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Perihelion precession and deflection of light by the gravitational object in general relativity

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In the present research note, we have derived the explicit expression for the perihelion precession and the deflection angle of a light by the gravitational object. The explicit expression for the trajectory of the planet orbiting around central object in the framework of Newtonian theory as well as general relativity has been discussed. It is studied the deflection of the light ray and gravitational lensing effects by the gravitational object in the weak field approximation. Using the gravitational lensing equation the magnification of the primary and secondary images has been derived. It is shown that the for small angle β, it is difficult to distinguish the magnifications of the primary and secondary images, while for largest value of the angle β the magnification of the primary image dominates will equal to the total magnification. It is also discussed the time delay of the light ray passing through the gravitational object. Finally, the classical expression for the energy of relativistic particle has been derived in the frame of the general relativity.

I. INTRODUCTION

Based on astronomical observations, in the early 1600s Kepler established that the orbit described by a planet in the solar system is an ellipse, with the Sun occupying one of its foci. In fact the Keplerian laws are derived within the framework of Newtonian theory. However, this theory does not explain orbital motion of some planets in solar system, in particular, motion of Mecuiry. That is why general relativity is a very successful theory of the gravitational field suggested by Einstein in 1915, whose predictions are in excellent agreement with a large number of astronomical observations and experiments performed at the scale of the Solar System. In particular, three fundamental tests of general relativity, the perihelion precession of planet Mercury [1, 2], the bending of light by the Sun [3, 4], and the radar echo delay experiment [5, 6] have all fully confirmed, within the range of observational/experimental errors, the predictions of Einstein's theory of gravity.

As we mentioned before that gravitational lensing is one of the most important tests of general relativity. It has been observed in distant astrophysical sources, however these observations are poorly controlled and it is uncertain how they constrain general relativity. The most precise tests are analogous to Eddington's 1919 experiment: they measure the deflection of radiation from a distant source by the Sun. The sources that can be most precisely analyzed are distant radio sources, in particular, some quasars are very strong radio sources. An important improvement in obtaining positional high accuracies was obtained by combining radio telescopes across Earth. The technique is called very long baseline interferometry (VLBI). With this technique radio observations couple the phase information of the radio signal observed in telescopes separated over large distances. Recently, these telescopes have measured the deflection of radio waves by the Sun to extremely high precision, confirming the amount of deflection predicted by general relativity aspect to the 0.03% level [7]. Launched in 2013, the Gaia spacecraft conducts a census of one billion stars in the Milky Way and measure their positions to an accuracy of 24 microarcseconds. Thus it also provides stringent new tests of gravitational deflection of light caused by the Sun which was predicted by General relativity.

In the present paper, we derive the expression for the perihelion shift of the planet in the framework of Newtonian theory and general relativity. The paper is organized as follows. In Sect. II, we provide in very detailed derivation of the perihelion of the planet orbiting around the gravitational object in the framework of both Newtonian theory and general relativity. In Sect. III, we discuss the deflection of light ray and gravitational lensing effect in the general relativity. Finally, in Sect. IV, we summarize obtained results.

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II. PERIHELION PRECESSION

The perihelion is the closest orbit of the planet to the central object. Due to its importance in many applications, the study of the motion of massive or massless particles in different geometries obtained as solutions of Einstein's gravitational field equations and of their extensions is a fundamental field of general relativity. The first exact solution of the vacuum field equations was the Schwarzschild solution, which can be used efficiently to explain all astronomical observations at the scale of the Solar System. The exact equation of motion in Schwarzschild geometry is highly nonlinear, and therefore to obtain the observable physical parameters approximate methods must be used. The first-order approximation of the equation of motion already gives the correct approximation of the perihelion precession of Mercury and of the deflection of light by the Sun. However, due to the importance of the problem many mathematical techniques for the study of the astrometric properties of the planetary motions and of the light might be extended in highest order approximations.

A. Newtonian approach

In Newtonian theory, the Lagrangian for a celestial object of mass m influenced by the central gravitational object of mass M at the equatorial plane is given by [8]

$$\mathscr{L} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + \frac{GMm}{r}\,, \tag{1}$$

where an overdot denoted the derivative with respect to time, G is the Newtonian gravitational constant. The constants of motion, namely, energy, E, and and angular momentum, L, of the celestial object can be easily found as

$$E = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - \frac{GMm}{r}\,, L = mr^2\dot{\phi}\,, \tag{2}$$

Using the above equations (2) and (2) one can obtain

$$\frac{1}{r^4}\left(\frac{dr}{d\phi}\right)^2 + \frac{1}{r^2} = \frac{2E}{mL^2} + \frac{2GM}{L^2 r}\,. \tag{3}$$

Keep in mind that the solution to the above equation represents the trajectory of the celestial object around central object. Hereafter intruding the new coordinate $u = 1/r$ and differentiating the equation (3), one can have [8]

$$\frac{d^2u}{d\phi^2} + u = \frac{GMm^2}{L^2}\,, \tag{4}$$

where u is a function of the angle ϕ, i.e $u = u(\phi)$. Note that equation (4) reduces to the standard harmonic oscillator equation for $(u - GMm^2/L^2)$ and the solution is given by $u - GMm^2/L^2 = A\cos(\phi - \phi_0)$, where A and ϕ_0 are the constants of integration, respectively. For simplicity, we set $\phi_0 = 0$ and $A = eGMm^2/L^2$. Consequently, the solution to equation (4) is simply can be found as [8]

$$u(\phi) = \frac{GMm^2}{L^2}\left(1 + e\cos\phi\right)\,, \tag{5}$$

where e is the eccentricity.

The precession of an orbit looks at the motion of the perihelion over time. The perihelion of an orbit is the point closest to the host body. In Figure 1 shows the perihelion of the point particle orbiting around central gravitational object. As one can see that in absence of eccentricity trajectory of point particle follows circular orbits, while for largest values of the eccentricity trajectory will be elliptical.

B. General relativistic approach

Now we focus on the same problem in the framework of the general relativity. To take general relativistic effect, we consider the Schwarzschild spacetime which is given by the following line element:

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right)d(ct)^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,, \tag{6}$$

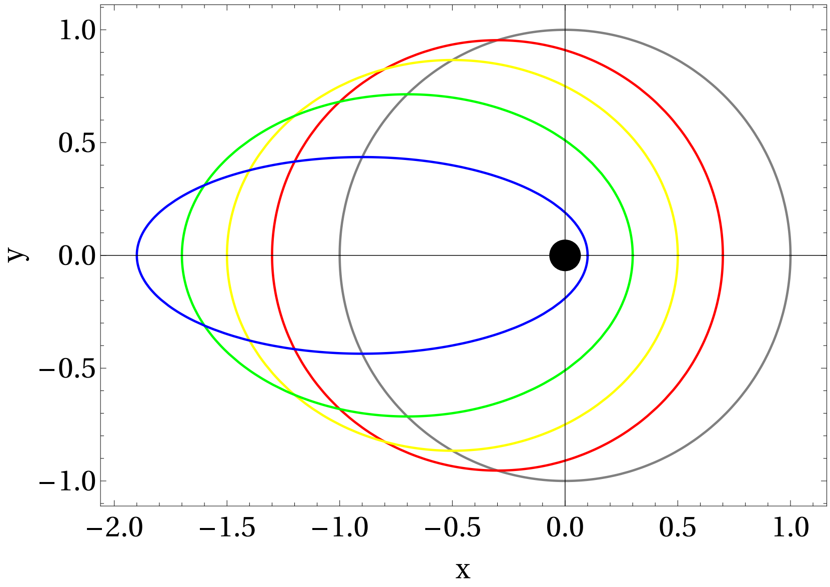

FIG. 1: The trajectories of particle for different values of eccentricity, in particular, $e = 0, 0.3, 0.5, 0.7, 0.9$.

where c is the speed of light, M is the total mass of the black hole and the virtual size of the black hole is represented by the Schwarzschild radius $r_g = 2GM/c^2$.

In fact the motion of test particle around the black hole is described by the geodesic equation:

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0 \,, \tag{7}$$

where $\dot{x}^\alpha = dx^\alpha/d\tau$ is the 4-velocity of test particle normalized as $\dot{x}_\alpha\dot{x}^\alpha = -1$, $\Gamma^\alpha_{\mu\nu}$ are the Christoffel symbols. Hereafter integrating equation (7) at the equatorial plane, one can obtain

$$\frac{dt}{d\tau} = \frac{E}{mc^2}\left(1 - \frac{2GM}{c^2r}\right)^{-1} \,, \qquad \frac{d\phi}{d\tau} = \frac{L}{mr^2} \,, \qquad \left(\frac{dr}{d(c\tau)}\right)^2 = \left(\frac{E}{mc^2}\right)^2 - \left(1 - \frac{2GM}{c^2r}\right)\left[1 + \left(\frac{L}{mcr}\right)^2\right] \,. \tag{8}$$

Taking into account equations in (8), and after introducing new variable $u = 1/r$, the radial equation read

$$\left(\frac{du}{d\phi}\right)^2 = \left(\frac{mc}{L}\right)^2\left[\left(\frac{E}{mc^2}\right)^2 - 1\right] + \frac{2GMm^2}{L^2}u - u^2 + \frac{2GM}{c^2}u^3 \,. \tag{9}$$

Hereafter differentiation equation (9), one can get

$$\frac{d^2u}{d\phi^2} + u = \frac{GMm^2}{L^2} + \frac{3GM}{c^2}u^2 \,. \tag{10}$$

Here the last term in equation (10) is responsible to general relativistic correction to the equation for the trajectory of test particle. On the other hand, from the mathematical point of view, the equation (10) represents the nonlinear harmonic oscillator equation and the solution is represented by the incomplete elliptic integral (See, e.g. []). Obviously, this solution is not allowed to find perihelion shift. That is why the semi-analytical approach can be performed to find approximate value of the perihelion shift. Let's introduce the expansion parameter $\epsilon = 3(GMm/cL)^2 << 1$ which allows to expand the solution in the form:

$$u(\phi) = u_0(\phi) + \epsilon u_1(\phi) + \mathcal{O}(\epsilon^2) \,. \tag{11}$$

Inserting the equation (11) into (10), one can obtain solution in the zeroth-order approximation in the form:

$$\frac{d^2u_0}{d\phi^2} + u_0 = \frac{GMm^2}{L^2} \,, \qquad \rightarrow \qquad u_0(\phi) = \frac{GMm^2}{L^2}\left(1 + e\cos\phi\right) \,, \tag{12}$$

which is the same as predicted in the Newtonian theory, while in the first-order approximation equation (10) reads

$$\frac{d^2u_1}{d\phi^2}+u_1=\frac{L^2}{GM}u_0^2=\frac{GMm^2}{L^2}\left(1+\frac{e^2}{2}+2e\cos\phi+\frac{e^2}{2}\cos 2\phi\right)\ . \tag{13}$$

The general solution to equation (13) can be found as $u_1(\phi)=A+B\phi\sin\phi+C\cos 2\phi$, which satisfies the following equation

$$\frac{d^2u_1}{d\phi^2}+u_1=A+2B\cos\phi-3C\cos 2\phi\ , \tag{14}$$

where A, B, and C are unknown constants found by comparing equations (13) and (14) in the form:

$$A=\frac{GMm^2}{L^2}\left(1+\frac{e^2}{2}\right)\ ,\qquad B=\frac{GMm^2e}{L^2}\ ,\qquad C=-\frac{GMm^2e^2}{6L^2}\ . \tag{15}$$

Before presenting the final result for u, one can use the following useful expression for small ϵ in the form:

$$\cos[\phi(1-\epsilon)]=\cos\phi\cos\epsilon\phi+\sin\phi\sin\epsilon\phi\simeq\cos\phi+\epsilon\phi\sin\phi+\mathcal{O}(\epsilon^2)\ . \tag{16}$$

Consequently, taking into account all facts above together with equation (16), the solution (11) can rewritten as

$$u(\phi)=\frac{GMm^2}{L^2}\left[1+e\cos\phi(1-\epsilon)\right]+\frac{\epsilon GMm^2}{L^2}\left[1+\frac{e^2}{3}\left(1+\sin^2\phi\right)\right]\ . \tag{17}$$

As the precession is the result of the orbit not being periodic in 2π, the precession must result from this term, and so can be calculated from it. Take the precession as $\delta\phi$. This then gives $u(0)=u(2\pi+\delta\phi)$. The perihelia first occurs at $\phi=0$, as was defined earlier. This means that the second perihelia will occur when the cos term generating the precession has gone through a full 2π. Then, one can obtain the following relation $2\pi=(2\pi+\delta\phi)(1-\epsilon)$ which means that

$$\delta\phi=\frac{2\pi\epsilon}{1-\epsilon}\simeq 2\pi\epsilon=\frac{6\pi G^2M^2m^2}{c^2L^2}\ . \tag{18}$$

Figure 2 illustrates the perihelion precession of point particle in general relativity for the different values of the eccentricity.

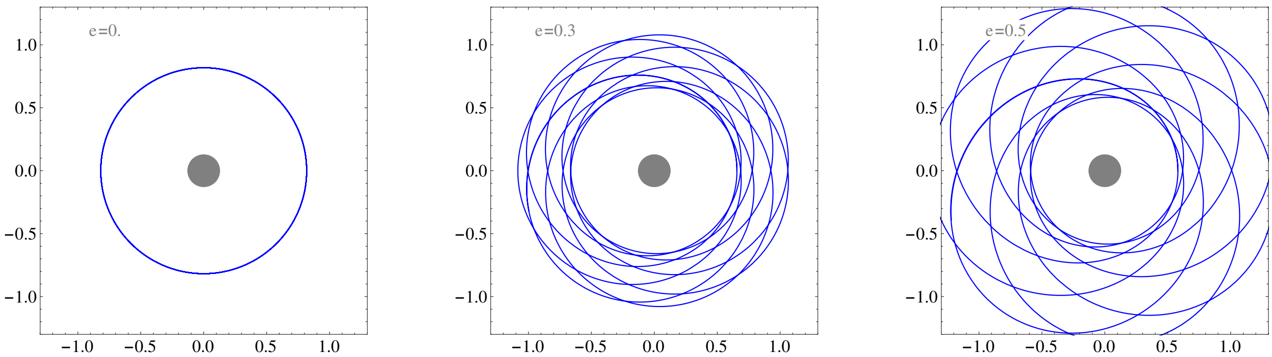

FIG. 2: The perihelion shift of point particle for different values of eccentricity.

III. DEFLECTION OF LIGHT

In the weak gravitational field approximation the metric tensor of the spacetime can be written as $g_{\alpha\beta}=\eta_{\alpha\beta}+h_{\alpha\beta}$ and $g^{\alpha\beta}=\eta^{\alpha\beta}-h^{\alpha\beta}$, where $\eta_{\mu\nu}$ is the metric tensor in flat space, $h_{\mu\nu}$ is a small perturbation $\eta_{\alpha\beta}=\eta^{\alpha\beta}$, $h_{\alpha\beta}=h^{\alpha\beta}$, $h_{\alpha\beta}h^{\alpha\beta}\to 0$. According to Ref. [9], the deflection angle of photon is defined as the difference between the directions of the incoming and of the outgoing light rays. We can write the expression of the deflection angle of the light ray: $\hat{\alpha}=\mathbf{e}_{\rm out}-\mathbf{e}_{\rm in}$, where $\mathbf{e}_{\rm in}$ and $\mathbf{e}_{\rm out}$ are the unit vectors along the spatial component momentum vector $\mathbf{p}$ of the "incoming" and "outgoing" photon, respectively, i.e. $\mathbf{e}=\mathbf{p}/p$, and $p=\sqrt{p_x^2+p_y^2+p_z^2}=p_z$. The explicit expression for the deflection angle is given by [9]

$$\hat{\alpha}_b=-\frac{1}{2}\int_{-\infty}^{\infty}\frac{d}{db}\left(h_{tt}+h_{zz}\right)dz\ , \tag{19}$$

with h_{tt} and h_{zz} in the Schwarzschild spacetime are [8]

$$h_{tt} = \frac{2GM}{c^2 r} \,, \qquad h_{ij} = \frac{2GM}{c^2 r}\hat{n}_i\hat{n}_j \,, \qquad h_{zz} = \frac{2GM}{c^2 r}\cos^2\theta \,, \tag{20}$$

where $\hat{n}_i$ is the component of the unit vector with the same direction as the radius vector $r_i = (x, y, z)$ and has the form $\hat{n}_i = (\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$. Using equations (19) and (20), the expression for the deflection angle of a light ray passing near a static compact object can be found as

$$\hat{\alpha}_b = -\frac{1}{2}\int_{-\infty}^{\infty}\frac{d}{db}\left[\frac{2GM}{c^2\sqrt{b^2+z^2}}\left(1+\frac{z^2}{b^2+z^2}\right)\right]dz = \frac{4GM}{c^2 b} \,, \tag{21}$$

where b is the impact parameter of the light ray. While in next leading approximation the deflection of angle can be found as

$$\hat{\alpha}_b = \frac{4GM}{c^2 b} + \pi\left(\frac{GM}{c^2 b}\right)^2 \,. \tag{22}$$

Now, we study the observational consequences of gravitational lensing, namely the magnification of image sources, Einstein cross (rings), time delay etc. For this purpose, one can use the lens equation which relates the angle β between the real position of the source and lens with respect to the observer, the angle θ between the apparent image of the source and the observer-lens axis, and the deflection angle α: $\beta = \theta - \Theta_0^2/\theta$, where $\Theta_0 = \sqrt{4GMD_{\rm ls}/c^2 D_{\rm l} D_{\rm s}}$ is the Einstein ring, with $D_{\rm s}$, $D_{\rm ls}$ and $D_{\rm l}$ being the distances between observer and the source, lens and the source, and observer and lens, respectively. Let us now consider the image magnification due to lensing. The solution of lens equation can be found as $\theta = \left(\beta \pm \sqrt{\beta^2 + 4\Theta_0^2}\right)/2$. The magnification of the image is identified as $\mu = |(\theta/\beta)(d\theta/d\beta)|$, while for individual images one can have

$$\mu_1 = \frac{1}{4}\left[\frac{y}{\sqrt{y^2+4}} + \frac{\sqrt{y^2+4}}{y} + 2\right] \,, \qquad \mu_2 = \frac{1}{4}\left[\frac{y}{\sqrt{y^2+4}} + \frac{\sqrt{y^2+4}}{y} - 2\right] \,, \qquad y = \beta/\Theta_0 \,, \tag{23}$$

where the sub indecies "1" and "2" denote the primary and secondary images of the source, respectively. The total $\mu = \mu_1 + \mu_2$ and the ratio $R = \mu_1/\mu_2$ of magnifications of images are given by

$$\mu = \frac{y^2+2}{y\sqrt{y^2+4}} \,, \qquad R = \left(\frac{\sqrt{y^2+4}+y}{\sqrt{y^2+4}-y}\right)^2 \,. \tag{24}$$

Figure 3 shows the magnification of the primary and secondary images of the source due to weak lensing: Figure 3 illustrates splitting of the first and the second images or Einstein cross due to the external magnetic field relatively the image that in plasma. In case of "positive parity", size of the Einstein cross is larger, while in case of "negative parity" is smaller than that of plasma case.

A variable source behind the lensing object produces an observable variable images. However, the source and the image will not necessarily vary simultaneously: in general, there will be a time delay between the two events and there are two contributions. First, there is a purely geometrical time delay. Second, there is a delay due to the potential of the lensing object or so-called Shapiro time delay. The total time delay that arises from both the geometry and the gravitational potential turns out to be

$$\Delta T = \frac{4GM}{c^3}(1+z)\left[\ln\left(\frac{\sqrt{y^2+4}+y}{\sqrt{y^2+4}-y}\right) + \frac{1}{2}y\sqrt{y^2+4}\right] \,. \tag{25}$$

where z is the redshift of the lensing object. To have an idea about the value of the time delay for the suppermassive black hole of mass $10^6 M_\odot$ can be estimated as

$$\Delta T \sim 2\times 10^{-4}\left(\frac{M}{10^6 M_\odot}\right)\mathrm{s} \,. \tag{26}$$

Dependence of the time delay due to the gravitational field from the position y for the value of the red-shift $z = 0$ and $z = 2.7$ is illustrated in Fig. 4. It is easy to see that with increasing of β angle the time delay light ray increases.

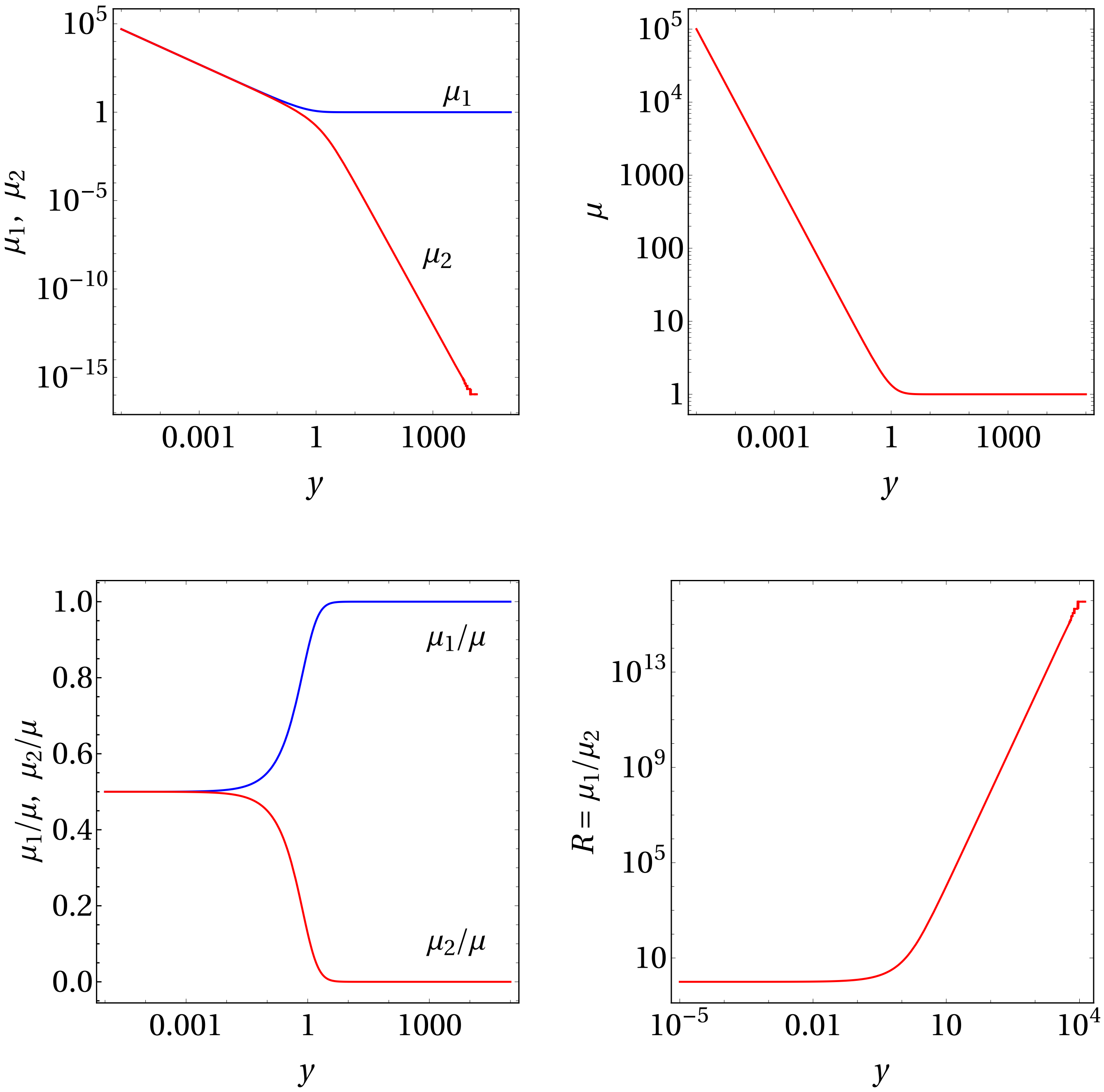

FIG. 3: The magnification of individual image (left) the total magnification (centre) and the ratio of magnifications (right) as the functions of fractinal angle $y = \beta/\Theta_0$.

A. From general relativity to special relativity

The Lagrangian for test particle of mass m is given by

$$\mathscr{L} = \frac{1}{2} m g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta , \qquad \dot{x}^\alpha = \frac{dx^\alpha}{d\lambda} , \tag{27}$$

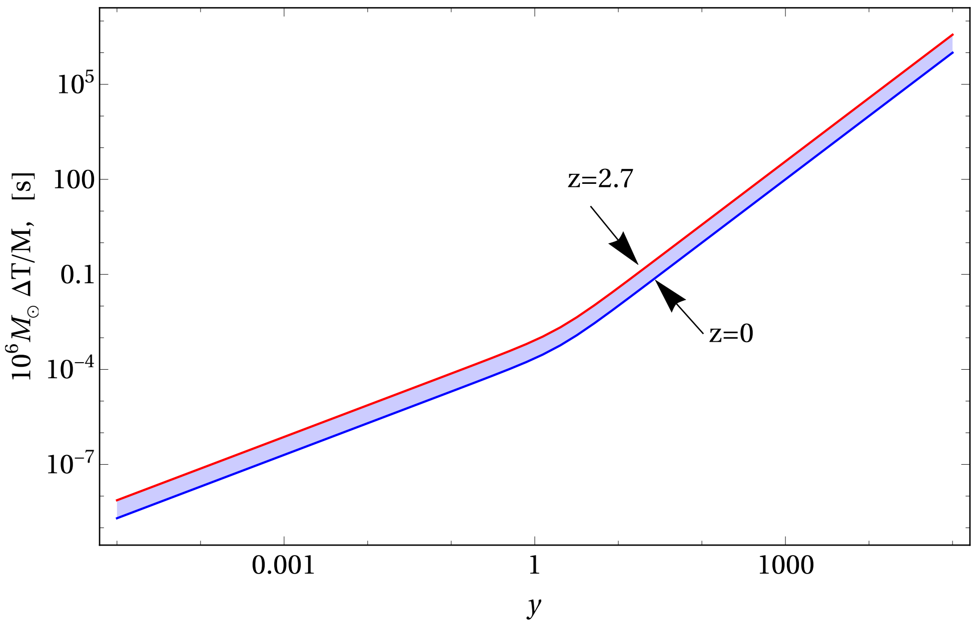

FIG. 4: Time delay ΔT of the light-ray as a function of the angle $y = \beta/\Theta_0$.

where $g_{\alpha\beta}$ is the metric tensor, λ is an affine parameter, $\dot{x}^\alpha$ is the four-velocity normalized as, $g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = -1$, that allows to write the following expression:

$$g_{tt}\dot{t}^2(1 - v^2/c^2) = -1 , \tag{28}$$

where v is the relative three-velocity of particle measured in a frame of a local observer defined as, $v^2 = v_{\hat{r}}^2 + v_{\hat{\theta}}^2 + v_{\hat{\phi}}^2$, and the orthonormal components are given as [8, 10, 11]

$$v_{\hat{r}} = \frac{dr}{dt}\sqrt{\frac{g_{rr}}{-g_{tt}}} , \qquad v_{\hat{\theta}} = \frac{d\theta}{dt}\sqrt{\frac{g_{\theta\theta}}{-g_{tt}}} , \qquad v_{\hat{\phi}} = \frac{d\phi}{dt}\sqrt{\frac{g_{\phi\phi}}{-g_{tt}}} . \tag{29}$$

The conserved quantities, namely, the energy and angular momentum of test particle are given by

$$P_t = -E = mc^2 g_{tt}\dot{t} , \qquad P_\phi = L = m g_{\phi\phi}\dot{\phi} . \tag{30}$$

Hereafter eliminating, $\dot{t}$, from equations (30) and (28), the classical expression for the energy of relativistic particle in the frame of GR can be expressed as [8, 10, 11]

$$E = \frac{mc^2\sqrt{-g_{tt}}}{\sqrt{1 - v^2/c^2}} , \qquad L = \frac{m v_\phi \sqrt{g_{\phi\phi}}}{\sqrt{1 - v^2/c^2}} . \tag{31}$$

On the other hand, using the normalization of the four-velocity of particle, one can obtain expression for the effective potential in the form:

$$V_{\text{eff}} \equiv -g_{tt}\left(1 + \frac{(L/mc)^2}{g_{\phi\phi}}\right) = \left(\frac{E}{mc^2}\right)^2 + g_{tt}\left(g_{rr}\dot{r}^2 + g_{\theta\theta}\dot{\theta}^2\right) . \tag{32}$$

The critical values for the conserved quantities can be found, by applying the conditions $\dot{r} = \ddot{r} = 0$ and $\dot{\theta} = \ddot{\theta} = 0$, as

$$E = \frac{mc^2\sqrt{-g_{tt}}}{\sqrt{1 - \frac{g'_{tt}}{g'_{\phi\phi}}\frac{g_{\phi\phi}}{g_{tt}}}} , \qquad L = \frac{mc\sqrt{g_{\phi\phi}}}{\sqrt{1 - \frac{g'_{tt}}{g'_{\phi\phi}}\frac{g_{\phi\phi}}{g_{tt}}}}\sqrt{\frac{g'_{tt}}{g'_{\phi\phi}}\frac{g_{\phi\phi}}{g_{tt}}} , \tag{33}$$

where prime denotes the derivative with respect to the radial coordinate.

Now, comparing the energy expressions (31) and (33), one can express the orbital velocity of particle in the form:

$$v = v_\phi = c\sqrt{\frac{g'_{tt}}{g'_{\phi\phi}}\frac{g_{\phi\phi}}{g_{tt}}} = c\sqrt{\frac{\partial_r \ln g_{tt}}{\partial_r \ln g_{\phi\phi}}}\ , \tag{34}$$

which can be easily determined for any spacetime geometry. It is easy to show that orbital velocity of particle at the photonsphere always equals to speed of the light, i.e. $v = c$. Because the position of the photonsphere can be found from the fact that the denominator of the energy expression (33) should be zero.

IV. CONCLUSIONS

In the present research note, we have derived the explicit expression for the perihelion precession and the deflection angle of a light by the gravitational object. The classical expression for the energy of relativistic particle has been derived in the frame of the general relativity.

The explicit expression for the trajectory of the planet orbiting around central object in the framework of Newtonian theory has been reproduced. The dependence of the shape of the trajectory of the planet from the eccentricity has been explicitly analyzed. It shown that with increasing the eccentricity the perihelion of the planet come close the central object. The same problem has been investigated in the framework of general relativity. It is shown that planet orbits around gravitational object with elliptic trajectory, however, perihelion of planet always shift in per round which is first predicted in general theory of relativity. It is also shown that trajectory of the planet can be expressed in terms of the elliptic integrals, however, it does not give information for perihelion shift of the planet orbiting around gravitational object.

It is studied the deflection of the light ray and gravitational lensing effects by the gravitational object in the weak field approximation. To involve general relativistic effects the Schwarzshild spacetime has been considered. The derivation of the expression for the deflection angle has been explicitly shown in the first and second order approximations. Using the gravitational lensing equation the magnification of the primary and secondary images has been derived. It is shown that the for small angle β, it is difficult to distinguish the magnifications of the primary and secondary images, while for largest value of the angle β the magnification of the primary image dominates will equal to the total magnification. It is also discussed the time delay of the light ray passing through the gravitational object.

Finally, it is investigated particle motion around the gravitational compact object described by the arbitrary spacetime. The analog of the special relativity has been shown by considering the pure general relativistic approach. It is shown that the orbital velocity in photonsphere is equal to the speed of light in arbitrary spacetime. We underline that test particle orbits at the ISCO position around the gravitational object with relativistic velocity, in particular, with a half of the speed of light, i.e. $v = c/2$ around the Scharzschild black hole.

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