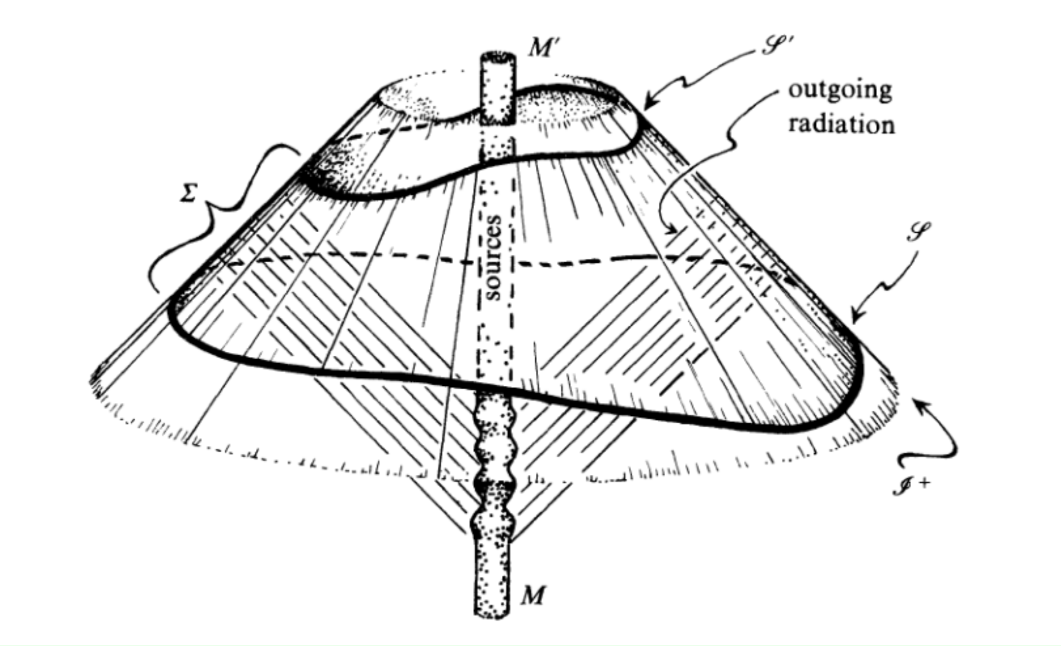
New Applications of Asymptotic Symmetries Involving Maxwell Fields



Pujian Mao

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In this thesis, several new aspects of asymptotic symmetries have been exploited.

Firstly, we have shown that the asymptotic symmetries can be enhanced to symplectic symmetries in three dimensional asymptotically Anti-de Sitter (AdS) space-time with Dirichlet boundary conditions. Such enhancement provides a natural connection between the asymptotic symmetries in the far region (*i.e.* close to the boundary) and the near-horizon region, which leads to a consistent treatment for both cases. The second investigation in three dimensional space-time is to study the Einstein-Maxwell theory including asymptotic symmetries, solution space and surface charges with asymptotically flat boundary conditions at null infinity. This model allows one to illustrate several aspects of the four dimensional case in a simplified setting. Afterwards, we give a parallel analysis of Einstein-Maxwell theory in the asymptotically AdS case.

Another new aspect consists in demonstrating a deep connection between certain asymptotic symmetry and soft theorem. Recently, a remarkable equivalence was found between the Ward identity of certain residual (large) $U(1)$ gauge transformations and the leading piece of the soft photon theorem. It is well known that the soft photon theorem includes also a sub-leading piece. We have proven that the large $U(1)$ gauge transformation responsible for the leading soft factor can also explain the sub-leading one.

In the last part of the thesis, we will investigate the asymptotic symmetries near the inner boundary. As a null hypersurface, the black hole horizon can be considered as an inner boundary. The near horizon symmetries create "soft" degrees of freedom. We have generalised such argument to isolated horizon and have shown that those "soft" degrees of freedom of an isolated horizon are equivalent to its electric multipole moments.

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The picture on the cover is taken from the book *Spinors and space-time* Vol. 2 page 426

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Contents

Chapter 1

Preface

One hundred years after the predictions of the existence of gravitational waves by Einstein [1, 2], it has been directly detected by LIGO [3] recently. This is definitely one of the most exciting discoveries of this year. The existence of gravitational waves was somewhat debatable up until 1960's. The main issue was that it was not clear if the radiation was just an artifact of linearization considering that the non-linearity was the most significant property of Einstein's general relativity. Due to the extreme complexity of the field equations of general relativity, certain methods of approximation are still needed. To achieve this, Bondi, van der Burg, and Metzner established an elegant framework of expansions for axi-symmetric isolated systems [4]. The basic idea of their treatment is that in a suitable coordinates system, the fields can be expanded[1] in inverse powers of a suitably defined radius. Then the equations of motion can be solved order by order when proper boundary conditions are set. Several important non-linear results have been demonstrated in their framework: the radiation is characterized by a single function of two variables called the *news function*; the mass of a system always decreases when there is news. In the same year, Sachs extended this framework to asymptotically flat spacetimes by removing the assumption of axi-symmetry [6]. Apart from an improved understanding of gravitational waves, a side-product of the analysis of [4,6] is fact that the *asymptotic symmetry group* which leaves invariant the boundary conditions for asymptotic flatness is not the Poincaré group. Actually this asymptotic symmetry group (the BMS group) is an infinite dimensional group and contains an infinite dimensional Abelian normal subgroup whose factor group is homogeneous orthochronous Lorentz group [7]. We refer to this Abelian normal subgroup of the BMS group as supertranslations. This terminology comes from the fact that the translations in Minkowski space are elements of this subgroup. Historically the enhancement from translations to supertranslations was a surprise and there were attempts to define consistent conditions to bring the asymptotic symmetry group back to the standard Poincaré group.

[1] The problem of convergence of this expansion was fixed by Friedrich 20 years later [5].

Unlike in special relativity, defining local conserved quantities is a quite subtle issue in general relativity. This is because the usual analysis that the existence of a symmetry group preserving the numerical value of the metric tensor is absent. A more reasonable question is how to make sense of the concept of the system's total mass (energy) and angular momentum. The crucial step is to set suitable boundary condition to have an isolated gravitational system. This was first achieved by Arnowitt, Deser, and Misner (ADM) [8–10] at spatial infinity. In the ADM formalism, energy-momentum and angular momentum of the gravitational system are well-defined because the asymptotic symmetry group at spatial infinity is the Poincaré group (see [11] for a comprehensive review) and the definition of the total charges are associated to those asymptotic symmetries [12, 13]. The obstacle at null infinity is that the asymptotic symmetry group, the BMS group, does not have any physically preferred Poincaré subgroup, but rather an infinite-dimensional family of them. This issue is often interpreted as an ambiguity in defining a satisfactory space of origins with respect to which angular momentum is to be measured [14].

In the past half centory, the BMS group has been studied intensively [15–23]. However, it is still not, we believe, well understood. In contrast, the asymptotic symmetry group of asymptotically Anti-de-Sitter (AdS) spacetime has a clear dual interpretation via the AdS/CFT correspondence [24–26]. The asymptotic symmetry group of the bulk spacetime is the global conformal group of the dual theory living on the boundary. A fascinating example of AdS/CFT arises from three dimensional AdS spacetime. Although three dimensional Einstein gravity admits no local degrees of freedom, it is shown to admit black hole solutions *e.g.* the BTZ black hole [27]. The asymptotic symmetry group of three dimensional AdS spacetime was shown to be the conformal group in two dimensions [28]. Moreover, Brown and Henneaux found a classical central charge in the canonical realization of this asymptotic symmetry algebra which is known as Brown-Henneaux central charge [28]. If one believes that there is a two dimensional CFT dual to three dimensional gravity and identifies the central charge of the CFT with the Brown-Henneaux central charge, the entropy of the CFT derived from the Cardy's formula [29] is in precise numerical agreement with the Bekenstein-Hawking entropy of the BTZ black hole [30].

Recently there has been renewed interest at null infinity of asymptotically flat spacetime. In the case of asymptotic flatness, the S-Matrix is the most important observable. The S-Matrix for massless particles, directly relating the data living on the past null infinity to those on the future null infinity, has recently received renewed attention and much progress has been achieved without starting from a local Lagrangian [31, 32]. The so-called on-shell method introduced in [31, 32] considerably simplifies the tree-level amplitudes of massless particles. In particular, it makes the soft theorem in Yang-Mills theory and gravitational theory transparent [33, 34].

The universal properties of the soft theorems are suggesting that some symmetries might be responsible. Very recently, Strominger and collaborators argued that the S-Matrix of massless particles should have BMS symmetry [35] and have found a deep connection between BMS supertranslations and Weinberg's soft graviton theorem [36]. Since the supertranslations can not preserve the Minkowski vacuum, they are spontaneously broken. It has been shown precisely in [36] that the soft graviton theorem is nothing but the Ward identity of supertranslations and the soft gravitons are just the Goldstone particles of the supertranslations.

Apart from the remarkable perspectives in the understanding of the soft theorem, the degenerate vacuum on a null boundary brings new physical degrees of freedom which are related to the spontaneously broken symmetries *i.e.* the supertranslations. The inequivalent vacua differ from one another by the creation or annihilation of soft gravitons. Interestingly the null infinity is not the only null boundary when a black hole is formed in the bulk spacetime. As a null hypersurface, the black hole horizon can serve as an inner boundary. A straightforward question would be what are the asymptotic symmetries of the near horizon region. Surprisingly, the near horizon symmetries also include a supertranslation part as shown in [37]. Taking into account of the near horizon supertranslations, a highly meaningful question arises that if stationary black holes are nearly bald due to the no-hair theorem [38]. Since the no-hair theorem is a basic assumption in the black hole information paradox issue, the emergence of new symmetries in the near horizon region may shed new insights on the black hole physics. Based on those facts, Hawking, Perry, and Strominger proposed that black holes can carry a large amount of soft hairs which gives the effective soft degrees of freedom and the complete information about the quantum states of those soft hairs are stored on a holographic plate [39]. Moreover, soft hairs have a description as quantum pixels in a holographic plate which lives on a two sphere at the future boundary of the horizon. They further argued that the effective number of soft hairs should be proportional to the area of the horizon in Planck units as it was the case in the string-theoretic black hole [40].

The aim of the present thesis is to exploit new asymptotic symmetries and new applications of asymptotic symmetries both at infinity and in the near horizon region, especially for systems involving Maxwell fields.

Outline of the thesis

This thesis contains four main parts and a few appendices. In part one, we review several methods to perform asymptotic analysis. Those are conformal compactification, Newman-Penrose formalism and metric formalism. The point is to establish a complete framework of physics on the conformal boundary and set up our notations. The original contribution in this part is deriving the surface charge in first order formalism, especially in Newman-Penrose formalism by cohomological techniques [41–43], which allows one to compute the surface charge directly from Newman-Penrose spin coefficients and tetrads. To achieve this, we introduce a Lagrangian multiplier to recast Newman-Penrose formalism in an action principle.

The second part of this thesis consists of three original results in 3 dimensions. Firstly, the asymptotic symmetries of AdS_3 with Dirichlet boundary conditions can be defined everywhere into the bulk space-time. Thus the asymptotic symmetries can be enhanced to sympletic symmetries in the bulk space-time. Such enhancement provides a natural connection between the symmetries in the far region (*i.e.* close to the boundary) and the near-horizon region which leads to a consistent treatment for both cases. The second investigation is to study three-dimensional Einstein-Maxwell theory (including asymptotic symmetries, solution space and surface charges) with asymptotically flat boundary conditions at null infinity. This model allows one to illustrate several aspects of the four dimensional case in a simplified setting. In the end of this part, we give a parallel analysis of Einstein-Maxwell theory in the asymptotically AdS case.

Part three presents a new connection between asymptotic symmetries and soft theorem. In [44], it was found that certain residual (large) gauge transformations are responsible for the leading piece in the soft photon theorem. It was well understood that the soft photon theorem includes a next-to-leading order. We notice that the fundamental ingredient to explain both terms in the soft photon theorem was only gauge invariance. That may lead one to think that the residual large gauge transformations responsible for the leading soft factor can also explain the sub-leading one. This is precisely what we will show in the third part of this thesis.

The last part is devoted to the original investigation of asymptotic symmetries near the inner boundary. As a null hypersurface, the black hole horizon can be considered as an inner boundary. Recently, Hawking, Perry, and Strominger argued that the near horizon symmetries create “soft” degree of freedom on the horizon [39, 45]. We generalize their argument to isolated horizon which is a more realistic resolution of black hole physics. It is further shown that those “soft” degree of freedom of an isolated horizon are equivalent to its electric multipole moments introduced in [46].

All these four parts of the thesis are supplemented by appendices showing details on computation.

During the realization of this thesis, the following research papers have been finished:

1. P. Mao, X. Wu, H. Zhang, “Soft hairs on isolated horizon implanted

by electromagnetic fields," Submitted to journal, `arXiv:1606.03226`.

2. E. Conde and P. Mao, "Comments on Asymptotic Symmetries and the Sub-leading Soft Photon Theorem," Submitted to journal, `arXiv:1605.09731`.

3. G. Compère, P. Mao, A. Seraj, M.M. Sheikh-Jabbari, "Symplectic and Killing Symmetries of AdS_3 Gravity: Holographic vs Boundary Gravitons," JHEP 01 (2016) 080, `arXiv:1511.06079`.

4. G. Barnich, P. -H. Lambert, P. Mao, "Three-dimensional asymptotically flat Einstein-Maxwell theory," Class. Quantum Grav. 32 (2015) 245001, `arXiv:1503.00856`.

Chapter 2

Background material

2.1 Conformal boundary

Originally, infinity is not part of spacetime. However the causal structure of spacetime is unchanged by a conformal transformation:

$$ds^2 \to d\tilde{s}^2 = \Omega^2 ds^2. \tag{2.1}$$

We can choose it in such a way that all points at infinity in the original metric are at finite affine parameter in the new metric. To achieve this, we must choose

$$\Omega \to 0. \tag{2.2}$$

In this case, infinity can be identified as those points for which $\Omega = 0$. These points are not part of the original spacetime but they can be added to it to yield a conformal boundary of spacetime.

Example: Minkowski space [47]

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin\theta^2 d\phi^2). \tag{2.3}$$

Let

$$\left\{ \begin{array}{ccc} u & = & t-r \\ v & = & t+r \end{array} \right\} \to ds^2 = -du\,dv + \frac{(u-v)^2}{4}(d\theta^2 + \sin\theta^2 d\phi^2). \tag{2.4}$$

Now set

$$\left\{ \begin{array}{cccc} u & = & \tan\tilde{U} & -\pi/2 < \tilde{U} < \pi/2 \\ v & = & \tan\tilde{V} & -\pi/2 < \tilde{V} < \pi/2 \end{array} \right\} \begin{array}{l} \text{with } \tilde{V} \geq \tilde{U} \\ \text{since } r \geq 0 \end{array} \tag{2.5}$$

In these coordinates,

$$ds^2 = \left(2\cos\tilde{U}\cos\tilde{V}\right)^{-2}\left[-4d\tilde{U}\,d\tilde{V} + \sin^2\left(\tilde{V} - \tilde{U}\right)(d\theta^2 + \sin\theta^2 d\phi^2)\right] \tag{2.6}$$

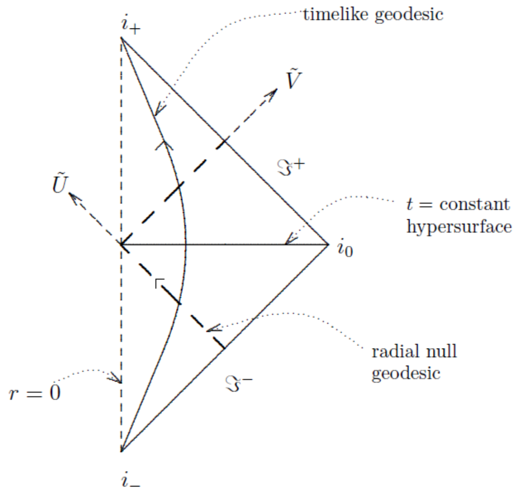

Figure 2.1: Each point represents a 2-sphere, except points on $r = 0$ and $i_0, i_\pm$. Light rays travel at 45^0 from $\Im^-$ through $r = 0$ and then out to $\Im^+$. This picture is taken from [47] page 44.

To approach ∞ in this metric we must take $\left|\tilde{U}\right| \to \pi/2$ or $\left|\tilde{V}\right| \to \pi/2$, so by choosing

$$\Omega = 2\cos\tilde{U}\cos\tilde{V} \tag{2.7}$$

we bring these points to finite affine parameter in the new metric

$$d\tilde{s}^2 = \Omega ds^2 = -4d\tilde{U}d\tilde{V} + \sin^2\left(\tilde{V} - \tilde{U}\right)\left(d\theta^2 + \sin\theta^2 d\phi^2\right) \tag{2.8}$$

We can now add the points at infinity. Taking the restriction $\tilde{V} \geq \tilde{U}$ into account, these are

$$\left.\begin{array}{rcl} \tilde{U} &=& -\pi/2 \\ \tilde{V} &=& \pi/2 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{c} r\to\infty \\ t \text{ finite} \end{array}\right\} \text{spatial infinity, } i_0$$

$$\left.\begin{array}{rcl} \tilde{U} &=& \pm\pi/2 \\ \tilde{V} &=& \pm\pi/2 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{c} t\to\pm\infty \\ r \text{ finite} \end{array}\right\} \begin{array}{l}\text{past and future} \\ \text{temporal infinity, } i_\pm\end{array}$$

$$\left.\begin{array}{rcl} \tilde{U} &=& -\pi/2 \\ |\tilde{V}| &\neq& \pi/2 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{c} r\to\infty \\ t\to-\infty \\ r+t \text{ finite} \end{array}\right\} \begin{array}{l}\text{past null infinity} \\ \Im^-\end{array}$$

$$\left.\begin{array}{rcl} |\tilde{U}| &\neq& \pi/2 \\ \tilde{V} &=& \pi/2 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{c} r\to\infty \\ t\to\infty \\ r-t \text{ finite} \end{array}\right\} \begin{array}{l}\text{future null infinity} \\ \Im^+\end{array}$$

Minkowski spacetime is conformally embedded in the new spacetime with metric $d\tilde{s}^2$ with boundary at $\Omega = 0$. Figure 2.1 is the Carter-Penrose diagram of Minkowski spacetime.

2.2 Asymptotic structure at null infinity

2.2.1 Asymptotic flatness at null infinity

We have introduced the conformal technique in general previously. Null infinity will be analyzed in details here. At first, we give a general definition of asymptote for a generic 4 dimensional manifold M with smooth C^∞ metric of Lorentz signature g_{ab} following [11]. By asymptotes of (M, g_{ab}) we mean a manifold $\tilde{M}$ with boundary I, together with a smooth Lorentz metric $\tilde{g}_{ab}$ on $\tilde{M}$, a smooth function Ω on $\tilde{M}$, and a diffeomorphism from M to $\tilde{M} - I$, satisfying the following conditions:

1) On M, $\tilde{g}_{ab} = \Omega^2 g_{ab}$.

2) At I, $\Omega = 0$, $\tilde{\nabla}_a \Omega \neq 0$, and $\tilde{g}^{ab}\tilde{\nabla}_a\Omega\tilde{\nabla}_b\Omega = 0$,where $\tilde{\nabla}_a$ denotes the gradient on $\tilde{M}$.

This $\tilde{g}_{ab}$ is called the unphysical metric (to distinguish it from the physical metric g_{ab}) , while I is called the boundary at null infinity. Note that the definition requires that the unphysical metric be defined and have Lorentz signature also at points of the boundary.

The definition represents the intuitive idea of “the attachment to the space-time manifold M of additional ideal points at null infinity”. The additional points are of course those of I, while the diffeomorphism inserts M in $\tilde{M}$; thus, M itself represents the physical space-time manifold. The first condition from the definition states that the conformal factor rescales the physical metric to the unphysical one. The first part of the second condition, together with the requirement that the unphysical metric being well-behaved on I, states that “infinity is far away in the physical space-time”. The second part of the second condition fixes the asymptotic behavior of Ω. Effectively, it states that Ω falls to zero as $\frac{1}{r}$. The third part of the second condition states essentially that I is a null hypersurface. Hence, we are working at null infinity. These remarks reflect the intuitive idea of “asymptotic flatness at null infinity”. The conformal boundary of Minkowski space we have discussed in the previous section is absolutely in consistent with this intuitive idea.

Another aspect of this issue as pointed in [11] is the question of whether or not the existence of a boundary at infinity is persistent. In other words, we are wondering if asymptotic flatness is stable against disturbance in the space-time, *e.g.* by emitting gravitational radiation. One would wish it to be true that such a disturbance could not result in destruction of the boundary.

Otherwise, the physically realistic space-time would not admit a non-trivial ($I \neq \emptyset$) asymptote, since such disturbance presumably exists in our world.

However, there is no definite proof to the persistent conditions *e.g.* see [11] for more discussions. Nevertheless, the evidence in favor of the present definition consists of some examples and the fact that the definition leads to quantities of apparent physical interest.

We next consider a somewhat different aspect of the definition. As stressed in [11], the ultimate goal is to describe the asymptotic structure of a physical space-time M, g_{ab} in terms of the local behavior at I of various fields. One would like to have some guarantee that statements about fields near I will actually say something about the physical space-time. What is needed essentially is a proof that, given the physical space-time, its asymptote is in some sense unique, for otherwise statements about I may refer only to the choice of asymptote. However there are indeed two distinct senses in which the asymptote is certainly not unique. Let M, g_{ab} be a space-time, $\tilde{M}$, $\tilde{g}_{ab}$, Ω an asymptote, and ω a smooth positive scalar field on M. Then $\tilde{M}$, $\omega^2\tilde{g}_{ab}$, $\omega\Omega$ is clearly also an asymptote. [Note: ω-factors so chosen that $\Omega^{-2}\tilde{g}_{ab}$ the physical metric, remains the same.] We call two asymptotes related in this way equivalent. Hence, one always has the freedom of an additional conformal transformation. We would refer to such conformal transformation as a gauge transformation from now on. The second non-uniqueness is the following: For C any closed subset of I, $\tilde{M} - C$, $\tilde{g}_{ab}$, Ω (the latter two fields now restricted to $\tilde{M} - C$) is also an asymptote. We call $\tilde{M}$, $\tilde{g}_{ab}$, Ω an extension of this one which means one can always remove part of the boundary.

It turns out that these two are the only ambiguities, at least for sufficiently well-behaved asymptotes. The sense in which equivalence and extension are the only ambiguities in selecting an asymptote is the following [11]:

***Theorem** 2. Let M, g_{ab} be a space-time. Then there exists a regular asymptote, $\tilde{M}$, $\tilde{g}_{ab}$, Ω, unique up to equivalence, which is maximal: Any other regular asymptote of M, g_{ab} is equivalent to one of which $\tilde{M}$ is an extension.*

2.2.2 Symmetries at null infinity

In this section, we will review the results in [11] about the symmetries at null infinity. To investigate the symmetries at null infinity, one needs to introduce the local geometry of null infinity first. Let M, g_{ab} be a space-time, and let $\tilde{M}$, $\tilde{g}_{ab}$, Ω be an asymptote. Denote by ι a diffeomorphic copy of the three-dimensional manifold I, and let ζ: $\iota \to \tilde{M}$ be the corresponding smooth mapping, so ζ sends ι to I diffeomorphically. This manifold ι represents I, detached from $\tilde{M}$; It will be convenient to describe the asymptotic structure in terms of it. We denote by $\zeta^\star$ the pullback: the operator, now to be defined, which carries certain fields from $\tilde{M}$ to ι. Let $n_a = \tilde{\nabla}_a \Omega$ be the normal vector of the null infinity. Then, $\zeta^\star n_a = 0$. Set $\underline{n}^a = \zeta^\star n^a$ and $\underline{g}_{ab} = \zeta^\star \tilde{g}_{ab}$. These two fields on ι essentially describe the universal geometry of this manifold. Furthermore, applying $\zeta^\star$ to $\tilde{g}_{ab} n^b$, one obtains $\underline{g}_{ab}\underline{n}^b = 0$. Thus, $\underline{g}_{ab}$ is not invertible; indeed, it is clear from the fact that I is a null surface and $\underline{g}_{ab}$ has the signature $(0,+,+)$. A final consequence follows from the Einstein's equation. It is convenient to define a combination $S_a^{\ b} = R_a^{\ b} - \dfrac{1}{6} R \delta_a^{\ b}$. Let $\tilde{L}_{ab} = \tilde{g}_{ac} S_b^{\ c}$. The behavior of the Ricci tensor under conformal transformation implies

$$\Omega \tilde{S}_{ab} + \mathcal{L}_{n^a} \tilde{g}_{ab} - f \tilde{g}_{ab} = \Omega^{-1} \tilde{L}_{ab}, \tag{2.9}$$

where $f = \Omega^{-1} n^a n_a$. Let us now suppose that the stress-energy vanishes asymptotically to order two which is a very weak supposition. Then, applying $\zeta^\star$ to (2.9), we obtain $\mathcal{L}_{\underline{n}} \underline{g}_{cd} = \underline{f}\, \underline{g}_{cd}$ where we have used $\underline{f} = \zeta^\star(f)$. Thus, $\underline{n}^a$ is a conformal Killing field for $\underline{g}_{ab}$.

The gauge-function ω on $\tilde{M}$ is represented, in terms of ι, by the (positive) function $\underline{\omega} = \zeta^\star \omega$ on this manifold. Applying $\zeta^\star$ to $\tilde{g}'_{ab} = \omega^2 \tilde{g}_{ab}$ and to $n'^a = \omega^{-1} n^a + \omega^{-2} \Omega \tilde{\nabla}^a \omega$, we obtain

$$\underline{g}'_{ab} = \underline{\omega}^2 \underline{g}_{ab}, \quad \underline{n}'^a = \underline{\omega}^{-1} \underline{n}^a. \tag{2.10}$$

That is to say, $\underline{g}_{ab}$, $\underline{n}^a$ and $\underline{g}'_{ab}$, $\underline{n}'^a$, related by (2.10) for some positive $\underline{\omega}$, are to represent the same geometrical situation. Applying $\zeta^\star$ to $f' = \omega^{-1} f + 2\omega^{-2} \mathcal{L}_n \omega + \omega^{-3} \Omega \tilde{\nabla}^m \omega \tilde{\nabla}_m \omega$ gives

$$\underline{f}' = \underline{\omega}^{-1} \underline{f} + 2 \underline{\omega}^{-2} \mathcal{L}_{\underline{n}} \underline{\omega}. \tag{2.11}$$

Set $\Gamma^{ab}{}_{cd} = \underline{n}^a \underline{n}^b \underline{g}_{cd}$. Then this tensor field $\Gamma^{ab}{}_{cd}$ is gauge-invariant. Its properties are the following: 1) $\Gamma^{ab}{}_{cd} = \Gamma^{(ab)}{}_{(cd)} \neq 0$ and $\Gamma^{a[b}{}_{cd} \Gamma^{e]f}{}_{gh} = 0$, 2) $\Gamma^{am}{}_{cm} = 0$ (ensures that $\underline{n}^a \underline{g}_{ab} = 0$), 3) whenever $w_c v^{[a} \Gamma^{b]c}{}_{de} \neq 0$, $w_a w_b v^c v^d \Gamma^{ab}{}_{cd}$ is positive (signature of $\underline{g}_{ab}$) and 4) whenever $v^{[a} \Gamma^{b]c}{}_{de} = 0$, $\mathcal{L}_v \Gamma^{ab}{}_{cd}$ is a multiple of $\Gamma^{ab}{}_{cd}$ (ensures thet $\underline{n}^a$ is a $\underline{g}_{ab}$-conformal Killing field). Thus, we may regard $\Gamma^{ab}{}_{cd}$ as representing the complete universal structure of ι in a gauge-invariant way. By an asymptotic geometry we shall mean a three-dimensional manifold ι with a tensor field $\Gamma^{ab}{}_{cd}$ satisfying the four properties above.

Set $\underline{f}' = 0$ in (2.11) to obtain $-2\mathcal{L}_n \ln \underline{\omega} = f$. Clearly, there always exists, at least locally, a positive $\underline{\omega}$ satisfying this equation. That is to say, by a gauge transformation we can always arrange locally to have $\mathcal{L}_{\underline{n}} \underline{g}_{cd} = 0$. Furthermore, $\underline{f} = 0$ is preserved by (2.11) when and only when $\mathcal{L}_n \underline{\omega} = 0$, *i. e.*, when and only when $\underline{\omega}$ is constant along the n-integral curves. For ι,

$\Gamma^{ab}{}_{cd}$ an asymptotic geometry, by a decomposition of $\Gamma^{ab}{}_{cd}$ we mean fields $\underline{n}^a$ and $\underline{g}_{ab}$ such that $\Gamma^{ab}{}_{cd} = \underline{n}^a\underline{n}^b\underline{g}_{cd}$ and $\mathcal{L}_{\underline{n}}\underline{g}_{cd} = 0$. What we have shown, then, is that every asymptotic geometry possesses, locally, a decomposition, and that it is unique up to a gauge transformation by $\underline{\omega}$ which is constant along the n-integral curves.

A symmetry on $(\iota,\ \Gamma^{ab}{}_{cd})$ is a diffeomorphism from ι to ι which sends $\Gamma^{ab}{}_{cd}$ to itself. An infinitesimal symmetry on $(\iota,\ \Gamma^{ab}{}_{cd})$ is a vector field ξ^a on ι satisfying $\mathcal{L}_\xi\Gamma^{ab}{}_{cd} = 0$. Under the bracket of vector fields, the infinitesimal symmetries, we denote by χ, have the structure of a Lie algebra. An alternative statement that ξ^a is an infinitesimal symmetry will be

$$\mathcal{L}_\xi\underline{g}_{ab} = 2\kappa\underline{g}_{ab}, \quad \mathcal{L}_\xi\underline{n}^a = -\kappa\underline{n}^a. \tag{2.12}$$

An infinitesimal symmetry ξ^a is called an infinitesimal supertranslation if ξ^a is proportional to $\underline{n}^a$. We denote the set of infinitesimal supertranslations by $\wp$. The terminology is motivated by the fact that the translations in Minkowski space give rise to elements of $\wp$. Let $\xi^a = \alpha\underline{n}^a$. Then $\mathcal{L}_\xi\underline{g}_{ab} = 0$, and $\mathcal{L}_\xi\underline{n}^a = -\mathcal{L}_n\alpha\underline{n}^a$. Thus, $\alpha\underline{n}^a$ is in $\wp$ if and only if $\mathcal{L}_n\alpha = 0$[1]. $\wp$ is a vector subspace of the vector space χ and, an infinite-dimensional subspace. One can further conclude that $\wp$ is even an abelian subalgebra and forms an ideal in Lie algebra of χ. This can be checked by the bracket of an infinitesimal symmetry ξ^a with an element of $\wp$

$$\mathcal{L}_\xi(\alpha\underline{n}^a) = \mathcal{L}_\xi\alpha\underline{n}^a + \alpha\mathcal{L}_\xi\underline{n}^a = (\mathcal{L}_\xi\alpha - \alpha\kappa)\underline{n}^a. \tag{2.13}$$

Now, we have "understood" the $\wp$-part of χ. What remains is to understand the rest of χ. This is accomplished as follows. Since $\wp$ is an ideal in χ, one can form the quotient algebra, $\chi/\wp$. This quotient algebra just represents the "rest" of χ; we wish, therefore, to understand its structure. It turns out that $\chi/\wp$ can be represented explicitly within ι. Fix a decomposition of $\Gamma^{ab}{}_{cd}$, let ξ^a be any infinitesimal symmetry, and set $\xi_a = \underline{g}_{ab}\xi^b$. Then, this ξ_a satisfies

$$\underline{n}^a\xi_a = 0, \quad D_{(a}\xi_{b)} = \kappa\underline{g}_{ab}, \quad \mathcal{L}_{\underline{n}}\xi_a = 0, \tag{2.14}$$

where D is the derivative compatible with $\underline{g}_{ab}$, *i.e.* $D_a\underline{g}_{ab} = 0$. We can further claim, conversely, that any ξ_a satisfying (2.14) is of the form $\underline{g}_{ab}\xi^b$ for some infinitesimal symmetry ξ^b. Let such a ξ_a be given. Then the first equation in (2.14) implies that $\xi_a = \underline{g}_{ab}\eta^b$ for some η^b; set $\xi^a = \eta^a + \alpha\underline{n}^a$. Then the second equation in (2.14) yields $\mathcal{L}_\xi\underline{g}_{ab} = 2\kappa\underline{g}_{ab}$ while the third yields $\mathcal{L}_\xi\underline{n}^a = -\mathcal{L}_{\underline{n}}\alpha\underline{n}^a$. This ξ^a will therefore be an infinitesimal symmetry if and only if α satisfies $\mathcal{L}_{\underline{n}}\alpha = \kappa$. But, we can always find some α satisfying this equation, *i.e.*, we can always find some infinitesimal symmetry ξ^a such

[1] Without choosing the decomposition, a supertranslation will lead to different constraint on α.

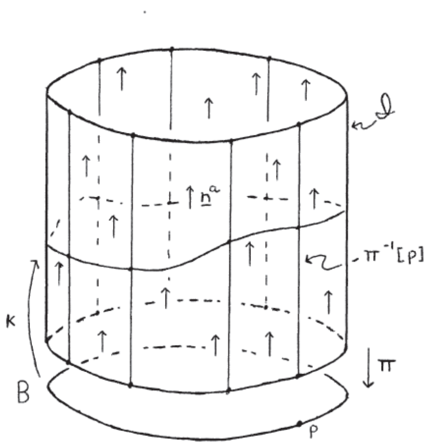

Figure 2.2: The base space is a two-sphere (circle in the figure). The mapping π acts vertically downward; the vertical lines in ι are integral curves of $\underline{n}^a$. This picture is taken from [11] page 27.

that $\xi_a = \underline{g}_{ab}\eta^b$ is our original solution of (2.14). Finally, we note that infinitesimal symmetries ξ^a and ρ^a differ by an infinitesimal supertranslation if and only if $\underline{g}_{ab}\xi^b = \underline{g}_{ab}\rho^b$,*i.e.*, if and only if ξ^a and ρ^a define the same solution of (2.14). Thus, a solution of (2.14) determines an element of χ up to addition of an arbitrary element of $\wp$ *i.e.*, the solutions of (2.14) realize the quotient algebra $\chi/\wp$.

We next introduce the base space B following [11] to have a better interpretation of χ from the geometrical point of view. Let ι, $\Gamma^{ab}{}_{cd}$ be an asymptotic geometry. A maximally extended integral curve τ of $\underline{n}^a$ is said to be almost closed if, for some point p of ι, τ reenters every sufficiently small neighborhood of p. Suppose an asymptotic geometry having no almost-closed integral curves of $\underline{n}^a$. Denote by B the set of all maximally extended integral curves of $\underline{n}^a$, and let π: $\iota \to B$ be the mapping which sends each point of ι to the integral curve on which it lies in B. Now, given an open set U in ι, such that no $\underline{n}$-integral curve passes through U more than once, and such that there are two of the coordinate functions which are constant along the $\underline{n}$-integral curves in U, after projecting these two coordinate functions to B by the mapping π, we obtain a chart in B based on $\pi[U]$. Thus, B becomes a two-dimensional manifold. When B is Hausdorff, we call it the base space of the asymptotic geometry. In this case, the mapping π is smooth, and the manifold ι is just $B \times R$. One can define a cross section of ι by a smooth mapping ε from B to ι, such that $\pi \circ \varepsilon$ is the identity on B. We can consider a cross section of ι represents a "lifting" of B back into ι such that each point p of B is sent to a point of the integral curve in ι which defines p.

With a base space, we are able to introduce the geometrical meaning of χ. Let $\alpha \underline{n}^a$ be a supertranslation on ι. Such α's on ι are precisely those of the form $\alpha = \pi^\star(\beta)$ for some scalar field β on the base space B. Thus, $\wp$ is essentially the same as the set of scalar fields on B. This representation of an element of $\wp$ by a field α depends of course, on the particular choice of decomposition of $\Gamma^{ab}{}_{cd}$. In order to have $\alpha' \underline{n}'^a = \alpha \underline{n}^a$, *i.e.* (2.10), we must set $\alpha' = \underline{\omega} \alpha$. Hence, the set $\wp$ is in fact the same as the set of scalar fields on B with dimension +1. The second equation in (2.14) is then just $\pi^\star$ applied to the conformal Killing equation for any vector μ_a on B with metric h_{ab}. Finally, the solutions of (2.14) are precisely the pullbacks of conformal Killing vectors on B. The Lie algebra $\chi/\wp$, then, is naturally isomorphic to the Lie algebra of conformal Killing fields on the base space B.

All of the remarks above complete the symmetries at null infinity. The Lie algebra of infinitesimal symmetries discussed in this section reproduce the BMS (Bondi-Metzner-Sachs) algebra originally derived in [4, 6, 7] where metric language was used.

2.2.3 Physical fields at null infinity

We have so far studied two notions: that of an asymptote and of an asymptotic geometry. These two merely provide a geometrical framework. The physics itself is to be characterized in terms of certain other fields which arise on ι from the various physical fields in the physical space-time. In this section, we will discuss such fields introduced in [11]. There are of course numerous possibilities, for there are numerous physical fields in general relativity. Rather than attempt to give an exhaustive list, we shall largely restrict consideration to the main one - the gravitational - and one other example - Maxwell.

Let M, g_{ab} be a space-time. By a Maxwell field on M, g_{ab}, we mean an antisymmetric tensor field F_{ab} satisfying

$$\nabla_{[a} F_{bc]} = 0, \quad \nabla_{[a} {}^\star F_{bc]} = 0, \tag{2.15}$$

where $\star$ denotes the dual: ${}^\star F_{ab} = \frac{1}{2} \epsilon_{abcd} F^{cd}$. Since we only focus on physics near the boundary, we may omit sources on the right hand side of (2.15), provided they vanish in a neighborhood of the boundary (or, still more generally, vanish to an appropriate asymptotic order).

Let $\tilde{M}$, $\tilde{g}_{ab}$, Ω be an asymptote of the physical space-time. We will call the Maxwell field asymptotically regular, with respect to this asymptote, if the fields $\tilde{F}_{ab} = F_{ab}$ and ${}^\star \tilde{F}_{ab} = {}^\star F_{ab}$ on M have smooth extension to I on $\tilde{M}$. We refer to F_{ab} and ${}^\star F_{ab}$ as the physical field; to $\tilde{F}_{ab}$ and ${}^\star \tilde{F}_{ab}$ as the unphysical ones.

Having now introduced Maxwell field which contribute to the stress-energy, we may now return to the question of to what asymptotic order that stress-energy should vanish. The stress-energy of Maxwell field is given by

$$T_{ab} = F_a{}^m F_{bm} + {}^\star F_a{}^{m\star} F_{bm}. \tag{2.16}$$

Replacing T_{ab} by the $\tilde{L}_{ab}$, and replacing physical fields everywhere by unphysical ones, we have

$$\tilde{L}_{am} = \Omega^4(\tilde{F}_a{}^b \tilde{F}_{bm} + {}^\star\tilde{F}_a{}^{b\star}\tilde{F}_{bm}), \tag{2.17}$$

with indices now raised and lowered with the unphysical metric. We conclude, therefore, that regular Maxwell field produces stress-energy vanishing asymptotically to order four.

Since the Maxwell field on M has a smooth extension to $\tilde{M}$, we can introduce the corresponding fields on ι given by $\underline{F}_{ab} = \zeta^\star(\tilde{F}_{ab})$ and ${}^\star\underline{F}_{ab} = \zeta^\star({}^\star\tilde{F}_{ab})$ (Note that the two stars on the right in the second equation have different meanings). We first note that, by definition of the dual, ${}^\star\tilde{F}_{am}n^m = \frac{1}{2}\tilde{g}_{am}\epsilon^{mcdq}n_q\tilde{F}_{cd}$. Applying $\zeta^\star$ to this equation, and to the analogous one obtained by interchange of $\tilde{F}_{ab}$ and ${}^\star\tilde{F}_{ab}$, we obtain

$${}^\star\underline{F}_{am}n^m = \frac{1}{2}\underline{g}_{am}\epsilon^{mcd}\underline{F}_{cd}, \quad \underline{F}_{am}n^m = -\frac{1}{2}\underline{g}_{am}\epsilon^{mcd\star}\underline{F}_{cd}. \tag{2.18}$$

(2.18), then, reflects in ι the fact that $\underline{F}_{ab}$ and ${}^\star\underline{F}_{ab}$ begin as mutual dual. Further applying $\zeta^\star$ to Maxwell's equations in terms of the unphysical fields yields

$$D_{[a}\underline{F}_{bc]} = 0, \quad D_{[a}{}^\star\underline{F}_{bc]} = 0. \tag{2.19}$$

Thus, a Maxwell field is described asymptotically by two fields, $\underline{F}_{ab}$ and ${}^\star\underline{F}_{ab}$ on ι, satisfying (2.18) and (2.19).

We turn now to the gravitational field. It turns out that one obtains four objects on ι in the gravitational case: a derivative operator, its curvature tensor, and two other fields. One can consider the derivative operator as the "potential" of the curvature tensor, and the curvature tensor as the potential of the two remaining fields.

Again, let M, g_{ab} be a space-time, and $\tilde{M}$, $\tilde{g}_{ab}$, Ω an asymptote. We start with the following observation. Let μ_b be a covariant vector field on ι. Then $\mu_b = \zeta^\star(\nu_b)$ for some ν_b on $\tilde{M}$, and this ν_b is uniquely determined up to addition of terms of the form $\alpha n_b + \Omega\tau_b$. But, in $\tilde{M}$, we have $\tilde{\nabla}_a(\alpha n_b + \Omega\tau_b) = \tilde{\nabla}_a\alpha n_b + \alpha\tilde{\nabla}_a n_b + n_a\tau_b + \Omega\tilde{\nabla}_a\tau_b$. Now choose the conformal factor such that $f = 0$ on I, accordingly n^a and $\tilde{g}_{ab}$ lead to a decomposition of $\Gamma^{ab}{}_{cd}$. Then $\zeta^\star[\tilde{\nabla}_a(\alpha n_b + \Omega\tau_b)] = 0$. Thus, $\zeta^\star(\tilde{\nabla}_a\nu_b)$ on ι depends only on the original field μ_b on ι. We define this field as $D_a\mu_b$. In this way, we obtain a derivative operator (on covariant vector fields, and hence on all tensor fields) on ι. We have immediately that $D_a\underline{n}^b = 0$ from the fact that $\mathcal{L}_{\underline{n}^a}\underline{g}_{bc} = 0$. Thus, $D_m\Gamma^{ab}{}_{cd} = 0$. This derivative operator is the first object of our gravitational fields.

The second field is obtained from the Einstein equation. Contracting (2.9) with n^b, we have

$$\tilde{S}_{ab}n^b + \tilde{\nabla}_a f = \Omega^{-2}\tilde{L}_{ab}n^b. \qquad (2.20)$$

Let the stress-energy vanish asymptotically to order three, and keep our gauge-choice $f = 0$ on I. Then, at points of I, $\tilde{\nabla}_a f$ is proportional to n_a, whence this gives that $\tilde{S}_a{}^b n_b$ will be proportional to n_a there. Set $\underline{S}_a{}^b = \zeta^\star(\tilde{S}_a{}^b)$. This is the second gravitational field, essentially the pullback of the unphysical Ricci tensor. Since $n^a\tilde{S}_a{}^b$ is a multiple of n^b, we have, applying $\zeta^\star$, that $\underline{n}^a\underline{S}_a{}^b = \sigma\underline{n}^b$ for some σ on ι. Set $\underline{S} = \underline{S}_m{}^m$, and $\underline{S}_{ab} = \underline{S}_a{}^m\underline{g}_{mb}$. Then $\underline{S}_{ab}\underline{n}^b = 0$, and $\underline{g}^{ab}\underline{S}_{ab} = \underline{S} - \sigma$. These are the algebraic properties of $\underline{S}_{ab}$. A differential property follows from (2.20). Taking the curl of (2.9), we obtain

$$\Omega\tilde{\nabla}_{[a}\tilde{S}_{b]c} + n_{[a}\tilde{S}_{b]c} + 2\tilde{\nabla}_{[a}\tilde{\nabla}_{b]}n_c - \tilde{\nabla}_{[a}f\tilde{g}_{b]c} = \tilde{\nabla}_{[a}(\Omega^{-1}\tilde{L}_{b]c}). \qquad (2.21)$$

The third term on the left hand side equals to $\tilde{R}_{abcd}n^d$. Inserting $\tilde{R}_{abcd} = \tilde{C}_{abcd} + \tilde{g}_{a[c}\tilde{S}_{d]b} - \tilde{g}_{b[c}\tilde{S}_{d]a}$, and using (2.20) to eliminate $\tilde{S}_{ab}n^b$ term, we get

$$\Omega\tilde{\nabla}_{[a}\tilde{S}_{b]c} + \tilde{C}_{abcd}n^d = \tilde{\nabla}_{[a}(\Omega^{-1}\tilde{L}_{b]c}) - \Omega^{-2}\tilde{g}_{c[a}\tilde{L}_{b]d}n^d. \qquad (2.22)$$

Contracting the Bianchi identity $\tilde{\nabla}_{[a}\tilde{R}_{bc]de} = 0$ once and eliminating the Ricci tensor by $\tilde{S}_{ab}$, one has

$$\tilde{\nabla}^m\tilde{C}_{abcm} + \tilde{\nabla}_{[a}\tilde{S}_{b]c} = 0. \qquad (2.23)$$

Contracting this again, it will be reduced to

$$\tilde{\nabla}^m\tilde{S}_{am} - \tilde{\nabla}_a\tilde{S}^m{}_m = 0. \qquad (2.24)$$

Eliminating the second term in (2.23) via (2.22), (2.23) can be written as

$$\tilde{\nabla}^m(\Omega^{-1}\tilde{C}_{abcm}) = -\Omega^{-2}\tilde{\nabla}_{[a}(\Omega^{-1}\tilde{L}_{b]c}) + \Omega^{-4}\tilde{g}_{c[a}\tilde{L}_{b]d}n^d. \qquad (2.25)$$

Finally, by inserting $\tilde{R}_{abcd} = \tilde{C}_{abcd} + \tilde{g}_{a[c}\tilde{S}_{d]b} - \tilde{g}_{b[c}\tilde{S}_{d]a}$ to the Bianchi identity, we obtain

$$\tilde{\nabla}_{[a}(\Omega^{-1}\tilde{C}_{bc]}{}^{de}) = 2\Omega^{-2}\delta_{[a}{}^{[d}\tilde{\nabla}_b(\Omega^{-1}\tilde{L}_{c]}{}^{e]}) - 2\Omega^{-4}\delta_{[a}{}^{[d}\delta_b{}^{e]}\tilde{L}_{c]m}n^m. \qquad (2.26)$$

These are the equations needed.

Assume the vanishing of the stress-energy to order four. We have $\tilde{\nabla}_{[a}(\tilde{S}_{b]}{}^c n_c) = \tilde{\nabla}_{[a}(\tilde{S}_{b]}{}^c)n_c + \tilde{S}_{[b}{}^c\tilde{\nabla}_{a]}n_c$. Evaluate on I. Then the left side vanishes by (2.21) and vanishing of the stress-energy to order four, while the second term on the right vanishes by (2.9). Hence, $\tilde{\nabla}_{[a}(\tilde{S}_{b]}{}^c)n_c = 0$. Since $\zeta^\star$ of the contraction of $\tilde{\nabla}_{[a}(\tilde{S}_{b]}{}^c)$ equals the contraction of $\zeta^\star$. But the former vanishes, by (2.24). Hence,

$$D_b(\underline{S}_a{}^b - \underline{S}\delta_a{}^b) = 0. \qquad (2.27)$$

In particular, contracting this equation with $\underline{n}^a$ and using $D_b\underline{n}^a = 0$ and $\underline{n}^a\underline{S}_a{}^b = \sigma\underline{n}^b$, we obtain $\underline{n}^a D_a(\underline{S} - \sigma) = 0$: That is, $\underline{S} - \sigma = \underline{g}^{ab}\underline{S}_{ab}$ is constant along the $\underline{n}$-integral curves.

The last two gravitational fields, as expected, come from the unphysical Weyl tensor. We first have

Theorem 3: *Let M, g_{ab} be a space-time, and $\tilde{M}$, $\tilde{g}_{ab}$, Ω an asymptote, such that the stress-energy vanishes asymptotically to order four, and such that the asymptotic geometry is Minkowskian. Then the unphysical Weyl tensor, $\tilde{C}_{abcd}$ vanishes at I.*

The complete proof is given in [11]. In order to obtain the remaining two gravitational fields, we must restrict consideration to those cases in which the unphysical Weyl tensor vanishes at I. Then, $\Omega^{-1}\tilde{C}_{abcd}$ is smooth up to and including I. Set $K^{ab} = \epsilon^{cmn}\epsilon^{dpq}\zeta^\star(\Omega^{-1}\tilde{C}_{abcd})$ and ${}^\star K^{ab} = \epsilon^{cmn}\epsilon^{dpq}\zeta^\star(\Omega^{-1\star}\tilde{C}_{abcd})$, where ${}^\star\tilde{C}_{abcd} = \frac{1}{2}\epsilon_{abmn}\tilde{g}^{mp}\tilde{g}^{nq}\tilde{C}_{pqcd}$ is the dual of the Weyl tensor. These are the remaining two gravitational fields.

We will derive some properties of K^{ab} and ${}^\star K^{ab}$. Since the Weyl tensor and its dual are trace-free, we have that K^{ab} and ${}^\star K^{ab}$ are also trace-free. Multiplying Ω^{-1} on the dual of Weyl tensor, applying $\zeta^\star$, and expressing the result in terms of the K's, we obtain

$$\underline{g}_{am}K^{mb} = -\epsilon_{amp}\underline{n}^{p\star}K^{mb}, \quad \underline{g}_{am}{}^\star K^{mb} = -\epsilon_{amp}\underline{n}^p K^{mb}. \tag{2.28}$$

These equations are analogous to (2.18) in the electromagnetic case. These are the only algebraic properties. Multiplying (2.22) by Ω^{-1} and applying $\zeta^\star$ leads to

$$D_{[a}\underline{S}_{b]}{}^c = \frac{1}{4}\epsilon_{abm}{}^\star K^{mc}. \tag{2.29}$$

The trace of this equation again gives (2.27). The final two differential equations come from (2.25) and (2.26). Again, we suppose that the stress-energy vanishes asymptotically to order four, and let $\tilde{L}_{ab} = \Omega^4\tilde{L}_{0ab}$, with $\tilde{L}_{0ab}$ finite on I. Then (2.26) can be written as

$$\tilde{\nabla}_{[a}(\Omega^{-1}\tilde{C}_{bc]de}) = 2\Omega\tilde{g}_{d[a}\tilde{\nabla}_b\tilde{L}_{0c]e} - 2\tilde{g}_{d[a}\tilde{g}_{b|e|]}\tilde{L}_{0c]m}n^m + 6\tilde{g}_{d[a}n_b\tilde{L}_{0c]e}, \tag{2.30}$$

where antisymmetrization over "de" is to be applied on the right. Now apply $\zeta^\star$ to this equation. The first two terms on the right give zero. Setting $\underline{L}_{ab} = \zeta^\star(\tilde{L}_{0ab})$, we then obtain

$$D_m K^{am} = -4\underline{n}^a(\underline{L}_{mn}\underline{n}^m\underline{n}^n). \tag{2.31}$$

Proceeding in the same way on (2.25), we obtain

$$D_m{}^\star K^{am} = 0. \tag{2.32}$$

Effectively, the asymptotic stress-energy, interpreted as the field $\underline{L}_{ab}$ on ι, is a source for the K's. (2.31) and (2.32) are the gravitational analogue of the electromagnetic wave equations (2.19).

There is, in fact, one more differential equation in this system which is a consequence of the vanishing of the unphysical Weyl tensor on I. For any k_c in the unphysical space-time, we have $\tilde{\nabla}_{[a}\tilde{\nabla}_{b]}k_c = \frac{1}{2}\tilde{R}_{abc}{}^{d}k_d$. Substituting the curvature tensor by the Weyl tensor, applying $\zeta^\star$, and using the fact that $\tilde{C}_{abcd}$ is vanishing on I, one gets

$$D_{[a}D_{b]}\underline{k}_c = \frac{1}{2}(\underline{g}_{c[a}\underline{S}_{b]}{}^{d} + \underline{S}_{c[a}\delta_{b]}{}^{d})\underline{k}_d, \tag{2.33}$$

where $\underline{k}_c = \zeta^\star(k_c)$. This equation holds for all fields $\underline{k}_c$ on ι. Hence, we recover the curvature tensor $\mathcal{R}_{abc}{}^{d}$ of the derivative operator D_a on ι, which is the tensor field in parentheses on the right hand side of (2.33).

2.3 Newman-Penrose formalism

The Newman-Penrose formalism [48] is a tetrad system with a null base $l, n, m, \overline{m}$ satisfying the orthogonality conditions $l \cdot m = l \cdot \overline{m} = n \cdot m = n \cdot \overline{m} = 0$ and the normalization conditions $l \cdot n = -m \cdot \overline{m} = 1$. The various Ricci rotation-coefficients, now called the spin coefficients, are designated by special symbols as following

$$\begin{aligned}
\kappa &= \omega_{311} = l^\nu m^\mu \nabla_\nu l_\mu, \quad \pi = -\omega_{421} = -l^\nu \bar{m}^\mu \nabla_\nu n_\mu, \\
\epsilon &= \frac{1}{2}(\omega_{211} - \omega_{431}) = \frac{1}{2}(l^\nu n^\mu \nabla_\nu l_\mu - l^\nu \bar{m}^\mu \nabla_\nu m_\mu), \\
\tau &= \omega_{312} = n^\nu m^\mu \nabla_\nu l_\mu, \quad \nu = -\omega_{422} = -n^\nu \bar{m}^\mu \nabla_\nu n_\mu, \\
\gamma &= \frac{1}{2}(\omega_{212} - \omega_{432}) = \frac{1}{2}(n^\nu n^\mu \nabla_\nu l_\mu - n^\nu \bar{m}^\mu \nabla_\nu m_\mu), \\
\sigma &= \omega_{313} = m^\nu m^\mu \nabla_\nu l_\mu, \quad \mu = -\omega_{423} = -m^\nu \bar{m}^\mu \nabla_\nu n_\mu, \\
\beta &= \frac{1}{2}(\omega_{213} - \omega_{433}) = \frac{1}{2}(m^\nu n^\mu \nabla_\nu l_\mu - m^\nu \bar{m}^\mu \nabla_\nu m_\mu), \\
\rho &= \omega_{314} = \bar{m}^\nu m^\mu \nabla_\nu l_\mu, \quad \lambda = -\omega_{424} = -\bar{m}^\nu \bar{m}^\mu \nabla_\nu n_\mu, \\
\epsilon &= \frac{1}{2}(\omega_{214} - \omega_{434}) = \frac{1}{2}(\bar{m}^\nu n^\mu \nabla_\nu l_\mu - \bar{m}^\nu \bar{m}^\mu \nabla_\nu m_\mu),
\end{aligned}$$

In the Newman-Penrose formalism, derivative operators D, Δ, δ are defined as $l^\mu\partial_\mu$, $n^\mu\partial_\mu$, $m^\mu\partial_\mu$ respectively. The ten independent components of the Weyl tensor are represented by five complex scalars,

$$\begin{aligned}
\Psi_0 &= -C_{abcd}l^a m^b l^c m^d, \\
\Psi_1 &= -C_{abcd}l^a n^b l^c m^d, \\
\Psi_2 &= -C_{abcd}l^a m^b \overline{m}^c n^d, \\
\Psi_3 &= -C_{abcd}l^a n^b \overline{m}^c n^d, \\
\Psi_4 &= -C_{abcd}n^a m^b n^c \overline{m}^d.
\end{aligned} \tag{2.34}$$

The Weyl tensor has the following form

$$
\begin{aligned}
C_{abcd} \quad = \quad & -\Psi_0\{n_a\overline{m}_b n_c\overline{m}_d\} - \Psi_1[\{l_a n_b n_c\overline{m}_d\} + \{n_a\overline{m}_b\overline{m}_c m_d\}] \\
& +\Psi_2[\{l_a m_b n_c\overline{m}_d\} + \{l_a n_b m_c\overline{m}_d\} - \{l_a n_b l_c n_d\} - \{m_a\overline{m}_b m_c\overline{m}_d\}] \\
& +\Psi_3[\{l_a n_b l_c m_d\} - \{l_a m_b m_c\overline{m}_d\}] - \Psi_4\{l_a m_b l_c m_d\} \\
& +\text{complex conjugates,} \qquad (2.35)
\end{aligned}
$$

where $\{abcd\}$ denotes

$$
\{abcd\} = abcd - abdc - bacd + badc + cdab - cdba - dcab + dcba.
$$

Finally the ten components of the Ricci tensor are defined in terms of four real and three complex scalars:

$$
\begin{aligned}
&\Phi_{00} = -\frac{1}{2}R_{11}, \;\; \Phi_{22} = -\frac{1}{2}R_{22}, \;\; \Phi_{02} = -\frac{1}{2}R_{33}, \;\; \Phi_{20} = -\frac{1}{2}R_{44}, \\
&\Phi_{11} = -\frac{1}{4}(R_{12} + R_{34}), \;\; \Phi_{01} = -\frac{1}{2}R_{13}, \;\; , \Phi_{12} = -\frac{1}{2}R_{23} \\
&\Lambda = \frac{1}{24}R = \frac{1}{12}(R_{12} - R_{34}), \;\; \Phi_{10} = -\frac{1}{2}R_{14}, \;\; \Phi_{21} = -\frac{1}{2}R_{24},
\end{aligned}
$$

while Λ is the cosmological constant.

When Maxwell field is coupled, the antisymmetric Maxwell-tensor is replaced by the three complex scalars

$$
\begin{aligned}
\phi_0 &= F_{ab}l^a m^b, \\
\phi_1 &= \frac{1}{2}F_{ab}(l^a n^b + \overline{m}^a m^b), \qquad (2.36) \\
\phi_2 &= F_{ab}\overline{m}^a n^b.
\end{aligned}
$$

Maxwell-tensor will be represented by

$$
\begin{aligned}
F_{\mu\nu} \quad = \quad & \phi_0[\overline{m}_\mu n_\nu - n_\mu\overline{m}_\nu] + \phi_1[n_\mu l_\nu - l_\mu n_\nu + m_\mu\overline{m}_\nu - \overline{m}_\mu m_\nu] \\
& +\phi_2[l_\mu m_\nu - m_\mu l_\nu] + \text{complex conjugates.} \qquad (2.37)
\end{aligned}
$$

The full Newman-Penrose equations for Einstein-Maxwell theory are listed as following:

- **Hypersurface equations**

$$
\begin{aligned}
\delta\rho-\bar{\delta}\sigma &= \rho(\bar{\alpha}+\beta)-\sigma(3\alpha-\bar{\beta})+(\rho-\bar{\rho})\tau+(\mu-\bar{\mu})\kappa-\Psi_1+\Phi_{01},\\
\delta\alpha-\bar{\delta}\beta &= (\mu\rho-\lambda\sigma)+\alpha\bar{\alpha}+\beta\bar{\beta}-2\alpha\beta+\gamma(\rho-\bar{\rho})+\epsilon(\mu-\bar{\mu})-\Psi_2\\
&\quad+\Phi_{11}+\Lambda,\\
\delta\lambda-\bar{\delta}\mu &= (\rho-\bar{\rho})\nu+(\mu-\bar{\mu})\pi+\mu(\alpha+\bar{\beta})+\lambda(\bar{\alpha}-3\beta)-\Psi_3+\Phi_{21},\\
\Delta\lambda-\bar{\delta}\nu &= -(\mu+\bar{\mu})\lambda-(3\gamma-\bar{\gamma})\lambda+(3\alpha+\bar{\beta}+\pi-\bar{\tau})\nu-\Psi_4, \quad (2.38)\\
\Delta\rho-\bar{\delta}\tau &= -(\rho\bar{\mu}+\sigma\lambda)+(\bar{\beta}-\alpha-\bar{\tau})\tau+(\gamma+\bar{\gamma})\rho+\nu\kappa-\Psi_2-2\Lambda,\\
\Delta\alpha-\bar{\delta}\gamma &= (\rho+\epsilon)\nu-(\tau+\beta)\lambda+(\bar{\gamma}-\bar{\mu})\alpha+(\bar{\beta}-\bar{\tau})\gamma-\Psi_3,\\
\delta\nu-\Delta\mu &= (\mu^2+\lambda\bar{\lambda})+(\gamma+\bar{\gamma})\mu-\bar{\nu}\pi+(\tau-3\beta-\bar{\alpha})\nu+\Phi_{22},\\
\delta\gamma-\Delta\beta &= (\tau-\beta-\bar{\alpha})\gamma+\mu\tau-\sigma\nu-\epsilon\bar{\nu}-\beta(\gamma-\bar{\gamma}-\mu)+\alpha\bar{\lambda}+\Phi_{12},\\
\delta\tau-\Delta\sigma &= (\sigma\mu+\rho\bar{\lambda})+(\tau+\beta-\bar{\alpha})\tau-(3\gamma-\bar{\gamma})\sigma-\kappa\bar{\nu}+\Phi_{02},\\
\delta\Delta-\Delta\delta &= -\nu D+(\tau-\bar{\alpha}-\beta)\Delta+\bar{\lambda}\bar{\delta}+(\mu-\gamma+\bar{\gamma})\delta,\\
\bar{\delta}\delta-\delta\bar{\delta} &= (\bar{\mu}-\mu)D+(\bar{\rho}-\rho)\Delta+(\beta-\bar{\alpha})\bar{\delta}+(\alpha-\bar{\beta})\delta,
\end{aligned}
$$

- **Radial equations**

$$
\begin{aligned}
D\rho-\bar{\delta}\kappa &= (\rho^2+\sigma\bar{\sigma})+(\epsilon+\bar{\epsilon})\rho-\bar{\kappa}\tau-\kappa(3\alpha+\bar{\beta}-\pi)+\Phi_{00},\\
D\sigma-\delta\kappa &= (\rho+\bar{\rho})\sigma+(3\epsilon-\bar{\epsilon})\sigma-\kappa(\tau+3\beta+\bar{\alpha}-\bar{\pi})+\Psi_0,\\
D\tau-\Delta\kappa &= (\tau+\bar{\pi})\rho+(\pi+\bar{\tau})\sigma+(\epsilon-\bar{\epsilon})\tau-(3\gamma+\bar{\gamma})\kappa+\Psi_1+\Phi_{01},\\
D\alpha-\bar{\delta}\epsilon &= (\rho+\bar{\epsilon}-2\epsilon)\alpha+\beta\bar{\sigma}-\bar{\beta}\epsilon-\kappa\lambda-\bar{\kappa}\gamma+(\epsilon+\rho)\pi+\Phi_{10},\\
D\beta-\delta\epsilon &= (\alpha+\pi)\sigma+(\bar{\rho}-\bar{\epsilon})\beta-(\mu+\gamma)\kappa-(\bar{\alpha}-\bar{\pi})\epsilon+\Psi_1, \quad (2.39)\\
D\gamma-\Delta\epsilon &= (\tau+\bar{\pi})\alpha+(\pi+\bar{\tau})\beta-(\epsilon+\bar{\epsilon})\gamma-(\gamma+\bar{\gamma})\epsilon+\tau\pi-\nu\kappa\\
&\quad+\Psi_2+\Phi_{11}-\Lambda,\\
D\lambda-\bar{\delta}\pi &= (\rho\lambda+\bar{\sigma}\mu)+\pi^2+(\alpha-\bar{\beta})\pi-\nu\bar{\kappa}-(3\epsilon-\bar{\epsilon})\lambda+\Phi_{20},\\
D\mu-\delta\pi &= (\bar{\rho}\mu+\sigma\lambda)+\pi\bar{\pi}-\mu(\epsilon+\bar{\epsilon})-\pi(\bar{\alpha}-\beta)-\nu\kappa+\Psi_2+2\Lambda,\\
D\nu-\Delta\pi &= (\pi+\bar{\tau})\mu+(\tau+\bar{\pi})\lambda+(\gamma-\bar{\gamma})\pi-(3\epsilon+\bar{\epsilon})\nu+\Psi_3+\Phi_{21},\\
\Delta D-D\Delta &= (\gamma+\bar{\gamma})D+(\epsilon+\bar{\epsilon})\Delta-(\tau+\bar{\pi})\bar{\delta}-(\bar{\tau}+\pi)\delta,\\
\delta D-D\delta &= (\beta+\bar{\alpha}-\bar{\pi})D+\kappa\Delta-\sigma\bar{\delta}-(\bar{\rho}+\epsilon-\bar{\epsilon})\delta,
\end{aligned}
$$

- **Bianchi identities**

$$
\begin{aligned}
&D\Psi_1-\bar{\delta}\Psi_0 = -3\kappa\Psi_2+(2\epsilon+4\rho)\Psi_1+(\pi-4\alpha)\Psi_0+D\Phi_{01}-\delta\Phi_{00}\\
&-2(\epsilon+\bar{\rho})\Phi_{01}-2\sigma\Phi_{10}+2\kappa\Phi_{11}+\bar{\kappa}\Phi_{02}-(\bar{\pi}-2\bar{\alpha}-2\beta)\Phi_{00}, \quad (2.40)\\
&D\Psi_2-\bar{\delta}\Psi_1 = -2\kappa\Psi_3+3\rho\Psi_2+(2\pi-2\alpha)\Psi_1-\lambda\Psi_0+\bar{\delta}\Phi_{01}-\Delta\Phi_{00}-2\tau\Phi_{10}\\
&-2(\alpha+\bar{\tau})\Phi_{01}+2\rho\Phi_{11}+\bar{\sigma}\Phi_{02}-(\bar{\mu}-2\gamma-2\bar{\gamma})\Phi_{00}-2D\Lambda, \quad (2.41)\\
&D\Psi_3-\bar{\delta}\Psi_2 = -\kappa\Psi_4+(2\rho-2\epsilon)\Psi_3+3\pi\Psi_2-2\lambda\Psi_1+D\Phi_{21}-\delta\Phi_{20}\\
&-2(\bar{\rho}-\epsilon)\Phi_{21}+2\mu\Phi_{10}-2\pi\Phi_{11}+\bar{\kappa}\Phi_{22}+(2\bar{\alpha}-2\beta-\bar{\pi})\Phi_{20}+2\bar{\delta}\Lambda, \quad (2.42)
\end{aligned}
$$

$$D\Psi_4 - \bar{\delta}\Psi_3 = (\rho - 4\epsilon)\Psi_4 + (4\pi + 2\alpha)\Psi_3 - 3\lambda\Psi_2 - \Delta\Phi_{20} + \bar{\delta}\Phi_{21}$$
$$+2(\alpha - \bar{\tau})\Phi_{21} + 2\nu\Phi_{10} - 2\lambda\Phi_{11} + \bar{\sigma}\Phi_{22} - (\bar{\mu} + 2\gamma - 2\bar{\gamma})\Phi_{20}, \quad (2.43)$$
$$\Delta\Psi_0 - \delta\Psi_1 = (4\gamma - \mu)\Psi_0 - (4\tau + 2\beta)\Psi_1 + 3\sigma\Psi_2 - D\Phi_{02} + \delta\Phi_{01}$$
$$+2(\bar{\pi} - \beta)\Phi_{01} - 2\kappa\Phi_{12} - \bar{\lambda}\Phi_{00} + 2\sigma\Phi_{11} + (\bar{\rho} + 2\epsilon - 2\bar{\epsilon})\Phi_{02}, \quad (2.44)$$
$$\Delta\Psi_1 - \delta\Psi_2 = \nu\Psi_0 + (2\gamma - 2\mu)\Psi_1 - 3\tau\Psi_2 + 2\sigma\Psi_3 + \Delta\Phi_{01} - \bar{\delta}\Phi_{02}$$
$$+2(\bar{\mu} - \gamma)\Phi_{01} - 2\rho\Phi_{12} - \bar{\nu}\Phi_{00} + 2\tau\Phi_{11} + (\bar{\tau} + 2\alpha - 2\bar{\beta})\Phi_{02} + 2\delta\Lambda, \quad (2.45)$$
$$\Delta\Psi_2 - \delta\Psi_3 = 2\nu\Psi_1 - 3\mu\Psi_2 + (2\beta - 2\tau)\Psi_3 + \sigma\Psi_4 - D\Phi_{22} + \delta\Phi_{21}$$
$$+2(\bar{\pi} + \beta)\Phi_{21} - 2\mu\Phi_{11} - \bar{\lambda}\Phi_{20} + 2\pi\Phi_{12} + (\bar{\rho} - 2\epsilon - 2\bar{\epsilon})\Phi_{22} - 2\Delta\Lambda, \quad (2.46)$$
$$\Delta\Psi_3 - \delta\Psi_4 = 3\nu\Psi_2 - (2\gamma + 4\mu)\Psi_3 + (4\beta - \tau)\Psi_4 + \Delta\Phi_{21} - \bar{\delta}\Phi_{22}$$
$$+2(\bar{\mu} + \gamma)\Phi_{21} - 2\nu\Phi_{11} - \bar{\nu}\Phi_{20} + 2\lambda\Phi_{12} + (\bar{\tau} - 2\alpha - 2\bar{\beta})\Phi_{22}. \quad (2.47)$$

- **Maxwell equations**

$$\Phi_{ab} = \phi_a\bar{\phi}_b, \quad (2.48)$$
$$\delta\phi_1 - \Delta\phi_0 = (\mu - 2\gamma)\phi_0 + 2\tau\phi_1 - \sigma\phi_2, \quad (2.49)$$
$$\delta\phi_2 - \Delta\phi_1 = -\nu\phi_0 + 2\mu\phi_1 + (\tau - 2\beta)\phi_2, \quad (2.50)$$
$$D\phi_1 - \bar{\delta}\phi_0 = (\pi - 2\alpha)\phi_0 + 2\rho\phi_1 - \kappa\phi_2, \quad (2.51)$$
$$D\phi_2 - \bar{\delta}\phi_1 = -\lambda\phi_0 + 2\pi\phi_1 + (\rho - 2\epsilon)\phi_2. \quad (2.52)$$

The standard Newman-Penrose prescription can always make the following choice:

$$\kappa = \pi = \epsilon = 0, \quad \rho = \bar{\rho}, \quad \tau = \bar{\alpha} + \beta.$$

The geometrical interpretation of such disposal is that l-vector forms a congruence of null geodesics with affine parameter and all the rest basis vectors $n, m, \bar{m}$ will be parallely propagated along l. Moreover, the congruence of the null geodesics will be hyper-surface orthogonal and l will be equal to the gradient of a scalar field. Thus let us choose a Bondi-like coordinate $(u, r, z, \bar{z})$ with $l = du$. This gives the tetrad system the following ansatz

$$l^\mu = [0, 1, 0, 0], \quad n^\mu = [1, U, X^A], \quad m^\mu = [0, \omega, L^A].$$
$$l_\mu = [1, 0, 0, 0], \quad n_\mu = [-U - X^A(\bar{\omega}L_A + \omega\bar{L}_A), 1, \omega\bar{L}_A + \bar{\omega}L_A],$$
$$m_\mu = [-X^A L_A, 0, L_A],$$

where $L^A L_A = 0, L^A\bar{L}_A = -1$.

We will focus on the pure gravity case with the constraint made above in this section. E. T. Newman and T. W. J. Unti have shown the general solutions of Newman-Penrose equations in [49] with asymptotically flat boundary condition. We present the solutions adapted to our convention as follows:

$$
\begin{aligned}
&\Psi_0 = \frac{\Psi_0^0}{r^5} + O(r^{-6}),\\
&\Psi_1 = \frac{\Psi_1^0}{r^4} - \frac{\bar{\eth}\Psi_0^0}{r^5} + O(r^{-6}),\\
&\Psi_2 = \frac{\Psi_2^0}{r^3} - \frac{\bar{\eth}\Psi_1^0}{r^4} + O(r^{-5}),\\
&\Psi_3 = \frac{\Psi_3^0}{r^2} - \frac{\bar{\eth}\Psi_2^0}{r^3} + O(r^{-4}),\\
&\Psi_4 = \frac{\Psi_4^0}{r} - \frac{\bar{\eth}\Psi_3^0}{r^2} + O(r^{-3}),\\
&\rho = -\frac{1}{r} - \frac{\sigma^0\overline{\sigma}^0}{r^3} + O(r^{-5}),\ \ \sigma = \frac{\sigma^0}{r^2} + O(r^{-4}),\ \ \tau = -\frac{\Psi_1^0}{r^3} + O(r^{-4}),\\
&\alpha = \frac{\alpha^0}{r} + \frac{\overline{\sigma}^0\overline{\alpha}^0}{r^2} + \frac{\sigma^0\overline{\sigma}^0\alpha^0}{r^3} + O(r^{-4}),\ \ \alpha^0 = \frac{1}{2}\bar{P}\partial\ln P,\\
&\beta = -\frac{\overline{\alpha}^0}{r} - \frac{\sigma^0\alpha^0}{r^2} - \frac{\sigma^0\overline{\sigma}^0\overline{\alpha}^0 + \frac{1}{2}\Psi_1^0}{r^3} + O(r^{-4}),\\
&\mu = \frac{\mu^0}{r} - \frac{\sigma^0\lambda^0 + \Psi_2^0}{r^2} + O(r^{-3}),\ \ \mu^0 = -\frac{1}{2}P\bar{P}\partial\bar{\partial}\ln P\bar{P}\\
&\lambda = \frac{\lambda^0}{r} - \frac{\overline{\sigma}^0\mu^0}{r^2} + O(r^{-3}),\ \ \lambda^0 = \dot{\overline{\sigma}}^0 + \overline{\sigma}^0(3\gamma^0 - \overline{\gamma}^0),\\
&\gamma = \gamma^0 - \frac{\Psi_2^0}{r^2} + O(r^{-3}),\ \ \gamma^0 = -\frac{1}{2}\partial_u\ln\bar{P},\\
&\nu = \nu^0 - \frac{\Psi_3^0}{r} + \frac{\bar{\eth}\Psi_2^0}{r^2} + O(r^{-3}),\ \ \nu^0 = \bar{\eth}(\gamma^0 + \overline{\gamma}^0) \qquad (2.53)\\
&X^A = O(r^{-3}),\ \ \omega = \frac{\bar{\eth}\sigma^0}{r} - \frac{\sigma^0\eth\overline{\sigma}^0 + \frac{1}{2}\Psi_2^0}{r^2} + O(r^{-3}),\\
&U = -r(\gamma^0 + \overline{\gamma}^0) + \mu^0 - \frac{\Psi_2^0 + \overline{\Psi}_2^0}{2r} + O(r^{-2}),\\
&L^z = -\frac{\sigma^0\bar{P}}{r^2} + O(r^{-4}),\ \ L^{\bar{z}} = \frac{P}{r} + \frac{\sigma^0\overline{\sigma}^0 P}{r^3} + O(r^{-4}),\\
&L_z = -\frac{r}{\bar{P}} + O(r^{-2}),\ \ L_{\bar{z}} = -\frac{\sigma^0}{P} + O(r^{-2}),\\
&\Psi_3^0 = \bar{\eth}\mu^0 - \eth\lambda^0,\ \ \Psi_4^0 = \bar{\eth}\nu^0 - \partial_u\lambda^0 - 4\gamma^0\lambda^0\\
&\Psi_2^0 - \overline{\Psi}_2^0 = \bar{\eth}^2\sigma^0 - \eth^2\overline{\sigma}^0 + \overline{\sigma}^0\overline{\lambda}^0 - \sigma^0\lambda^0\\
&\partial_u\Psi_0^0 + (\gamma^0 + 5\overline{\gamma}^0)\Psi_0^0 = \eth\Psi_1^0 + 3\sigma^0\Psi_3^0\\
&\partial_u\Psi_1^0 + 2(\gamma^0 + 2\overline{\gamma}^0)\Psi_1^0 = \eth\Psi_2^0 + 2\sigma^0\Psi_3^0\\
&\partial_u\Psi_2^0 + 3(\gamma^0 + \overline{\gamma}^0)\Psi_2^0 = \eth\Psi_3^0 + \sigma^0\Psi_4^0\\
&\partial_u\Psi_3^0 + 2(2\gamma^0 + \overline{\gamma}^0)\Psi_3^0 = \eth\Psi_4^0\\
&\partial_u\mu^0 = -2(\gamma^0 + \overline{\gamma}^0)\mu^0 + \bar{\eth}\eth(\gamma^0 + \overline{\gamma}^0),\ \ \partial_u\alpha^0 = -2\gamma^0\alpha^0 - \bar{\eth}\overline{\gamma}^0.
\end{aligned}
$$

Table 2.1: Spin and conformal weights

	$\eth$	∂_u	γ^0	ν^0	μ^0	σ^0	λ^0	Ψ_4^0	Ψ_3^0	Ψ_2^0	Ψ_1^0	Ψ_0^0	$\mathcal{Y}$
s	1	0	0	−1	0	2	−2	−2	−1	0	1	2	−1
w	−1	−1	−1	−2	−2	−1	−2	−3	−3	−3	−3	−3	1

The "eth" operator is given by

$$\eth\eta = P\bar{P}^{-s}\bar{\partial}(\bar{P}^s\eta) = P\bar{\partial}\eta^s + sP\bar{\partial}\ln\bar{P}\eta = P\bar{\partial}\eta + 2s\overline{\alpha}^0\eta,$$
$$\bar{\eth}\eta = \bar{P}P^s\partial(P^{-s}\eta) = \bar{P}\partial\eta^s - s\bar{P}\partial\ln P\eta = \bar{P}\partial\eta - 2s\alpha^0\eta.$$

where s is the spin weights of the field η.

Szekeres gave an interpretation of the different Weyl scalars at large distances in [50]: Ψ_2^0 is a "Coulomb" term, representing the gravitational monopole of the source; Ψ_1^0 and Ψ_3^0 are ingoing and outgoing "longitudinal" radiation terms; Ψ_0^0 and Ψ_4^0 are ingoing and outgoing "transverse" radiation terms. This can be understood as a translation in Newman-Penrose formalism of the physical fields at null infinity in the previous section. The electromagnetic analogue will be: ϕ_1^0 is a "Coulomb" term, representing the electromagnetic monopole of the source; ϕ_0^0 and ϕ_2^0 are ingoing and outgoing radiation terms.

As a first order formalism, the system has both diffeomorphism and local Lorentz rotation invariant. The infinitesimal transformation on the tetrad vectors and spin coefficients are given by

$$\delta e_a^\mu = \xi^\rho\partial_\rho e_a^\mu - e_a^\rho\partial_\rho\xi^\mu - \Lambda_a^{\ b}e_b^\mu, \tag{2.54}$$
$$\delta\omega_{abc} = \xi^\rho\partial_\rho\omega_{abc} + e_c^\mu\partial_\mu\Lambda_{ab} - \omega_{dbc}\Lambda_a^{\ d} - \omega_{adc}\Lambda_b^{\ d} - \omega_{abd}\Lambda_c^{\ d}, \tag{2.55}$$

where ξ^μ is a spacetime vector generating the infinitesimal diffeomorphism transformation while $\Lambda_a^{\ b}$'s are the components of Lorentz group elements.

The transformations preserving this solution space are specified by

$$\begin{aligned}
&\xi^u = f,\ \ \partial_u f = \frac{1}{2}(\eth\mathcal{Y} + \overline{\eth\mathcal{Y}}) + f(\gamma^0 + \overline{\gamma}^0) + \frac{1}{2}(\Omega + \bar{\Omega}),\\
&\xi^z = Y - \frac{\bar{P}\eth f}{r} + \frac{\sigma^0\bar{P}\bar{\eth}f}{r^2} + O(r^{-3}),\ \ \xi^{\bar{z}} = \bar{Y} - \frac{P\bar{\eth}f}{r} + \frac{\overline{\sigma}^0 P\eth f}{r^2} + O(r^{-3}),\\
&\xi^r = -r\partial_u f + \frac{1}{2}\bar{\Delta}f - \frac{\bar{\eth}\sigma^0\bar{\eth}f + \eth\overline{\sigma}^0\eth f}{r} + O(r^{-2}),\\
&\Lambda^{21} = \partial_u f + O(r^{-3}),\\
&\Lambda^{32} = \frac{\bar{\eth}f}{r} - \frac{\overline{\sigma}^0\eth f}{r^2} + \frac{\sigma^0\overline{\sigma}^0\bar{\eth}f}{r^3} + O(r^{-4}),\\
&\Lambda^{42} = \frac{\eth f}{r} - \frac{\sigma^0\bar{\eth}f}{r^2} + \frac{\sigma^0\overline{\sigma}^0\eth f}{r^3} + O(r^{-4}),
\end{aligned}$$

$$
\begin{aligned}
\Lambda^{31} &= (\gamma^0+\overline{\gamma}^0)\overline{\eth}f - \overline{\eth}\partial_u f + \frac{\lambda^0\eth f + \mu^0\overline{\eth}f}{r}\\
&-\frac{\overline{\sigma}^0\mu^0\eth f + \sigma^0\lambda^0\overline{\eth}f}{r^2} - \frac{\Psi_2^0\overline{\eth}f}{2r^2} + O(r^{-3}),\\
\Lambda^{41} &= (\gamma^0+\overline{\gamma}^0)\eth f - \eth\partial_u f + \frac{\overline{\lambda}^0\overline{\eth}f + \overline{\mu}^0\eth f}{r}\\
&-\frac{\sigma^0\overline{\mu}^0\overline{\eth}f + \overline{\sigma}^0\overline{\lambda}^0\eth f}{r^2} - \frac{\overline{\Psi}_2^0\eth f}{2r^2} + O(r^{-3}),\\
\Lambda^{43} &= \frac{1}{2}(\eth\mathcal{Y} - \overline{\eth\mathcal{Y}}) + \bar{Y}\bar{\partial}\ln\bar{P} - Y\partial\ln P + f(\overline{\gamma}^0-\gamma^0) + \frac{1}{2}(\Omega-\bar{\Omega})\\
&\quad + \frac{2\alpha^0\eth f - 2\overline{\alpha}^0\overline{\eth}f}{r} + \frac{2\overline{\sigma}^0\overline{\alpha}^0\eth f - 2\sigma^0\alpha^0\overline{\eth}f}{r^2} + O(r^{-3}),
\end{aligned}
\tag{2.56}
$$

where $\mathcal{Y} = \dfrac{\bar{Y}}{P}$. The components of the Lorentz rotation are determined by the asymptotic Killing vector completely. Thus the whole asymptotic symmetry is characterized by the asymptotic Killing vector who forms the extended BMS algebra including Weyl transformation introduced in [21].

The transformation properties of the fields can be worked out directly

$$
\begin{aligned}
&\delta P = \Omega P,\ \ \delta\mu^0 = \mu^0(\Omega+\bar{\Omega}) - \frac{1}{2}\overline{\eth}\eth(\Omega+\bar{\Omega}),\\
&\delta\nu^0 = \nu^0\bar{\Omega} - \frac{1}{2}\overline{\eth}\partial_u(\Omega+\bar{\Omega}),\ \ \delta\gamma^0 = \frac{1}{2}\partial_u\overline{\Omega},\\
&\delta\sigma^0 = [Y\partial + \bar{Y}\bar{\partial} + f\partial_u + \partial_u f + 2\Lambda_0^{43}]\sigma^0 - \eth^2 f,\\
&\delta\lambda^0 = [Y\partial + \bar{Y}\bar{\partial} + f\partial_u + \partial_u f - 2\Lambda_0^{43}]\lambda^0 - \partial_u\overline{\eth}^2 f + (\overline{\gamma}^0 - 3\gamma^0)\overline{\eth}^2 f,\\
&\delta\Psi_0^0 = [Y\partial + \bar{Y}\bar{\partial} + f\partial_u + 3\partial_u f + 2\Lambda_0^{43}]\Psi_0^0 + 4\Psi_1^0\eth f,\\
&\delta\Psi_1^0 = [Y\partial + \bar{Y}\bar{\partial} + f\partial_u + 3\partial_u f + \Lambda_0^{43}]\Psi_1^0 + 3\Psi_2^0\eth f,\\
&\delta\Psi_2^0 = [Y\partial + \bar{Y}\bar{\partial} + f\partial_u + 3\partial_u f]\Psi_2^0 + 2\Psi_3^0\eth f,\\
&\delta\Psi_3^0 = [Y\partial + \bar{Y}\bar{\partial} + f\partial_u + 3\partial_u f - \Lambda_0^{43}]\Psi_3^0 + \Psi_4^0\eth f,\\
&\delta\Psi_4^0 = [Y\partial + \bar{Y}\bar{\partial} + f\partial_u + 3\partial_u f - 2\Lambda_0^{43}]\Psi_4^0,
\end{aligned}
$$

with the help of

$$
\begin{aligned}
&\partial_u\eth\mathcal{Y} = 2\mathcal{Y}\overline{\nu}^0,\ \ \partial_u\overline{\eth\mathcal{Y}} = 2\overline{\mathcal{Y}}\nu^0,\ \ \eth\nu^0 = \overline{\eth}\overline{\nu}^0,\ \ \overline{\eth}\eth\mathcal{Y} = 2\mu^0\mathcal{Y},\\
&[\overline{\eth},\partial_u]\eta^s = 2(\overline{\gamma}^0\eth\eta^s + s\eth\gamma^0\eta^s),\ \ [\eth,\partial_u]\eta^s = 2(\gamma^0\overline{\eth}\eta^s - s\overline{\eth}\overline{\gamma}^0\eta^s),\\
&[\overline{\eth},\eth]\eta^s = -2s\mu^0\eta^s,\ \ \eth^2 f = P\bar{\partial}\eth f + 2\overline{\alpha}^0\eth f,\\
&\partial_u^2 f = \mathcal{Y}\overline{\nu}^0 + \overline{\mathcal{Y}}\nu^0 + \partial_u(f\gamma^0 + f\overline{\gamma}^0) + \frac{1}{2}\partial_u(\Omega+\bar{\Omega})\\
&Y\partial\eta^s + \bar{Y}\bar{\partial}\eta^s + s\Lambda_0^{43}\eta^s = \mathcal{Y}\eth\eta^s + \overline{\mathcal{Y}}\overline{\eth}\eta^s + \frac{1}{2}(\eth\mathcal{Y} - \overline{\eth\mathcal{Y}})\eta^s\\
&\qquad\qquad + f(\overline{\gamma}^0 - \gamma^0)\eta^s + \frac{1}{2}(\Omega-\bar{\Omega})\eta^s.
\end{aligned}
$$

One can see directly Λ_0^{43} shows the spin weight and $-\partial_u f$ shows the conformal weight.

2.4 Charges in the first order formalism

We start with the Cartan action

$$S[e_a^\mu, \omega^{bc}{}_\nu] = \frac{1}{16\pi G}\int d^4xe(R^{ab}{}_{\mu\nu}e_a^\mu e_b^\nu) \tag{2.57}$$

Since, the cosmological constant will not contribute to the charge, we neglect it for simplicity.

Let ∇ be the spacetime covariant derivative and D be the Lorentz covariant derivative defined by

$$D_\mu A^a = \partial_\mu A^a + \omega^{ab}{}_\mu A_b \tag{2.58}$$

The covariant derivative of the tetrad will be given as $D_\mu e_a^\nu = -\Gamma^\nu{}_{\mu\rho}e_a^\rho$. Where $\Gamma^\nu_{\mu\rho}$ is a metric connection satisfying $\nabla g_{\mu\nu} = 0$ and $\omega^{ab}{}_\mu = e^{a\nu}\nabla_\mu e_\nu^b$. The curvature two form is given by

$$R^{ab}{}_{\mu\nu} = \partial_\mu\omega^{ab}{}_\nu + \omega^a{}_{c\mu}\omega_\nu^{cb} - (\mu \leftrightarrow \nu) \tag{2.59}$$

The variation of the action is

$$\begin{aligned} 16\pi G\delta S &= \int d^4x\{e[D_\mu\delta\omega^{ab}{}_\nu - (\mu\leftrightarrow\nu)]e_a^\mu e_b^\nu + e[2R_\mu^a - e_\mu^a R]\delta e_a^\mu\} \\ &= \int d^4x\{\partial_\mu[\delta\omega^{ab}{}_\nu e(e_a^\mu e_b^\nu - e_b^\mu e_a^\nu)] \\ &\quad +\delta\omega^{ab}{}_\nu D_\rho(ee_a^\nu e_b^\rho - ee_b^\nu e_a^\rho) + e[2R_\mu^a - e_\mu^a R]\delta e_a^\mu\}. \end{aligned} \tag{2.60}$$

By dropping the total derivative, one gets the equation of motion as

$$16\pi G\frac{\delta\mathcal{L}}{\delta e_a^\mu} = e[2R_\mu^a - e_\mu^a R], \tag{2.61}$$

$$16\pi G\frac{\delta\mathcal{L}}{\delta\omega^{ab}{}_\nu} = D_\rho(ee_a^\nu e_b^\rho - ee_b^\nu e_a^\rho). \tag{2.62}$$

The second EOM can be adapted to

$$e\big[e_b^\rho e_a^\tau(\Gamma^\nu{}_{\rho\tau} - \Gamma^\nu{}_{\tau\rho}) + (\Gamma^\rho{}_{\tau\rho} - \Gamma^\rho{}_{\rho\tau})(e_b^\tau e_a^\nu - e_b^\nu e_a^\tau)\big], \tag{2.63}$$

which is equivalent to $\Gamma^\nu{}_{\rho\tau} - \Gamma^\nu{}_{\tau\rho} = 0$ on-shell, where the fact $\partial_\rho e = e\tilde{\Gamma}^\tau_{\tau\rho}$ has been used. And $\tilde{\Gamma}^\nu_{\mu\rho} = \frac{1}{2}g^{\nu\tau}[\partial_\mu g_{\tau\rho} + \partial_\rho g_{\tau\mu} - \partial_\tau g_{\mu\rho}]$ is the Christoffel and it is related to metric connection by $\Gamma^\nu{}_{\mu\rho} = \tilde{\Gamma}^\nu{}_{\mu\rho} + \frac{1}{2}(T_\rho{}^\nu{}_\mu + T_\mu{}^\nu{}_\rho + T^\nu{}_{\mu\rho})$ and $T^\nu_{\mu\rho} = \Gamma^\nu{}_{\mu\rho} - \Gamma^\nu{}_{\rho\mu}$ is the torsion.

We continue to derive the surface charge. The gauge transformation of the fields are given by

$$\delta e_a^\mu = \xi^\rho \partial_\rho e_a^\mu - e_a^\rho \partial_\rho \xi^\mu - \Lambda_a{}^b e_b^\mu \tag{2.64}$$
$$\delta \omega^{ab}{}_\mu = \xi^\rho \partial_\rho \omega^{ab}{}_\mu + \omega^{ab}{}_\rho \partial_\mu \xi^\rho + \partial_\mu \Lambda^{ab} - \omega^{ac}{}_\mu \Lambda^b{}_c - \omega^{cb}{}_\mu \Lambda^a{}_c . \tag{2.65}$$

According to the cohomological techniques [41–43], the $n-1$ current is defined by

$$S^\mu = \frac{1}{16\pi G}\left[e\xi^\mu R - 2e\xi^\rho R^\mu_\rho + 2(\xi^\rho \omega^{ab}{}_\rho + \Lambda^{ab}) D_\tau (e e_a^\mu e_b^\tau)\right] \tag{2.66}$$

Acting with the homotopy operator defined in [41–43] on the $n-1$ current, we get the $n-2$ current as

$$k^{[\mu\nu]}_{\xi\Lambda} = \frac{1}{2}\delta e_c^\sigma \frac{\partial}{\partial \partial_\nu e_c^\sigma} S^\mu + \frac{1}{2}\delta \omega^{cd}{}_\sigma \frac{\partial}{\partial\partial_\nu \omega^{cd}{}_\sigma} S^\mu - (\mu \leftrightarrow \nu). \tag{2.67}$$

Using

$$\frac{\partial}{\partial\partial_\nu \omega^{cd}{}_\sigma} R^{ab}{}_{\lambda\rho} = \delta_c^a \delta_d^b (\delta_\lambda^\nu \delta_\rho^\sigma - \delta_\rho^\nu \delta_\lambda^\sigma), \tag{2.68}$$

one gets

$$\frac{\partial}{\partial\partial_\nu \omega^{cd}{}_\sigma} R^\mu_\rho = e_c^\mu (e_d^\sigma \delta_\rho^\nu - e_d^\nu \delta_\rho^\sigma)$$
$$\frac{\partial}{\partial\partial_\nu \omega^{cd}{}_\sigma} R = e_c^\nu e_d^\sigma - e_c^\sigma e_d^\nu . \tag{2.69}$$

Finally the $n-2$ current will be given as

$$k^{[\mu\nu]}_{\xi\Lambda} = \frac{e}{16\pi G}\Big\{\delta\omega^{ab}{}_\rho [e_a^\mu e_b^\nu \xi^\rho + 2\xi^\mu e_b^\rho e_a^\nu] + (2\delta e_a^\mu e_b^\nu - e_a^\mu e_b^\nu e_\tau^c \delta e_c^\tau)(\xi^\rho \omega^{ab}{}_\rho + \Lambda^{ab})\Big\} - (\mu \leftrightarrow \nu). \tag{2.70}$$

It can be written in the integrable and non-integrable part

$$k^{[\mu\nu]}_{\xi\Lambda} = \delta K^{\mu\nu}_{\xi,\Lambda} - K^{\mu\nu}_{\delta_\xi,\delta_\Lambda} + \Theta^{\mu\nu} - (\mu \leftrightarrow \nu). \tag{2.71}$$

where

$$K^{\mu\nu}_{\xi,\Lambda} = \frac{e}{16\pi G} e_a^\mu e_b^\nu (\xi^\rho \omega^{ab}{}_\rho + \Lambda^{ab})$$
$$\Theta^{\mu\nu} = \frac{e}{8\pi G} \xi^\mu e_a^\nu e_b^\rho \delta\omega^{ab}{}_\rho .$$

As shown in Appendix A.1, the $n-2$ current derived from first order formalism is completely equivalent to metric formalism formulated in [41].

To recast Newman-Penrose formalism in an action principle, one needs to include a Lagrangian multiplier. The action is formulated as

$$S[e^\mu_a, \omega_{abc}, R_{abcd}, \lambda^{abcd}] = \frac{1}{16\pi G}\int d^4x\, e\big[\frac{1}{2}R_{abcd}(\eta^{ac}\eta^{bd} - \eta^{ad}\eta^{bc}) - R_{abcd}\lambda^{abcd}$$
$$+(\lambda^{abcd} - \lambda^{abdc})(e^\mu_c\partial_\mu\omega_{abd} + \omega_{afc}\omega^f{}_{bd} + \omega_{abf}e^\mu_c e^\nu_d\partial_\mu e^f_\nu)\big]. \qquad (2.72)$$

The variation of the action is

$$16\pi G\delta S = \int d^4x\{e[\frac{1}{2}(\eta^{ac}\eta^{bd} - \eta^{ad}\eta^{bc}) - \lambda^{abcd}]\delta R_{abcd}$$
$$-e[R_{abcd} - (e^\mu_c\partial_\mu\omega_{abd} + \omega_{afc}\omega^f{}_{bd} + \omega_{abf}e^\mu_c e^\nu_d\partial_\mu e^f_\nu - e^\mu_d\partial_\mu\omega_{abc}$$
$$-\omega_{afd}\omega^f{}_{bc} - \omega_{abf}e^\mu_d e^\nu_c\partial_\mu e^f_\nu)]\delta\lambda^{abcd}$$
$$[e(\lambda^{abcd} - \lambda^{abdc})(e^\mu_c e^\nu_d\partial_\mu e^f_\nu + \omega_c{}^f{}_d) - D_\mu\big(ee^\mu_c(\lambda^{abcf} - \lambda^{abfc})\big)]\delta\omega_{abf}$$
$$+\{e(\lambda^{abhc} - \lambda^{abch})[\partial_\tau\omega_{abc} + \omega_{abf}e^\nu_c(\partial_\tau e^f_\nu - \partial_\nu e^f_\tau)] + e^h_\nu e^f_\tau\partial_\mu[e(\lambda^{abcd} - \lambda^{abdc})e^\mu_c e^\nu_d]$$
$$-ee^h_\tau[R - R_{abcd}\lambda^{abcd} + (\lambda^{abcd} - \lambda^{abdc})(e^\mu_c\partial_\mu\omega_{abd} + \omega_{afc}\omega^f{}_{bd} + \omega_{abf}e^\mu_c e^\nu_d\partial_\mu e^f_\nu)]\}\delta e^\tau_h$$
$$+\partial_\mu[e(\lambda^{abch} - \lambda^{abhc})(e^\mu_c\delta\omega_{abh} - \omega_{abf}e^\mu_c e^f_\tau\delta e^\tau_h)]\big\}. \qquad (2.73)$$

Dropping the total derivative, one gets the equation of motion as

$$16\pi G\frac{\delta\mathcal{L}}{\delta R_{abcd}} = e[\frac{1}{2}(\eta^{ac}\eta^{bd} - \eta^{ad}\eta^{bc}) - \lambda^{abcd}],$$
$$16\pi G\frac{\delta\mathcal{L}}{\delta\lambda^{abcd}} = -e[R_{abcd} - (e^\mu_c\partial_\mu\omega_{abd} + \omega_{afc}\omega^f{}_{bd} + \omega_{abf}e^\mu_c e^\nu_d\partial_\mu e^f_\nu - (c\leftrightarrow d))],$$
$$16\pi G\frac{\delta\mathcal{L}}{\delta\omega_{abf}} = e(\lambda^{abcd} - \lambda^{abdc})(e^\mu_c e^\nu_d\partial_\mu e^f_\nu + \omega_c{}^f{}_d) - D_\mu[ee^\mu_c(\lambda^{abcf} - \lambda^{abfc})],$$
$$16\pi G\frac{\delta\mathcal{L}}{\delta e^\tau_h} = e(\lambda^{abhc} - \lambda^{abch})[\partial_\tau\omega_{abc} + \omega_{abf}e^\nu_c(\partial_\tau e^f_\nu - \partial_\nu e^f_\tau)]$$
$$+e^h_\nu e^f_\tau\partial_\mu[e(\lambda^{abcd} - \lambda^{abdc})e^\mu_c e^\nu_d]$$
$$-ee^h_\tau[R - R_{abcd}\lambda^{abcd} + (\lambda^{abcd} - \lambda^{abdc})$$
$$\times(e^\mu_c\partial_\mu\omega_{abd} + \omega_{afc}\omega^f{}_{bd} + \omega_{abf}e^\mu_c e^\nu_d\partial_\mu e^f_\nu)]$$

On-shell those equations are totally equivalent to Cartan formalism.

The gauge transformation on ω, R and λ are given by

$$\delta\omega_{abc} = \xi^\rho\partial_\rho\omega_{abc} + e^\mu_c\partial_\mu\Lambda_{ab} - \omega_{dbc}\Lambda_a{}^d - \omega_{adc}\Lambda_b{}^d - \omega_{abd}\Lambda_c{}^d,$$
$$\delta\lambda^{abcd} = \xi^\rho\partial_\rho\lambda^{abcd} - \lambda^{fbcd}\Lambda^a{}_f - \lambda^{afcd}\Lambda^b{}_f - \lambda^{abfd}\Lambda^c{}_f - \lambda^{abcf}\Lambda^d{}_f$$
$$\delta R_{abcd} = \xi^\rho\partial_\rho R_{abcd} - R_{fbcd}\Lambda_a{}^f - R_{afcd}\Lambda_b{}^f - R_{abfd}\Lambda_c{}^f - R_{abcf}\Lambda_d{}^f. \qquad (2.74)$$

Via those gauge transformations, the $n-1$ current is given by

$$\begin{aligned} S^\mu &= \Lambda_{ab}\partial_\nu[e(\lambda^{abcf}-\lambda^{abfc})e^\mu_c e^\nu_f]+\xi^\tau\partial_\nu[e(\lambda^{abcf}-\lambda^{abfc})\omega_{abd}e^\mu_c e^\nu_f e^d_\tau] \\ &\quad +e(\lambda^{abcf}-\lambda^{abfc})e^\tau_f(\xi^\mu e^\nu_c-\xi^\nu e^\mu_c)\partial_\nu(\omega_{abd}e^d_\tau)+\text{non-derivative terms.} \end{aligned} \tag{2.75}$$

Acting the homotopy operator, the $n-2$ current can be computed as

$$k^{[\mu\nu]}_{\xi\Lambda}=\delta K^{\mu\nu}_{\xi,\Lambda}-K^{\mu\nu}_{\delta_\xi,\delta_\Lambda}+\Theta^{\mu\nu}-(\mu\leftrightarrow\nu). \tag{2.76}$$

where

$$\begin{aligned} K^{\mu\nu}_{\xi,\Lambda} &= \frac{e}{32\pi G}(\lambda^{abfc}-\lambda^{abcf})e^\mu_f e^\nu_c(\xi^\rho\omega_{abd}e^d_\rho+\Lambda_{ab}) \\ \Theta^{\mu\nu} &= \frac{e}{16\pi G}\xi^\mu(\lambda^{abcf}-\lambda^{abfc})\xi^\mu e^\nu_c e^\tau_f\delta(\omega_{abd}e^d_\tau). \end{aligned}$$

Inserting the EOM $\lambda^{abcd}=\frac{1}{2}(\eta^{ac}\eta^{bd}-\eta^{ad}\eta^{bc})$, we find the $n-2$ current is exactly the same as Cartan formalism (2.71).

Insert the solution (2.53) and the symmetry parameters(2.56) in (2.76), the $n-2$ current in the Newman-Penrose formalism can be computed easily as

$$\begin{aligned} 8\pi G k^{ur} &= \delta\Big\{\frac{1}{P\bar P}[f(\Psi^0_2+\sigma^0\lambda^0)-\frac{1}{2}(\Omega+\bar\Omega)\sigma^0\bar\sigma^0+\mathcal{Y}(\sigma^0\eth\bar\sigma^0+\frac{1}{2}\eth(\sigma^0\bar\sigma^0) \\ &\quad +\Psi^0_1)]\Big\}+\frac{1}{P\bar P}\Big\{\frac{1}{2}\sigma^0\bar\sigma^0\delta\eth\mathcal{Y}-(\Psi^0_2+\lambda^0\sigma^0)\delta f+\eth\bar\sigma^0\eth f\delta\ln P \\ &\quad -f\lambda^0\delta\sigma^0+f\lambda^0\sigma^0\delta\ln P-f\Psi^0_2\delta\ln P-fP\eth\bar\sigma^0\delta\bar\partial\ln\bar P\Big\}+c.c., \end{aligned} \tag{2.77}$$

$$\begin{aligned} 8\pi G k^{zr} &= -\delta\Big\{\frac{1}{P}[\overline{\mathcal{Y}}\overline{\Psi}^0_2+f\overline{\Psi}^0_3+\frac{1}{2}\overline{\mathcal{Y}}(\lambda^0\sigma^0-\overline{\lambda}^0\overline{\sigma}^0) \\ &\quad +\frac{1}{2}\bar\eth\sigma^0(\overline{\eth\mathcal{Y}}-\eth\mathcal{Y}+\bar\Omega-\Omega) \\ &\quad -\frac{1}{2}\sigma^0\bar\eth(\overline{\eth\mathcal{Y}}-\eth\mathcal{Y}+\bar\Omega+\Omega)+\overline{\lambda}^0\bar\eth f]\Big\} \\ &\quad +\frac{1}{P}\Big\{2\eth f\mu^0\delta\ln\bar P-\eth f\delta\mu^0+\overline{\Psi}^0_3\delta f+\overline{\lambda}^0\delta\eth f \\ &\quad -\bar\eth\sigma^0[\delta\Lambda^{43}_0-(\gamma^0-\overline{\gamma}^0)\delta f]-\sigma^0\delta\bar\eth\partial_u f+\sigma^0\delta[\bar\eth f(\gamma^0+\overline{\gamma}^0)] \\ &\quad +\sigma^0(\gamma^0+\overline{\gamma}^0)\bar\eth f\delta\ln P+(\sigma^0\bar\eth f-\overline{\mathcal{Y}}\sigma^0\overline{\sigma}^0)\delta(\gamma^0+\overline{\gamma}^0) \\ &\quad -\sigma^0\bar\eth f(\gamma^0+\overline{\gamma}^0)\delta\ln P\bar P-2\mu^0\bar\eth f\delta\ln P\bar P+\frac{1}{2}P\overline{\Delta}f\delta\bar\partial\ln\bar P \\ &\quad +\overline{\mathcal{Y}}[\lambda^0\sigma^0\delta\ln P+\overline{\lambda}^0\overline{\sigma}^0\delta\ln\bar P-\lambda^0\delta\sigma^0-\overline{\lambda}^0\delta\overline{\sigma}^0 \\ &\quad -\Psi^0_2\delta\ln P-\overline{\Psi}^0_2\delta\ln\bar P-P\eth\overline{\sigma}^0\delta\bar\partial\ln\bar P-P\bar\eth\sigma^0\delta\partial\ln P]\Big\}. \end{aligned} \tag{2.78}$$

A $n-3$ current

$$\eta^{urz} = r[\delta(\frac{\mathcal{Y}\sigma^0}{P}) + \frac{1}{2}\bar{\partial} f \delta \ln P\bar{P} - \frac{1}{2} f \delta \bar{\partial} \ln P \bar{\partial}] + \delta(\frac{\overline{\mathcal{Y}}\sigma^0\overline{\sigma}^0}{2P}), \tag{2.79}$$
$$\eta^{zr\bar{z}} = 0, \tag{2.80}$$

has been dropped.

2.5 Metric formalism

The metric in $(+,-,-,-)$ signature is defined via $g_{ab} = l_a n_a + n_a l_a - m_a \overline{m}_b - \overline{m}_a m_b$ from the Newman-Penrose formalism. The ansatz we have taken for the tetrad $l, n, m, \overline{m}$ leads to the Newman-Unti gauge [49, 51] in the metric formalism as

$$g^{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & W & V^B \\ 0 & V^B & g^{AB} \end{pmatrix}. \tag{2.81}$$

Under this gauge, the solution of Einstein equation is related to (2.53) by

$$\begin{aligned}
W &= 2(U - \omega\overline{\omega}) = -2r\partial_u\tilde{\varphi} + 2e^{-2\varphi}\partial\bar{\partial}\tilde{\varphi} - \frac{\Psi_2^0 + \overline{\Psi}_2^0}{r} + O(r^{-2}),\\
V^z &= X^z - \omega\bar{L}^z - \overline{\omega}L^z = -\frac{\bar{P}\overline{\eth}\sigma^0}{r^2} + O(r^{-3}),\\
g^{zz} &= -L^z\bar{L}^z - \bar{L}^z L^z = \frac{2\bar{P}^2\sigma^0}{r^3} + O(r^{-4}),\\
g^{z\bar{z}} &= -L^z\bar{L}^{\bar{z}} - \bar{L}^z L^{\bar{z}} = -\frac{P\bar{P}}{r^2} + O(r^{-4}),
\end{aligned} \tag{2.82}$$

where $\tilde{\varphi} = -\frac{1}{2}\ln P\bar{P}$. As fully discussed in [51], the Newman-Unti coordinate is related to the BMS one by the changing of the radial coordinate

$$r_{BMS} = r - \frac{\sigma^0\overline{\sigma}^0}{2r} + O(r^{-2}). \tag{2.83}$$

Thus the solution (2.82) is connected to the one derived in BMS gauge [21] by such a transformation.

The transformations preserving the solution space (2.82) asymptotically was derived in [51, 52]. We list them as following

$$\begin{aligned}
\xi^u &= f,\ \ \partial_u f = \frac{1}{2}(\eth\mathcal{Y} + \overline{\eth\mathcal{Y}}) + f(\gamma^0 + \overline{\gamma}^0) + \frac{1}{2}(\Omega + \bar{\Omega}),\\
\xi^z &= Y - \frac{\bar{P}\eth f}{r} + \frac{\sigma^0\bar{P}\overline{\eth} f}{r^2} + O(r^{-3}),\\
\xi^{\bar{z}} &= \bar{Y} - \frac{P\overline{\eth} f}{r} + \frac{\overline{\sigma}^0 P\eth f}{r^2} + O(r^{-3}),\\
\xi^r &= -r\partial_u f + \frac{1}{2}\bar{\Delta} f - \frac{\overline{\eth}\sigma^0\overline{\eth} f + \eth\overline{\sigma}^0\eth f}{r} + O(r^{-2}).
\end{aligned} \tag{2.84}$$

This is in consistent with (2.56) by turning off the Lorentz rotation.

Lastly, we come to the would-be conserved current associated to the asymptotic Killing vector (2.84). When restraining ourself by turning off the Weyl transformation (*i.e.* $\Omega = 0$), the current can be computed from the metric formalism as

$$\begin{aligned}
k^{ur} =& \frac{1}{P\bar{P}}\big\{\delta[f(\Psi_2^0 + \sigma^0\lambda^0) + \mathcal{Y}(\sigma^0\eth\overline{\sigma}^0 + \frac{1}{2}\eth(\sigma^0\overline{\sigma}^0) + \Psi_1^0)] - f\lambda^0\delta\sigma^0 \\
&+ \delta[f(\overline{\Psi}_2^0 + \overline{\sigma}^0\overline{\lambda}^0) + \overline{\mathcal{Y}}(\overline{\sigma}^0\overline{\eth}\sigma^0 + \frac{1}{2}\overline{\eth}(\sigma^0\overline{\sigma}^0) + \overline{\Psi}_1^0)] - f\overline{\lambda}^0\delta\overline{\sigma}^0\big\}, \\
k^{zr} =& -\frac{1}{P}\big\{\delta[\overline{\mathcal{Y}}\overline{\Psi}_2^0 + f\overline{\Psi}_3^0 + \frac{1}{2}\overline{\mathcal{Y}}(\lambda^0\sigma^0 - \overline{\lambda}^0\overline{\sigma}^0) + \frac{1}{2}\overline{\eth}\sigma^0(\overline{\eth}\overline{\mathcal{Y}} - \eth\mathcal{Y}) \\
&- \frac{1}{2}\sigma^0\overline{\eth}(\overline{\eth}\overline{\mathcal{Y}} - \eth\mathcal{Y}) + \overline{\lambda}^0\overline{\eth}f] + \overline{\mathcal{Y}}\big[\lambda^0\delta\sigma^0 + \overline{\lambda}^0\delta\overline{\sigma}^0\big]\big\}.
\end{aligned} \tag{2.85}$$

If we only focus on the case of u-independent P and $\bar{P}$, the current (2.85) just reproduces the one obtained in [51, 53] explicitly.

Chapter 3

Applications in 3 dimensional space-time

Although it admits no propagating degrees of freedom ("bulk gravitons"), three dimensional Einstein gravity is known to admit black holes [27, 54], particles [55, 56], wormholes [57–59] and boundary dynamics [28, 60, 61]. Moreover, it can arise as a consistent subsector of higher dimensional matter-gravity theories, see e.g. [62, 63]. Therefore, three-dimensional gravity in the last three decades has been viewed as a simplified and fruitful setup to analyze and address issues related to the physics of black holes and quantum gravity.

In three dimensions the Riemann tensor is completely specified in terms of the Ricci tensor, except at possible defects, and hence all Einstein solutions with generic cosmological constant are locally maximally symmetric. The fact that AdS_3 Einstein gravity can still have a nontrivial dynamical content was first discussed in the seminal work of Brown and Henneaux [28, 64]. There, it was pointed out that one may associate nontrivial conserved charges, defined at the AdS_3 boundary, to diffeomorphisms which preserve prescribed (Brown-Henneaux) boundary conditions. These diffeomorphisms and the corresponding surface charges obey two copies of the Virasoro algebra and the related bracket structure may be viewed as a Dirac bracket defining (or arising from) a symplectic structure for these "boundary degrees of freedom" or "boundary gravitons". It was realized that the Virasoro algebra should be interpreted in terms of a holographic dictionary with a conformal field theory [30]. These ideas found a more precise and explicit formulation within the celebrated AdS_3/CFT_2 dualities in string theory [65]. Many other important results in this context have been obtained [63, 66–78].

3.1 Symplectic symmetries

A recent proposal in [79] has shown that the asymptotic symmetries of dS_3 with Dirichlet boundary conditions defined as an analytic continuation of the Brown-Henneaux symmetries to the case of positive cosmological constant [80] can be defined everywhere into the bulk spacetime. A similar result is expected to follow for AdS_3 geometries by analytical continuation, however, few details were given in [79] (see also [81, 82] for related observations). In this work, we revisit the Brown-Henneaux analysis from the first principles and show that the surface charges and the associated algebra and dynamics can be defined not only on the circle at spatial infinity, but also on any circle inside of the bulk obtained by a smooth deformation which does not cross any geometric defect or topological obstruction. This result is consistent with the expectation that if a dual 2d CFT exists, it is not only "defined at the boundary", but it is defined in a larger sense from the AdS bulk.

Our derivation starts with the set of Bañados geometries [70] which constitute all locally AdS_3 geometries with Brown-Henneaux boundary conditions. We show that the invariant presymplectic form [83] (but not the Lee-Wald presymplectic form [84]) vanishes in the entire bulk spacetime. The charges defined from the presymplectic form are hence conserved everywhere, i.e. they define sympletic symmetries, and they obey an algebra through a Dirac bracket, which is isomorphic to two copies of the Virasoro algebra. In turn, this Dirac bracket defines a lower dimensional non-trivial symplectic form, the Kirillov-Kostant symplectic form for coadjoint orbits of the Virasoro group [85]. In that sense the boundary gravitons may be viewed as *holographic gravitons*: they define a lower dimensional dynamics inside of the bulk. Similar features were also observed in the near-horizon region of extremal black holes [86, 87].

Furthermore, we will study in more detail the extremal sector of the phase space. Boundary conditions are known in the decoupled near-horizon region of the extremal BTZ black hole which admit a chiral copy of the Virasoro algebra [73]. Here, we extend the notion of decoupling limit to more general extremal metrics in the Bañados family and show that one can obtain this (chiral) Virasoro algebra as a limit of the bulk symplectic symmetries, which are defined from the asymptotic AdS_3 region all the way to the near-horizon region. We discuss two distinct ways to take the near-horizon limit: at finite coordinate radius (in Fefferman-Graham coordinates) and at wiggling coordinate radius (in Gaussian null coordinates), depending upon the holographic graviton profile at the horizon. We will show that these two coordinate systems lead to the same conserved charges and are therefore equivalent up to a gauge choice. Quite interestingly, the vector fields defining the Virasoro symmetries take a qualitatively different form in both coordinate systems which are also distinct from all previous ansatzes for near-horizon symmetries [73, 79, 86–89].

In [76] it was noted that Bañados geometries in general have (at least) two global $U(1)$ Killing vectors (defined over the whole range of the Bañados coordinate system). We will study the conserved charges $J_\pm$ associated with these two Killing vectors. We will show that these charges commute with the surface charges associated with symplectic symmetries (the Virasoro generators). We then discuss how the elements of the phase space may be labeled using the $J_\pm$ charges. This naturally brings us to the question of how the holographic gravitons may be labeled through representations of Virasoro group, the Virasoro coadjoint orbits, e.g. see [85,90]. The existence of Killing horizons in the set of Bañados geometries was studied in [76]. We discuss briefly that if the Killing horizon exists, its area defines an entropy which together with $J_\pm$, satisfies the first law of thermodynamics.

3.1.1 Symplectic symmetries in Fefferman-Graham coordinates

The AdS_3 Einstein gravity is described by the action and equations of motion,

$$S = \frac{1}{16\pi G}\int d^3x\sqrt{-g}(R + \frac{2}{\ell^2}), \qquad R_{\mu\nu} = -\frac{2}{\ell^2}g_{\mu\nu}. \tag{3.1}$$

As discussed in the introduction, all solutions are locally AdS_3 with radius ℓ. To represent the set of these solutions, we adopt the Fefferman-Graham coordinate system[1] [67, 91, 92],

$$g_{rr} = \frac{\ell^2}{r^2}, \qquad g_{ra} = 0, \qquad a = 1, 2, \tag{3.2}$$

where the metric reads

$$ds^2 = \ell^2\frac{dr^2}{r^2} + \gamma_{ab}(r, x^c)\, dx^a\, dx^b. \tag{3.3}$$

Being asymptotically locally AdS_3, close to the boundary $r \to \infty$ one has the expansion $\gamma_{ab} = r^2 g^{(0)}_{ab}(x^c) + \mathcal{O}(r^0)$ [67]. A variational principle is then defined for a subset of these solutions which are constrained by a boundary condition. Dirichlet boundary conditions amount to fixing the boundary metric $g^{(0)}_{ab}$. The Brown-Henneaux boundary conditions [28] are Dirichlet boundary conditions with a fixed flat boundary metric,

$$g^{(0)}_{ab} dx^a dx^b = -dx^+ dx^-, \tag{3.4}$$

[1] We will purposely avoid to use the terminology of Fefferman-Graham *gauge* which would otherwise presume that leaving the coordinate system by any infinitesimal diffeomorphism would be physically equivalent in the sense that the associated canonical generators to this diffeomorphism would admit zero Dirac brackets with all other physical generators. Since this coordinate choice precedes the definitions of boundary conditions, and therefore the definition of canonical charges, the *gauge* terminology is not appropriate.

together with the periodic identifications $(x^+, x^-) \sim (x^+ + 2\pi, x^- - 2\pi)$ which identify the boundary metric with a flat cylinder (the identification reads $\phi \sim \phi + 2\pi$ upon defining $x^\pm = t/\ell \pm \phi$). Other relevant Dirichlet boundary conditions include the flat boundary metric with no identification (the resulting solutions are usually called "Asymptotically Poincaré AdS_3"), and the flat boundary metric with null orbifold identification $(x^+, x^-) \sim (x^+ + 2\pi, x^-)$ which is relevant to describing near-horizon geometries [73, 76, 93].[2]

The set of all solutions to AdS_3 Einstein gravity with flat boundary metric was given by Bañados [70] in the Fefferman-Graham coordinate system. The metric takes the form

$$ds^2 = \ell^2 \frac{dr^2}{r^2} - \Big(r dx^+ - \ell^2 \frac{L_-(x^-) dx^-}{r}\Big)\Big(r dx^- - \ell^2 \frac{L_+(x^+) dx^+}{r}\Big) \qquad (3.5)$$

where $L_\pm$ are two single-valued arbitrary functions of their argument. The determinant of the metric is $\sqrt{-g} = \frac{\ell}{2r^3}(r^4 - \ell^4 L_+ L_-)$ and the coordinate patch covers the radial range $r^4 > \ell^4 L_+ L_-$. These coordinates are particularly useful in stating the universal sector of all AdS_3/CFT_2 correspondences since the expectation values of holomorphic and anti-holomorphic components of the energy-momentum tensor of the CFT can be directly related to $L_\pm$ [28, 66].

The constant $L_\pm$ cases correspond to better known geometries [27,54,56]: $L_+ = L_- = -1/4$ corresponds to AdS_3 in global coordinates, $-1/4 < L_\pm < 0$ correspond to conical defects (particles on AdS_3), $L_- = L_+ = 0$ correspond to massless BTZ and generic positive values of $L_\pm$ correspond to generic BTZ geometry of mass and angular momentum respectively equal to $(L_+ + L_-)/(4G)$ and $\ell(L_+ - L_-)/(4G)$. The selfdual orbifold of AdS_3 [93] belongs to the phase space with null orbifold identification and $L_- = 0, L_+ \neq 0$.

We would now like to establish that the set of Bañados metrics (3.5) together with a choice of periodic identifications of $x^\pm$ forms a well-defined on-shell phase space. To this end, we need to take two steps: specify the elements in the tangent space of the on-shell phase space and then define the presymplectic structure over this phase space. Given that the set of all solutions are of the form (3.5), the on-shell tangent space is clearly given by metric variations of the form

$$\delta g = g(L + \delta L) - g(L)\,, \qquad (3.6)$$

where $\delta L_\pm$ are arbitrary single-valued functions. The vector space of all on-shell perturbations δg can be written as the direct sum of two types of perturbations: those which are generated by diffeomorphisms and those which are not, and that we will refer to as *parametric perturbations.*

[2] Other boundary conditions which lead to different symmetries were discussed in [94–96].

As for the presymplectic form, there are two known definitions for Einstein gravity: the one $\boldsymbol{\omega}^{LW}$ by Lee-Wald [84] (see also Crnkovic and Witten [97]) and invariant presymplectic form $\boldsymbol{\omega}^{inv}$ as defined in [83].[3] The invariant presymplectic form is determined from Einstein's equations only, while the Lee-Wald presymplectic form is determined from the Einstein-Hilbert action, see [98] for details. Upon explicit evaluation, we obtain that the invariant presympletic form exactly vanishes *on-shell* on the phase space determined by the set of metrics (3.5), that is,

$$\boldsymbol{\omega}^{inv}[\delta g, \delta g; g] \approx 0. \tag{3.7}$$

On the contrary, the Lee-Wald presymplectic form is equal to a boundary term

$$\begin{aligned} \boldsymbol{\omega}^{LW}[\delta g, \delta g; g] &\approx -d\boldsymbol{E}[\delta g, \delta g; g], \\ \star\boldsymbol{E}[\delta g, \delta g; g] &= \frac{1}{32\pi G}\delta g_{\mu\alpha} g^{\alpha\beta}\delta g_{\nu\beta} dx^\mu \wedge dx^\nu \end{aligned} \tag{3.8}$$

Indeed, the two presymplectic forms are precisely related by this boundary term [83], as reviewed in appendix.

As mentioned earlier, the most general form of on-shell perturbations preserving Fefferman-Graham coordinates is of the form (3.6). Among them there are perturbations generated by an infinitesimal diffeomorphism along a vector field χ. The components of such vector field are of the form

$$\chi^r = r\,\sigma(x^a), \qquad \chi^a = \varepsilon^a(x^b) - \ell^2 \partial_b\, \sigma \int_r^\infty \frac{dr'}{r'} \gamma^{ab}(r', x^a) \tag{3.9}$$

where $\sigma(x^a)$ and $\varepsilon^a(x^b)$ are constrained by the requirement $\delta g^{(0)}_{ab} \equiv \mathcal{L}_{\vec{\varepsilon}}\, g^{(0)}_{ab} + 2\sigma g^{(0)}_{ab} = 0$. That is, $\vec{\varepsilon} \equiv (\varepsilon_+(x^+), \varepsilon_-(x^-))$ is restricted to be a conformal Killing vector of the flat boundary metric and σ is defined as the Weyl factor in terms of $\vec{\varepsilon}$.

One can in fact explicitly perform the above integral for a given Bañados metric and solve for $\sigma(x)$ to arrive at

$$\begin{aligned} \chi = &-\frac{r}{2}(\varepsilon'_+ + \varepsilon'_-)\partial_r + \Big(\varepsilon_+ + \frac{\ell^2 r^2 \varepsilon''_- + \ell^4 L_- \varepsilon''_+}{2(r^4 - \ell^4 L_+ L_-)}\Big)\partial_+ \\ &+ \Big(\varepsilon_- + \frac{\ell^2 r^2 \varepsilon''_+ + \ell^4 L_+ \varepsilon''_-}{2(r^4 - \ell^4 L_+ L_-)}\Big)\partial_-, \end{aligned} \tag{3.10}$$

where $\varepsilon_\pm$ are two arbitrary single-valued periodic functions of $x^\pm$ and possibly of the fields $L_+(x^+)$, $L_-(x^-)$, and the *prime* denotes derivative w.r.t. the argument. As we see,

[3] More precisely, $\boldsymbol{\omega}$ is a $(2;2)$ form i.e. a two-form on the manifold and a two-form in field space. For short we call $\boldsymbol{\omega}$ a presymplectic form, and given any spacelike surface Σ, $\Omega = \int_\Sigma \boldsymbol{\omega}$ the associated presymplectic structure, which is the (possibly degenerate) $(0;2)$ form. A non-degenerate $(0;2)$ form defines a symplectic structure. We also refer to $\boldsymbol{\omega}$ as a *bulk* presymplectic form.

1. χ is a *field-dependent* vector field. That is, even if the two arbitrary functions $\varepsilon_\pm$ are field independent, it has explicit dependence upon $L_\pm$: $\chi = \chi(\varepsilon_\pm; L_\pm)$.

2. The vector field χ is defined in the entire coordinate patch spanned by the Bañados metric, not only asymptotically.

3. Close to the boundary, at large r, χ reduces to the Brown-Henneaux asymptotic symmetries [28]. Also, importantly, at large r the field-dependence of χ drops out if one also takes $\varepsilon_\pm$ field-independent.

The above points bring interesting conceptual and technical subtleties, as compared with the better known Brown-Henneaux case, that we will fully address.

The above vector field can be used to define a class of on-shell perturbations, $\delta_\chi g_{\mu\nu} \equiv \mathcal{L}_\chi g_{\mu\nu}$. It can be shown that

$$\delta_\chi g_{\mu\nu} = g_{\mu\nu}(L_+ + \delta_\chi L_+, L_- + \delta_\chi L_-) - g_{\mu\nu}(L_+, L_-), \tag{3.11}$$

where

$$\begin{aligned} \delta_\chi L_+ &= \varepsilon_+ \partial_+ L_+ + 2L_+ \partial_+ \varepsilon_+ - \frac{1}{2}\partial_+^3 \varepsilon_+, \\ \delta_\chi L_- &= \varepsilon_- \partial_- L_- + 2L_- \partial_- \varepsilon_- - \frac{1}{2}\partial_-^3 \varepsilon_-. \end{aligned} \tag{3.12}$$

As it is well-known in the context of AdS_3/CFT_2 correspondence [62,65] the variation of $L_\pm$ under diffeomorphisms generated by χ is the same as the variation of a 2d CFT energy-momentum tensor under generic infinitesimal conformal transformations. Notably, the last term related to the central extension of the Virasoro algebra is a quantum anomalous effect in a 2d CFT while in the dual AdS_3 gravity it appears classically.

The vector field χ determines *symplectic symmetries* as defined in [86] (they were defined as asymptotic symmetries everywhere in [79]). The reason is that the invariant presymplectic form contracted with the Lie derivative of the metric with respect to the vector vanishes on-shell,

$$\boldsymbol{\omega}^{inv}[g; \delta g, \mathcal{L}_\chi g] \approx 0, \tag{3.13}$$

which is obviously a direct consequence of (3.7), while $\mathcal{L}_\chi g$ does not vanish. Then according to (A.33), the charges associated to symplectic symmetries can be defined over any closed codimension two surface $\mathcal{S}$ (circles in 3d) anywhere in the bulk. Moreover, as we will show next, the surface charge associated to χ is non-vanishing and integrable. That is, the concept of "symplectic symmetry" extends the notion of "asymptotic symmetry" inside the bulk.

A direct computation gives the formula for the infinitesimal charge one-forms as defined by Barnich-Brandt [41], see appendix, as

$$\boldsymbol{k}^{BB}_\chi[\delta g; g] = \hat{\boldsymbol{k}}_\chi[\delta g; g] + d\boldsymbol{B}_\chi[\delta g; g], \tag{3.14}$$

where

$$\begin{aligned} \hat{\boldsymbol{k}}_\chi[\delta g; g] \;\; = \;\; & \frac{\ell}{8\pi G}\big[\varepsilon_+(x^+, L_+(x^+))\delta L_+(x^+)dx^+ \\ & -\varepsilon_-(x^-, L_-(x^-))\delta L_-(x^-)dx^-\big], \end{aligned} \tag{3.15}$$

is the expected result and

$$\boldsymbol{B}_\chi = \frac{\ell(\varepsilon'_+ + \varepsilon'_-)(L_+\delta L_- - L_-\delta L_+)}{32\pi G(r^4 - L_+L_-)},$$

is an uninteresting boundary term which vanishes close to the boundary and which drops after integration on a circle.

Now, since the Lee-Wald presymplectic form does not vanish, the Iyer-Wald [99] surface charge one-form is not conserved in the bulk. From the general theory, it differs from the Barnich-Brandt charge by the supplementary term $\boldsymbol{E}[\delta g, \mathcal{L}_\chi g; g]$, see (3.8). In Fefferman-Graham coordinates we have $\mathcal{L}_\chi g_{r\mu} = 0$ therefore $E_+ = E_- = 0$ and only E_r is non-vanishing. In fact we find $E_r = O(r^{-6})$ which depends upon $L_\pm(x^\pm)$. Since $\boldsymbol{E}$ is clearly not a total derivative, the Iyer-Wald charge is explicitly radially dependent which is expected since χ does not define a symplectic symmetry for the Lee-Wald presymplectic form.

We shall therefore only consider the invariant presymplectic form and Barnich-Brandt charges here. The standard charges are obtained by considering $\varepsilon_\pm$ to be field-independent. In that case the charges are directly integrable, see also the general analysis of appendix A.2. We define the left and right-moving stress-tensors as $T = \frac{c}{6}L_+(x^+)$ and $\bar{T} = \frac{c}{6}L_-(x^-)$ where $c = \frac{3\ell}{2G}$ is the Brown-Henneaux central charge. The finite surface charge one-form then reads

$$\boldsymbol{Q}_\chi[g] \equiv \int_{\bar{g}}^{g} \boldsymbol{k}_\chi[\delta g; g] = \frac{1}{2\pi}(\varepsilon_+(x^+)T(x^+)dx^+ - \varepsilon_-(x^-)\bar{T}(x^-)dx^-). \tag{3.16}$$

Here we chose to normalize the charges to zero for the zero mass BTZ black hole $\bar{g}$ for which $L_\pm = 0$.[4] In AdS_3/CFT_2, the functions $T, \bar{T}$ are interpreted as components of the dual stress-energy tensor. In the case of periodic identifications leading to the boundary cylinder (asymptotically global AdS_3), we are led to the standard Virasoro charges

$$Q_\chi[g] = \int_S \boldsymbol{Q}_\chi[g] = \frac{\ell}{8\pi G}\int_0^{2\pi} d\phi\,\big(\varepsilon_+(x^+)L_+(x^+) + \varepsilon_-(x^-)L_-(x^-)\big), \tag{3.17}$$

[4] As we will discuss in section 3.1.4, the zero mass BTZ can only be used as a reference to define charges over a patch of phase space connected to it. For other disconnected patches, one should choose other reference points.

where $\phi \sim \phi + 2\pi$ labels the periodic circle S. The charges are manifestly defined everywhere in the bulk in the range of the Bañados coordinates.

Let us finally extend the Bañados geometries beyond the coordinate patch covered by Fefferman-Graham coordinates and comment on the existence of singularities. In the globally asymptotically AdS case, the charges (3.17) are defined by integration on a circle. Since the charges are conserved in the bulk, one can arbitrarily smoothly deform the integration circle and the charge will keep its value, as long as we do not reach a physical singularity or a topological obstruction. Now, if one could deform the circle in the bulk to a single point, the charge would vanish which would be a contradiction. Therefore, the geometries with non-trivial charges, or "hair", are singular in the bulk or contain non-trivial topology which would prevent the circle at infinity to shrink to zero. In the case of global AdS_3 equipped with Virasoro hair, the singularities would be located at defects, where the geometry would not be well-defined. Such defects are just generalizations of other well known defects. For example, in the case of conical defects we have an orbifold-type singularity (deficit angle) and for the BTZ black hole, the singularities arise from closed time-like curves (CTC) which are located behind the locus $r = 0$ in BTZ coordinates [54]. Removal of the CTC's creates a topological obstruction which is hidden behind the inner horizon of the BTZ geometry.

The algebra of conserved charges is defined from the Dirac bracket

$$\{Q_{\chi_1}, Q_{\chi_2}\} = -\delta_{\chi_1} Q_{\chi_2}, \tag{3.18}$$

where the charges have been defined in appendix.

Let us denote the charge associated with the vector $\chi_n^+ = \chi(\varepsilon_+ = e^{inx^+}, \varepsilon_- = 0)$ by L_n and the charge associated with the vector $\chi_n^- = \chi(\varepsilon_+ = 0, \varepsilon_- = e^{inx^-})$ by $\bar{L}_n$. From the definition of charges (3.17) and the transformation rules (3.12), we directly obtain the charge algebra

$$\begin{aligned} \{L_m, L_n\} &= (m-n)L_{m+n} + \frac{c}{12}m^3\delta_{m+n,0}, \\ \{\bar{L}_m, L_n\} &= 0, \\ \{\bar{L}_m, \bar{L}_n\} &= (m-n)\bar{L}_{m+n} + \frac{c}{12}m^3\delta_{m+n,0}, \end{aligned} \tag{3.19}$$

where

$$c = \frac{3\ell}{2G}, \tag{3.20}$$

is the Brown-Henneaux central charge. These are the famous two copies of the Virasoro algebra. In the central term there is no contribution proportional to m as a consequence of the choice of normalization of the charges to zero for the massless BTZ black hole.

In fact, the algebra represents, up to a central extension, the algebra of symplectic symmetries. There is however one subtlety. The symplectic symmetry generators χ are field dependent and hence in computing their bracket we need to "adjust" the Lie bracket by subtracting off the terms coming from the variations of fields within the χ vectors [21,100]. Explicitly,

$$\big[\chi(\varepsilon_1;L),\chi(\varepsilon_2;L)\big]_* = \big[\chi(\varepsilon_1;L),\chi(\varepsilon_2;L)\big]_{L.B} - \Big(\delta^L_{\varepsilon_1}\chi(\varepsilon_2;L) - \delta^L_{\varepsilon_2}\chi(\varepsilon_1;L)\Big), \tag{3.21}$$

where the variations δ^L_ε are defined as

$$\delta^L_{\varepsilon_1}\chi(\varepsilon_2;L) = \delta_{\varepsilon_1}L\,\frac{\partial}{\partial L}\chi(\varepsilon_2;L). \tag{3.22}$$

This is precisely the bracket which lead to the representation of the algebra by conserved charges in the case of field-dependent vector fields. We call $[,]_*$ the *adjusted bracket.* Here the field dependence is stressed by the notation $\chi(\varepsilon;L)$. We also avoided notational clutter by merging the left and right sectors into a compressed notation, $\varepsilon = (\varepsilon_+,\varepsilon_-)$ and $L=(L_+,L_-)$.

Using the adjusted bracket, one can show that symplectic symmetry generators form a closed algebra

$$\big[\chi(\varepsilon_1;L),\chi(\varepsilon_2;L)\big]_* = \chi(\varepsilon_1\varepsilon_2' - \varepsilon_1'\varepsilon_2;L). \tag{3.23}$$

Upon expanding in modes $\chi^\pm_n$, one obtains two copies of the Witt algebra

$$\begin{aligned}
\big[\chi^+_m,\chi^+_n\big]_* &= (m-n)\chi^+_{m+n},\\
\big[\chi^+_m,\chi^-_n\big]_* &= 0,\\
\big[\chi^-_m,\chi^-_n\big]_* &= (m-n)\chi^-_{m+n},
\end{aligned} \tag{3.24}$$

which is then represented by the conserved charges as the centrally extended algebra (3.19).

We discussed in the previous subsections that the phase space of Bañados geometries admits a set of non-trivial tangent perturbations generated by the vector fields χ. Then, there exists finite coordinate transformations (obtained by "exponentiating the χ's") which map a Bañados metric to another one. That is, there are coordinate transformations

$$x^\pm \to X^\pm = X^\pm(x^\pm,r)\,, \qquad r\to R = R(x^\pm,r), \tag{3.25}$$

with $X^\pm, R$ such that the metric $\tilde g_{\mu\nu} = g_{\alpha\beta}\frac{\partial x^\alpha}{\partial X^\mu}\frac{\partial x^\beta}{\partial X^\nu}$ is a Bañados geometry with appropriately transformed $L_\pm$. Such transformations change the physical charges. They are not gauge transformations but are instead solution or charge generating transformations.

Here, we use the approach of Rooman-Spindel [71]. We start by noting that the technical difficulty in "exponentiating" the χ's arise from the fact that χ's are field dependent and hence their form changes as we change the functions $L_\pm$, therefore the method discussed in section 3.3 of [87] cannot be employed here. However, this feature disappears in the large r regime. Therefore, if we can find the form of (3.25) at large r we can read how the $L_\pm$ functions of the two transformed metrics should be related. Then, the subleading terms in r are fixed such that the form of the Bañados metric is preserved. This is guaranteed to work as a consequence of Fefferman-Graham's theorem [91]. From the input of the (flat) boundary metric and first subleading piece (the boundary stress-tensor), one can in principle re-contruct the entire metric.

It can be shown that the finite coordinate transformation preserving (3.2) is

$$\begin{aligned} x^+ \to X^+ &= h_+(x^+) + \frac{\ell^2}{2r^2}\frac{h''_-}{h'_-}\frac{h'_+}{h_+} + \mathcal{O}(r^{-4}), \\ r \to R &= \frac{r}{\sqrt{h'_+ h'_-}} + \mathcal{O}(r^{-1}), \\ x^- \to X^- &= h_-(x^-) + \frac{\ell^2}{2r^2}\frac{h''_+}{h'_+}\frac{h'_-}{h_-} + \mathcal{O}(r^{-4}), \end{aligned} \tag{3.26}$$

where $h_\pm(x^\pm + 2\pi) = h_\pm(x^\pm) \pm 2\pi$, $h_\pm$ are monotonic ($h'_\pm > 0$) so that the coordinate change is a bijection. At leading order (in r), the functions $h_\pm$ parametrize a generic conformal transformation of the boundary metric.

Acting upon the metric by the above transformation one can read how the functions $L_\pm$ transform:

$$L_+(x^+) \quad \to \tilde{L}_+ = {h'_+}^2 L_+ - \frac{1}{2}S[h_+; x^+], \tag{3.27}$$

$$L_-(x^-) \quad \to \tilde{L}_- = {h'_-}^2 L_- - \frac{1}{2}S[h_-; x^-], \tag{3.28}$$

where $S[h; x]$ is the Schwarz derivative

$$S[h(x); x] = \frac{h'''}{h'} - \frac{3h''^2}{2h'^2}. \tag{3.29}$$

It is readily seen that in the infinitesimal form, where $h_\pm(x) = x^\pm + \varepsilon_\pm(x)$, the above reduced to (3.12). It is also illuminating to explicitly implement the positivity of $h'_\pm$ through

$$h'_\pm = e^{\Psi_\pm}, \tag{3.30}$$

where $\Psi_\pm$ are two real single-valued functions. In terms of Ψ fields the Schwarz derivative takes a simple form and the expressions for $\tilde{L}_\pm$ become

$$\begin{aligned} \tilde{L}_+[\Psi_+, L_+] &= e^{2\Psi_+} L_+(x^+) + \frac{1}{4}\Psi_+'^2 - \frac{1}{2}\Psi_+'', \\ \tilde{L}_-[\Psi_-, L_-] &= e^{2\Psi_-} L_-(x^-) + \frac{1}{4}\Psi_-'^2 - \frac{1}{2}\Psi_-''. \end{aligned} \tag{3.31}$$

This reminds the form of a Liouville stress-tensor and dovetails with the fact that AdS_3 gravity with Brown-Henneaux boundary conditions may be viewed as a Liouville theory [93] (see also [78] for a recent discussion).

We finally note that not all functions $h_\pm$ generate new solutions. The solutions to $\tilde{L}_+ = L_+$, $\tilde{L}_- = L_-$ are coordinate transformations which leave the fields invariant: they are finite transformations whose infinitesimal versions are generated by the isometries. There are therefore some linear combinations of symplectic symmetries which do not generate any new charges. These "missing" symplectic charges are exactly compensated by the charges associated with the Killing vectors that we will discuss in section 3.1.3.

3.1.2 Symplectic symmetries in Gaussian null coordinates

In working out the symplectic symmetry generators, their charges and their algebra we used Fefferman-Graham coordinates which are very well adapted to the holographic description. One may wonder if similar results may be obtained using different coordinate systems. This question is of interest because the symplectic symmetries (3.10) were obtained as the set of infinitesimal diffeomorphisms which preserved the Fefferman-Graham condition (3.2) and one may wonder whether the whole phase space and symplectic symmetry setup is dependent upon that particular choice.

Another coordinate frame of interest may be defined a Gaussian null coordinate system,

$$g_{rr} = 0, \qquad g_{ru} = -1, \qquad g_{r\phi} = 0, \tag{3.32}$$

in which ∂_r is an everywhere null vector field. We note along the way that the $\ell \to \infty$ limit can be made well-defined in this coordinate system after a careful choice of the scaling of other quantities [74]. This leads to the BMS_3 group and phase space.

The set of all locally AdS_3 geometries subject to Dirichlet boundary conditions with flat cylindrical boundary metric in such coordinate system takes the form

$$ds^2 = \left(-\frac{r^2}{\ell^2} + 2\ell M(u^+, u^-)\right) du^2 - 2du dr + 2\ell J(u^+, u^-) du d\phi + r^2 d\phi^2, \tag{3.33}$$

where $u^\pm = u/\ell \pm \phi$. Requiring (3.33) to be solutions to AdS_3 Einstein's equations (3.1) implies

$$\ell M(u^+, u^-) = L_+(u^+) + L_-(u^-), \quad J(u^+, u^-) = L_+(u^+) - L_-(u^-). \tag{3.34}$$

As in the Fefferman-Graham coordinates, one may then view the set of metrics g in (3.33) and generic metric perturbations within the same class δg (i.e. metrics with $L_\pm \to L_\pm + \delta L_\pm$) as members of an on-shell phase space and its tangent space. Since the coordinate change between the two Fefferman-Graham and Gaussian null coordinate systems is field dependent, the presymplectic form cannot be directly compared between the two. After direct evaluation, we note here that both the Lee-Wald and the invariant presymplectic forms vanish on-shell

$$\boldsymbol{\omega}^{LW}[\delta g,\delta g;g]\approx 0,\qquad \boldsymbol{\omega}^{inv}[\delta g,\delta g;g]\approx 0,\tag{3.35}$$

since the boundary term which relates them vanishes off-shell, $\boldsymbol{E}[\delta g,\delta g;g]=0$. This implies in particular that the conserved charges defined from either presymplectic form (either Iyer-Wald or Barnich-Brandt charges) will automatically agree.

The phase space of metrics in Gaussian null coordinate system (3.33) is preserved under the action of the vector field ξ

$$\begin{aligned}\xi &= \frac{1}{2}\Big\{\ell(Y_+ + Y_-)\partial_u + \big((Y_+ - Y_-) - \frac{\ell}{r}(Y_+' - Y_-')\big)\partial_\phi \\ &\quad + \left(-r(Y_+' + Y_-') + \ell(Y_+'' + Y_-'') - \frac{\ell^2}{r}(L_+ - L_-)(Y_+' - Y_-')\right)\partial_r\Big\},\end{aligned}\tag{3.36}$$

where $Y_+ = Y_+(u^+)$, $Y_- = Y_-(u^-)$. More precisely, we have

$$\delta_\xi g = \mathcal{L}_\xi g_{\mu\nu}(L_+,L_-) = g_{\mu\nu}(L_+ + \delta_\xi L_+, L_- + \delta_\xi L_-) - g_{\mu\nu}(L_+,L_-),\tag{3.37}$$

where

$$\delta_\xi L_\pm = Y_\pm \partial_\pm L_\pm + 2L_\pm Y_\pm' - \frac{1}{2}Y_\pm''',\tag{3.38}$$

stating that (Fourier modes of) $L_\pm$ are related to generators of a Virasoro algebra.

It is easy to show that the surface charge one-forms are integrable $\boldsymbol{k}_\xi[\delta g;g]=\delta(\boldsymbol{Q}_\xi[g])$ in the phase space. The surface charge one-forms are determined up to boundary terms. It is then convenient to subtract the following subleading boundary term at infinity,

$$\boldsymbol{B}_\xi = \frac{\ell^2}{32\pi G}\left(\frac{1}{r}(L_+ - L_-)(Y_+ + Y_-)\right)\tag{3.39}$$

so that the total charge $\boldsymbol{Q}'_\xi$ is given by the radius independent expression

$$\boldsymbol{Q}'_\xi \equiv \boldsymbol{Q}_\xi - \boldsymbol{d}\boldsymbol{B}_\xi = \frac{\ell}{8\pi G}(L_+Y_+du^+ - L_-Y_-du^-).\tag{3.40}$$

The two sets of Virasoro charges can then be obtained by integration on the circle spanned by ϕ. They obey the centrally extended Virasoro algebra under the Dirac bracket as usual, as a consequence of (3.38). Since the result is exact, the Virasoro charges and their algebra is defined everywhere in the bulk. The symplectic symmetry generators ξ are field dependent (i.e. they explicitly depend on $L_\pm$), and hence their algebra is closed once we use the adjusted bracket defined in subsection 3.1.1. Also note that in the reasoning above we did not use the fact that ϕ is periodic until the very last step where the Virasoro charges are defined as an integral over the circle. If instead the coordinate ϕ is not periodic, as it is relevant to describe AdS_3 with a planar boundary, the Virasoro charge can be replaced by *charge densities*, defined as the one-forms (3.40).

We conclude this section with the fact that both phase spaces constructed above in Fefferman-Graham coordinates and Gaussian null coordinates are spanned by two holomorphic functions and their symmetry algebra and central extension are the same. This implies that there is a one-to-one map between the two phase spaces, and therefore the corresponding holographic dynamics (induced by the Dirac bracket) is not dependent upon choosing either of these coordinate systems. We shall return to this point in the discussion section.

3.1.3 The two Killing symmetries and their charges

So far we discussed the symplectic symmetries of the phase space. These are associated with non-vanishing metric perturbations which are degenerate directions of the on-shell presymplectic form. A second important class of symmetries are the Killing vectors which are associated with vanishing metric perturbations. In this section we analyze these vector fields, their charges and their commutation relations with the symplectic symmetries. We will restrict our analysis to the case of asymptotically globally AdS_3 where ϕ is 2π-periodic. We use Fefferman-Graham coordinates for definiteness but since Killing vectors are geometrical invariants, nothing will depend upon this specific choice.

Killing vectors are vector fields along which the metric does not change. All diffeomorphisms preserving the Fefferman-Graham coordinate system are generated by the vector fields given in (3.10). Therefore, Killing vectors have the same form as χ's, but with the extra requirement that $\delta L_\pm$ given by (3.12) should vanish. Let us denote the functions $\varepsilon_\pm$ with this property by $K_\pm$ and the corresponding Killing vector by ζ (instead of χ). Then, ζ is a Killing vector if and only if

$$K_+''' - 4L_+K_+' - 2K_+L_+' = 0, \qquad K_-''' - 4L_-K_-' - 2K_-L_-' = 0. \quad (3.41)$$

These equations were thoroughly analyzed in [76] and we only provide a summary of the results relevant for our study here. The above linear third order differential equations have three linearly independent solutions and hence Bañados geometries in general have six (local) Killing vectors which form an $sl(2,\mathbb{R})\times sl(2,\mathbb{R})$ algebra, as expected. The three solutions take the form $K_+ = \psi_i\psi_j$, $i,j=1,2$ where $\psi_{1,2}$ are the two independent solutions to the second order Hill's equations

$$\psi'' = L_+(x^+)\psi \tag{3.42}$$

where $L_+(x^+ + 2\pi) = L_+(x^+)$. Therefore, the function K_+ functionally depends upon L_+ but not on L'_+, i.e. $K_+ = K_+(L_+)$. This last point will be crucial for computing the commutation relations and checking integrability as we will shortly see. The same holds for the right moving sector. In general, ψ_i are not periodic functions under $\phi \sim \phi + 2\pi$ and therefore not all six vectors described above are global Killing vectors of the geometry. However, Floquet's theorem [101] implies that the combination $\psi_1\psi_2$ is necessarily periodic. This implies that Bañados geometries have at least two global Killing vectors. Let us denote these two global Killing vectors by $\zeta_\pm$,

$$\zeta_+ = \chi(K_+, K_- = 0; L_\pm), \qquad \zeta_- = \chi(K_+ = 0, K_-; L_\pm), \tag{3.43}$$

where χ is the vector field given in (3.10). These two vectors define two global $U(1)$ isometries of Bañados geometries.

The important fact about these global $U(1)$ isometry generators is that they commute with *each* symplectic symmetry generator χ (3.10): Since the vectors are field-dependent, one should use the adjusted bracket (3.21) which reads explicitly as

$$\big[\chi(\varepsilon;L),\zeta(K;L)\big]_* = \big[\chi(\varepsilon;L),\zeta(K;L)\big]_{L.B.} - \Big(\delta^L_\varepsilon\zeta(K;L) - \delta^L_K\chi(\varepsilon;L)\Big),$$

where the first term on the right-hand side is the usual Lie bracket. Since $K = K(L)$, the adjustment term reads as

$$\delta^L_\varepsilon\zeta(K(L);L) = \delta_\varepsilon L\,\frac{\partial}{\partial L}\zeta(K;L) + \zeta(\delta^L_\varepsilon K;L), \tag{3.44}$$

$$\delta^L_K\chi(\varepsilon;L) = \delta_K L\,\frac{\partial}{\partial L}\chi(\varepsilon;L) = 0 \tag{3.45}$$

where we used the fact that ζ,χ are linear in their first argument as one can see from (3.10) and we used Killing's equation. We observe that we will get only one additional term with respect to the previous computation (3.23) due to the last term in (3.44). Therefore,

$$\big[\chi(\varepsilon;L),\zeta(K(L);L)\big]_* = \zeta(\varepsilon\,K' - \varepsilon'\,K;L) - \zeta(\delta^L_\varepsilon K;L). \tag{3.46}$$

Now the variation of Killing's condition (3.41) implies that

$$(\delta K)''' - 4L(\delta K)' - 2L'\delta K = 4\delta L K' + 2(\delta L)'K.$$

Then, recalling (3.12) and using again (3.41) we arrive at

$$\delta^L_\varepsilon K = \varepsilon\,K' - \varepsilon'\,K, \tag{3.47}$$

and therefore

$$\big[\chi(\varepsilon; L), \zeta(K(L); L)\big]_* = 0. \tag{3.48}$$

The above may be intuitively understood as follows. ζ being a Killing vector field does not transform L, while a generic χ transforms L. Now the function K is a specific function of the metric, $K = K(L)$. The adjusted bracket is defined such that it removes the change in the metric and only keeps the part which comes from Lie bracket of the corresponding vectors as if L did not change.

It is interesting to compare the global Killing symmetries and the symplectic symmetries. The symplectic symmetries are given by (3.10) and determined by functions $\varepsilon_\pm$. The functions $\varepsilon_\pm$ are field independent, so that they are not transformed by moving in the phase space. On the other hand, although the Killing vectors have the same form (3.10), their corresponding functions $\varepsilon_\pm$ which are now denoted by $K_\pm$, are field dependent as a result of (3.41). Therefore the Killing vectors differ from one geometry to another. Accordingly if we want to write the Killing vectors in terms of the symplectic symmetry Virasoro modes $\chi_n^\pm$ (3.24), we have

$$\zeta_+ = \sum_n c_n^+(L_+)\chi_n^+, \qquad \zeta_- = \sum_n c_n^-(L_-)\chi_n^-. \tag{3.49}$$

For example for a BTZ black hole, one can show using (3.41) that the global Killing vectors are $\zeta_\pm = \chi_0^\pm$ while for a BTZ black hole with Virasoro hair or "BTZ Virasoro descendant", which is generated by the coordinate transformations in section 3.1.1, it is a complicated combination of Virasoro modes. For the case of global AdS$_3$ with $L_\pm = -\frac{1}{4}$ (but not for its descendents), (3.41) implies that there are six global Killing vectors which coincide with the subalgebras $\{\chi^+_{1,0,-1}\}$ and $\{\chi^-_{1,0,-1}\}$ of symplectic symmetries.

We close this part by noting the fact that although we focused on single-valued K functions, one may readily check that this analysis and in particular (3.48) is true for any K which solves (3.41). Therefore, all six generators of local $sl(2,\mathbb{R}) \times sl(2,\mathbb{R})$ isometries commute with symplectic symmetry generators χ (3.10). This is of course expected as all geometries (3.5) are locally $sl(2,\mathbb{R}) \times sl(2,\mathbb{R})$ invariant. We shall discuss this point further in section 3.1.7.

Similarly to the Virasoro charges (3.15), the infinitesimal charges associated to Killing vectors can be computed using (A.32), leading to

$$\delta J_+ = \frac{\ell}{8\pi G}\int_0^{2\pi} d\phi\; K_+(L_+)\delta L_+, \qquad \delta J_- = \frac{\ell}{8\pi G}\int_0^{2\pi} d\phi\; K_-(L_-)\delta L_- \tag{3.50}$$

Integrability of Killing charges. Given the field dependence of the K-functions, one may inquire about the integrability of the charges $J_\pm$ over the phase space. In appendix A.2.2, we find the necessary and sufficient condition for the integrability of charges associated with field dependent vectors. However, in the present case, the integrability of $J_\pm$ can be directly checked as follows

$$\delta_1(\delta_2 J) = \frac{\ell}{8\pi G}\oint \delta_1 K(L)\,\delta_2 L = \frac{\ell}{8\pi G}\oint \frac{\partial K}{\partial L}\,\delta_1 L\,\delta_2 L, \tag{3.51}$$

and therefore $\delta_1(\delta_2 J) - \delta_2(\delta_1 J) = 0$.

Having checked the integrability, we can now proceed with finding the explicit form of charges through an integral along a suitable path over the phase space connecting a reference field configuration to the configuration of interest. However, as we will see in section 3.1.4, the Bañados phase space is not simply connected and therefore one cannot reach any field configuration through a path from a reference field configuration. As a result, the charges should be defined independently over each connected patch of the phase space. In section 3.1.4 we will give the explicit form of charges over a patch of great interest, i.e. the one containing BTZ black hole and its descendants. We then find a first law relating the variation of entropy to the variation of these charges.

Algebra of Killing and symplectic charges. We have already shown in the beginning of this section that the adjusted bracket between generators of respectively symplectic and Killing symmetries vanish. If the charges are correctly represented, it should automatically follow that the corresponding charges L_n, J_+ (and $\bar{L}_n, J_-$) also commute:

$$\{J_\pm, L_n\} = \{J_\pm, \bar{L}_n\} = 0. \tag{3.52}$$

Let us check (3.52). By definition we have

$$\{J_+, L_n\} = -\delta_K L_n, \tag{3.53}$$

where one varies the dynamical fields in the definition of L_n with respect to the Killing vector K. Since K leaves the metric unchanged, we have $\delta_K L_+(x^+) = 0$ and therefore directly $\delta_K L_n = 0$. Now, let us also check that the bracket is anti-symmetric by also showing

$$\{L_n, J_+\} \equiv -\delta_{L_n} J_+ = 0. \tag{3.54}$$

This is easily shown as follows:

$$\begin{aligned}
\delta_{L_n} J_+ &= \frac{\ell}{8\pi G}\int_0^{2\pi} d\phi\ K_+ \delta_{\varepsilon_n^+} L \\
&= \frac{\ell}{8\pi G}\int_0^{2\pi} d\phi\ K_+ (\varepsilon_n^+ L'_+ + 2L_+ \varepsilon_n^{+\prime} - \frac{1}{2}\varepsilon_n^{+\prime\prime\prime}) \\
&= \frac{\ell}{8\pi G}\int_0^{2\pi} d\phi\ (-L'_+ K_+ - 2L_+ K'_+ + \frac{1}{2}K'''_+)\varepsilon_{+,n} = 0
\end{aligned} \tag{3.55}$$

after using (3.12), integrating by parts and then using (3.41). The same reasoning holds for J_- and $\bar{L}_n$.

In general, the Bañados phase space only admits two Killing vectors. An exception is the descendants of the vacuum AdS_3 which admit six globally defined Killing vectors. In that case, the two $U(1)$ Killing charges are $J_\pm = -\frac{1}{4}$ and the other four $\frac{SL(2,\mathbb{R})}{U(1)} \times \frac{SL(2,\mathbb{R})}{U(1)}$ charges are identically zero. In the case of the decoupled near-horizon extremal phase space defined in section 3.1.5 we will have four global Killing vectors with the left-moving $U(1)_+$ charge J_+ arbitrary, but the $SL(2,\mathbb{R})_-$ charges all vanishing $J_-^a = 0$, $a = +1, 0, -1$.

3.1.4 Phase space as Virasoro coadjoint orbits

As discussed in the previous sections, one can label each element of the phase space in either Fefferman-Graham coordinates or Gaussian null coordinates, described respectively by (3.5) and (3.33), by its symplectic charges $L_n, \bar{L}_n$ and its global commuting Killing charges $J_\pm$. Moreover, the phase space functions $L_\pm$ transform under the coadjoint action of the Virasoro algebra, see (3.12). Hence, we are led to the conclusion that the phase space forms a reducible representation of the Virasoro group composed of distinct Virasoro coadjoint orbits.

Construction of Virasoro coadjoint orbits has a long and well-established literature, see e.g. [90] and references therein. In this literature the $\delta L_\pm = 0$ (i.e. (3.41)) equation is called the stabilizer equation [85] and specifies the set of transformations which keeps one in the same orbit. The stabilizer equation and classification of its solutions is hence the key to the classification of Virasoro coadjoint orbits. Since an orbit is representation of the Virasoro group it might as well be called a conformal multiplet. The elements in the same orbit/conformal multiplet may be mapped to each other upon the action of coordinate transformations (3.27). Explicitly, a generic element/geometry in the same orbit (specified by $\tilde{L}_\pm$) is related to a single element/geometry with $L_\pm$ given as (3.31) for arbitrary periodic functions $\Psi_\pm$. One can hence classify the orbits by the set of periodic functions $L_\pm(x^\pm)$ which may not be mapped to each other through (3.31). One may also find a specific $L_\pm$, the *representative of the orbit*, from which one can generate the entire orbit by conformal transformations (3.31). In the language of a dual 2d CFT, each orbit may be viewed as a primary operator together with its conformal descendants. Each geometry is associated with (one or many) primary operators or descendants thereof, in the dual 2d CFT. From this discussion it also follows that there is no regular coordinate transformation respecting the chosen boundary conditions, which moves us among the orbits.

Let us quickly summarize some key results from [90]. In order to avoid notation clutter we focus on a single sector, say the + sector (which we refer to as left-movers). One may in general distinguish two classes of orbits: those where a constant representative exists and those where it doesn't. The constant L_+ representatives correspond to the better studied geometries, e.g. see [76, 102] for a review. They fall into four categories:

- *Exceptional orbits* $\mathcal{E}_n$ with representative $L = -n^2/4,\ n = 1, 2, 3, \cdots$. The $\mathcal{E}_1 \times \mathcal{E}_1$ orbit admits global AdS_3 as a representative and therefore corresponds to the vacuum Verma module in the language of a 2d CFT on the cylinder. For $n \geq 2$, $\mathcal{E}_n \times \mathcal{E}_n$ is represented by an n-fold cover of global AdS_3.

- *Elliptic orbits* $\mathcal{C}(\nu)$, with representative $L = -\nu^2/4,\ 0 < \nu < 1$. The geometries with elliptic orbit representatives correspond to conic spaces, particles on AdS_3 [56] and geometries in this orbit may be viewed as "excitations" (descendants) of particles on AdS_3.

- *Hyperbolic orbits* $\mathcal{B}_0(b)$, with representative $L = b^2/4$, where b is generic real number $b > 0$. The geometries with both $L_\pm = b_\pm^2/4$ are BTZ black holes.

- *Parabolic orbit* $\mathcal{P}_0^+$, with representative $L = 0$. The geometries associated with $\mathcal{P}_0^+ \times \mathcal{B}_0(b)$ orbits correspond to the extremal BTZ. The $\mathcal{P}_0^+ \times \mathcal{P}_0^+$ orbit corresponds to AdS_3 in the Poincaré patch and its descendants, which in the dual 2d CFT corresponds to vacuum Verma module of the CFT on 2d plane.

The non-constant representative orbits, come into three categories, the generic *hyperbolic* orbits $\mathcal{B}_n(b)$ and two *parabolic* orbits $\mathcal{P}_n^\pm$, $n \in \mathbb{N}$. Geometries associated with these orbits are less clear and understood. This question has been addressed in [103].

To summarize, if we only focus on the labels on the orbits, the $\mathcal{E}_n, \mathcal{P}_n^\pm$ orbits have only an integer label, the $\mathcal{C}(\nu)$ is labeled by a real number between 0 and 1, and the hyperbolic ones $\mathcal{B}_n(b)$ with an integer and a real positive number.

As shown in (3.55), all the geometries associated with the same orbit have the same $J_\pm$ charges. In other words, $J_\pm$ do not vary as we make coordinate transformations using χ diffeomorphisms (3.10); $J_\pm$ are "orbit invariant" quantities. One may hence relate them with the labels on the orbits, explicitly, J_+ should be a function of b or ν for the hyperbolic or elliptic orbits associated to the left-moving copy of the Virasoro group and J_- a similar function of labels on the right-moving copy of the Virasoro group.

The Bañados phase space has a rich topological structure. It consists of different disjoint patches. Some patches (labeled by only integers) consist of only one orbit, while some consist of a set of orbits with a continuous parameter. On the other hand, note that the conserved charges in covariant phase space methods are defined through an integration of infinitesimal charges along a path connecting a reference point of phase space to a point of interest. Therefore, the charges can be defined only over the piece of phase space simply connected to the reference configuration. For other patches, one should use other reference points. In this work we just present explicit analysis for the $\mathcal{B}_0(b_+) \times \mathcal{B}_0(b_-)$ sector of the phase space. Since this sector corresponds to the family of BTZ black holes of various mass and angular momentum and their descendants, we call it the BTZ sector. Note that there is no regular coordinate transformation respecting the chosen boundary conditions, which moves us among the orbits. In particular for the BTZ sector, this means that there is no regular coordinate transformation which relates BTZ black hole geometries with different mass and angular momentum, i.e. geometries with different $b_\pm$.

We now proceed with computing the charges $J_\pm$ for an arbitrary field configuration in the BTZ sector of the phase space. Since the charges are integrable, one can choose any path from a reference configuration to the desired point. We fix the reference configuration to be the massless BTZ with $L_\pm = 0$. We choose the path to pass by the constant representative $L_\pm$ of the desired solution of interest $\tilde{L}_\pm(x^\pm)$. Let us discuss J_+ (the other sector follows the same logic). Then the charge is defined as

$$J_+ = \int_\gamma \delta J_+ = \int_0^{\tilde{L}_+} \delta J_+ = \int_0^{L_+} \delta J_+ + \int_{L_+}^{\tilde{L}_+} \delta J_+. \tag{3.56}$$

We decomposed the integral into two parts: first the path *across* the orbits, between constant representatives $L_+ = 0$ and L_+ and second the path along (within) a given orbit with representative L_+. Since the path along the orbit does not change the values $J_\pm$ ($\delta_\chi J_\pm = 0$), the second integral is zero. Accordingly, the charge is simply given by

$$J_+ = \frac{\ell}{8\pi G} \int_0^{L_+} \oint d\varphi K_+(L) \delta L \tag{3.57}$$

where L_+ is a constant over the spacetime. Solving (3.41) for constant $L_\pm$ and assuming periodicity of ϕ, we find that $K_\pm = \textit{const}$. Therefore the Killing vectors are $\partial_\pm$ up to a normalization constant, which we choose to be 1. Hence $K_+(L) = 1$, and

$$J_+ = \frac{\ell}{4G} L_+, \qquad J_- = \frac{\ell}{4G} L_-. \tag{3.58}$$

Therefore the Killing charges are a multiple of the Virasoro zero mode of the constant representative.

Since the BTZ descendants are obtained through a finite coordinate transformation from the BTZ black hole, the descendants inherit the causal structure and other geometrical properties of the BTZ black hole. We did not prove that the finite coordinate transformation is non-singular away from the black hole Killing horizon but the fact that the Virasoro charges are defined all the way to the horizon gives us confidence that there is no singularity between the horizon and the spatial boundary. The geometry of the Killing horizon was discussed in more detail in [76].

The area of the outer horizon defines a geometrical quantity which is invariant under diffeomorphisms. Therefore the BTZ descendants admit the same area along the entire orbit. The angular velocity and surface gravity are defined geometrically as well, given a choice of normalization at infinity. This choice is provided for example by the asymptotic Fefferman-Graham coordinate system which is shared by all BTZ descendants. Therefore these chemical potentials $\tau_\pm$ are also orbit invariant and are identical for all descendants and in particular are constant. This is the zeroth law for the BTZ descendant geometries.

One may define more precisely $\tau_\pm$ as the chemical potentials conjugate to $J_\pm$ [103]. Upon varying the parameters of the solutions we obtain a linearized solution which obeys the first law

$$\delta S = \tau_+ \delta J_+ + \tau_- \delta J_- . \tag{3.59}$$

This first law is an immediate consequence of the first law for the BTZ black hole since all quantities are geometrical invariants and therefore independent of the orbit representative. In terms of $L_\pm$, the constant representatives of the orbits in the BTZ sector, one has (3.58) and [65]

$$\tau_\pm = \frac{\pi}{\sqrt{L_\pm}} \tag{3.60}$$

and the entropy takes the usual Cardy form

$$S = \frac{\pi}{3} c(\sqrt{L_+} + \sqrt{L_-}). \tag{3.61}$$

One can also write the Smarr formula in terms of orbit invariants as

$$S = 2(\tau_+ J_+ + \tau_- J_-). \tag{3.62}$$

The only orbits which have a continuous label (necessary to write infinitesimal variations) and which admit a bifurcate Killing horizon are the hyperbolic orbits [76,103]. The extension of the present discussion to generic hyperbolic orbits (and not just for the BTZ sector) has been discussed in [103].

3.1.5 Extremal phase space and decoupling limit in Fefferman-Graham coordinates

We define the "extremal phase space" as the subspace of the set of all Bañados geometries (equipped with the invariant presymplectic form) with the restriction that the right-moving function L_- vanishes identically. The Killing charge J_- is therefore identically zero. Also, perturbations tangent to the extremal phase space obey $\delta L_- = 0$ but δL_+ is an arbitrary left-moving function.

A particular element in the extremal phase space is the extremal BTZ geometry with $M\ell = J$. It is well-known that this geometry admits a decoupled near-horizon limit which is given by the self-dual spacelike orbifold of AdS_3 [93]

$$ds^2 = \frac{\ell^2}{4}\left(-r^2dt^2 + \frac{dr^2}{r^2} + \frac{4|J|}{k}(d\phi - \frac{r}{2\sqrt{|J|/k}}dt)^2\right), \quad \phi \sim \phi + 2\pi, \tag{3.63}$$

where $k \equiv \frac{\ell}{4G}$. A Virasoro algebra exists as asymptotic symmetry in the near-horizon limit and this Virasoro algebra has been argued to be related to the asymptotic Virasoro algebra defined close to the AdS_3 spatial boundary [73]. Since these asymptotic symmetries are defined at distinct locations using boundary conditions it is not entirely obvious that they are uniquely related. Now, using the concept of symplectic symmetries which extend the asymptotic symmetries to the bulk spacetime, one deduces that the extremal black holes are equipped with one copy of Virasoro hair. The Virasoro hair transforms under the action of the Virasoro symplectic symmetries, which are also defined everywhere outside of the black hole horizon.

One subtlety is that the near-horizon limit is a decoupling limit obtained after changing coordinates to near-horizon comoving coordinates. We find two interesting ways to take the near-horizon limit. In Fefferman-Graham coordinates the horizon is sitting at $r = 0$ and it has a constant angular velocity $1/\ell$ *independently of the Virasoro hair*. Therefore taking a near-horizon limit is straightforward and one readily obtains the near-horizon Virasoro symmetry. It is amusing that the resulting vector field which generates the symmetry differs from the ansatz in [73], as well as the original Kerr/CFT ansatz [89] and the newer ansatz for generic extremal black holes [79, 87]. The difference is however a vector field which is pure gauge, i.e. charges associated with it are zero.

A second interesting way to take the near-horizon limit consists in working with coordinates such that the horizon location depends upon the Virasoro hair. This happens in Gaussian null coordinates. Taking the near-horizon limit then requires more care. This leads to a yet different Virasoro ansatz for the vector field which is field dependent. After working out the details, a chiral half of the Virasoro algebra is again obtained, which also shows the equivalence with the previous limiting procedure.

The general metric of the extremal phase space of AdS_3 Einstein gravity with Brown-Henneaux boundary conditions and in the Fefferman-Graham coordinate system is given by

$$ds^2 = \frac{\ell^2}{r^2}dr^2 - r^2 dx^+ dx^- + \ell^2 L(x^+) dx^{+2}, \qquad x^\pm = t/\ell \pm \phi, \qquad \phi \sim \phi + 2\pi \tag{3.64}$$

where we dropped the + subscript, $L_+ = L$. It admits two global Killing vectors: ∂_- and ζ_+ defined in subsection 3.1.3. In the case of the extremal BTZ orbit, the metrics (3.64) admit a Killing horizon at $r = 0$ which is generated by the Killing vector ∂_- [76].

One may readily see that a diffeomorphism $\chi(\epsilon_+, \epsilon_- = 0)$ defined from (3.10) with arbitrary $\epsilon_+(x^+)$, namely

$$\chi_{ext} = \frac{\ell^2 \epsilon_+''}{2r^2}\partial_- + \epsilon_+ \partial_+ - \frac{1}{2} r \epsilon_+' \partial_r, \tag{3.65}$$

is tangent to the phase space. Indeed, it preserves the form of the metric (3.64). Remarkably, the field dependence, i.e. the dependence on L_+, completely drops out in χ_{ext}. Note however that although χ_{ext} is field independent, the Killing vector ζ_+ is still field dependent. From the discussions of section 3.1.1 it immediately follows that χ_{ext} generates symplectic symmetries.

One may then take the decoupling limit

$$t \to \frac{\ell \tilde{t}}{\lambda}, \qquad \phi \to \phi + \Omega_{ext}\frac{\ell \tilde{t}}{\lambda}, \qquad r^2 \to 2\ell^2 \lambda \tilde{r}, \qquad \lambda \to 0 \tag{3.66}$$

where $\Omega_{ext} = -1/\ell$ is the constant angular velocity at extremality. As a result $x^+ \to \phi$ and $x^- \to 2\frac{\tilde{t}}{\lambda} - \phi$. Functions periodic in x^+ are hence well-defined in the decoupling limit while functions periodic in x^- are not. Therefore, the full Bañados phase space does not admit a decoupling limit. Only the extremal part of the Bañados phase space does. Also, since $\frac{\tilde{t}}{\lambda}$ is dominant with respect to ϕ in the near-horizon limit, the coordinate x^- effectively decompactifies in the limit while x^+ remains periodic.

In this limit the metric (3.64) and symplectic symmetry generators (3.65) become

$$\frac{ds^2}{\ell^2} = \frac{d\tilde{r}^2}{4\tilde{r}^2} - 4\tilde{r} d\tilde{t} d\phi + L(\phi) d\phi^2 \tag{3.67}$$

$$\chi_{ext} = \frac{\epsilon''(\phi)}{8\tilde{r}}\partial_{\tilde{t}} - \tilde{r}\epsilon'(\phi)\partial_{\tilde{r}} + \epsilon(\phi)\partial_\phi, \tag{3.68}$$

where we dropped again the + subscript, $\varepsilon_+ = \varepsilon$. As it is standard in such limits, this geometry acquires an enhanced global $SL(2,\mathbb{R})_- \times U(1)_+$ isometry [75, 76]. The $sl(2,\mathbb{R})_-$ Killing vectors are given as

$$\xi_1 = \frac{1}{2}\partial_{\tilde{t}}, \quad \xi_2 = \tilde{t}\partial_{\tilde{t}} - \tilde{r}\partial_{\tilde{r}}, \quad \xi_3 = [(2\tilde{t}^2 + \frac{L}{8\tilde{r}^2})\partial_{\tilde{t}} + \frac{1}{2\tilde{r}}\partial_\phi - 4\tilde{t}\tilde{r}\partial_{\tilde{r}}]. \tag{3.69}$$

and obey the algebra

$$[\xi_1, \xi_2] = \xi_1, \qquad [\xi_1, \xi_3] = 2\xi_2, \qquad [\xi_2, \xi_3] = \xi_3, \tag{3.70}$$

The $u(1)_+$ is still generated by ζ_+.

As it is explicitly seen from the metric (3.67), absence of Closed Timelike Curves (CTC) requires $L(\phi) \geq 0$. This restricts the possibilities for orbits which admit a regular decoupling limit. The obvious example is the extremal BTZ orbit for which the decoupling limit is a near-horizon limit. Representatives of these orbits are the extremal BTZ black holes with $L_+ \geq 0$ constant and the near-horizon metric (3.67) is precisely the self-dual orbifold (3.63) after recognizing $J = \frac{\ell}{4G} L = \frac{c}{6} L$ and setting $\tilde{t} = \sqrt{L_+} t/4$ and $\tilde{r} = r$.[5]

From the analysis provided in [90] one can gather that all orbits other than the hyperbolic $\mathcal{B}_0(b)$ and the parabolic $\mathcal{P}_0^+$ orbits, admit a function $L(\phi)$ which can take negative values. The corresponding geometries therefore contain CTCs. The only regular decoupling limit is therefore the near-horizon limit of generic extremal BTZ (including massless BTZ [104]). Therefore, the near-horizon extremal phase space is precisely the three-dimensional analogue of the phase space of more generic near-horizon extremal geometries discussed in [86, 87].

Under the action of χ_{ext} above, one has

$$\delta_\chi L(\phi) = \epsilon L(\phi)' + 2L(\phi)\epsilon' - \frac{1}{2}\epsilon''' \tag{3.71}$$

in the decoupling limit. With the mode expansion $\epsilon = e^{in\phi}$, one may define the symplectic symmetry generators l_n which satisfy the Witt algebra,

$$i[l_m, l_n] = (m - n) l_{m+n}. \tag{3.72}$$

The surface charge is integrable and given by

$$H_\chi[\Phi] = \frac{\ell}{8\pi G} \oint d\phi \, \epsilon(\phi) L(\phi). \tag{3.73}$$

Moreover, one may show that the surface charges associated to the $SL(2, \mathbb{R})_-$ Killing vectors, J_-^a, vanish. Interestingly, we find that the $\tilde{t}$ and $\tilde{r}$ components of χ_{ext} (3.68) do not contribute to the surface charges. The various ansatzes described in [73, 79, 87, 89] which differ precisely by the $\partial_{\tilde{t}}$ term are therefore physically equivalent to the one in (3.68).

One may also work out the algebra of charges H_n associated with $\epsilon = e^{in\phi}$:

$$\{H_m, H_n\} = (m - n) H_{m+n} + \frac{c}{12} m^3 \delta_{m+n,0}, \tag{3.74}$$

[5] For the case of the massless BTZ, one should note that there are two distinct near-horizon limits; the first leads to null self-dual orbifold of AdS_3 and the second to the pinching AdS_3 orbifold [104].

where c is the usual Brown-Henneaux central charge.

The charge J_+ associated with the Killing vector ζ_+ commutes with the H_n's, as discussed in general in section 3.1.3. Following the analysis of section 3.1.4, one may associate an entropy S and chemical potential τ_+ which satisfy the first law and Smarr relation

$$\delta S = \tau_+ \delta J_+ \,, \qquad S = 2\tau_+ J_+ . \tag{3.75}$$

These are the familiar laws of "near horizon extremal geometry (thermo)dynamics" presented in [105, 106].

3.1.6 Extremal phase space and and near horizon limit in Gaussian null coordinates

Let us now consider the analogue extremal phase space but in Gaussian null coordinates. It is defined from the complete phase space discussed in section 3.1.2 by setting the right-moving function $L_- = 0$. The metric is

$$ds^2 = (-\frac{r^2}{\ell^2} + 2L_+(u^+))du^2 - 2dudr + 2\ell L_+(u^+)dud\phi + r^2 d\phi^2, \tag{3.76}$$

where $u^\pm = u/\ell \pm \phi$, $\phi \sim \phi + 2\pi$. It depends upon a single function $L_+(u^+)$. One may analyze the isometries of metrics (3.76). The Killing vectors are within the family of ξ's (3.36) with $\delta_\xi L_\pm = 0$ (*cf.* (3.38)). Since $L_- = 0$ in this family, there are three local Killing vectors associated with solutions of $Y_-''' = 0$, i.e. $Y_- = 1, u^-, (u^-)^2$. The first Killing vector is $\xi_1 = \partial_- = \frac{1}{2}(\ell\partial_u - \partial_\phi)$. The other two are not globally single-valued but we will display them for future use,

$$\begin{aligned} \xi_2 &= u^-\partial_- + \frac{\ell}{2r}\partial_\phi - \frac{1}{2}(r - \frac{\ell^2 L_+}{r})\partial_r, \\ \xi_3 &= (u^-)^2\partial_- + u^-\frac{\ell}{r}\partial_\phi + \big[\ell - u^-(r - \frac{\ell^2 L_+}{r})\big]\partial_r. \end{aligned} \tag{3.77}$$

Together they form an $sl(2,\mathbb{R})$ algebra (3.70). There is also a global $U(1)_+$ associated with the Y_+ functions, which is the periodic solution to $\delta_\xi L_+ = 0$.

The set of geometries (3.76) together with $\xi(Y_+, Y_- = 0)$ (*cf.* (3.36)) form a phase space, elements of which fall into the Virasoro coadjoint orbits. Orbits are labeled by J_+. We consider for simplicity only the extremal BTZ orbit. The above geometries then have a Killing horizon at variable radius $r = r_H(u^+)$, unlike the Fefferman-Graham coordinate system studied in the previous section. The function $r_H(u^+)$ is defined from the function $L_+(u^+)$ through

$$\ell\frac{dr_H}{du^+} + r_H^2 = \ell^2 L_+(u^+)\,. \tag{3.78}$$

This Killing horizon is generated by the Killing vector ∂_-. Requiring the function r_H to be real imposes a constraint on the Virasoro zero mode $\int_0^{2\pi} du^+ L_+(u^+) \geq 0$ which is obeyed in the case of the hyperbolic $\mathcal{B}_0(b)$ orbit. It is notable that upon replacing $r_H = \ell\frac{\psi'}{\psi}$, (3.78) exactly reduces to Hill's equation $\psi'' = L_+\psi$.

Let us now perform the following near-horizon limit,

$$r = r_H(u^+) + \varepsilon\hat{r}, \qquad u = \frac{\hat{u}}{\varepsilon}, \qquad \phi = \hat{\phi} + \Omega_{ext}\frac{\hat{u}}{\varepsilon}, \qquad \varepsilon \to 0 \tag{3.79}$$

where $\Omega_{ext} = -\frac{1}{\ell}$ is the extremal angular velocity. In this limit $u^+ = \hat{\phi}$ is kept finite. The metric takes the form

$$ds^2 = -2d\hat{r}d\hat{u} - 4\hat{r}\frac{r_H(\hat{\phi})}{\ell}d\hat{u}d\hat{\phi} + r_H^2(\hat{\phi})d\hat{\phi}^2. \tag{3.80}$$

Note also that $r_H(\hat{\phi})$ is a function of $L_+(\hat{\phi})$, as is given in (3.78). For constant r_H the metric (3.80) is the self-dual AdS$_3$ metric. In general, it admits a $SL(2,\mathbb{R})_- \times U(1)_+$ global isometry. The explicit form of generators of $SL(2,\mathbb{R})_-$ are obtained from (3.77) upon the limit (3.79) as

$$\begin{aligned} \xi_1 &= \frac{\ell}{2}\partial_{\hat{u}}, \quad \xi_2 = \hat{u}\partial_{\hat{u}} - \hat{r}\partial_{\hat{r}} + \frac{\ell}{2r_H(\hat{\phi})}\partial_{\hat{\phi}}, \\ \xi_3 &= \frac{2\hat{u}^2}{\ell}\partial_{\hat{u}} + (\ell - \frac{4\hat{u}\hat{r}}{\ell})\partial_{\hat{r}} + \frac{2\hat{u}}{r_H(\hat{\phi})}\partial_{\hat{\phi}}. \end{aligned} \tag{3.81}$$

Let us now analyze the presymplectic form and the corresponding charges. To this end, we first recall that we obtained in section 3.1.2 that both the Lee-Wald and the invariant presymplectic form vanish on-shell for the general case. Therefore, both presymplectic structures also vanish for the special case $L_- = 0$. All transformations that preserve the phase space are therefore either symplectic symmetries or pure gauge transformations, depending on whether or not they are associated with non-vanishing conserved charges.

The symplectic symmetry vector field generators $\hat{\xi}$ may naively be defined from (3.36), where we set $L_- = Y_- = 0$ and take the above near horizon limit. Doing so we obtain:

$$\hat{\xi} = (Y - \frac{\ell}{2r_H}Y')\partial_{\hat{\phi}} - \frac{1}{\varepsilon}\left(r_H Y - \frac{\ell Y'}{2}\right)'\partial_{\hat{r}},$$

where $Y = Y(\hat{\phi}), r_H = r_H(\hat{\phi})$ and primes denotes derivatives with respect to $\hat{\phi}$. Since this vector field admits a diverging $1/\varepsilon$ term, it is not well-defined in the near-horizon limit. Moreover, this vector field does not generate perturbations tangent to the near-horizon phase space. In doing the near-horizon change of coordinates, it is required to change the generator of symplectic symmetries. One may check that a term like $f(\hat{\phi})\partial_{\hat{r}}$ for both Barnich-Brandt or Iyer-Wald charges is pure gauge since it does not contribute to the charges. Therefore, the problematic $1/\varepsilon$ term may be dropped from $\hat{\xi}$ to obtain

$$\hat{\xi} = (Y - \frac{\ell}{2r_H}Y')\partial_{\hat{\phi}}\,. \tag{3.82}$$

In fact, the vector field (3.82) is the correct vector field in the near-horizon phase space since $\mathcal{L}_\xi g_{\mu\nu}$ is tangent to the phase space (3.80) with the transformation law

$$\delta_\xi r_H = r_H\partial_{\hat{\phi}}Y + Y\partial_{\hat{\phi}}r_H - \frac{\ell}{2}\partial^2_{\hat{\phi}}Y\,. \tag{3.83}$$

This transformation law is consistent with the definition (3.78) and the Virasoro transformation law (3.38). It is tricking that the resulting symplectic symmetry generator (3.82) takes a quite different form from (3.68) as well as all other ansatzes in the literature [73, 79, 87, 89].

Using the expansion in modes $Y = e^{in\hat{\phi}}$ we define the resulting vector field l_n. Since the vector field is field-dependent, we should use the "adjusted bracket" defined in section 3.1.1. Doing so, we obtain the Witt algebra

$$i[l_m, l_n]_* = (m-n)l_{m+n}. \tag{3.84}$$

One may then check that the surface charges associated with $\hat{\xi}$ are integrable, using the integrability condition for general field dependent generators, *cf.* discussions of Appendix A.2.2. For the surface charges the Barnich-Brandt and Iyer-Wald prescriptions totally agree since the invariant and Lee-Wald presymplectic forms coincide off-shell. We then obtain

$$Q_{\hat{\xi}} = \frac{1}{8\pi G}\int \left(\frac{r_H^2}{\ell}Y - r_H Y'\right)d\hat{\phi}. \tag{3.85}$$

After adding a boundary term $\boldsymbol{dB}_{\hat{\xi}}$ where,

$$\boldsymbol{B}_{\hat{\xi}} = \frac{1}{8\pi G}r_H Y, \tag{3.86}$$

to the integrand and after using (3.78), we find the standard Virasoro charge

$$Q_{\hat{\xi}} = \frac{\ell}{8\pi G}\int L_+(\hat{\phi})Y(\hat{\phi})d\hat{\phi}. \tag{3.87}$$

We have therefore shown that the near-horizon Virasoro symplectic symmetry can be directly mapped to the Brown-Henneaux asymptotic symmetry at the boundary of AdS_3.

3.1.7 Discussion and outlook

We established that the set of all locally AdS_3 geometries with Brown-Henneaux boundary conditions form a phase space whose total symmetry group is *in general* a direct product between the left and right sector and between $U(1)$ Killing and Virasoro symplectic symmetries quotiented by a compact $U(1)$:

$$\left(U(1)_+ \times \frac{Vir_+}{U(1)_+}\right) \times \left(U(1)_- \times \frac{Vir_-}{U(1)_-}\right). \tag{3.88}$$

Elements of the phase space are solutions with two copies of “Virasoro hair” which can have two different natures: either Killing symmetry charges or symplectic symmetry charges. One special patch of the phase space consists of the set of descendants of the global AdS_3 vacuum, where the two compact $U(1)$'s are replaced with two $SL(2,\mathbb{R})$'s with compact $U(1)$ subgroup:

$$\left(SL(2,\mathbb{R})_+ \times \frac{Vir_+}{SL(2,\mathbb{R})_+}\right) \times \left(SL(2,\mathbb{R})_- \times \frac{Vir_-}{SL(2,\mathbb{R})_-}\right). \tag{3.89}$$

In the case of the phase space with Poincaré AdS boundary conditions, the $U(1)$'s are instead non-compact.

In the case of the decoupling (near-horizon) limit of extremal black holes, the (let say) right sector is frozen to $L_- = 0$ in order to be able to define the decoupling limit. In the limit the $U(1)_-$ isometry is enhanced to $SL(2,\mathbb{R})_-$ and the $U(1)_-$ subgroup decompactifies. The exact symmetry group of the near-horizon phase space is a direct product of the left-moving Killing and left-moving non-trivial symplectic symmetries, isomorphic to Virasoro group quotiented by a compact $U(1)_+$,

$$SL(2,\mathbb{R})_- \times \left(U(1)_+ \times \frac{Vir_+}{U(1)_+}\right). \tag{3.90}$$

The global Killing $SL(2,\mathbb{R})_-$ charges are fixed to zero, and there is no right-moving symplectic symmetry. We studied two particular decoupling limits which realized this symmetry. Taking the decoupling limit in Fefferman-Graham coordinates leads to zooming at fixed coordinate horizon radius while taking the decoupling limit in Gaussian null coordinates amounts to zooming on a wiggling horizon radius. We noticed that both decoupling limits lead to the same charge algebra. In principle it should also be possible to have geometries associated with

$$\left(SL(2,\mathbb{R})_+ \times \frac{Vir_+}{SL(2,\mathbb{R})_+}\right) \times \left(U(1)_- \times \frac{Vir_-}{U(1)_-}\right),$$

where the representative of the left-movers is fixed to have $L_+ = -1/4$.

Orbits and Killing charges. The above obviously parallels the construction of Virasoro coadjoint orbits where the group that quotients the Virasoro group is the "stabilizer group" [85, 90]. The stabilizer group, as intuitively expected, appears as the Killing isometry algebra of the locally AdS_3 geometries. Importantly for making connection with Virasoro orbits, the Killing vectors commute with the Virasoro symmetries. Their associated conserved charges $J_\pm$ therefore label individual orbits. There are, nonetheless, other options for the stabilizer group besides compact $U(1)$ and $SL(2,\mathbb{R})$ which are, in general, labeled by n-fold cover of these stabilizer groups. This will lead to an extra integer label which being discrete, is not covered in the analysis of the type we presented here. This may be associated with a topological charge [103].

Relationship with asymptotic symmetries. It is well-known that all geometries with Brown-Henneaux boundary conditions admit two copies of the Virasoro group as asymptotic symmetry group [28]:

$$Vir_+ \times Vir_- \,. \tag{3.91}$$

In the case of the vacuum AdS_3 orbit, the global asymptotic $SL(2,\mathbb{R})_+ \times SL(2,\mathbb{R})_-$ subgroup of the Virasoro group exactly coincides with the $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ isometries with constant charges and the asymptotic symmetries reduce to (3.89). For generic orbits, only a $U(1)_+ \times U(1)_-$ subgroup of the $SL(2,\mathbb{R})_+ \times SL(2,\mathbb{R})_-$ is an isometry while the remaining generators are symplectic symmetries, which matches with (3.88). The novelty is that the conserved charges are not defined at infinity only, they are defined at finite radius.

Symplectic charges and the Gauss law. The electric charge of a set of electrons can be computed as the integral of the electric flux on an enclosing surface. It was observed some time ago that Killing symmetries lead to the same property for gravity [107]. The total mass of a set of isolated masses at equilibrium can be obtained by integrating the Killing surface charge on an enclosing surface. This property arises after viewing gravity as a gauge theory on the same footing as Maxwell theory. Here, we generalized the result of [107] to symplectic symmetries. Given a configuration with a symplectic symmetry and given a surface in a given homology class, one can define associated symplectic charge which is conserved upon smoothly deforming the surface.

On the presymplectic form. We reviewed the definition of the Lee-Wald and the invariant presymplectic forms and noticed that only the invariant one was vanishing on-shell in both Fefferman-Graham and Gaussian null coordinates. This enabled us to define symplectic symmetries on any closed circle which encloses all geometrical and topological defects. Together with the Killing symmetries, they extend the asymptotic symmetries of Brown-Henneaux in the bulk spacetime with identical results in both coordinate systems. However, the Lee-Wald presymplectic structure is equal on-shell to a boundary term in Fefferman-Graham coordinates. A natural question is whether a suitable boundary term can be added to the Lee-Wald presymplectic structure which fits among the known ambiguities in order that it vanishes exactly on-shell. We expect that it would be possible, but for our purposes the existence of an on-shell vanishing presymplectic structure was sufficient.

Coordinate independence and gauge transformations. Every structure we could find in Fefferman-Graham coordinates could be mapped onto the same structure in Gaussian null coordinates. We therefore expect that there is a gauge transformation between these coordinate systems which can be defined in the bulk spacetime. On general grounds, we expect that one could enhance the set of metrics with additional gauge transformation redundancy and incorporate more equivalent coordinate systems. Such a procedure would however not add any physics classically since the physical phase space and charges would be left invariant. The advantage of either Fefferman-Graham or Gaussian null coordinates is that their only admissible coordinate transformations (which preserve the coordinates) are the physical symplectic and Killing symmetries. In that sense, they allow to express the phase space in a fixed gauge.

Generalization to other boundary conditions. Boundary conditions alternative to Dirichlet boundary conditions exist for AdS_3 Einstein gravity [94–96]. Our considerations directly apply to these boundary conditions as well. As an illustration, the semi-direct product of Virasoro and Kaç-Moody asymptotic symmetries found for chiral boundary conditions [95] can be extended to symplectic symmetries in the corresponding phase space. Indeed, it is easy to check that both the Lee-Wald and the invariant symplectic structures vanish for arbitrary elements in, and tangent to, the phase space. The BTZ black holes equipped with Virasoro and Kaç-Moody charges can be qualified as BTZ black holes with Virasoro and Kaç-Moody hair all the way to the horizon.

3.2 Three-dimensional asymptotically flat Einstein-Maxwell theory

The original studies of four-dimensional asymptotically flat spacetimes at null infinity [4, 6, 108] and their extensions to include the electromagnetic field [109,110] rely on an expansion in inverse powers of the radial coordinate

r for the metric components or the spin and tetrad coefficients. In order to guarantee a self-consistent solution space, some of these expansions need well-chosen gaps so as to prevent the appearance of logarithmic terms in r.

In more recent investigations, this assumption has been relaxed. More general consistent solution spaces have been proposed that involve double series with inverse powers and logarithms in r from the very beginning. Details on such "polyhomogeneous spacetimes" can be found for instance in [111–113].

Another non trivial aspect of 4d spacetimes with non trivial asymptotics at Scri is that charges associated to the asymptotic symmetry transformations, even though well-defined, are neither conserved nor integrable [114]. Furthermore, when considering a local version of the asymptotic symmetry algebra [20, 21], the associated current algebra acquires a field dependent central extension [22, 53].

In contrast, three-dimensional asymptotically flat Einstein gravity at null infinity is much easier, in the sense that the expansion in inverse powers of r of the general solution with non trivial asymptotics can be shown not to admit logarithms and to truncate after the leading order terms [21]. The symmetry algebra [115] is $\mathfrak{bms}_3$, the charges are conserved, integrable (and also r independent [79]), while their algebra involves a constant central extension [116], closely related to the one for asymptotically anti-de Sitter spacetimes [28].

The purpose of the present section is to study three-dimensional Einstein-Maxwell theory with asymptotically flat boundary conditions at null infinity. This model allows one to illustrate several aspects of the four dimensional case in a simplified setting. On the one hand, there is a clear physical reason for the occurrence of logarithms as such a term is needed in the time component of the gauge potential in order to generate electric charge. This term leads to a self-consistent polyhomogeneous solution space that includes the charged analog of particle [55] and cosmological solutions [74, 117–119]. The latter correspond to the flat space limit of the three-dimensional charged rotating asymptotically anti-de Sitter black holes [120]. On the other hand, the asymptotic symmetry algebra is a Virasoro-Kac-Moody type algebra that extends the $\mathfrak{bms}_3$ algebra of the purely gravitational case. The associated surface charges turn out to be neither conserved nor integrable due to the presence of electromagnetic news. Furthermore the algebra of surface charges now involves a field dependent central charge that persists when switching off the news.

3.2.1 Asymptotic symmetries

To work out the asymptotic symmetries, we follow closely the original literature [7] and adapt it to the current context. More generally, for the Einstein-Yang-Mills system in all dimensions greater than 3, this problem has been addressed recently in detail in a unified way both for flat and anti-de Sitter backgrounds in [121]. In this approach, the gauge fixing condition in the definition of asymptotic flat spacetimes fix the radial dependence of gauge parameters completely, while the fall-off conditions fix the temporal dependence. In the current set-up, the fall-off conditions on A_u are more relaxed as compared to those considered in section 5.5 of [121] in order to account for non-vanishing electric charge. As a consequence, the time dependence of the electromagnetic gauge parameter is no longer fixed, unless one switches off the news.

In order to define asymptotic flatness of the three-dimensional Einstein-Maxwell at future null infinity, coordinates u, r, ϕ are used together with the gauge fixing ansatz

$$g_{\mu\nu} = \begin{pmatrix} Ve^{2\beta} + r^2U^2 & -e^{2\beta} & -r^2U \\ -e^{2\beta} & 0 & 0 \\ -r^2U & 0 & r^2 \end{pmatrix}, \quad A_r = 0, \tag{3.92}$$

where U, β, V and A_u, A_ϕ are functions of u, r, ϕ. Suitable fall-off conditions that allow for non-vanishing electric charge are

$$\begin{aligned} U = o(r^{-1}), \quad V = o(r), \quad \beta = o(r^0), \\ A_u = O(\ln \frac{r}{r_0}), \quad A_\phi = O(\ln \frac{r}{r_0}), \end{aligned} \tag{3.93}$$

where r_0 is a constant radial scale.

The gauge structure of Einstein-Maxwell theory can be described as follows. Gauge parameters are pairs (ξ^μ, ϵ) consisting of a vector field $\xi^\mu \partial_\mu$ and a scalar ϵ. A generating set of gauge symmetries can be chosen as

$$-\delta_{(\xi,\epsilon)} g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu}, \quad -\delta_{(\xi,\epsilon)} A_\mu = \mathcal{L}_\xi A_\mu + \partial_\mu \epsilon. \tag{3.94}$$

When the gauge parameters are field dependent, as will be the case for the parameters of asymptotic symmetries below, the commutator of gauge transformations contains additional terms:

$$\left[\delta_{(\xi_1,\epsilon_1)}, \delta_{(\xi_2,\epsilon_2)}\right] (g_{\mu\nu}, A_\mu) = \delta_{[(\xi_1,\epsilon_1),(\xi_2,\epsilon_2)]_M} (g_{\mu\nu}, A_\mu), \tag{3.95}$$

where the Lie (algebroid) bracket for field dependent gauge parameters is defined through

$$\begin{aligned} &[(\xi_1, \epsilon_1), (\xi_2, \epsilon_2)]_M = (\hat{\xi}, \hat{\epsilon}), \\ &\hat{\xi} = [\xi_1, \xi_2] + \delta_{(\xi_1,\epsilon_1)} \xi_2 - \delta_{(\xi_2,\epsilon_2)} \xi_1 \ , \ \hat{\epsilon} = \xi_1(\epsilon_2) + \delta_{(\xi_1,\epsilon_1)} \epsilon_2 - (1 \leftrightarrow 2). \end{aligned} \tag{3.96}$$

Gauge transformations preserving asymptotically flat configurations are explicitly worked out in Appendix **A.3.1**. They are determined by gauge parameters depending linearly and homogeneously on arbitrary functions $T(\phi), Y(\phi), E(u,\phi)$ according to

$$\begin{aligned}
\xi^u &= f = T + uY', \\
\xi^\phi &= Y - f' \int_r^\infty \frac{e^{2\beta}}{r'^2} dr' = Y - \frac{f'}{r} + o(r^{-2}), \\
\xi^r &= -r\partial_\phi \xi^\phi + rUf' = -rY' + f'' + o(1), \\
\epsilon &= E(u,\phi) + f' \int_r^\infty \frac{e^{2\beta} A_\phi}{r'^2} dr' = E(u,\phi) + O(\frac{\ln \frac{r}{r_0}}{r}),
\end{aligned} \tag{3.97}$$

where dot and prime denote u and ϕ derivatives, respectively.

Consider then the "$\mathfrak{bms}_3$/Maxwell" Lie algebra consisting of triples $s = (T, Y, E)$ with bracket

$$[s_1, s_2] = \left(\hat{T}, \hat{Y}, \hat{E}\right), \tag{3.98}$$

where

$$\hat{T} = Y_1 T_2' + T_1 Y_2' - (1 \leftrightarrow 2),\ \hat{Y} = Y_1 Y_2' - (1 \leftrightarrow 2),\ \hat{E} = Y_1 E_2' + f_1 \dot{E}_2 - (1 \leftrightarrow 2). \tag{3.99}$$

This is the asymptotic symmetry algebra of the system in the following sense:

When equipped with the modified bracket (3.96)*, the parameters* (3.97) *of the residual gauge symmetries form a representation of the Lie algebra* (3.98)*.*

The proof, following the one originally worked out in [21], is sketched in Appendix **A.3.2**.

3.2.2 Solution space

In this section, we present the polyhomogeneous solution space for our model, following mainly [6, 112, 122].

We start from the Einstein-Maxwell Lagrangian density in three dimensions

$$\mathcal{L} = \frac{\sqrt{-g}}{16\pi G}(R - F^2), \tag{3.100}$$

with equations of motion

$$\partial_\nu(\sqrt{-g}F^{\mu\nu}) = 0, \quad L_{\mu\nu} := G_{\mu\nu} - T_{\mu\nu} = 0, \tag{3.101}$$

where $T_{\mu\nu} = 2F_{\mu\rho}F_\nu{}^\rho - \frac{1}{2}g_{\mu\nu}F^2$.

The detailed analysis in Appendix **A.3.3** then yields the following results: given the ansatz

$$A_\phi = \alpha(u,\phi) \ln \frac{r}{r_0} + A^0_\phi(u,\phi) + \sum_{m=1}^{\infty} \sum_{n=0}^{m} \frac{A_{mn}(u,\phi)(\ln \frac{r}{r_0})^n}{r^m}, \tag{3.102}$$

at a fixed time u_0, the general solution to the Einstein-Maxwell system in three dimensions with the prescribed asymptotics is completely determined in terms of the initial data $A^0_\phi(u_0,\phi), A_{mn}(u_0,\phi)$, the news functions $A^0_u(u,\phi)$ and integration functions $\omega(\phi)$, $\lambda(\phi)$, $\theta(\phi)$, $\chi(\phi)$ according to

$$\begin{cases} \alpha = -\omega - u\lambda', \quad N = \chi + u\theta' \\ \beta = -\frac{\alpha^2}{2r^2} + \sum_{m=1}^{\infty}\sum_{n=0}^{m} \frac{\beta_{mn}(\ln\frac{r}{r_0})^n}{r^{m+2}}, \\ U = \frac{4\lambda\alpha\ln\frac{r}{r_0}+2\lambda\alpha-N}{2r^2} + \sum_{m=1}^{\infty}\sum_{n=0}^{m} \frac{U_{mn}(\ln\frac{r}{r_0})^n}{r^{m+2}}, \\ A_u = -\lambda\ln\frac{r}{r_0} + A^0_u + \frac{\alpha'}{r} + \sum_{m=1}^{\infty}\sum_{n=0}^{m} \frac{B_{mn}(\ln\frac{r}{r_0})^n}{r^{m+1}}, \\ V = 2\lambda^2\ln\frac{r}{r_0} + \theta + \frac{2\alpha\lambda'-2\lambda\alpha'}{r} + \sum_{m=1}^{\infty}\sum_{n=0}^{m} \frac{V_{mn}(\ln\frac{r}{r_0})^n}{r^{m+1}}, \end{cases} \tag{3.103}$$

where the functions $\beta_{mn}, U_{mn}, B_{mn}, V_{mn}$ are determined recursively in terms of the initial data, the news, the integration functions and their ϕ derivatives. In particular,

$$\dot{A}^0_\phi = -\lambda' + (A^0_u)'. \tag{3.104}$$

Furthermore, the leading parts of the metric and electromagnetic gauge potentials are given by

$$\begin{aligned} ds^2 = {} & [2\lambda^2\ln\frac{r}{r_0} + \theta + O(r^{-1})]du^2 - [2+O(r^{-2})]dudr \\ & -[4\lambda\alpha\ln\frac{r}{r_0} + 2\lambda\alpha - \chi - u\theta' + O(r^{-1}\ln\frac{r}{r_0})]dud\phi + r^2d\phi^2, \end{aligned} \tag{3.105}$$

$$A_\phi = \alpha\ln\frac{r}{r_0} + A^0_\phi + O(r^{-1}\ln\frac{r}{r_0}), \; A_u = -\lambda\ln\frac{r}{r_0} + A^0_u + O(r^{-1}). \tag{3.106}$$

Asymptotic symmetries transform solutions to solutions. This allows one to work out the transformation properties of the functions characterising asymptotic solution space:

$$\begin{aligned} \mathcal{L}_\xi g_{uu}: & \; -\delta\theta = Y\theta' + 2(\theta-\lambda^2)Y' - 2Y''', \\ & \; -\delta\lambda = Y\lambda' + \lambda Y', \\ \mathcal{L}_\xi g_{u\phi}: & \; -\delta\chi = Y\chi' + 2(\chi - 2\omega\lambda)Y' + T\theta' + 2(\theta-\lambda^2)T' - 2T''', \\ & \; -\delta\omega = Y\omega' + \omega Y' + T\lambda' + \lambda T', \\ \mathcal{L}_\xi A_u + \dot{\epsilon}: & \; -\delta A^0_u = Y(A^0_u)' + (A^0_u + \lambda)Y' + f\dot{A}^0_u + \dot{E}, \\ \mathcal{L}_\xi A_\phi + \epsilon': & \; -\delta A^0_\phi = Y(A^0_\phi)' + (A^0_\phi - \alpha)Y' + f\dot{A}^0_\phi + A^0_u f' + E'. \end{aligned} \tag{3.107}$$

3.2.3 Surface charge algebra

Associating charges to asymptotic symmetries in general relativity is a notoriously subtle question. The approach followed here consists in deriving conserved co-dimension 2 forms in the linearized theory that can be shown to be uniquely associated, up to standard amiguities, to the exact symmetries of the background [41, 42, 123]. When using these expressions for asymptotic symmetries in the full theory, neither conservation nor integrability is guaranteed [22, 53, 83, 114].

More concretely, using the general expressions for the linearized Einstein-Maxwell system derived in [124], the surface charge one form of the linearized theory reduces to

$$\oint_{S^\infty} \not{\delta} k_{\xi,\varepsilon} = -\delta \oint_{S^\infty} K_{\xi,\varepsilon} + \oint_{S^\infty} K_{\delta\xi,\delta\varepsilon} - \oint_{S^\infty} \xi \cdot \Theta, \tag{3.108}$$

where

$$\begin{aligned} K_{\xi,\varepsilon} &= (dx^{n-2})_{\mu\nu} \frac{\sqrt{-g}}{16\pi G} [\nabla^\mu \xi^\nu - \nabla^\nu \xi^\mu + 4F^{\mu\nu}(\xi^\sigma A_\sigma + \epsilon)], \\ \Theta &= (dx^{n-1})_\mu \frac{\sqrt{-g}}{16\pi G} [\nabla_\sigma \delta g^{\mu\sigma} - \nabla^\mu \delta g^\nu_\nu + 4F^{\sigma\mu}\delta A_\sigma], \end{aligned} \tag{3.109}$$

and S^∞ is the circle at constant $u = u_0$ and $r = R \to \infty$. At this stage, we have used already that expressions in $\oint_{S^\infty} \not{\delta} k_{\xi,\varepsilon}$ that are proportional to the exact generalized Killing equations vanish,

$$\begin{aligned} \frac{1}{16\pi G} \oint_{S^\infty} \delta g^\nu_\rho (\nabla^\mu \xi^\rho + \nabla^\rho \xi^\mu) \sqrt{-g} (d^{n-2}x)_{\mu\nu} = 0, \\ \frac{1}{4\pi G} \oint_{S^\infty} g^{\mu\rho} \delta A^\nu (\mathcal{L}_\xi A_\rho + \partial_\rho \varepsilon) \sqrt{-g} (d^{n-2}x)_{\mu\nu} = 0, \end{aligned} \tag{3.110}$$

when evaluated for solutions and asymptotic symmetry parameters (3.97). As in four-dimensional asymptotically flat pure Einstein gravity [114], the remaining expression then splits into an integrable part and a non-integrable part proportional to the electromagnetic news,

$$\oint_{S^\infty} \not{\delta} k_{\xi,\varepsilon} = \delta Q_s + \Theta_s, \tag{3.111}$$

with

$$\begin{aligned} Q_s[g, A] &= \frac{1}{16\pi G} \int_0^{2\pi} d\phi \Big[\theta T + Y(\chi + 4\lambda A^0_\phi) + 4\lambda E \Big], \\ \Theta_s[\delta g, \delta A; g, A] &= \frac{1}{4\pi G} \int_0^{2\pi} d\phi f A^0_u \delta\lambda. \end{aligned} \tag{3.112}$$

It follows that $(8G)^{-1}\theta, (8G)^{-1}(\chi + 4\lambda A^0_\phi), (2G)^{-1}\lambda$ can be interpreted as the mass, angular momentum and electric charge aspect, respectively.

Applying now the proposal of [22, 53] for the modified bracket of the integrable part of the charges,

$$\{Q_{s_1}, Q_{s_2}\} = -\delta_{s_2} Q_{s_1} + \Theta_{s_2}[-\delta_{s_1} g, -\delta_{s_1} A; g, A], \tag{3.113}$$

gives

$$\{Q_{s_1}, Q_{s_2}\} = Q_{[s_1,s_2]} + K_{s_1,s_2}, \tag{3.114}$$

where

$$K_{s_1,s_2} = \frac{1}{8\pi G}\int_0^{2\pi} d\phi\ \Big[Y_1' T_2'' - 2\lambda f_1 \dot{E}_2 - \lambda^2 T_1 Y_2' - (1 \leftrightarrow 2)\Big]. \tag{3.115}$$

It is then straightforward to check that the field dependent central extension satisfies the generalised cocycle condition

$$K_{[s_1,s_2],s_3} - \delta_{s_3} K_{s_1,s_2} + \text{cyclic}\ (1,2,3) = 0. \tag{3.116}$$

3.2.4 Switching off the news

In the analysis above, we are in an unusual situation where the asymptotic symmetry algebra depends arbitrarily on time through the dependence of E on u. This can be fixed by requiring the electromagnetic news function to vanish, $A^0_u = 0$. Since asymptotic symmetries need to preserve this condition, we find from (3.107) that $E = \bar{E} - \int_{u_0}^u du' \lambda Y'$. The asymptotic symmetry algebra then becomes time independent, but field dependent since the last of (3.99) gets replaced by

$$\hat{\bar{E}} = Y_1 \bar{E}_2' - T_1 Y_2' \lambda - (1 \leftrightarrow 2). \tag{3.117}$$

Charges become integrable and conserved: the second, non integrable part vanishes while in the first line of (3.112), E, A^0_ϕ get replaced by $\bar{E}, \bar{A}^0_\phi$. In order to see this, one has to go back to (3.108) where the second term now contributes to remove the u-dependent terms when using that, on shell, $E = \bar{E}(\phi) - u\lambda(\phi)Y'$, and $A^0_\phi = \bar{A}^0_\phi(\phi) - u\lambda'$. Finally, the field dependent central charge becomes

$$K_{s_1,s_2} = \frac{1}{8\pi G}\int_0^{2\pi} d\phi\ \left[Y_1' T_2'' + \lambda^2 T_1 Y_2' - (1 \leftrightarrow 2)\right]. \tag{3.118}$$

3.2.5 Discussion

Apart from its intrinsic interest, one might hope that the elaborate symmetry structure and the explicit solution of the three-dimensional Einstein-Maxwell system with non trivial asymptotics at Scri presented here could be suitably tuned so as to have applications in the context of holographic condensed matter models in 1+1 dimensions. Indeed, the Einstein-Maxwell system with various backgrounds, asymptotics, and additional scalar or form fields is ubiquitous in this context, see for instance [125–127], and more specifically [128–130] in three bulk dimensions. From the viewpoint of symmetries as well, this is quite reasonable since the $\mathfrak{bms}_3$ algebra is isomorphic to $\mathfrak{gca}_2$, the Galilean conformal algebra in 2 dimensions [131, 132].

Independently of such speculations, let us compare the three-dimensional results derived here to those of the four dimensional case. First, we note that in the four dimensional Einstein-Maxwell system, one imposes the conditions $A_r = 0$ and $A_u = O(r^{-1})$ (see e.g. [109] and [133] section II.C for a detailed discussion of pure electromagnetism). As shown in [121], from the viewpoint of asymptotic symmetries, the absence of a term in A_u of order zero in r^{-1} also guarantees a time independent symmetry algebra similar to the one discussed here in three dimensions, but with an additional arbitrary dependence on the supplementary polar angle. In four dimensions, the electromagnetic news nevertheless persists since it is encoded in different components of the vector potential.

Concerning the algebra of charges, there does not exist, to our knowledge, a complete study of the Einstein-Maxwell system in the four-dimensional case. That is the reason why we compare the rest of the results here to the purely gravitational ones in four dimensions.

As recalled in the introduction, in four dimensions, self-consistent asymptotically flat solution spaces at Scri including charged black hole solutions have been constructed in spaces involving integer powers of $1/r$.

The simplest solution in three dimensions is the flat limit of the charged BTZ black hole. It is characterized by $\omega = 0 = A^0_\phi = A_{mn} = A^0_u$ and $\theta = 8GM$, $\chi = 8GJ$, $\lambda = 2GQ$, where M, J, Q are constants that, according to (3.112), are interpreted as the mass, angular momentum and electric charge of the solution. Let us recall that the uncharged solutions with $Q = 0$ describe three dimensional cosmologies [74, 118, 119] when $M \geq 0$ and spinning particles, i.e., the angular defects and excesses of [55], when $M < 0$.

In both cases, it is the news, electromagnetic in the former and gravitational in the latter, that is responsible for the non-integrability and non conservation of the charges. In the latter case, there is no (field-dependent) central extension, unless one admits singular symmetry generators at null infinity and considers a local extension of the $\mathfrak{bms}_4$ algebra including superrotations [21, 22, 53]. It disappears when switching off the news, as this reduces superrotations to standard Lorentz rotations. In the former case, superrotations always exist and are globally well-defined at null infinity. The field dependent central extension persists even after switching off the electromagnetic news. To our knowledge, this is the first example in the context of asymptotic symmetries where there is a field dependent term in the symmetry algebra and in the central extension of the algebra of conserved and integrable charges.

The first field independent term in (3.118) exists also for pure gravity in three dimensions and is well understood from a cohomological point of view [116,134]. It has also been used in an argument pertaining to the Bekenstein-Hawking entropy of the three-dimensional cosmological solutions [135, 136], modeled on the one in [30] for the BTZ black holes.

The second field dependent term involving the electric charge aspect, is novel and much less understood. It certainly deserves further study, both from the viewpoint of Lie algebroid cohomology and from a physical perspective.

A final comment will be given on the nature of the solutions considered here. As in the majority of the papers on the subject since the pioneering work by Bondi et al. [4], the solutions are constructed as formal power series in the radial coordinate. In the polyhomogeneous case, there has been an investigation of convergence and existence of such solutions for linear massless higher spin fields on Minkowski spacetime as a preliminary study for the gravitational problem [137]. Addressing this question is clearly relevant in this set-up as well, but beyond the scope of the current work. We just note that the asymptotic symmetry algebra itself is not very sensitive to the details of solution space, as it is based solely on (3.119), (3.93) and the absence of news in later considerations.

3.3 Three-dimensional asymptotically AdS Einstein-Maxwell theory

As already pointed in the discussion previously, the Einstein-Maxwell system is ubiquitous for the understanding of holographic condensed matter models in 1+1 dimensions. Since the standard AdS/CFT dictionary [24–26] requires an asymptotically AdS boundary, it is natural to expect the extension of the analysis in the previous section to the AdS case. This is precisely what we will show in the coming section. To achieve this, we will follow closely the procedure of the previous section.

3.3.1 Asymptotic symmetries

We use coordinates (u, r, ϕ) and the gauge fixing ansatz

$$g_{\mu\nu} = \begin{pmatrix} Ve^{2\beta} + r^2U^2 & -e^{2\beta} & -r^2U \\ -e^{2\beta} & 0 & 0 \\ -r^2U & 0 & r^2 \end{pmatrix}, \quad A_r = 0, \tag{3.119}$$

with U, β, V and A_u, A_ϕ functions of u, r, ϕ. Suitable fall-off conditions that allow for non-vanishing electric charge are

$$\begin{aligned} U = o(r^{-1}), \quad V = -\frac{r^2}{l^2} + o(r), \quad \beta = o(r^0), \\ A_u = O(\ln\frac{r}{l}), \quad A_\phi = O(\ln\frac{r}{l}). \end{aligned} \tag{3.120}$$

The gauge structure of Einstein-Maxwell theory can be described as follows. Gauge parameters are pairs (ξ^μ, ϵ) consisting of a vector field $\xi^\mu\partial_\mu$

and a scalar ϵ. A generating set of gauge symmetries can be chosen as

$$-\delta_{(\xi,\epsilon)} g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu}, \quad -\delta_{(\xi,\epsilon)} A_\mu = \mathcal{L}_\xi A_\mu + \partial_\mu \epsilon. \tag{3.121}$$

When the gauge parameters are field dependent, one finds that

$$\left[\delta_{(\xi_1,\epsilon_1)}, \delta_{(\xi_2,\epsilon_2)}\right](g_{\mu\nu}, A_\mu) = \delta_{[(\xi_1,\epsilon_1),(\xi_2,\epsilon_2)]_M}(g_{\mu\nu}, A_\mu), \tag{3.122}$$

where the Lie (algebroid) bracket for field dependent gauge parameters is defined through

$$\begin{aligned}
&[(\xi_1,\epsilon_1),(\xi_2,\epsilon_2)]_M = (\hat{\xi},\hat{\epsilon}),\\
&\hat{\xi} = [\xi_1,\xi_2] + \delta_{(\xi_1,\epsilon_1)}\xi_2 - \delta_{(\xi_2,\epsilon_2)}\xi_1 \ , \ \hat{\epsilon} = \xi_1(\epsilon_2) + \delta_{(\xi_1,\epsilon_1)}\epsilon_2 - (1 \leftrightarrow 2).
\end{aligned} \tag{3.123}$$

Gauge transformations preserving asymptotically flat configurations are determined by gauge parameters depending linearly and homogeneously on arbitary functions $f(u,\phi), Y(u,\phi), E(u,\phi)$ according to

$$\begin{aligned}
&\xi^u = f,\\
&\xi^\phi = Y - f'\int_r^\infty \frac{e^{2\beta}}{r'^2}dr' = Y - \frac{f'}{r} + o(r^{-2}),\\
&\xi^r = -r\partial_\phi \xi^\phi + rUf' = -rY' + f'' + o(1),\\
&\epsilon = E + f'\int_r^\infty \frac{e^{2\beta}A_\phi}{r'^2}dr' = E + O(\frac{\ln\frac{r}{l}}{r}),
\end{aligned} \tag{3.124}$$

with $\dot{f} = Y'$ and $f' = l^2\dot{Y}$, where dot and prime denote u and ϕ derivatives, respectively. They are worked out precisely in Appendix A.4.1.

Consider the "/Maxwell" Lie algebra consisting of triples $s = (f, Y, E)$ with bracket

$$[s_1, s_2] = \left(\hat{f}, \hat{Y}, \hat{E}\right), \tag{3.125}$$

with

$$\hat{f} = f_1\dot{f}_2 + Y_1 f_2' - (1 \leftrightarrow 2),\ \hat{Y} = f_1\dot{Y}_2 + Y_1 Y_2' - (1 \leftrightarrow 2),\ \hat{E} = f_1\dot{E}_2 + Y_1 E_2' - (1 \leftrightarrow 2). \tag{3.126}$$

This is the asymptotic symmetry algebra of the system in the following sense:

When equipped with the modified bracket (3.123), the parameters (3.124) of the residual gauge symmetries form a realization of the Lie algebra (3.125). The asymptotic symmetry algebra is a Virasoro-Kac-Moody type algebra that extends the two copies of Virasoro algebra of the purely gravitational case.

3.3.2 Solution space

We start from the Einstein-Maxwell Lagrangian density in three dimensions

$$\mathcal{L} = \frac{\sqrt{-g}}{16\pi G}(R + \frac{2}{l^2} - F^2), \tag{3.127}$$

with equations of motion

$$\partial_\nu(\sqrt{-g}F^{\mu\nu}) = 0, \quad L_{\mu\nu} := G_{\mu\nu} - \frac{g_{\mu\nu}}{l^2} - T_{\mu\nu} = 0, \tag{3.128}$$

where $T_{\mu\nu} = 2F_{\mu\rho}F_\nu{}^\rho - \frac{1}{2}g_{\mu\nu}F^2$.

Given the ansatz

$$\begin{aligned} A_\phi \;\; = \;\; & \alpha(u,\phi)\ln\frac{r}{l} + A^0_\phi(u,\phi) + \frac{A_1(u,\phi)}{r} \\ & + \sum_{m=1}^{\infty}\sum_{n=1}^{m}\left[\frac{\tilde{A}_{\phi m}(u,\phi)}{r^{2m}} + \frac{\bar{A}_{\phi m}(u,\phi)}{r^{2m+1}} + \frac{\tilde{A}_{mn}(\ln\frac{r}{l})^n}{r^{2m}} + \frac{\bar{A}_{mn}(\ln\frac{r}{l})^n}{r^{2m+1}}\right], \end{aligned} \tag{3.129}$$

at a fixed time u_0, the general solution to the Einstein-Maxwell system in three dimensions with the prescribed asymptotics is calculated in Appendix A.4.3. The solution space is completely determined in terms of the initial data $\alpha(u_0,\phi)$, $A^0_\phi(u_0,\phi)$, $A_1(u_0,\phi)$, $\tilde{A}_{\phi m}(u_0,\phi)$, $\bar{A}_{\phi m}(u_0,\phi)$ and integration functions $\lambda(u,\phi)$, $A^0_u(u,\phi)$, $\theta(u,\phi)$, $N(u,\phi)$ as follows:

$$\begin{cases} \beta = -\frac{\alpha^2}{2r^2} + \frac{2\alpha A_1}{3r^3} + \sum\limits_{m=1}^{\infty}\sum\limits_{n=0}^{m}\left[\frac{\tilde{\beta}_{mn}(\ln\frac{r}{l})^n}{r^{2m+2}} + \frac{\bar{\beta}_{mn}(\ln\frac{r}{l})^n}{r^{2m+3}}\right], \\ U = \frac{4\lambda\alpha\ln\frac{r}{l}+2\lambda\alpha-N}{2r^2} + \frac{4\lambda A_1+2\alpha\alpha'}{3r^3} + \sum\limits_{m=1}^{\infty}\sum\limits_{n=0}^{m}\left[\frac{\tilde{U}_{mn}(\ln\frac{r}{l})^n}{r^{2m+2}} + \frac{\bar{U}_{mn}(\ln\frac{r}{l})^n}{r^{2m+3}}\right], \\ A_u = -\lambda\ln\frac{r}{l} + A^0_u + \frac{\alpha'}{r} + \sum\limits_{m=1}^{\infty}\sum\limits_{n=0}^{m}\left[\frac{\tilde{B}_{mn}(\ln\frac{r}{l})^n}{r^{2m}} + \frac{\bar{B}_{mn}(\ln\frac{r}{l})^n}{r^{2m+1}}\right], \\ V = -\frac{r^2}{l^2} + 2(\lambda^2 + \frac{\alpha^2}{l^2})\ln\frac{r}{l} + \theta + \frac{2\alpha\lambda'-2\lambda\alpha'}{r} + \frac{8\alpha A_1}{3rl^2} - \frac{4\lambda^2\alpha^2(\ln\frac{r}{l})^2}{r^2} \\ \quad + \sum\limits_{m=1}^{\infty}\sum\limits_{n=0}^{m}\left[\frac{\tilde{V}_{mn}(\ln\frac{r}{l})^n}{r^{2m}} + \frac{\bar{V}_{mn}(\ln\frac{r}{l})^n}{r^{2m+1}}\right], \end{cases} \tag{3.130}$$

where the functions $\tilde{A}_{mn}, \bar{A}_{mn}, \tilde{\beta}_{mn}, \bar{\beta}_{mn}, \tilde{U}_{mn}, \bar{U}_{mn}, \tilde{B}_{mn}, \bar{B}_{mn}, \tilde{V}_{mn}, \bar{V}_{mn}$ are determined recursively in terms of the initial data, the integration functions and their ϕ derivatives. In particular,

$$\begin{aligned}
&\dot{\alpha} + \lambda' = 0, \\
&\dot{\lambda} + \frac{\alpha'}{l^2} = 0, \\
&\dot{A}^0_\phi + \lambda' - (A^0_u)' + \frac{A_1}{l^2} = 0, \\
&\dot{N} = \theta' + \frac{2\alpha\alpha'}{l^2} + \frac{4\lambda A_1}{l^2} \\
&\dot{\theta} = \frac{1}{l^2}N' - \frac{6\alpha\lambda'}{l^2} - \frac{4\alpha A_1}{l^2}.
\end{aligned}$$

Note that the logarithmic term leads to a self-consistent polyhomogeneous solution space with a finely chosen logarithmic expansion. The series with inverse of r split into even powers and odd powers and the series with logarithms in r are completely determined by the series of the inverse powers of r.

Furthermore, the leading parts of the metric and Maxwell fields are given by

$$\begin{aligned}
ds^2 = &[-\frac{r^2}{l^2} + 2(\lambda^2 + \frac{\alpha^2}{l^2})\ln\frac{r}{l} + \theta + \frac{\alpha^2}{l^2} + O(r^{-1})]du^2 \\
&-2[1 - \frac{\alpha^2}{r^2} + O(r^{-3})]dudr \\
&\quad -[4\lambda\alpha\ln\frac{r}{l} + 2\lambda\alpha - N + O(r^{-1})]dud\phi + r^2 d\phi^2, \qquad (3.131)
\end{aligned}$$

$$A_\phi = \alpha\ln\frac{r}{r_0} + A^0_\phi + O(r^{-1}),\ A_u = -\lambda\ln\frac{r}{r_0} + A^0_u + O(r^{-1}) \qquad (3.132)$$

Asymptotic symmetries transform solutions to solutions. This allows one to work out the transformation properties of the functions characterising asymptotic solution space.

$$\begin{aligned}
\mathcal{L}_\xi A_u + \dot{\epsilon}: \ & -\delta\lambda = f\dot{\lambda} + \lambda\dot{f} - Y\dot{\alpha} - \alpha\dot{Y}, \\
& -\delta A^0_u = f\dot{A}^0_u + Y(A^0_u)' + A^0_u\dot{f} + A^0_\phi\dot{Y} + \lambda Y' + \dot{E}, \\
\mathcal{L}_\xi A_\phi + \epsilon': \ & -\delta\alpha = Y\alpha' + \alpha Y' - f\lambda' - \lambda f', \\
& -\delta A^0_\phi = f\dot{A}^0_\phi + Y(A^0_\phi)' + A^0_u f' + A^0_\phi Y' - \alpha Y' + E', \\
\mathcal{L}_\xi g_{uu}: \ & -\delta\theta = Y\theta' + 2(\theta - \lambda^2 - \frac{\alpha^2}{l^2})Y' + f\dot{\theta} + 2(N - \lambda\alpha)\dot{Y} - 2Y''', \\
\mathcal{L}_\xi g_{u\phi}: \ & -\delta N = YN' + 2(N + 2\alpha\lambda)Y' + f\dot{N} + 2(\theta - \lambda^2)f' - 2f'''.
\end{aligned} \qquad (3.133)$$

3.3.3 Surface charge algebra

Using the general expressions derived in [41,123,124], the surface charge one form reduces to

$$\oint_{S^\infty} \not{\delta} k_{\xi,\varepsilon} = -\delta \oint_{S^\infty} K_{\xi,\varepsilon} + \oint_{S^\infty} K_{\delta\xi,\delta\varepsilon} - \oint_{S^\infty} \xi \cdot \Theta, \tag{3.134}$$

where

$$\begin{aligned} K_{\xi,\varepsilon} &= (dx^{n-2})_{\mu\nu} \frac{\sqrt{-g}}{16\pi G} [\nabla^\mu \xi^\nu - \nabla^\nu \xi^\mu + 4F^{\mu\nu}(\xi^\sigma A_\sigma + \epsilon)], \\ \Theta &= (dx^{n-1})_\mu \frac{\sqrt{-g}}{16\pi G} [\nabla_\sigma \delta g^{\mu\sigma} - \nabla^\mu \delta g^\nu_\nu + 4F^{\sigma\mu} \delta A_\sigma], \end{aligned} \tag{3.135}$$

and S^∞ is the circle at constant $u = u_0$ and $r = R \to \infty$. At this stage, we have used already that expressions in $\oint_{S^\infty} \not{\delta} k_{\xi,\varepsilon}$ that are proportional to the exact generalised Killing equations vanish,

$$\begin{aligned} &\frac{1}{16\pi G} \oint_{S^\infty} \delta g^\nu_\rho (\nabla^\mu \xi^\rho + \nabla^\rho \xi^\mu) \sqrt{-g} (d^{n-2}x)_{\mu\nu} = 0, \\ &\frac{1}{4\pi G} \oint_{S^\infty} g^{\mu\rho} \delta A^\nu (\mathcal{L}_\xi A_\rho + \partial_\rho \varepsilon) \sqrt{-g} (d^{n-2}x)_{\mu\nu} = 0, \end{aligned} \tag{3.136}$$

when evaluated for solutions and asymptotic symmetry parameters (3.124). As in four-dimensional asymptotically flat pure Einstein gravity [114], the remaining expression then splits into an integrable part and a non-integrable part

$$\oint_{S^\infty} \not{\delta} k_{\xi,\varepsilon} = \delta Q_s + \Theta_s, \tag{3.137}$$

with

$$\begin{aligned} Q_s[g, A] &= \frac{1}{16\pi G} \int_0^{2\pi} d\phi \Big[f(\theta - \frac{\alpha^2}{l^2} - \frac{\alpha A^0_\phi}{l^2}) + Y(N + 4\lambda A^0_\phi) + 4\lambda E \Big], \\ \Theta_s[\delta g, \delta A; g, A] &= \frac{1}{4\pi G} \int_0^{2\pi} d\phi\, f \left[A^0_u \delta\lambda + \frac{A^0_\phi \delta\alpha}{l^2} \right]. \end{aligned} \tag{3.138}$$

The associated surface charges turn out to be neither conserved nor integrable.

Applying the proposal of [22,53] for the modified bracket of the integrable part of the charges,

$$\{Q_{s1}, Q_{s2}\} = -\delta_{s2} Q_{s1} + \Theta_{s2}[-\delta_{s1} g, -\delta_{s1} A; g, A], \tag{3.139}$$

gives

$$\{Q_{s1}, Q_{s2}\} = Q_{[s_1,s_2]} + K_{s_1,s_2}, \tag{3.140}$$

where

$$\begin{aligned} K_{s_1,s_2} &= \frac{1}{8\pi G}\int_0^{2\pi} d\phi \,\big[Y_1' f_2'' - 2\lambda f_1 \dot{E}_2 - (\lambda^2 + \frac{\alpha^2}{l^2}) f_1 Y_2' \\ &\quad + 2(\alpha A_u^0 - \lambda A_\phi^0)\frac{f_1 f_2'}{l^2} - (1 \leftrightarrow 2)\big]. \end{aligned} \tag{3.141}$$

It is then straightforward to check that the field dependent central extension satisfies the generalised cocycle condition

$$K_{[s_1,s_2],s_3} - \delta_{s_3} K_{s_1,s_2} + \text{cyclic } (1,2,3) = 0. \tag{3.142}$$

3.3.4 Comparison with the asymptotically flat case

The interest of Einstein-Maxwell system in 3 dimensions has been well stressed in the previous section. Here, we summarize promptly by comparing the main result between the asymptotically flat and asymptotically AdS case. From (3.103) and (3.130), one can recognize immediately that the solution space of the AdS case is much more complicated than the flat case. The main difference between those two cases is the fact that there is no natural analog of the news function to characterize electromagnetic radiation in the AdS case. Thus a physical interpretation is missing for a non-conserved nor integrable surface charges (3.138). An alternative approach was proposed in [138, 139] with integrability condition a priori.

Nevertheless, with a time-like boundary, in any case, one needs an additional boundary condition to make the evolution well-defined and it turns out that the natural choice would be the reflective boundary condition [140–142]. Such constraint leads a "no outgoing radiation condition". It would be interesting to investigate the consequence of the reflective boundary condition in our prescription and its relation with the integrability condition of [138, 139] elsewhere.

Chapter 4

Applications in Quantum ElectroDynamics amplitudes

Ordinary flat-space Quantum Field Theories without mass gap are peculiar in the sense that the structure of null infinity is accessible to the massless excitations of the theory. The perturbative S-matrix of many of these theories, like gravity or Yang-Mills[1] turns out to be simpler than expected. Since the S-matrix is an object who inherently lives at the boundary of space-time, a fair question to pose is whether this simplicity might be somehow connected to the structure of null infinity.

A more down-to-earth question that we can pose in these theories is whether something particular happens in the scattering of very low-energy massless quanta, since this limit only makes sense for zero-mass particles. That is actually an old subject of study to which many people contributed.

Although not the first in chronological order, Weinberg showed quite generally in [143, 144] that, for theories with long-range interactions mediated by a spin-s boson ($s = 1, 2$), when emitting one of these bosons with very low frequency, the tree-level scattering amplitude develops a pole whose residue is given by the universal formula:

$$M_{n+1}\big(p_1, \ldots, p_n, \{q; \epsilon^{\pm s}\}\big) = S^{(0)}(\epsilon^{\pm s}, p_k, q) M_n(p_1, \ldots, p_n) + \mathcal{O}\left(\omega^0\right) \,, \tag{4.1}$$

where $\omega = q_0$ and $\epsilon^{\pm s}$ are respectively the energy and polarization tensor of the soft boson, and

$$S^{(0)} = \sum_{k=1}^{n} g_k \, \frac{(p_k \cdot \epsilon^{\pm})^s}{p_k \cdot q} \tag{4.2}$$

is called a soft factor, with g_k being the cubic couplings controlling the emission of the soft particle from the external legs. For the case of $s = 2$, equation (4.1) has come to be known as Weinberg's soft graviton theorem.

[1] Of course Yang-Mills is a confining theory, but the mass gap arises non-perturbatively. It is the S-matrix of perturbative Yang-Mills the one that is simple.

In reality, long before Weinberg; Low [145] and Gell-Mann and Goldberger [146] had realized that the scattering of light by arbitrary targets displayed universal properties through the next-to-leading order in a low-frequency expansion. Such a result was put in amplitude form by Low [147], essentially predating the soft theorem (4.1) for photons with the formula:

$$M_{n+1}(p_1,\ldots,p_n,\{q;\epsilon^{\pm}\}) = \left(S^{(0)} + S^{(1)}\right) M_n(p_1,\ldots,p_n) + \mathcal{O}\left(\omega^1\right) \ , \quad (4.3)$$

where now there appears a sub-leading soft factor that involves the angular momentum operator $J^{\mu\nu}$:

$$S^{(1)} = \sum_{k=1}^{n} e_k \, \frac{q_\mu \epsilon^{\pm}_\nu \, J^{\mu\nu}_k}{p_k \cdot q} \ . \quad (4.4)$$

The original derivation by Low (typically referred to as Low's theorem) works for spinless sources, but it was later extended to Maxwell theory coupled to generic sources by [148, 149][2]. While many other works investigated in the past the issue of sub-leading, multiple as well as loop corrections to soft limits of photons, gluons and gravitons, that line of research was somewhat abandoned before a new spark reignited the field. That spark was provided by the recent conjecture (or tree-level discovery) of a sub-sub-leading soft theorem for gravitons by Cachazo and Strominger [34]. The main interest of their work comes however not so much from the discovery of the sub-sub-leading piece, but rather from the reasons that prompted them to look for it. This connects with our initial question about the structure of null infinity.

The relations (4.1) and (4.3) for tree-level amplitudes are maybe not too spectacular – after all they are expected in the classical theory – and to call them theorems sounds a bit excessive. What is exciting about them is the fact that they can be argued not to get loop corrected (this is a subtle issue, whose discussion is out of the scope of this note concerned just with the tree level[3]). That may suggest that behind these relations there is some symmetry which does not become anomalous at the quantum level. This is precisely the idea that Strominger and collaborators have been trying to put forward in a series of papers whose origins can be traced back to 2013 [35, 153]. The basic idea is that the soft theorems are nothing but the Ward identities of asymptotic symmetries, that can be interpreted as spontaneously broken symmetries whose Goldstone particle is the soft boson. Let us sketch the reasoning behind this idea.

[2] An interesting revisiting of these old works in a more modern language can be found in the initial sections of [150].

[3] For completeness, and restricting ourselves just to photons, we mention that loop modifications do appear at the sub-leading level when massless sources are considered [151, 152]. The expectation though is that within a framework where IR divergencies can be decoupled, the soft theorems will hold.

In S-matrix language, a symmetry is just a relation between matrix elements $\langle \text{out}'|\text{in}'\rangle = \langle \text{out}|\text{in}\rangle$, where the *in* and *out* states have been transformed as $|\text{in}'/\text{out}'\rangle = \mathrm{U}^{\text{in}/\text{out}}|\text{in}/\text{out}\rangle$. The operators implementing the symmetry must verify $U^{\text{out}\dagger}U^{\text{in}} = 1$. If this symmetry is generated by a charge Q (*i.e.* $U^{\text{in}/\text{out}} = e^{i\theta\, Q^{\text{in}/\text{out}}}$), the associated Ward identity reads as

$$\langle \text{out}|\mathrm{Q}^{\text{out}} - \mathrm{Q}^{\text{in}}|\text{in}\rangle = 0 \,. \tag{4.5}$$

The charge for a spontaneously broken symmetry must act non-linearly on the states – otherwise it would annihilate the vacuum – so it can be decomposed into linear and non-linear pieces $Q = Q_{\text{L}} + Q_{\text{NL}}$. The Ward identity for a broken charge becomes

$$\langle \text{out}|\mathrm{Q}_{\text{NL}}^{\text{out}} - \mathrm{Q}_{\text{NL}}^{\text{in}}|\text{in}\rangle = -\langle \text{out}|\mathrm{Q}_{\text{L}}^{\text{out}} - \mathrm{Q}_{\text{L}}^{\text{in}}|\text{in}\rangle \,. \tag{4.6}$$

Neglecting issues about a proper, non-divergent definition of a broken charge, if Q_{NL} creates zero-momentum Goldstone bosons, equation (4.6) looks very much like (4.1) or (4.3). To make a more precise identification, we need to pinpoint which is the symmetry allegedly responsible for the soft theorem and its associated charge.

From here on we restrict ourselves to the case of Maxwell theory minimally coupled[4] to a general source (for definiteness, one can think of scalar Quantum ElectroDynamics). We know that in this theory the emission of a soft photon is controlled by the two universal terms of (4.3), the leading and sub-leading soft-photon factors. In [44] (see also [155–159]) it was found that certain residual (large) gauge transformations, which become a global symmetry at null infinity, are responsible for the leading soft factor, and the authors showed how to translate the Ward identity for this asymptotic symmetry into (4.1). Regarding the sub-leading order, the work [160] derived from the known sub-leading soft factor (4.4) the form that the charge of the asymptotic symmetry should have, although the origin and nature of the symmetry could not be identified.

The purpose of this chapter is to shed light into the symmetry behind the sub-leading soft photon factor. If we search for inspiration in the old works on the soft photon theorem [145–149], we notice that the fundamental ingredient to explain both terms in (4.3) was gauge invariance. That may lead one to think that *no new symmetry* needs to be invoked. Or in other words, that the residual large gauge transformations responsible for the leading soft factor can also explain the sub-leading one.

Now, if one is to believe that no new symmetry needs to be invoked, the only charge that we have is the one associated to the residual large gauge transformations. Then the only place where the sub-leading (in ω) soft

[4] Soft theorems can be modified in the presence of non-minimal couplings, as it happens for instance for the sub-sub-leading soft graviton theorem [154].

theorem can come from seems to be the sub-leading (in the inverse radial coordinate $1/r$) term in the charge. This is precisely what we will show in the next pages.

4.1 The solution space for Maxwell theory

Having a good control over the solution space of four-dimensional Maxwell theory coupled to an external source theory will allow us to make a clearer discussion of the asymptotic symmetries present in our system. The solution space of source-free Maxwell equations has been discussed in [161,162] in the Newman-Penrose formalism. Contrary to what these old works were doing, here we will favour the use of the gauge field A_μ over the field strength $F_{\mu\nu}$, as gauge transformations will play an important role later on. Our analysis is similar in spirit to that in [163], where the asymptotic structure at null infinity of three-dimensional Einstein-Maxwell theory was worked out.

Our theory lives in Minkowski space-time, which has two null "boundaries", past null infinity $\Im^-$, and future null infinity $\Im^+$. They are better appreciated when adopting advanced or retarded coordinates respectively. In what follows we concentrate on $\Im^+$, although everything can be similarly repeated on $\Im^-$. We introduce retarded spherical coordinates with the following change of coordinates:

$$u = t - r\,, \quad r = \sqrt{x^i x_i}\,, \quad x^1 + i x^2 = \frac{2rz}{1+z\bar{z}}\,, \quad x^3 = r\,\frac{1-z\bar{z}}{1+z\bar{z}}\,. \tag{4.7}$$

Minkowski space-time becomes

$$\mathrm{d}s^2 = -\mathrm{d}u^2 - 2\mathrm{d}u\,\mathrm{d}r + 2r^2\gamma_{z\bar{z}}\mathrm{d}z\mathrm{d}\bar{z}\,, \qquad \gamma_{z\bar{z}} = \frac{2}{(1+z\bar{z})^2}\,. \tag{4.8}$$

The piece $2r^2\gamma_{z\bar{z}}\mathrm{d}z\mathrm{d}\bar{z}$ is just the metric of the round sphere S^2, and $\Im^+$ is precisely the submanifold $r = \infty$ in the retarded spherical coordinates, with topology $S^2 \times \mathbb{R}$. The sphere at $u = \pm\infty$ is denoted by $\Im^+_\pm$. In the bulk we have a gauge field A_μ and matter. We will avoid talking about the kind of matter we have (*e.g.* scalar, fermionic) by introducing just a conserved current J_μ. As [44], we choose the following (radial) gauge and asymptotic conditions for the gauge fields and the current

$$\begin{aligned} &A_r = 0\ ,\ \ A_u = \mathcal{O}(r^{-1})\ ,\ \ A_z = \mathcal{O}(1), \\ &J_r = 0\ ,\ \ J_u = \mathcal{O}(r^{-2})\ ,\ \ J_z = \mathcal{O}(r^{-2})\ . \end{aligned} \tag{4.9}$$

Notice that we have used the ambiguities of a conserved current[5] to set the radial component of the current to zero. This is consistent with working in

[5] When a conserved current is derived from a global symmetry, it is naturally defined up to the equivalence $J^\mu \sim J^\mu + \nabla_\nu k^{[\mu\nu]}$, so it makes more sense to consider equivalence classes of currents $[J^\mu]$ [41].

the radial gauge. More specifically, let us assume the following ansatz for the $\frac{1}{r}$-expansion of the gauge field

$$A_u = \frac{A_u^0(u,z,\bar{z})}{r} + \mathcal{O}\Big(\frac{1}{r^2}\Big) \ , \qquad A_{z(\bar{z})} = A^0_{z(\bar{z})}(u,z,\bar{z}) + \sum_{m=1}^{\infty} \frac{A^m_{z(\bar{z})}(u,z,\bar{z})}{r^m} \ , \tag{4.10}$$

and the current

$$J_u = \frac{J_u^0(u,z,\bar{z})}{r^2} + \mathcal{O}\left(\frac{1}{r^3}\right) \ , \qquad J_{z(\bar{z})} = \frac{J^0_{z(\bar{z})}(u,z,\bar{z})}{r^2} + \sum_{m=1}^{\infty} \frac{J^m_{z(\bar{z})}(u,z,\bar{z})}{r^{m+2}} \ . \tag{4.11}$$

The reason for not specifying further the expansions of the u-components of A_μ and J_μ is that they are determined by the equations of motion. Actually, let us be more precise about what boundary data we need to specify in order to characterize a solution. For this purpose it is convenient to arrange all Maxwell equations in Minkowski space-time (4.8) with a conserved source J^μ as follows:

- One hypersurface equation:

$$\nabla_\mu F^{\mu u} = J^u \ , \tag{4.12}$$

- Two standard equations:

$$\nabla_\mu F^{\mu z} = J^z \ , \tag{4.13}$$
$$\nabla_\mu F^{\mu \bar{z}} = J^{\bar{z}} \ , \tag{4.14}$$

- The current conservation equation:

$$\nabla_\mu J^\mu = 0 \ . \tag{4.15}$$

- One supplementary equation:

$$\nabla_\mu F^{\mu r} = J^r \ . \tag{4.16}$$

The reason for splitting equations in this way is that the hypersurface equation contains no derivative of the fields with respect to the retarded time u while the standard equations contain $\partial_u A_z$, $\partial_u A_{\bar{z}}$. When (4.12)-(4.15) are satisfied, the electromagnetic Bianchi equation $\nabla_\nu[\nabla_\mu F^{\mu\nu} - J^\nu] = 0$ reduces to $\partial_r[\sqrt{-g}(\nabla_\mu F^{\mu r} - J^r)] = 0$. This implies that we can just impose $\nabla_\mu F^{\mu r} = J^r$ at order $\mathcal{O}(r^0)$, and all the sub-leading orders will automatically vanish. Thus it is called the supplementary equation.

Let us start by solving for the u-components of the fields. From the current conservation equation we get

$$J_u = \frac{J_u^0(u,z,\bar{z})}{r^2} - \frac{1}{r^2}\int_r^{+\infty} \mathrm{d}r' \left[\gamma_{z\bar{z}}^{-1}(\partial_z J_{\bar{z}} + \partial_{\bar{z}} J_z)\right] \ , \tag{4.17}$$

while integrating the hypersurface equation yields

$$A_u = \frac{A_u^0(u,z,\bar{z})}{r} - \int_r^{+\infty} \mathrm{d}r' \, \frac{1}{r'^2} \int_{-\infty}^{r'} \mathrm{d}r'' \left[\gamma_{z\bar{z}}^{-1}(\partial_z \partial_{r''} A_{\bar{z}} + \partial_{\bar{z}} \partial_{r''} A_z)\right] \ . \tag{4.18}$$

The standard equations control the time evolution of the coefficients of $A^m_{z(\bar{z})}(u,z,\bar{z})$. We can see that (4.14) reduces to

$$2\partial_u A^1_z = \partial_z A^0_u + \partial_z[\gamma^{-1}_{z\bar{z}}(\partial_z A^0_{\bar{z}} - \partial_{\bar{z}} A^0_z)] + J^0_z \ , \tag{4.19}$$

$$\partial_u A^m_z = \frac{J^{m-1}_z}{2m} - \frac{m-1}{2} A^{m-1}_z - \frac{\partial_z[\gamma^{-1}_{z\bar{z}}(\partial_{\bar{z}} A^{m-1}_z)]}{m} \ , \quad (m \geq 2) \ . \tag{4.20}$$

Hence all the coefficients in the expansion of A_z have uniquely determined retarded time derivatives except the leading A^0_z. We will refer to $\partial_u A^0_z$ as the electromagnetic "*news*" since it reflects the propagation of electromagnetic waves. The other standard equation (4.13) has an analogous structure (just change $z \rightleftharpoons \bar{z}$ above), giving another "*news*" $\partial_u A^0_{\bar{z}}$. Let us remark here that the equation for the time evolution of A^1_z is different from those for A^m_z $(m \geq 2)$. We will come back to this point later on.

Finally, the supplementary equation (4.16) gives the time evolution of the integration constant $A^0_u(u,z,\bar{z})$ as

$$\partial_u A^0_u = \gamma^{-1}_{z\bar{z}} \partial_u(\partial_z A^0_{\bar{z}} + \partial_{\bar{z}} A^0_z) + J^0_u \ . \tag{4.21}$$

To summarize, we have shown that the general solution to the Maxwell system in four-dimensional Minkowski space-time (4.8) with the prescribed asymptotics (4.9) is completely determined in terms of the initial data $A^0_u(u_0,z,\bar{z})$, $A^m_z(u_0,z,\bar{z}), A^m_{\bar{z}}(u_0,z,\bar{z})$ $(m \geq 1)$, the functions $A^0_z(u,z,\bar{z})$, $A^0_{\bar{z}}(u,z,\bar{z})$ and the current. The latter is characterized by the source functions $J^0_u(u,z,\bar{z})$, $J^m_z(u,z,\bar{z}), J^m_{\bar{z}}(u,z,\bar{z})$ $(m \geq 0)$.

4.2 Charges

The radial gauge condition that we imposed in (4.9) leaves residual gauge transformations of the form

$$\delta A_z = \partial_z \varepsilon(z,\bar{z}) \ , \qquad \delta A_{\bar{z}} = \partial_{\bar{z}} \varepsilon(z,\bar{z}) \ . \tag{4.22}$$

It is clear from the analysis in the previous section that (4.22) does not spoil the well-definition of the boundary problem . Therefore, since the equations of motion are not affected, these transformations can be interpreted as symmetries at null infinity, or symmetries of the S-matrix following [44, 153]. The associated charge is [41, 44, 133]:

$$Q_{\varepsilon_{\text{out}}} = \int_{\Im^+_-} \mathrm{d}z\mathrm{d}\bar{z}\, \gamma_{z\bar{z}}\, \varepsilon(z,\bar{z})\, r^2\, F_{ru} = -\int_{\Im^+} \mathrm{d}z\mathrm{d}\bar{z}\mathrm{d}u\, \gamma_{z\bar{z}}\, \varepsilon(z,\bar{z})\, r^2\, \partial_u F_{ru} \ , \tag{4.23}$$

where in the second equality we assumed $A^0_u|_{\Im^+_+} = 0$, meaning that in the far future the system contains no bulk electric charge. This integration by parts is convenient to later express everything in terms of the "news" functions

$\partial_u A^0_z$, $\partial_u A^0_{\bar{z}}$, which can be achieved using (4.21) and (4.19)-(4.20). Let us plug the expansion (4.10) in (4.18), yielding:

$$r^2 \partial_u F_{ru} = -\partial_u A^0_u + \frac{\gamma^{-1}_{z\bar{z}} \partial_u (\partial_z A^1_{\bar{z}} + \partial_{\bar{z}} A^1_z)}{r} + \sum_{m=1}^{\infty} \frac{\gamma^{-1}_{z\bar{z}} \partial_u (\partial_z A^{m+1}_{\bar{z}} + \partial_{\bar{z}} A^{m+1}_z)}{r^{m+1}} , \tag{4.24}$$

This clearly gives a $\frac{1}{r}$-expansion of the charge (4.23):

$$Q_{\varepsilon_{\rm out}} = Q^{(0)}_{\varepsilon_{\rm out}} + \frac{Q^{(1)}_{\varepsilon_{\rm out}}}{r} + \sum_{m=2}^{\infty} \frac{Q^{(m)}_{\varepsilon_{\rm out}}}{r^m} . \tag{4.25}$$

Using (4.21), we can see that the leading (constant in r) piece,

$$Q^{(0)}_{\varepsilon} = -\int_{\mathcal{I}^+_-} {\rm d}z{\rm d}\bar{z}\, \gamma_{z\bar{z}}\, \varepsilon(z,\bar{z})\, A^0_u = -\int_{\mathcal{I}^+} {\rm d}z{\rm d}\bar{z}{\rm d}u\, \epsilon \left[\partial_u(\partial_z A^0_{\bar{z}} + \partial_{\bar{z}} A^0_z) + \gamma_{z\bar{z}} J^0_u\right] , \tag{4.26}$$

gives the charge that was identified in [44] as responsible for the leading soft photon theorem. We want to investigate here the sub-leading piece $Q^{(1)}_{\varepsilon}$. Using (4.19) we can immediately write

$$\begin{aligned} Q^{(1)}_{\varepsilon_{\rm out}} &= \int_{\mathcal{I}^+} {\rm d}z{\rm d}\bar{z}{\rm d}u\, \varepsilon\, \partial_u \left(\partial_z A^1_{\bar{z}} + \partial_{\bar{z}} A^1_z\right) \\ &= \int_{\mathcal{I}^+} {\rm d}z{\rm d}\bar{z}{\rm d}u\, \varepsilon \left[\partial_z \partial_{\bar{z}} A^0_u + \frac{1}{2}(\partial_z J^0_{\bar{z}} + \partial_{\bar{z}} J^0_z)\right] . \end{aligned} \tag{4.27}$$

In order to have everything in terms of "news", we massage the integral as follows:

$$\begin{aligned} Q^{(1)}_{\varepsilon_{\rm out}} &= \int_{\mathcal{I}^+} {\rm d}z{\rm d}\bar{z}{\rm d}u \left[\partial_u \left(u\, \varepsilon\, \partial_z \partial_{\bar{z}} A^0_u\right) - u\, \varepsilon\, \partial_z \partial_{\bar{z}} \partial_u A^0_u - \frac{1}{2}\left(\partial_z \varepsilon\, J^0_{\bar{z}} + \partial_{\bar{z}} \varepsilon\, J^0_z\right)\right] \\ &= \int_{\mathcal{I}^+} {\rm d}z{\rm d}\bar{z}{\rm d}u \Big[u \left(D^2_z D^z \varepsilon\, \partial_u A^0_{\bar{z}} + D^2_{\bar{z}} D^{\bar{z}} \varepsilon\, \partial_u A^0_z\right) - \frac{1}{2} u \left(D_z D_{\bar{z}} \varepsilon + D_{\bar{z}} D_z \varepsilon\right) J^0_u \\ &\qquad - \frac{1}{2}\left(D_z \varepsilon\, J^0_{\bar{z}} + D_{\bar{z}} \varepsilon\, J^0_z\right)\Big] , \end{aligned} \tag{4.28}$$

where we have used again (4.21), and in the second step we have dropped the boundary term $\int_{\mathcal{I}^+} {\rm d}z{\rm d}\bar{z}{\rm d}u\, \partial_u \left(u\, \varepsilon\, \partial_z \partial_{\bar{z}} A^0_u\right) = -\lim_{u \to -\infty} u \int_{\mathcal{I}^+_-} {\rm d}z{\rm d}\bar{z}\, \partial_z \partial_{\bar{z}} \varepsilon\, A^0_u = 0$, which holds when bracketed between *in* and *out* states. This is simply a consequence of $\langle {\rm out}| Q^{(0)}_{\varepsilon} |{\rm in}\rangle = 0$. The notation $D_{z(\bar{z})}$ is for the two-dimensional covariant derivative on the sphere. If one now calls

$$Y^z = D^z \varepsilon , \qquad Y^{\bar{z}} = D^{\bar{z}} \varepsilon , \tag{4.29}$$

we see that the sub-leading charge $Q^{(1)}_{\varepsilon_{\rm out}}$ is exactly the one that was written in [160]! This is the charge that one gets when translating the sub-leading soft photon theorem into an S-matrix Ward identity. Therefore, we reached the announced conclusion that the sub-leading soft photon theorem is a consequence of same symmetry responsible for the leading one.

4.3 Soft theorems

For completeness, we quickly review the results in [44] and [160], in order to appreciate better the interplay between leading and sub-leading terms in the present context. We just aim at sketching the physical picture without getting into the more mathematical details, which have been already worked out in the previous references.

We decompose the leading and sub-leading charges into a piece containing the "news" and the rest, containing the sources. These are respectively the non-linear and linear pieces that we mentioned around equation (4.6). For notational brevity, we momentarily suppress the *out* label.

$$Q^{(0)}_{\rm NL} = \int_{\Im^+} \mathrm{d}u\mathrm{d}^2z\,\varepsilon\,\partial_u\left(\partial_z A^0_{\bar z} + \partial_{\bar z} A^0_z\right) \ , \tag{4.30}$$

$$Q^{(1)}_{\rm NL} = \int_{\Im^+} \mathrm{d}z\mathrm{d}\bar z\mathrm{d}u\left[u\left(D_z^2 D^z\varepsilon\,\partial_u A^0_{\bar z} + D_{\bar z}^2 D^{\bar z}\varepsilon\,\partial_u A^0_z\right)\right] \ , \tag{4.31}$$

$$Q^{(0)}_{\rm L} = \int_{\Im^+} \mathrm{d}u\mathrm{d}^2z\,\gamma_{z\bar z}\,\varepsilon\,J^0_u \ , \tag{4.32}$$

$$Q^{(1)}_{\rm L} = -\frac{1}{2}\int_{\Im^+} \mathrm{d}z\mathrm{d}\bar z\mathrm{d}u\left[u\left(D_z D_{\bar z}\varepsilon + D_{\bar z} D_z\varepsilon\right)J^0_u + \left(D_z\varepsilon\,J^0_{\bar z} + D_{\bar z}\varepsilon\,J^0_z\right)\right] \ . \tag{4.33}$$

Let us now write a more convenient form of the soft-photon theorems. We can parametrize null momenta by their energy and direction on the sphere as

$$p_{k\,\mu} = \frac{\omega_k}{1+w_k\bar w_k}\left(1+w_k\bar w_k, w_k+\bar w_k, i(\bar w_k - w_k), 1-w_k\bar w_k\right) \ , \tag{4.34}$$

$$q_\mu = \frac{\omega_q}{1+w\bar w}\left(1+w\bar w, w+\bar w, i(\bar w - w), 1-w\bar w\right) \ . \tag{4.35}$$

Similarly we parametrize polarization tensors in flat space-time indices as[6]

$$\epsilon^-_\mu(q) = \frac{1}{\sqrt 2}\left(\bar w, 1, -i, -\bar w\right) \ , \qquad \epsilon^+_\mu(q) = \frac{1}{\sqrt 2}\left(w, 1, i, -w\right) \ . \tag{4.36}$$

Now we particularize to an outgoing negative(positive)-helicity soft photon for the leading (sub-leading) soft theorem, for reasons that will become apparent below. Other cases can be treated in a similar manner. We can

[6] When $q = (1,0,0,1)$, the polarization tensors are $\epsilon^\pm = \frac{1}{\sqrt 2}(0,1,\pm i,0)$. Just rotate these to get them in a general frame.

rewrite the corresponding leading and sub-leading pieces of the soft-photon theorem (4.3) as

$$\lim_{\omega_q\to 0}\langle \text{out}|\omega_q\, \mathfrak{a}_+(q)|\text{in}\rangle = \frac{1+|w|^2}{\sqrt{2}}\sum_{k=1}^{n}\frac{e_k}{w-w_k}\,\langle \text{out}|\text{in}\rangle\ , \tag{4.37}$$

$$\lim_{\omega_q\to 0}\partial_{\omega_q}\langle \text{out}|\omega_q\, \mathfrak{a}_-(q)|\text{in}\rangle =$$

$$\sum_{k=1}^{n}\frac{e_k}{\sqrt{2}}\left(\frac{1+w\,\bar{w}_k}{\bar{w}-\bar{w}_k}\partial_{\omega_k}+\frac{(1+|w_k|^2)(w-w_k)}{\omega_k(\bar{w}-\bar{w}_k)}\partial_{w_k}\right)\langle \text{out}|\text{in}\rangle\ , \tag{4.38}$$

where e_k is the electric charge of the k-th particle, and $\mathfrak{a}_{+(-)}(q)$ the annihilation operator, which creates outgoing negative(positive)-helicity soft photons with momentum q. Moreover, for simplicity and following [160], we have assumed scalar matter so that the angular momentum operator is simply $J_k^{\mu\nu}=-i\left(p_k^{\mu}\frac{\partial}{\partial p_{k\,\nu}}-p_k^{\nu}\frac{\partial}{\partial p_{k\,\mu}}\right)$, without extra helicity terms.

To see how (4.37) and (4.3) arise from (4.6), one just needs to plug in there the (*out*) charges (4.30)-(4.33) (and their analogues for the *in* charges), and make a concrete choice of $\varepsilon(z,\bar{z})$. For the cases under consideration, a convenient choice is

$$\varepsilon(z,\bar{z})=\frac{1}{w-z}\ . \tag{4.39}$$

In order to obtain the proper action of the charges on the *out* states, one has to define canonical commutation relations at infinity [133]. For the "news" fields, it is enough to perform a stationary-phase approximation of the gauge-field mode expansion:

$$A^0_{z(\bar{z})}(x)=-\frac{i}{8\pi^2}\frac{\sqrt{2}}{1+z\bar{z}}\int_0^{\infty}\mathrm{d}\omega_q\left[\mathfrak{a}_{+(-)}(\omega_q\hat{x})\,e^{-i\omega_q u}-\mathfrak{a}^{\dagger}_{-(+)}(\omega_q\hat{x})\,e^{i\omega_q u}\right]\ , \tag{4.40}$$

where the creation and annihilation operators satisfy the standard commutation relations. Then, using the simple Fourier relations (defining $F(u)=\int_{-\infty}^{\infty}\mathrm{d}\omega\, e^{i\omega u}\tilde{F}(\omega)$):

$$\begin{aligned}\int_{-\infty}^{\infty}\mathrm{d}u\,\partial_u F(u) &= 2\pi i\lim_{\omega\to 0}\left[\omega\tilde{F}(\omega)\right]\ ,\\ \int_{-\infty}^{\infty}\mathrm{d}u\,u\,\partial_u F(u) &= -2\pi\lim_{\omega\to 0}\left[\partial_\omega\left(\omega\tilde{F}(\omega)\right)\right]\ ,\end{aligned} \tag{4.41}$$

and the special form that that $\partial_{\bar{z}}\varepsilon=-2\pi\delta^2(z-w)$ takes for the choice (4.39), we obtain for the non-linear pieces of the charges[7]:

$$\langle \text{out}|Q^{(0)}_{\text{NL}}|\text{in}\rangle = \frac{1}{2}\frac{\sqrt{2}}{1+|w|^2}\lim_{\omega_q\to 0}\langle \text{out}|\omega_q\,\mathfrak{a}_+(q)|\text{in}\rangle \ , \tag{4.42}$$

$$\langle \text{out}|Q^{(1)}_{\text{NL}}|\text{in}\rangle = -\frac{\sqrt{2}\,i}{4\gamma_{w\bar{w}}}\,D^2_w\left[\frac{1}{1+|w|^2}\lim_{\omega_q\to 0}\partial_{\omega_q}\langle \text{out}|\omega_q\,\mathfrak{a}_-(q)|\text{in}\rangle\right] \ , \tag{4.43}$$

where we are denoting $\gamma_{w\bar{w}} = \frac{2}{(1+|w|^2)^2}$. Regarding the linear pieces, restricting ourselves to scalar charged (with charge Q_e) matter: $\Phi = \sum_{m=0}^{\infty}\frac{\Phi^m(u,z,\bar{z})}{r^{m+1}}$ with current $J^0_\mu = iQ_e(\bar{\Phi}^0\partial_\mu\Phi^0 - \Phi^0\partial_\mu\bar{\Phi}^0)$ at leading order[8], we just need to use the boundary canonical commutation relation [160]:

$$[\bar{\Phi}^0(u,z,\bar{z}),\Phi^0(u',w,\bar{w})] = \frac{i}{4}\gamma_{w\bar{w}}\,\Theta(u'-u)\delta^2(z-w) \ , \tag{4.44}$$

to see that

$$\langle \text{out}|Q^{(0)}_{\text{L}}|\text{in}\rangle = \sum_{k=1}^{n} -\frac{e_k}{2(w-w_k)}\langle \text{out}|\text{in}\rangle \ , \tag{4.45}$$

$$\langle \text{out}|Q^{(1)}_{\text{L}}|\text{in}\rangle = \sum_{k=1}^{n} -\frac{\pi i\, e_k}{2}\left(\partial_w(\gamma^{-1}_{w\bar{w}}\delta(w-w_k))\partial_{\omega_q} + \gamma^{-1}_{w\bar{w}}\,\omega_q^{-1}\delta(w-w_k)\partial_{w_k}\right)\langle \text{out}|\text{in}\rangle \ . \tag{4.46}$$

Assembling all these expressions, it is immediate to recover the leading soft theorem (4.37) from (4.6). To also recover the sub-leading (4.3), we just need to use the extra identity:

$$\begin{aligned} &D^2_w\left[\frac{1}{1+|w|^2}\left(\frac{1+w\,\bar{w}_k}{\bar{w}-\bar{w}_k}\partial_{\omega_k} + \frac{(1+|w_k|^2)(w-w_k)}{\omega_k(\bar{w}-\bar{w}_k)}\partial_{w_k}\right)\right] \\ &\quad = -2\pi\,\gamma_{w\bar{w}}\left(\partial_w\left(\gamma^{-1}_{w\bar{w}}\delta(w-w_k)\right)\partial_{\omega_k} + \gamma^{-1}_{w\bar{w}}\,\omega_q^{-1}\delta(w-w_k)\partial_{w_k}\right) \ . \end{aligned} \tag{4.47}$$

We recall that we have only paid attention to the *out* part of (4.6). The analysis of the *in* part can be carried out analogously, up to an anti-podal identification. These details have been laid out in [44, 160], and are not needed for our only-illustrative purposes of this section, which does not contain new results.

[7] We are just keeping the anti-holomorphic parts of the the charges (4.30)-(4.33), meaning those containing only $\partial_{\bar{z}}\varepsilon$. In particular one has to split (4.32) via $J^0_u \to \frac{1}{2}J^0_u + \frac{1}{2}J^0_u$, which is essentially the same arbitrary separation we have for the first term in (4.33). Otherwise one has to introduce some extra factors of 2, that arise from a proper treatment of the radiative phase space [44, 156].

[8] Recall from (4.9) we are taking the current to have zero radial component. This can be done by adding a total derivative to the usual current as $J^\mu = iQ_e(\bar{\Phi}\nabla^\mu\Phi - \Phi\nabla^\mu\bar{\Phi}) + \nabla_\nu k^{[\mu\nu]}$, with an anti-symmetric two-tensor whose only non-zero component is $k^{[ur]} = \frac{1}{\sqrt{-g}}\int \mathrm{d}r\,\sqrt{-g}(\overline{\Phi}\partial_r\Phi - \Phi\partial_r\overline{\Phi})$. Therefore $J^0_r = 0$.

4.4 Brief summary and open questions

We hope to have convinced the reader familiar with the works [44] and [160] that the connection between the leading soft photon theorem (4.1) and the asymptotic symmetry associated to residual large gauge transformations of the Maxwell system can be extended to the sub-leading order. In other words, one does not need to invoke a new asymptotic symmetry for explaining Low's theorem.

By properly studying the asymptotic structure of Maxwell equations, what we showed is that the charge associated to the asymptotic symmetry (4.22) can be expanded in powers of $\frac{1}{r}$. This induces a similar expansion for the Ward identities of such charge. It was already established in [44] that the $\mathcal{O}(r^0)$ Ward identity was equivalent to the leading soft photon theorem. Our finding here is that the $\mathcal{O}(r^{-1})$ Ward identity produces the sub-leading soft-photon theorem. In particular, the $\mathcal{O}(r^{-1})$ term of the charge (4.23) matches the one conjectured in [160].

The charge (4.23) actually contains an infinite number of charges, parametrised by the function $\varepsilon(z, \bar{z})$. A curious observation is that the function that naturally gives the form of the leading soft theorem for an outgoing photon with negative helicity, namely (4.39), gives the sub-leading soft theorem for an outgoing photon with positive helicity (differentiated twice with respect to the angular direction of the soft photon).

Two simple questions immediately come to mind. The first one is that, in light of the procedure here, it seems one should get an infinite number of soft theorems, by just considering the remaining $\mathcal{O}(r^{-n})$ $(n \geq 2)$ orders of the charge. We can see from (4.24) and (4.20) that this is not the case. There is no way to massage these sub-sub-leading charges the way we did in (4.28) without running into infinities. It is amusing to see this way in which the classical Maxwell conspires for the soft photon theorem to stop at sub-leading order.

A more intuitive way to understand this is to notice that the boundary fields, defined as those appearing at leading order in the different components of the field-strength [44], are only A_u^0, $A_{z(\bar{z})}^0$ and $A_{z(\bar{z})}^1$. We should interpret that the other fields do not really "live at the boundary". Therefore, as soon as A_z^m $(m \geq 2)$ appears in the charge, which happens at sub-sub-leading order, we cannot rewrite it in terms of only boundary fields. See the appendix A.5 for yet another perspective on this.

The second question is to apply the line of reasoning presented in this manuscript to the case of gravity, where the soft theorem stretches to sub-sub-leading order. It was shown in [36] that the leading soft factor can be understood as the Ward identity of BMS supertranslations [4,6]. While there have been several works exploring the possibility of having new symmetries (like superrotations [20–22,134]) explaining the sub-leading orders [164–167], it would be very interesting to see to which degree only supertranslations can determine the behavior of soft gravitons.

Chapter 5

Four-dimensional near horizon Einstein-Maxwell theory

We have shown an explicit example in application of asymptotic symmetry in scattering amplitudes in the previous chapter. Such a scenario has been extended and succeeded in different gauge theories [44,153,155,157–160,168–171] and gravity [35, 36, 164, 172, 173] recently, which gives the asymptotic symmetry a new lease of life. Apart from the fruitful result, the application in gravitational theory [35, 36] is highly sensitive to the existence of black holes in the bulk. On the symmetry level, the asymptotic symmetry group at future null infinity(BMS$^+$) and the asymptotic symmetry group at past null infinity(BMS$^-$) are isomorphic to each other but they are independent symmetries on different null infinities. To be a symmetry of the S-matrix, a canonical relation between BMS$^+$ and BMS$^-$ is required to act on both incoming and outgoing state. Only in a finite neighborhood of the Minkowski spacetime, a canonical identification between elements of BMS$^+$ and BMS$^-$ can be achieved [174]. The presence of black holes, though the spacetime is still asymptotically flat, will definitely challenge this identification. At the quantum level, the unitarity is not guaranteed due to black hole formation, which is known as the information paradox [175].

Nevertheless, the obstacle of the black hole formation does not block the progress of understanding the asymptotic symmetries, but sheds new insights to black hole physics [37, 39, 45, 176–182]. As a null hypersurface, the black hole horizon can be treated as an inner boundary. If one starts to believe that the soft particles create new degrees of freedom at the null infinity, one natural and interesting question arises immediately that if there can be new degrees of freedom on the black hole horizon. However, the well-known uniqueness theorem [38] tells us that the non-zero conserved charge of a stationary black hole in Einstein-Maxwell theory should be only M, J and Q. A stationary horizon seems insensitive to the possible new degrees of freedom. As also pointed in [39], the dynamical process such as black hole formation/evaporation will cause the changing of vacuum state of gravity. This means that one needs to investigate non-stationary process instead of just considering stationary black hole. In order to do so, the isolated horizon should be a better candidate for understanding the near horizon physics. Isolated horizon was introduced in [183], which describes later state of black holes. All dynamical processes in the neighborhood of the horizon have been almost settled down, but the space-time far away from the horizon can be still dynamical. Consequently, the isolated horizon framework serves as a more realistic resolution of black hole physics and is playing an important role in numerical simulations. Though the exact symmetry of an isolated horizon is not rich enough [184], it was shown in [46] that the isolated horizon structure has some ambiguity which is supposed to count for the recently revealed near horizon supertranslation in [37]. The action of a supertranslation on the isolated horizon can map one set of isolated horizon structure into another one which is equivalent to a dynamical process *e.g.* radiation crossing the horizon. This can be understood as a isolated horizon analogue of memory effect at null infinity [185]. Hence, it would be highly meaningful to have a fully asymptotic analysis of an isolated horizon.

In this chapter, the asymptotic structure of Einstein-Maxwell system near the isolated horizon will be worked out explicitly. We will show a tractable solution space and asymptotic symmetries of Einstein-Maxwell theory near the isolated horizon $\mathcal{H}$. The asymptotic symmetries consist of supertranslation, superrotation, and the asymptotic $U(1)$ symmetry which is an expected enhancement of the new discovery in [37]. Those symmetries form a closed algebra. The local conserved current can be constructed associated to the asymptotic symmetries. To have a concrete physical interpretation, we would restrain ourself in a special class of the solution space which is the approximation of a Schwarzschild black hole surrounded by electromagnetic fields. In such a physical specification, one can define an infinite number of conserved charges with respect to the asymptotic $U(1)$ symmetry. According to the terminology of [39], apart from the electric charge, we would call them as soft electric hairs. It's further shown that those soft hairs are equivalent to the electric multipole moments of isolated horizons defined in [46]. The supertranslation charges can be also introduced but they are vanishing except the zero mode. Remarkably, the presence of Maxwell fields does not affect the zero mode of the supertranslation charge. This is a direct evidence that the soft electric hairs are implanted by soft photons who contain no energy. The existence of soft hairs reveals new dynamical degrees of freedom meaning that isolated horizons with different soft electric hairs should be considered as different physical states. This may inspire a direct counting of horizon degrees of freedom.

5.1 Solution space of Einstein-Maxwell theory near isolated horizon

The concept of isolated horizons was introduced to approximate event horizons of black holes at late stages of gravitational collapse and of black hole mergers when back-scattered radiation falling into the black hole can be neglected [183].

In the present section, we will use the Newman-Penrose formalism as introduced in the preliminary. Let us choose n to be the normal vector[1] of the isolated horizon $\mathcal{H}$. According to [184, 186], the definition of a generic Isolated Horizon in Newman-Penrose formalism is

Definition 1: *A 3-dimensional null sub-manifold $\mathcal{H}$ is called an isolated horizon if*
(1) $\mathcal{H}$ is diffeomorphic to the product $S^2 \times \Re$, where S^2 is a 2-dimensional space-like manifold, and the fibers of the projection

$$\Pi : S^2 \times \Re \to S^2$$

are null curves in $\mathcal{H}$;
(2) the expansion of its normal vector n vanishes everywhere on $\mathcal{H}$;
(3) Einstein's equations hold on $\mathcal{H}$ and the stress-tensor satisfies $\Phi_{22}\hat{=}\Phi_{12}\hat{=}\Phi_{21}\hat{=}0$, where $\hat{=}$ means on horizon $\mathcal{H}$ only;
(4) the entire geometry of $\mathcal{H}$ and the gauge potential A_μ are stationary, i.e. time-independent.

To admit an isolated horizon structure, the horizon data will have several constraints. Since $\mathcal{H}$ is null, the normal vector n is the generator of null geodesic on the horizon $\mathcal{H}$. Hence, the spin coefficients relation

$$\nabla_n n = -(\gamma + \overline{\gamma})n + \nu m + \overline{\nu m}$$

gives $\nu\hat{=}0$. $\kappa_{(n)} := -(\gamma + \overline{\gamma})$ will be called surface gravity of $\mathcal{H}$. The orthogonality and normalization conditions of the tetrad system now uniquely specify a two-parameter subset of geodesic of a null geodesic congruence by displacement vectors $m, \overline{m}$. From the definition of the spin coefficients, requirement (2) implies $\mu = -\overline{m}\nabla_m n\hat{=}0$; (3) fixes the complex scalar $\phi_2\hat{=}0$; the last condition requires the Lie derivative of all spin coefficients and gauge fields alone n vanish on the horizon. Moreover, $\kappa_{(n)}$ is a constant on the horizon. Those constraints are derived explicitly in [184, 186].

In a Bondi-like coordinate $(u, r, z, \bar{z})$, the horizon will be located at $r = 0$. We will further choose the following gauge and boundary conditions on

[1] Our convention of the normal vector is different from [184, 186] to be consistent with the notation at null infinity. To compare with [184, 186], one just needs to exchange l and n.

Maxwell potential A_μ as $A_r = 0, A_u \hat{=} 0$. The full solution of Newman-Penrose equations has been derived in Appendix A.6. We summarize as following:

A solution of Einstein-Maxwell theory in the neighborhood of an isolated horizon will be specified by the initial data $\Psi_0(r,z,\bar{z})$, $\phi_0(r,z,\bar{z})$ *and the data on* S^2 *including its metric* $q_{ab} = \frac{2dzd\bar{z}}{P(z,\bar{z})\bar{P}(z,\bar{z})}$, *its extrinsic curvature* $\tau(z,\bar{z})$, *electromagnetic fields* $A_u(z,\bar{z})$, $A_z(z,\bar{z})$[2].

The near horizon metric in $(+,-,-,-)$ signature and gauge fields are given by

$$\begin{aligned}
ds^2 = & [4\gamma_0 r + (\Psi_2^0 + \overline{\Psi}_2^0 + 2\phi_1^0\overline{\phi}_1^0)r^2 + O(r^3)]du^2 + 2dudr \\
& +[4\frac{\overline{\tau}_0}{\bar{P}}r + \frac{2\overline{\sigma}_0}{\bar{P}}(\overline{\Psi}_1^0 + \overline{\phi}_0^0\phi_1^0 - \tau_0)r^2 + O(r^3)]dudz \\
& +[4\frac{\tau_0}{P}r + \frac{2\sigma_0}{P}(\Psi_1^0 + \phi_0^0\overline{\phi}_1^0 - \overline{\tau}_0)r^2 + O(r^3)]dud\bar{z} \\
& +[\frac{2\overline{\sigma}_0}{\bar{P}^2}r - \frac{4\overline{\Psi}_0^0}{\bar{P}^2}r^2 + O(r^3)]dz^2 + [\frac{2\sigma_0}{P^2}r - \frac{4\Psi_0^0}{P^2}r^2 + O(r^3)]d\bar{z}^2 \\
& +2[-\frac{1}{P\bar{P}} - \frac{2\sigma_0\overline{\sigma}_0}{P\bar{P}}r^2 + O(r^3)]dzd\bar{z}, \qquad (5.1)
\end{aligned}$$

$$A_u = (\phi_1^0 + \overline{\phi}_1^0)r + (\bar{\eth}\phi_0^0 + \eth\overline{\phi}_0^0)r^2 + O(r^3), \qquad (5.2)$$

$$A_z = A_z^0 + \frac{\overline{\phi}_0^0}{\bar{P}}r + \frac{1}{2}(\frac{\overline{\phi}_0^1}{\bar{P}} - \frac{\phi_0^0\sigma_0}{\bar{P}})r^2 + O(r^3), \qquad (5.3)$$

where $\sigma_0 = \frac{1}{2\gamma_0}[\tau_0^2 - \eth\tau_0]$.

5.2 Asymptotic symmetries and conserved current near the isolated horizon

The asymptotic behavior of our solution (5.1) is consistent with the boundary choice of [37], which means the Bondi-like coordinate is compatible with such boundary condition. In the following, we will work out the asymptotic symmetry of Einstein-Maxwell theory near isolated horizon with the gauge

[2]The local existence for the characteristic initial value has been proven in [187] in a neighborhood in the future of the light cone for vacuum Einstein equations. Using similar method, it is not difficult to extend the local existence to Einstein-Maxwell equations with the conditions we have set on Maxwell fields.

and boundary condition given as

$$g_{rr} = g_{rz} = g_{r\bar{z}} = A_r = 0, \quad g_{ur} = 1, \tag{5.4}$$
$$g_{uu} = 4\gamma_0 r + O(r^2), \quad g_{uz} = O(r), \quad g_{u\bar{z}} = O(r), \quad \partial_u g_{ab} = O(r^2), \tag{5.5}$$
$$A_u = O(r), \quad A_z = O(1), \quad A_{\bar{z}} = O(1), \quad \partial_u A_\mu = O(r). \tag{5.6}$$

The gauge condition (5.4) will fix the asymptotic Killing vector and asymptotic $U(1)$ symmetry parameter ζ up to an integration constant

$$\begin{cases} \xi^u = f(u,z,\bar{z}), \\ \xi^r = R_r(u,z,\bar{z}) - r\partial_u f + \partial_z f \int dr\ (g^{zz}g_{uz} + g^{z\bar{z}}g_{u\bar{z}}) + \partial_{\bar{z}} f \int dr\ (g^{\bar{z}\bar{z}}g_{u\bar{z}} + g^{z\bar{z}}g_{uz}), \\ \xi^z = Y^z(u,z,\bar{z}) - \partial_z f \int dr\ g^{zz} - \partial_{\bar{z}} f \int dr\ g^{z\bar{z}}, \\ \xi^{\bar{z}} = Y^{\bar{z}}(u,z,\bar{z}) - \partial_z f \int dr\ g^{z\bar{z}} - \partial_{\bar{z}} f \int dr\ g^{\bar{z}\bar{z}}, \\ \zeta = \zeta_0(u,z,\bar{z}) + \int dr\ (A_z\partial_{\bar{z}} f + A_{\bar{z}}\partial_z f). \end{cases} \tag{5.7}$$

Then, the boundary condition (5.5) and (5.6) will fix the u dependence of the integration constants. Hence, the asymptotic Killing vector and asymptotic $U(1)$ symmetry parameter ζ are further constrained to

$$\begin{cases} \xi^u = T(z,\bar{z}), \\ \xi^r = \partial_z T \int dr\ (g^{zz}g_{uz} + g^{z\bar{z}}g_{u\bar{z}}) + \partial_{\bar{z}} T \int dr\ (g^{\bar{z}\bar{z}}g_{u\bar{z}} + g^{z\bar{z}}g_{uz}) = O(r^2), \\ \xi^z = Y(z) - \partial_z T \int dr\ g^{zz} - \partial_{\bar{z}} T \int dr\ g^{z\bar{z}} = Y(z) + O(r), \\ \xi^z = \bar{Y}(\bar{z}) - \partial_{\bar{z}} T \int dr\ g^{\bar{z}\bar{z}} - \partial_z T \int dr\ g^{z\bar{z}} = \bar{Y}(\bar{z}) + O(r), \\ \zeta = \zeta(z,\bar{z}) + \int dr\ (A_z\partial_{\bar{z}} T + A_{\bar{z}}\partial_z T) = \zeta(z,\bar{z}) + O(r). \end{cases} \tag{5.8}$$

Near horizon, the complete asymptotic symmetries of Einstein-Maxwell form a closed algebra as

$$[(\xi_1,\zeta_1),(\xi_2,\zeta_2)]_M = (\hat{\xi},\hat{\zeta}), \tag{5.9}$$
$$\hat{\xi} = [\xi_1,\xi_2], \quad \hat{\zeta} = \xi_1(\zeta_2) - \xi_2(\zeta_1). \tag{5.10}$$

By turning off Maxwell fields, our asymptotic symmetry algebra will recover the one found in [37,188]. Following the strategy of [53], one can compute the conserved current associated to the asymptotic symmetries and the current is derived by[3]

$$\begin{aligned} \mathcal{J}^u_{\xi,\zeta} \quad = \quad & -\frac{1}{P\bar{P}}\Big\{2T\gamma_0 + Y[\frac{\overline{\tau}_0}{\bar{P}} + (\phi_1^0 + \overline{\phi}_1^0)A_z^0] \\ & + \bar{Y}[\frac{\tau_0}{P} + (\phi_1^0 + \overline{\phi}_1^0)A_{\bar{z}}^0] + \zeta(\phi_1^0 + \overline{\phi}_1^0)\Big\}, \end{aligned} \tag{5.11}$$

[3] We have set $8\pi G = c = 1$ and the Lagrangian of Einstein-Maxwell theory is $\mathcal{L} = \frac{1}{2}\sqrt{g}(R + \frac{1}{2}F^2)$.

$$\mathcal{J}^z = 0, \tag{5.12}$$

with a locally well-defined current algebra

$$\delta_{(\xi_2,\zeta_2)}\mathcal{J}^u_{(\xi_1,\zeta_1)} = \mathcal{J}^u_{(\hat{\xi},\hat{\zeta})} + \partial_a L^{[ua]}_{(\xi_1,\zeta_2),(\xi_2,\zeta_2)}, \tag{5.13}$$

while

$$L^{[uz]}_{(\xi_1,\zeta_2),(\xi_2,\zeta_2)} = Y_2\mathcal{J}^u_{(\xi_1,\zeta_1)}. \tag{5.14}$$

The asymptotic current can be adapted into a more consistent way

$$\mathcal{J}^u_{\xi,\zeta} = \frac{1}{P\bar{P}}\ \left[\xi \cdot \omega + \xi \cdot \mathcal{A}(\phi_1^0 + \overline{\phi}_1^0) + \zeta(\phi_1^0 + \overline{\phi}_1^0)\right], \tag{5.15}$$

where the extrinsic curvature one-form on the horizon is defined by

$$\omega_a \hat{=} l_b \nabla_a n^b \hat{=} -2\gamma_0 l_a + \tau_0 \overline{m}_a + \overline{\tau}_0 m_a$$

and $\xi \hat{=} Tn + \mathcal{Y}m + \overline{\mathcal{Y}}\overline{m}$ is the asymptotic Killing vector while

$$\mathcal{Y} = \frac{\bar{Y}(\bar{z})}{P},\ \ \overline{\mathcal{Y}} = \frac{Y(z)}{\bar{P}},\ \ \mathcal{A} = PA^0_{\bar{z}}\overline{m} + \bar{P}A^0_z m.$$

Amazingly, the current associated to supertranslation gets no contribution from electromagnetic fields. This reveals the fact that the electromagnetic fields become soft on the isolated horizon and do not contribute to black hole energy which is consistent with the statement in [39]. The superrotation parts indeed have been modified from the polarization of the soft photons.

5.3 Soft electric hairs on isolated horizon

In this section we will deal with a special case of solution with more concrete physical interpretation. The isolated horizon is spherically symmetric and has unit radius, *i.e.* $P = \bar{P} = \frac{1}{\sqrt{2}}(1 + z\bar{z})$ and $\tau_0 = 0$. Electromagnetic fields are not vanishing near the horizon. This case can be considered as the later state of electromagnetic fields around a Schwarzschild black hole with mass $M = \frac{1}{2}$. Back-scattered radiation can be neglected but wave can still radiate in the region far away from the horizon, eventually will be scattered at null infinity. This choice will lead to a globally well-defined horizon charge from the conserved current with respect to asymptotic $U(1)$ symmetry and supertranslation. The charge associated to asymptotic $U(1)$ symmetry will be deduced to

$$\mathcal{Q}^{\mathcal{H}}_\zeta = \int_{S^2} d\Omega^2 \zeta(\phi_1^0 + \overline{\phi}_1^0), \tag{5.16}$$

where $d\Omega^2$ is the unit spherical surface element. By expanding the asymptotic $U(1)$ symmetry parameter ζ in spherical harmonics

$$\zeta = \sum_{h=0}^{\infty} \sum_{g=-h}^{h} \zeta_{h,g} Y_{h,g}(z, \bar{z}),$$

one can further define the modes of the horizon charge as

$$\mathcal{Q}^{\mathcal{H}}_{\zeta_{h,g}} = \int_{S^2} d\Omega^2 (\phi_1^0 + \overline{\phi}_1^0) Y_{h,g}. \tag{5.17}$$

Interestingly, the modes we have defined in (5.17) are equivalent to the electric multipoles introduced in [46]. The supertranslation charges will vanish except the zero mode which becomes a combination proportional to the surface gravity multiplying the horizon area [37, 176]. This also has a counterpart from the mass multipoles in [46], where the mass monopole M_0 is the only non-zero mass multipoles of a spherically symmetric isolated horizon. Thus, one immediate application of the soft electric hairs is to capture the electric multipoles information on the horizon[4]. This indeed confirms that the later stage of a black hole collapse carries infinite numbers of soft electric hairs. As we have shown in the previous section that the electromagnetic fields become soft on the isolated horizon, the modes (5.17) can be understood as soft photons located on the horizon during the dynamical process before the isolated horizon formed. As shown in Chapter 5.1, the Maxwell fields are fully characterized by the electromagnetic fields A_z^0 and the real part of ϕ_1^0. To track the whole information of the electromagnetic fields, one still needs to consult to the local information from the superrotation current.

The isolated horizon is admitted by a black hole who itself is in equilibrium but whose exterior contains radiation (*i.e.* the whole spacetime is not yet stationary). The huge amount of classical charges we have introduced on the isolated horizon have nothing to violate the no hair theorem because it is only valid when the whole spacetime is stationary. One may wonder whether the charges will act on a quantum state trivially or not, which would be equivalent to ask the asymptotic $U(1)$ symmetry is spontaneously broken or not. As far as we can understand, it's not necessary to have the asymptotic $U(1)$ symmetry spontaneously broken. The reason is that the system is already in equilibrium around the isolated horizon. There will be no longer photon or soft photon reaching the horizon. Thus, the system has no interaction and becomes a free theory. Hence, the quantum states are supposed to be the engine states of the quantized operators $\mathcal{Q}^{\mathcal{H}}_{h,g}$. This can be also seen from the fact that the horizon charges, we have defined in

[4]In [189], multipoles are considered as required parameters in the construction of the ensemble to calculate the black hole entropy.

(5.16), do not include a soft part compared to the one defined in [44]. However, this symmetry will be broken by the appearance of radiation crossing the horizon. It will be highly interesting to investigate the charges on a dynamical horizon [190]. One could expect that it will be very similar to the case of null infinity where the asymptotic charges have both the hard part and the soft part. Accordingly these charges will act on quantum states non-trivially.

5.4 Supertranslation and foliation of an isolated horizon

We would like to have a deeper investigation of the action of the supertranslation on the horizon. A supertranslation will preserve the induced horizon metric, which is degenerate, and the null normal vector n. But the extrinsic curvature one-form will be transformed like the gradient of a scalar

$$\omega'_a \to \omega_a - 2\gamma_0 df. \tag{5.18}$$

Such a transformation is related to the ambiguity of the foliation of an isolated horizon which was discussed in [46, 186].

Let's consider a non-extremal($\gamma_0 \neq 0$) isolated horizon $(\mathcal{H}, n)$. We will ignore the Maxwell fields and focus only on the horizon geometry determined by the induced metric q_{ab} and the induced derivative operator D in this section. Then a fixed cross-section S of $\mathcal{H}$ can be treated as a leaf of a foliation u =constant such that $n^a D_a u = 1$ and the normal l_a of this foliation can be set as $l_a = D_a u$ with $l_a n^a = 1$. A projection operator $\tilde{q}_a^b$ on the leaves of the foliation is defined by $\tilde{q}_a^b = \delta_a^b - l_a n^b$. Since q_{ab} is degenerate, D can not be fully determined by q_{ab}. But on the cross-section S, one has a unique (torsion-free) derivative operator $\hat{D}$ compatible with $\hat{q}_{ab}$ the projection of q_{ab} on S. To determine the derivative operator D on the horizon, one only needs to specify its action on l_a[5]. Let's define $S_{ab} := D_a l_b$ who satisfies $S_{ab} n^b \hat{=} -\omega^a$ on the horizon. Then the horizon geometry is completely specified by the triplet $(q_{ab}, \omega_a, S_{ab})$.

Suppose the triplet $(q_{ab}, \omega_a, S_{ab})$ is given on the horizon with one foliation u =constant, hence on cross-section S, the free data $(\hat{q}_{ab}, \hat{\omega}_a, \hat{S}_{ab})$ can be derived by the projection operator from the horizon $\mathcal{H}$ as

$$\hat{q}_{ab} = q_{ab}, \tag{5.19}$$

$$\hat{\omega}_a = \omega_a + 2\gamma_0 l_a, \tag{5.20}$$

$$\hat{S}_{ab} = S_{ab} + l_a \omega_b + \omega_a l_b + 2\gamma_0 l_a l_b. \tag{5.21}$$

Let's now consider another cross section S' which does not belong to the same foliation. One can choose u' =constant as the corresponding foliation.

[5]The action of D on n^a is $D_a n^b \hat{=} \omega_a n^b$.

Let $f = u' - u$ and $\mathcal{L}_n f = 0$. The two sets of free data are related by

$$\hat{q}_{ab} = \hat{q'}_{ab}, \tag{5.22}$$

$$\hat{\omega}_a = \hat{\omega'}_a - 2\gamma_0 df, \tag{5.23}$$

$$\hat{S}_{ab} = \hat{S'}_{ab} + D_a D_b f. \tag{5.24}$$

This is the ambiguity of choosing foliation of an isolated horizon. This ambiguity can be also understood in an inverse way that the same set of free data can create different foliation. The difference of ω_a matches precisely the transformation under a supertranslation on the horizon given in (5.18) in the beginning of this section.

The study of null infinity reveals an fascinating relation between the supertranslation on the null infinity and the memory effect. It seems quite promising that there should exist the analogue of the memory effect on the horizon. The memory effect is non-zero change of asymptotic shear, which is caused by some dynamical processes. Such result has been realized by a supertranslation at null infinity recently in [185]. Similar things also happen on a horizon. As discussed previously, the foliation of an isolated horizon has some ambiguities and different foliations are related by supertranslation on the horizon. A foliation of an isolated horizon can be connected to another one by a dynamical process before it formed. Since the foliation of dynamical horizon is unique [191], the final foliation of isolated horizon is fixed by continuity condition, *i.e.* different foliation corresponds to different dynamical process of black hole. This is quite similar to what happens at null infinity. It would be definitely worthwhile to investigate such an effect elsewhere.

Appendix A

Details on computations

A.1 Comparison of the charges between first order formalism and metric formalism

To compare with the results in the metric formalism in [41], we are allowed to use any on-shell result and reducibility parameters to show the equivalence. So we have the following results

$$\begin{aligned}
&\omega^{ab}{}_{\mu} = e^a_\tau \nabla_\mu e^{b\tau}, \\
&\Lambda^{ab} = \omega^{ba}{}_{\rho}\xi^\rho + e^b_\rho e^a_\tau \nabla^\rho \xi^\tau, \\
&\nabla_\tau \xi_\rho + \nabla_\rho \xi_\tau = 0. \qquad (A.1)
\end{aligned}$$

From (A.1), one can deduce to

$$\omega^{ab}{}_{\alpha}\xi^\alpha + \Lambda^{ab} = e^b_\rho e^a_\tau \nabla^\rho \xi^\tau, \qquad (A.2)$$

$$\delta\omega^{ab}{}_{\rho} = \delta e^a_\tau \nabla_\rho e^{b\tau} + e^a_\tau \nabla_\rho \delta e^{b\tau} + e^a_\alpha \delta\Gamma^\alpha_{\rho\tau} e^{b\tau}. \qquad (A.3)$$

Thus, the $n-2$ current will be simplified as

$$\begin{aligned}
k^{[\mu\nu]}_{\xi\Lambda} = {} & \frac{e}{16\pi G}\big\{[\delta e^a_\tau \nabla_\rho e^{b\tau} + e^a_\tau \nabla_\rho \delta e^{b\tau} + e^a_\alpha \delta\Gamma^\alpha_{\rho\tau} e^{b\tau}][e^\mu_a e^\nu_b \xi^\rho + 2\xi^\mu e^\rho_b e^\nu_a] \\
& + (2\delta e^\mu_a e^a_\sigma \nabla^\nu \xi^\sigma - \nabla^\nu \xi^\mu e^a_\rho \delta e^\rho_a)\big\} - (\mu \leftrightarrow \nu). \qquad (A.4)
\end{aligned}$$

with the help of $\delta(e^a_\alpha e^\beta_a) = 0$ and $\nabla(e^{a\alpha} e^\beta_a) = 0$, the current can be further reduced to

$$\begin{aligned}
k^{[\mu\nu]}_{\xi\Lambda} = {} & \frac{e}{16\pi G}\big\{\xi^\rho \nabla_\rho(\delta e^\mu_a e^{a\nu}) + 2\xi^\mu \nabla_\rho(\delta e^\nu_a e^{a\rho}) + e^a_\alpha \delta\Gamma^\alpha_{\rho\tau} e^{b\tau}[e^\mu_a e^\nu_b \xi^\rho + 2\xi^\mu e^\rho_b e^\nu_a] \\
& + (2\delta e^\nu_a e^a_\rho \nabla^\rho \xi^\mu - \nabla^\nu \xi^\mu e^a_\rho \delta e^\rho_a)\big\} - (\mu \leftrightarrow \nu), \\
= {} & \frac{e}{16\pi G}\big\{\nabla_\rho(\xi^\rho \delta e^\mu_a e^{a\nu}) + 2\nabla_\rho(\xi^\mu \delta e^\nu_a e^{a\rho}) + [\delta\Gamma^\mu_{\rho\tau} g^{\nu\tau}\xi^\rho + 2\delta\Gamma^\nu_{\rho\tau} g^{\rho\tau}\xi^\mu] \\
& - \nabla^\nu \xi^\mu e^a_\rho \delta e^\rho_a\big\} - (\mu \leftrightarrow \nu) \qquad (A.5)
\end{aligned}$$

Since the term $\nabla_\rho(\xi^\rho \delta e^\mu_a e^{a\nu}) + \nabla_\rho(\xi^\mu \delta e^\nu_a e^{a\rho}) + \nabla_\rho(\xi^\nu \delta e^\rho_a e^{a\mu}) - (\mu \leftrightarrow \nu)$ does not change the equivalence class, the current can be deduced to

$$\begin{aligned}
k^{[\mu\nu]}_{\xi\Lambda} &= \frac{e}{16\pi G}\big\{\nabla_\rho(\xi^\mu \delta e^\nu_a e^{a\rho}) - \nabla_\rho(\xi^\nu \delta e^\rho_a e^{a\mu}) + [\delta\Gamma^\mu_{\rho\tau} g^{\nu\tau}\xi^\rho + 2\delta\Gamma^\nu_{\rho\tau} g^{\rho\tau}\xi^\mu] \\
&\quad -\nabla^\nu \xi^\mu e^a_\rho \delta e^\rho_a\big\} - (\mu \leftrightarrow \nu) \\
&= \frac{e}{16\pi G}\big\{\nabla_\rho(\xi^\mu \delta e^\nu_a e^{a\rho}) + \nabla_\rho(\xi^\mu \delta e^\rho_a e^{a\nu}) + [\delta\Gamma^\mu_{\rho\tau} g^{\nu\tau}\xi^\rho + 2\delta\Gamma^\nu_{\rho\tau} g^{\rho\tau}\xi^\mu] \\
&\quad -\nabla^\nu \xi^\mu e^a_\rho \delta e^\rho_a\big\} - (\mu \leftrightarrow \nu) \\
&= \frac{e}{16\pi G}\big\{\nabla_\rho[\xi^\mu \delta(e^\nu_a e^{a\rho})] + [\delta\Gamma^\mu_{\rho\tau} g^{\nu\tau}\xi^\rho + 2\delta\Gamma^\nu_{\rho\tau} g^{\rho\tau}\xi^\mu] \\
&\quad -\nabla^\nu \xi^\mu e^a_\rho \delta e^\rho_a\big\} - (\mu \leftrightarrow \nu) \qquad (A.6)
\end{aligned}$$

We have the relation between δe, $\delta\Gamma$ and h as

$$\begin{aligned}
&h^{\alpha\beta} = -\delta g^{\alpha\beta} = -\delta(e^{\alpha a} e^\beta_a), \\
&h = -2 e^a_\alpha \delta e^\alpha_a, \\
&\delta\Gamma^\alpha{}_{\rho\tau} = \frac{1}{2}(\nabla_\rho h^\alpha_\tau + \nabla_\tau h^\alpha_\rho - \nabla^\alpha h_{\rho\tau}),
\end{aligned}$$

Insert all these relations to current, one has

$$\begin{aligned}
k^{[\mu\nu]}_{\xi\Lambda} &= \frac{e}{16\pi G}\big\{ - \nabla_\rho[\xi^\mu h^{\nu\rho}] + \frac{1}{2} h \nabla^\nu \xi^\mu + \delta\Gamma^\mu_{\rho\tau} g^{\nu\tau}\xi^\rho + 2\delta\Gamma^\nu_{\rho\tau} g^{\rho\tau}\xi^\mu\big\} \\
&\quad -(\mu \leftrightarrow \nu), \\
&= \frac{e}{16\pi G}\big\{\xi^\nu \nabla_\sigma h^{\mu\sigma} + \frac{1}{2} h \nabla^\nu \xi^\mu - h^{\nu\sigma}\nabla_\sigma \xi^\mu + \frac{1}{2}(\nabla_\rho h^\mu_\tau + \nabla_\tau h^\mu_\rho \\
&\quad -\nabla^\mu h_{\rho\tau}) g^{\nu\tau}\xi^\rho + (\nabla_\rho h^\nu_\tau + \nabla_\tau h^\nu_\rho - \nabla^\nu h_{\rho\tau}) g^{\rho\tau}\xi^\mu\big\} - (\mu \leftrightarrow \nu), \\
&= \frac{e}{16\pi G}\big\{\xi^\nu \nabla_\sigma h^{\mu\sigma} + \frac{1}{2} h \nabla^\nu \xi^\mu - h^{\nu\sigma}\nabla_\sigma \xi^\mu + \xi_\sigma \nabla^\nu h^{\mu\sigma} + \xi^\nu \nabla^\mu h \\
&\quad -2\xi^\nu \nabla_\sigma h^{\mu\sigma}\big\} - (\mu \leftrightarrow \nu), \\
&= \frac{e}{16\pi G}\big\{\xi^\nu \nabla^\mu h - \xi^\nu \nabla_\sigma h^{\mu\sigma} + \xi_\sigma \nabla^\nu h^{\mu\sigma} + \frac{1}{2} h \nabla^\nu \xi^\mu - h^{\nu\sigma}\nabla_\sigma \xi^\mu\big\} \\
&\quad -(\mu \leftrightarrow \nu). \qquad (A.7)
\end{aligned}$$

Compare to eq.(6.23) of the [41], it is exactly the same. It can be also shown the equivalence from the integrable form (2.71).

A.2 Conserved charges for field dependent vectors

In this appendix, we provide with the formalism of conserved charges in Einstein gravity in the case of field dependent vectors like those in (3.10) and establish that the expression of charges obtained from covariant phase space methods [99] or cohomological methods [83] apply to this case as well. We also discuss the integrability of charge variations in the case of field dependent vectors. We will keep the spacetime dimension arbitrary since no special feature arises in three dimensions.

A.2.1 Expression for the charges

Field dependence and the Iyer-Wald charge. Assume we have a vector χ which is a function of the dynamical fields Φ such as the metric. In our example, the metric dependence reduces to $\chi = \chi(L_+, L_-)$. We call this a field dependent vector. We want to find the corresponding charge $\delta \boldsymbol{Q}_\chi$ and the integrability condition for such vectors. We proceed using the approach of Iyer-Wald [99] and carefully keep track of the field dependence. We adopt the convention that $\delta\Phi$ are Grassman even. First define the Noether current associated to the vector χ as

$$\boldsymbol{J}_\chi[\Phi] = \boldsymbol{\Theta}[\delta_\chi\Phi, \Phi] - \chi \cdot \boldsymbol{L}[\Phi], \tag{A.8}$$

where $\boldsymbol{L}[\Phi]$ is the Lagrangian (as a top form), and $\boldsymbol{\Theta}[\delta_\chi\Phi, \Phi]$ is equal to the boundary term in the variation of the Lagrangian, i.e $\delta \boldsymbol{L} = \frac{\delta L}{\delta \Phi}\delta\Phi + d\boldsymbol{\Theta}[\delta\Phi, \Phi]$. Using the Noether identities one can then define the on-shell vanishing Noether current as $\frac{\delta L}{\delta\Phi}\mathcal{L}_\chi\Phi = d\boldsymbol{S}_\chi[\Phi]$. It follows that $\boldsymbol{J}_\chi + \boldsymbol{S}_\chi$ is closed off-shell and therefore $\boldsymbol{J}_\chi \approx d\boldsymbol{Q}_\chi$, where $\boldsymbol{Q}_\chi$ is the Noether charge density (we use the symbol $\approx$ to denote an on-shell equality). Now take a variation of the above equation

$$\begin{aligned} \delta \boldsymbol{J}_\chi &= \delta\boldsymbol{\Theta}[\delta_\chi\Phi, \Phi] - \delta(\chi\cdot\boldsymbol{L}) \\ &= \delta\boldsymbol{\Theta}[\delta_\chi\Phi, \Phi] - \chi\cdot\delta\boldsymbol{L} - \delta\chi\cdot\boldsymbol{L} \\ &\approx \delta\boldsymbol{\Theta}[\delta_\chi\Phi, \Phi] - \chi\cdot d\boldsymbol{\Theta}[\delta\Phi, \Phi] - \delta\chi\cdot\boldsymbol{L}\,. \end{aligned} \tag{A.9}$$

Using the Cartan identity $\mathcal{L}_\chi\boldsymbol{\sigma} = \chi\cdot d\boldsymbol{\sigma} + d(\chi\cdot\boldsymbol{\sigma})$ valid for any vector χ and any form $\boldsymbol{\sigma}$, we find

$$\delta \boldsymbol{J}_\chi = \Big(\delta\boldsymbol{\Theta}[\delta_\chi\Phi, \Phi] - \delta_\chi\boldsymbol{\Theta}[\delta\Phi, \Phi]\Big) + d(\chi\cdot\boldsymbol{\Theta}[\delta\Phi, \Phi]) - \delta\chi\cdot\boldsymbol{L}. \tag{A.10}$$

The important point here is that

$$\delta\boldsymbol{\Theta}[\delta_\chi\Phi, \Phi] = \delta^{[\Phi]}\boldsymbol{\Theta}[\delta_\chi\Phi, \Phi] + \boldsymbol{\Theta}[\delta_{\delta\chi}\Phi, \Phi]\,, \tag{A.11}$$

where we define $\delta^{[\Phi]}$ to act only on the explicit dependence on dynamical fields and its derivatives, but not on the implicit field dependence in χ. Therefore, we find

$$\begin{aligned} \delta \boldsymbol{J}_\chi &= \Big(\delta^{[\Phi]}\boldsymbol{\Theta}[\delta_\chi\Phi, \Phi] - \delta_\chi\boldsymbol{\Theta}[\delta\Phi, \Phi]\Big) + d(\chi\cdot\boldsymbol{\Theta}[\delta\Phi, \Phi]) + \Big(\boldsymbol{\Theta}[\delta_{\delta\chi}\Phi, \Phi] - \delta\chi\cdot\boldsymbol{L}\Big) \\ &= \boldsymbol{\omega}^{LW}[\delta\Phi\,,\,\delta_\chi\Phi\,;\,\Phi] + d(\chi\cdot\boldsymbol{\Theta}[\delta\Phi, \Phi]) + \boldsymbol{J}_{\delta\chi}\,, \end{aligned} \tag{A.12}$$

where

$$\boldsymbol{\omega}^{LW}[\delta\Phi\,,\,\delta_\chi\Phi\,;\,\Phi] = \delta^{[\Phi]}\boldsymbol{\Theta}[\delta_\chi\Phi, \Phi] - \delta_\chi\boldsymbol{\Theta}[\delta\Phi, \Phi], \tag{A.13}$$

is the Lee-Wald presymplectic form [84]. Note that the variation acting on $\boldsymbol{\Theta}[\delta_\chi\Phi, \Phi]$, only acts on the explicit field dependence. This is necessary in order for $\boldsymbol{\omega}^{LW}[\delta\Phi\,,\,\delta_\chi\Phi\,;\,\Phi]$ to be bilinear in its variations. Reordering the terms we find

$$\begin{aligned}\boldsymbol{\omega}^{LW}[\delta\Phi\,,\,\delta_\chi\Phi\,;\,\Phi] &= \delta\boldsymbol{J}_\chi - \boldsymbol{J}_{\delta\chi} - d(\chi\cdot\boldsymbol{\Theta}[\delta\Phi,\Phi]) \\ &= \delta^{[\Phi]}\boldsymbol{J}_\chi - d(\chi\cdot\boldsymbol{\Theta}[\delta\Phi,\Phi]).\end{aligned} \tag{A.14}$$

If $\delta\Phi$ solves the linearized field equations, then $\boldsymbol{J}_\chi \approx d\boldsymbol{Q}_\chi$ implies $\delta^{[\Phi]}\boldsymbol{J}_\chi \approx d(\delta^{[\Phi]}\boldsymbol{Q}_\chi)$. As a result we obtain

$$\boldsymbol{\omega}^{LW}[\delta\Phi\,,\,\delta_\chi\Phi\,;\,\Phi] \quad \approx d\boldsymbol{k}^{IW}_\chi[\delta\Phi;\Phi] \tag{A.15}$$

where $\boldsymbol{k}^{IW}_\chi$ is the Iyer-Wald surface charge form

$$\boldsymbol{k}^{IW}_\chi = \Big(\delta^{[\Phi]}\boldsymbol{Q}_\chi - \chi\cdot\boldsymbol{\Theta}[\delta\Phi,\Phi]\Big). \tag{A.16}$$

Therefore the infinitesimal charge associated to a field dependent vector and a codimension two, spacelike compact surface S is defined as the Iyer-Wald charge

$$\delta H_\chi = \oint_S \boldsymbol{k}^{IW}_\chi[\delta\Phi;\Phi] = \oint_S \Big(\delta^{[\Phi]}\boldsymbol{Q}_\chi - \chi\cdot\boldsymbol{\Theta}[\delta\Phi,\Phi]\Big). \tag{A.17}$$

The key point in the above expression is that the variation does not act on χ. One may rewrite the charge as

$$\delta H_\chi = \oint_S \Big(\delta\boldsymbol{Q}_\chi - \boldsymbol{Q}_{\delta\chi} - \chi\cdot\boldsymbol{\Theta}[\delta\Phi,\Phi]\Big). \tag{A.18}$$

From the above, there is an additional term in the Iyer-Wald charge in the case of field dependent vectors.

Field dependence and the Barnich-Brandt charge. There is another definition of the presymplectic structure which leads to a consistent covariant phase space framework. This is the so-called invariant presymplectic form [83] defined through Anderson's homotopy operator [43]:

$$\begin{aligned}\boldsymbol{\omega}^{inv}[\delta_1\Phi,\delta_2\Phi;\Phi] &= -\frac{1}{2}I^n_{\delta_1\Phi}\left(\delta_2\Phi^i\frac{\delta\boldsymbol{L}}{\delta\Phi^i}\right) - (1\leftrightarrow 2), \\ I^n_{\delta\Phi} &\equiv \left(\delta\Phi^i\frac{\partial}{\Phi^i_{,\mu}} - \delta\Phi^i\partial_\nu\frac{\partial}{\Phi^i_{,\nu\mu}} + \delta\Phi_{,\nu}\frac{\partial}{\partial\Phi_{,\nu\mu}}\right)\frac{\partial}{\partial dx^\mu}.\end{aligned} \tag{A.19}$$

The invariant presymplectic form only depends on the equations of motion of the Lagrangian and is therefore independent on the addition of boundary terms in the action. This presymplectic structure differs from the Lee-Wald presymplectic structure by a specific boundary term $\boldsymbol{E}$

$$\boldsymbol{\omega}^{inv}[\delta_1\Phi,\delta_2\Phi;\Phi] = \boldsymbol{\omega}^{LW}[\delta_1\Phi,\delta_2\Phi;\Phi] + d\boldsymbol{E}[\delta_1\Phi,\delta_2\Phi,\Phi], \tag{A.20}$$

where $\boldsymbol{E}$ is given by [83, 98]

$$\boldsymbol{E}[\delta_1\Phi, \delta_2\Phi, \Phi] = -\frac{1}{2} I^{n-1}_{\delta_1\Phi} \boldsymbol{\Theta}[\delta_2\Phi, \Phi] - (1 \leftrightarrow 2). \tag{A.21}$$

Here, $\boldsymbol{\Theta}[\delta\Phi, \Phi]$ is defined as $I^n_{\delta\Phi}\boldsymbol{L}$, which agrees with the Lee-Wald prescription and Anderson's homotopy operator for a $n-1$ form is given for second order theories by

$$I^{n-1}_{\delta\Phi} \equiv \left(\frac{1}{2}\delta\Phi^i \frac{\partial}{\partial\Phi^i_{,\nu}} - \frac{1}{3}\delta\Phi^i\partial_\rho \frac{\partial}{\partial\Phi^i_{,\rho\nu}} + \frac{2}{3}\delta\Phi^i_{,\rho}\frac{\partial}{\partial\Phi^i_{,\rho\nu}}\right)\frac{\partial}{\partial dx^\nu}. \tag{A.22}$$

The identity (A.20) follows from $[\delta, I^p_{\delta\Phi}] = 0$ and the equalities

$$0 \leqslant p < n: \quad I^{p+1}_{\delta\Phi} d + dI^p_{\delta\Phi} = \delta, \tag{A.23}$$

$$p = n: \quad \delta\Phi^i \frac{\delta}{\delta\Phi^i} + dI^n_{\delta\Phi} = \delta. \tag{A.24}$$

The presymplectic structure evaluated on the field transformation generated by the (possibly field-dependent) vector field χ, $\boldsymbol{\omega}^{inv}[\delta_1\Phi, \delta_\chi\Phi; \Phi]$, is defined from a contraction as

$$\boldsymbol{\omega}^{inv}[\delta_1\Phi, \delta_\chi\Phi; \Phi] = (\partial_{(\mu)}\delta_\chi\Phi)\frac{\partial}{\delta_2\Phi^i_{(\mu)}}\boldsymbol{\omega}^{inv}[\delta_1\Phi, \delta_2\Phi; \Phi]. \tag{A.25}$$

It then follows from (A.20) that

$$\boldsymbol{\omega}^{inv}[\delta\Phi, \delta_\chi\Phi; \Phi] = \boldsymbol{\omega}^{LW}[\delta\Phi\,,\, \delta_\chi\Phi\,;\, \Phi] + d\boldsymbol{E}[\delta\Phi, \delta_\chi\Phi, \Phi]. \tag{A.26}$$

Inserting (A.15) from the above analysis, we find

$$\boldsymbol{\omega}^{inv}[\delta\Phi\,,\, \delta_\chi\Phi\,;\, \Phi] \approx d\boldsymbol{k}^{BB}_\chi[\delta\Phi; \Phi] \tag{A.27}$$

where $\boldsymbol{k}^{BB}_\chi$ is the Barnich-Brandt surface charge form,

$$\boldsymbol{k}^{BB}_\chi[\delta\Phi; \Phi] = \delta^{[\Phi]}\boldsymbol{Q}_\chi - \chi \cdot \boldsymbol{\Theta}[\delta\Phi, \Phi] + \boldsymbol{E}[\delta\Phi, \delta_\chi\Phi, \Phi]. \tag{A.28}$$

After evaluation on a codimension two, spacelike compact surface S, the infinitesimal charge is

$$\delta H_\chi \equiv \oint_S \boldsymbol{k}^{BB}_\chi[\delta\Phi; \Phi] = \oint_S \Big(\delta^{[\Phi]}\boldsymbol{Q}_\chi - \chi \cdot \boldsymbol{\Theta}[\delta\Phi, \Phi] + \boldsymbol{E}[\delta\Phi, \delta_\chi\Phi, \Phi]\Big). \tag{A.29}$$

This formula is identical to the standard Barnich-Brandt formula, which is therefore valid even when χ has an implicit field dependence.

The Barnich-Brandt surface charge form can be alternatively defined as $\boldsymbol{k}^{BB}_\chi[\delta\Phi; \Phi] = I^{n-1}_{\delta\Phi}\boldsymbol{S}_\chi[\Phi]$ where $\boldsymbol{S}_\chi$ is the on-shell vanishing Noether current defined earlier. Here the formalism requires that the homotopy operator only acts on the explicit field dependence in $\boldsymbol{S}_\chi[\Phi]$ but not on the possible implicit field dependence in χ. Otherwise the commutation relations (A.23) would not be obeyed. (Also, if the operator $I^{n-1}_{\delta\Phi}$ acts anyways on the field-dependence in χ, the resulting terms will vanish on-shell by definition of $\boldsymbol{S}_\chi[\Phi]$.) One can then show that this definition is equivalent on-shell to $\boldsymbol{k}^{BB}_\chi[\delta\Phi;\Phi] = I^{n-1}_\chi \boldsymbol{\omega}^{inv}[\delta\Phi\,,\,\delta_\chi\Phi\,;\,\Phi]$ where the homotopy operator I^{n-1}_χ obeys $dI^{n-2}_\chi + I^{n-1}_\chi d = 1$ [83, 98]. For the purposes of this homotopy operator, χ is considered as a field by itself and the implicit field dependence in Φ is irrelevant. One always obtains the same expression (A.28).

A special feature of the cohomological formalism is that the presymplectic form is not identically closed in the sense that

$$\delta_1^{[\Phi]}\boldsymbol{\omega}^{inv}[\delta_2\Phi,\delta_3\Phi,\Phi] + (2,3,1) + (3,1,2) = d\big[\delta_1^{[\Phi]}\boldsymbol{E}[\delta_2\Phi,\delta_3\Phi,\Phi] + (2,3,1) + (3,1,2)\big] \quad \text{(A.30)}$$

is a boundary term, not zero. A prerequisite in order to have a well-defined charge algebra is that in the phase space

$$\oint \Big(\delta_1^{[\Phi]}\boldsymbol{E}[\delta_2\Phi,\delta_\chi\Phi,\Phi] + \delta_2^{[\Phi]}\boldsymbol{E}[\delta_\chi\Phi,\delta_1\Phi,\Phi] + \delta_\chi\boldsymbol{E}[\delta_1\Phi,\delta_2\Phi,\Phi]\Big) = 0 \quad \text{(A.31)}$$

This condition will be obeyed for the phase spaces considered here.

In $3d$ Einstein theory, the charges are given explicitly by

$$\begin{aligned}\mathbf{k}^{Einstein}_\chi = \frac{\sqrt{-g}}{8\pi G}(d^{n-2}x)_{\mu\nu}\Big\{&\chi^\nu\nabla^\mu h - \chi^\nu\nabla_\sigma h^{\mu\sigma} + \chi_\sigma\nabla^\nu h^{\mu\sigma} \\ &+ \frac{1}{2}h\nabla^\nu\chi^\mu - h^{\rho\nu}\nabla_\rho\chi^\mu + \frac{\alpha}{2}h^{\sigma\nu}(\nabla^\mu\chi_\sigma + \nabla_\sigma\chi^\mu)\Big\},\end{aligned} \quad \text{(A.32)}$$

where $\alpha = 0$ according to the definition of Iyer-Wald and $\alpha = +1$ according to the definition by Barnich-Brandt. Here $(d^{n-2}x)_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha}dx^\alpha$ in 3 dimensions. The last prescription also coincides with the one of Abbott-Deser [12]. In the case of Killing symmetries, there is no difference between the Iyer-Wald and Barnich-Brandt or Abbott-Deser charges. However, there is a potential difference for symplectic symmetries.

Equations (A.15) or (A.27) relates the charges computed on different surfaces. Consider the infinitesimal charges (A.17) or (A.29) evaluated on two different codimension two, spacelike compact surface $\mathcal{S}_1$ and $\mathcal{S}_2$. Denote a surface joining these two by Σ. Then taking the integral of (A.15) or (A.27) over Σ and using Stokes' theorem, one obtains

$$\delta H_\chi\Big|_{\mathcal{S}_2} - \delta H_\chi\Big|_{\mathcal{S}_1} = \int_\Sigma \boldsymbol{\omega}[\delta\Phi\,,\,\delta_\chi\Phi\,;\,\Phi]. \quad \text{(A.33)}$$

Killing symmetries ($\delta_\chi\Phi \approx 0$) or symplectic symmetries ($\boldsymbol{\omega}[\delta\Phi\,,\,\delta_\chi\Phi\,;\,\Phi] \approx 0$, $\delta_\chi\Phi \neq 0$) therefore lead to conserved charges.

A.2.2 Integrability of charges

In order the charge perturbation defined in (A.17) or (A.29) to be the variation of a finite charge $H_\chi[\Phi]$ defined over any field configuration Φ connected to the reference configuration $\bar{\Phi}$ in the phase space, it should satisfy integrability conditions. More precisely, integrability implies that the charges defined as $H_\chi = \int_{\bar{\Phi}}^{\Phi} \delta H_\chi$ along a path γ over the phase space does not depend upon γ. In the absence of topological obstructions in the phase space, it amounts to the following integrability conditions

$$I \equiv \delta_1 \delta_2 H_\chi - \delta_2 \delta_1 H_\chi = 0. \tag{A.34}$$

which can be conveniently written as

$$I \equiv \delta_1^{[\Phi]} \delta_2 H_\chi + \delta_2 H_{\delta_1 \chi} - (1 \leftrightarrow 2) = 0. \tag{A.35}$$

Using (A.29) in the first term we note that the Noether charge term drops by anti-symmetry in $(1 \leftrightarrow 2)$. We obtain

$$I = \oint \left(\delta_1^{[\Phi]} \boldsymbol{E}[\delta_2 \Phi, \delta_\chi \Phi; \Phi] - \chi \cdot \delta_1^{[\Phi]} \boldsymbol{\Theta}[\delta_2 \Phi; \Phi] \right) + \delta_2 H_{\delta_1 \chi} - (1 \leftrightarrow 2). \tag{A.36}$$

We can then use the cocyle condition (A.31) to obtain

$$\begin{aligned} I = & \oint \left(-\delta_\chi \boldsymbol{E}[\delta_1 \Phi, \delta_2 \Phi; \Phi] - \chi \cdot \delta_1^{[\Phi]} \boldsymbol{\Theta}[\delta_2 \Phi; \Phi] + \chi \cdot \delta_2^{[\Phi]} \boldsymbol{\Theta}[\delta_1 \Phi; \Phi] \right) \\ & + \delta_2 H_{\delta_1 \chi} - \delta_1 H_{\delta_2 \chi}. \end{aligned}$$

We can replace δ_χ by δ_χ^Φ or $\mathcal{L}_\chi$ in the first term. With the help of Cartan identity $\mathcal{L}_\chi = d\chi \cdot + \chi \cdot d$ and using the definition of the invariant presymplectic form (A.20) we finally obtain

$$I = -\oint \chi \cdot \boldsymbol{\omega}^{inv}[\delta_1 \Phi, \delta_2 \Phi; \Phi] - \left(\delta_1 H_{\delta_2 \chi} - \delta_2 H_{\delta_1 \chi} \right) = 0\,. \tag{A.37}$$

The term in parentheses arises due to the field dependence of vectors.

By dropping the $\boldsymbol{E}$ term, one obtains the integrability condition for field dependent vectors according to the definition of charges of Iyer-Wald. The result is simply

$$I = -\oint \chi \cdot \boldsymbol{\omega}^{LW}[\delta_1 \Phi, \delta_2 \Phi; \Phi] - \left(\delta_1 H_{\delta_2 \chi} - \delta_2 H_{\delta_1 \chi} \right) = 0\,. \tag{A.38}$$

A.3 3 dimensional asymptotically flat case

A.3.1 Residual symmetries

Gauge parameters ξ^μ that preserve the metric ansatz depend on two arbitrary functions $T(\phi), Y(\phi)$:

- $\mathcal{L}_\xi g_{rr} = 0$ implies $\partial_r \xi^u = 0$ and so $\xi^u = f(u,\phi)$,
- $\mathcal{L}_\xi g_{r\phi} = 0$ implies $\partial_r \xi^\phi = \dfrac{e^{2\beta}}{r^2}\partial_\phi f$ and so $\xi^\phi = Y(u,\phi) - \int_r^\infty dr' \dfrac{e^{2\beta}}{r'^2}\partial_\phi f$,
- $\mathcal{L}_\xi g_{\phi\phi} = 0$ implies $\xi^r = -r(\partial_\phi \xi^\phi - U\partial_\phi f)$,
- $\mathcal{L}_\xi g_{u\phi} = o(r)$ implies $\partial_u Y = 0$ and so $Y = Y(\phi)$,
- $\mathcal{L}_\xi g_{ur} = o(r^0)$ implies $\partial_u f = \partial_\phi Y$ and so $f = T(\phi) + u\partial_\phi Y$,
- $\mathcal{L}_\xi g_{uu} = o(r)$ implies no further conditions.

The gauge parameter ϵ preserving the gauge and fall-off conditions of the gauge potentials depends on an arbitrary function $E(u,\phi)$ according to

- $\mathcal{L}_\xi A_r + \partial_r \varepsilon = 0$ implies $\varepsilon = E(u,\phi) + \partial_\phi \xi^u \int_r^\infty \frac{e^{2\beta}A_\phi}{r'^2} dr'$,
- $\mathcal{L}_\xi A_u + \partial_u \varepsilon = O(\ln \frac{r}{r_0}) = \mathcal{L}_\xi A_\phi + \partial_\phi \varepsilon$ imply no further conditions.

A.3.2 Asymptotic symmetry algebra

We want to show that the gauge parameters (3.97), when equipped with the bracket (3.96), provide a representation of the Lie algebra (3.98). By evaluating $\mathcal{L}_\xi g_{\mu\nu}$, we find

$$\begin{cases} -\delta\beta = \xi^\alpha \partial_\alpha \beta + \frac{1}{2}\big[\partial_u f + \partial_r \xi^r + \partial_\phi f U\big], \\ -\delta U = \xi^\alpha \partial_\alpha U + U\big[\partial_u f + \partial_\phi f U - \partial_\phi \xi^\phi\big] - \partial_u \xi^\phi - \partial_r \xi^\phi V + \partial_\phi \xi^r \frac{e^{2\beta}}{r^2}, \end{cases} \tag{A.39}$$

while $-\delta A_\phi = \xi^\alpha \partial_\alpha A_\phi + A_\alpha \partial_\phi \xi^\alpha + \partial_\phi \epsilon$. It follows that

$$\begin{cases} \delta_1 \xi_2^u = 0, \\ \delta_1(\partial_r \xi_2^\phi) = \partial_\phi f_2 \frac{e^{2\beta}}{r^2} 2\delta_1\beta, \\ \delta_1 \xi_2^r = -r\big[\partial_\phi(\delta_1 \xi_2^\phi) - \partial_\phi f_2 \delta_1 U\big], \\ \delta_1(\partial_r \epsilon_2) = -\frac{1}{r^2}\left(\partial_\phi f_2 e^{2\beta}\delta_1 A_\phi + \partial_\phi f_2 e^{2\beta} A_\phi 2\delta_1\beta\right). \end{cases} \tag{A.40}$$

Direct computation then shows that

$$\begin{aligned} &\partial_r \hat{\xi}^u = \partial_r \hat{f} = 0, \qquad \partial_u \hat{f} = \partial_\phi \hat{Y}, \qquad \hat{f} = \hat{T} + u\partial_\phi \hat{Y}, \\ &\partial_r \hat{\xi}^\phi = \frac{e^{2\beta}\hat{f}}{r^2}, \qquad \lim_{r\to\infty} \hat{\xi}^\phi = \hat{Y}, \\ &\hat{\xi}^r = -\partial_\phi \hat{\xi}^\phi + U\partial_\phi \hat{f}, \\ &\partial_r \hat{\epsilon} = -\frac{\partial_\phi \hat{f} e^{2\beta} A_\phi}{r^2}, \qquad \lim_{r\to\infty} \hat{\epsilon} = \hat{E}, \end{aligned}$$

which proves the result since these conditions determine uniquely gauge parameters (3.97) where (T, Y, E) have been replaced by $(\hat{T}, \hat{Y}, \hat{E})$.

A.3.3 Solution space

The equations of motion can be organized as as follows

$$\partial_\nu(\sqrt{-g}F^{u\nu}) = 0, \quad (A.41)$$
$$\partial_\nu(\sqrt{-g}F^{\phi\nu}) = 0, \quad (A.42)$$
$$\partial_\nu(\sqrt{-g}F^{r\nu}) = 0, \quad (A.43)$$
$$L_{r\alpha} = G_{r\alpha} - T_{r\alpha} = 0, \quad (A.44)$$
$$L_{\phi\phi} = G_{\phi\phi} - T_{\phi\phi} = 0, \quad (A.45)$$
$$L_{u\phi} = G_{u\phi} - T_{u\phi} = 0, \quad (A.46)$$
$$L_{uu} = G_{uu} - T_{uu} = 0. \quad (A.47)$$

When equations (A.41) and (A.42) hold, the electromagnetic Bianchi equation reduces to $\partial_r[\partial_\nu(\sqrt{-g}F^{r\nu})] = 0$. This means that if $\partial_\nu(\sqrt{-g}F^{r\nu}) = 0$ for some constant r, it vanishes for all r. The gravitational Bianchi identities can be written as

$$0 = 2\sqrt{-g}\nabla_\nu G^\nu_\mu = 2\partial_\nu(\sqrt{-g}L^\nu_\mu) + \sqrt{-g}L_{\rho\sigma}\partial_\mu g^{\rho\sigma} + 2\sqrt{-g}\nabla_\nu T^\nu_\mu. \quad (A.48)$$

When (A.41)-(A.44) are satisfied and $\mu = r$ in (A.48), one gets $L_{\phi\phi}\partial_r g^{\phi\phi} = 0$ which implies $L_{\phi\phi} = 0$. In this case, the remaining Bianchi identities reduce to $2\partial_\nu(\sqrt{-g}L^\nu_\phi) = 0 = 2\partial_\nu(\sqrt{-g}L^\nu_u)$. The first one gives $\partial_r(rL_{u\phi}) = 0$. This means that if $rL_{u\phi} = 0$ for some fixed r, it vanishes everywhere. Finally, when $L_{u\phi} = 0$, the last Bianchi identity reads $\partial_r(rL_{uu}) = 0$. Thus the only non-vanishing term of rL_{uu} is the constant one.

Accordingly, the equations of motions are solved in the following order:

- 4 main equations: $L_{rr} = 0, \partial_\nu(\sqrt{-g}F^{u\nu}) = 0, L_{r\phi} = 0, L_{ru} = 0$,
- 1 standard equation: $\partial_\nu(\sqrt{-g}F^{\phi\nu}) = 0$,
- 3 supplementary equations: $\partial_\nu(\sqrt{-g}F^{r\nu}) = 0, L_{u\phi} = 0, L_{uu} = 0$,
- 1 trivial equation: $L_{\phi\phi} = 0$.

Starting with $L_{rr} = 0$, we have $g_{rr} = 0$, $R_{rr} = 2\frac{\partial_r\beta}{r}$, $T_{rr} = \frac{2}{r^2}(F_{r\phi})^2$. Hence $\partial_r\beta = \frac{1}{r}(F_{r\phi})^2$ and thus $\beta = \beta_0(u,\phi) - \int_r^\infty dr' \;\frac{1}{r'}(F_{r\phi})^2$ with β_0 an integration constant. The fall-off condition $\beta = o(r^0)$ puts β_0 to zero and thus,

$$\beta = -\int_r^\infty dr \;\frac{1}{r'}(F_{r\phi})^2. \quad (A.49)$$

Consider now the equation $\partial_\nu(\sqrt{-g}F^{u\nu}) = 0$. Explicitly, this equation reads $\partial_r(re^{2\beta}F^{ur}) + \partial_\phi(re^{2\beta}F^{u\phi}) = 0$. Defining

$$m := e^{2\beta}F^{ur} = -e^{-2\beta}(F_{ur} - UF_{r\phi}), \quad (A.50)$$

and using $e^{2\beta}F^{u\phi} = -\frac{1}{r^2}F_{r\phi}$, this equation of motion is a first order differential equation for m,

$$\partial_r(rm) = \frac{\partial_\phi F_{r\phi}}{r} \Longrightarrow m = \frac{-\lambda - \int_r^\infty dr' \quad \frac{\partial_\phi F_{r\phi}}{r'}}{r}, \tag{A.51}$$

with $\lambda(u,\phi)$ a constant of integration.

For $L_{r\phi} = 0$, we have $g_{r\phi} = 0$, $R_{r\phi} = -\partial_{\phi r}\beta + \frac{\partial_\phi \beta}{r} - r^2 e^{-2\beta}\partial_r\beta\partial_r U + \frac{3}{2}re^{-2\beta}\partial_r U + \frac{r^2}{2}e^{-2\beta}\partial_{rr}U$, $T_{r\phi} = 2F_{r\phi}m$. Defining

$$n := \frac{r^2}{2}e^{-2\beta}\partial_r U, \tag{A.52}$$

$R_{r\phi} = -\partial_{\phi r}\beta + \frac{\partial_\phi\beta}{r} + \left(\partial_r + \frac{1}{r}\right)n$, the equation is a first order differential equation for n,

$$\begin{aligned}\partial_r n + \frac{n}{r} &= 2F_{r\phi}m + \partial_{r\phi}\beta - \frac{\partial_\phi\beta}{r} \\ \Longrightarrow n &= \frac{N - 2\int_r^\infty dr' \quad r'(2F_{r\phi}m + \partial_{r\phi}\beta - \frac{\partial_\phi\beta}{r})}{2r},\end{aligned} \tag{A.53}$$

with $N(u,\phi)$ an integration constant. As a consequence of the fall-off condition on U, we end up with

$$U = -\int_r^\infty dr' \quad (\frac{2e^{2\beta}}{r'^2}n). \tag{A.54}$$

For $L_{ur} = 0$, we have $G_{ru} = -\frac{1}{2}g_{ru}(R_{\phi\phi}g^{\phi\phi} + 2R_{r\phi}g^{r\phi} + R_{rr}g^{rr})$, $R_{\phi\phi} = re^{-2\beta}(\partial_r V + 2\partial_\phi U) - 2\partial_{\phi\phi}\beta + r^2 e^{-2\beta}\partial_{\phi r}U - 2(\partial_\phi\beta)^2 - \frac{e^{-4\beta}}{2}r^4(\partial_r U)^2$ and $-2r^2(T_{ru}g^{ru} + T_{r\phi}g^{r\phi} + \frac{1}{2}T_{rr}g^{rr}) = 2r^2m^2$. This gives

$$\partial_r V = 2re^{2\beta}m^2 + \frac{2e^{2\beta}\partial_{\phi\phi}\beta}{r} - r\partial_{\phi r}U + \frac{2e^{2\beta}(\partial_\phi\beta)^2}{r} + \frac{1}{2}e^{-2\beta}r^3(\partial_r U)^2 - 2\partial_\phi U, \tag{A.55}$$

and

$$\begin{aligned}V = \theta - \int_r^\infty dr'\Big(2re^{2\beta}m^2 + \frac{2e^{2\beta}\partial_{\phi\phi}\beta}{r} - r\partial_{\phi r}U + \frac{2e^{2\beta}(\partial_\phi\beta)^2}{r} + \\ + \frac{1}{2}e^{-2\beta}r^3(\partial_r U)^2 - 2\partial_\phi U\Big),\end{aligned} \tag{A.56}$$

with $\theta(u,\phi)$ a constant of integration.

For $j \geq 0$, we have

$$\partial_r[r^i \ln^j r] = \begin{cases} \sum_{k=0}^{j} C_{ijk} r^{i-1} \ln^k r, & i \neq 0, \\ j r^{-1} \ln^{j-1} r, & i = 0, \end{cases} \tag{A.57}$$

$$\int r^i \ln^j r \; dr = \begin{cases} \sum_{k=0}^{j} D_{ijk} r^{i+1} \ln^k r, & i \neq -1, \\ \dfrac{\ln^{j+1} r}{j+1}, & i = -1, \end{cases} \tag{A.58}$$

for some coefficients C_{ijk} and D_{ijk}, and up to constants for the integrations. Consider then series $\mathcal{S}^n$ with elements of the form

$$\sum_{i \leq -n,\, 0 \leq j \leq -i-n} s_{ij}(u, \phi) r^i \ln^j r, \tag{A.59}$$

with $n \geq 0$. These series satisfy $\mathcal{S}^{n+1} \subset \mathcal{S}^n$, $\mathcal{S}^n * \mathcal{S}^m \subset \mathcal{S}^{n+m}$, $(\mathcal{S}^n)' \subset \mathcal{S}^{n+1}$. For integration however, $\int dr \; \mathcal{S}^{n+1} \subset \mathcal{S}^n$, up to constants for $n \neq 0$ and the divergent logarithmic term for $n = 0$.

The ansatz (3.102) belongs to $\mathcal{S}^0$, up to the divergent logarithmic term proportional to $\alpha(u, \phi)$. This implies $F_{r\phi} \in \mathcal{S}^1$ and β in (A.49), because of the absence of the constant, belongs to $\mathcal{S}^2$ with all coefficients determined by the coefficients $\alpha(u, \phi), A_{mn}(u, \phi)$ of (3.102).

In the same way, from (A.51), it follows that $m \in \mathcal{S}^1$ with all coefficients determined by those of (3.102) and the integration function λ.

For U, we have in a first stage that n belongs to $\mathcal{S}^0$ and is determined by the data in (3.102) and the integration constants λ, N. For U itself, it follows from (A.54), that it belongs to $\mathcal{S}^1$, with no new integration constant because of the assumed fall-off.

Finally, it follows from (A.56) that V belongs to $\mathcal{S}^0$, up to a logarithmic divergence, with coefficients determined by the data in (3.102) and the integration functions $\alpha, \lambda, N, \theta$.

In summary, by integrating m in r in order to get A_u and making the α dependence explicit, we find that all main equations are solved as

$$m = -\frac{\lambda}{r} - \frac{\alpha'}{r^2} + \sum_{m=1}^{\infty} \sum_{n=0}^{m} \frac{m_{mn} (\ln \frac{r}{r_0})^n}{r^{m+2}}, \tag{A.60}$$

$$\beta = -\frac{\alpha^2}{2r^2} + \sum_{m=1}^{\infty} \sum_{n=0}^{m} \frac{\beta_{mn} (\ln \frac{r}{r_0})^n}{r^{m+2}}, \tag{A.61}$$

$$U = \frac{4\lambda\alpha \ln \frac{r}{r_0} + 2\lambda\alpha - N}{2r^2} + \sum_{m=1}^{\infty} \sum_{n=0}^{m} \frac{U_{mn} (\ln \frac{r}{r_0})^n}{r^{m+2}}, \tag{A.62}$$

$$A_u = -\lambda \ln \frac{r}{r_0} + A_u^0 + \frac{\alpha'}{r} + \sum_{m=1}^{\infty}\sum_{n=0}^{m} \frac{B_{mn}(\ln \frac{r}{r_0})^n}{r^{m+1}}, \tag{A.63}$$

$$V = 2\lambda^2 \ln \frac{r}{r_0} + \theta + \frac{2\alpha\lambda' - 2\lambda\alpha'}{r}, \tag{A.64}$$

$$+ \sum_{m=1}^{\infty}\sum_{n=0}^{m} \frac{V_{mn}(\ln \frac{r}{r_0})^n}{r^{m+1}}, \tag{A.65}$$

where $m_{mn}, \beta_{mn}, U_{mn}, V_{mn}, B_{mn}$ are determined by $\alpha(u,\phi)$, $A_{mn}(u,\phi)$, the integration constants $\lambda(u,\phi), N(u,\phi)$ and their ϕ derivatives.

The standard equation determines the u evolution of α, A_ϕ^0 and $A_{mn}(u,\phi)$. Indeed, $\partial_\nu(\sqrt{-g}F^{\phi\nu}) = \partial_u(re^{2\beta}F^{\phi u}) + \partial_r(re^{2\beta}F^{\phi r}) = 0$. Since $e^{2\beta}F^{\phi r} = Um + \frac{1}{r^2}(F_{u\phi} + VF_{r\phi})$, $e^{2\beta}F^{\phi u} = \frac{1}{r^2}F_{r\phi}$, we get

$$\partial_u F_{r\phi} = -r^2\left(\partial_r + \frac{1}{r}\right)\left[Um + \frac{1}{r^2}(F_{u\phi} + VF_{r\phi})\right],$$

which is a differential equation governing the u-dependence of $F_{r\phi}$ and thus of A_ϕ. In terms of coefficients, we get

$$\dot{\alpha} = -\lambda', \quad \dot{A}_\phi^0 = -\lambda' + (A_u^0)', \quad \dot{A}_{mn+1} = \frac{(2m+1)\dot{A}_{mn} + X_{mn}}{2(n+1)}, \tag{A.66}$$

where $A_{mn+1} = 0$ when $n = m$ and X_{mn} is a linear combination of α, A_ϕ^0, A_{mn}, integration functions $\lambda, A_u^0, N, \theta$ and their ϕ derivative.

The first supplementary equation reads explicitly $0 = \partial_\nu(\sqrt{-g}F^{r\nu}) = \partial_u(re^{2\beta}F^{ru}) + \partial_\phi(re^{2\beta}F^{r\phi})$. Since $e^{2\beta}F^{ru} = -m = \frac{\lambda}{r} + O(r^{-2})$ and $e^{2\beta}F^{r\phi} = \left[Um - \frac{1}{r^2}\left(F_{u\phi} + \frac{V}{r}F_{r\phi}\right)\right] = O(r^{-2})$, $\lim_{r\to\infty}\partial_\nu(\sqrt{-g}F^{r\nu}) = 0$ implies $\dot{\lambda} = 0$ so that $\lambda = \lambda(\phi)$. The first of equation (A.66) then implies $\alpha = -\omega(\phi) - u\lambda'$.

For the second supplementary equation, $L_{u\phi} = 0$, we have $L_{u\phi} = \frac{1}{2r}(\theta' - \dot{N}) + O(r^{-2})$. Hence, $\lim_{r\to\infty}(rL_{u\phi}) = 0$ implies $\dot{N} = \theta'$.

For the last supplementary equation $L_{uu} = 0$, we have $L_{uu} = \frac{\dot{\theta}}{r} + O(r^{-2})$. $\lim_{r\to\infty}(rL_{uu}) = 0$ then implies $\partial_u\theta = 0$ and thus $\theta = \theta(\phi)$ and then also that $N = \chi(\phi) + u\theta'$.

A.4 3 dimensional asymptotically AdS case

A.4.1 Residual symmetries

Gauge parameters ξ^μ that preserve the metric ansatz depend on two arbitary functions $T(\phi), Y(\phi)$:

- $\mathcal{L}_\xi g_{rr} = 0$ implies $\partial_r \xi^u = 0$ and so $\xi^u = f(u,\phi)$,
- $\mathcal{L}_\xi g_{r\phi} = 0$ implies $\partial_r \xi^\phi = \dfrac{e^{2\beta}}{r^2}\partial_\phi f$ and so $\xi^\phi = Y(u,\phi) - \int_r^\infty dr' \dfrac{e^{2\beta}}{r'^2}\partial_\phi f$,
- $\mathcal{L}_\xi g_{\phi\phi} = 0$ implies $\xi^r = -r(\partial_\phi \xi^\phi - U\partial_\phi f)$,
- $\mathcal{L}_\xi g_{u\phi} = o(r)$ implies $\partial_u Y = \dfrac{1}{l^2}\partial_\phi f$,
- $\mathcal{L}_\xi g_{ur} = o(r^0)$ implies $\partial_u f = \partial_\phi Y$,
- $\mathcal{L}_\xi g_{uu} = o(r)$ implies no further conditions.

The gauge parameter ϵ preserving the gauge and fall-off conditions of the gauge potentials depends on an arbitrary function $E(u,\phi)$ according to

- $\mathcal{L}_\xi A_r + \partial_r \varepsilon = 0$ implies $\varepsilon = E(u,\phi) + \partial_\phi \xi^u \int_r^\infty \frac{e^{2\beta}A_\phi}{r'^2} dr'$,
- $\mathcal{L}_\xi A_u + \partial_u \varepsilon = O(\ln \frac{r}{l}) = \mathcal{L}_\xi A_\phi + \partial_\phi \varepsilon$ imply no further conditions.

A.4.2 Asymptotic symmetry algebra

We want to show that the gauge parameters (3.124), when equipped with the bracket (3.123), provide a representation of the Lie algebra (3.125). By evaluating $\mathcal{L}_\xi g_{\mu\nu}$, we find

$$\begin{cases} -\delta\beta = \xi^\alpha \partial_\alpha \beta + \frac{1}{2}\big[\partial_u f + \partial_r \xi^r + \partial_\phi f U\big], \\ -\delta U = \xi^\alpha \partial_\alpha U + U\big[\partial_u f + \partial_\phi f U - \partial_\phi \xi^\phi\big] - \partial_u \xi^\phi - \partial_r \xi^\phi V + \partial_\phi \xi^r \frac{e^{2\beta}}{r^2}, \end{cases} \tag{A.67}$$

while $-\delta A_\phi = \xi^\alpha \partial_\alpha A_\phi + A_\alpha \partial_\phi \xi^\alpha + \partial_\phi \epsilon$. It follows that

$$\begin{cases} \delta_1 \xi_2^u = 0, \\ \delta_1(\partial_r \xi_2^\phi) = \partial_\phi f_2 \frac{e^{2\beta}}{r^2} 2\delta_1 \beta, \\ \delta_1 \xi_2^r = -r\big[\partial_\phi(\delta_1 \xi_2^\phi) - \partial_\phi f_2 \delta_1 U\big], \\ \delta_1(\partial_r \epsilon_2) = -\frac{1}{r^2}\left(\partial_\phi f_2 e^{2\beta}\delta_1 A_\phi + \partial_\phi f_2 e^{2\beta} A_\phi 2\delta_1\beta\right). \end{cases} \tag{A.68}$$

Direct computation then shows that

$$\begin{aligned} &\partial_r \hat{\xi}^u = \partial_r \hat{f} = 0, \qquad \partial_u \hat{f} = \partial_\phi \hat{Y}, \qquad \partial_\phi \hat{f} = l^2 \partial_\phi \hat{Y}, \\ &\partial_r \hat{\xi}^\phi = \frac{e^{2\beta}\hat{f}}{r^2}, \qquad \lim_{r\to\infty} \hat{\xi}^\phi = \hat{Y}, \\ &\hat{\xi}^r = -\partial_\phi \hat{\xi}^\phi + U\partial_\phi \hat{f}, \\ &\partial_r \hat{\epsilon} = -\frac{\partial_\phi \hat{f} e^{2\beta} A_\phi}{r^2}, \qquad \lim_{r\to\infty} \hat{\epsilon} = \hat{E}, \end{aligned}$$

which proves the result since these conditions determine uniquely gauge parameters (3.124) where (T,Y,E) have been replaced by $(\hat{T},\hat{Y},\hat{E})$.

A.4.3 Solution space

The equations of motion can be organized as as follows

$$\partial_\nu(\sqrt{-g}F^{u\nu}) = 0, \tag{A.69}$$
$$\partial_\nu(\sqrt{-g}F^{\phi\nu}) = 0, \tag{A.70}$$
$$\partial_\nu(\sqrt{-g}F^{r\nu}) = 0, \tag{A.71}$$
$$L_{r\alpha} = G_{r\alpha} - \frac{g_{r\alpha}}{l^2} - T_{r\alpha} = 0, \tag{A.72}$$
$$L_{\phi\phi} = G_{\phi\phi} - \frac{g_{\phi\phi}}{l^2} - T_{\phi\phi} = 0, \tag{A.73}$$
$$L_{u\phi} = G_{u\phi} - \frac{g_{u\phi}}{l^2} - T_{u\phi} = 0, \tag{A.74}$$
$$L_{uu} = G_{uu} - \frac{g_{uu}}{l^2} - T_{uu} = 0. \tag{A.75}$$

When equations (A.69) and (A.70) hold, the electromagnetic Bianchi equation reduces to $\partial_r[\partial_\nu(\sqrt{-g}F^{r\nu})] = 0$. This means that if $\partial_\nu(\sqrt{-g}F^{r\nu}) = 0$ for some constant r, it vanishes for all r. The gravitational Bianchi identities can be written as

$$0 = 2\sqrt{-g}\nabla_\nu G^\nu_\mu = 2\partial_\nu(\sqrt{-g}L^\nu_\mu) + \sqrt{-g}L_{\rho\sigma}\partial_\mu g^{\rho\sigma} + 2\sqrt{-g}\nabla_\nu T^\nu_\mu. \tag{A.76}$$

When (A.69)-(A.72) are satisfied and $\mu = r$ in (A.76), one gets $L_{\phi\phi}\partial_r g^{\phi\phi} = 0$ which implies $L_{\phi\phi} = 0$. In this case, the remaining Bianchi identities reduce to $2\partial_\nu(\sqrt{-g}L^\nu_\phi) = 0 = 2\partial_\nu(\sqrt{-g}L^\nu_u)$. The first one gives $\partial_r(rL_{u\phi}) = 0$. This means that if $rL_{u\phi} = 0$ for some fixed r, it vanishes everywhere. Finally, when $L_{u\phi} = 0$, the last Bianchi identity reads $\partial_r(rL_{uu}) = 0$. Thus the only non-vanishing term of rL_{uu} is the constant one.

Accordingly, the equations of motions are solved in the following order:

- 4 main equations: $L_{rr} = 0, \partial_\nu(\sqrt{-g}F^{u\nu}) = 0, L_{r\phi} = 0, L_{ru} = 0$,
- 1 standard equation: $\partial_\nu(\sqrt{-g}F^{\phi\nu}) = 0$,
- 3 supplementary equations: $\partial_\nu(\sqrt{-g}F^{r\nu}) = 0, L_{u\phi} = 0, L_{uu} = 0$,
- 1 trivial equation: $L_{\phi\phi} = 0$.

Starting with $L_{rr} = 0$, we have $g_{rr} = 0$, $R_{rr} = 2\frac{\partial_r\beta}{r}$, $T_{rr} = \frac{2}{r^2}(F_{r\phi})^2$. Hence $\partial_r\beta = \frac{1}{r}(F_{r\phi})^2$ and thus $\beta = \beta_0(u,\phi) - \int_r^\infty dr' \ \frac{1}{r'}(F_{r\phi})^2$ with β_0 an integration constant. The fall-off condition $\beta = o(r^0)$ puts β_0 to zero and thus,

$$\beta = -\int_r^\infty dr \ \ \frac{1}{r'}(F_{r\phi})^2. \tag{A.77}$$

Consider now the equation $\partial_\nu(\sqrt{-g}F^{u\nu}) = 0$. Explicitly, this equation reads $\partial_r(re^{2\beta}F^{ur}) + \partial_\phi(re^{2\beta}F^{u\phi}) = 0$. Defining

$$m := e^{2\beta}F^{ur} = -e^{-2\beta}(F_{ur} - UF_{r\phi}), \tag{A.78}$$

and using $e^{2\beta}F^{u\phi} = -\frac{1}{r^2}F_{r\phi}$, this equation of motion is a first order differential equation for m,

$$\partial_r(rm) = \frac{\partial_\phi F_{r\phi}}{r} \Longrightarrow m = \frac{-\lambda - \int_r^\infty dr' \ \ \frac{\partial_\phi F_{r\phi}}{r'}}{r}, \tag{A.79}$$

with $\lambda(u,\phi)$ a constant of integration.

For $L_{r\phi} = 0$, we have $g_{r\phi} = 0$, $R_{r\phi} = -\partial_{\phi r}\beta + \frac{\partial_\phi\beta}{r} - r^2e^{-2\beta}\partial_r\beta\partial_r U + \frac{3}{2}re^{-2\beta}\partial_r U + \frac{r^2}{2}e^{-2\beta}\partial_{rr}U$, $T_{r\phi} = 2F_{r\phi}m$. Defining

$$n := \frac{r^2}{2}e^{-2\beta}\partial_r U, \tag{A.80}$$

$R_{r\phi} = -\partial_{\phi r}\beta + \frac{\partial_\phi\beta}{r} + \left(\partial_r + \frac{1}{r}\right)n$, the equation is a first order differential equation for n,

$$\partial_r n + \frac{n}{r} = 2F_{r\phi}m + \partial_{r\phi}\beta - \frac{\partial_\phi\beta}{r}$$
$$\Longrightarrow n = \frac{N - 2\int_r^\infty dr' \ \ r'(2F_{r\phi}m + \partial_{r\phi}\beta - \frac{\partial_\phi\beta}{r})}{2r}, \tag{A.81}$$

with $N(u,\phi)$ an integration constant. As a consequence of the fall-off condition on U, we end up with

$$U = -\int_r^\infty dr' \ \ (\frac{2e^{2\beta}}{r'^2}n). \tag{A.82}$$

For $L_{ur} = 0$, we have $G_{ru} = -\frac{1}{2}g_{ru}(R_{\phi\phi}g^{\phi\phi} + 2R_{r\phi}g^{r\phi} + R_{rr}g^{rr})$, $g_{ru} = -e^{2\beta}$, $R_{\phi\phi} = re^{-2\beta}(\partial_r V + 2\partial_\phi U) - 2\partial_{\phi\phi}\beta + r^2e^{-2\beta}\partial_{\phi r}U - 2(\partial_\phi\beta)^2 - \frac{e^{-4\beta}}{2}r^4(\partial_r U)^2$ and $-2r^2(T_{ru}g^{ru} + T_{r\phi}g^{r\phi} + \frac{1}{2}T_{rr}g^{rr}) = 2r^2m^2$. This gives

$$\partial_r V = -\frac{2re^{2\beta}}{l^2} + 2re^{2\beta}m^2 + \frac{2e^{2\beta}\partial_{\phi\phi}\beta}{r} - r\partial_{\phi r}U + \frac{2e^{2\beta}(\partial_\phi\beta)^2}{r} + \frac{1}{2}e^{-2\beta}r^3(\partial_r U)^2 - 2\partial_\phi U, \tag{A.83}$$

and

$$V = \theta - \int_r^\infty dr' \Big(-\frac{2re^{2\beta}}{l^2} + 2re^{2\beta}m^2 + \frac{2e^{2\beta}\partial_{\phi\phi}\beta}{r} - r\partial_{\phi r}U + \frac{2e^{2\beta}(\partial_\phi\beta)^2}{r} + \\ + \frac{1}{2}e^{-2\beta}r^3(\partial_r U)^2 - 2\partial_\phi U\Big), \tag{A.84}$$

with $\theta(u,\phi)$ a constant of integration.

In summary, with the ansatz

$$
\begin{aligned}
A_\phi &= \alpha(u,\phi)\ln\frac{r}{l} + A^0_\phi(u,\phi) + \frac{A_1(u,\phi)}{r} \\
&+ \sum_{m=1}^{\infty}\sum_{n=1}^{m}\left[\frac{\tilde{A}_{\phi m}(u,\phi)}{r^{2m}} + \frac{\bar{A}_{\phi m}(u,\phi)}{r^{2m+1}} + \frac{\tilde{A}_{mn}(\ln\frac{r}{l})^n}{r^{2m}} + \frac{\bar{A}_{mn}(\ln\frac{r}{l})^n}{r^{2m+1}}\right]
\end{aligned}
\tag{A.85}
$$

by integrating m in r in order to get A_u, we find that all main equations are solved as

$$
\begin{aligned}
&\beta = -\frac{\alpha^2}{2r^2} + \frac{2\alpha A_1}{3r^3} + \sum_{m=1}^{\infty}\sum_{n=0}^{m}\left[\frac{\tilde{\beta}_{mn}(\ln\frac{r}{l})^n}{r^{2m+2}} + \frac{\bar{\beta}_{mn}(\ln\frac{r}{l})^n}{r^{2m+3}}\right], \\
&m = -\frac{\lambda}{r} - \frac{\alpha'}{r^2} + \frac{A_1'}{2r^3} + \sum_{m=1}^{\infty}\sum_{n=0}^{m}\left[\frac{\tilde{m}_{mn}(\ln\frac{r}{l})^n}{r^{2m+2}} + \frac{\bar{m}_{mn}(\ln\frac{r}{l})^n}{r^{2m+3}}\right], \\
&U = \frac{4\lambda\alpha\ln\frac{r}{r_0} + 2\lambda\alpha - N}{2r^2} + \frac{4\lambda A_1 + 2\alpha\alpha'}{3r^3} \\
&+ \sum_{m=1}^{\infty}\sum_{n=0}^{m}\left[\frac{\tilde{U}_{mn}(\ln\frac{r}{l})^n}{r^{2m+2}} + \frac{\bar{U}_{mn}(\ln\frac{r}{l})^n}{r^{2m+3}}\right], \\
&A_u = -\lambda\ln\frac{r}{l} + A^0_u + \frac{\alpha'}{r} + \sum_{m=1}^{\infty}\sum_{n=0}^{m}\left[\frac{\tilde{B}_{mn}(\ln\frac{r}{l})^n}{r^{2m}} + \frac{\bar{B}_{mn}(\ln\frac{r}{l})^n}{r^{2m+1}}\right], \\
&V = -\frac{r^2}{l^2} + 2(\lambda^2 + \frac{\alpha^2}{l^2})\ln\frac{r}{l} + \theta + \frac{2\alpha\lambda' - 2\lambda\alpha'}{r} + \frac{8\alpha A_1}{3rl^2} - \frac{4\lambda^2\alpha^2(\ln\frac{r}{l})^2}{r^2} \\
&+ \sum_{m=1}^{\infty}\sum_{n=0}^{m}\left[\frac{\tilde{V}_{mn}(\ln\frac{r}{l})^n}{r^{2m}} + \frac{\bar{V}_{mn}(\ln\frac{r}{l})^n}{r^{2m+1}}\right],
\end{aligned}
\tag{A.86}
$$

where $\tilde{\beta}_{mn}, \bar{\beta}_{mn}, \tilde{U}_{mn}, \bar{U}_{mn}, \tilde{B}_{mn}, \bar{B}_{mn}, \tilde{V}_{mn}, \bar{V}_{mn}$ are determined by $\alpha(u,\phi)$, $A_1(u_0,\phi)$, $\tilde{A}_{\phi m}(u_0,\phi)$, $\bar{A}_{\phi m}(u_0,\phi)$, $\tilde{A}_{mn}, \bar{A}_{mn}$ the integration constants $\lambda(u,\phi), N(u,\phi)$ and their ϕ derivatives.

The standard equation determines $\tilde{A}_{mn}, \bar{A}_{mn}$ and the u evolution of α, A^0_ϕ $A_1(u_0,\phi)$, $\tilde{A}_{\phi m}$, $\bar{A}_{\phi m}$. Indeed, $\partial_\nu(\sqrt{-g}F^{\phi\nu}) = \partial_u(re^{2\beta}F^{\phi u}) + \partial_r(re^{2\beta}F^{\phi r}) = 0$. Since $e^{2\beta}F^{\phi r} = Um + \frac{1}{r^2}(F_{u\phi} + VF_{r\phi})$, $e^{2\beta}F^{\phi u} = \frac{1}{r^2}F_{r\phi}$, we get

$$
\frac{\partial_u A_\phi}{r^2} - \frac{2\partial_u\partial_r A_\phi}{r} = \partial_r\left[rUm + \frac{V}{r}\partial_r A_\phi - \frac{\partial_\phi A_u}{r}\right]. \tag{A.87}
$$

We can see from the RHS of (A.87) directly that $-\frac{2\lambda^2\alpha\ln\frac{r}{l}}{r^2}$ and the leading of $[-\frac{r}{l^2} + 2(\lambda^2 + \frac{\alpha^2}{l^2})\frac{\ln\frac{r}{l}}{r} - 4\lambda^2\alpha^2\frac{(\ln\frac{r}{l})^2}{r^3}]\partial_r A_\phi$ on every inverse order of r do not have counter terms on the LHS. Thus they must be canceled by themselves. We can solve out $\tilde{A}_{mm} = -\frac{\alpha(2\alpha^2)^m}{2m}$. Since $\tilde{A}_{mm}$ has been fixed, the u derivative of $\tilde{A}_{mm}$ from the LHS will fix $\bar{A}_{mm}$ terms by $\lambda, \lambda', \alpha, \alpha', A_1, A_1'$. Then the u derivative of $\bar{A}_{mm}$ will fix $\tilde{A}_{mm-1}$, and so on. Finally, all $\tilde{A}_{mn}, \bar{A}_{mn}$ terms will be fixed by initial data α, A^0_ϕ, $\tilde{A}_{\phi m}$, $\bar{A}_{\phi m}$, the integration constant λ, θ, N, the ϕ derivative of them and the u derivative of the integration constants which are not known at this point. But it will be fixed by the supplementary equations later. In the end, $\tilde{A}_{mn}, \bar{A}_{mn}$ terms will be fixed by initial data, integration constants and their ϕ derivative.

The u evolution of the initial data will be given by

$$\dot{\alpha} = -\lambda', \quad \dot{A}^0_\phi = -\lambda' + (A^0_u)' - \frac{A_1}{l^2}, \quad \dot{\tilde{A}}_{\phi m} = X_m, \quad \dot{\bar{A}}_{\phi m} = Y_m \tag{A.88}$$

where X_m and Y_m are linear combination of $\alpha(u, \phi), A^0_\phi, A_1, \tilde{A}_{\phi m}, \bar{A}_{\phi m}$, integration functions $\theta, \lambda, A^0_u, N$ and their ϕ derivative.

The first supplementary equation reads explicitly $0 = \partial_\nu(\sqrt{-g}F^{r\nu}) = \partial_u(re^{2\beta}F^{ru}) + \partial_\phi(re^{2\beta}F^{r\phi})$. Since $e^{2\beta}F^{ru} = -m = \frac{\lambda}{r} + O(r^{-2})$ and $e^{2\beta}F^{r\phi} = \left[Um - \frac{1}{r^2}\left(F_{u\phi} + \frac{V}{r}F_{r\phi}\right)\right] = \frac{\alpha}{l^2} + O(r^{-2})$, $\lim_{r\to\infty}\partial_\nu(\sqrt{-g}F^{r\nu}) = 0$ implies $\dot{\lambda} = -\frac{\alpha'}{l^2}$.

For the second supplementary equation, $L_{u\phi} = 0$, we have

$$L_{u\phi} = \frac{1}{r}\left[\frac{1}{2}\theta' - \frac{1}{2}\dot{N} + 2\lambda(A^0_u)' - 2\lambda\lambda' - 2\lambda\dot{A}^0_\phi + \frac{\alpha\alpha'}{l^2}\right] + O(r^{-2}). \tag{A.89}$$

Hence, $\lim_{r\to\infty}(rL_{u\phi}) = 0$ implies $\dot{N} = \theta' + 4\lambda(A^0_u)' - 4\lambda\lambda' - 4\lambda\dot{A}^0_\phi + \dfrac{2\alpha\alpha'}{l^2}$.

For the last supplementary equation $L_{uu} = 0$, we have

$$L_{uu} = \frac{1}{r}\left[-\frac{1}{2}\dot{\theta} + \frac{1}{2l^2}N' - \frac{2\alpha(A^0_u)'}{l^2} + \frac{2\alpha\dot{A}^0_\phi}{l^2} + \frac{\alpha\dot{\alpha}}{l^2}\right] + O(r^{-2}). \tag{A.90}$$

$\lim_{r\to\infty}(rL_{uu}) = 0$ then implies $\dot{\theta} = \dfrac{1}{l^2}N' - \dfrac{4\alpha(A^0_u)'}{l^2} + \dfrac{4\alpha\dot{A}^0_\phi}{l^2} + \dfrac{2\alpha\dot{\alpha}}{l^2}$.

A.5 Newman-Penrose charges of linear Maxwell theory

As proposed in [44], it is very suggesting to think of the charge controlling the leading soft photon theorem as a generalization of the usual electric charge, so as to make it dependent on the angle at the S^2 at infinity. In the late 60's Newman and Penrose [162] discovered new conserved charges for several theories possessing massless particles. These charges cannot be associated to bulk divergenceless vectors as usual; instead they are expressed as surface integrals at infinity. For the Maxwell theory, of course one of these Newman-Penrose charges is the electric charge.

It would seem reasonable to think that the charge in control of the sub-leading soft photon theorem should be an angle-dependent generalization of the other Newman-Penrose charges. It was hinted in [160] that one should consider generalizing the "sub-leading" charge in a multipole expansion of the electromagnetic field, namely dipole charge. In order to investigate this possibility, we will review the main results of the Maxwell theory part in [162] in this appendix.

First we collect some mathematical results. We define the $\eth, \bar{\eth}$ operators as

$$\eth\eta = \gamma_{z\bar{z}}^{-\frac{1}{2}}\,\partial_{\bar{z}}\eta + s\,\eta\,\partial_{\bar{z}}\gamma_{z\bar{z}}^{-\frac{1}{2}}\ , \qquad \bar{\eth}\eta = \gamma_{z\bar{z}}^{-\frac{1}{2}}\,\partial_{z}\eta - s\,\eta\,\partial_{z}\gamma_{z\bar{z}}^{-\frac{1}{2}}\ , \tag{A.91}$$

where s is the spin weight of the field η, meaning that it has the commutation relation[1] $[\eth, \bar{\eth}]\eta = -s\eta$. An important property of $\eth$ and $\bar{\eth}$ is their action on the spherical harmonics $Y_{l,m}$ ($l = 0, 1, 2, \dots\ ; m = -l, \dots, l$). Defining the spin s spherical harmonics as

$${}_sY_{l,m} = \begin{cases} \sqrt{\dfrac{(l-s)!}{2(l+s)!}}\eth^s Y_{l,m} & (0 \le s \le l) \\ (-1)^s\sqrt{\dfrac{(l+s)!}{2(l-s)!}}\bar{\eth}^{-s} Y_{l,m} & (-l \le s \le 0) \end{cases}\ , \tag{A.92}$$

the following relations can be deduced:

$$\begin{aligned} &\int \mathrm{d}z\mathrm{d}\bar{z}\,\gamma_{z\bar{z}}\,{}_sY_{l,m}\,\eth^{l-s+1}\eta = 0\ , && \int \mathrm{d}z\mathrm{d}\bar{z}\,\gamma_{z\bar{z}}\,{}_s\bar{Y}_{l,m}\,\bar{\eth}^{l-s+1}\zeta = 0\ , \\ &\bar{\eth}\eth{}_sY_{l,m} = -\frac{1}{2}(l-s)(l+s+1){}_sY_{l,m}\ , && \int \mathrm{d}z\mathrm{d}\bar{z}\,\gamma_{z\bar{z}}\,A\eth B = -\int \mathrm{d}z\mathrm{d}\bar{z}\,\gamma_{z\bar{z}}\,B\eth A\ . \end{aligned} \tag{A.93}$$

where η and ζ have spin weight $-l-1$ and $l+1$ respectively, while $A\eth B$ has no spin weight. These properties allow for a compact definition of conserved charges below.

Following [162], the Maxwell-tensors are replaced by three complex scalars:

$$\phi_0 = F_{r\bar{z}}\frac{\gamma_{z\bar{z}}^{-\frac{1}{2}}}{r}\ , \quad \phi_1 = \frac{1}{2}(F_{ru} + F_{z\bar{z}}\frac{\gamma_{z\bar{z}}^{-1}}{r^2})\ , \quad \phi_2 = \frac{\gamma_{z\bar{z}}^{-\frac{1}{2}}}{r}(F_{zu} - \frac{1}{2}F_{zr})\ , \tag{A.94}$$

With these quantities, the vacuum Maxwell equations can be organized in the Newman-Penrose formalism as

$$\begin{aligned} \partial_r(r^2\phi_1) &= r\bar{\eth}\phi_0 \qquad (\partial_u - \frac{1}{2}\partial_r - \frac{1}{2r})\phi_0 = \frac{\eth\phi_1}{r}\ , \\ \partial_r(r\phi_2) &= r\bar{\eth}\phi_1\ , \qquad (\partial_u - \frac{1}{2}\partial_r - \frac{1}{r})\phi_1 = \frac{\eth\phi_2}{r}\ . \end{aligned} \tag{A.95}$$

[1] There is no factor 2 compared to [162] since we are working on a unit sphere.

Assuming the ansatz (4.10), the solution is

$$\begin{aligned}
\phi_0 &= \sum_{n=0}^{\infty} \frac{\phi_0^n(u,z,\bar{z})}{r^{n+3}} \,, & \partial_u \phi_1^0 &= \eth \phi_2^0 \,, \\
\phi_1 &= \frac{\phi_1^0}{r^2} - \sum_{n=0}^{\infty} \frac{\bar{\eth}\phi_0^n(u,z,\bar{z})}{(n+1)r^{n+3}} \,, & \partial_u \phi_0^0 &= \eth \phi_1^0 \,, \\
\phi_2 &= \frac{\phi_2^0}{r} - \frac{\bar{\eth}\phi_1^0}{r^2} + \sum_{n=0}^{\infty} \frac{\bar{\eth}^2 \phi_0^n(u,z,\bar{z})}{(n+1)(n+2)r^{n+3}} \,, & \partial_u \phi_0^{n+1} &= -\frac{n+2}{2}\phi_0^n - \frac{\eth\bar{\eth}\phi_0^n}{n+1} \,.
\end{aligned} \tag{A.96}$$

Confirming the analysis that we did in Section 4.1, we see that, in the end, $\phi_2^0(u,z,\bar{z})$ is the news function of this system, associated to electromagnetic radiations. To be more specific, the concrete translation to the expressions in the main text is

$$\phi_0^n = -(n+1)\gamma_{z\bar{z}}^{-\frac{1}{2}} A_{\bar{z}}^{n+1} \,, \quad \phi_1^0 = -\frac{1}{2}[A_u^0 - \gamma_{z\bar{z}}^{-1}(\partial_z A_{\bar{z}}^0 - \partial_{\bar{z}} A_z^0)] \,, \quad \phi_2^0 = -\gamma_{z\bar{z}}^{-\frac{1}{2}} \partial_u A_z^0 \,, \tag{A.97}$$

where it is understood that n is a non-negative integer. Notice that ϕ_i^0 ($i = 0,1,2$) only involve the boundary fields A_u^0, $A_{z(\bar{z})}^0$ and $A_{z(\bar{z})}^1$. The physical meaning of ϕ_1^0 and ϕ_0^n was already identified in [161]. The real and imaginary parts of ϕ_1^0 are the electric and magnetic charges respectively. Similarly, real and imaginary parts of $\bar{\eth}\phi_0^n$ are the electric and magnetic multipoles. In particular, the dipole corresponds to $\bar{\eth}\phi_0^0$. Moreover, the spin weights of ϕ_2^0, ϕ_1^0 and ϕ_0^n are respectively $-1, 0, 1$. Then, using (A.96) and (A.93), we can construct conserved quantities

$$\partial_u \int \mathrm{d}z\mathrm{d}\bar{z}\, \gamma_{z\bar{z}}\, {}_0Y_{0,0}\, \phi_1^0 = 0 \,, \tag{A.98}$$

$$\partial_u \int \mathrm{d}z\mathrm{d}\bar{z}\, \gamma_{z\bar{z}}\, {}_0Y_{l,m}\, \bar{\eth}\phi_0^l = 0 \,. \quad (l = 0,1,2,\ldots) \,. \tag{A.99}$$

Notice that these conserved quantities all follow from the different orders of ϕ_1. When $l = 0$ we can see that $\int \mathrm{d}z\mathrm{d}\bar{z}\, {}_0Y_{0,0}\, \bar{\eth}\phi_0^0$ is automatically vanishing. This means that none of the Newman-Penrose charges correspond to dipole charge. Therefore, to us it does not seem possible to interpret (4.28) as a generalization of dipole charge.

As a final remark, let us on comment on the fact that the Newman-Penrose construction for Einstein gravity is very similar. But instead of the three boundary fields that we had in (A.94) (*i.e.* the "news" ϕ_2^0, plus ϕ_1^0, and ϕ_0^0), there we will have four boundary fields, namely the Bondi "news" and other three (Ψ_2^0, Ψ_1^0 and Ψ_0^0 in the standard Newman-Penrose notation) coming from the Weyl tensor. This clearly hints at the fact that in Einstein gravity there is an extra sub-sub-leading order in the soft theorem [34].

A.6 Near horizon solution space of Einstein-Maxwell theory

Under the Bondi gauge choice, the Newman-Penrose equations can be solved out recursively. Firstly, the solutions on the horizon can be worked out through hypersurface equations (2.38), Bianchi identities (2.45)-(2.47) and Maxwell equations (2.49)-(2.50). Apart from Ψ_0, ϕ_0 who have to be given at any order of r as initial data, the asymptotic r dependence of all the rest variables can be calculated by the radial equations (2.39), Bianchi identities (2.40)-(2.43) and Maxwell equations (2.51)-(2.52). Finally, (2.37) will determine Maxwell potential A_μ under the gauge condition we have chosen.

The full solutions are listed up to order $O(r^3)$ as following:

$$\begin{aligned}\gamma &= \gamma_0 + \gamma_1 r + \gamma_2 r^2 + O(r^3), \quad \gamma_0 \text{ is a real constant.}, \\ &\gamma_1 = \alpha_0\tau_0 + \beta_0\overline{\tau}_0 + \Psi_2^0 + \phi_1^0\overline{\phi}_1^0, \\ &\gamma_2 = \frac{1}{2}(\tau_0\alpha_1 + \tau_1\alpha_0 + \overline{\tau}_0\beta_1 + \overline{\tau}_1\beta_0 + \Psi_2^1 + \phi_1^0\overline{\phi}_1^0 + \phi_1^1\overline{\phi}_1^0),\end{aligned} \tag{A.100}$$

$$\begin{aligned}\tau &= \tau_0 + \tau_1 r + \tau_2 r^2 + O(r^3), \quad \tau_1 = \Psi_1^0 + \phi_0^0\overline{\phi}_1^0, \\ &\tau_2 = \frac{1}{2}(\tau_0\rho_1 + \tau_1\rho_0 + \overline{\tau}_0\sigma_1 + \overline{\tau}_1\sigma_0\Psi_1^1 + \phi_0^1\overline{\phi}_0^0\phi_0^0\overline{\phi}_0^1),\end{aligned} \tag{A.101}$$

$$\begin{aligned}\rho &= \rho_0 + \rho_1 r + \rho_2 r^2 + O(r^3), \\ &\rho_0 = \frac{1}{4\gamma_0}[\Psi_2^0 + \overline{\Psi}_2^0 - \eth\overline{\tau}_0 - \bar{\eth}\tau_0 + 2\tau_0\overline{\tau}_0], \quad \rho_1 = \phi_0^0\overline{\phi}_0^0, \\ &\rho_2 = \frac{1}{2}(\sigma_0\overline{\sigma}_1 + \sigma_1\overline{\sigma}_0 + 2\rho_0\rho_1 + \phi_0^1\overline{\phi}_0^0 + \phi_0^0\overline{\phi}_0^1),\end{aligned} \tag{A.102}$$

$$\sigma = \sigma_0 + \sigma_1 r + \sigma_2 r^2 + O(r^3), \quad \sigma_0 = \frac{1}{2\gamma_0}[\tau_0^2 - \eth\tau_0], \quad \sigma_1 = \Psi_0^0, \tag{A.103}$$

$$\sigma_2 = \frac{1}{2}(2\sigma_0\rho_1 + 2\sigma_1\rho_0 + \Psi_0^1),$$

$$\alpha = \alpha_0 + \alpha_1 r + \alpha_2 r^2 + O(r^3), \quad \alpha_0 = \frac{1}{2}(\overline{\tau}_0 + \overline{\delta}\ln P), \quad \alpha_1 = \phi_1^0\overline{\phi}_0^0, \tag{A.104}$$

$$\alpha_2 = \frac{1}{2}(\rho_0\alpha_1 + \rho_1\alpha_0 + \overline{\sigma}_0\beta_1 + \overline{\sigma}_1\beta_0 + \phi_1^1\overline{\phi}_0^0 + \phi_1^0\overline{\phi}_0^1),$$

$$\beta = \beta_0 + \beta_1 r + \beta_2 r^2 + O(r^3), \quad \beta_0 = \frac{1}{2}(\tau_0 - \delta\ln\bar{P}), \quad \beta_1 = \Psi_1^0, \tag{A.105}$$

$$\beta_2 = \frac{1}{2}(\alpha_0\sigma_1 + \alpha_1\sigma_0 + \Psi_1^1),$$

$$\mu = \Psi_2^0 r + \frac{1}{2}\Psi_2^1 r^2 + O(r^3), \tag{A.106}$$

$$\lambda = \frac{1}{2}(\mu_1\overline{\sigma}_0 + \phi_2^1\overline{\phi}_0^0)r^2 + O(r^3), \tag{A.107}$$

$$\nu = \frac{1}{2}(\mu_1\overline{\tau}_0 + \Psi_3^1 + \phi_2^1\overline{\phi}_1^0)r^2 + O(r^3), \tag{A.108}$$

$$\phi_0 = \phi_0^0 + \phi_0^1 r + O(r^2), \tag{A.109}$$

$$\phi_1 = \phi_1^0 + \phi_1^1 r + \phi_1^2 r^2 + O(r^3), \quad \phi_1^1 = \overline{\eth}\phi_0^0 - \tau_0\phi_0^0, \tag{A.110}$$
$$\phi_1^2 = \frac{1}{2}(\overline{\eth}\phi_0^1 - \overline{\tau}_0\phi_0^1 - 2\alpha_1\phi_0^0 + 2\rho_1\phi_1^0),$$

$$\phi_2 = \overline{\eth}\phi_2^0 r + \frac{1}{2}\overline{\eth}\phi_1^1 r^2 + O(r^3), \tag{A.111}$$

$$\Psi_0 = \Psi_0^0 + \Psi_0^1 + O(r^2), \tag{A.112}$$

$$\Psi_1 = \Psi_1^0 + \Psi_1^1 r + \Psi_1^2 r^2 + O(r^3), \tag{A.113}$$
$$\Psi_1^0 = \overline{\eth}\sigma_0 - \eth\rho_0 - \sigma_0\overline{\tau}_0 + \rho_0\tau_0 + \phi_0^0\overline{\phi}_1^0,$$
$$\Psi_1^1 = \overline{\eth}\sigma_1 - \eth\rho_1 - \sigma_1\overline{\tau}_0 + \rho_1\tau_0 - \sigma_0\overline{\tau}_1 + \rho_0\tau_1 + \phi_0^1\overline{\phi}_1^0 + \phi_0^0\overline{\phi}_1^1,$$
$$\Psi_1^2 = \overline{\eth}\sigma_2 - \eth\rho_2 + \rho_0\tau_2 + \rho_2\tau_0 + \rho_1\tau_1 - 2(\alpha_2 - \overline{\beta}_2)\sigma_0 - 2(\alpha_1 - \overline{\beta}_1)\sigma_1,$$
$$= -\overline{\tau}_2\sigma_0 - \overline{\tau}_1\sigma_1 - \overline{\tau}_0\sigma_2 + \phi_0^0\overline{\phi}_1^2 + \phi_0^1\overline{\phi}_1^1 + \phi_0^2\overline{\phi}_1^0,$$

$$\Psi_2 = \Psi_2^0 + \Psi_2^1 r + \Psi_2^2 r^2 + O(r^3), \tag{A.114}$$
$$\Psi_2^0 = \phi_1^0\overline{\phi}_1^0 - \frac{R}{4} + \frac{1}{2}(\overline{\eth}\tau_0 - \eth\overline{\tau}_0),$$
$$\Psi_2^1 = \phi_1^1\overline{\phi}_1^0 + \phi_1^0\overline{\phi}_1^1 + \overline{\eth}\beta_1 - \eth\alpha_1 + \alpha_0\overline{\alpha}_1 + \beta_0\overline{\beta}_1 - \alpha_0\beta_1 - \alpha_1\beta_0,$$
$$\Psi_2^2 = \overline{\eth}\beta_2 - \eth\alpha_2 + \rho_1\mu_1 + \rho_0\mu_2 - \sigma_0\lambda_2 + \alpha_0\overline{\alpha}_2 + \alpha_1\overline{\alpha}_1 + \beta_0\overline{\beta}_2 + \beta_1\overline{\beta}_1,$$
$$= -\alpha_0\overline{\beta}_2 - \alpha_2\beta_0 - 2\alpha_1\beta_1 + \phi_1^0\overline{\phi}_1^2 + \phi_1^1\overline{\phi}_1^1 + \phi_1^2\overline{\phi}_1^0,$$

$$\Psi_3 = \Psi_3^1 r + \Psi_3^2 r^2 + O(r^3), \quad \Psi_3^1 = \overline{\eth}\mu_1 + \mu_1\overline{\tau}_0 + \phi_2^1\overline{\phi}_1^0, \tag{A.115}$$
$$\Psi_3^2 = \overline{\eth}\mu_2 - \eth\lambda_2 + \mu_1\overline{\tau}_1 + \mu_2\overline{\tau}_0 - \tau_0\lambda_2 + \phi_2^1\overline{\phi}_1^1 + \phi_2^2\overline{\phi}_1^0,$$

$$\Psi_4 = \Psi_4^2 r^2 + O(r^3), \tag{A.116}$$
$$\Psi_4^2 = \overline{\eth}\nu_2 - \frac{1}{2}\phi_2^1(\overline{\eth\phi}_1^0 - 2\overline{\tau}_0\overline{\phi}_1^0 + 2\gamma_0\overline{\phi}_0^0) - 2\gamma_0\lambda_2 + \overline{\tau}_0\nu_2,$$

$$X^z = \tau_0\bar{P}r + \frac{1}{2}(\tau_1\bar{P} + \overline{\tau}_0\sigma_0\bar{P})r^2 + O(r^3), \tag{A.117}$$

$$X^{\bar{z}} = \overline{\tau}_0 P r + \frac{1}{2}(\overline{\tau}_1 P + \tau_0\overline{\sigma}_0 P)r^2 + O(r^3), \tag{A.118}$$

$$\omega = -\tau_0 r + \frac{1}{2}(\sigma_0\overline{\omega}_0 - \tau_1)r^2 + O(r^3), \tag{A.119}$$

$$U = -2\gamma_0 r + \frac{1}{2}(\tau_0\overline{\omega}_1 + \overline{\tau}_0\omega_1 - \gamma_1 - \overline{\gamma}_1)r^2 + O(r^3), \tag{A.120}$$

$$L^z = \sigma_0\bar{P}r + \frac{1}{2}\sigma_1\bar{P}r^2 + O(r^3), \quad L^{\bar{z}} = P + \frac{1}{2}\sigma_0\overline{\sigma}_0 P r^2 + O(r^3), \tag{A.121}$$

$$\overline{L}^z = \bar{P} + \frac{1}{2}\overline{\sigma}_0\sigma_0\bar{P}r^2 + O(r^3), \quad \overline{L}^{\bar{z}} = \overline{\sigma}_0 P r + \frac{1}{2}\overline{\sigma}_1 P r^2 + O(r^3), \tag{A.122}$$

$$L_z = -\frac{1}{\bar{P}} - \frac{\sigma_0\overline{\sigma}_0}{2\bar{P}}r^2 + O(r^3), \quad L_{\bar{z}} = \frac{\sigma_0}{P}r + \frac{\sigma_1}{2P}r^2 + O(r^3), \tag{A.123}$$

$$\overline{L}_z = \frac{\overline{\sigma}_0}{\bar{P}}r + \frac{\overline{\sigma}_1}{2\bar{P}}r^2 + O(r^3), \quad \overline{L}_{\bar{z}} = -\frac{1}{P} - \frac{\sigma_0\overline{\sigma}_0}{2P}r^2 + O(r^3), \tag{A.124}$$

$$A_u = (\phi_1^0 + \overline{\phi}_1^0)r + (\phi_1^1 + \overline{\phi}_1^1 + \tau_0\phi_0^0 + \overline{\tau}_0\overline{\phi}_0^0)r^2 + O(r^3), \tag{A.125}$$

$$A_z = A_z^0 + \frac{\overline{\phi}_0^0}{\bar{P}} r + \frac{1}{2}(\frac{\overline{\phi}_0^1}{\bar{P}} - \frac{\phi_0^0 \sigma_0}{\bar{P}}) r^2 + O(r^3), \;\; \partial_{\bar{z}} A_z^0 - \partial_z A_{\bar{z}}^0 = \frac{\overline{\phi}_1^0 - \phi_1^0}{P\bar{P}},$$

where $\tau_0, P, \phi_1^0, A_z^0$ are arbitrary functions depending on $(z, \bar{z})$ only, and $R = 2(\delta\bar{\delta}\ln P + \bar{\delta}\delta\ln\bar{P} - 2\bar{\delta}\ln P\delta\ln\bar{P}) = 2\bar{P}P\partial_z\partial_{\bar{z}}\ln\bar{P}P$ is the scalar curvature of S^2. τ_0 is the extrinsic curvature of the S^2 on the horizon. The $\eth$ operator is defined by $\eth\eta = \delta\eta + s\delta\ln\bar{P}\eta$ and $\bar{\eth}\eta = \bar{\delta}\eta - s\bar{\delta}\ln P\eta$ where s is the spin weight of the field η. The time evolution of Ψ_0 and ϕ_0 are fully controlled by (321.e) and (332).

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