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# Foliation of the Kottler–Schwarzschild–de Sitter spacetime by flat spacelike hypersurfaces

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**Abstract** There exist Kruskal like coordinates for the Reissner–Nordstrom (RN) black hole spacetime which are regular at coordinate singularities. Non-existence of such coordinates for the extreme RN black hole spacetime has already been shown. Also the Carter coordinates available for the extreme case are not manifestly regular at the coordinate singularity, therefore, a numerical procedure was developed to obtain free fall geodesics and flat foliation for the extreme RN black hole spacetime. The Kottler–Schwarzschild–de Sitter (KSSdS) spacetime geometry is similar to the RN geometry in the sense that, like the RN case, there exist non-singular coordinates when there are two distinct coordinate singularities. There are no manifestly regular coordinates for the extreme KSSdS case. In this paper foliation of all the cases of the KSSdS spacetime by flat spacelike hypersurfaces is obtained by introducing a non-singular time coordinate.

**Keywords** Black hole, Foliation, Coordinate singularity

## 1 Introduction

In General Relativity one often needs slicing of a spacetime by a sequences of hypersurfaces which is a foliation. A lot of work has been done on foliation by hypersurfaces of zero mean extrinsic curvature called *maximal slicing* [1; 2; 3; 4; 5; 6; 7] and by hypersurfaces of constant mean extrinsic curvature or *CMC-slicing* [8; 9; 10; 11; 12; 13; 14; 15; 16]. There has been a significant work to obtain foliation by hypersurfaces of zero intrinsic curvature called *flat foliation* [17; 18; 19; 20; 21] as well. It is known that spherically symmetric static spacetimes admit flat foliations [19; 20; 21] and their uniqueness is also known [22]. Qadir et al. have obtained foliations of the Schwarzschild and RN spacetimes by

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flat spacelike hypersurfaces [23]. As the analogue of the Kruskal coordinates does not exist for the extreme RN spacetime and Carter's coordinates available for this geometry are not manifestly regular at the coordinate singularity, a numerical procedure is developed to use Carter's coordinates to construct free-fall geodesics and a complete flat foliation of the extreme RN spacetime [24].

Here we present foliation of the KSSdS spacetime by flat spacelike hypersurfaces (the KSSdS cosmologies and their CMC-slicing has been discussed and presented in detail in [16]). Instead of following the procedures similar to those for the RN and extreme RN spacetimes [23; 24], we have introduced a non-singular time coordinate to get rid of the coordinate singularities. This removes the coordinate singularities from the equations giving flat foliating hypersurfaces and enables us to obtain foliations of all the cases of the KSSdS spacetime in a much simpler way. In the following section we review the earlier work to obtain the differential equation satisfied by flat spherically symmetric hypersurfaces. In Sect. 3, foliation of the KSSdS spacetime by flat spacelike hypersurfaces is presented and in the last section conclusion on our work and some comments on foliation of the Schwarzschild-anti-de Sitter spacetime by flat spacelike hypersurfaces are given.

## 2 Flat hypersurfaces admitted by spherically symmetric static spacetimes

Consider the following spherically symmetric static spacetime metric

$$ds^2 = -e^{v(r)} dt^2 + e^{\lambda(r)} dr^2 + r^2 d\Omega^2, \quad (1)$$

where

$$d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2. \quad (2)$$

Using spherical symmetry to take  $\theta$  and  $\phi$  constant, an arbitrary hypersurface in explicit form can be given as

$$t = F(r). \quad (3)$$

The induced 3-metric (of the hypersurfaces) is then

$$ds_3^2 = (e^{\lambda(r)} - e^{v(r)} F'^2) dr^2 + r^2 d\Omega^2. \quad (4)$$

For the induced metric to be flat a *necessary* but *not sufficient* condition, namely the Ricci scalar,  $R = 0$ , implies

$$\frac{r(-\lambda' e^\lambda + v' e^v F'^2 + 2e^v F' F'')}{(e^\lambda - e^v F'^2)^2} + \frac{1 - e^\lambda + e^v F'^2}{e^\lambda - e^v F'^2} = 0, \quad (5)$$

where  $'$  represents the derivative with respect to  $r$ . Using the substitution

$$g^2(r) = \frac{1}{e^\lambda - e^v F'^2}, \quad (6)$$

Eq. (5) becomes

$$2r g g' + g^2 - 1 = 0, \quad (7)$$

and we have the general solution

$$g^2(r) = 1 - \frac{k}{r}, \quad (8)$$

where  $k$  is an arbitrary constant with dimensions of length. The induced metric now takes the form

$$ds_3^2 = \frac{dr^2}{1 - \frac{k}{r}} + r^2 d\Omega^2. \quad (9)$$

The above metric, Eq. (9), of the hypersurfaces is flat, i.e. all the components of the Riemann curvature tensor are zero (which is the necessary and sufficient condition for the hypersurfaces to be flat), only if  $k = 0$  or in other words only if  $g^2(r) = 1$ . Then, from Eqs. (3) and (6), the flat spherically symmetric hypersurfaces are uniquely given as [22]

$$t = F(r) = \int e^{\frac{\lambda-v}{2}} \sqrt{1 - e^{-\lambda}} dr. \quad (10)$$

The mean extrinsic curvature,  $K$ , of these hypersurfaces is

$$K = e^{\left(\frac{v+\lambda}{2}\right)} \left( \frac{v'e^v}{2\sqrt{1-e^v}} - \frac{2\sqrt{1-e^v}}{r} \right), \quad (11)$$

and the Hamiltonian constraint gives

$$R + K^2 - K_{ab}K^{ab} = \frac{2(K^2 - e^v)}{r^2} - \frac{2v'e^v}{r}, \quad (12)$$

(here for flat hypersurfaces  $R = 0$ ).

### 3 Foliation of the KSSdS spacetime by flat spacelike hypersurfaces

The KSSdS metric in gravitational units ( $c = G = 1$ ) is given by [16]

$$ds^2 = -V(r)dt^2 + V^{-1}(r)dr^2 + r^2 (d\theta^2 + \sin^2\theta d\phi^2), \quad (13)$$

where

$$V(r) = 1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}, \quad (14)$$

and the cosmological constant  $\Lambda$  and  $m$  are positive. In the limit where  $\Lambda$  goes to zero, the spacetime metric tends to the Schwarzschild metric and in the limit where  $m$  goes to zero, the metric becomes de Sitter.

There are three possible cases depending on the value of  $C$ , where  $C = 9m^2\Lambda$ .

*Case 1* If  $C < 1$ , we call it usual black hole. In this case we have two horizons, namely the *black hole horizon*,  $r_b$ , and the *cosmological horizon*,  $r_c$ , which satisfy

$$2m < r_b < 3m < \frac{1}{\sqrt{\Lambda}} < r_c < \frac{3}{\sqrt{\Lambda}}. \quad (15)$$

The function  $V(r)$  is zero at these horizons and is positive in the interval  $(r_b, r_c)$ . The spacetime can be covered by two coordinate patches: one valid in the region  $0 < r < r_c$  and the other in the region  $r_b < r < \infty$ .

*Case 2* If  $C = 1$ , the *black hole horizon* and the *cosmological horizon* coincide at  $3m$  and we have an extreme black hole with maximal mass  $m = \frac{1}{3\sqrt{\Lambda}}$  and maximal size  $r_b = r_c = 3m$ . In this case no non-singular coordinates are available that remove both singularities simultaneously.

*Case 3* If  $C > 1$ , there are no horizons and we have a naked singularity case.

Now substituting  $e^{v(r)} = e^{-\lambda(r)} = V(r)$  in Eq. (10), where  $V(r)$  is given by Eq. (14), we obtain the equation satisfied by flat spacelike hypersurfaces for the KSSdS spacetime as

$$t = \int \frac{\sqrt{\frac{2m}{r} + \frac{\Lambda r^2}{3}}}{\frac{2m}{r} + \frac{\Lambda r^2}{3} - 1} dr + t_0, \quad (16)$$

where  $t_0$  is the constant of integration and its different values correspond to different flat hypersurfaces in  $(t, r)$  coordinates. Solving Eq. (16) gives the required flat foliating spacelike hypersurfaces. It is not difficult to obtain numerical solution for Case 3, as there are no coordinate singularities. One can also try to obtain results in Case 1 (like the usual RN case [23]) by solving Eq. (16) numerically separately in two coordinate patches and matching the solution at a point between  $r_b$  and  $r_c$ . For the extremal case one could try to follow the procedure adopted for the extreme RN case [24]. However, instead of following these procedures to get rid of the coordinate singularities, we first write Eq. (16) (by adding and subtracting 1 in the numerator of the integral and simplifying) as

$$t = \int \frac{dr}{\sqrt{\frac{2m}{r} + \frac{\Lambda r^2}{3}} + 1} + \int \frac{dr}{\frac{2m}{r} + \frac{\Lambda r^2}{3} - 1} + t_0, \quad (17)$$

or

$$t = \int \frac{dr}{\sqrt{\frac{2m}{r} + \frac{\Lambda r^2}{3}} + 1} - \int \frac{dr}{V(r)} + t_0. \quad (18)$$

This motivates to introduce a non-singular time coordinate,  $T$ , given by

$$dT = dt + \int \frac{dr}{V(r)}. \quad (19)$$

**Fig. 1** Flat foliating hypersurfaces in  $(T, r)$  coordinates for  $C < 1$  (the usual black hole case). The hypersurfaces labeled as 1, 2 and 3 correspond to the values of the foliating parameter,  $T_0 = -1, 0$  and 1 respectively.  $\Lambda = m = 3/7$  for all hypersurfaces

**Fig. 2** Flat foliating hypersurfaces in  $(T, r)$  coordinates for  $C = 1$  (the extreme black hole case). The hypersurfaces labeled as 1, 2 and 3 correspond to the values of the foliating parameter,  $T_0 = -1, 0$  and 1 respectively.  $\Lambda = 1$  and  $m = 1/3$  for all hypersurfaces

**Fig. 3** Flat foliating hypersurfaces in  $(T, r)$  coordinates for  $C > 1$  (the naked singularity case). The hypersurfaces labeled as 1, 2 and 3 correspond to the values of the foliating parameter,  $T_0 = -1, 0$  and 1 respectively.  $\Lambda = 3$  and  $m = 1/2$  for all hypersurfaces

**Fig. 4** Flat foliating hypersurfaces in  $(t, r)$  coordinates for  $C > 1$  (the naked singularity case). The hypersurfaces labeled as 1, 2 and 3 correspond to the values of the foliating parameter,  $t_0 = -1, 0$  and 1 respectively.  $\Lambda = 3$  and  $m = 1/2$  for all hypersurfaces. Notice that the behaviour of the corresponding hypersurfaces in  $(T, r)$  coordinates in Fig. 3 is essentially the same. This shows that the results obtained in  $(T, r)$  coordinates are not artifact of the coordinate transformation

Now using Eq. (19) in Eq. (18) to obtain the expression for the flat hypersurfaces in  $(T, r)$  coordinates as

$$T = \int \frac{dr}{1 + \sqrt{\frac{2m}{r} + \frac{\Lambda r^2}{3}}} + T_0, \quad (20)$$

where  $T_0$  is the constant of integration and its different values give different flat hypersurfaces. The numerical solution of Eq. (20) for  $C$  greater than, equal to and less than one are obtained and displayed in Figs. 1, 2 and 3 respectively. In order to see that the results are not artifact of the coordinate transformation, the numerical solutions of Eq. (16) are also obtained in Case 3 (when there are no coordinate singularities) and displayed in Fig. 4. The mean extrinsic curvature,  $K$ , of these hypersurfaces is

$$K = \frac{\frac{m}{r^3 + \frac{2\Lambda}{3}}}{\sqrt{\frac{2m}{r^3 + \frac{\Lambda}{3}}}}. \quad (21)$$

## 4 Conclusion

Foliation of the RN spacetime was obtained by Qadir et al. [23]. They introduced a numerical procedure to deal with the extreme RN case, but the method was very sensitive near the coordinate singularity and resulted in the form of kinks in the graphs [24]. Like the RN spacetime, there are also three cases of the KSSdS spacetime (namely, the usual, extreme and naked singularity). In this paper foliation of the KSSdS spacetime by flat spacelike hypersurfaces is obtained by introducing a

coordinate transformation that removes the singularity from the equation of the flat hypersurfaces in all the cases. In case of the naked singularity we have obtained flat hypersurfaces both in the original  $(t, r)$  coordinates and in the transformed  $(T, r)$  coordinates. The results show that our procedure works well and the hypersurfaces obtained in all the cases are not artifact of the transformation. It will be interesting to apply our procedure to the RN spacetime and compare the results with the earlier results. It is expected that our procedure will remove the kinks appearing in the graphs obtained earlier [24].

The Schwarzschild-anti-de Sitter spacetime has the same form as given by Eqs. (13) and (14) for the KSSdS spacetime, but the cosmological constant,  $\Lambda$ , now takes negative values [25]. Replacing  $\Lambda$  by  $-\Lambda$  and following the same procedure as discussed in Sect. 3 for the KSSdS spacetime, we obtain the following expression for the flat hypersurfaces admitted by the Schwarzschild-anti-de Sitter spacetime

$$T = \int \frac{dr}{1 + \sqrt{\frac{2m}{r} - \frac{\Lambda r^2}{3}}} + T_0, \quad (\Lambda > 0). \quad (22)$$

Notice that the term inside the square root becomes negative for  $r > \left(\frac{6m}{\Lambda}\right)^{1/3}$ , and restricts the solution in that region. Therefore, a direct application of our procedure does not work for the Schwarzschild-anti-de Sitter spacetime. However, it will be interesting to try to construct some other non-singular coordinates in this case and also explore the significance of the barrier at  $r = \left(\frac{6m}{\Lambda}\right)^{1/3}$ .

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## References

1. F. Estabrook H. Wahlquist S. Christensen B. DeWitt L. Smarr E. Tsiang (1973) *Phys. Rev. D* **7** 2814
2. J.E. Marsden F.J. Tipler (1980) *Phys. Rep.* **66** 109
3. R. Beig N.O' Murchadha (1998) *Phys. Rev. D* **57** 4728
4. L. Smarr J.W. York Jr (1978) *Phys. Rev. D* **17** 2529
5. D.M. Eardley L. Smarr (1978) *Phys. Rev. D* **19** 2239
6. A. Lichnerowich (1944) *J. Math. Pure Appl.* **23** 37
7. J. York (1972) *Phys. Rev. Lett.* **28** 1082
8. A. Goddard (1977) *Commun. Math. Phys.* **54** 279
9. A. Goddard (1977) *Math. Proc. Cambridge Philos. Soc.* **82** 489
10. A. Goddard (1977) *Gen. Relativ. Gravit.* **8** 525
11. D.R. Brill F. Flaherty (1976) *Commun. Math. Phys.* **50** 157
12. D.R. Brill J.M. Cavallo J.A. Isenberg (1980) *J. Math. Phys.* **21** 2789
13. A. Pervez A. Qadir A.A. Siddiqui (1995) *Phys. Rev. D* **51** 4598
14. A.D. Rendall (1996) *Helv. Phys. Acta* **69** 490
15. A. Qadir A.A. Siddiqui (1999) *J. Math. Phys.* **40** 5883

16. Beig, R., Heinzle, J.M.: Commun. Math. Phys. **260**, 673 (2005). (ArXiv: gr-qc/0501020, v2, 9 Feb. 2005)
17. P. Kraus F. Wilczek (1995) *Nucl. Phys. B* **433** 403
18. S. Corley T. Jacobson (1998) *Phys. Rev. D* **57** 6269
19. Hussain, V., Qadir, A., Siddiqui, A.A.: Phys. Rev. D **65**, 027501 (2002). (ArXiv: gr-qc/0110068, v3, 20 Apr. 2002)
20. Iriondo, M., Malec, E., Murchadha, N.O'.: Phys. Rev. D **54**, 4792 (1996). (ArXiv: gr-qc/9503030, v1, 16 Mar. 1995)
21. J. Guven N.O'. Murchadha (1999) *Phys. Rev. D* **60** 104015
22. R. Beig A.A. Siddiqui (2007) *Class. Quantum Gravit.* **24** 5435
23. A. Qadir A.A. Siddiqui (2006) *Int. J. Mod. Phys. D* **15** 1419
24. A. Qadir A.A. Siddiqui (2002) *Nuovo Cimento B* **117** 909
25. N. Cruz M. Olivares J.R. Villanueva (2005) *Class. Quantum Gravit.* **22** 1167