ON THE SPIN AND POLARIZATION EFFECTS IN THE THEORY OF SYNCHROTRON RADIATION

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(Presented by A. A. Sokolov)

As is well known, synchrotron radiation appears to be strongly polarized. In particular, in the classical approximation, 7/8 of the total radiated intensity belongs to the σ-component (the electrical vector of the radiation field which is along the radius towards the center of the trajectory) and 1/8 to the π-component (the electrical vector of the radiation field which is almost perpendicular to the plane of the orbit [1]). This deduction was experimentally verified by F. A. Korolev and his co-workers [2].

In the present paper, we attempted to study the influence of the electron spin orientation on the polarization and the intensity of the radiation when the electron is moving within a constant and uniform magnetic field. During the study of spin effects, it is convenient to split the Dirac equation solution into two states which characterize the spin orientation either with or against the direction of motion (longitudinal polarization) or with or against the field, i.e., in our problem the almost perpendicular polarization.

The Dirac equation

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi, \quad \hat{H} = c(\boldsymbol{\alpha}\mathbf{P}) + \varrho_3 m_0 c^2, \tag{1}$$

where

$$\mathbf{P} = \mathbf{p} - \frac{e}{c}\mathbf{A}, \quad A_1 = -\frac{1}{2}yH, \quad A_2 = \frac{1}{2}xH, \quad A_3 = 0, \tag{2}$$

describing the motion of the electron within a constant and uniform magnetic field has the following solution:

$$\psi_{1,3} = e^{-i\varepsilon cKt} \frac{e^{ik_3 z}}{\sqrt{L}} \frac{e^{i(l-1)\varphi}}{\sqrt{2\pi}} f_{1,3}(\varrho); \quad \psi_{2,4} = e^{-i\varepsilon cKt} \frac{e^{ik_3 z}}{\sqrt{L}} \frac{e^{il\varphi}}{\sqrt{2\pi}} f_{2,4}(\varrho); \tag{3}$$

$$f_{1,2,3,4} = \sqrt{2\gamma} \begin{vmatrix} C_1 I_{n-1,s}(\varrho) \\ iC_2 I_{n,s}(\varrho) \\ C_3 I_{n-1,s}(\varrho) \\ iC_4 I_{n,s}(\varrho) \end{vmatrix}. \qquad (4)$$

In this equation $\varrho = \gamma r^2$, $\gamma = e_0 H/2ch$, $e_0 = -e > 0$ is the elementary charge; $E = \varepsilon c\hbar K = \varepsilon ch\sqrt{k_0^2 + k_3^2 + 4\gamma n}$; $n = l + s = 0, 1 \ldots$ are the principal, $l = 0, \pm 1 \ldots -\infty \leqslant l \leqslant n$ are the azimuthal, and $s = 0, 1$ are the radial quantum numbers. The quantity $\varepsilon = \pm 1$ characterizes the sign of the energy, while the function $I_{n,s}(\varrho)$ is connected to the Laguerre polynomial $Q_s^l(\varrho)$ relationship

$$I_{n,s}(\varrho) = \frac{1}{\sqrt{n!s!}} e^{-\frac{1}{2}\rho} \varrho^{\frac{n-s}{2}} Q_s^{n-s}(\varrho). \qquad (5)$$

For a unique determination of the C_μ coefficients, we must, in addition to establishing the validity of the Dirac equation (1) and the normalization condition, subject the wave function to another additional condition which characterizes the direction of the spin and the value of polarization.

When studying the longitudinal polarization, this additional condition is best formulated by the condition for the preservation of the spin projection on the direction of motion

$$(\boldsymbol{\sigma}\mathbf{P})\psi = \hbar\tilde{k}\tilde{\zeta}\psi. \qquad (6)$$

The operator $(\boldsymbol{\sigma}\mathbf{P})$ represents the temporal component $T_{\mu 4}$ of the electron polarization pseudovector having the general form

$$T_{\mu\nu} = \frac{1}{2}\{P_\nu \sigma_\mu + \sigma_\mu P_\nu\}, \qquad (7)$$

where $\sigma_\mu\{\sigma, i\varrho_1\}$ is the spin 4 pseudovector, and $P_4 = (\hat{H} - e\varphi)\frac{1}{c}$ is the fourth component of the generalized momentum which in our case has the scalar potential $\varphi = 0$ [4].

During the study of the almost perpendicular polarization, it is convenient to subject the wave equation to the following condition [5]:

$$\{m_0 c\sigma_3 + \varrho_2 [\boldsymbol{\sigma}\mathbf{P}]_3\}\psi = \hbar k\zeta\psi. \qquad (8)$$

The operator on the left-hand side of this equation represents the component F_{124} of the third order polarization tensor:

$$F_{\mu\nu\lambda} = \frac{1}{2}\{P_\lambda \alpha_{\mu\nu} + \alpha_{\mu\nu} P_\lambda\}, \qquad (9)$$

where $\alpha_{23} = \rho_3\sigma_1$, $\alpha_{14} = i\rho_2\sigma_1$, etc. are the tensors of the characteristic magnetic and electric momenta.

Note that the operators on the left side of equations (6) and (8) are constants of motion, i.e., are consistent with the Hamiltonian $\hat{H}$. Within these equations the quantities $\tilde{\zeta}$, ζ are equal to ± 1 and characterize the two possible directions of the spin. The quantities $\tilde{k}$ and k are equal to

$$\tilde{k} = \sqrt{K^2 - k_0^2}, \quad k = \sqrt{K^2 - k_3^2}. \tag{10}$$

During the study of the longitudinal polarization ($\tilde{\zeta} = \pm 1$) the coefficients of equation (4) will be equal [6]:

$$\left.\begin{aligned} C_1 &= \tilde{\zeta}\tilde{a}\tilde{A}, \\ C_2 &= \tilde{a}\tilde{B}, \\ C_3 &= \varepsilon\tilde{b}\tilde{A}, \\ C_4 &= \varrho\tilde{\zeta}\tilde{b}\tilde{B}, \end{aligned}\right\} \tag{11}$$

where

$$\left.\begin{aligned} \tilde{a} &= \sqrt{\frac{1}{2}\left(1 + \varepsilon\frac{k_0}{K}\right)}, \\ \tilde{b} &= \sqrt{\frac{1}{2}\left(1 - \varepsilon\frac{k_0}{K}\right)}, \\ \tilde{A} &= \sqrt{\frac{1}{2}\left(1 + \tilde{\zeta}\frac{k_3}{\tilde{k}}\right)}, \\ \tilde{B} &= \sqrt{\frac{1}{2}\left(1 - \tilde{\zeta}\frac{k_3}{\tilde{k}}\right)}. \end{aligned}\right\} \tag{12}$$

In particular, the results from the last quoted equations with $\varepsilon = 1$ agree with those of another paper [6]. During the study of the perpendicular polarization [8] we must put:

$$\left.\begin{aligned} C_1 &= aA, \\ C_2 &= -\zeta bB, \\ C_3 &= bA, \\ C_4 &= \zeta aB, \end{aligned}\right\} \tag{13}$$

where

$$\left.\begin{aligned}
A &= \sqrt{\frac{1}{2}\left(1+\zeta\frac{k_0}{k}\right)}, \\
B &= \sqrt{\frac{1}{2}\left(1-\zeta\frac{k_0}{k}\right)}, \\
a &= \frac{1}{2}\left\{\sqrt{1+\varepsilon\frac{k_3}{K}}+\right. \\
&\quad \left.+\varepsilon\zeta\sqrt{1-\varepsilon\frac{k_3}{K}}\right\}, \\
b &= \frac{1}{2}\left\{\sqrt{1+\varepsilon\frac{k_3}{K}}-\right. \\
&\quad \left.-\varepsilon\zeta\sqrt{1-\varepsilon\frac{k_3}{K}}\right\}.
\end{aligned}\right\} \tag{14}$$

Utilizing in what follows the method developed in reference [1] (see also [7]), one can find intensities of the synchrotron radiation of the π- and σ-components which take into account the direction of electron spin.

During the study of the longitudinal electron polarization (see the additional condition (6)), the intensity of radiation connected with the change in the spin orientation does not depend on the direction of the initial spin (with or against the direction of motion). In the case of the almost perpendicular polarization, the radiated intensity may depend on the initial direction of the spin (with or against the field):

$$\left.\begin{aligned}
W_\sigma^{\uparrow\uparrow} &= W^{cl}\left\{\frac{7}{8}-\xi\left(\frac{25\sqrt{3}}{12}+\zeta\right)+\right. \\
&\quad \left.+\xi^2\left[\frac{335}{18}+\frac{245\sqrt{3}}{48}\zeta\right]+\dots\right\}, \\
W_\sigma^{\uparrow\downarrow} &= W^{cl}\left\{\xi^2\frac{1}{18}\right\}, \\
W_\pi^{\uparrow\uparrow} &= W^{cl}\left\{\frac{1}{8}-\xi\frac{5\sqrt{3}}{24}+\xi^2\frac{25}{18}+\dots\right\}, \\
W^{\uparrow\downarrow} &= W^{cl}\cdot\xi^2\frac{23}{18}\left\{1+\zeta\frac{105\sqrt{3}}{184}\right\}.
\end{aligned}\right\} \tag{15}$$

Here

$$W^{cl} = \frac{2}{3}\frac{e^2c}{R^2}\left(\frac{E}{m_0c^2}\right)^4; \quad \xi = \frac{3}{2}\frac{\hbar}{mcR}\left(\frac{E}{m_0c^2}\right)^2.$$

Arrows indicate the relative direction of the spin in the initial and final state, where $\zeta = 1$ indicates the initial spin oriented along the field, while $\zeta = -1$ is against the field.

Let us emphasize that the expression for the radiated intensity contains terms proportional to $\hbar$, depending on the initial direction of the spin, which disappear after averaging over the final spin states. Furthermore, it is clear from these equations that as a result of the radiation the spin will tend to orient against the field ($\zeta = -1$).

Let us investigate the probability of quantum transitions for the change of spin orientation per unit time. We then find that with the longitudinal polarization this probability does not depend on the initial spin orientation with or against the velocity direction:

$$w^{\rightleftarrows} = \frac{5\sqrt{3}}{36} \frac{e_0^2}{\hbar c} \frac{c}{R} \frac{E}{m_0 c^2} \xi^2 \frac{7}{9} \,. \quad (16)$$

In the case of the almost perpendicular polarization, however, the result depends essentially on the initial spin orientation with or against the magnetic field:

$$w^{\uparrow\downarrow} = \frac{5\sqrt{3}}{36} \frac{e_0^2}{\hbar c} \frac{c}{R} \frac{E}{m_0 c^2} \xi^2 \left(1 + \zeta \frac{8\sqrt{3}}{15}\right). \quad (17)$$

Consequently, because of the synchrotron radiation the electron spin must acquire an orientation which is dominantly against the direction of the magnetic field.

For a quantitative estimate of this effect we investigated the statistical change in the number of electrons with a given spin orientation. Let n_1 be the number of electrons with spin directed against the field ($\zeta = -1$), and n_2 be the number of electrons with the spin oriented along the field ($\zeta = 1$). Then for the changes of these quantities per second we have

$$\frac{dn_1}{dt} = n_2 w_{21} - n_1 w_{12}, \quad (18)$$

under the condition of the conservation of the total number of particles

$$n_1 + n_2 = n_0. \quad (19)$$

In this equation w_{12} and w_{21} are the probabilities found from equation (17) if one substitutes $\zeta = -1$ and $\zeta = 1$, respectively.

By integrating equation (18) we obtain:

$$\left.\begin{aligned} n_1 &= \frac{w_{21} n_0 - (w_{21} n_{20} - w_{12} n_{10})\, e^{-t/\tau}}{w_{12} + w_{21}} = \frac{(15 + 8\sqrt{3})\, n_0 - [15\,(n_{20} - n_{10}) + 8\sqrt{3}\, n_0]\, e^{-t/\tau}}{30}, \\ n_2 &= \frac{w_{12} n_0 + (w_{21} n_{20} - w_{12} n_{10})\, e^{-t/\tau}}{w_{12} + w_{21}} = \frac{(15 - 8\sqrt{3})\, n_0 + [15\,(n_{20} - n_{10}) + 8\sqrt{3}\, n_0]\, e^{-t/\tau}}{30}, \end{aligned}\right\} \quad (20)$$

where n_{10} and n_{20} are the initial values of n_1 and n_2 and the lifetime τ has the following value:

$$\begin{aligned} \tau &= (w_{12} + w_{21})^{-1} = \\ &= \left[\frac{5\sqrt{3}}{8} \frac{\hbar}{m_0 c R} \left(\frac{E}{m_0 c^2}\right)^5 \frac{e_0^2}{m_0 c R^2}\right]^{-1} = \\ &= \left[\frac{5\sqrt{3}}{8} \frac{m_0 c e_0^2}{\hbar} \left(\frac{E}{m_0 c^2}\right)^2 \left(\frac{H}{H_0}\right)^3\right]^{-1}; \end{aligned} \quad (21)$$

$H_0 = m_0^2 c^3/e^2\hbar = 4.67 \cdot 10^{-13}$ Oe. Assuming, in particular, that $H = 10^4$ Oe and E = 1 GeV, we get $\tau \sim$ 1 hour.

At time $t \gg \tau$ the ratio n_1/n_2 tends to the limiting value

$$\frac{n_1}{n_2} = \frac{w_{21}}{w_{12}} = \frac{15+8\sqrt{3}}{15-8\sqrt{3}} \qquad (22)$$

independent of the initial distribution of the electrons over the spin states. It is clear from the last expression that this limiting value of the polarization may consist of about 95 percent of the component with $\zeta = -1$ (i.e., with the spin oriented against the field).

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DISCUSSION

Yu. F. Orlov

Does your equation apply for arbitrary particles energies? Did you take into account $s \to s'$, $s \neq 0$ transitions (where s is the radial quantum number)?

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For this ideal case (constant and uniform field) our formula is applicable for arbitrary energies. The $s \to s'$ transitions are taken care of by means of summation over all values of s'.