237** (1984) 397;
C.R. Ordoñez and M.A. Rubin, "Graviton Dominance in Quantum Kaluza-Klein Theory," Nucl. Phys. **B 260** (1985) 456;
R. Kantowski and K.A. Milton, "Scalar Casimir Energies in $M^4 \times S^N$ for Even N," Phys. Rev. **D35** (1987) 549;
D. Birmingham, R. Kantowski and K.A. Milton, "Scalar and Spinor Casimir Energies in Even Dimensional Kaluza-Klein Spaces of the Form $M(4) \times S(N_1) \times S(N_2) \times \cdots$," Phys. Rev. **D38** (1988) 1809;

M. Minamitsuji, “The Casimir effect in rugby-ball type flux compactifications,” J. Phys. A **A41** (2008) 164060.

[6.38] D. M. Ghilencea, D. Hoover, C. P. Burgess and F. Quevedo, “Casimir energies for 6D supergravities compactified on T(2)/Z(N) with Wilson lines,” JHEP **0509** (2005) 050 [arXiv:hep-th/0506164];
E. Elizalde, M. Minamitsuji and W. Naylor, “Casimir effect in rugby-ball type flux compactifications,” Phys. Rev. D **75**, 064032 (2007) [arXiv:hep-th/0702098].

[6.39] E. Witten, “New *gauge* theories in six dimensions,” JHEP **9801** (1998) 001 [Adv. Theor. Math. Phys. **2** (1998) 61] [arXiv:hep-th/9710065];
A. Hanany and A. Zaffaroni, “Branes and six dimensional supersymmetric theories,” Nucl. Phys. B **529** (1998) 180 [arXiv:hep-th/9712145];
M. Alishahiha and Y. Oz, “Supergravity and 'new' six-dimensional gauge theories,” Phys. Lett. B **495** (2000) 418 [arXiv:hep-th/0008172].

[6.40] D. Hoover and C. P. Burgess, “Ultraviolet Sensitivity in Higher Dimensions,” JHEP **0601** (2006) 058 [arXiv:hep-th/0507293];
“UV Sensitivity in Supersymmetric Large Extra Dimensions: The Ricci-flat Case,” Nucl. Phys. B **772** (2007) 175 [arXiv:hep-th/0504004].

[6.41] J. Schwinger, “On Gauge Invariance and Vacuum Polarization” Phys. Rev. 82 (1951) 664;
B.S. De Witt, “Dynamical Theory of Groups and Fields,” in *Relativity, Groups and Topology*, B.S. De Witt and C. De Witt, (New York, Gordon and Breach, 1965);

P. B. Gilkey, "The Spectral Geometry Of A Riemannian Manifold," J. Diff. Geom. **10**, 601 (1975);

A. O. Barvinsky and G. A. Vilkovisky, "The Generalized Schwinger-Dewitt Technique In Gauge Theories And Quantum Gravity," Phys. Rept. **119** (1985) 1;

D.M. McAvity and H. Osborn, "A DeWitt expansion of the heat kernel for manifolds with a boundary" *Class. Quant. Grav.* **8** (1991) 603–638.

D. V. Vassilevich, "Heat kernel expansion: User's manual," Phys. Rept. **388**, 279 (2003) [hep-th/0306138].

[6.42] M. Cvetic, G. W. Gibbons, C. N. Pope, "A String and M theory origin for the Salam-Sezgin model," Nucl. Phys. **B677** (2004) 164-180. [hep-th/0308026];

T. G. Pugh, E. Sezgin, K. S. Stelle, "D=7 / D=6 Heterotic Supergravity with Gauged R-Symmetry," JHEP **1102** (2011) 115. [arXiv:1008.0726 [hep-th]].

[6.43] M. Cicoli, C. P. Burgess, F. Quevedo, "Anisotropic Modulus Stabilisation: Strings at LHC Scales with Micron-sized Extra Dimensions," [arXiv:1105.2107 [hep-th]].

[6.44] C. P. Burgess, A. Maharana, F. Quevedo, "Uber-naturalness: unexpectedly light scalars from supersymmetric extra dimensions," JHEP **1105** (2011) 010. [arXiv:1005.1199 [hep-th]].

[6.45] N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, "The hierarchy problem and new dimensions at a millimeter," Phys. Lett. B **429** (1998) 263 [arXiv:hep-ph/9803315];

I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, "New dimensions at a millimeter to a Fermi and superstrings at a TeV," Phys. Lett. B **436** (1998) 257 [arXiv:hep-ph/9804398].

[6.46] C. P. Burgess, J. Matias and F. Quevedo, "MSLED: A Minimal supersymmetric large extra dimensions scenario," Nucl. Phys. B **706** (2005) 71 [arXiv:hep-ph/0404135];

[6.47] S. Hannestad and G. G. Raffelt, "Stringent neutron star limits on large extra dimensions," Phys. Rev. Lett. **88** (2002) 071301 [arXiv:hep-ph/0110067]; "New supernova limit on large extra dimensions," Phys. Rev. Lett. **87** (2001) 051301 [arXiv:hep-ph/0103201].

[6.48] S. Cullen and M. Perelstein, "SN1987A constraints on large compact dimensions," Phys. Rev. Lett. **83** (1999) 268 [arXiv:hep-ph/9903422];
V. D. Barger, T. Han, C. Kao and R. J. Zhang, "Astrophysical constraints on large extra dimensions," Phys. Lett. B **461** (1999) 34 [arXiv:hep-ph/9905474];
C. Hanhart, D. R. Phillips, S. Reddy and M. J. Savage, "Extra dimensions, SN1987a, and nucleon-nucleon scattering data," Nucl. Phys. B **595** (2001) 335 [arXiv:nucl-th/0007016];

[6.49] P. Callin and C. P. Burgess, "Deviations from Newton's law in supersymmetric large extra dimensions," Nucl. Phys. B **752** (2006) 60 [arXiv:hep-ph/0511216].

[6.50] C. M. Will, "The confrontation between general relativity and experiment," Living Rev. Rel. **4** (2001) 4 [arXiv:gr-qc/0103036]; Living Rev. Rel. **9** (2005) 3 [arXiv:gr-qc/0510072];

E. G. Adelberger, J. H. Gundlach, B. R. Heckel, S. Hoedl and S. Schlamminger, "Torsion balance experiments: A low-energy frontier of particle physics," Prog. Part. Nucl. Phys. **62** (2009) 102.

[6.51] E. G. Adelberger, B. R. Heckel and A. E. Nelson, "Tests of the gravitational inverse-square law," Ann. Rev. Nucl. Part. Sci. **53**, 77 (2003) [arXiv:hep-ph/0307284];
C. D. Hoyle, D. J. Kapner, B. R. Heckel, E. G. Adelberger, J. H. Gundlach, U. Schmidt and H. E. Swanson, "Sub-millimeter tests of the gravitational inverse-square law," Phys. Rev. D **70** (2004) 042004 [arXiv:hep-ph/0405262].

[6.52] I. Antoniadis and K. Benakli, "Large dimensions and string physics in future colliders," Int. J. Mod. Phys. A **15** (2000) 4237 [arXiv:hep-ph/0007226].
I. Antoniadis, E. Kiritsis, J. Rizos and T. N. Tomaras, "D-branes and the standard model," Nucl. Phys. B **660** (2003) 81.
D. Lust, S. Stieberger and T. R. Taylor, "The LHC String Hunter's Companion," Nucl. Phys. B **808** (2009) 1 [arXiv:0807.3333 [hep-th]];
D. Lust, O. Schlotterer, S. Stieberger and T. R. Taylor, "The LHC String Hunter's Companion (II): Five-Particle Amplitudes and Universal Properties," Nucl. Phys. B **828** (2010) 139 [arXiv:0908.0409 [hep-th]];

[6.53] M. Williams, C. P. Burgess, A. Maharana, F. Quevedo, "New Constraints (and Motivations) for Abelian Gauge Bosons in the MeV-TeV Mass Range," [arXiv:1103.4556 [hep-ph]].

[6.54] G. F. Giudice, R. Rattazzi, J. D. Wells, "Quantum gravity and extra dimensions at high-energy colliders," Nucl. Phys. **B544** (1999) 3-38. [hep-ph/9811291];

T. Han, J. D. Lykken, R. -J. Zhang, "On Kaluza-Klein states from large extra dimensions," Phys. Rev. **D59** (1999) 105006. [hep-ph/9811350];

J. L. Hewett, "Indirect collider signals for extra dimensions," Phys. Rev. Lett. **82** (1999) 4765 [arXiv:hep-ph/9811356];

G. F. Giudice and A. Strumia, "Constraints on extra dimensional theories from virtual graviton exchange," Nucl. Phys. B **663** (2003) 377 [arXiv:hep-ph/0301232].

[6.55] D. Atwood, C. P. Burgess, E. Filotas, F. Leblond, D. London and I. Maksymyk, "Supersymmetric large extra dimensions are small and/or numerous," Phys. Rev. D **63** (2001) 025007 [arXiv:hep-ph/0007178];

J. L. Hewett and D. Sadri, "Supersymmetric extra dimensions: Gravitino effects in selectron pair production," Phys. Rev. D **69** (2004) 015001 [arXiv:hep-ph/0204063];

[6.56] G. Azuelos, P. H. Beauchemin, C. P. Burgess, "Phenomenological constraints on extra dimensional scalars," J. Phys. G **G31** (2005) 1-20. [hep-ph/0401125];

P. H. Beauchemin, G. Azuelos, C. P. Burgess, "Dimensionless coupling of bulk scalars at the LHC," J. Phys. G **G30** (2004) N17. [hep-ph/0407196].

[6.57] S. Dimopoulos, G. L. Landsberg, "Black holes at the LHC," Phys. Rev. Lett. **87** (2001) 161602. [hep-ph/0106295];

M. Cavaglia, S. Das, R. Maartens, "Will we observe black holes at LHC?," Class. Quant. Grav. **20** (2003) L205-L212. [hep-ph/0305223];

S. Hossenfelder, S. Hofmann, M. Bleicher, H. Stoecker, "Quasistable black holes at CERN LHC," Phys. Rev. **D66** (2002) 101502. [hep-ph/0109085];

A. Chamblin, G. C. Nayak, "Black hole production at CERN LHC: String balls and black holes from pp and lead-lead collisions," Phys. Rev. **D66** (2002) 091901. [hep-ph/0206060];

L. Anchordoqui, H. Goldberg, "Black hole chromosphere at the CERN LHC," Phys. Rev. **D67** (2003) 064010. [hep-ph/0209337];

B. Koch, M. Bleicher, S. Hossenfelder, "Black hole remnants at the LHC," JHEP **0510** (2005) 053. [hep-ph/0507138];

P. Meade, L. Randall, "Black Holes and Quantum Gravity at the LHC," JHEP **0805** (2008) 003 [arXiv:0708.3017 [hep-ph]];

I. Mocioiu, Y. Nara, I. Sarcevic, "Hadrons as signature of black hole production at the LHC," Phys. Lett. **B557** (2003) 87-93. [hep-ph/0301073];

G. Dvali, M. Redi, "Black Hole Bound on the Number of Species and Quantum Gravity at LHC," Phys. Rev. **D77** (2008) 045027. [arXiv:0710.4344 [hep-th]];

T. G. Rizzo, "Black hole production at the LHC: Effects of Voloshin suppression," JHEP **0202** (2002) 011. [hep-ph/0201228].

[6.58] D. Amati, M. Ciafaloni, G. Veneziano, "Superstring Collisions at Planckian Energies," Phys. Lett. **B197** (1987) 81; "Classical and Quantum Gravity Effects from Planckian Energy Superstring Collisions," Int. J. Mod. Phys. **A3** (1988) 1615-1661; "Can Space-Time Be Probed Below the String Size?," Phys. Lett. **B216** (1989) 41;

I. J. Muzinich, M. Soldate, "High-Energy Unitarity of Gravitation and

Strings," Phys. Rev. **D37** (1988) 359;
H. L. Verlinde, E. P. Verlinde, "Scattering at Planckian energies," Nucl. Phys. **B371** (1992) 246-268. [hep-th/9110017];
D. N. Kabat, M. Ortiz, "Eikonal quantum gravity and Planckian scattering," Nucl. Phys. **B388** (1992) 570-592. [hep-th/9203082];
S. B. Giddings, D. J. Gross, A. Maharana, "Gravitational effects in ultrahigh-energy string scattering," Phys. Rev. **D77** (2008) 046001. [arXiv:0705.1816 [hep-th]];
S. B. Giddings, M. Schmidt-Sommerfeld, J. R. Andersen, "High energy scattering in gravity and supergravity," Phys. Rev. **D82** (2010) 104022. [arXiv:1005.5408 [hep-th]].

[6.59] S. Chatrchyan *et al.* [CMS Collaboration], "Search for Large Extra Dimensions in the Diphoton Final State at the Large Hadron Collider," arXiv:1103.4279 [hep-ex];
G. Aad *et al.* [ATLAS Collaboration], "Search for New Physics in Dijet Mass and Angular Distributions in pp Collisions at $\sqrt{s} = 7$ TeV Measured with the ATLAS Detector," [arXiv:1103.3864 [hep-ex]].

[6.60] R. Franceschini, G. F. Giudice, P. P. Giardino, P. Lodone, A. Strumia, "LHC bounds on large extra dimensions," [arXiv:1101.4919 [hep-ph]].

[6.61] A. Albrecht, C. P. Burgess, F. Ravndal, C. Skordis, "Natural quintessence and large extra dimensions," Phys. Rev. **D65** (2002) 123507. [astro-ph/0107573];
K. Kainulainen, D. Sunhede, "Dark energy from large extra dimensions," Phys. Rev. **D73** (2006) 083510. [astro-ph/0412609].

[6.62] R. Blandford and S. A. Teukolsky, "Arrival-time analysis for a pulsar in a binary system," Astrophys. J. **205** (1976) 580;

T. Damour and N. Deruelle, "General relativistic celestial mechanics of binary systems I. The post-Newtonian motion," Ann. Inst. Henri Poincaré **43** (1985) 107-132;

T. Damour and N. Deruelle, "General relativistic celestial mechanics of binary systems II. The post-Newtonian timing formula," Ann. Inst. Henri Poincaré **44** (1986) 263-292;

T. Damour, J. H. Taylor, "Strong field tests of relativistic gravity and binary pulsars," Phys. Rev. **D45** (1992) 1840-1868;

Clare Burrage, Anne-Christine Davis and Douglas J. Shaw "Detecting Chameleons: The Astronomical Polarization Produced by Chameleon-like Scalar Fields" Phys. Rev. **D79** (2009) 044028 [arXiv:0809.1763v2 [astro-ph]].

[6.63] M. W. Horbatsch, C. P. Burgess, "Model-Independent Comparisons of Pulsar Timings to Scalar-Tensor Gravity," [arXiv:1107.3585 [gr-qc]];

M. W. Horbatsch, C. P. Burgess, "Semi-Analytic Stellar Structure in Scalar-Tensor Gravity," [arXiv:1006.4411 [gr-qc]].

[6.64] Cecile Robilliard, Remy Battesti, Mathilde Fouche, Julien Mauchain, Anne-Marie Sautivet, Francois Amiranoff, Carlo Rizzo, "No light shining through a wall : new results from a photoregeneration experiment," Phys. Rev. Lett. **99** (2007) 190403 [arXiv:0707.1296v4 [hep-ex]];

A. S. Chou, W. Wester, A. Baumbaugh, H. R. Gustafson, Y. Irizarry-Valle, P. O. Mazur, J. H. Steffen, R. Tomlin, X. Yang, J. Yoo, "Search for

axion-like particles using a variable baseline photon regeneration technique" Phys. Rev. Lett. **100** (2008) 080402 [arXiv:0710.3783v2 [hep-ex]];
Pierre Pugnat et al. "Results from the OSQAR photon-regeneration experiment: No light shining through a wall," Phys. Rev. **D78** (2008) 092003 ;
Mathilde Fouch et al. "Search for photon oscillations into massive particles," Phys. Rev. bf D78 (2008) 032013 [arXiv:0808.2800v1 [hep-ex]];
A. Afanasev, O.K. Baker, K.B. Beard, G. Biallas, J. Boyce, M. Minarni, R. Ramdon, M. Shinn, P. Slocum, "New Experimental limit on Optical Photon Coupling to Neutral, Scalar Bosons," Phys. Rev. Lett. **101** (2008) 120401 [arXiv:0806.2631v1 [hep-ex]];
Klaus Ehret et al. "New ALPS results on hidden-sector lightweights " Phys. Lett. **B689** (2010), 149 155.
Philippe Brax and Clare Burrage, "Atomic Precision Tests and Light Scalar Couplings" Phys. Rev. **D83** (2011) 035020 [arXiv:1010.5108v1 [hep-ph]].

[6.65] J. Matias and C. P. Burgess, "MSLED, neutrino oscillations and the cosmological constant," JHEP **0509** (2005) 052 [arXiv:hep-ph/0508156];

[6.66] G. R. Dvali and A. Y. Smirnov, "Probing large extra dimensions with neutrinos," Nucl. Phys. B **563**, 63 (1999);
N. Arkani-Hamed, S. Dimopoulos, G. R. Dvali and J. March-Russell, "Neutrino masses from large extra dimensions," Phys. Rev. D **65**, 024032 (2002).

[6.67] A. K. Das and O. C. W. Kong, "On neutrino masses and mixings from extra dimensions," Phys. Lett. B **470**, 149 (1999);

R. N. Mohapatra, S. Nandi and A. Perez-Lorenzana, “Neutrino masses and oscillations in models with large extra dimensions,” Phys. Lett. B **466**, 115 (1999);
A. Lukas, P. Ramond, A. Romanino and G. G. Ross, “Solar neutrino oscillation from large extra dimensions,” Phys. Lett. B **495**, 136 (2000);
K. R. Dienes, “Extra dimensions and their implications for neutrino physics,” Int. J. Mod. Phys. A **16S1C**, 925 (2001);
R. N. Mohapatra and A. Perez-Lorenzana, “Three flavour neutrino oscillations in models with large extra dimensions,” Nucl. Phys. B **593**, 451 (2001);
C. S. Lam, “Large mixing induced by the strong coupling with a single bulk neutrino,” Phys. Rev. D **65** (2002) 053009;
C. S. Lam and J. N. Ng, “Neutrino oscillations via the bulk,” Phys. Rev. D **64** (2001) 113006;
A. De Gouvea, G. F. Giudice, A. Strumia and K. Tobe, “Phenomenological implications of neutrinos in extra dimensions,” Nucl. Phys. B **623**, 395 (2002);
K. R. S. Balaji, A. S. Dighe and R. N. Mohapatra, “Neutrino magnetic moment, large extra dimensions and high energy cosmic neutrino spectra,” [hep-ph/0202267];
J. Maalampi, V. Sipilainen and I. Vilja, “A scheme with two large extra dimensions confronted with neutrino physics,” Phys. Rev. D **67**, 113005 (2003);
Q. H. Cao, S. Gopalakrishna and C. P. Yuan, “Collider signature of bulk neutrinos in large extra dimensions,” Phys. Rev. D **70**, 075020 (2004).

[6.68] V. Balasubramanian, P. Berglund, J. P. Conlon and F. Quevedo, "Systematics of moduli stabilisation in Calabi-Yau flux compactifications," JHEP **0503**, 007 (2005) [arXiv:hep-th/0502058];
J. P. Conlon, F. Quevedo and K. Suruliz, "Large-volume flux compactifications: Moduli spectrum and D3/D7 soft supersymmetry breaking," JHEP **0508** (2005) 007 [hep-th/0505076];

[6.69] C.P. Burgess and L. van Nierop, "Sculpting the Extra Dimensions: Inflation from Codimension-2 Brane Back-reaction" [arXiv:1108.2553]

Chapter 7

Summary and conclusions

In this thesis we developed the tools to calculate the back reaction of codimension-2 branes. In general, bulk fields tend to diverge at the presence of branes in codimension-2 or higher. In order to describe brane couplings to these divergent fields, the parameters in the brane action need to be renormalized at the classical level.

When the branes couple to a flat direction of the bulk physics, generically the flat direction is lifted and the corresponding modulus in the extra dimensions is stabilized. This stabilization is of particular interest in gauged chiral supergravity, where the zero mode has overlap with the volume in the extra dimensions. We have shown that this mechanism can lead to exponentially large volume in the extra dimensions, from only a moderate hierarchy of the brane parameters.

An important brane contribution that we include is a magnetic coupling to a stabilizing bulk Maxwell flux. Although the corresponding brane term is subdominant in a derivative expansion, it can be the dominant contribution to the low energy potential. The reason for this is that the dominant term — the brane tension — can drop out of the low energy theory. The contribution from the magnetic coupling allows a relaxation of the flux quantization condition in the bulk, by allowing some of the flux to localize at the branes (at a low energy cost).

In the regime of small perturbations around rugby ball solutions that are sourced by branes with constant, equal tension we find both the effective 4 dimensional potential, and the bulk response to the perturbations. In the case of a simple Einstein-Maxwell-scalar system, we correct the probe-brane results by including the brane localized flux, which is competitive with the probe-brane results. The more interesting case is with a supersymmetric bulk, where we find that the size of the potential is very generally set by the *derivative* of the brane tension with respect to the dilaton.

Finally, we use that observation to construct a model with a 6 dimensional supersymmetric bulk coupled to completely nonsupersymmetric branes. We find that for an appropriate choice of brane-bulk couplings the cosmological constant can be made on the order of magnitude of the observed value. The importance of this result is that we do so in a technically natural way: once we choose the parameters in our model, they only get corrections through quantum effects that are of the same order (or smaller) than the classical values.

7.0.3 Outlook

Although we have found interesting new results from including back reaction of branes on their surrounding spacetime, we have only scratched the surface. The results in this thesis are all dealing with maximally symmetric configurations in the brane directions. In order to explore the cosmology of this type of models, work on a timedependent generalization has begun (7.1) but is far from done. In order to make contact with a more fundamental theory, it would also be worthwhile to have a string-theoretic construction of this system.

In addition we believe that our techniques carry over to codimension-2 objects in string theory, as long as they are applied to a sufficiently long-distance limit. In particular, the renormalization procedure presupposes that some new physics interjects to regularize the bulk divergences before the weak coupling approximation breaks down. Still, the results for F-theory from chapter (3) are evidence that our approach to codimension-2 objects is quite broadly applicable. We are using this to investigate the possibility to obtain de Sitter solutions in higher dimensional supergravities (7.2). This has not yet lead to new de Sitter solutions, but our results do challenge some of the no-go results in the literature (7.3)

Bibliography

[7.1] Leo van Nierop and C.P. Burgess, "Sculpting the Extra Dimensions: Inflation from Codimension-2 Brane Back-reaction" [arXiv:1108.2553]

[7.2] C.P. Burgess, Anshuman Maharana, L. van Nierop, A. A. Nizami, F. Quevedo "On Brane Back-Reaction and de Sitter Solutions in Higher-Dimensional Supergravity" [arXiv:1109.0532]

[7.3] G.W. Gibbons, in F. del Aguila, J.A. de Azcárraga, L.E. Ibáñez (eds), "Supersymmetry, supergravity and related topics". World Scientific, Singapore (1985).

J. Maldacena and C. Nunez, "Supergravity description of field theories on curved manifolds and a no go theorem," Int. J. Mod. Phys. **A16** (2001) 822-855 [arXiv:hep-th/0007018].

D. H. Wesley, "Oxidised cosmic acceleration," JCAP **0901** (2009) 041, [arXiv:0802.3214 [hep-th]].

P.J. Steinhardt and D. Wesley, "Dark Energy, Inflation and Extra Dimensions," Phys. Rev. **D79** 104026 (2009).

Appendix A

Appendix for chapter 2

A.1 Some Properties of Bessel Functions

This appendix summarizes a few properties of modified Bessel functions which are used in the main text. The modified Bessel functions are linearly independent solutions to the differential equation

$$z^2 y'' + z y' - (z^2 + \nu^2) y = 0 \,, \tag{A.1}$$

with $I_\nu(z)$ chosen to be regular at $z = 0$ and $K_\nu(z)$ chosen to fall off to zero as $z \to \infty$. They are defined in terms of ordinary Bessel functions, $J_\nu(z)$, and Hankel functions, $H_\nu^{(1)}(z)$, by

$$I_\nu(z) = i^{-\nu} J_\nu(iz) \quad \text{and} \quad K_\nu(z) = \frac{\pi}{2}\, i^{\nu+1} H_\nu^{(1)}(iz)\,. \tag{A.2}$$

The expansion of these functions for small argument is used in the text. For $0 < z \ll \sqrt{\nu + 1}$ it is given by

$$I_\nu(z) \simeq \frac{1}{\Gamma(\nu+1)} \left(\frac{z}{2}\right)^\nu \,, \quad K_0(z) \simeq -\ln\left(\frac{z}{2}\right) - \gamma$$

$$\text{and} \quad K_\nu(z) \simeq \frac{\Gamma(\nu)}{2} \left(\frac{2}{z}\right)^\nu \quad \text{if} \quad \nu > 0\,. \tag{A.3}$$

The asymptotic form at large z is similarly given (for $z \gg \left|\nu^2 - \frac{1}{4}\right|$) by

$$I_\nu(z) \simeq \frac{1}{\sqrt{2\pi z}}\, e^z \quad \text{and} \quad K_\nu(z) \simeq \sqrt{\frac{\pi}{2z}}\, e^{-z}\,. \tag{A.4}$$

The energy integral encountered in the main text can be evaluated explicitly, using the following Bessel-function identities

$$K'_\nu = -K_{\nu-1} - \frac{\nu K_\nu}{z} = -K_{\nu+1} + \frac{\nu K_\nu}{z}\,, \tag{A.5}$$

which imply in particular $K'_0 = -K_1$, $K'_1 = -K_0 - K_1/z = -K_2 + K_1/z$ and $K'_2 = -K_1 - 2K_2/z$. Repeated application of these shows that

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{1}{2}\, z^2\Big(K_0^2 - K_1^2\Big)\right] = z\, K_0^2 \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{1}{2}\, z^2\Big(K_1^2 - K_0\, K_2\Big)\right] = z\, K_1^2\,, \tag{A.6}$$

and so

$$z\left(K_0^2 + K_1^2\right) = \frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{1}{2}\, z^2 K_0\Big(K_0 - K_2\Big)\right]\,. \tag{A.7}$$

A.2 Classical Divergences in Brane Couplings

This appendix summarizes the derivation of the renormalization of the codimension-2 couplings encountered in the text, with an emphasis on identifying its domain of validity.

Consider to this end the following bulk-brane quadratic action for a single real scalar field,

$$S = -\frac{1}{2}\int \mathrm{d}^4x\mathrm{d}^2y\, \left[\partial_M\phi\, \partial^M\phi + m_B^2\phi^2\right] + \frac{1}{2}\int d^4x\; \lambda_2\phi^2\,. \tag{A.8}$$

(The unusual sign for the brane term is chosen to be consistent with its use in the main text.) The exact propagator, $G(x,y;x',y')$, for this theory satisfies the differential equation

$$\Big[\partial_{M}\partial^{M} - m_{B}^2 + \lambda_2\delta^2(y)\Big]G(x,y;x',y') = i\delta^4(x-x')\delta^2(y-y')\,, \tag{A.9}$$

while the propagator in the absence of the brane coupling, $D(x,y;x',y')$, instead satisfies

$$\Big[\partial_{M}\partial^{M} - m_{B}^2\Big]D(x,y;x',y') = i\delta^4(x-x')\delta^2(y-y')\,. \tag{A.10}$$

It is useful to regard these as the position-basis representation of two abstract operators, G and D, so that $G(x,y;x',y') = \langle x,y|G|x',y'\rangle$ (and similarly for D). In this case the above relations can be written $G^{-1} = D^{-1} - iV$, where $\langle x,y|V|x',y'\rangle = \lambda_2\delta^2(y)\delta^4(x-x')\delta^2(y-y')$. Multiplying on the left by D and on the right by G then allows this to be written as $G = D + iDVG$, whose position-basis expression is equivalent to the integral equation

$$G(x,y;x',y') = D(x,y;x',y') + i\lambda_2\int \mathrm{d}^4\hat{x}\; D(x,y;\hat{x},0)G(\hat{x},0;x',y')\,. \tag{A.11}$$

After Fourier transforming the translation-invariant x^μ directions

$$G(x,y;x',y') = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\; G_p(y;y')\, e^{ip\cdot(x-x')}\,, \tag{A.12}$$

eq. (A.11) becomes the exact statement

$$G_p(y;y') = D_p(y;y') + i\lambda_2\, D_p(y;0)G_p(0;y')\,. \tag{A.13}$$

Since this no longer involves convolutions it may be solved explicitly. Specializing first to $y = 0$ implies $G_p(0;y') = D_p(0;y')/[1 - i\lambda_2 D_p(0;0)]$, which when re-substituted into eq. (A.13) gives

$$G_p(y;y') = D_p(y;y') + i\lambda_2 \, \frac{D_p(y;0)D_p(0;y')}{1 - i\lambda_2 D_p(0;0)} \,. \tag{A.14}$$

Notice that no approximations have been made that implicitly restrict us to small λ_2.

The problem with the solution, eq. (A.14), is that the quantity $D_p(0;0)$ diverges, and this observation lies at the root of the need for renormalization. The expression for $D_p(y;y')$ may be explicitly constructed as the following mode sum, using polar coordinates $\{y^m\} = \{r,\theta\}$ in the transverse dimensions, with $r = 0$ representing the brane position:

$$D_p(r,\theta;r',\theta') = -i \sum_{n=-\infty}^{\infty} e^{in(\theta-\theta')} \int_0^\infty \left(\frac{q\mathrm{d}q}{2\pi}\right) \frac{1}{p^2+q^2+m_B^2} \, J_n(qr)\, J_n(qr') \,, \tag{A.15}$$

where[1] $p^2 = p_\mu p^\mu$. To isolate the divergence in $D_p(0;0)$ evaluate at $r = r' = 0$ and use $J_n(0) = \delta_{n0}$ to get

$$D_p^\Lambda(0;0) = -i \int_0^\Lambda \left(\frac{q\mathrm{d}q}{2\pi}\right) \frac{1}{p^2+q^2+m_B^2} = -\frac{i}{2\pi}\, \ln\left(\frac{\Lambda}{P}\right) + O\left(\frac{P^2}{\Lambda^2}\right) \,, \tag{A.16}$$

where $P^2 = p^2 + m_B^2$.

Renormalization may also be performed without resorting to an expansion in powers of λ_2. The goal is to redefine $\lambda_2 = \bar\lambda_2(\Lambda) \to \bar\lambda_2(\mu)$ in such a way as to absorb the divergence in $D^\Lambda(0;0)$:

$$\frac{\lambda_2}{1 - i\lambda_2 D_p^\Lambda(0;0)} \equiv \frac{\bar\lambda_2(\mu)}{1 - i\bar\lambda_2(\mu) D_p^\mu(0;0)} \,, \tag{A.17}$$

[1]The generalization of this expression to the case where the transverse geometry has a conical defect at the brane position is given in ref. (2.19).

or, equivalently

$$\frac{1}{\bar{\lambda}_2(\Lambda)} \equiv \frac{1}{\bar{\lambda}_2(\mu)} + i\Big[D_p^{\Lambda}(0;0) - D_p^{\mu}(0;0)\Big] = \frac{1}{\bar{\lambda}_2(\mu)} + \frac{1}{2\pi}\ln\left(\frac{\Lambda}{\mu}\right) , \qquad \text{(A.18)}$$

in agreement with the usage in the main text.

A.3 Higher codimension

In this appendix we examine how the arguments of §2 change for a Higgs living in a $(4+n)$-dimensional bulk coupled to a codimension-n brane, with $n \geq 3$.

We divide the discussion into a derivation of how the brane couplings renormalize in arbitrary codimension, and then examine the energy density that governs the size of the resulting scalar expectation value.

A.3.1 Coupling renormalization

We start with a discussion of brane coupling renormalization. The main complication in the higher-codimension case is the appearance of power-law divergences, with all of the pitfalls and complications which these entail for the low-energy description (2.29).

Consider the $(n+4)$-dimensional scalar field

$$S = -\int \mathrm{d}^4x\,\mathrm{d}^ny \left[\frac{1}{2}(\partial_M\phi\,\partial^M\phi) + \frac{1}{2}m_B^2\phi^2 + \delta^n(y)\,V_b(\phi)\right] , \qquad \text{(A.19)}$$

with brane potential

$$V_b = -\frac{1}{2}\lambda_2\phi^2 + \frac{1}{4}\lambda_4\phi^4 , \qquad \text{(A.20)}$$

living in a flat space-time with metric

$$\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + \mathrm{d}r^2 + r^2\gamma_{ab}(\theta)\mathrm{d}\theta^a\mathrm{d}\theta^b . \qquad \text{(A.21)}$$

Here the θ^a are coordinates for the $n-1$ angular directions, whose total volume we denote by $\varpi = \int \mathrm{d}^{(n-1)}\theta\sqrt{\gamma}$. We focus for simplicity on spherically symmetric solutions (independent of the angular directions), although this assumption is not crucial (since higher modes in the angular directions are regular at $r=0$).

As for codimension 2, the relation between the propagator, G, in the presence of the brane coupling, and the propagator, D, in its absence, is

$$G_k(y;y') = D_k(y;y') + i\lambda_2 \frac{D_k(y;0)D_k(0;y')}{1 - i\lambda_2 D_k(0;0)}\,, \tag{A.22}$$

and as before the need for renormalization may be traced to the divergence in $D_k(0;0)$. The nature of this divergence can be divined from the mode sum giving the propagator, D, in the absence of brane couplings

$$\left[\Box - m_B^2\right] D(x,y;x',y') = i\delta^4(x-x')\delta^n(y-y')\,, \tag{A.23}$$

which, in brane-Fourier space,

$$D(x,y;x',y') = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\, D_p(y;y')\, e^{ip\cdot(x-x')}\,, \tag{A.24}$$

has as solution

$$D_p(x;x') = -i\int_0^\infty \frac{q^{n-1}\mathrm{d}q}{\varpi}\frac{1}{p^2+m_B^2+q^2}\left[\frac{1}{(qr)^\nu}J_\nu(qr)\right]\left[\frac{1}{(qr')^\nu}J_\nu(qr')\right] + \cdots\,, \tag{A.25}$$

with $\nu = (n-2)/2$. The ellipses in this last equation represent those terms involving the nontrivial angular modes.

Using the asymptotic form for J_ν in the limit $qr \ll 1$: $J_\nu(qr) =$

$(qr)^\nu/[\nu!2^\nu] + \mathcal{O}(qr)$, we find

$$D_p(r=0;r'=0) = -\frac{i}{(\nu!)^2\, 2^{2\nu}} \int_0^\infty \frac{q^{n-1}\mathrm{d}q}{\varpi} \frac{1}{m_B^2+p^2+q^2}\,, \tag{A.26}$$

which diverges as a power of the UV cutoff, Λ, as

$$\begin{aligned} D_p^{\tilde\Lambda}(0;0) &= -\frac{i}{(\nu!)^2\, 2^{2\nu}} \int_0^{\tilde\Lambda} \frac{q^{n-1}\mathrm{d}q}{\varpi} \frac{1}{m_B^2+p^2+q^2}\,. \qquad \text{(A.27)} \\ &= -\frac{i}{\varpi\,(\nu!)^2\, 2^{2\nu}} \left[\frac{q^n}{nP^2} - \frac{q^{n+2}}{(n+2)P^4}\; {}_2F_1\left(1, \frac{n+2}{2}; \frac{n+4}{2}, -\frac{q^2}{P^2}\right)\right]_0^{\tilde\Lambda}, \end{aligned}$$

where $P^2 = m_B^2 + p^2$ and ${}_2F_1(a,b;c;z)$ denotes the Hypergeometric function.

Our focus is on even n, $n = 2m$, in which case the hypergeometric function can be simplified to the following terminating series

$${}_2F_1(1, m+1, m+2, z) = -(m+1)z^{-(m+1)}\left[\log(1-z) + \sum_{j=1}^{m} \frac{z^j}{j}\right]. \tag{A.28}$$

Using this in the expression of the brane-brane propagator for even codimensions, we get

$$\begin{aligned} D_p^{\tilde\Lambda}(0;0) &= \frac{i2^{2-m}}{\varpi[\Gamma(m)]^2}\left[\frac{(-)^m}{2}P^{2(m-1)}\log\left(1+\frac{q^2}{P^2}\right)\right. \\ &\qquad \left. + \frac{1}{2}\sum_{j=1}^{m-1}\frac{(-)^{j-m}}{j}\, q^{2j}P^{2(m-1-j)}\right]_0^{\tilde\Lambda}. \end{aligned} \tag{A.29}$$

For even codimension, $n = 2m$, we redefine $\Lambda^2 = \tilde\Lambda^2 + P^2$, leading to

$$D_p^{\Lambda} = -\frac{i2^{2-m}}{\varpi[\Gamma(m)]^2}(-P^2)^{m-1}\left[\log\Lambda + \sum_{j=1}^{m-1}\frac{1}{2j}\left(1-\frac{\Lambda^2}{P^2}\right)^j\right] + (\text{finite})\,. \tag{A.30}$$

For odd codimensions, a similar argument gives

$$D_p^\Lambda = -\frac{i2^{2-n}}{\varpi[\Gamma(n/2)]^2}\sum_{j=0}^{[n/2-1]}(-)^j\frac{P^{2j}\Lambda^{n-2-2j}}{n-2-2j}+(\text{finite}), \tag{A.31}$$

where $[n/2-1]$ denotes the largest integer smaller than $n/2-1$.

Renormalization proceeds as for codimension two, with the requirement that

$$\frac{\lambda_2(\Lambda)}{1-i\lambda_2(\Lambda)D_k^\Lambda(0,0)}=\frac{\lambda_2(\mu)}{1-i\lambda_2(\mu)D_k^\mu(0,0)}, \tag{A.32}$$

where μ is the renormalization scale, leading to the following expression,

$$\frac{1}{\lambda_2(\Lambda)}=\frac{1}{\bar{\lambda}_2(\mu)}+i\left(D_k^\Lambda-D_k^\mu\right). \tag{A.33}$$

The divergence of propagator on the brane also induces divergences in the expression of the 4-point function, which should be absorbed by a renormalization of λ_4,

$$\begin{aligned} G^{(4)}_{k_1,k_2,k_3,k_4}(y_1;y_2;y_3;y_4) &= -6i\,\lambda_4\left[\prod_{i=1}^4 G^{(2)}_{k_i}(y_i;0)\right]\delta^4\!\left(\sum_i k_i\right) \\ &= -6i\,\lambda_4\left[\prod_{i=1}^4\frac{D_{k_i}(y_i;0)}{1-i\lambda_2 D_{k_i}(0;0)}\right]\delta^4\!\left(\sum_i k_i\right). \end{aligned} \tag{A.34}$$

The quantity $\lambda_4/(1-i\lambda_2 D_{k_i}(0,0))^4$ is finite if λ_4 is renormalized in the following way

$$\lambda_4(\Lambda)=\frac{\bar{\lambda}_4}{\left(1+i\bar{\lambda}_2\left(D_k^\Lambda-D_k^\mu\right)\right)^4}. \tag{A.35}$$

Similar expressions can be found for higher-point couplings.

A.3.2 Boundary condition and energy density

We now turn to the classical solutions for $\phi(r)$, and the boundary conditions which communicate the information of the brane potential to the bulk theory. Just as in the main text the singular form of the bulk solutions require us to regularize the boundary condition by evaluating it at $r = \epsilon$ rather than at $r = 0$. Smooth results are obtained as $\epsilon \to 0$ once the bare couplings are eliminated in terms of the renormalized couplings.

The classical solution to the bulk field equation that vanishes far from the brane is

$$\phi(r) = \bar{\phi}\, \frac{K_\nu(m_{B} r)}{(m_{B} r)^\nu}\,. \tag{A.36}$$

Integrating the equation of motion over the brane, we obtain the boundary condition

$$\varpi \epsilon^{n-1} \phi'_\epsilon = -\lambda_2 \phi_\epsilon + \lambda_4 \phi_\epsilon^3\,. \tag{A.37}$$

The energy density for such a field configuration is similarly given by

$$\begin{aligned} \mathcal{H} &= v \int_\epsilon^\infty r^{n-1} \mathrm{d}r \left[\frac{1}{2} (\partial_r \phi)^2 + \frac{1}{2}\, m_{B}^2 \phi^2 \right] + U_b(\phi(\epsilon)) \\ &= v \frac{m_{B}^2}{2} \bar{\phi}^2\, \epsilon^{n+1}\, (m_{B}\epsilon)^{-n}\, K_\nu(m_{B}\epsilon) K_{\nu+1}(m_{B}\epsilon) + U_b(\phi(\epsilon))\,. \end{aligned} \tag{A.38}$$

In general both of these last equations become finite once expressed in terms of renormalized quantities, although the cancellation becomes more regularization dependent in the higher-codimension case due to the appearance there of power-law divergences rather than logarithms. Rather than working this through in complete generality, we restrict ourselves here to an illustrative calculation for codimension three.

A.3.3 Codimension-3

For a codimension-3 brane the divergent part of the brane-brane propagator goes as

$$D_p^\Lambda = -\frac{2i\Lambda}{\pi\varpi}\,, \tag{A.39}$$

and so the divergent part of the boundary condition (A.37) cancels identically if $2\Lambda/\pi = \epsilon$. The leading order part of the boundary condition becomes

$$\bar{\phi}\left(m_B + \frac{\varpi}{\bar{\lambda}_2} - \mu + \frac{\pi\varpi^3\bar{\lambda}_4}{2\bar{\lambda}_2^4 m_B^2}\,\bar{\phi}^2\right) = 0\,, \tag{A.40}$$

where to simplify the notation we rescale $\mu \to \pi\mu/2$. The system has solution $\bar{\phi} = 0$ as well as

$$\bar{\phi}^2 = -\left(\frac{2\bar{\lambda}_2^4 m_B^2}{\pi\varpi^3\bar{\lambda}_4}\right) m_{\rm eff}\,, \tag{A.41}$$

although the second solution is only possible when

$$m_{\rm eff} = \left(\frac{\varpi}{\bar{\lambda}_2} - \mu + m_B\right) < 0\,. \tag{A.42}$$

These conclusions are consistent with the form of the energy density, which in this case is

$$\mathcal{H} = \left(\frac{\pi\varpi}{4m_B^2}\right) m_{\rm eff}\,\bar{\phi}^2 + \frac{\bar{\lambda}_4}{4}\left(\frac{\varpi}{\bar{\lambda}_2 m_B}\right)^4 \left(\frac{\pi}{2}\right)^2 \bar{\phi}^4\,. \tag{A.43}$$

Notice that the criterion for having a nonzero v.e.v. in this case depends more strongly on m_B, relative to the codimension-2 case.

A similar argument can be made for higher codimensions. Notice that for codimension-4 and higher, the propagator includes sub-leading divergences which should also be renormalized. Doing so, we recover a finite energy density with slightly different criteria on having a nonzero v.e.v.

A.4 Bulk Goldstone modes

A natural worry arises when the Higgs is regarded as a bulk scalar while the Standard Model gauge bosons are confined to a brane. Since the bulk $SU(2) \times U(1)$ rotations are not gauged, their spontaneous breaking might be expected to bulk Goldstone modes, corresponding to KK towers of bulk scalar modes whose lightest members are massless (or with masses set by the KK scale, if the global symmetries are broken by boundary conditions). Since only three of these 4D KK states are eaten by the Higgs mechanism, the remainder could survive and generate a potentially dangerous large number of light states. In this section, we show that only three massless Goldstone modes are produced, all of which are eaten by the gauge fields on the brane.

We start with the argument in a nutshell: when choosing a specific vacuum, such as the unitary gauge choice of the main text, one expects Goldstone modes connecting to nearby vacua. Since all vacua have the same profile in the extra dimensions, the Goldstone modes also share this profile. The modes with the smallest energy cost have only momentum along the brane directions, and so are effectively already four-dimensional. These modes turn out to be the self-localized states of those components of the Higgs doublet that do not acquire a v.e.v.

To see this explicitly we repeat the calculation of the light states in section 2.3.3, for the Higgs doublet H. The equation of motion analogous to Eq. (2.32) is

$$\left[\frac{1}{r^2}\left((r\partial_r)^2 + \partial_\theta^2\right) + \partial_\mu\partial^\mu - m_B^2\right] H = \frac{\delta_+(r)}{2\pi r}\left(-\lambda_2 + \lambda_4 H^\star H\right) H\,. \qquad \text{(A.44)}$$

This equation may be linearized around the vacuum by setting

$$H = \begin{pmatrix} 0 \\ \varphi(r) \end{pmatrix} + \sum_{n\geq 0}^{\infty} \begin{pmatrix} \zeta_1^n(r) + i\zeta_2^n(r) \\ \chi^n(r) + i\zeta_3^n(r) \end{pmatrix} \sin(n\theta)\,, \tag{A.45}$$

with $\varphi(r) = \bar{\phi} K_0(m_B r)$ and where we introduce an infinite tower of excitation modes along the angular direction. Each one of these modes satisfies the equations of motion

$$\left[\frac{1}{r}\partial_r\left(r\partial_r\right) - \frac{n^2}{r^2} - k^2\right]\chi^n = \frac{\delta_+(r)}{2\pi r}\left(-\lambda_2 + 3\lambda_4\varphi^2(r)\right)\chi^n \tag{A.46}$$

$$\left[\frac{1}{r}\partial_r\left(r\partial_r\right) - \frac{n^2}{r^2} - k^2\right]\zeta_i^n = \frac{\delta_+(r)}{2\pi r}\left(-\lambda_2 + \lambda_4\varphi^2(r)\right)\zeta_i^n\,, \tag{A.47}$$

where, as before, $k^2 = m_B^2 - \omega^2$.

The field χ^0 is the 'physical' self-localized state, discussed in the main text, and has a mass as calculated in Eq. (2.36) with (2.37). The same goes through for the zero mode of the other fields ζ_i^0, but taking into account the different factor of λ_4 between equations (A.46) and (A.47), their masses are given by

$$\omega_{\zeta^0}^2 = m_B^2\left[1 - e^{-4\pi/\lambda_{\zeta^0}}\right]\,, \tag{A.48}$$

with now

$$\frac{1}{\lambda_{\zeta^0}} = \frac{1}{\lambda_{2\star}}\left[1 + \frac{4\pi^2\bar{\lambda}_4\bar{\phi}^2}{\lambda_{2\star}^3}\right]\,. \tag{A.49}$$

In the broken phase, $\bar{\phi}$ is given by Eq. (2.29) which leads to

$$\frac{1}{\lambda_{\zeta^0}} = 0\,, \tag{A.50}$$

showing there are three massless 4D Goldstone modes, ζ_i^0. The bulk profile of these modes is enforced by the boundary condition imposed on the brane, and as argued in section 2.4.1, choosing unitary gauge on the brane removes these three massless states as they become 'eaten' by the brane gauge fields.

Turning now to the infinite tower of angular dependent modes ($n \neq 0$), the profile of these modes is now of the form $\chi^n, \zeta_i^n = N_i^n K_n(kr)$, where N_i^n is the normalization constant and we expect k to be determined by the the boundary condition (2.34) which takes the form

$$2\pi\, r\partial_r \begin{pmatrix} \chi^n(r) \\ \zeta_i^n(r) \end{pmatrix}\Bigg|_\epsilon = \left(-\lambda_2 + \begin{pmatrix} 3 \\ 1 \end{pmatrix} \lambda_4 \varphi(r)^2\right) \begin{pmatrix} \chi^n \\ \zeta_i^n \end{pmatrix}\Bigg|_\epsilon . \tag{A.51}$$

In the limit $\epsilon \to 0$, this reduces to

$$2^n \pi N_i^n (k\epsilon)^{-n} \left(n! + \frac{(n-1)!}{\log(\epsilon\, m_B e^\gamma/2)}\right) = \mathcal{O}\left((k\epsilon)^{-n+2}\right) . \tag{A.52}$$

We see we must have $N_i^n = 0$ if these modes are to remain bounded, and so there are therefore no light modes of this form having $\omega < m_B$. All the remaining excitations along the radial direction form a Kaluza-Klein tower of states starting at the bulk mass m_B and are thus harmless. There are therefore only three massless states ζ_i^0 playing the role of four-dimensional Goldstone modes, one self-localized massive mode (χ) with mass $0 < m < m_B$ and a tower of Higgs excitations with mass higher than the bulk mass.

Appendix B

appendix for chapter 4

B.1 Brane fluxes and flux quantization

To see how to interpret the parameter Φ_b, rewrite the brane flux term as a regularized 6D integral weighted by a scalar function $s(\rho)$ whose support is nonzero only in a short interval $|\rho - \rho_b| < \varepsilon$ away from the brane, and is normalized so that $\int \mathrm{d}^2x \sqrt{g_2}\; s = 1$. That is,

$$S_{\rm flux} \;=\; \frac{\Phi_b}{2} \int \mathrm{d}^6x \sqrt{-g_6}\; s\, \epsilon^{mn} \mathcal{F}_{mn} = \Phi_b \int \mathrm{d}^6x \sqrt{-g_4}\; s\, F_{\rho\theta}\,. \tag{B.1}$$

Then the $\delta\mathcal{A}_\theta$ Maxwell equation becomes

$$\partial_\rho\Big(\sqrt{-g_6}\; F^{\rho\theta} - \Phi_b \sqrt{-g_4}\; s\Big) = 0\,, \tag{B.2}$$

which integrates to give

$$e^{4W}\Big(e^{-B}\mathcal{A}_\theta' - \Phi_b s\Big) = \mathcal{Q}\,. \tag{B.3}$$

This is the bulk solution found in the text away from the brane, where $s = 0$.

Imagine now integrating this to obtain $\mathcal{A}_\theta(\rho)$ in the vicinity of the brane at $\rho_b = 0$, using for s a simple step function: $s = 1/(\pi\varepsilon^2)$ for $\rho < \varepsilon$ and $s = 0$ for $\rho > \varepsilon$. Assuming $W \simeq W_b$ is approximately constant and $e^B \simeq \rho$ for $\rho < \varepsilon$, the solution satisfying $\mathcal{A}_\theta(0) = 0$ is

$$\mathcal{A}_\theta(\rho) = \frac{1}{2}\left(\mathcal{Q}e^{-4W_b} + \frac{\Phi_b}{\pi\varepsilon^2}\right)\rho^2\,, \tag{B.4}$$

and so at $\rho = \varepsilon$ in particular

$$\mathcal{A}_\theta(\varepsilon) = \frac{\Phi_b}{2\pi} + \mathcal{O}(\varepsilon^2)\,. \tag{B.5}$$

The junction condition for $\mathcal{A}'_\theta$ at $\rho = \varepsilon$ can also be seen by subtracting the solution, eq. (B.3) evaluated at $\rho < \varepsilon$ — where $s = 1/(\pi\varepsilon^2)$ — from the same solution evaluated at $\rho > \varepsilon$ — where $s = 0$. Since the RHS is the same in both cases we get the following jump discontinuity across $\rho = \varepsilon$:

$$\Big[e^{-B}\mathcal{A}'_\theta\Big]_{\rho=\varepsilon-}^{\rho=\varepsilon+} = -\frac{\Phi_b}{\pi\varepsilon^2}\,. \tag{B.6}$$

This can be related to the derivative of the brane action with respect to $\mathcal{A}_\theta$ by rewriting eq. (B.1) as

$$S_{\text{flux}} = \Phi_b \int \mathrm{d}^6x\,\sqrt{-g_4}\;s\,F_{\rho\theta} = \frac{2\pi\Phi_b}{\pi\varepsilon^2}\int \mathrm{d}^4x\,\sqrt{-g_4}\;\mathcal{A}_\theta(\varepsilon)\,, \tag{B.7}$$

and so (keeping in mind the relative sign between the tension and flux terms)

$$\Big[e^{-B}\mathcal{A}'_\theta\Big]_{\rho=\varepsilon-}^{\rho=\varepsilon+} = +\frac{1}{2\pi}\left(\frac{\partial T_b}{\partial\mathcal{A}_\theta}\right)\,, \tag{B.8}$$

as stated in ref. (4.15).

B.2 Rugby-ball response

This section provides the explicit solutions for the properties of rugby ball solutions as functions of the assumed (shared) brane tension, and in particular computes the response to small changes in its value.

Rugby ball configurations solve the field equations

$$\mathcal{R}_{MN} + \partial_M\phi\,\partial_N\phi + \kappa^2\mathcal{F}_{MP}\mathcal{F}_N{}^P - \left[\frac{\kappa^2}{8}\,\mathcal{F}_{PQ}\mathcal{F}^{PQ} - \frac{\kappa^2\Lambda}{2}\right] g_{MN} = 0\,, \tag{B.9}$$

and

$$\nabla_M\mathcal{F}^{MN} = 0\,, \tag{B.10}$$

subject to the ansatz

$$\mathrm{d}s^2 = \hat{g}_{\mu\nu}\,\mathrm{d}x^\mu\mathrm{d}x^\nu + \mathrm{d}\rho^2 + \alpha^2L^2\sin^2\left(\frac{\rho}{L}\right)\mathrm{d}\theta^2 \tag{B.11}$$

$$\mathcal{F}_{\rho\theta} = \alpha\mathcal{Q}L\sin\left(\frac{\rho}{L}\right), \tag{B.12}$$

with $\phi = \varphi_0$ constant. The bulk field equations give the 2D and 4D curvature scalars as

$$-\mathcal{R}_{(2)} = \frac{2}{L^2} = \kappa^2\left(\frac{3\mathcal{Q}^2}{2} + \Lambda\right)\,, \tag{B.13}$$

and

$$\mathcal{R}_{(4)} = \hat{\mathcal{R}} = 2\kappa^2\left(\frac{\mathcal{Q}^2}{2} - \Lambda\right). \tag{B.14}$$

The gauge potential corresponding to eq. (B.12) is

$$\mathcal{A}^\pm = \alpha\mathcal{Q}L^2\left[\pm1 - \cos\left(\frac{\rho}{L}\right)\right]\,\mathrm{d}\theta \pm \frac{\Phi_\pm}{2\pi}\,\mathrm{d}\theta\,, \tag{B.15}$$

where the $\pm$ sign indicates the solution for the northern or southern hemisphere, since $\mathcal{A}$ must evaluate to the brane flux at the corresponding pole. Requiring the difference between these two solutions near the equator to be a well-defined gauge transformation, $gA^+ - gA^- = \mathrm{d}\Omega$, implies the constants $\mathcal{Q}$ and L must be related by

$$g\mathcal{Q} = \frac{N}{2\,\alpha L^2}\,, \tag{B.16}$$

where we define $N = n - g\Phi_{\rm tot}/2\pi$.

Eqs. (B.13) and (B.16) determine the constants $\mathcal{Q}$ and L in terms of α and Λ, with solutions

$$\frac{1}{L^2} = \frac{8\,\alpha^2 g^2}{3N^2\kappa^2}\left[1 \pm \sqrt{1 - \left(\frac{3\,N^2\kappa^4\Lambda}{8\,\alpha^2 g^2}\right)}\right] = \frac{1}{2L_{\rm min}^2}\left[1 \pm \sqrt{1 - \left(\frac{3\,N^2\kappa^4\Lambda}{8\,\alpha^2 g^2}\right)}\right]\,, \tag{B.17}$$

and

$$\mathcal{Q} = \frac{N}{2\alpha g L^2} = \frac{4\alpha g}{3N\kappa^2}\left[1 \pm \sqrt{1 - \left(\frac{3\,N^2\kappa^4\Lambda}{8\,\alpha^2 g^2}\right)}\right]\,. \tag{B.18}$$

These provide two solutions for L and $\mathcal{Q}$ for each given value of α and Λ, satisfying $L^2 \geq L^2_{\rm min} = 3N^2\kappa^2/16\,\alpha^2 g^2$. Starting with the lower sign in eq. (B.17) the radius L falls from $L \to \infty$ to $L = \sqrt{2}\;L_{\rm min}$ as Λ climbs from 0 to $\Lambda_{\rm max} = 8\,\alpha^2 g^2/3N^2\kappa^4$. On this branch $\Lambda \ll \Lambda_{\rm max}$ implies $1/L^2 \simeq \kappa^2\Lambda/2$. Then switching to the branch corresponding to the upper sign has L fall from $\sqrt{2}\;L_{\rm min}$ to $L_{\rm min}$ as Λ recedes from $\Lambda_{\rm max}$ back to zero. There are no real solutions with $\Lambda > \Lambda_{\rm max}$, or with $L < L_{\rm min}$.

For each of these solutions the last equation, eq. (B.14), gives the on-brane curvature, $\hat{\mathcal{R}}$. There is a choice $\Lambda = \Lambda_f$, for which $\hat{\mathcal{R}}$ vanishes, given by $\Lambda_f = \mathcal{Q}^2/2$. For this choice L and $\mathcal{Q}$ become

$$\frac{1}{L_f(\alpha)} = \frac{2\,\alpha g}{N\kappa} \quad \text{and} \quad \mathcal{Q}_f(\alpha) = \frac{2\,\alpha g}{N\kappa^2}\,, \tag{B.19}$$

and so

$$\Lambda_f = \frac{\mathcal{Q}_f^2}{2} = \frac{2\,\alpha^2 g^2}{N^2\kappa^4}\,. \tag{B.20}$$

Because $L_{\rm min} < L_f < \sqrt{2}\;L_{\rm min}$ we see that this solution lies on the branch corresponding to the upper sign of eq. (B.17). In particular

$$\kappa^2 \mathcal{Q}_f^2 L_f^2 = \kappa^2 \left(\frac{2\alpha g}{N\kappa^2}\right)^2 \left(\frac{N\kappa}{2\alpha g}\right)^2 = 1\,. \tag{B.21}$$

Notice that the semiclassical approximation requires the curvature to remain small compared with the relevant energy scales, and so in 6D requires $\mathcal{R}^3$ to be much smaller than Λ or $\mathcal{Q}^2$. Because $\mathcal{R} \simeq \kappa^2\Lambda$ and $\kappa^2\mathcal{Q}^2$ this requires $\kappa^3\Lambda$ and $\kappa^3\mathcal{Q}^2$ must both be much smaller than unity. So for $\Lambda \sim \mathcal{Q}^2 \sim \alpha^2 g^2/N^2\kappa^4$ the semiclassical limit implies $\alpha^2 g^2/N^2\kappa \ll 1$. This in turn ensures $\kappa/L_f^2 \ll 1$, showing that this value of L_f lies within the classical limit.

The above expressions can be used to check the linearized analysis performed in the main text. To this end, suppose we start with $\alpha = \alpha_0$, with $\Lambda = \Lambda_0 = \Lambda_f(\alpha_0)$ chosen so that $\hat{\mathcal{R}} = 0$ for this value of α. Then we change the brane tension (but not the brane flux), and so also α, without also adjusting Λ. Choosing the upper sign, the radius and magnetic flux become

$$\frac{1}{L^2} = \frac{8\alpha^2 g^2}{3N^2\kappa^2}\left[1 + \sqrt{1 - \left(\frac{3N^2\kappa^4\Lambda_0}{8\alpha^2 g^2}\right)}\right] = \left(\frac{2\alpha^2}{3\alpha_0^2}\right)\frac{1}{L_0^2}\left[1 + \sqrt{1 - \frac{3\alpha_0^2}{4\alpha^2}}\right], \tag{B.22}$$

and

$$\mathcal{Q} = \frac{4\alpha g}{3N\kappa^2}\left[1 + \sqrt{1 - \left(\frac{3N^2\kappa^4\Lambda_0}{8\alpha^2 g^2}\right)}\right] = \left(\frac{2\alpha}{3\alpha_0}\right)\mathcal{Q}_0\left[1 + \sqrt{1 - \frac{3\alpha_0^2}{4\alpha^2}}\right]. \tag{B.23}$$

The last equality in these two equations is obtained by using eq. (B.20) to trade Λ_0 for α_0, and then using eqs. (B.19) to express the result in terms of the values L_0 and $\mathcal{Q}_0$ that correspond to $\alpha = \alpha_0$. These equations show how the values of L and $\mathcal{Q}$ adjust to compensate for the change of α. The on-brane curvature similarly changes, and is given by

$$\begin{aligned} \hat{\mathcal{R}} &= \kappa^2(\mathcal{Q}^2 - 2\Lambda_0) \\ &= \frac{32\alpha_0^2 g^2}{9N^2\kappa^2}\left[\sqrt{1-\frac{3\alpha_0^2}{4\alpha^2}} + 1 - \frac{3\alpha_0^2}{2\alpha^2}\right], \end{aligned} \tag{B.24}$$

which vanishes as $\alpha \to \alpha_0$, as it must. For $\alpha = \alpha_0 + \Delta\alpha$, then $\alpha_0^2/\alpha^2 \simeq 1 - 2\Delta\alpha/\alpha_0$ and so

$$\hat{\mathcal{R}} \simeq \left(\frac{16\alpha_0^2 g^2}{N^2\kappa^2}\right)\left(\frac{\Delta\alpha}{\alpha_0}\right) = \frac{4}{L_0^2}\left(\frac{\Delta\alpha}{\alpha_0}\right) = -\frac{2\kappa^2\Delta T}{\pi\alpha_0 L_0^2} = -8\kappa_4^2\Delta T\,, \tag{B.25}$$

which uses the matching condition $1 - \alpha = 4GT = \kappa^2 T/2\pi$ in the form $\Delta\alpha = -\kappa^2\Delta T/2\pi$, as well as the definition of the 4D gravitational coupling: $\kappa^2 = 4\pi\alpha_0 L_0^2\,\kappa_4^2$. Defining the 4D potential, $V_{\rm eff}$, by $\hat{\mathcal{R}} = -4\kappa_4^2\,V_{\rm eff}$ gives the expected result

$$V_{\rm eff} \simeq 2\Delta T\,. \tag{B.26}$$

The factor of 2 arises because a change of α requires an equal change of tension for *both* branes if it is to preserve the rugby-ball form.

B.3 Misaligned currents

This Appendix uses a simple model to track the implications that arise if the external current happens not to be aligned precisely with the lightest mode of the system. When this happens errors can arise in the identification of quantities like low-energy masses, but this section argues that these are generically suppressed by powers of the light mass divided by heavier masses.

Consider then the toy 4D lagrangian

$$\frac{\mathcal{L}}{\sqrt{-g}} = -\frac{1}{2\kappa_4^2}\,\mathcal{R} - \frac{1}{2}\Big[(\partial\varphi)^2 + (\partial\chi)^2\Big] - V(\varphi,\chi)\,, \tag{B.27}$$

whose potential is given by

$$V = V_0 + \frac{1}{2}\Big(m^2\varphi^2 + M^2\chi^2\Big) + J(\varphi + \zeta\,\chi)\,, \tag{B.28}$$

with masses assumed to satisfy $m \ll M$. Here φ is meant as the analog of the KK would-be zero mode in the main text, while χ is representative of some other, more massive, KK mode. The goal is to ascertain the extent to which our method of determining the low-energy mass would be thrown off by a small coupling — parameterized here by ζ — of the external current to a heavy state.

The classical equations of motion for the scalar fields are

$$\Box\,\varphi - m^2\varphi = J \quad \text{and} \quad \Box\,\chi - M^2\chi = \zeta\,J\,, \tag{B.29}$$

while the Einstein equation reads

$$\mathcal{R}_{\mu\nu} + \kappa_4^2\Big(\partial_\mu\varphi\,\partial_\nu\varphi + \partial_\mu\chi\,\partial_\nu\chi\Big) + \kappa_4^2 V = 0\,, \tag{B.30}$$

and so

$$\frac{\mathcal{R}}{4\kappa_4^2} = -V = -V_0 - \frac{1}{2}\Big(m^2\varphi^2 + M^2\chi^2\Big) - J(\varphi + \zeta\,\chi)\,. \tag{B.31}$$

Evaluated at the particular solutions

$$J = -m^2\varphi \quad \text{and} \quad \chi = -\frac{\zeta J}{M^2} = \left(\frac{\zeta m^2}{M^2}\right)\varphi\,, \tag{B.32}$$

this last equation gives

$$\mathcal{F}(\varphi) := \frac{\mathcal{R}}{4\kappa_4^2} = -V_0 + \frac{1}{2}\,m^2\varphi^2\left[1 + \left(\frac{\zeta m}{M}\right)^2\right]\,. \tag{B.33}$$

In terms of $\mathcal{F}(\varphi)$ the method of the main text gives the low-energy scalar potential as

$$V_{\text{eff}}(\varphi) := \varphi \int \frac{\mathrm{d}\varphi}{\varphi^2} \, \mathcal{F}(\varphi) \,. \tag{B.34}$$

For $\mathcal{F}(\varphi) = A + B\varphi + \frac{1}{2}\, C\varphi^2$ the integral evaluates to[1]

$$V_{\text{eff}} = -A + B\varphi \ln \varphi + \frac{1}{2}\, C\varphi^2 + D\varphi \,, \tag{B.35}$$

where D is the integration constant, and so when applied to the above toy model this gives

$$V_{\text{eff}}(\varphi) = V_0 + \frac{1}{2}\, m^2 \varphi^2 \left[1 + \left(\frac{\zeta m}{M} \right)^2 \right] . \tag{B.36}$$

This expression correctly identifies the value of the potential at its minimum to be V_0, and — provided $\zeta \lesssim \mathcal{O}(1)$ — gives the correct mass for the field φ, up to corrections of relative order m^2/M^2.

B.4 When brane fluxes and tensions compete

This Appendix briefly discusses another kind of competition, which would arise if $\delta T_b(\varphi_0)$ and $\delta \Phi_b(\varphi_0)$ at the same brane were not minimized by the same scalar configuration. A simple representative in this category is

$$T_{\scriptscriptstyle N}(\varphi_0) = T + T_{\scriptscriptstyle N0} + \frac{T_{\scriptscriptstyle N2}}{2}\, (\varphi_0 - v_{\scriptscriptstyle T})^2 \quad \text{and} \quad \Phi_{\scriptscriptstyle N}(\varphi_0) = \Phi + \Phi_{\scriptscriptstyle N0} + \frac{\Phi_{\scriptscriptstyle N2}}{2}\, (\varphi_0 - v_\phi)^2 \,, \tag{B.37}$$

together with $\delta T_{\scriptscriptstyle S}(\varphi_0) = \delta \Phi_{\scriptscriptstyle S}(\varphi_0) = 0$, so the 'south' brane plays no role in the stabilization of φ_0.

[1]The singular form of $V''_{\text{eff}}(0)$ when $B \neq 0$ corresponds to the pathological case where brane fluxes and tensions are not extremized for the same value of φ, discussed in more detail in Appendix B.4.

In this case because the flux is irrelevant for determining $\varphi_\star$, its value is simply $\varphi_\star = v_T$. The warping difference is also insensitive to Φ_b and so becomes $W_N - W_S = (3\kappa^2 T_{N0}/20\pi\alpha)$, while $\delta L/L = (3\kappa^2 \varrho_{\rm eff}/8\pi\alpha)$ with

$$\begin{aligned} \varrho_{\rm eff} &= T_{N0} - \mathcal{Q}\Phi_{N0} - \frac{\mathcal{Q}\Phi_{N2}}{2}(v_T - v_\phi)^2 \\ \text{and} \quad m_\varphi^2 &= \frac{T_{N2}}{f^2} + \frac{\mathcal{Q}\Phi_{N2}}{f^2} \lim_{\varphi \to v_T} \left(\frac{\varphi - v_\phi}{\varphi - v_T} \right) . \end{aligned} \tag{B.38}$$

Clearly, the expression for the mass is singular when $v_T \neq v_\phi$. The reason for the singularity, is that for this choice of brane there is no solution satisfying the *ansatz* with which we work. The obstruction lies with the Maxwell field, which we choose to lie in the $\mathcal{F}_{\rho\theta}$ direction only. However, the perturbation that gets excited by moving φ_0 away from equilibrium, if we do not stabilize with an external current, necessarily gets a time dependent Maxwell field. But a changing magnetic field induces an electric field, so the $\mathcal{F}_{\rho t}$ and $\mathcal{F}_{\theta t}$ components cannot both remain zero.

To see that the Maxwell field must acquire a time dependence, consider a perturbation, $\delta\varphi$, that oscillates about the background v_T,

$$\varphi = v_T + \delta\varphi(\rho)\, e^{-imt} . \tag{B.39}$$

In the flux condition, eq. (4.56), the brane fluxes now have a part that is linear in $\delta\varphi$ that becomes proportional to e^{-imt}. If we now assume that $\delta\mathcal{Q}$ does not acquire any time dependence, we get a contradiction: δB has a part proportional to e^{-imt} according to the flux condition, but in matching it with the brane in eq. (4.37), if $\delta\mathcal{Q}$ is time independent the right hand side is either constant or proportional to $(\Phi - v_T)^2 \propto e^{-2imt}$. This shows that the matching conditions cannot be satisfied unless the magnetic field becomes time-dependent.

Appendix C

Appendix for chapter 5

C.1 Flux quantization with brane fluxes

To see how to interpret the parameter Φ_b, rewrite the brane flux term as a regularized 6D integral weighted by a scalar function $s(\rho)$ whose support is nonzero only in a short interval $|\rho - \rho_b| < \varepsilon$ away from the brane, and is normalized so that $\int \mathrm{d}^2x \sqrt{g_2}\; s = 1$. That is,

$$S_{\text{flux}} = \frac{\Phi_b}{2} \int \mathrm{d}^4x \sqrt{-g_6} e^{-\phi}\; s\, \epsilon^{mn} \mathcal{F}_{mn} = \Phi_b \int \mathrm{d}^6x \sqrt{-g_4} e^{-\phi}\; s\, \mathcal{F}_{\rho\theta}. \quad \text{(C.1)}$$

Then the $\delta\mathcal{A}_\theta$ Maxwell equation becomes

$$\partial_\rho \Big(e^{-\phi} \sqrt{-g_6}\; \mathcal{F}^{\rho\theta} - e^{-\phi} \Phi_b \sqrt{-g_4}\; s \Big) = 0\,, \quad \text{(C.2)}$$

which integrates to give

$$\Big(e^{-B} \mathcal{A}_\theta' - \Phi_b s \Big) = \mathcal{Q} e^{\phi}\,. \quad \text{(C.3)}$$

This is the bulk solution found in the text away from the brane, where $s = 0$.

Imagine now integrating this to obtain $\mathcal{A}_\theta(\rho)$ in the vicinity of the brane at $\rho_b = 0$, using for s a simple step function: $s = 1/(\pi\varepsilon^2)$ for $\rho < \varepsilon$ and $s = 0$ for $\rho > \varepsilon$. Assuming $W \simeq W_b$ is approximately constant and $e^B \simeq \rho$ for $\rho < \varepsilon$, we set $\mathcal{A}_\theta(0) = 0$ and integrate tof find $\mathcal{A}_\theta(\varepsilon)$,

$$\mathcal{A}_\theta(\varepsilon) = \frac{\Phi_b}{\pi\varepsilon^2}\left[\frac{1}{2}\rho^2\right]_0^\varepsilon + \mathcal{Q}\int_0^\varepsilon \mathrm{d}\rho\rho e^\phi = \frac{\Phi_b}{2\pi} + \mathcal{Q}\int_0^\varepsilon \mathrm{d}\rho\rho e^\phi\,, \tag{C.4}$$

and so as long as e^ϕ diverges less fast than ρ^{-2} at the brane we have

$$\lim_{\varepsilon\to 0}\mathcal{A}_\theta(\varepsilon) = \frac{\Phi_b}{2\pi}\,. \tag{C.5}$$

The junction condition for $\mathcal{A}_\theta'$ at $\rho = \varepsilon$ can also be seen by subtracting the solution, eq. (C.3) evaluated at $\rho < \varepsilon$ — where $s = 1/(\pi\varepsilon^2)$ — from the same solution evaluated at $\rho > \varepsilon$ — where $s = 0$. Since the RHS is the same in both cases we get the following jump discontinuity across $\rho = \varepsilon$:

$$\Big[e^{-B}\mathcal{A}_\theta'\Big]_{\rho=\varepsilon-}^{\rho=\varepsilon+} = -\frac{\Phi_b}{\pi\varepsilon^2}\,. \tag{C.6}$$

This can be related to the derivative of the brane action with respect to $\mathcal{A}_\theta$ by rewriting eq. (C.1) as

$$S_{\text{flux}} = \Phi_b\int \mathrm{d}^6x\,\sqrt{-g_4}\;s\,\mathcal{F}_{\rho\theta} = \frac{2\pi\Phi_b}{\pi\varepsilon^2}\int \mathrm{d}^4x\,\sqrt{-g_4}\;\mathcal{A}_\theta(\varepsilon)\,, \tag{C.7}$$

and so (keeping in mind the relative sign between the tension and flux terms)

$$\Big[e^{-B}\mathcal{A}_\theta'\Big]_{\rho=\varepsilon-}^{\rho=\varepsilon+} = +\frac{1}{2\pi}\left(\frac{\partial T_b}{\partial\mathcal{A}_\theta}\right)\,, \tag{C.8}$$

as stated in ref. (5.32).

C.2 Alternative currents

In this section, we check that the details of the current are not important for the stabilization of φ. Define for comparison purposes the current

$$S_{_J} = -\int \mathrm{d}^6x\,\sqrt{-g}\;J\,e^{\phi}\,. \tag{C.9}$$

This choice keeps the scale invariance of the bulk action intact. Compared to the current used in the main body, the changes to the linearized equations of motion arise only in the ϕ and δB equations. Here we write only the contributions to these equations due to the current:

$$\begin{aligned} (\sin x\,\delta\phi')' &= \cdots - \kappa^2 J L^2 e^{\varphi_0}\sin x \\ \frac{\left(\sin^2 x\,\delta B'\right)'}{\sin^2 x} &= \cdots - \kappa^2 J e^{\varphi_0}\,, \end{aligned} \tag{C.10}$$

where the δB term was present before but lacked the factor e^{φ_0}. By contrast the $\delta\phi$ contribution given above didn't exist for the current used in the main text.

The resulting change to the perturbations that is a consequence of the current only is

$$\begin{aligned} (\delta\phi)_{_J} &= \kappa^2 J L^2 e^{\varphi_0}\ln|\sin x| \\ (\delta B)_{_J} &= \kappa^2 J L^2 e^{\varphi_0}\left[-\mathcal{H}_2(x) + \frac{1}{2}\left(1 + x\cot x\right)\right]. \end{aligned} \tag{C.11}$$

The resulting changes in the matching conditions to the brane are as follows:

$$\begin{aligned} \frac{\delta\mathcal{Q}}{\mathcal{Q}} + \kappa^2 J L^2 e^{\varphi_0} &= \frac{\kappa^2}{4\pi\alpha}\Big(T_{_N}' + T_{_S}'\Big) \\ \frac{1}{2}\,\kappa^2 J L^2 e^{\varphi_0} &= -\frac{\kappa^2}{4\pi\alpha}\left[\delta T_{_N} + \delta T_{_S} - \mathcal{Q}\Phi_{_N} - \mathcal{Q}\Phi_{_S} + \frac{1}{2}\left(T_{_N}' + T_{_S}'\right)\right]. \end{aligned} \tag{C.12}$$

Relative to the main text, the current appearing in the first of these equations is new, and in the second equation it is half the size as found in the main text (apart from the trivial scaling by e^{φ_0} throughout). The resulting 4D curvature is unchanged because it is the combination

$$L^2\hat{R} = 2\left(\frac{\delta\mathcal{Q}}{\mathcal{Q}}\right) - 2\kappa^2 JL^2 e^{\varphi_0} = \frac{\kappa^2}{2\pi\alpha}\Big(T'_{_N} + T'_{_S}\Big) - 4\kappa^2 JL^2 e^{\varphi_0}\,. \tag{C.13}$$

Here the current contribution is twice the result of the main text, so with the current being only half as large for a given φ_0, the curvature remains unchanged.

Using the corresponding current in the 4 dimensional theory — *i.e.* using $\sqrt{-\hat{g}}\; j$ — yields in general a different Einstein-frame effective potential. However, $V_{_{EF}}(\varphi_\star)$ and $V''_{_{EF}}(\varphi_\star)$ agree. Since the coefficient of the kinetic term is unchanged, neither is the mass of the dilaton and the cosmological constant. This shows that we can extract the properties at the stationary point reliably, even though the shape of the potential away from this point can depend on the detailed definition of the current that is used. This reflects a general property: the detailed form of a scalar potential can be varied (as always) by performing a field redefinition, though any dependence on the field variables used ultimately drops from any physical prediction.

C.3 Linearization around the rugby ball

This appendix computes the linearization of the field equations about the rugby ball solutions, with

$$\begin{aligned} e^B &= e^{B_0}(1+\delta B) = e^{-\varphi_0/2}\alpha L \sin\left(\frac{\hat{\rho}}{L}\right)(1+\delta B) \\ W &= \delta W \qquad \text{and} \qquad \phi = \varphi_0 + \delta\phi \end{aligned} \tag{C.14}$$

$$\mathcal{F}_{\rho\theta} = (\mathcal{Q}+\delta\mathcal{Q})e^{\phi+B-4W} = \mathcal{Q}e^{\varphi_0+B_0}\left(1+\frac{\delta\mathcal{Q}}{\mathcal{Q}}+\delta B+\delta\phi-4\delta W\right).$$

Using these in the d'Alembertian for the dilaton gives

$$\begin{aligned}\Box\phi &= \frac{1}{\sqrt{-g}}\,\partial_M\left(\sqrt{-g}\,g^{MN}\partial_N\phi\right) = \frac{e^{\varphi_0}}{\sqrt{-g}}\,\partial_{\hat\rho}\left(\sqrt{-g}\,\partial_{\hat\rho}\phi\right)\\ &= \frac{e^{\varphi_0}}{\sin(\hat\rho/L)}\,\partial_{\hat\rho}\left[\sin\left(\frac{\hat\rho}{L}\right)\partial_{\hat\rho}(\delta\phi)\right],\end{aligned} \tag{C.15}$$

where we use $\partial_\rho\phi = \partial_\rho(\delta\phi)$ to allow the use of the background metric. Similarly,

$$\begin{aligned}\delta\left(\frac{2g_R^2}{\kappa^2}\,e^{\phi}-\frac{1}{2}\,\kappa^2\mathcal{Q}^2\,e^{\phi}e^{-8W}\right) &= \left(\frac{2g_R^2}{\kappa^2}-\frac{\kappa^2}{2}\,\mathcal{Q}^2\right)e^{\varphi_0}\delta\phi\\ &\quad -e^{\varphi_0}\mathcal{Q}^2\kappa^2\left(\frac{\delta\mathcal{Q}}{\mathcal{Q}}-4\delta W\right)\\ &= -e^{\varphi_0}\mathcal{Q}^2\kappa^2\left(\frac{\delta\mathcal{Q}}{\mathcal{Q}}-4\delta W\right),\end{aligned} \tag{C.16}$$

where the second line uses the rugby ball condition for the background value of $\mathcal{Q}$. With these the dilaton equation becomes

$$\frac{\partial_{\hat\rho}\left[\sin(\hat\rho/L)\,\partial_{\hat\rho}(\delta\phi)\right]}{\sin(\hat\rho/L)} = -\mathcal{Q}^2\kappa^2\left(\frac{\delta\mathcal{Q}}{\mathcal{Q}}-4\delta W\right) = -\frac{1}{L^2}\left(\frac{\delta\mathcal{Q}}{\mathcal{Q}}-4\delta W\right), \tag{C.17}$$

which is the form used in the main text.

Similarly, the linearization of the Einstein equation, eq. (5.8), uses

$$\begin{aligned}\delta\left(\frac{g_R^2e^{\phi}}{\kappa^2}\right) &= \frac{1}{4}\,\kappa^2\mathcal{Q}^2e^{\varphi_0}\delta\phi = \frac{e^{\varphi_0}}{4L^2}\,\delta\phi\\ \delta\left(\kappa^2\mathcal{Q}^2e^{\phi-8W}\right) &= \kappa^2\mathcal{Q}^2e^{\varphi_0}\left(\frac{2\delta\mathcal{Q}}{\mathcal{Q}}+\delta\phi-8\delta W\right) = \frac{e^{\varphi_0}}{L^2}\left(\frac{2\delta\mathcal{Q}}{\mathcal{Q}}+\delta\phi-8\delta W\right)\\ \delta(B')^2 &= 2\partial_\rho B_0\,\partial_\rho(\delta B) = \frac{2e^{\varphi_0}}{L^2}\,\cot\left(\frac{\hat\rho}{L}\right)\partial_{\hat\rho}(\delta B).\end{aligned} \tag{C.18}$$

Since J is perturbatively small, the background metric can be used to simplify the current term in the action,

$$\delta(\sqrt{-g}\, J) = \delta(\sqrt{-\hat{g}}\, e^{4W+B}\, J) = \sqrt{-\hat{g}}\, e^{-\varphi_0} \alpha L \sin\left(\frac{\hat{\rho}}{L}\right) J\,, \qquad \text{(C.19)}$$

ensuring that J appears as a new contribution $\kappa^2 J$ to the Einstein equations, eq. (5.8). Finally, the derivative terms for δB become

$$\delta B'' + \frac{2}{L} \cot\left(\frac{\hat{\rho}}{L}\right) B' = e^{\varphi_0} \frac{\partial_{\hat{\rho}}\left[\sin^2(\hat{\rho}/L)\partial_{\hat{\rho}}(\delta B)\right]}{\sin^2(\hat{\rho}/L)}\,. \qquad \text{(C.20)}$$

Putting this all together yields

$$\begin{aligned} \frac{\partial_{\hat{\rho}}\left[\sin^2(\hat{\rho}/L)\,\partial_{\hat{\rho}}(\delta B)\right]}{\sin^2(\hat{\rho}/L)} &= -\frac{1}{L^2}\left[\delta\phi + \frac{3}{2}\left(\frac{\delta\mathcal{Q}}{\mathcal{Q}}\right) - 6\delta W + \kappa^2 J L^2 e^{-\varphi_0}\right] \\ &\qquad -\frac{4}{L} \cot\left(\frac{\hat{\rho}}{L}\right)\partial_{\hat{\rho}} W \qquad \text{(C.21)} \\ \frac{\partial_{\hat{\rho}}\left[\sin^2(\hat{\rho}/L)\,\partial_{\hat{\rho}}(\delta B)\right]}{\sin^2(\hat{\rho}/L)} &= -\frac{1}{L^2}\left[\delta\phi + \frac{3}{2}\left(\frac{\delta\mathcal{Q}}{\mathcal{Q}}\right) - 6\delta W + \kappa^2 J L^2 e^{-\varphi_0}\right] - \partial_{\hat{\rho}}^2 W\,, \end{aligned}$$

and

$$\hat{R} = -4e^{\varphi_0}\left[\frac{2W}{L^2} + \frac{1}{L}\cot\left(\frac{\hat{\rho}}{L}\right)\partial_{\hat{\rho}} W + \partial_{\hat{\rho}}^2 W\right] + \frac{2e^{\varphi_0}}{L^2}\left(\frac{\delta\mathcal{Q}}{\mathcal{Q}}\right) - 2\kappa^2 J\,, \qquad \text{(C.22)}$$

which are the equations solved in the main body.

C.4 Some useful integrals

This appendix evaluates the integrals $\mathcal{M}_i$ and $\mathcal{H}_i$ encountered in the main text.

Evaluation of $\overline{\mathcal{M}}$

This section evaluates the constant encountered in the φ_1 perturbation. The integrals of interest are

$$\begin{aligned}
\mathcal{M}_1(x) &= \int_0^x \mathrm{d}y \ \sin^2 y \ \ln\left|\frac{1-\cos y}{\sin y}\right| \\
\mathcal{M}_2(x) &= \int_0^x \mathrm{d}y \ \frac{\mathcal{M}_1(y)}{\sin^2 y} \\
\text{and} \quad \overline{\mathcal{M}} &= \int_0^\pi \mathrm{d}x \ \sin x \, \mathcal{M}_2(x) \, .
\end{aligned} \tag{C.23}$$

First of all, notice that the logarithm in the first line is antisymmetric under $y \to \pi - y$, while $\sin^2 y$ is symmetric. This means that the first integral integrates to 0 if $x = \pi$: that is, $\mathcal{M}_1(\pi) = 0$. These observations justify the following manipulations:

$$\begin{aligned}
\mathcal{M}_1(\pi - x) &= \mathcal{M}_1(\pi) - \int_{\pi-x}^{\pi} \mathrm{d}z \ \sin^2 z \ln\left|\frac{1-\cos z}{\sin z}\right| \\
&= -\int_0^x \mathrm{d}y \ \sin^2(\pi - y) \ \ln\left|\frac{1-\cos(\pi-y)}{\sin(\pi-y)}\right| \\
&= \int_0^x \mathrm{d}y \ \sin^2 y \ \ln\left|\frac{1-\cos y}{\sin y}\right| \\
&= \mathcal{M}_1(x) \, .
\end{aligned} \tag{C.24}$$

The same manipulations applied to $\mathcal{M}_2$ then give:

$$\begin{aligned}
\mathcal{M}_2(\pi - x) &= \mathcal{M}_2(\pi) - \int_{\pi-x}^{\pi} \mathrm{d}z \ \frac{\mathcal{M}_1(z)}{\sin^2 z} \\
&= \mathcal{M}_2(\pi) - \int_0^x \mathrm{d}y \ \frac{\mathcal{M}_1(\pi-y)}{\sin^2(\pi-y)} \\
&= \mathcal{M}_2(\pi) - \int_0^x \mathrm{d}y \ \frac{\mathcal{M}_1(y)}{\sin^2 y} \\
&= \mathcal{M}_2(\pi) - \mathcal{M}_2(x) \, .
\end{aligned} \tag{C.25}$$

Numerical evaluation of $\mathcal{M}_2(\pi)$ is complicated by the weak convergence of the integral near π. It can be evaluated more efficiently by using the above expressions to relate it to $\mathcal{M}_2(\pi/2)$. That is, numerical integration gives $\mathcal{M}_2(\pi/2) = -0.5$ to within the numerical (Maple 11) precision. Using this, we find

$$\mathcal{M}_2(\pi) = 2\mathcal{M}_2(\pi/2) = -1\,. \tag{C.26}$$

Hence $\mathcal{M}_2$ satisfies

$$\mathcal{M}_2(\pi - x) = -1 - \mathcal{M}_2(x)\,. \tag{C.27}$$

To evaluate $\overline{\mathcal{M}}$, use

$$\begin{aligned}
\overline{\mathcal{M}} &= \int_0^{\pi/2} \mathrm{d}x\ \sin x\,\mathcal{M}_2(x) + \int_{\pi/2}^{\pi} \mathrm{d}x\ \sin x\,\mathcal{M}_2(x) \\
&= \int_0^{\pi/2} \mathrm{d}x\ \sin x\,\mathcal{M}_2(x) + \int_0^{\pi/2} \mathrm{d}x\ \sin(\pi - x)\,\mathcal{M}_2(\pi - x) \\
&= \int_0^{\pi/2} \mathrm{d}x\ \sin x\,\mathcal{M}_2(x) + \int_0^{\pi/2} \mathrm{d}x\ \sin x\Big[-1 - \mathcal{M}_2(x)\Big] \\
&= -\int_0^{\pi/2} \mathrm{d}x\ \sin x = -1\,.
\end{aligned} \tag{C.28}$$

Evaluation of $\overline{\mathcal{H}}$

Recall the definitions,

$$\begin{aligned}
\mathcal{H}_1(x) &= \int_0^x \mathrm{d}y\ \sin^2 y\ \ln|\sin y| \\
\mathcal{H}_2(x) &= \int_0^x \mathrm{d}y\ \frac{\mathcal{H}_1(y)}{\sin^2 y} \\
\bar{\mathcal{H}} &= \int_0^{\pi} \mathrm{d}y\ \sin y\,\mathcal{H}_2(y)\,.
\end{aligned} \tag{C.29}$$

In this case numerical evaluation gives (Maple 11):

$$\mathcal{H}_1(\pi) = \frac{\pi}{4}\,(1 - \ln 4)\,. \tag{C.30}$$

Similar numerical integration to evaluate $\mathcal{H}_2(\phi)$ is complicated by the apparent singularity at the endpoints caused by the factors of $1/\sin^2 y$ in the integrand. These can be dealt with by repeating the arguments of the previous section, which in this case give

$$\begin{aligned}
\mathcal{H}_1(\pi - x) &= \mathcal{H}_1(\pi) - \int_{\pi-x}^{\pi} \mathrm{d}y\ \sin^2 y\ \ln|\sin y| \\
&= \mathcal{H}_1(\pi) - \int_0^x \mathrm{d}y\ \sin^2 y\ \ln|\sin y| \\
&= \mathcal{H}_1(\pi) - \mathcal{H}_1(x)\,.
\end{aligned} \tag{C.31}$$

Next consider the following symmetry properties of $\mathcal{H}_2$:

$$\begin{aligned}
\mathcal{H}_2\left(\frac{\pi}{2} + x\right) &= \mathcal{H}_2\left(\frac{\pi}{2}\right) + \int_{\pi/2}^{\pi/2+x} \mathrm{d}z\ \frac{\mathcal{H}_1(z)}{\sin^2 z} \\
&= \mathcal{H}_2\left(\frac{\pi}{2}\right) - \int_{\pi/2}^{\pi/2-x} \mathrm{d}y\ \frac{\mathcal{H}_1(\pi - y)}{\sin^2(\pi - y)} \\
&= \mathcal{H}_2\left(\frac{\pi}{2}\right) + \int_{\pi/2-x}^{\pi/2} \mathrm{d}y\ \frac{\mathcal{H}_1(\pi) - \mathcal{H}_1(y)}{\sin^2 y}\,,
\end{aligned} \tag{C.32}$$

and simplify using

$$\begin{aligned}
\int_{\pi/2-x}^{\pi/2} \mathrm{d}y\ \frac{\mathcal{H}_1(y)}{\sin^2 y} &= \int_0^{\pi/2} \mathrm{d}y\ \frac{\mathcal{H}_1(y)}{\sin^2 y} - \int_0^{\pi/2-x} \mathrm{d}y\ \frac{\mathcal{H}_1(y)}{\sin^2 y} \\
&= \mathcal{H}_2\left(\frac{\pi}{2}\right) - \mathcal{H}_2\left(\frac{\pi}{2} - x\right)\,,
\end{aligned} \tag{C.33}$$

to get

$$\begin{aligned}
\mathcal{H}_2\left(\frac{\pi}{2} + x\right) &= \mathcal{H}_2\left(\frac{\pi}{2} - x\right) + \mathcal{H}_1(\pi) \int_{\pi/2-x}^{\pi/2} \frac{\mathrm{d}y}{\sin^2 y} \\
&= \mathcal{H}_2\left(\frac{\pi}{2} - x\right) + \mathcal{H}_1(\pi)\ \cot\left(\frac{\pi}{2} - x\right)\,.
\end{aligned} \tag{C.34}$$

The evaluation of $\overline{\mathcal{H}}$ now proceeds, with

$$\begin{aligned}
\overline{\mathcal{H}} &= \int_0^{\pi} \mathrm{d}x \;\sin x\, \mathcal{H}_2(x) \\
&= \int_0^{\pi/2} \mathrm{d}x \;\left\{ \sin x\, \mathcal{H}_2(x) + \sin\left(\frac{\pi}{2} + x\right)\, \mathcal{H}_2\left(\frac{\pi}{2} + x\right) \right\} \\
&= \int_0^{\pi/2} \mathrm{d}x \;\left\{ \sin x\, \mathcal{H}_2(x) + \sin\left(\frac{\pi}{2} - x\right) \left[\mathcal{H}_2\left(\frac{\pi}{2} - x\right) + \mathcal{H}_1(\pi)\, \cot\left(\frac{\pi}{2} - x\right) \right] \right\} \\
&= 2\int_0^{\pi/2} \mathrm{d}x \;\sin x\, \mathcal{H}_2(x) + \mathcal{H}_1(\pi) \int_0^{\pi/2} \mathrm{d}x \;\cos\left(\frac{\pi}{2} - x\right) \\
&= 2\int_0^{\pi/2} \mathrm{d}x \;\sin x\, \mathcal{H}_2(x) + \mathcal{H}_1(\pi)\,. \qquad \text{(C.35)}
\end{aligned}$$

This can now be integrated numerically without problems, giving (to ten decimal places) a result consistent with $\overline{\mathcal{H}} = -2 + \ln 4$.

Appendix D

Appendix for chapter 6

D.1 Localized brane fluxes

An important role is played by brane-localized flux in the discussion of the main text, and in particular the choice of large values for Φ_b. In this appendix we use a simple but explicit model of microscopic brane dynamics to explore how reasonable these choices might be. In particular, one might worry that microscopic details (like flux quantization) of brane-localized flux could obstruct its role in the relaxation mechanism for the low-energy curvature.

Ideally this question should be addressed within string theory, which provides the most likely UV completion. However we are handicapped by the lack of a controlled derivation of 6D gauged chiral supergravity from an explicit string vacuum (see however (6.42; 6.43)). Instead, as a first step we model the codimension-2 brane with localized flux as a very small cylindrical codimension-1 brane situated at $\rho = \epsilon$ which surrounds the position of the codimension-2 brane at $\rho = 0$, along the lines of refs. (6.10; 6.24). We regard this brane, together with a suitably smooth interior configuration for $\rho < \epsilon$, as a specific UV completion of the codimension-2 brane. Although this is unlikely to be a realistic microscopic realization of brane flux localization, it has the advantage of allowing an explicit examination of many of the consistency issues involved.

Consider therefore the following codimension-1 brane action,

$$S_5 = -\int_{\rho=\epsilon} \mathrm{d}^5x\sqrt{-g_5}\,\Big[Z_1(\phi)D_m\sigma D^m\sigma + T_1(\phi)\Big]\,, \tag{D.1}$$

describing a small cylinder at radius $\rho = \epsilon$, where σ is a brane-localized Stückelberg field whose covariant derivative is

$$D_m\sigma = \partial_m\sigma + g_b\mathcal{A}_m\,. \tag{D.2}$$

This is invariant under the gauge transformations

$$\mathcal{A}_m \to \mathcal{A}_m - \frac{1}{g}\,\partial_m\Omega \quad \text{and} \quad \sigma \to \sigma + \frac{g_b}{g}\,\Omega\,. \tag{D.3}$$

Here g denotes the bulk gauge coupling while g_b denotes a corresponding brane gauge coupling.

The presence of a field like σ is important for stabilizing the size of the codimension-1 brane at a small but nonzero radius (6.10; 6.11; 6.24). For ϵ sufficiently small the codimension-1 brane becomes effectively a codimension-2 brane, whose action can be found by dimensional reduction. Having a finite codimension-2 brace action in this limit generally requires the quantities

$$t_1(\phi) := \epsilon\,T_1(\phi) \quad \text{and} \quad z_1(\phi) := \epsilon\,Z_1(\phi)\,, \tag{D.4}$$

remain finite in the limit of small ϵ.

In the region exterior to the brane, $\rho \geq \epsilon$, and in the presence of any bulk matter fields having charge g, the single-valuedness of the gauge group element, $e^{i\Omega}$, requires $\Omega(\theta + 2\pi) - \Omega(\theta) = 2\pi s$ for some integer s. The Stückelberg field can also wind nontrivially as a function of θ if its target space should be a circle,

$$\sigma(\theta + 2\pi) - \sigma(\theta) = 2\pi n f \,, \tag{D.5}$$

for some nonzero integer n, where $2\pi f$ denotes the circumference of the target-space circle. This boundary condition is consistent with gauge transformations provided

$$\begin{aligned} \sigma_\Omega(\theta + 2\pi) - \sigma_\Omega(\theta) &= \sigma(\theta + 2\pi) + \frac{g_b}{g}\,\Omega(\theta + 2\pi) - \sigma(\theta) - \frac{g_b}{g}\,\Omega(\theta) \\ &= 2\pi n f + 2\pi s\,\frac{g_b}{g} \,, \end{aligned} \tag{D.6}$$

is also an integer multiple of $2\pi f$. This is automatically true if the brane gauge coupling is quantized in units of the bulk gauge coupling: $g_b = kfg$ for some integer k. In this case because $\sigma_\Omega(\theta + 2\pi) - \sigma_\Omega(\theta) = 2\pi(n + sk)f$ differs from $\sigma(\theta + 2\pi) - \sigma(\theta) = 2\pi n f$, large gauge transformations (those with $s \neq 0$) map different choices for σ boundary conditions into one another.

Brane equation of motion

The equation of motion on the brane is

$$\partial_m \Big[\sqrt{-g_5}\; Z_1(\phi) D^m \sigma \Big] = 0 \,, \tag{D.7}$$

and we are interested in solutions that depend on θ only. Since none of the bulk fields that appear in this equation of motion depend on θ this simplifies to

$$\partial_\theta D_\theta \sigma = 0 \,, \tag{D.8}$$

which has as solution

$$D_\theta \sigma = \partial_\theta \sigma + kfg\, \mathcal{A}_\theta = C \,, \tag{D.9}$$

where C is independent of θ. However, since all of the bulk fields depend only on the radial coordinate ρ, they do not depend on any of the five on-brane directions (including θ), and so C can depend on any of them. In particular, C can be a function of the dilaton, ϕ.

The requirement that σ be single-valued up to integer multiples of $2\pi f$ then means that

$$2\pi f n = \sigma(\theta + 2\pi) - \sigma(\theta) = \oint \mathrm{d}\theta\, \partial_\theta \sigma = 2\pi C - kfg \oint_{\rho=\epsilon} \mathcal{A}_\theta\, \mathrm{d}\theta\,, \qquad \text{(D.10)}$$

which implies that C is given in terms of the flux,

$$\Phi := \oint_{\rho=\epsilon} \mathcal{A}_\theta\, \mathrm{d}\theta\,, \qquad \text{(D.11)}$$

by

$$C = \left(n + \frac{kg}{2\pi}\, \Phi \right) f\,. \qquad \text{(D.12)}$$

Notice that the transformation of Φ under large gauge transformations (*i.e.* those with $s \neq 0$) ensures that C is invariant even though $n \to n + ks$.

We now specify in more detail the system interior to the codimension-1 brane, with the goal of deriving a second relationship between C and Φ, from which we may eliminate C. At first sight one might worry that any expression for Φ won't be gauge invariant, since Φ transforms under large gauge transformations. However once we match through to a smooth inner configuration the noncontractible loop whose topology underlies the existence of large gauge transformations disappears. The nontrivial large gauge transformations necessarily become singular somewhere once they are extended into the interior region.

The cylinder's interior

We next specify the interior of the cylindrical brane, which we require to be everywhere smooth. We keep the action in the interior the same as in the bulk, apart from only one change: we take the dilaton potential to be

$$V = V_0 \, e^{\phi} \,, \tag{D.13}$$

for a general constant V_0. If we write $V_0 = 2g_R^2/\kappa^4$, we effectively choose the value of $g_R = g_R^{\rm in}$ interior to the cylinder to differ from its value on the outside.

We take the interior solution to be the Salam-Sezgin solution,

$$\mathrm{d}s^2 = e^{2W} \hat{g}_{\mu\nu} \, \mathrm{d}x^\mu \mathrm{d}x^\nu + e^{-\varphi_{\rm in}} \Big[\mathrm{d}\hat{\rho}^2 + e^{2B} \, \mathrm{d}\theta^2 \Big] \,, \tag{D.14}$$

with

$$\phi = \varphi_{\rm in} \,, \qquad W = W_{\rm in} \quad \text{and} \quad \mathcal{F}_{\rho\theta} = \mathcal{Q}_{\rm in} e^{\varphi_{\rm in}/2} e^{B-4W} \,, \tag{D.15}$$

where $\varphi_{\rm in}$ and $W_{\rm in}$ are constants, and

$$e^B = \ell_{\rm in} \sin \left(\frac{\hat{\rho} - \hat{\rho}_c}{\ell_{\rm in}} \right) . \tag{D.16}$$

The center of the interior geometry is located at $\hat{\rho} = \hat{\rho}_c$, which need not be $\hat{\rho} = 0$ due to our choice that the codimension-1 brane is located at $\rho = \epsilon$ for both the exterior and interior geometries.

Like in the exterior geometry the equations of motion still imply

$$\ell_{\rm in} = \frac{\kappa}{2 g_R^{\rm in}} \,, \qquad \mathcal{Q}_{\rm in} = \pm \sqrt{2V_0} = \pm \frac{2 g_R^{\rm in}}{\kappa^2} \qquad \text{and} \qquad \hat{g}_{\mu\nu} = \eta_{\mu\nu} \,, \tag{D.17}$$

which shows that we can dial the value of V_0 to achieve any desired flux for the interior gauge field. Choosing a gauge with $\mathcal{A}_\theta(\rho_c) = 0$, with all other components of $\mathcal{A}_M$ vanishing, the gauge fields become

$$\begin{aligned} \mathcal{F}_{\rho\theta} &= \mathcal{Q}_{\rm in} e^{-4W_{\rm in}+\varphi_{\rm in}/2} \left[\ell_{\rm in} \sin\left(\frac{\hat\rho - \hat\rho_c}{\ell_{\rm in}}\right)\right] \\ \text{and so} \quad \mathcal{A}_\theta(\rho) &= \mathcal{Q}_{\rm in} e^{-4W_{\rm in}} \ell_{\rm in}^2 \left[1 - \cos\left(\frac{\hat\rho - \hat\rho_c}{\ell_{\rm in}}\right)\right]. \end{aligned} \quad \begin{aligned} &(D.18) \\ &(D.19) \end{aligned}$$

Matching conditions

Continuity of the metric and dilaton at the brane location, $\rho = e^{-\varphi_{\rm in}/2}\hat\rho = \epsilon$, implies

$$\begin{aligned} \varphi_{\rm in} = \phi(\epsilon) &= \phi_b \\ W_{\rm in} = W(\epsilon) &= W_b \\ \text{and} \quad e^{-\varphi_{\rm in}/2} \ell_{\rm in} \sin\left(\frac{\hat\epsilon - \hat\rho_c}{\ell_{\rm in}}\right) = e^{B_b} &= \alpha_b \epsilon \,, \end{aligned} \quad (D.20)$$

where $\hat\epsilon = \epsilon\, e^{\varphi_{\rm in}/2}$ and ϕ_b, B_b and W_b are the (regulated) values of the dilaton and warping at the brane in the exterior bulk solution. From this we find the value of the gauge field at the brane is,

$$\mathcal{A}_\theta(\epsilon) \simeq \mathcal{Q}_{\rm in} e^{-4W_{\rm in}} \ell_{\rm in}^2 \left[1 - \sqrt{1 - \left(\frac{\alpha_b \epsilon\, e^{\phi_b/2}}{\ell_{\rm in}}\right)^2}\,\right]. \quad (D.21)$$

Next we impose the jump discontinuity of the gauge field across the brane position. For the above interior and exterior solutions and brane action, this reads

$$\mathcal{Q}_{\rm in} - \mathcal{Q}_{\rm out} = -\frac{e^{B+4W}}{\sqrt{-g_5}} \frac{\delta S_5}{\delta \mathcal{A}_\theta} \simeq \frac{2 g_b C Z_1 e^{4W_b}}{\alpha_b \epsilon}. \quad (D.22)$$

Using this to trade $\mathcal{Q}_{\rm in}$ for $\mathcal{Q}_{\rm out}$ in the gauge potential, eq. (D.21), we see that when $\alpha_b \epsilon\, e^{\phi_b/2} \ll \ell_{\rm in}$ the result for $\mathcal{A}_\theta(\epsilon)$ is proportional to

$$(\alpha_b \epsilon)^2\, \mathcal{Q}_{\rm in} = (\alpha_b \epsilon)^2\, \mathcal{Q}_{\rm out} + \alpha_b \epsilon \left(2 g_b C Z_1 e^{4W_b} \right). \tag{D.23}$$

Although the first term on the right-hand side vanishes in the codimension-2 limit where $\epsilon \to 0$, the second term need not because the finiteness of the dimensionally reduced codimension-2 action obtained from S_5 requires $z_1 = \lim_{\epsilon \to 0} \epsilon\, Z_1$ be finite in this limit. In terms of this the brane-localized flux becomes

$$\Phi = \oint_{\rho=\epsilon} \mathcal{A}_\theta \, {\rm d}\theta = 2\pi \mathcal{A}_\theta(\epsilon) = 2\pi \alpha_b g_b C z_1\, e^{\phi_b}. \tag{D.24}$$

Combining this last result with eq. (D.12) allows us to solve for C, giving the quantization condition

$$\frac{C}{f} \left(1 - \alpha_b g_b^2 z_1\, e^{\phi_b} \right) = n. \tag{D.25}$$

Equivalently, using this to eliminate C from the flux gives

$$\frac{\Phi}{2\pi} = \frac{n f \alpha_b g_b z_1\, e^{\phi_b}}{1 - \alpha_b g_b^2 z_1\, e^{\phi_b}}. \tag{D.26}$$

Notice that although this expression is quantized in the sense that it is proportional to an integer, it is also ϕ_b-dependent through the quantity $z_1(\phi_b)\, e^{\phi_b}$. Furthermore, the regime of weak coupling and small derivatives has $g_b^2 z_1\, e^{\phi_b} \ll 1$ and so we may approximate the denominator by unity, leading to a contribution to Φ that is proportional to $z_1\, e^{\phi_b}$. (Intriguingly, if $g_b^2 z_1\, e^{\phi_b}$ were instead large we would find the ϕ_b-independent result $\Phi \to -2\pi n f / g_b$, and so $g\Phi/2\pi \to -n/k$ would be quantized at rational values.)

In the special case that the brane does not break the bulk classical scale invariance then $T_1 \propto e^{\phi_b/2}$ and $Z_1 \propto e^{-\phi_b/2}$, so writing $\epsilon = \hat{\epsilon}\, e^{-\phi_b/2}$ we see that $t_1 = \epsilon\, T_1$ is ϕ_b-independent and $z_1 = \epsilon\, Z_1 \propto e^{-\phi_b}$, as expected. This means Φ

is independent of ϕ_b, as is also argued to be true for the scale invariant case in the main text.

On the other hand, the case of most interest in the main text is where $\Phi = \mu\, e^{\phi_b}$, which corresponds to choosing z_1 to be ϕ_b-independent. The above calculation then gives the coefficient, μ, as

$$\mu = 2\pi n\, \alpha_b g_b z_1 f = 2\pi n k\, \alpha_b g z_1 f^2\,. \tag{D.27}$$

In particular, we seek situations where $g_R\mu$ is very large, while keeping $g_R\mu\, e^{\phi_b}$ small. This we can arrange in several ways: (i) by making f very large (so σ takes values on a very large circle); (ii) by making the integers k and/or n very large; or (iii) by making $g_R g z_1$ large. All of these choices come down to including a lot of current on the codimension-1 brane, as one might expect. Large n means a very high gradient in σ, which can be interpreted as a lot of particles in the current. Large k means a comparatively large coupling, g_b, which gives σ a large charge. Finally, large f gives both a large brane charge and a large gradient.

The main worry with these choices would be if they would indicate a failure of the low-energy derivative expansion, on whose validity the entire calculation rests. However since μ appears systematically in the brane action only through the combination $\mu\, e^{\phi_b}$ this expansion appears to be under control provided only that this product be small. e^{ϕ_b} also appears without factors of μ in the bulk action, but extremely small values of e^{ϕ_b} are there under control because this is the small quantity that controls the bulk loop expansion. In particular, there seems to be no consistency restriction on how large the parameter f can be.

D.2 The view from 4 dimensions

In this section, we ask what the scalar potential is that lifts the flat direction parameterized by φ_0, as would be seen from the perspective of a brane-localized 4D observer. To do so we draw heavily on the results of (6.13), which computes this potential for geometries that are perturbatively close to the rugby-ball geometries.[1]

Writing the brane action as

$$S_b = -\int \mathrm{d}^4x\sqrt{-g_4}\; L_b = -\int \mathrm{d}^4x\sqrt{-g_4}\left(T_b + \frac{1}{2}\,\Phi_b\epsilon^{mn}F_{mn}\right)\,, \qquad \text{(D.28)}$$

we calculate the low-energy 4D effective potential in the special case that the tensions satisfy $T_b = \overline{T} + \delta T_b$, where δT_b is much smaller than the (positive) average tension, $\overline{T}$. We further assume the background rugby-ball geometry satisfies

$$\frac{ng_R}{g} = 1 - \frac{\kappa^2\overline{T}}{2\pi}\,, \qquad \text{(D.29)}$$

so that no background brane-localized flux is present.

We compute the response of the bulk to deviations $\delta T_b = T_b - \overline{T}$ by linearizing the bulk equations in δT_b and $\delta\Phi_b$, obtaining the general solutions as a function of the parameter φ_0 that labels the orbits of the bulk scaling symmetry. The brane-bulk matching conditions define the boundary conditions that are then used to eliminate the integration constants in terms of brane properties. This allows the calculation of the stabilized value $\varphi_0 = \varphi_\star$ and the energy cost for deviations of φ_0 from $\varphi_\star$. In this way the features of the low-energy 4D potential can be mapped out (6.13). (We emphasize that this linearization is not required for the arguments of the main text, for which the general exact classical solutions are known. This limit is simply one for

[1] Beware the notational change, as ref. (6.13) denotes L_b by T_b; denotes T_b by τ_b; and denotes Φ_b by $\Phi_b e^{-\phi}$.

which we can explicitly calculate the view in the 4D effective theory, to check our general results.)

Writing

$$\delta L_b = \left(T_b - \overline{T}\right) - \mathcal{Q}\Phi_b\, e^{\varphi_0} = \delta T_b - \mathcal{Q}\Phi_b\, e^{\varphi_0}\,, \tag{D.30}$$

(which uses the result that $W = 0$ for the unperturbed rugby-ball geometry), the stationary point, $\varphi_\star$, for the scalar zero mode turns out to be given by (6.13)

$$0 = \sum_b \left(\delta L_b + \frac{1}{2}\,\delta L_b' - \mathcal{Q}\Phi_b\, e^{\varphi_\star}\right)\,, \tag{D.31}$$

where prime denotes differentiation with respect to ϕ. When δT_b and Φ_b are both independent of φ_0 for all of the branes, then this simplifies to[2]

$$0 = \sum_b \left(\delta T_b - 2\mathcal{Q}\Phi_b\, e^{\varphi_\star}\right)\,, \tag{D.32}$$

with solution

$$e^{\varphi_\star} = \frac{\sum_b \delta T_b}{\sum_b 2\mathcal{Q}\Phi_b}\,. \tag{D.33}$$

The Jordan frame potential of the low-energy effective 4D theory is shown in ref. (6.13) to satisfy

$$(e^{\varphi} V_{JF})' = \frac{1}{2}\, e^{\varphi} \sum_b \left(\delta L_b + \frac{3}{2}\,\delta L_b' - \mathcal{Q}\Phi_b\right)\,, \tag{D.34}$$

which for ϕ-independent δT_b and Φ_b can be integrated to give

$$V_{JF}(\varphi) = Ce^{-\varphi} + \frac{1}{2}\sum_b \left(\delta T_b - \mathcal{Q}\Phi_b\, e^{\varphi}\right)\,, \tag{D.35}$$

with C an integration constant. The corresponding Einstein-frame potential

[2]Recall $\delta L_b'$ denotes differentiation with respect to ϕ with $\mathcal{A}_M$ and g_{MN} fixed, and because $\mathcal{A}_M$ and g_{MN} depend on φ_0 this is *not* the same as differentiating δL_b with respect to φ_0.

is

$$V_{EF}(\varphi) = e^{2(\varphi-\varphi_\star)}\, V_{JF} = Ce^{\varphi-2\varphi_\star} + \frac{1}{2}\sum_b \Big(\delta T_b e^{2(\varphi-\varphi_\star)} - \mathcal{Q}\Phi_b e^{3\varphi-2\varphi_\star}\Big)\,. \quad \text{(D.36)}$$

The integration constant C is set by demanding that V'_{EF} vanishes at $\varphi = \varphi_\star$:

$$Ce^{-\varphi_\star} = \sum_b \left(-\delta T_b + \frac{3}{2}\, \mathcal{Q}\Phi_b\, e^{\varphi_\star}\right)\,, \quad \text{(D.37)}$$

leading to the full Einstein-frame potential

$$V_{EF} = \left(\frac{1}{2}\sum_b \delta T_b\right)\Big(e^{2\psi} - 2\, e^{\psi}\Big) + \left(\frac{1}{2}\sum_b \mathcal{Q}\Phi_b e^{\varphi_\star}\right)\Big(3\, e^{\psi} - e^{3\psi}\Big)\,, \quad \text{(D.38)}$$

where $\psi := \varphi - \varphi_\star$.

The 4D on-brane curvature obtained from the full 6D field equations agrees (by construction) with the curvature obtained from the 4D Einstein equations with V_{EF} evaluated at $\varphi_\star$. Using (D.38), we find

$$V_\star := V_{EF}(\varphi_\star) = \frac{1}{2}\sum_b \Big(-\delta T_b + 2\, \mathcal{Q}\Phi_b\, e^{\varphi_\star}\Big)\,, \quad \text{(D.39)}$$

which vanishes by virtue of the stabilization condition defining $\varphi_\star$, eq. (D.32). Notice that this condition is also equivalent to the linearized version of the warped flux-quantization condition, eq. (6.36)

$$\begin{aligned} 0 &= \delta\left[\frac{(\alpha_+\alpha_-)^{1/2}}{g_R} + \frac{1}{2\pi}\left(\frac{\Phi_o}{\mathcal{W}_o^2} + \frac{\Phi_f}{\mathcal{W}_f^2}\right) e^{\varphi_\star}\right] \\ &= \frac{1}{2g_R}\sum_b \delta\alpha_b + \frac{1}{2\pi}\sum_b \delta\Phi_b\, e^{\varphi_\star} \\ &= \frac{\kappa^2}{4\pi g_R}\sum_b \Big(\delta T_b - 2\, \mathcal{Q}\Phi_b\, e^{\varphi_\star}\Big)\,, \end{aligned} \quad \text{(D.40)}$$

which uses $\delta\alpha_b = \kappa^2 \delta L_b/2\pi = \kappa^2(\delta T_b - \mathcal{Q}\Phi_b)/2\pi$, as well as the unperturbed rugby-ball relation $\mathcal{Q} = 2g_{R}/\kappa^2$.

Bulk loop corrections to the brane action

In general, bulk loops induce a ϕ-dependence to the brane action, and so generate a nonzero curvature for the on-brane directions. The most UV-sensitive contributions come when very heavy bulk particles circulate in the loop, and because these involve only very short wavelengths they generate local corrections to the brane action. We now argue that these UV-sensitive bulk loops contribute only to $V_\star$ at order m_{KK}^4, where $m_{KK} \simeq 1/r \simeq \mathcal{V}_2^{-1/2}$ is the Kaluza-Klein scale.

Loops involving comparatively long-wavelength states at the KK scale need not generate only local effects on the branes, but also only give rise to contributions to the low-energy vacuum energy that are of order $\delta V_\star \sim m_{KK}^4$ (and so are not larger than the UV loops we examine below). Because $e^{2\phi}$ is the loop-counting parameter in the bulk, an estimate for the size of the UV loop-generated curvature can be found by repeating the above arguments, but now writing T_b and Φ_b as a series in powers of $e^{2\phi}$, rather than taking them to be ϕ-independent.

To this end we write

$$\begin{aligned} \delta T_b &= \delta T_b^{(0)} + \delta T_b^{(1)} e^{2\phi} + \cdots \\ \Phi_b &= \Phi_b^{(0)} + \Phi_b^{(1)}\, e^{2\phi} + \cdots \,, \end{aligned} \tag{D.41}$$

where $T_b^{(1)}$ and $T_b^{(0)}$ are ϕ-independent and similar in size, as are $\Phi_b^{(1)}$ and $\Phi_b^{(0)}$. With these choices we have

$$\begin{aligned} \delta L_b &= \delta T_b^{(0)} - \mathcal{Q}\Phi_b^{(0)}\, e^{\varphi_0} + \delta T_b^{(1)}\, e^{2\varphi_0} - \mathcal{Q}\Phi_b^{(1)}\, e^{3\varphi_0} \\ \text{and so} \quad \delta L_b' &= 2\,\delta T_b^{(1)}\, e^{2\varphi_0} - 2\mathcal{Q}\Phi_b^{(1)}\, e^{3\varphi_0}\,. \end{aligned} \tag{D.42}$$

The condition defining the stationary point, $\varphi_\star$, is given by

$$\begin{aligned} 0 &= \sum_b \left(\delta L_b + \frac{1}{2}\, \delta L_b' - \mathcal{Q}\Phi_b \right) \\ &= \sum_b \Big(\delta T_b^{(0)} - 2\mathcal{Q}\Phi_b^{(0)}\, e^{\varphi_\star} + 2\, \delta T_b^{(1)}\, e^{2\varphi_\star} - 3\mathcal{Q}\Phi_b^{(1)}\, e^{3\varphi_\star} \Big) . \end{aligned} \tag{D.43}$$

At lowest order this is solved by $\varphi_\star^{(0)}$ satisfying eq. (D.33), and to next-to-leading order the correction, $\delta e^{\varphi_\star}$, satisfies

$$\delta e^{\varphi_\star} \sum_b 2\mathcal{Q}\Phi_b^{(0)} = \sum_b \Big(2\, \delta T_b^{(1)}\, e^{2\varphi_\star^{(0)}} - 3\mathcal{Q}\Phi_b^{(1)}\, e^{3\varphi_\star^{(0)}} \Big) . \tag{D.44}$$

If $\delta T_b^{(1)}$ and $\mathcal{Q}\Phi_b^{(1)}$ are similar in size then only the first term on the right-hand-side of this last expression dominates. We keep both here because our interest in the main text is in the case where $\delta T_b^{(1)}$ and $\mathcal{Q}\Phi_b^{(1)}\, e^{\varphi_\star^{(0)}}$ are similar in size.

The corrected Jordan-frame potential due to the brane perturbations then solves eq. (D.34), or

$$(e^{\varphi} V_{JF})' = e^{\varphi} \sum_b \left[\frac{1}{2}\, \delta T_b^{(0)} - \mathcal{Q}\Phi_b^{(0)}\, e^{\varphi} + 2\, \delta T_b^{(1)}\, e^{2\varphi} - \frac{5}{2}\, \mathcal{Q}\Phi_b^{(1)}\, e^{3\varphi} \right] , \tag{D.45}$$

which integrates to

$$V_{JF} = C e^{-\varphi} + \sum_b \left[\frac{1}{2}\, \delta T_b^{(0)} - \frac{1}{2}\, \mathcal{Q}\Phi_b^{(0)}\, e^{\varphi} + \frac{2}{3}\, \delta T_b^{(1)}\, e^{2\varphi} - \frac{5}{8}\, \mathcal{Q}\Phi_b^{(1)}\, e^{3\varphi} \right] , \tag{D.46}$$

with C an integration constant, as before.

The Einstein frame potential, $V_{EF} = e^{2(\varphi - \varphi_\star)} V_{JF}$, similarly is

$$V_{EF} \;=\; C e^{\varphi - 2\varphi_\star} + \sum_b \left[\frac{1}{2}\, \delta T_b^{(0)}\, e^{2(\varphi-\varphi_\star)} - \frac{1}{2}\, \mathcal{Q}\Phi_b^{(0)}\, e^{3\varphi - 2\varphi_\star} \right.$$

$$+\frac{2}{3}\,\delta T_b^{(1)}\,e^{4\varphi-2\varphi_\star}-\frac{5}{8}\,\mathcal{Q}\Phi_b^{(1)}\,e^{5\varphi-2\varphi_\star}\Big]\,. \tag{D.47}$$

As before, enforcing $V'_{EF}(\varphi_\star)=0$ fixes C, giving

$$Ce^{-\varphi_\star}=\sum_b\left[-\delta T_b^{(0)}+\frac{3}{2}\,\mathcal{Q}\Phi_b^{(0)}\,e^{\varphi_\star}-\frac{8}{3}\,\delta T_b^{(1)}\,e^{2\varphi_\star}+\frac{25}{8}\,\mathcal{Q}\Phi_b^{(1)}\,e^{3\varphi_\star}\right]. \tag{D.48}$$

The full next-to-leading Einstein-frame potential then is

$$\begin{aligned} V_{EF} \;=\; & \left(\frac{1}{2}\sum_b \delta T_b^{(0)}\right)\left(e^{2\psi}-2\,e^{\psi}\right)+\left(\frac{1}{2}\,e^{\varphi_\star}\sum_b \mathcal{Q}\Phi_b^{(0)}\right)\left(3\,e^{\psi}-e^{3\psi}\right)\\ & +\left(\frac{2}{3}\,e^{2\varphi_\star}\sum_b \delta T_b^{(1)}\right)\left(e^{4\psi}-4\,e^{\psi}\right)\\ & +\left(\frac{5}{8}\,e^{3\varphi_\star}\sum_b \mathcal{Q}\Phi_b^{(1)}\right)\left(5\,e^{\psi}-e^{5\psi}\right), \end{aligned} \tag{D.49}$$

with $\psi=\varphi-\varphi_\star$. Evaluating this at $\varphi_\star$, and using (D.43), we find

$$V_\star=V_{EF}(\varphi_\star)=-e^{2\varphi_\star}\sum_b\delta T_b^{(1)}+e^{3\varphi_\star}\sum_b\mathcal{Q}\Phi_b^{(1)}\,. \tag{D.50}$$

This is clearly of order $e^{2\varphi_\star}$ if $\delta T_b^{(0)}$, $\delta T_b^{(1)}$, $\Phi_b^{(0)}e^{\varphi_\star}$ and $\Phi_b^{(1)}e^{\varphi_\star}$ are all of the same order. Recalling that the flux-quantization condition relates $\varphi_\star$ to the stabilized extra-dimensional radius by $r_\star$ by $e^{\varphi_\star}\simeq\mathcal{O}(1/r_\star^2)$, we see that the loop-corrected brane action gives a result of order $1/r_\star^4$, which is also similar to the size of a generic bulk Casimir energy.

D.3 No-go results

There are a number of famous no-go results, that superficially appear to contradict our results. In this appendix we describe two of these, describing why they do not represent real obstructions.

D.3.1 Weinberg's no-go theorem

The best-known obstruction to finding a relaxation mechanism that sets the cosmological constant to zero is due to Weinberg (6.3). His is a general objection to using scale invariance to solve the cosmological constant problem. Although his argument is phrased quite generally, it is easier to describe the issues within a simple toy model.

Why at first sight scale invariance seems to help

At first sight, scale invariance provides a very attractive way to approach why the vacuum energy might be zero. To see why, consider the following simple scale-invariant toy theory:

$$S = -\int \mathrm{d}^4x \, \sqrt{-g} \, \left(\frac{1}{2} \, \partial_\mu \chi \, \partial^\mu \chi + \overline{\psi} \gamma^\mu \partial_\mu \psi + \lambda \, \chi^4 + g \overline{\psi} \psi \chi \right) . \tag{D.51}$$

This action is invariant under the rigid rescalings $\psi \to \zeta^{-3/2} \psi$, $\chi \to \zeta^{-1} \chi$ together with $g_{\mu\nu} \to \zeta^2 g_{\mu\nu}$, although this symmetry is anomalous and so does not survive quantization. However at the classical level it restricts the potential to only have a quartic term, and ensures the scale-invariant point, $\chi = 0$, is a solution to $V'(\chi = 0) = 0$. Because scale invariance precludes the existence of any dimensionful parameters, it also guarantees that the potential vanishes at this scale-invariant minimum: $V(\chi = 0) = 0$.

Suppose we put aside (for now) the anomaly in scale invariance, and ask whether the fact that $V' = V = 0$ is automatically satisfied means that scale invariance can help solve the cosmological constant problem. At first sight the answer is 'no', because having $V = 0$ when $\chi = 0$ is not in itself sufficient. It is insufficient because not only does scale invariance ensure $V = 0$; it makes *all* masses zero. After all, the cosmological constant problem is the puzzle of why the effective scalar potential is minimized at a value that is much smaller than the other nonzero masses in the problem.

A more promising attempt might be to consider the case where $\lambda = 0$. In this case all values of χ are equally good as vacua, and for all of these except $\chi = 0$ the mass of the fermion ψ is nonzero, $m = g\chi$, because the scale invariance is spontaneously broken. Since it is broken masses can be nonzero, but notice that the potential energy is nevertheless still minimized (trivially, since $V = 0$) at zero. Scale invariance guarantees that $V = 0$ remains true even once scale invariance is broken, because all values of χ are related to one another by a symmetry (scale transformations), and so V must have the same value for all of them (and so must in particular vanish, because $V = 0$ for the scale-invariant point where $\chi = 0$).

Phrased this way, spontaneously broken scale invariance sounds like a promising approach to having vanishing vacuum energy while still having nonzero masses.

Weinberg's objection

Weinberg's objection to the above argument is that, although promising, scale-invariance in itself cannot solve the cosmological constant problem, even assuming that it could be made not anomalous. That is because scale invariance can never preclude quantum corrections from generating a nonzero scalar potential, like $\lambda\,\chi^4$ which we've seen is completely scale invariant. And if such a potential is generated, the only minimum is again the scale invariant point, $\chi = 0$, for which all masses vanish.

The problem with scale invariance is not that quantum corrections raise the minimum of the potential from $V = 0$; it is that quantum corrections generically lift the flat direction and make the scale-invariant point the only minimum.

Relevance to the 6D model

Weinberg's analysis is not specific to four dimensions, and applies equally well to extra-dimensional theories that are scale invariant. It is particularly pertinent for the supergravity models discussed in the main text, for which the bulk enjoys a classical scaling symmetry. Although the analog of the scale-invariant point may seem less clear in the extra-dimensional model, it is $\chi = e^{\varphi_0}$ that plays the role described above, since this transforming multiplicatively under a scale transformation rather than shifting. This shows that having a potential minimized only at the scale invariant point corresponds in the extra-dimensional model to having a runaway potential that is only minimized for infinitely large values of the dilaton, ϕ.

And in the main text we've also seen that in the special case where the branes couple to ϕ in the scale-invariant way, the generic form for the classical low-energy potential is $V_{\rm eff} = A\, e^{2\varphi_0}$, revealing the generic runaway Weinberg's argument requires.

But nothing in this argument precludes finding the minima obtained in the main text. For more general kinds of brane-dilaton couplings the shape of the potential is more complicated since its form is no longer dictated by scale invariance. Nothing forces it to be minimized only at the scale invariant point in this case.

Furthermore, nothing in the argument says *how large* the corrections to the potential have to be. In the 6D model described above, supersymmetry in the bulk generically acts to suppress the size of quantum corrections, regardless of whether or not these are scale invariant.

So this argument, while true, doesn't preclude the behaviour found in the main text.

D.3.2 What is the 4D mechanism?

Another general objection to the kind of calculation presented here asks what the ultimate mechanism looks like in 4 dimensions. That is, even though the full theory is extra-dimensional, why can't I ask what the perspective is of a 4D brane-localized observer? After all, if there is a mechanism at work in 4D, this could be more widely useful than a particular higher-dimensional example.

The basic response to this question is that the underlying mechanism at work in the brane back-reaction is higher-dimensional, and cannot be simply seen in a purely 4D framework involving only a small number of 4D fields. Ultimately, this is why the KK scale must be as low as sub-eV energies in order to be relevant to the observed Dark Energy density: if it were higher the extra dimensions could have been integrated out and we would be back to the unsolved problem of understanding why the Dark Energy is small in four dimensions.

Of course the world *does* appear four-dimensional below the KK scale, and in this energy range a 4D observer must be able to understand what is going on. But when the KK scale is as low as the Dark Energy scale, $\rho \simeq m_{KK}^4$, there really is also no cosmological constant problem in 4D since ρ is as big as the largest UV scale — *i.e.* m_{KK} — would suggest it should be. The essence of the SLED mechanism is that above the KK scale the gravitational response of the vacuum *must* be understood in 6D, even though all non-gravitational physics remains 4D (because it is localized on the brane).

Arguments why a 4D mechanism is necessary

But a more subtle objection[3] asks why a thought experiment cannot be performed that allows the vacuum energy to be understood within a 4D effective theory, even if the cosmological constant were larger than m_{KK}. After

[3]We thank Nima Arkani-Hamed for making this argument to us.

all, one can imagine adiabatically changing the underlying parameters of the model in such a way as to generate an effective 4D cosmological constant, $\mathcal{L}_{\rm eff} = -\sqrt{-g}\; A$, with $A > m_{KK}^4$. If so, because energy cannot be directly extracted from ρ, no consistency issue would preclude us from analyzing the theory in an effective 4D approximation, provided the Hubble scale remains small enough: $H^2 \simeq A/M_p^2 \ll m_{KK}^2$. (If H were to become larger than m_{KK} then sufficient energy could be extracted from the time-dependent geometry to excite KK modes and force us outside of the domain of the effective 4D description.)

In this picture, it seems we are again forced to be able to understand what keeps ρ from being large purely within a 4D context.

But again it is the scale invariance that saves the day. As you adiabatically manipulate the underlying parameters in the 6D theory, what is generated is a potential, $V(\varphi_0)$, for the entire flat direction rather than just a cosmological constant. Since the flat direction partially involves the extra-dimensional metric, general covariance precludes generating just a φ_0-independent constant.

So instead of getting a constant like $\mathcal{L}_{\rm eff} = -\sqrt{-g}\; A$ one instead gets a potential like $\mathcal{L}_{\rm eff} = -\sqrt{-g}\; A\, e^{a\varphi_0}$, where a is order unity. But if A rises above m_{KK}^4, then not only does V rise above m_{KK}^4, but also so does its derivative, V'. Once this is true a 4D description is no longer possible, because the equation of motion for φ_0 implies that having V' this large generates a time derivative $\dot{\varphi}_0$ that is equally large, which provides an energy source that can generate KK modes.

The upshot is that there is no effective 4D understanding of the cosmological constant problem; but this does not mean that no solution exists, it simply means that the KK scale cannot be much larger than the observed Dark Energy density. It also means that the existence of a light field, φ_0, in the low-energy theory is a crucial part of the story, making it unavoidable that there be a scalar-tensor gravity in the long-wavelength limit.