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Discrete and Lie Quantum Symmetries derived from Classical Projective Geometry

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Abstract. A bottom-up process based on quantum Lie symmetries and related Riemannian spaces led us to line and Complex geometry in order to represent various aspects of the real projective spaces P^3 and P^5. Two major aspects, the 15-dim automorphisms of the Plücker-Klein quadric – e.g. given by Lorentz transformations or the Dirac algebra – and their generator incidence structure, lead us further to the Fano space PG(3,2) and related Segre manifolds. Vice versa, top-down geometrical reduction of SU∗(4)≅SL(2,$\mathbb{H}$) and PG(3,2) as background patterns allows for stepwise derivation of well-known kinematic and unitary Lie symmetries. SL(2,$\mathbb{H}$) thus represents actions of the Dirac algebra and relates to the quaternion one-sphere $S^1_{\mathbb{H}}$, $\mathbb{H}P^1$ and the Hopf map $\pi : S^7 \rightarrow S^4$. The related Riemannian space AII explains a vacuum structure by photons as Goldstone particles, the next reduction to the Hermitean symmetric space CI (when associated with line coordinates) yields SU(3) and links to string theory concepts like Kähler and Calabi-Yau manifolds. PG(3,2) is related to S_{15} and Clifford algebra models, octonion and sedenion models as well as various aspects known from algebraic geometry. This indicates that most of the standard quantum rep theory has subsidiary character only. As applications, we finally discuss briefly relations between physical action and this incidence structure, the entanglement of two photons, and with respect to quantum computing SU(4) acting on 2-qubit states.

1 Introduction

This paper is intended as a sequel to ongoing work reported at the ICGTMP34 [1] and at the ISQS28 [2]. The first paper outlines the anchoring of the 15-dim Lie algebra su(4) – which we are going to discuss below in more detail – in the hadron spectrum and in experimental data, and how we are led to su(4) and various, associated real forms. Here, we skip these details and start the discussion right from the 15-dim su(4) Lie algebra and two resulting Riemannian spaces AII and CI (see [1] and references). In the second (ISQS28) paper, we have discussed the rep theory of the 15-dim automorphisms of the Plücker-Klein quadric M_4^2 and identified Lorentz transformations *based on point coordinate transformations*, so that special relativity emerged as 'a gauge theory' of this quadric.

Here, we want to focus on further background details of this 15-dim algebra, especially we want to discuss some discrete background structure[1] (see [3] and [4]) which again forces the research direction towards projective geometry (PG). This viewpoint structures the background threads as well as it discriminates geometry from rep theory in the sense of Klein's 'Erlanger Programm'.

So in sec. 2, we summarize some relevant facts from the papers above. In sec. 3, we use the PG(3,2) context to discuss further discrete symmetries, mostly related to incidence geometry, to connect to one specific Clifford algebra, and we give an argument to rule out octonion discussions. The final secs. 4 and 5 address few physical applications and aspects as well as ongoing work from the viewpoint of PG.

[1]Having given an invited plenary talk at the QTS 13 in Yerevan few weeks later, we discuss details of the relativistic and kinematical aspects, especially action and metrics, in the QTS proceedings. The introductory sec. 2 is similar.

2 Basic Aspects of Line and Complex Geometry

It has been a central requirement to find a non-compact symmetry structure isomorphic to the established rep theory of the Dirac algebra in relativistic quantum mechanics and quantum field theory (QFT) on the one hand. On the other hand, we wanted to be able to explain the success of SU(4) and the respective rep theory in the low-energy spectrum and in current algebra [1]. This led us to the 15-dim operator algebras su(4) and su∗(4), their origin and the quest to identify reps and transformations in the spectrum by Lie theory. In the background, besides comparing to experiments, this invoked deeper formal and theoretical quests with respec to correct and *overall* consistent physical identifications of masses, energy, and even *physical* associations of 'particles', or resonances as well as rep theory and transformations/transitions in general. Moreover, while running through usual text books, everybody knows the respective author's freedom to choose phase factors, so we felt free to formulate the question and the relevant toolkit a little more freely.

2.1 Linearized Approaches

Our anchor points were the 15-dim Lie algebras su(4) and su∗(4), both comprising the 10-dim compact USp(4) as common maximal compact subgroup, and both even yielding an algebra decomposition without dimensional deficits [5] [3] according to $5 \oplus 10$, where 10 itself contains the very interesting 4-dim Lie subalgebra locally isomorphic to su(2)⊕u(1) [3] [4], so that $15 \longrightarrow 5 \oplus 10 \longrightarrow 5 \oplus 6 \oplus 4 = 5 \oplus 6 \oplus 3 \oplus 1$. The algebraic viewpoint allows for realizations in terms of all three associative division algebras: the quaternions, the complex and the real numbers. So we chose twofold quaternionic (direct) product reps, $Q_{\alpha\beta} := q_\alpha \times q_\beta$, $q_0 = \mathbb{1}$, $q_i \in \mathbb{H}$, $0 \leq \alpha, \beta \leq 3$, $1 \leq i, j \leq 3$, as well as their embedding in $\mathbb{C}_{4\times4}$ to compare to text books and publications, and to handle the phases overall uniquely and correctly.

A quaternionic rep features twofold quaternion 'spinors' and the transformation group SL(2,$\mathbb{H}$). This, however, when discussing rep theory, can be seen as a projective transformation over the skew quaternion field $\mathbb{H}$, and we may stress various known analytic as well as algebraic approaches. Realizing e.g. the rep theory of $\mathbb{H}P^1$ by real numbers, one is lead naturally to the Hopf map $\pi : S^7 \rightarrow S^4$ of the quaternion one-sphere to $\mathbb{H}P^1$, the quaternion projective space (see e.g. [6], sec. 9.4.1, or [4], secs. III.A and III.B). From here, however, it is easy to drift or even sink into the depths of e.g. principal and fibre bundles, Hopf algebra theory, or associated topological ideas – mostly without explicit physical identifications.

Our main discussion, however, used complex reps of this algebra and the bundle reps, and it is obvious that this 15-dim Lie algebra is only a subset 'of the full algebra' $M_{4\times4}(\mathbb{C})$. In a first step, one can thus check complex reps of the 15-dim algebras $su(n, m)$ where $n + m = 4$, su(4) being the most compact real form, and their respective (double) coverings of associated orthogonal compact and non-compact symmetry groups SO(n',m') where $n' + m' = 6$. Some well-known low-dimensional coverings thus emerge naturally (see e.g. [4], or [1] as well as the underlying text books [5], or [7]). An investigation of various phase definitions, however, further screwed up the physical identifications, when the complex groups as well as the covered orthogonal groups occurred in various different contexts and associated various physical identifications. So the only way out was to start from scratch, find physical meaningful and unique identifications of the reps, and perform the calculations step by step on the same (unique) identifications – just according to the very idea of Klein's Erlanger Programm.

2.2 Geometric Analysis and Physical Identifications

To treat this framework formally, we have followed the usual reasoning of Riemannian spaces (see e.g. [7]), which gave a lot of insight by using the spaces AII and CI [4], [2]. However, it also raised the usual problems with spontaneous symmetry breaking, i.e. the interpretation of nonlinear field reps and Goldstone bosons, and their physical identifications. But especially the Goldstone aspect yielded the important clue because instead of using 5 massless bosons and a remaining symplectic unitary symmetry, we have attached a wellknown rep which intrinsically fulfils both requirements – the photon! We know a 5-dim massless rep of the photon in terms of special linear (line) Complexe, and the symplectic symmetry is essentially the very basic symmetry of line Complexe, usually both represented by so-called '6-vectors'. This suggests the second decomposition of the 10-dim Lie algebra into a 6-dim linear algebra and a 4-dim Lie algebra as 'gauge group' if we can give geometrical and physical meaning to the two 6-vector reps and the four remaining generators of the symmetry group(s) which we will do in the next section.

To summarize the concept: It is geometry which allows *to fix the phases* globally if we identify the 5-dim coset rep by a special linear Complex, the photon, in terms of classical line geometry. Note that we discuss Lie symmetries vs. quantum reps, i.e. *infinitesimal* transformations. In this setting, the derivation of the 'two' electromagnetic invariants from the line Complex is given in [8], [9] II.A, i.e. $I_1 \sim \vec{B}^2 - \vec{E}^2$, $I_2 \sim \vec{E} \cdot \vec{B}$, with respect to the identification of affine fields $\vec{E}$ and $\vec{B}$, or Minkowski's 4-vector symbolism

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$F^{\mu\nu}$. Geometrically, these fields span the plane orthogonal to the force, and the Poynting vector $\vec{S} \sim \vec{E}\times\vec{B}$ describes the axis of the special Complex (which itself doesn't belong to the Complex!). This scenario can be considered as a derived, special case of Pfaff's equations (see [9], IV.A) whereas the integral discussions are given in [10]. The AII Goldstone picture $15 \longrightarrow 5 \oplus 10$ thus 'explains' the masslessness of the photon and stabilizes a compact symplectic isometry – a thourough mass identification in affine terms, however, with respect to SU(4) matter reps is to appear. A second important consequence of such a photon identification yields the photon relation to the vacuum structure as well as its occurrence as Bremsstrahlung (or even Cerenkov radiation at higher energies) which can be seen within the full 15-dim algebra. Transforming reps by the dim 10-Lie algebra usp(4) describes the internal symmetry of the 6-dim reps, they allow to gauge the coordinate systems of the CI scheme using the remaining four generators, and general 15-dim transformations associate 5-dim photon reps to the 6-dim matter fields.

Thus we have identified the 6-dim reps with regular line Complexe (or null systems) by the antisymmetric operator reps in dual P^3, or physically with matter fields when spanned by linear fundamental Complexe in P^5. Some of the arguments with respect to spin and handedness are given in [2]. So even based on this fibred/linearized approach, it is line geometry (represented by skew Lie algebra operators) as part of spatial projective geometry of P^3 and P^5 which determines the playground as well as the reps and the respective symmetry groups[2]. However, the reader should keep in mind that this approach yields a linearized version in the sense that in P^5 we have to use *quadratic* Complexe in order to describe quartics in P^3 and treat the general case of Cayley-Klein metrics in P^3, not the specialized affine approach by an absolute plane intrinsic to Weyl's gauge approach to quantum theory and QFT[3] (see the concepts in [11]). In other words, the rep by groups and Geometric Analysis may serve to answer the question of how to decompose a quadratic Complex in P^3 into two *linear* constituents (here in terms of two 6-vectors) infinitesimally, but in order to handle the general cases one has to use the PG of P^3 and P^5.

2.3 Line and Line Complex Reps

Having that said, in our case, we find the 5-dim coset rep $A \sim \{iQ_{01}, iQ_{03}, Q_{k2}\}$ (see [3], [4], and references) which can be used to discuss various real forms by matrix reps, i.e. by switching from $\exp A$ to the physical rep $\exp iA$ and by extracting a commutative phase 'i', or based on the full 15-dim algebra and the rôle of the complex structure of the CI space and (the non-commutative'phase') Q_{02} within the 4-dim subalgebra. While $\exp A \sim \{\mathbb{1}, iQ_{01}, iQ_{03}, Q_{k2}\}$ can be rewritten as $\sim Q_{02} \times \{iQ_{01}, Q_{02}, iQ_{03}, Q_{k0}\}$ (which formally generates SO(4,2)), we find Q_{02} associated to the 6-vector rep of the photon as well as to the matter 6-vector reps below and their su(2)⊕u(1) symmetry generators.

One can switch directly to the SO(3,3) symmetry of the real Plücker coordinates by simply using a relative (commuting) complexification of the two quaternion algebras $\{Q_{0i}\}$ and $\{Q_{k0}\}$, or exercise a covering by SU(2,2). Thus we can associate the Plücker rep of a line in P^3, or a point on the Plücker-Klein quadric M_4^2 in P^5, respectively, and accordingly we see (projective) line geometry as driving background geometry, and Lie theory as necessary but derived consequence for complex rep theory of dual P^3.

The matter identification can be based on the 6-dim CI space which is spanned by $\{iQ_{j1}, iQ_{k3}\}$ while $iQ_{k1} = Q_{02} iQ_{k3}$ and $\{iQ_{j1}, iQ_{k3}\} \sim (aQ_{02} + b)_k \, iQ_{k3} = (a - bQ_{02})_k \, iQ_{k1}$. Again it is important to note the relative complexification of the two 3-dim algebras iQ_{j1} and iQ_{k3} by Q_{02} with $Q_{02}^2 = -\mathbb{1}$, so the Hermitean CI rep has to be identified with the 6 Klein coordinates while preserving a Hermitean quadric and a compact unitary symmetry. So in both cases – the 5-dim AII case with $\{Q_{0i}$ and $Q_{02}Q_{k0} = Q_{k0}$ and the 6-dim case CI with iQ_{j1} and $-Q_{02} iQ_{k1} = iQ_{k1} Q_{02} = iQ_{k3}$ – the complexification is more than a simple 'i'. The special geometrical rôle of Q_{02} can be seen below in fig. 1.

To compare to projective geometry, we can switch to six classical line coordinates $p_{\alpha\beta} = x_\alpha y_\beta - y_\alpha x_\beta$ where quaternary coordinates x_α, y_β denote the two points spanning the line by $p_{\alpha\beta}$ (see e.g. [2]). The 'new' senary Klein coordinates $x_\aleph$, $1 \le \aleph \le 6$, are then given by [12]

$$\begin{array}{llcll} p_{01} = x_1 + I x_2\,, & p_{23} = x_1 - I x_2\,, & & 2x_1 = p_{01} + p_{23}\,, & 2I\,x_2 = p_{01} - p_{23}\,, \\ p_{02} = x_3 + I x_4\,, & p_{31} = x_3 - I x_4\,, & \text{or} & 2x_3 = p_{02} + p_{31}\,, & 2I\,x_4 = p_{02} - p_{31}\,, \\ p_{03} = x_5 + I x_6\,, & p_{12} = x_5 - I x_6\,, & & 2x_5 = p_{03} + p_{12}\,, & 2I\,x_6 = p_{03} - p_{12}\,, \end{array} \tag{1}$$

and the Plücker quadric $P = p_{01}p_{23} + p_{02}p_{31} + p_{03}p_{12}$ reads as $P = \sum_\aleph x_\aleph^2$, an SO(6) invariant quadric. When 'omitting' the I, it yields again the known SO(3,3) invariant quadric. We have chosen a capital

[2] As a side aspect, one can introduce 'i's (phases) to see how the respective textbooks modify the this 15-dim algebra, and using their results, one can try to recover the background geometry and connect to known theorems and aspects of PG.

[3] In [9] II.F, we have shown that this step using $x_0 = 1$ decomposes 6-vectors $p_{\alpha\beta}$ into *two* 3-vectors, and one may introduce an *additional* 'parity' operator to distinguish the 3-dim polar from the 3-dim axial part.

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I to preserve the freedom *to interpret* this imaginary unit. In Klein's original work [13] I is simply the imaginary unit 'i' in $\mathbb{C}$. Here, however, we have the possibility to choose $I = Q_{02}$, too. This allows to interpret the second CI-operator triplet $iQ_{k3} = -Q_{02}iQ_{k1}$ as an imaginary copy within the Complex structure, the six Klein coordinates map to two conjugate pairs of complexified triples $\{z_k iQ_{k1}\}$, $z_k \in \mathbb{C}$, i.e. SU(3) emerges naturally as its invariance group when casting the three operators into a triplet 'vector' X and building $X^+X = \overline{z}_k z_k$ [12]. Moreover, due to the homogeneous character of the Plücker coordinates, this coordinatization by three homogeneous complex coordinates can be interpreted as $\mathbb{C}P^2$ which on one hand leads to Calabi-Yau structures[4], on the other hand, we can apply Study's machinery on ternary coordinates [15] as well as the discussion of planar conics [16] which are beyond scope here. The related fact that we we have used P^3 and dual P^3 as construction scheme identifies the linear transformations not only by Lie generators but – being a 15-dim projective space, too – leads to further considerations by Segre manifolds, e.g. $S_{(3+1)(3+1)-1} \sim S_{15}$.

In a later stage, when departing from this fibred decomposition and discussing quadratic Complexe instead of two linear Complexe, one is of course led to Study's work [17], dualizing 'translations' and 'rotations' in Plücker's sense (or forces and force pairs) in terms of Dynamen in P^7. This enforces the rep theory of M_6^2 and S^7 – however based on geometry and without associating octonionic discussions – which is also beyond scope here and postponed.

2.4 The Plücker-Klein Quadric M_4^2

The 6 Plücker coordinates represent the linear base elements of the '6-vectors' in P^5. The 4-dim space of lines in P^3 corresponds to the points of the Plücker-Klein quadric M_4^2 in P^5. Starting over from this quadric as a central 'tool', one can of course ask for the automorphisms of this quadric. Geometrically, if we map points of the quadric automorphically in P^5, we also map lines to lines in P^3. This is a basic building block of PG, moreover, according to the basic notions of quantum theories, we can thus map linear reps to linear reps while preserving incidence, so identifications and transformation theory of P^5 should simplify rep theory of P^3, and yield deeper insight by identifying reps and mapping them to physical aspects, especially when investigating reps of dual P^3 (i.e. 'the operators') inifinitesimally and globally.

An explicit approach has been discussed in [2] using (point) coordinate transformations of general collineations[5] $a_{\alpha\beta}$ in P^3. Thus, rewriting Plücker coordinates by 2×2-determinants $p_{\alpha\beta}$, the Plücker-Klein quadric remained invariant under 'Lorentz transformations', and we have defined the expressions

$$M^{\mu\nu}_{\alpha\beta} := \begin{vmatrix} a_{\mu\alpha} & a_{\mu\beta} \\ a_{\nu\alpha} & a_{\nu\beta} \end{vmatrix}, \quad [\,\mathfrak{A}\cdot\mathfrak{B}\,] = [\,\mathfrak{A}_{\alpha\beta}\cdot\mathfrak{B}_{\gamma\delta}\,] = M^{01}_{\alpha\beta}M^{23}_{\gamma\delta} + M^{02}_{\alpha\beta}M^{31}_{\gamma\delta} + \ldots + M^{12}_{\alpha\beta}M^{03}_{\gamma\delta} \tag{2}$$

for the 6-vector product $[\,\mathfrak{A}\cdot\mathfrak{B}\,]$ so that the transformed quadric P' finally reads as

$$P' = p_{01}\,p_{23}\,[\,\mathfrak{A}_{01}\cdot\mathfrak{A}_{23}\,] + p_{02}\,p_{31}\,[\,\mathfrak{A}_{02}\cdot\mathfrak{A}_{31}\,] + p_{03}\,p_{12}\,[\,\mathfrak{A}_{03}\cdot\mathfrak{A}_{12}\,]\,. \tag{3}$$

Here, the 2×2-determinant rep both of the $p_{\alpha\beta}$ as well as of $M^{\mu\nu}_{\alpha\beta}$ reflect immediately further discrete symmetries just by their definitions as determinants. Focussing on the Ms, in eq. (2) it is straightforward (for planar reps) to prove invariance under the generators of the dihedral groups ('Diedergruppe') D_2 and D_4,

$$r_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \, r_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \, s_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \, s_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \begin{matrix} r_2 = -r_0, & r_3 = -r_1, \\ s_2 = -s_0, & s_3 = -s_1, \end{matrix}$$

D_2 being isomorphic to the Klein four-group, and $\mathbb{Z}_2 \times \mathbb{Z}_2$. So these symmetry properties[6] as well as the involutions $x \to -x$ and $x \to x^{-1}$ are automatically related to binary line calculations in (point based) P^3, and their occurrence in (linear) physics shouldn't astonish. Although especially the analysis people – due to working with functions and unique mappings – want to avoid ambiguities, it is important to note the intrinsic feature of P^3 line geometry of being *quadratic* and thus associating *two* lines per space point. To make point-line mappings unique in P^3, one has to consider Congruences (or 'Strahlensysteme')!

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[4] see e.g. a discussion of the Dolgachev invariant (N. Yui, Calabi-Yau workshop, Mainz, June 2025, or [14]).

[5] A discussion with focus on further relativistic aspects of the related symmetries has been given at the QTS 13 conference in Yerevan; the proceedings will be published in the Bulgarian Journal of Physics.

[6] The D_2 case exchanges binary point reps on one line, the D_4 case on two lines, a quadrangle.

3 PG(3,2) as a Further Bridgehead to Algebraic Geometry

When investigating the Lie product $A * B := [A, B]$ in the context of linear and totally geodesic submanifolds [4], an important tool has been the classification by Lie triple systems (LTS) [7] ch. IV § 7, i.e. the closure of a subset under $X * (Y * Z)$ within the full Lie algebra. Recall moreover, that we have the common Lie algebra constraint given by the Jacobi identity[7], $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ which maps to $X * (Y * Z) + Y * (Z * X) + Z * (X * Y) = 0$. Again, we thus find several aspects which can be generalized: On one hand, we may change the product rule $*$, e.g. replace the (binary) Lie product by a Clifford product (which in case of matrix reps yields a simple multiplication of the reps) or by other, even by weird product rules. On the other hand, one may think about the occurrence (or modifications) of triple products to discuss alternative algebras, associators, octonions, or specific Clifford and division algebra constructions, e.g. by involving involutions like conjugations or transpositions. Whereas we have given a classification of the LTS of su∗(4) in [3], sec. 2, which naturally play an important rôle in relativity as totally geodesic submanifolds, we have already pointed out, that we may arrange the Lie algebra generators in triples and in a 3-dim pattern in dual space [4] (see fig. 1).

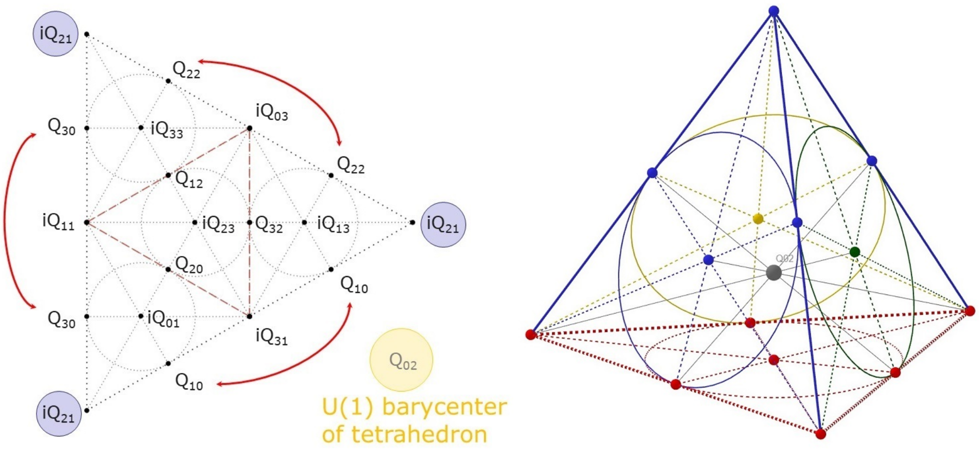

Figure 1: Operator identifications within su∗(4) versus PG(3,2) [4]. Note the essential rôle of Q_{02} in the tetrahedral barycenter and the fact that the operator triples along the 'lines' of PG(3,2) multiply to $\sim \mathbb{1}$!

This induced an additional entry point to discuss advanced discrete symmetries because our 3-dim construction (see [4] and fig. 1) is isomorphic to the Fano space PG(3,2), and moreover the phase freedom (and the discussion of the various su(4) real forms) coincides with PG(3,$\mathbb{K}$) where the field $\mathbb{K}$ is GF(2) underlying this finite projective space. This adds an additional thread with respect to discrete symmetries and incidence structures on dual spaces, and we thus want to approach particle and resonance interpretations usually given by root spaces and Regge trajectories. A pictorial rep has been given by [4] in the context of qquaternionic multiplication. Note however, that this is an associative structure inasfar as inner automorphisms of su∗(4) change the operator sets, i.e. the linear 'coordinatization' is *not unique* if we are to discuss quantum theory as usual in terms of *bilinear forms*. So octonions – being

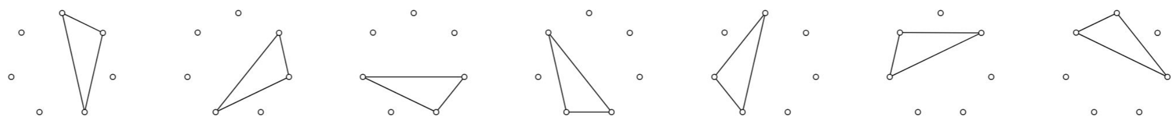

Figure 2: Decomposition of the complete Heawood graph into triangles.

special *planar* constructions only (see [3], sec. IV in conjunction with the Fano plane and the Steiner triple system S(2,3,7)) – are ruled out. Another beautiful publication on using octonions throughout the

[7] Note, that this may also be interpreted as a constraint on the submanifolds associated to the respective LTS.

ages [18] presents the operator (product) mapping in fig. 2. Such structures, however, can be *derived* from a mapping 7×7 ([19], examples I.18ff., p. 6 and I.26ff.) and *incidence* structures represented by graph theory which shifts the discussion to projective aspects of incidence (instead of metric) discussions. This is much closer to PG as metric structures are not (yet) available without having introduced absolute elements, their invariance groups and a Cayley-Klein metric. The graph given by fig. 3 (a) then relies on adjacency ([19] I.28) of 3-subsets. They lead to seven 3-subsets which form the blocks of a balanced incomplete block design, and the incidence graph thus shows the element-block incidence structure. Also a discussion of (classical) configurations (as introduced by Reye) is given at [19] p. 14/15, and the (symmetry) notation is introduced, e.g. the Fano plane as 7_3, the Pappus configuration 9_3, and the Desargues configuration 10_3. The discussion of desarguesian vs. nondesarguesian planes is started in [19] VII.2 by the fact that projective planes PG(2,q) are the classical *and the only* examples of *finite* projective *planes* in which the theorem of Desargues is valid. A discussion of ternary operations and nondesarguesian *planar* examples follows. So coming back to our 15-dim PG(3,2) operator pattern, we have $n > 2$, i.e. desarguesian and *non-planar* projective structures. With respect to incidence structures, we may use the (discrete) aspects of Steiner triple systems STS(15) treated by [19] exhaustively, and again we do not have to use approaches relying on (finite dimensional) division algebras but we discuss *derived* operator properties otherwise embedded in higher and advanced geometries like e.g. transformation spaces, or Segre manifolds.

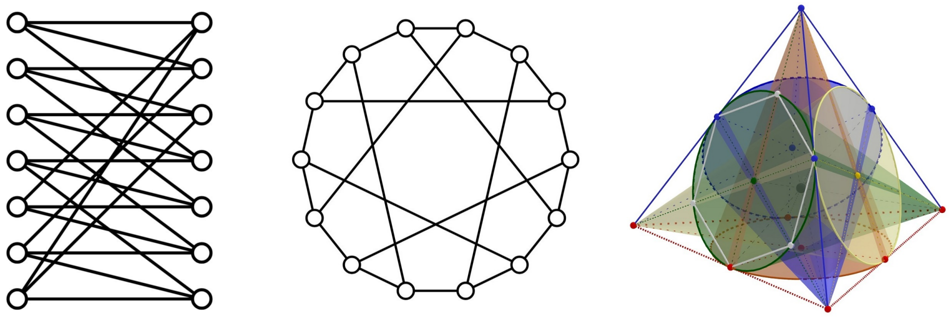

Figure 3: (a) left: Incidence graph rep of 7 symbols and 3-subsets; (b) mid: Heawood graph and Hamiltonian cycle of (a); (c) right: misleading composition of PG(3,2) and classical tetrahedral geometry.

There is, however, a direct link to using sedenions as representation [20] which connects to QFT approaches given e.g. in [21], and – starting from Kirkman's old 'Fifteen Schoolgirls problem' posed around 1850 (see e.g. [22]) – there is ongoing research in the context of Steiner and Kirkman triple systems [23]. For our discussions of PG(3,2), an important and very rich source is the book of Hirschfeld [24] which focusses solely on PG(3,2) and the very rich environment related to this symmetry. Here, it is important to mention the relations to PG(5,2) and the hyperbolic quadric $\mathcal{H}_5$ (see [24], ch. 17.5) which is another starting point for further investigations on discrete symmetries of QFT.

For our purposes, and connecting to Clifford evangelism with respect to quantum reps, besides sedenions we want to mention the connection to ideal theory and idempotents discussed by Lounesto [25] (see ch. 4 for the Pauli case and ch. 10 for the Dirac case). There, the 'spinorial' left ideals and the primitive idempotents were extracted from matrix reps [25] ch. 10.2

$$\begin{pmatrix} \psi_1 & 0 & 0 & 0 \\ \psi_2 & 0 & 0 & 0 \\ \psi_3 & 0 & 0 & 0 \\ \psi_4 & 0 & 0 & 0 \end{pmatrix}, \quad f = \frac{1}{2}(1+\gamma_0)\,\frac{1}{2}(1+i\gamma_1\gamma_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

the lhs describing the Dirac spinor and the rhs being an example of a square matrix spinor $\psi \in \mathrm{Mat}(4,\mathbb{C})f$ corresponding to a primitive idempotent f. So starting from P^3 transformations and the 4×4 matrix reps, it is obvious that we may represent the 'spinors' and idempotents by quaternary sums of $Q_{\alpha\beta}$, according to the position of the matrix elements represented by the canonical rep $E_{\alpha\beta}$, a zero matrix with one 1 at the $\alpha\beta$-position which then relates to Clifford language and 'spinorial' reps in relativistic quantum mechanics and QFT. In the equation above, $f = \frac{1}{4}\,(Q_{00} + iQ_{03} + iQ_{30} - Q_{33})$, and a calculation of the additional 15 generators as well as the extraction of idempotents is straightforward.

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As final thoughts – closing this section – we want to mention some misleading metric vs. projective aspects as given by the tetrahedron in fig. 3 (c). Throughout various discussions in the past years, people tended to apply geometric concepts known from classical metric or projective geometry, e.g. to understand *the line* $\{Q_{10}, Q_{22}, Q_{32}\}$ as a planar conic (or spatial quadric), or 'to complete' *the line* $\{Q_{10}, Q_{20}, Q_{30}\}$ by three 'additional' intersection points between $\{iQ_{01}, iQ_{11}\}$, $\{iQ_{01}, iQ_{21}\}$ and $\{iQ_{01}, iQ_{31}\}$ into a hexagon, thus obviously motivated by the $D(1,1)$ rep of an su(3) root diagram, the idea of ladder operators, or even reflections in Clifford algebras. Whereas in usual projective and metric geometries (*in point space*) in the cases of e.g. $\{Q_{22}, iQ_{13}\}$ or $\{Q_{10}, iQ_{01}\}$, one might expect sign changes with respect to the 'conic center' or with axis coordinatizations of the line segments $\overline{Q_{10}\, iQ_{01}}$ and $\overline{Q_{22}\, iQ_{13}}$, and there is no $-Q_{10}$ or $-Q_{22}$ at the virtual 'intersection points' but iQ_{11} and iQ_{31}. Due to commutativity, the resulting operator products are the same, and 'at best' one may interpret the plane itself as 'a hexagon' with vertices $\{Q_{10}, iQ_{31}, Q_{20}, iQ_{11}, Q_{30}, iQ_{21}\}$, and $\{Q_{10}, iQ_{21}, Q_{22}, iQ_{30}, Q_{32}, iQ_{31}, \}$, respectively, by neglecting metric ideas. So with respect to discrete symmetries and combinatorics in quantum theory, it is essential to understand the incidence structures and their reflections in (linear) operator structures.

4 Applications

Finally, we want to sketch briefly some applications:

- Well-known classical cases (which in addition explain the occurence of the 15 linear operators) are Reye's and Kummer's 16_6 configuration related to (rep) dimensions 4 to 6 of the coordinate sets, however, this is one small aspect of incidences known for almost two centuries. Having mentioned Pappus' 9_3 and Desargues' 10_3 above, a broad overview has been given in [26] III AB 5a.
- A further, simple application meets the two different reps of the linear Complex in P^3 while considering light and the double-slit experiment. Lines being dual to lines, due to the antisymmetric reps by P^3 point- and plane-coordinates x, y and u, v, $p_{\alpha\beta} = x_\alpha y_\beta - y_\alpha x_\beta$ and $\overline{p}_{\alpha\beta} = u_\alpha v_\beta - v_\alpha u_\beta$, yield $p_{\alpha\beta} = \overline{p}_{\alpha+3\,\beta+3}$ which may also be interpreted as line 'conjugation'. So one and the same point of the Plücker-Klein quadric M_4^2 in P^5 relates to *one and the same line* in P^3 describing the axis of a special Complex. This axis be seen as the carrier of a point set, or equivalently as an axis of a plane pencil. The affine P^3 picture of the experiment using an orthogonal plane which approaches the slit (see the $\vec{E}$-/$\vec{B}$-discussion above) allows for 'plane wave' approaches using 3-vector calculus. But both the coset identification as well as the description by special line Complexe suggest the correct 'particle' rep as well as the related transformation group to be taken in P^5! So the 'wave-particle dualism' in P^3 results from the two *equal*, but conjugate possibilities to represent P^5 special linear Complexe of M_4^2 -reps by points or by dual/'caustic' plane reps, respectively, i.e. as an artefact of inadequately chosen reps in P^3 to represent the underlying irrep in P^5.
- A last application can be treated briefly and fast: Nowadays treating qubits in quantum computing, people either discuss tensor products of Pauli spin matrices or of quaternions, dependent on their reasoning or starting points in spin physics, Clifford algebras, or .... Now the discussions above show, that both reps are inadequate to discuss photon reps and interactions. In the compact case, transformations of a direct product of two Pauli matrices are nowadays treated by SU(4) ('gates'), so in order to treat photon physics correctly (even at low energies) one has to switch from Klein to Plücker coordinates in order to minimize computing errors and the correction algorithms. In the case of quaternions, transformations of the direct product are covered by SL(2,$\mathbb{H}$), and again this transformation group covers SO(5,1), and *not* SO(3,3) of photon reps and related interactions.

5 Outlook

Using line and Complex reps along the reasoning discussed so far, the Lie operators as symmetry generators can be treated and seen from different viewpoints. The easiest approach given by linear spatial point transformations $f_j(x_i) = a_i^{(j)} x_i$ and operator reps $L_{ij} \sim x_i\partial_j - x_j\partial_i$ in conjunction with polar arguments, i.e. objects in tangent spaces. Thus, we have *derived* known infinitesimal symmetries seen within the standard model – note however, that we do not argue to replace the standard model but to use PG to gain insight into its postulated symmetry structures because we know a priori of the infinitesimal character of the operators. Line geometry provides a way *to derive* them consistently in accordance with kinematical reasoning from PG whereas effective SU(4) and (chiral) SU(2)×SU(2) spin-isospin emerge from low-energy Lie theory and Dirac symmetry arguments. In addition, we have given a few arguments on not to forget *the additional discrete incidence structures* of line and Complex geometry. Essentially, this paves the path to looking into more sophisticated structures of the transformation operators and associated structures, as we here have only started by pointing to incidences and e.g. Segre and Veronese manifolds. So ongoing work with B. Fauser is related to investigating especially aspects of the Klein bijection $P^3 \longleftrightarrow P^5$, linear and non-linear Complex geometry and related invariant theory, as well as incidence geometry.

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