

The Time Dependent Schrödinger Equation Revisited I¹: Quantum field and classical Hamilton-Jacobi routes to Schrödinger's wave equation

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Abstract. The time dependent Schrödinger equation is frequently “derived” by postulating the energy $E \rightarrow i\hbar \frac{\partial}{\partial t}$ and momentum $\vec{p} \rightarrow \frac{\hbar}{i} \vec{\nabla}$ operator relations. In the present paper we review the quantum field theoretic route to the Schrödinger wave equation which treats time and space as parameters, not operators. Furthermore, we recall that a classical (nonlinear) wave equation can be derived from the classical action via Hamiltonian-Jacobi theory. By requiring the wave equation to be linear we again arrive at the Schrödinger equation, without postulating operator relations. The underlying philosophy is operational: namely “a particle is what a particle detector detects.” This leads us to a useful physical picture combining the wave (field) and particle paradigms which points the way to the time-dependent Schrödinger equation.

1. Introduction

Prof. J. Briggs presented an interesting paper [1] at the “Time-Dependent Phenomena in Quantum Mechanics” Conference on the time-dependent Schrödinger equation (TDSE). His thesis was that time is not an operator in quantum mechanics, but space is an operator; and therefore the TDSE

$$i\hbar \frac{\partial \Psi(\vec{r}, t)}{\partial t} = H(\vec{r}, t)\Psi(\vec{r}, t) \quad (1)$$

is a strange mixture of “classical” (C number) time and quantum (operator) space. In the paper entitled “The Derivation of the Time-Dependent Schrödinger Equations,” [1] the authors say:

“In previous papers [see [1]] fundamental dissatisfaction with the conventional ‘derivations’ of the TDSE given in quantum mechanics textbooks was expressed [...] It is important to recognize that when time enters the Schrödinger equation in the form of a time-dependent interaction potential, the time is a classical variable arising from the solution of the classical Maxwell, Newton (or equivalent) equations. From this point of view the TDSE is a mixed quantum-classical equation, the interaction potential representing the influence of an exterior classical time-dependent force on a quantum system.”

The essence of Briggs’ perspective is that time in e.g. Maxwell’s equations is a classical variable, unlike the time in the TDSE which is somehow different.

¹ It is a pleasure to dedicate this paper to Prof. Manfred Kleber whose excellence as a scholar, physicist, and friend binds us to him and brings us together.

Stimulated by his paper we here present another perspective following a quantum field theoretic [2] route to the Schrödinger wave equation. We then present an alternative (De Broglie + Hamilton-Jacobi) [3] route to the TDSE which provides yet another way into ordinary wave mechanics. In neither case do we postulate operator relations like $\vec{p} \rightarrow \frac{\hbar}{i}\vec{\nabla}$, and $E \rightarrow i\hbar\frac{\partial}{\partial t}$. Fig. 1 conveys the message of the present paper.

Most of what we present here is “well known to those who know it well.” However, in view of discussions at this conference and elsewhere, it seems that a tutorial review blending quantum field theory, quantum measurement theory, and Bohmian mechanics is useful.

The philosophy we present is essentially that of Heisenberg: focus on what we measure. Thus, we regard a particle as that which a particle detector detects. An explicit example of this approach is given in the preprint included as Appendix A. In this way, we are led to think of the $|x\rangle$ state as resulting from a detector located at x which detects a particle at that point. This allows us to keep a field theoretical description (and mental picture) for the particle while thinking of a “point” detector which undergoes Lorentz boosts, etc. This perspective is especially useful in quantum optics as is discussed in a second paper [6] on the TDSE.

II. The Quantum Field Route to the Time Dependent Schrödinger Wave Equation

Consider a free spinless particle of mass m , e.g. a π meson or an α particle, in a one D. The system is described by the state vector $|\Psi(t)\rangle$ where t is the time parameter. The wave function is then given by

$$\langle x | \Psi(t) \rangle = \Psi(x, t) \quad (2)$$

where the position eigenvector $|x\rangle$ is given by the action of the creation operator (see Appendix A) on the vacuum, that is

$$|x\rangle = \hat{\psi}^\dagger(x) |0\rangle \quad (3)$$

with

$$\hat{\psi}^\dagger(x) = \frac{1}{\sqrt{2\pi}} \sum_p \hat{c}_p^+ e^{\frac{ipx}{\hbar}} \quad (4)$$

where $p = \hbar k$ is the momentum of the particle and \hat{c}_p^+ is the corresponding creation operator.

The time evolution of a free particle can be gleaned by considering the simple Lorentz boost operation. That is for a spin zero particle as described by the Lorentz group we have:

$$\langle \vec{p} | U(t) | \Psi \rangle = e^{-ip_0 t} \Psi(\vec{p}), \quad (5)$$

where $p_0 = \sqrt{p^2 c^2 + m_0^2 c^4}$ and t is the usual (proper) time. Furthermore, for a nonrelativistic particle $p_0 \cong \frac{p^2}{2m} + m_0 c^2$ and from Eq's (3-5) we have

$$|\Psi(t)\rangle = \frac{1}{\sqrt{2\pi}} \sum_p e^{-i\frac{p^2}{2m_0} + i\frac{px}{\hbar}} |p\rangle, \quad (6)$$

where $|p\rangle = \hat{c}_p^+ |0\rangle$.

It then follows that

$$\Psi(x, t) = \frac{1}{\sqrt{2\pi}} \sum_p e^{-i\frac{p^2}{2m_0} t + i\frac{px}{\hbar}} |p\rangle \quad (7)$$

where $|p\rangle = \hat{c}_p^+ |0\rangle$, and therefore

$$i\hbar \frac{\partial}{\partial t} \Psi(x, t) = \frac{1}{\sqrt{2\pi}} \sum_p E_p e^{-i\frac{E_p}{\hbar} t + i\frac{px}{\hbar}} |p\rangle \quad (8)$$

where $E_p = p^2/2m_0$. Finally we note that Eq(8) can be written as

$$i\hbar \frac{\partial}{\partial t} \Psi(x, t) = \left(-\frac{\hbar^2 \nabla_x^2}{2m} \right) \Psi(x, t). \quad (9)$$

Thus we arrive at the Schrödinger wave equation (9) in which the time and space derivatives arise naturally without attributing operator character to neither momentum and energy nor space and time.

One point should be emphasized: the wave function $\Psi(x, t)$ is nothing more (nor less) than the probability amplitude for detecting the particle described by the state $|\Psi\rangle$ at position x . That is, $\langle x|\Psi\rangle = \langle 0|\hat{\psi}(x)|\Psi\rangle$ where $\hat{\psi}(x)$ destroys the particle at x , where a point particle detection is placed. This is further discussed in the conclusion and in the appendix.

III. The Classical Hamilton-Jacobi Route to the Schrödinger Wave Equation

Next, we follow David Bohm [3] and apply Hamilton-Jacobi (H-J) theory of classical mechanics to obtain a classical wave equation for the quantity

$$\Psi(\vec{r}, t) = R e^{iS/\hbar}, \quad (10)$$

where $R = \sqrt{\rho}$, and the classical action

$$S(\vec{r}, t) = S(\vec{r}_0, t_0) + \int L(\vec{r}, \dot{\vec{r}}, t) dt, \quad (11)$$

where \hbar is a constant, which will turn out to be Planck's constant, and

$$\vec{p} = \vec{\nabla} S, \quad H = -\frac{\partial}{\partial t} S. \quad (12)$$

Using the classical H-J equations together with the continuity equation

$$\frac{\partial}{\partial t} S + \frac{(\vec{\nabla} S)^2}{2m} = 0, \quad (\text{Hamilton-Jacobi}) \quad (13a)$$

$$\frac{\partial}{\partial t} \rho + \vec{\nabla} \cdot \left(\rho \frac{\vec{\nabla} S}{m} \right) = 0, \quad \text{continuity} \quad (13b)$$

we find that Eq's (14) are equivalent to the "classical wave function" given by

$$i\hbar \frac{\partial}{\partial t} \Psi = -\frac{\hbar^2}{2m} \vec{\nabla}^2 \Psi + Q\Psi \quad (14)$$

Where the so called quantum potential, Q , is given by

$$Q = \frac{\hbar^2}{2m} \frac{(\vec{\nabla} S)^2}{R}. \quad (15)$$

Comparing the classical wave equation with the quantum (Schrödinger) wave equation we see that the difference is the $Q\Psi$ term in Eq (15); that is, the difference between the classical and quantum wave equation is the quantum potential we must throw out term to make the classical to quantum transition, see Appendix B. Then, as before we have the Schrödinger wave equation.

IV. Conclusions

The message is clear: The time in Schrödinger equation is the same as it is in Newtonian physics. Time is a parameter but *not* an operator in both the top down, quantum field theory derivation, and the bottom up, classical Hamiltonian-Jacobi derivation, see Fig. 1. The same can be said for momentum. In both cases, we find that it is as if we end up replacing $\vec{p} \rightarrow \frac{\hbar}{i} \vec{\nabla}$, just as in the textbook derivations of the Schrödinger wave equation. However, we are led naturally to this result, it is not an ansatz or a postulate.

It is almost a century since the advent of quantum mechanics. The conventional assignments

$$E \rightarrow i\hbar \frac{\partial}{\partial t} \quad (16)$$

and

$$\vec{p} \rightarrow \frac{\hbar}{i} \vec{\nabla} \quad (17)$$

should be motivated from quantum field and/or classical physics. When confusion arises, it is better to regard both time and space as simple *c*-number parameters. Further support for this approach comes from an operational measurement theory [4] perspective. From that vantage point, the key expression

$$\Psi(\vec{r}, t) = \langle 0 | \hat{\psi}(\hat{\vec{r}}) | \Psi(t) \rangle \quad (18)$$

is regarded as representing the “matrix element” for exciting a photodetector at \vec{r} , see Eq’s (A.7) and (A.8) in the appendix.

It is also noted that from this “a-particle-is-what-a-particle-detector-detects” perspective we could equally well regard the detector as being boosted. That is, the quantum field $|\Psi\rangle$ need not be visualized as describing a particle. The nice features of the Lorentz group (e.g., not using $i\hbar\dot{\psi} = H\psi$) applies to the field $|\Psi\rangle$ interacting with the point-like detector.

Finally, we endorse the Briggs program but from a different perspective. The most interesting affect of his approach is, in my opinion, that it provides a nice example of what Schrödinger calls “statistical time”. Robert Scully [5] in his book “The Demon and the Quantum” expounds on Schrödinger’s position as follows:

“Schrödinger explains that the very notion of past and future comes from statistical reasoning:

“To my view the ‘statistical theory of time’ has an even stronger bearing on the philosophy of time than the theory of relativity may, or so I believe, assert that physical theory in this present stage strongly suggests the indestructibility of Mind by Time.”

The statistical time concept that entropy = time’s arrow, has deep and fascinating implications. It therefore behooves us to try to understand entropy ever more deeply. Entropy not only explains the arrow of time, it also explains its existence; it is “time”. This was one of the first observations relating information (entropy) to our actual experience in nature. Clausius and Boltzmann were intrigued by this idea.”

In another paper [6], we present two other approaches to the TDSE. In the first instance, we follow a quantum optical [7] approach, and then a classical approach based on Maxwell’s Equations.

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$$\begin{array}{c}
 \boxed{\langle 0 | \hat{\psi}(\vec{r}) | \Psi(t) \rangle} \\
 \Downarrow \\
 \boxed{i\hbar \frac{\partial \Psi(\vec{r}, t)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi(\vec{r}, t)} \\
 \Updownarrow \\
 \boxed{\sqrt{\rho} e^{iS/\hbar}}
 \end{array}$$

Figure 1. Top Down: The time dependent Schrödinger wave equation follows from quantum field theory in which $\hat{\psi}(\vec{r}, t)$, the operator which annihilates a particle at \vec{r} , sandwiched between the vacuum and the state of a particle at time t is the wave function of that particle. Physically, we may think of a point particle detector at \vec{r} which is excited by the annihilation of the particle at that position; i.e., $\Psi(\vec{r}, t)$ is the probability amplitude for detecting a particle at \vec{r} .

Bottom Up: The time dependent Schrödinger equation follows from classical Hamiltonian-Jacobi theory where ρ is the particle density and S is the action, by requiring that our wave function be linear in $\Psi(\vec{r}, t)$.

Appendix A.

The Operational Approach to Measurement Yields $\langle 0 | \hat{\psi}(r) | \Psi(t) \rangle = \Psi(x, t)$.

The analysis of an array of particle detectors, i.e., atom detectors as in Fig 2 of [4]b, to demonstrate particle interference is carried out in Appendix A. In particular, the probability amplitude for triggering a detector of \vec{r} is shown in A.7 and A.8 to go as $\langle 0 | \hat{\psi}(r) | \Psi(t) \rangle$.

The main point of this Appendix is to demonstrate by worked (physical) example how naturally the definition

$$|x\rangle = \psi^+(x)|0\rangle ,$$

arises in measurement theory. This is in complete accord with the usual quantum field and many body perspective in which the many particle state is written in terms of

$$\langle 0 | \hat{\psi}(\vec{r}_n) \dots \hat{\psi}(\vec{r}_2) \hat{\psi}(\vec{r}_1) | n_1, n_2, \dots, n_j, \dots \rangle \quad (\text{A1})$$

and $\hat{\psi}^+(\vec{r}_i)$ is discussed as the operator which creates a particle at \vec{r}_i . This is a most useful tool in the many body problem. However, in modern quantum optical experiments the positions \vec{r}_i are the positions of photo/atom detectors as is discussed at length in [4]b (see also [6]).

References

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