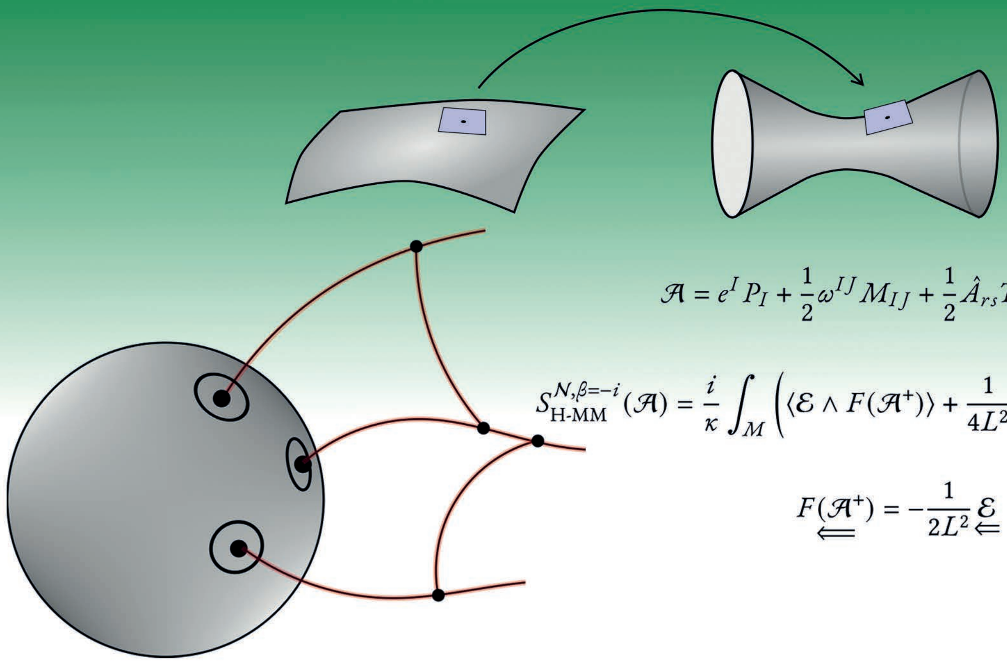


$$\mathcal{A} = e^I P_I + \frac{1}{2}\omega^{IJ} M_{IJ} + \frac{1}{2}\hat{A}_{rs}T^{rs} + \Psi^\alpha_r Q^r_\alpha$$

$$S^{\mathcal{N},\beta=-i}_{\text{H-MM}}(\mathcal{A}) = \frac{i}{\kappa}\int_M \left(\langle \mathcal{E}\wedge F(\mathcal{A}^+)\rangle + \frac{1}{4L^2}\langle\mathcal{E}\wedge\mathcal{E}\rangle\right) + S^{\text{OSp}}_{\text{CS}}(\mathcal{A}^+)$$

$$\underset{\Leftarrow}{F(\mathcal{A}^+)} = -\frac{1}{2L^2}\underset{\Leftarrow}{\mathcal{E}}$$

FAU Forschungen, Reihe B, Medizin, Naturwissenschaft, Technik 40

Konstantin Konrad Christoph Eder

Super Cartan geometry and loop quantum supergravity

FAU University Press

Konstantin Konrad Christoph Eder

Super Cartan geometry and loop quantum supergravity

FAU Forschungen, Reihe B
Medizin, Naturwissenschaft, Technik
Band 40

Herausgeber der Reihe:
Wissenschaftlicher Beirat der FAU University Press

Konstantin Konrad Christoph Eder

Super Cartan geometry and loop quantum supergravity

Erlangen
FAU University Press
2022

Bibliografische Information der Deutschen Nationalbibliothek:
Die Deutsche Nationalbibliothek verzeichnet diese Publikation in der Deutschen Nationalbibliografie; detaillierte bibliografische Daten sind im Internet über http://dnb.d-nb.de abrufbar.

Bitte zitieren als
Eder, Konstantin Konrad Christoph. 2022. *Super Cartan geometry and loop quantum supergravity*. FAU Forschungen, Reihe B, Medizin, Naturwissenschaft, Technik Band 40. Erlangen: FAU University Press. DOI: 10.25593/978-3-96147-530-8.

Der vollständige Inhalt des Buchs ist als PDF über den OPUS-Server der Friedrich-Alexander-Universität Erlangen-Nürnberg abrufbar:
https://opus4.kobv.de/opus4-fau/home

Verlag und Auslieferung:
FAU University Press, Universitätsstraße 4, 91054 Erlangen

Druck: docupoint GmbH

ISBN: 978-3-96147-529-2 (Druckausgabe)
eISBN: 978-3-96147-530-8 (Online-Ausgabe)
ISSN: 2198-8102
DOI: 10.25593/978-3-96147-530-8

Super Cartan geometry and loop quantum supergravity

Super-Cartan-Geometrie und Schleifenquantensupergravitation

Der Naturwissenschaftlichen Fakultät
der Friedrich-Alexander-Universität
Erlangen-Nürnberg

zur

Erlangung des Doktorgrades Dr. rer. nat.

vorgelegt von

Konstantin Konrad Christoph Eder, M.Sc.

aus Erlangen

Als Dissertation genehmigt
von der Naturwissenschaftlichen Fakultät
der Friedrich-Alexander-Universität Erlangen-Nürnberg

Tag der mündlichen Prüfung:	11. Februar 2022
Vorsitzender des Promotionsorgans:	Prof. Dr. Wolfgang Achtziger
Gutachter:	Prof. Dr. Hanno Sahlmann Prof. Dr. Dorothea Bahns Prof. Dr. Laurent Freidel

Contents

Abstract

The present thesis is dedicated to questions at the interface of the perhaps currently best known approaches toward quantum gravity, namely loop quantum gravity and superstring theory. Combining gravity with the principle of local supersymmetry leads to supergravity which, in certain cases, turns out to arise in terms of a low-energy limit of superstring theory. In this thesis, we want to deal with mathematical and physical aspects of (extended) supergravity theories in four spacetime dimensions and applications in the framework of loop quantum gravity such as quantum dynamics, boundary theory as well as classical and quantum cosmology.

To this end, in the beginning, we will study a mathematically rigorous approach toward geometric supergravity also commonly known as the Castellani-D'Auria-Fré approach by introducing the notion of a super Cartan geometry in an enriched category of supermanifolds. In fact, considering enriched categories turns out to be mandatory in order to, among other things, consistently implement the anticommutative nature of (classical) fermionic fields in mathematical physics. Furthermore, within this category, we will study mathematical aspects of super gauge theory and fiber bundle theory and analyze the parallel transport map associated to super connection forms.

We then turn towards applications of these methods in the framework of loop quantum gravity. First, the canonical analysis of pure $D = 4$, $\mathcal{N} = 1$ Holst supergravity will be performed using real Asthekar-Barbero variables and the existing formalism for the Hilbert space, representation, etc. A compact expression for the so-called SUSY constraint operator will be derived. Moreover, in this context, we will propose a specific regularization procedure and derive explicit expressions for its action on spin network states. This is important to investigate the dynamics in the quantum theory and to find physical states.

Next, the Cartan geometric approach toward (extended) supergravity in the presence of boundaries will be discussed. In particular, based on newer developments in this field, we will derive the Holst variant of the MacDowell-Mansouri action for $\mathcal{N} = 1$ and $\mathcal{N} = 2$ pure anti-de Sitter supergravity in $D = 4$ for arbitrary Barbero-Immirzi parameters. This action plays a crucial role if one imposes supersymmetry invariance at the boundary. We will also discuss the chiral limit of the theory, which turns out to possess some very special properties such as the manifest invariance of the resulting action under an enlarged gauge symmetry. Moreover, we will show that demanding supersymmetry invariance at the boundary yields a unique boundary term corresponding to a super Chern-Simons theory with $\mathrm{OSp}(\mathcal{N}|2)$ gauge group. These results provide a step towards the quantum description of supersymmetric black holes in the framework of loop quantum gravity.

Using the observations made in the chiral theory, we will finally study a class of symmetry reduced models of $\mathcal{N} = 1$ chiral supergravity. In fact, the enlarged gauge symmetry turns out to be essential as it allows for nontrivial fermionic contributions in the symmetry reduced super Ashtekar connection even if one imposes spatial isotropy. We will then also quantize the theory in terms of representations of a graded variant of the holonomy-flux *-algebra which yields a natural state space. Finally, the remaining dynamical constraints will be implemented in the quantum theory. For a certain subclass of these models, we show explicitly that the (graded) commutator of the supersymmetry constraints exactly reproduces the classical Poisson relations. In particular, the trace of the commutator between the so-called left and right SUSY constraint reproduces the Hamilton constraint operator. Finally, we consider the dynamics of the theory and compare it to a quantization using standard variables and standard minisuperspace techniques.

Zusammenfassung

Die vorliegende Arbeit widmet sich Fragen, die sich im Schnittbereich der vielleicht derzeit bekanntesten Ansätze zur Quantengravitation befinden, nämlich der Schleifenquantengravitation und der Superstringtheorie. Die Kombination der Gravitation mit dem Prinzip der lokalen Supersymmetrie führt zur Supergravitation, die sich in bestimmten Fällen als ein Niederenergie-Limes der Superstringtheorie herausstellt. In dieser Arbeit wollen wir uns mit mathematischen und physikalischen Aspekten (erweiterter) Supergravitationstheorien in vier Raumzeitdimensionen und Anwendungen im Rahmen der Schleifenquantengravitation wie der Quantendynamik, der Randtheorie sowie der klassischen und Quantenkosmologie beschäftigen.

Zu diesem Zweck werden wir zu Beginn einen mathematisch rigorosen Ansatz zur geometrischen Supergravitation untersuchen, der auch allgemein als Castellani-D'Auria-Fré-Ansatz bekannt ist, indem wir den Begriff einer Super-Cartan-Geometrie in einer angereicherten Kategorie von Supermannigfaltigkeiten einführen. Tatsächlich erweist sich die Betrachtung angereicherter Kategorien als zwingend notwendig, um u.a. die antikommutative Natur (klassischer) fermionischer Felder in der mathematischen Physik konsistent zu implementieren. Darüber hinaus werden wir innerhalb dieser Kategorie mathematische Aspekte der Super-Eichtheorie und Faserbündeltheorie untersuchen und den, zu Super-Zusammenhangsformen assoziierten, Paralleltransport analysieren.

Als nächstes wenden wir uns Anwendungen dieser Methoden im Rahmen der Schleifenquantengravitation zu. Zunächst wird die kanonische Analyse der reinen $D = 4, \mathcal{N} = 1$ Holst-Supergravitation unter Verwendung reeller Asthekar-Barbero-Variablen sowie des bestehenden Formalismus für den Hilbertraum, der Darstellung etc. durchgeführt. Es wird ein kompakter Ausdruck für den sogenannten SUSY-Constraint-Operators abgeleitet. Darüber hinaus werden wir in diesem Zusammenhang ein spezifisches Regularisierungsverfahren vorschlagen und explizite Ausdrücke für dessen Wirkung auf Spin-Netzwerk-Zuständen herleiten. Dies ist wichtig, um die Dynamik in der Quantentheorie zu untersuchen und physikalische Zustände zu identifizieren.

Als nächstes wird der Cartan-geometrische Ansatz zur (erweiterten) Supergravitation in Gegenwart von Rändern diskutiert. Insbesondere werden wir, basierend auf neueren Entwicklungen in diesem Feld, die Holst-Variante der MacDowell-Mansouri-Wirkung für die reine $\mathcal{N} = 1$ und $\mathcal{N} = 2$ Anti-de Sitter-Supergravitation in $D = 4$ für beliebige Barbero-Immirzi-Parameter herleiten. Diese Wirkung spielt eine entscheidende Rolle, wenn man Supersymmetrie-Invarianz am Rand fordert. Wir werden auch den chiralen Grenzwert der Theorie diskutieren, der einige sehr spezielle Eigenschaften besitzt, wie zum Beispiel die manifeste Invarianz der resultierenden Wirkung unter einer erweiterten Eichsymmetrie. Außerdem werden wir zeigen, dass die Forderung nach Supersymmetrie-

Invarianz am Rand einen eindeutigen Randterm ergibt, der einer Super-Chern-Simons-Theorie mit $\mathrm{OSp}(\mathcal{N}|2)$-Eichgruppe entspricht. Diese Ergebnisse stellen einen Schritt in Richtung einer Quantenbeschreibung von supersymmetrischen Schwarzen Löchern im Rahmen der Schleifenquantengravitation dar.

Unter Verwendung der Beobachtungen, die in der chiralen Theorie gemacht wurden, werden wir schließlich eine Klasse Symmetrie-reduzierter Modelle der $\mathcal{N} = 1$ chiralen Supergravitation untersuchen. Tatsächlich erweist sich die erweiterte Eichsymmetrie als essentiell, da sie nicht-triviale fermionische Beiträge im Symmetrie-reduzierten Super-Ashtekar-Zusammenhang erlaubt, selbst bei Forderung räumlicher Isotropie. Sodann wenden wir uns der Quantisierung der Theorie unter Studium der Darstellung einer gradierten Variante der Holonomie-Fluss *-Algebra zu, die einen natürlichen Zustandsraum ergibt. Schließlich werden die dynamischen Constraints in der Quantentheorie implementiert. Für eine bestimmte Unterklasse dieser Modelle zeigen wir explizit, dass der (gradierte) Kommutator der SUSY-Constraints die klassischen Poisson-Relationen exakt reproduziert. Insbesondere reproduziert die Spur des Kommutators zwischen dem sogenannten Links- und Rechts-SUSY-Constraint den Hamilton-Constraint-Operator. Schließlich betrachten wir die Dynamik der Theorie und vergleichen sie mit einer Quantisierung unter Verwendung von Standardvariablen und Standard-Minisuperspace-Techniken.

Publications connected to the thesis

Important parts of the present thesis have already been published:

[1] K. Eder, "Super fiber bundles, connection forms, and parallel transport," J. Math. Phys. **62** (2021), 063506 doi:10.1063/5.0044343 [arXiv:2101.00924 [math.DG]].

[2] K. Eder and H. Sahlmann, "$\mathcal{N} = 1$ Supergravity with loop quantum gravity methods and quantization of the SUSY constraint," Phys. Rev. D **103** (2021) no.4, 046010 doi:10.1103/ PhysRevD.103.046010 [arXiv:2011.00108 [gr-qc]].

[3] K. Eder and H. Sahlmann, "Holst-MacDowell-Mansouri action for (extended) supergravity with boundaries and super Chern-Simons theory," JHEP **07** (2021), 071 doi:10.1007/ JHEP07(2021)071 [arXiv:2104.02011 [gr-qc]].

[4] K. Eder and H. Sahlmann, "Supersymmetric minisuperspace models in self-dual loop quantum cosmology," JHEP **21** (2020), 064 doi:10.1007/JHEP03(2021)064 [arXiv:2010.15629 [gr-qc]].

Parts have also been published as a preprint:

[5] K. Eder, "Super Cartan geometry and the super Ashtekar connection," [arXiv:2010.09630 [gr-qc]] (submitted).

Therefore, there is some textual overlap with the above publications:

- The content of Section 2.1 and Sections 2.4-2.8 in Chapter 2 as well as the content of Sections 3.3-3.6 in Chapter 3 have been reproduced more or less verbatim from [1], with the permission of AIP Publishing.
- The content of Section 2.2, Sections 3.1-3.2 and Section 6.3 as well as certain parts of Section 2.3 and 5.5 have been reproduced from [5].
- The content of Chapter 4 has been reproduced from [2], with the permission of the American Physical Society.
- The content of Sections 5.1-5.4 and Section 5.8 in Chapter 5 has been reproduced from [3], with the permission of Springer-Nature.
- The content of Section 6.1 and Sections 6.4-6.7 in Chapter 6 has been reproduced from [4], with the permission of Springer Nature.

Acknowledgements

At this point, I would like to take the opportunity to thank all the people who accompanied and supported me on the way to my doctorate.

First and foremost, I would like to thank my supervisor Hanno Sahlmann. His invaluable support during all the stages of my doctorate has contributed significantly to the success of this work. In particular, I want to thank him for always taking time to answer all of my questions both on and beyond the doctorate. He has always been available for all of my concerns and especially so when things got though. Furthermore, I would like to thank him for his support for the successful application for a PhD scholarship, as well as for his support with the seminar talks where I presented the results.

I wish to thank Thomas Thiemann and Jerzy Lewandowski for the valuable discussions during and outside my seminar talks where I presented parts of my work. In addition, I would like to thank them as well as Robert Helling for their support with the applications for a postdoctoral position and for writing recommendation letters.

I would like to thank Lee Smolin and Tilman Wurzbacher for helpful correspondence at an early stage of this work. Furthermore, I would like to thank Enno Keßler for helpful discussions.

I want to thank the German Academic Scholarship Foundation (Studienstiftung des Deutschen Volkes) for intellectual and financial support without which this thesis could not have been written. In this context, I also would like to thank the Elite Graduate Program Theoretical and Mathematical Physics (TMP) and the Elite Network of the State of Bavaria. The unique orientation of this study program with its focus on topics in the intersection of mathematics and physics created ideal conditions to deal with the complex subject area which this present thesis is based on. I really profited a lot from it.

I wish to thank all the members and colleagues at the institute of quantum gravity at the FAU Erlangen-Nuremberg for the wonderful time I had there and the numerous beautiful experiences. Furthermore, I would like thank Madeleine for helping with the proofreading of the thesis.

Finally, my deepest thanks go to my family and especially my parents for their constant and unconditional support throughout the entire time of my studies and doctorate. Without you, all of this would not have been possible!

1. Introduction

1.1. Quantum gravity

According to our current understanding of the fundamemtal laws of nature, the physics of the macro- and microcosm is described in terms of two fundamental theories namely the *general theory of relativity* (or general relativity for short) and *quantum field theory* (QFT), respectively. General relativity, gradually discovered by Einstein, and published in its final form in 1915, led to a completely new understanding of gravity. It is based on the revolutionary idea that gravity is not just a force but the incarnation of geometry of space and time. Despite its huge successes and numerous experimental verifications inter alia achieved over the past decade, there also exist various phenomena which indicate its incompleteness. For instance, as an immediate consequence of the theory, singularities appear in the interior of black holes which turn out to be physically inconsistent. Moreover, cosmological models unavoidably lead to infinite energy densities as one follows the evolution equations of general relativity backwards in time from the present state of the universe. Since the curvature radius close to the singularities becomes smaller than Planck length

$$l_p = \sqrt{\frac{\hbar G}{c^3}} \tag{1.1}$$

this strongly suggests, that such a theory needs a unification of both general relativity and quantum theory, that is, a *quantum theory of gravity*.

One possible candidate for such a theory is string theory. According to string theory, it is expected that all fundamental particles can be understood as certain excitations of one-dimensional strings. Also the graviton, i.e. the hypothetical fundamental particle mediating the gravitational interaction, is part of the spectrum of closed strings. Hence, this gave a first hint towards unification of the four fundamental forces of nature. However, it turns out that a consistent theory incorporating fermionic particle species and excluding negative mass states requires the incorporation of *supersymmetry* (SUSY) and even higher spacetime dimensions. As a consequence, it follows from supersymmetry that there exist five possible versions of a superstring theory, two of them, called IIA- and IIB-superstring theory, containing the same low-energy particle spectrum as certain one-dimensional compactifications of the unique maximal 11-dimensional supergravity theory. Moreover, there are various duality relations connecting the different string theories. Since its original discovery, superstring theory attained a lot of interest and many intriguing results have been achieved such as a consistent microscopic description of the entropy of supersymmetric (charged) black holes [6–13] or a possible concrete realization of the holographic principle via the famous AdS/CFT-correspondence [14–16].

Another approach towards the formulation of a theory of quantum gravity which the present thesis will be focused on is *loop quantum gravity* (LQG) (see e.g. [17, 18] and references therein). Loop quantum gravity is a program originally based on canonical quantization of variables introduced by Sen, Ashtekar, Immirzi and Barbero [19–22] for Einstein gravity. These variables have the remarkable property that they embed the phase space of gravity in that of Yang-Mills theory. It was pointed out in [23] that all these variables can be obtained from an action that differs from the Palatini action of first-order Einstein gravity by a certain topological term defined by an operator on the Lie algebra of the structure group. This modification of the gravitational action is thus one of the foundations of the theory. While LQG is much less ambitious then string theory in terms of unification, it has very interesting results to its credit, such as a kinematical representation that carries a unitary representation of spatial diffeomorphisms explicitly incorporating the important principle of background independence in general relativity [17, 24–27], quantization of spatial geometry [28–32], as well as a path integral formulation in terms of so-called spin foams (see, e.g., [33, 34] and references therein). Moreover, within this theory, a consistent microscopic description of the black hole entropy for various physical four-dimensional (charged) black holes has been achieved [35–45]. Finally, adapting techniques from the full theory to a symmetry reduced setting, dynamical cosmological [46–48] (see also references in [49–51]) and black hole models [52–58] have been developed that are able to resolve the singularities one encounters in the classical theory.

1.2. The topic of this thesis

The present thesis is devoted to the question of how to bring together the two different approaches towards a formulation of a quantum theory of gravity, namely superstring theory and LQG, and to relate results and ideas achieved in these different theories. To this end, we will combine standard quantization techniques of LQG with the concept of supersymmetry. This also brings LQG closer to ideas of unification. In fact, even without consideration of string theory, there are hints that a unification of the gauge theory sector and gravity necessitate some form of SUSY. Supersymmetry is a new kind of symmetry that arose in the context of the famous results of Coleman-Mandula [59] and Haag-Lopuszanski-Sohnius [60] who were looking for symmetries of interacting QFTs that can have a nontrivial mixture with spacetime symmetries. As a consequence, it turns out that supersymmetries form a certain kind of Lie algebras $(\mathfrak{g}, [\cdot,\cdot])$ that carry an internal $\mathbb{Z}_2$-grading $\mathfrak{g}=\mathfrak{g}_0\oplus\mathfrak{g}_1$ also called *Lie superalgebras* that split into even and odd part $\mathfrak{g}_0$ and $\mathfrak{g}_1$ also referred to as the *bosonic* and *fermionic* part of $\mathfrak{g}$, respectively, where generators of the latter turn out transform as spin-$\frac{1}{2}$ fermions. Consequently, according to Wigner's classification theorem (see for instance [61] for a mathematical sophisticated approach), this implies that QFTs incorporating this new kind of symmetry necessarily need to contain an equal number of bosonic and fermionic degrees of freedom. Thus, supersymmetry yields a unified description of bosonic and fermionic particle species, that is, both force and matter particles, and therefore seems to be a natural candidate for the search for a unified field theory.

Combining the principle of local supersymmetry with gravity leads to *supergravity* (SUGRA). As mentioned above, supergravity theories also naturally arise in terms of low-energy limits of certain superstring theories. In the framework of LQG, the study of supergravity theories also has a long history. For instance, in [62], Jacobson introduced a chiral variant of the real $\mathcal{N} = 1$ Poincaré supergravity action using Ashtekar's self-dual connection variables which soon after has been extended by Fülöp [63] to anti-de Sitter supergravity including a cosmological constant. Canonical supergravity with *real* Asthekar-Barbero has been considered in [64, 65]. Generalizations to higher spacetime dimensions have been studied by Bodendorfer et al. [66–69] introducing new kind of variables different but similar to those of Ashekar-Barbero as usually applied in LQG in the context of four spacetime dimensions.

In this thesis, we want to investigate various physical and mathematical aspects of classical supergravity and study applications in the framework of LQG and loop quantum cosmology (LQC). To this end, we will focus on (extended) supergravity theories in $D = 4$ spacetime dimensions. In particular, we are interested in a reformulation of the corresponding canonical theory such that it preserves as much as possible the geometrical structure underlying SUGRA. In this context, we will also talk about an appropriate description of boundary theories in the framework of supergravity which are compatible with the principle local supersymmetry.

For this purpose, in Chapter 3, we will study a mathematically rigorous approach towards *geometric supergravity*. In fact, supergravity turns out to have a very intriguing geometrical interpretation allowing to store all the physical degrees of freedom of the theory in a single connection. As a consequence, SUGRA attains a structure quite similar to Yang-Mills gauge theories. This is the starting point of the so-called *group geometric approach* to SUGRA initiated by Ne'eman and Regge [70] and further developed by Castellani-D'Auria-Fré [71, 72] to include extended and higher dimensional supergravity theories. In the following, we want to study this approach in a mathematically rigorous manner using and extending tools in supergeometry discussed in detail in Chapter 2 and which results have been published in [1]. In particular, we will introduce an appropriate notion of a *super Cartan geometry* that consistently incorporates the anticommutative nature of (classical) fermionic fields which turns out to be a crucial property in the context of supergravity. In fact, as we will see, in order to resolve the fermionic degrees of freedom, this requires the inclusion of an additional parametrizing supermanifold leading to the concept of so-called *relative supermanifolds*. This is based on an idea first formulated by Schmitt in [73] and developed more systematically in [74, 75]. Interestingly, as will be explained in detail in Section 3.6, within this formalism, it turns out that one can provide a concrete link to the description of (classical) anticommutative fermionic fields in perturbative algebraic QFT (pAQFT) [76, 77].

Furthermore, working in the category of relative supermanifolds, we will construct and analyze the parallel transport map associated to super connection 1-forms defined on parametrized principal super fiber bundles. Moreover, explicit expressions of this map will be derived which will be useful for concrete physical applications to be discussed in Chapter 5 as well as Chapter 6. Finally, the induced parallel transport map corresponding to induced covariant derivatives on associated super vector bundles will be discussed. In this context, we will also relate our results to similar constructions on super vector bundles in the algebro-geometric approach in [78, 79].

We then turn next towards applications of these methods to loop quantum supergravity (LQSG). To this end, in Chapter 4, we will first address the canonical analysis of $D = 4$, $\mathcal{N} = 1$ Poincaré supergravity using real Ashtekar-Barbero variables starting with the corresponding Holst action of supergravity as first introduced by Tsuda [65]. In particular, we will work with half-densitized fermionic fields. Furthermore, a compact expression of the so-called *SUSY constraint* will be derived. The SUSY constraint plays a major role in canonical supergravity theories, akin to the role of the Hamiltonian constraint in non-supersymmetric generally covariant theories governing the dynamics of the theory. The canonical analysis of Poincaré supergravity with real Asthekar-Barbero-variables has been studied the first time in [64, 65]. However, these considerations did not include a full consistent treatment of half-densitized fermionic fields as proposed by Thiemann in [80] in order to solve the reality conditions to be satisfied by the Rarita-Schwinger field. The canonical analysis in arbitrary higher spacetime dimensions has been considered in [67]. But, for $D = 4$, the variables used there turn out to be different from the standard ones usually applied in LQG.

We will then devote ourselves to the proper implementation of the SUSY constraint operator in the quantum theory which, so far, has not been considered in the literature. To this end, we will propose a specific regularization procedure adapted to the classical SUSY constraint and derive for the first time a compact expression of the corresponding constraint operator using loop quantum gravity methods. Moreover, explicit expressions of the action of the resulting operator will be obtained. This is important as it is the first step on the way of analyzing the Dirac algebra generated by supersymmetry and Hamiltonian constraint in the quantum theory and for finding physical states in the full theory. We also discuss some qualitative properties of such solutions of the SUSY constraint. The results have been published in [2].

Chapter 5 is then devoted to classical and quantum description of (extended) anti-de Sitter supergravity theories in $D = 4$ spacetime dimensions. In particular, in this context, we will address the question of how to properly include boundary terms to the theory that are compatible with local supersymmetry at the boundary. This is important as this provides a first step towards the description of inner boundaries in the framework of LQSG as well as applications in the quantum description of supersymmetric black holes. This, among other things, may open up the possibility to compare results of entropy computations in LQG and superstring theory and thus to gain deeper insights into the relationship between these different approaches. To this end, adapting techniques developed in [81–83] to the case of a finite Barbero-Immirzi parameter β, we will consider the most general ansatz of boundary terms that are compatible with the symmetries of the bulk Lagrangian and such that the full theory is invariant under SUSY at the boundary. In this way, it turns out that the boundary term, at least in the cases of $\mathcal{N}=1$ and $\mathcal{N}=2$ extended SUGRA, is in fact uniquely fixed by this requirement and the resulting action of the full theory acquires an intriguing structure taking the form a Yang-Mills-type action where the contraction over internal indices of the structure group given by the $\mathcal{N}$-extended super anti-de Sitter group $\mathrm{OSp}(\mathcal{N}|4)$ is carried out via a *β-deformed inner product*. The results have been published in [3].

We then also consider the chiral limit of the theory corresponding to a purely imaginary Barbero-Immirzi parameter $\beta=\pm i$. This limit turns out have interesting properties such as, in particular, the manifest invariance of the resulting action under an enlarged gauge symmetry given by the orthosymplectic supergroup $\mathrm{OSp}(\mathcal{N}|2)_{\mathbb{C}}$ for both cases $\mathcal{N}=1$ and $\mathcal{N}=2$. In particular, it follows that the boundary theory which, as explained above, is uniquely fixed by the requirement of SUSY invariance at the boundary, takes the form of a super Chern-Simons action with gauge supergroup $\mathrm{OSp}(\mathcal{N}|2)_{\mathbb{C}}$. Moreover, the equations of motion (EOM) of the full theory yield boundary conditions coupling bulk and boundary degrees of freedom which turn out to be in complete analogy to the classical bosonic theory.

For $\mathcal{N}=1$ and without consideration of the boundary theory, the existence of an enlarged gauge symmetry of the chiral theory has been first observed by Fülöp in the seminal paper [63] while studying the constraint algebra generated by the Gauss and left SUSY constraint. In this way, it turns out that the constraint algebra has the structure of a graded Lie algebra leading to some kind of a graded generalization of Ashtekar's self-dual variables also called the *super Asthekar connection*. Using the Cartan geometric description of AdS supergravity, we will provide a conceptual and geometric explanation of the observations of [63] studying the chiral structure of the underlying super anti-de Sitter algebra yielding an interpretation of the super Ashtekar connection in terms of a *generalized super Cartan connection*. Using this connection, this paves the way towards a new approach to non-perturbative quantum supergravity in which parts of SUSY as well as the underlying geometrical structure of covariant SUGRA are kept manifest. In fact, as seen in Chapter 4, the canonical formulation of SURGA theories using real variables generically yield very complicated constraints which themselves, when going over to the quantum theory, are plagued by quantization ambiguities. This also makes any attempts to compare LQG with other approaches to quantum gravity much more difficult. However, this turns out to be resolved, at least partially, in the context of the chiral theory since, by quantizing this theory adapting tools of standard LQG, the left-handed part of the SUSY constraint is already implemented in a manifest way by simply imposing gauge invariance.

Explicitly making use of the gauge-theoretic structure of the canonical chiral theory, we will derive a *graded* analog of the classical *holonomy-flux algebra* in a mathematically rigorous way. To this end, we will, in particular, employ the parallel transport map as constructed in Chapter 2 induced by the super Ashtekar connection. As a consequence, in order to consistently incorporate the anticommutative nature of the fermionic fields, besides embedded graphs, it follows that the corresponding inductive family is labeled by an additional parametrization supermanifold. These results provide a mathematically consistent framework to study the manifest approach to loop quantum supergravity. In fact, existing results in this direction [63, 84–86] are rather formal and do not take into account the issue of how to consistently model the anticommuative nature of the (classical) fermionic fields. Based on these observations, we will then sketch the quantization of the theory adapting techniques from standard LQG. As we will see, the resulting kinematical state space carries a structure which shares many similarities with the kinematical Hilbert space obtained via the standard quantization scheme in LQG coupled to fermions [67, 80, 87].

In Chapter 6, we will finally go over to the application of the results obtained in the previous chapter in the framework of spatially symmetry reduced models and quantum cosmology which results have been published in [4]. For this purpose, we will first develop the theory of symmetry reduced super connections forms providing a general scheme towards symmetry reduction of supersymmetric field theories with local gauge symmetry associated to a gauge supergroup. These methods will then be used in the context of chiral $\mathcal{N}=1$ supergravity. More precisely, we exploit the enlarged $\mathrm{OSp}(1|2)_{\mathbb{C}}$-gauge symmetry of the theory and derive a general class of homogenenous and isotropic super connection forms. In fact, the enlarged gauge symmetry turns out to be crucial to allow for isotropic connections that contain nontrivial fermionic contributions. Moreover, the fermionic part of the connection turns out to coincide with the ansatz as derived by other means by D'Eath et al. in [88–90].

These results will then be used to derive symmetry reduced expressions for the constraints of the canonical chiral theory and study the constraint algebra. Moreover, mimicking the standard procedure in loop quantum cosmology and using the explicit form of the super holonomies as derived in Chapter 2, we will motivate the graded holonomy-flux algebra of the symmetry reduced classical theory. In this context, it follows, using the symmetry reduced form of the reality conditions, that the graded algebra can be equipped with a consistent *-relation. The quantum theory is then constructed choosing a *Ashtekar-Lewandowski-type representation* of the *-algebra. As it turns out, requiring the representation to define an even morphism of graded *-algebras already fixes uniquely the inner product on the graded kinematical Hilbert space extending and generalizing results obtained in the context of the purely bosonic theory in [91]. We will then finally also study the dynamics of the resulting quantum theory. In particular, for a specific subclass of the symmetry reduced models, we will show explicitly that the essential part of the quantum constraint algebra exactly reproduces the classical Poisson relations. More precisely, we will show that the anticommutator between the so-called left and right SUSY constraint operator exactly reproduces the Hamiltonian constraint operator. As a last step, we will consider the semi-classical limit of the theory and compare the results with those obtained by different means in [88–90].

In the following, the thesis is subdivided into two parts: The first two chapters deal with the mathematical rigorous approach toward geometric supergravity and, in this context, introduce essential mathematical methods such as the category of relative supermanifolds as well as bundles, connection forms and Cartan geometries defined in this category. Moreover, the parallel transport map will be constructed. The second part, treated in the Chapters 4-6, then focuses on physical applications in the framework of loop quantum supergravity. There, many mathematical details will be dropped in order to simplify the notation and to make it easier accessible for the reader. In particular, we will not explicitly mention the underlying parametrization supermanifold except in Section 5.5 as well as Sections 6.3 and 6.5.1 in the context of the construction of the graded holonomy-flux algebra and the symmetry reduction of chiral supergravity where the parametrization turns out be essential.

A list of important symbols as well as an overview of our choice of conventions concerning indices, physical constants etc. used in the main text can be found in the List of symbols, notations and conventions.

2. Supergeometry

2.1. Introduction

Over the last fifty years, many different approaches have been developed in order to formulate the notion of a supermanifold. The first and probably most popular one is the so-called *algebro-geometric approach* introduced by Berezin, Kostant and Leites [92, 93] which borrows techniques from algebraic geometry. It is based on the interesting observation that ordinary smooth manifolds can equivalently be described in terms of the structure sheaf of smooth functions defined on the underlying topological space. In this framework, it follows that supersmooth functions f are locally of the form

$$f = f_{\emptyset} + f_1\theta^1 + \ldots + f_n\theta^n + \ldots + f_{1\cdots n}\theta^1\cdots\theta^n \tag{2.1}$$

where θ^i for $i = 1, \ldots, n$ are anticommuting Grassmann variables and, for any ordered multi-index $\underline{I}$ of length $\leq n$, $f_{\underline{I}}$ is an ordinary smooth function. This approach is very elegant and, in particular, avoids the introduction of superfluous (unphysical) degrees of freedom. Nevertheless, its definition turns out to be very abstract since, roughly speaking, points in this framework are implicitly encoded in the underlying structure sheaf of supersmooth functions. This makes this approach less accessible for physicists for concrete applications.

Hence, another approach to supermanifolds, the so-called *concrete approach*, was initiated by DeWitt [94] and Rogers [95, 96], and studied even more systematically by Tuynman in [97], defining them similar to ordinary smooth manifolds in terms of a topological space of points, i.e., a topological manifold that locally looks a flat superspace (see Appendix C for a brief review). However, as it turns out, this definition has various ambiguities in formulating the notion of a point which, in contrast to the algebro-geometric approach, leads to too many unphysical degrees of freedom.

It was then found by Molotkov [98] and further developed by Sachse [99, 100] that both approaches can be regarded as two sides of the same coin. In that framework, at least in the finite-dimensional setting, it follows that Rogers-DeWitt supermanfolds can be interpreted in terms of a particular kind of a functor constructed out of a algebro-geometric supermanifold. This functorial intepretation then resolved the ambiguities arising in the Rogers-DeWitt approach and also opened the way towards a generalization of the theory to infinite-dimensional supermanifolds.

Another caveat, both in the algebraic and Rogers-DeWitt approach, is the appropriate description of anticommuting fermionic fields. In fact, it turns out that the pullback of superfields to the underlying ordinary smooth manifold are purely commutative (bosonic). In fact, roughly speaking, restricting a generic supersmooth function f

to the body of a supermanifold amounts to setting $\theta^i = 0$ in the expansion (2.1) so that f reduces to an ordinary smooth function $f_\emptyset$ which, in particular, is commutative (bosonic). This seems, however, incompatible in various constructions in physics. For instance, in the Castellani-D'Auria-Fré approach [71, 72], a geometric approach to supergravity (see Section 3.4), by the so-called *rheonomy principle*, physical degrees of freedom are supposed to be completey determined by their pullback.

Furthermore, as we will also see in Section 2.7.1, from a mathematical point of view, this issue also appears in the context of the parallel transport map corresponding to super connection 1-forms. In fact, it follows that in the ordinary category of supermanifolds, both in the algebraic and concrete approach, the parallel transport cannot be used in order to compare different fibers of the bundle, in contrast to the classical theory. A resolution has been proposed by Schmitt [73]. There, motivated by the Molotkov-Sachse approach to supermanifold theory [98, 99], superfields on parametrized supermanifolds are considered. Since, a priori, this additional parametrizing supermanifold is chosen arbitrarily, one then has to ensure that these superfields transform covariantly under change of parametrization. This idea has been studied rigorously for instance by Hack et. al. in [101] considering relative supermanifolds which are well-known in the algebraic approach [74] (see also [75]). As it turns out, superfields on these supermanifolds indeed have the required properties, i.e., in the sense of Molotkov-Sachse, they behave functorially under change of parametrization. Moreover, as we will show explicitly later in Section 2.2, in this framework, it follows that fermionic fields have the interpretation in terms of functionals on supermanifolds which is in strong similarity to other approaches such as in the context of pAQFT [76, 77].

In this chapter, we want to provide the mathematical rigorous foundations for the study of gauge theories on (relative) supermanifolds. In particular, we will study the parallel transport map corresponding to super connection forms defined on (relative) principal super fiber bundles. The parallel transport map, or the associated holonomies, have been considered in the context of covariant derivatives on super vector bundles in the algebro-geometric approach in [78, 79]. The theory of super fiber bundles and connection forms in the concrete approach has been developed in [97]. In the algebraic category, a precise definition of principal bundles and connection forms has been given in [102]. In what follows, we will generalize the considerations of [97] to the relative category and, in particular, define super connection forms on relative principal super fiber bundles. We will then use this formalism in order to construct the corresponding parallel transport map and study some of its important properties. Moreover, we will analyze the precise relation between the algebraic and concrete approach and show explicitly that both approaches are in fact equivalent. To this end, we will employ the functor of points technique which will be discussed in detail in Section 2.2. Moreover, studying the induced parallel transport map on associated super vector bundles, this enables us to compare the results with those obtained in [78, 79] in the algebraic setting. For an investigation of the geodesic flow on super Riemannian manifolds see [103].

The structure of this chapter is as follows: At the beginning, we will give a detailed introduction to the theory of supermanifolds and super Lie groups and establish a concrete link between these various approaches via the functor of points prescription. Then, in Section 2.4, we will summarize various important aspects of super fiber bundle theory in the concrete approach to supermanifold theory. To this end, we will mostly follow [97]. However, in contrast to [97], we will use the concept of *formal bundle atlases* which is very well-known in the classical theory (see e.g. [104, 105] and references therein) and turns out to be even applicable in the context of supermanifolds. In Section 2.5, we will then introduce the concept of relative supermanifolds and define principal connections and super connection one-forms. In Section 2.6, we will compare the theory of principal bundles and connection forms both in the algebraic and concrete approach and show that both approaches are equivalent. These results will then be used in the last Sections 2.7.1 and 2.7.2 in order to construct the parallel transport map. Moreover, for a particular subclass of super Lie groups, a concrete formula for this map will be derived making it easier accessible for physical applications.

A list of important symbols as well as an overview of our choice of conventions concerning indices, physical constants etc. can be found in the List of symbols, notations and conventions.

2.2. Three roads towards a theory of supermanifolds

2.2.1. Algebro-geometric supermanifolds

In the following, let us briefly review the basic definition of algebro-geometric supermanifolds. For a review of the Rogers-De Witt approach to supermanifold theory, we refer to Appendix C. The algebro-geometric approach is based on the observation that ordinary smooth manifolds can equivalently be described in terms of locally ringed spaces. To this end, one notes that any smooth manifold canonically yields the locally ringed space (M, C^{∞}_M) which is locally isomorphic to some $(V, C^{\infty}_{\mathbb{R}^n}|_V)$ with $V \subseteq \mathbb{R}^n$ open. In fact, it turns that all smooth manifold M can be described this way. That is, if $(M, \mathcal{O}_M)$ is a locally ringed space with $\mathcal{O}_M$ a sheaf on M such that $(M, \mathcal{O}_M)$ is locally isomorphic to some $(V, C^{\infty}_{\mathbb{R}^n}|_V)$ with $V \subseteq \mathbb{R}^n$ open. Then, M can be given the structure of smooth manifold in a unique way such that $\mathcal{O}_M \cong C^{\infty}_M$. Even more, it follows that both categories are in fact equivalent (for a proof see, e.g., [117]).

Based on this idea, one defines supermanifolds as some sort of locally super ringed spaces generalizing appropriately the notion of a smooth function. Hence, a so-called *supersmooth function* or *superfield* f on the superspace $\mathbb{R}^{m|n} = \mathbb{R}^m \oplus \mathbb{R}^n$ is defined as a function of the form

$$f = \sum_{\underline{I}} f_{\underline{I}} \theta^{\underline{I}} \tag{2.2}$$

with $f_{\underline{I}}$ ordinary smooth functions on $\mathbb{R}^m$ for any *ordered* multi-index $\underline{I} = (i_1, \ldots, i_k)$ of length $0 \leq |\underline{I}| = k \leq n$ such that $i_1 < i_2 < \ldots < i_k$ where $\theta^{\underline{I}} := \theta^{i_1} \cdots \theta^{i_k}$ with θ^{i_j} being odd Grassmann-variables. In the following, we will follow very closely [106] for the definition of algebro-geometric supermanifolds and the construction of the functor of points (see also Appendix A for our choice of conventions in super linear algebra as well as Appendix B for summary of important aspects of category theory and algebraic geometry). Therefore, we will omit most of the proofs.

Definition 2.2.1. An *algebro-geometric supermanifold* of dimension (m, n) is a locally super ringed space $\mathcal{M} = (M, \mathcal{O}_M)$ that is locally isomorphic to the superspace $\mathbb{R}^{m|n}$. More precisely, $(M, \mathcal{O}_M)$ consists of a topological space M which is Hausdorff and second countable as well as a sheaf $\mathcal{O}_M$ over M of super commutative rings called *structure sheaf* such that, for any $x \in M$, the stalk $\mathcal{O}_{M,x}$ is a local super ring. Moreover, for $x \in M$, there exists an open neighborhood $U \subset M$ of x as well as an isomorphism $\phi_U = (|\phi_U|, \phi_U^\sharp)$ of ordinary locally ringed spaces

$$\phi_U = (|\phi_U|, \phi_U^\sharp) : (U, \mathcal{O}_M|_U) \to (|\phi_U|(U), C^\infty_{\mathbb{R}^m}|_{|\phi_U|(U)} \otimes \bigwedge [\theta^1, \ldots, \theta^n]) \tag{2.3}$$

such that $\phi_U^\sharp : C^\infty_{\mathbb{R}^m}|_{|\phi_U|(U)} \otimes \bigwedge [\theta^1, \ldots, \theta^n] \to \mathcal{O}_M|_U$, in addition, is an even morphism of sheaves of superalgebras. The tuple (U, ϕ_U) is called a local *chart* or *superdomain* around x. A family $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in \Upsilon}$ of charts is called an *atlas* of $(M, \mathcal{O}_M)$ if $\bigcup_{\alpha \in \Upsilon} U_\alpha = M$.

A morphism $f = (|f|, f^\sharp) : (M, \mathcal{O}_M) \to (N, \mathcal{O}_N)$ of algebro-geometric supermanifolds is a morphism of the underlying ordinary locally ringed spaces such that $f^\sharp : \mathcal{O}_N \to f_* \mathcal{O}_M$ also is an even morphism of superalgebras. Algebro-geometric supermanifolds together with morphisms between them form a category $\mathbf{SMan}_{\mathrm{Alg}}$ called the *category of algebro-geometric supermanifolds*.

Remark 2.2.2. Choosing a chart (U, ϕ_U) of an algebro-geometric supermanifold $(M, \mathcal{O}_M)$, this induces local coordinates $(t^{\sharp i}, \theta^{\sharp j})$ on $\mathcal{M}|_U := (U, \mathcal{O}_M|_U)$ via $t^{\sharp i} := \phi_U^\sharp(t^i)$ and $\theta^{\sharp j} := \phi_U^\sharp(\theta^j)$ $\forall i = 1, \ldots, m$, $j = 1, \ldots, m$ where $\dim (M, \mathcal{O}_M) = (m, n)$. Moreover, any function $f \in \mathcal{O}_M|_U$ is of the form

$$f = \sum_{\underline{I}} f_{\underline{I}} \theta^{\sharp \underline{I}} \tag{2.4}$$

where, for any ordered mulit-index $\underline{I} = (i_1, \ldots, i_k)$ of length $0 \leq k \leq n$, $\theta^{\sharp \underline{I}} := \theta^{\sharp i_1} \cdots \theta^{\sharp i_k}$ and $f_{\underline{I}} = \phi_U^{\sharp}(g_{\underline{I}})$ for some smooth function $g_{\underline{I}} \in C^{\infty}(|\phi_U|(U))$.

Any supermanifold naturally contains an ordinary smooth manifold as a submanifold. To see this, for any algebro-geometric supermanifold $\mathcal{M} = (M, \mathcal{O}_M)$ and $U \subset M$ open, consider the set $\mathcal{J}_M(U) := \{f \in \mathcal{O}_M(U) | f \text{ is nilpotent}\}$. It then follows that $\mathcal{J}_M(U)$ is an ideal in $\mathcal{O}_M(U)$ yielding another sheaf $U \mapsto \mathcal{J}_M(U)$. Hence, one can construct the quotient sheaf $\mathcal{O}_M/\mathcal{J}_M$ whose sections locally have the structure of an ordinary smooth functions. This yields a locally ringed space

$$M_0 := (M, \mathcal{O}_M/\mathcal{J}_M) \tag{2.5}$$

which is a submanifold and has the structure of an ordinary smooth manifold. Before we continue, let us mention a central result in the theory of algebro-geometric supermanifolds as it will appear quite frequently in the discussion in what follows. It states that morphisms are uniquely characterized via the pullback of a basis of global sections. To this end, recall that, for a section $f \in \mathcal{O}(\mathcal{M}) := \mathcal{O}_M(M)$, the *value* $f(x) \equiv \mathrm{ev}_x(f)$ of f at $x \in M$ is defined as the unique real number such that $f - f(x)$ is not invertible in any open neighborhood of x in M. This induces a morphism $\mathrm{ev}_x \in \mathrm{Hom}_{\mathbf{SAlg}}(\mathcal{O}(\mathcal{M}), \mathbb{R})$ defined as

$$\mathrm{ev}_x(f) := f(x),\ \forall f \in \mathcal{O}(\mathcal{M}) \tag{2.6}$$

called the *evaluation morphism at* $x \in M$.

Theorem 2.2.3 (Global Chart Theorem [106])**.** *Let $\mathcal{M} = (M, \mathcal{O}_M)$ be an algebro-geometric supermanifold and $\mathcal{U}^{m|n} = (U, C^{\infty}_U) \subseteq \mathbb{R}^{m|n}$ be a superdomain with $U \subseteq \mathbb{R}^m$ open. There is a bijective correspondence between supermanifold morphisms $\psi : \mathcal{M} \to \mathcal{U}^{m|n}$ and tuples $(t^{\sharp i}, \theta^{\sharp j})$ of global sections of $\mathcal{O}_M$ with $t^{\sharp i}$ even and $\theta^{\sharp j}$ odd, $i = 1, \ldots, m$ and $j = 1, \ldots, n$, such that $(t^{\sharp 1}(x), \ldots, t^{\sharp m}(x)) \in U\ \forall x \in M$.*

Proof. It follows that, if $\phi : \mathcal{M} \to \mathcal{N}$ is a morphism of supermanifolds, then $\phi^{\sharp}(g)(y) = g(|\phi|(y))$ for any $g \in \mathcal{O}(\mathcal{N})$ and $y \in N$ where $\mathcal{N} = (N, \mathcal{O}_N)$. It is clear, by restricting the global sections t^i and θ^j of $\mathbb{R}^{m|n}$ to the superdomain $\mathcal{U}^{m|n}$, that their respective pullback $t^{\sharp i} := \psi^{\sharp}(t^i)$ and $\theta^{\sharp j} := \psi^{\sharp}(\theta^j)$ w.r.t. a morphism $\psi : \mathcal{M} \to \mathcal{U}^{m|n}$ indeed satisfy the properties as stated in the theorem. The inverse direction follows from the local triviality property of supermanifolds. □

Example 2.2.4 (The split functor (see also Example C.11))**.** Typical examples of a supermanifolds are obtained via their strong relationship to vector bundles. Let $V \to E \xrightarrow{\pi}$

M be a real vector bundle over an m-dimensional manifold with typical fiber given by a vector space V of dimensional n. This naturally yields a locally ringed space setting

$$\mathbf{S}(E,M) := (M,\Gamma(\bigwedge E^*)) \tag{2.7}$$

where $\Gamma(\bigwedge E^*)$ denotes the space of of smooth sections of the exterior bundle $\bigwedge E^*$. Since $\Gamma(\bigwedge E^*) \cong \bigwedge\Gamma(E)^*$ naturally carries a $\mathbb{Z}_2$-grading, it follows that it has the structure of a sheaf of local super rings, that is, $\mathbf{S}(E,M)$ defines an algebro-geometric supermanifold of dimension (m,n) also called a *split supermanifold*. A morphism $(\phi,f):(E,M)\rightarrow(F,N)$ between two vector bundles induces a morphism $\mathbf{S}(\phi,f):\mathbf{S}(E,M)\rightarrow\mathbf{S}(F,N)$ between the corresponding split supermanifolds. Hence, this yields a functor

$$\mathbf{S}:\ \mathbf{Vect}_{\mathbb{R}}\rightarrow\mathbf{SMan}_{\mathrm{Alg}} \tag{2.8}$$

from the category of real vector bundles to the category of algebro-geometric supermanifolds which we call the *split functor*. In case that the vector bundle $(E,M)=(M\times V,M)$ is trivial, $\mathbf{S}(E,M)$ will be called *globally split* and we also simply write $\mathbf{S}(V,M)\equiv\mathbf{S}(E,M)$. It is a general result due to Batchelor [107] that any algebro-geometric supermanfold is isomorphic to a split supermanifold of the form (2.7), i.e., (2.8) is surjective on objects. However, the split functor is *not* full, i.e., not every morphism $f:\ \mathbf{S}(E,M)\rightarrow\mathbf{S}(F,N)$ between split manifolds arises from a morphism between the respective vector bundles $(E,M),(F,N)\in\mathbf{Ob}(\mathbf{Vect}_{\mathbb{R}})$. Hence, the structure of morphisms between supermanifolds, in general, turns out to be much richer than for ordinary vector bundles. This is of utmost importance in modelling for instance supersymmetry transformations to be discussed in Section 3.4.

As a next step, we want to describe a relation between algebro-geometric and Rogers-DeWitt supermanifolds. A very elegant way in describing this relationship is given by the so-called *functors of point* approach. It is a general technique in algebraic geometry which can be used in order to give, a priori, very abstract objects a more concrete reinterpretation making proofs much easier in certain instances. As explained already in the introduction to this chapter, a general "issue" concerning algebro-geometric supermanifolds is the lack of points. In fact, in contrast to ordinary smooth manifolds, the points of the underlying topological space do not suffice to uniquely characterize the sections of the structure sheaf. As it turns out, this can be cured by studying the morphisms between them.

Definition 2.2.5. Let $\mathcal{M}$ be an algebro-geometric supermanifold. The *functor of points of* $\mathcal{M}$ is defined as the covariant functor $\mathcal{M}:\ \mathbf{SMan}_{\mathrm{Alg}}^{\mathrm{op}}\rightarrow\mathbf{Set}$ on the opposite category $\mathbf{SMan}_{\mathrm{Alg}}^{\mathrm{op}}$ associated to $\mathcal{M}$ which on objects $\mathcal{T}\in\mathbf{Ob}(\mathbf{SMan}_{\mathrm{Alg}})$ is defined as

$$\mathcal{M}(\mathcal{T}) := \mathrm{Hom}(\mathcal{T},\mathcal{M}) \tag{2.9}$$

also called the $\mathcal{T}$*-point of* $\mathcal{M}$ and for morphisms $f \in \mathrm{Hom}(\mathcal{T},\mathcal{S})$, the corresponding morphism $\mathcal{M}(f) \in \mathrm{Hom}(\mathcal{M}(\mathcal{S}),\mathcal{M}(\mathcal{T}))$ is given by

$$\mathcal{M}(f): \mathcal{M}(\mathcal{S}) \to \mathcal{M}(\mathcal{T}),\ g \mapsto g \circ f \tag{2.10}$$

Hence, the functor of points of $\mathcal{M}$ coincides with the *partial* Hom-*functor* $h_{\mathcal{M}} := \mathrm{Hom}(\mathcal{M},\cdot)$ on $\mathbf{SMan}_{\mathrm{Alg}}^{\mathrm{op}}$.

If $\phi: \mathcal{M} \to \mathcal{N}$ is a morphism between algebro-geometric supermanifolds, this yields a map $\phi_{\mathcal{T}}: \mathcal{M}(\mathcal{T}) \to \mathcal{N}(\mathcal{T})$ between the associated $\mathcal{T}$-points by setting $\phi_{\mathcal{T}}(f) := \phi \circ f$ $\forall f \in \mathcal{M}(\mathcal{T}) = \mathrm{Hom}(\mathcal{T},\mathcal{M})$. By definition, it then follows that for any morphism $f: \mathcal{S} \to \mathcal{T}$ one has

$$\mathcal{N}(f) \circ \phi_{\mathcal{T}}(g) = \phi_{\mathcal{T}}(g) \circ f = \phi \circ g \circ f = \phi_{\mathcal{S}} \circ \mathcal{M}(f)(g) \tag{2.11}$$

$\forall g \in \mathcal{M}(\mathcal{T})$, that is, the following diagram is commutative

$$\begin{array}{ccc} \mathcal{M}(\mathcal{T}) & \xrightarrow{\mathcal{M}(f)} & \mathcal{M}(\mathcal{S}) \\ \downarrow{\scriptstyle \phi_{\mathcal{T}}} & & \downarrow{\scriptstyle \phi_{\mathcal{S}}} \\ \mathcal{N}(\mathcal{T}) & \xrightarrow{\mathcal{N}(f)} & \mathcal{N}(\mathcal{S}) \end{array}$$

Hence, a morphism $\phi: \mathcal{M} \to \mathcal{N}$ induces a natural transformation between the associated functor of points. This poses the question whether all natural transformations arise in this way. This is an immediate consequence of the following well-known lemma.

Lemma 2.2.6 (Yoneda Lemma). *Let C be a category and $F: C \to \mathbf{Set}$ be a functor. Then, for any object $X \in \mathbf{Ob}(C)$, the assignment $\eta \mapsto \eta_X(\mathrm{id}_X)$ yields a bijective correspondence between natural transformations $\eta: \mathrm{Hom}(X,\cdot) \to F$ and the set $F(X) \in \mathbf{Ob}(\mathbf{Set})$.*

Applied to our concrete situation, this implies that for the functor of points $h_{\mathcal{M}}: \mathbf{SMan}_{\mathrm{Alg}}^{\mathrm{op}} \to \mathbf{Set}$ and $h_{\mathcal{N}}: \mathbf{SMan}_{\mathrm{Alg}}^{\mathrm{op}} \to \mathbf{Set}$ associated to algebro-geometric supermanifolds $\mathcal{M}$ and $\mathcal{N}$, one has a bijective correspondence between natural transformations between $h_{\mathcal{M}}$ and $h_{\mathcal{N}}$ and elements in $\mathrm{Hom}(\mathcal{N},\cdot)(\mathcal{M}) = \mathrm{Hom}(\mathcal{M},\mathcal{N})$. In particular, the supermanifolds $\mathcal{M}$ and $\mathcal{N}$ are isomorphic iff the associated functor of points are naturally isomorphic.

We next want to find an equivalent description of the $\mathcal{T}$-points of an algebro-geometric supermanifold $\mathcal{M}$ purely in terms of global sections of the structure sheaf $\mathcal{O}_M$. Consider therefore the set $\mathrm{Spec}_{\mathbb{R}}(\mathcal{O}(\mathcal{M})) := \mathrm{Hom}_{\mathbf{SAlg}}(\mathcal{O}(\mathcal{M}),\mathbb{R})$ called the *real spectrum of* $\mathcal{O}(\mathcal{M}) = \mathcal{O}_M(M)$. Since, a morphism $\phi: \mathcal{O}(\mathcal{M}) \to \mathbb{R}$ in the real spectrum is always surjective, it follows that the kernel $\ker(\phi)$ yields a maximal ideal in $\mathcal{O}(\mathcal{M})$, i.e., an element of the *maximal spectrum*

$$\mathrm{MaxSpec}_{\mathbb{R}}(\mathcal{O}(\mathcal{M})) := \{I \subset \mathcal{O}(\mathcal{M}) | \, I \text{ is a maximal ideal}\} \tag{2.12}$$

In fact, it follows that all maximal ideals in $\mathcal{O}(\mathcal{M})$ are of this form. This is a direct consequence of the super version of the classical "Milnor's exercise" [106].

Proposition 2.2.7 (Super Milnor's exercise)**.** *For an algebro-geometric supermanifold $\mathcal{M}$ all the maximal ideals in $\mathcal{O}(\mathcal{M})$ are of the form $\mathfrak{J}_x := \ker(\mathrm{ev}_x : \mathcal{O}(\mathcal{M}) \rightarrow \mathbb{R})$ for some $x \in M$, where $\mathrm{ev}_x \in \mathrm{Hom}_{\mathbf{SAlg}}(\mathcal{O}(\mathcal{M}), \mathbb{R})$ is the evaluation morphism at x (Eq. (2.6)).*

Proof. Let $I \subset \mathcal{O}_M$ be a maximal ideal. On $M_0 := (M, \mathcal{O}_M/\mathcal{J}_M)$ consider the the subset $j^\sharp(I) \subseteq C^\infty(M_0)$ of $C^\infty(M_0)$, where $j^\sharp : \mathcal{O}_M \rightarrow \mathcal{O}_M/\mathcal{J}_M \cong C^\infty_{M_0}$ is the pullback of the canonical embedding $j : M_0 \hookrightarrow \mathcal{M}$. Since $j^\sharp$ is a surjective morphism of super rings and $1 \notin I$, it follows that $j^\sharp(I)$ is a maximal ideal in $C^\infty(M_0)$. By the classical Milnor's exercise, we thus have $j^\sharp(I) := \ker(\mathrm{ev}_x : C^\infty(M_0) \rightarrow \mathbb{R})$ for some $x \in M$. Hence, $I \subseteq \mathfrak{J}_x$ implying $I = \mathfrak{J}_x$ by maximality of I. □

Hence, according to Prop. 2.2.7, we are allowed to identify the real spectrum with $\mathrm{MaxSpec}_{\mathbb{R}}(\mathcal{O}(\mathcal{M}))$ and even obtain a bijection $\Upsilon : M \xrightarrow{\sim} \mathrm{Spec}_{\mathbb{R}}(\mathcal{O}(\mathcal{M}))$ via

$$M \ni x \overset{\sim}{\mapsto} (\mathrm{ev}_x \colon \mathcal{O}(\mathcal{M}) \rightarrow \mathbb{R}) \in \mathrm{Spec}_{\mathbb{R}}(\mathcal{O}(\mathcal{M})) \overset{\sim}{\mapsto} \ker(\mathrm{ev}_x) \in \mathrm{MaxSpec}_{\mathbb{R}}(\mathcal{O}(\mathcal{M})) \tag{2.13}$$

We want to define a topology on $\mathrm{Spec}_{\mathbb{R}}(\mathcal{O}(\mathcal{M}))$ such that Υ becomes a homeomorphism. To this end, note that any section $f \in \mathcal{O}(\mathcal{M})$ canonically induces a morphsim $\phi_f : \mathrm{Spec}_{\mathbb{R}}(\mathcal{O}(\mathcal{M})) \rightarrow \mathbb{R}$ by setting

$$\phi_f(\mathrm{ev}_x) := \mathrm{ev}_x(f) = f(x) \tag{2.14}$$

Hence, let us endow $\mathrm{Spec}_{\mathbb{R}}(\mathcal{O}(\mathcal{M}))$ with the *Gelfand topology* which is defined as the coarsest topology such that the maps ϕ_f for all $f \in \mathcal{O}(\mathcal{M})$ are continuous. A basis of this topology is generated by open subsets of the form

$$\phi_f^{-1}(B_\epsilon(x_0)) = \{\mathrm{ev}_x \in \mathrm{Spec}_{\mathbb{R}}(\mathcal{O}(\mathcal{M})) | \, |(\mathrm{ev}_x - \mathrm{ev}_{x_0})(f)| < \epsilon\} \tag{2.15}$$

for some $f \in \mathcal{O}(\mathcal{M})$ and $B_\epsilon(x_0) \subset M$ an open ball of radius ϵ around $x_0 \in M$. It then follows immediately that the map $\Psi : M \xrightarrow{\sim} \mathrm{Spec}_{\mathbb{R}}(\mathcal{O}(\mathcal{M}))$ is continuous w.r.t. this topology, since

$$\Psi^{-1}(\phi_f^{-1}(B_\epsilon(x_0))) = |f|^{-1}(B_\epsilon(f(x_0))) \tag{2.16}$$

is open in M as $|f| : M \to \mathbb{R}$ is continuous. In fact, Ψ is even a homeomorphism. To see this, consider a closed subset $X \subseteq M$ and let $\mathfrak{p}_X$ be the ideal in $\mathcal{O}(\mathcal{M})$ defined as the set of all sections $f \in \mathcal{O}(\mathcal{M})$ vanishing on X. Using a partition of unity argument, it follows that

$$X = \{x \in M \,|\, f(x) = 0,\ \forall f \in \mathfrak{p}_X\} \tag{2.17}$$

and thus

$$\Psi(X) = \bigcap_{f \in \mathfrak{p}_X} \phi_f^{-1}(\{0\}) \tag{2.18}$$

i.e., $\Psi(X)$ is closed in $\mathrm{Spec}_{\mathbb{R}}(\mathcal{O}(\mathcal{M}))$ proving that Ψ is indeed a homeomorphism.

Theorem 2.2.8. *Let $\mathcal{M}$ and $\mathcal{N}$ be algebro-geometric supermanifolds. Then, their exists a bijective correspondence between the set* $\mathrm{Hom}(\mathcal{M}, \mathcal{N})$ *of morphisms of algebro-geometric supermanifolds and the set* $\mathrm{Hom}_{\mathbf{SAlg}}(\mathcal{O}(\mathcal{N}), \mathcal{O}(\mathcal{M}))$ *of superalgebra morphisms between the superalgebras of global sections of the respective structure sheaves.*

Sketch of Proof. One direction is immediate, i.e., that the pullback of a supermanifold morphism $\phi : \mathcal{M} \to \mathcal{N}$ induces a morphism $\phi^\sharp : \mathcal{O}(\mathcal{N}) \to \mathcal{O}(\mathcal{M})$ of the respective structure sheaves. The proof of the inverse direction uses a standard tool in algebraic geometry called *localization of rings*. See [106] for more details. □

Hence, according to this theorem, in the following, we will identify the $\mathcal{T}$-point $\mathcal{M}(\mathcal{T})$ of an algebro-geometric supermanifold $\mathcal{M}$ with $\mathrm{Hom}(\mathcal{O}(\mathcal{M}), \mathcal{O}(\mathcal{T}))$. For instance, let us consider the $\mathcal{T}$-points $\mathbb{R}^{m|n}(\mathcal{T}) = \mathrm{Hom}(\mathcal{O}(\mathbb{R}^{m|n}), \mathcal{O}(\mathcal{T}))$ of the superspace $\mathbb{R}^{m|n}$. By the Global Chart Theorem 2.2.3, this set can be identified with

$$\begin{aligned}\mathbb{R}^{m|n}(\mathcal{T}) &\cong \{(t^1, \ldots, t^m, \theta^1, \ldots, \theta^n) \,|\, t^i \in \mathcal{O}(\mathcal{T})_0,\ \theta^j \in \mathcal{O}(\mathcal{T})_1\} \\ &= \mathcal{O}(\mathcal{T})_0^m \oplus \mathcal{O}(\mathcal{T})_1^n = (\mathcal{O}(\mathcal{T}) \otimes \mathbb{R}^{m|n})_0\end{aligned} \tag{2.19}$$

For $\mathcal{J}(\mathcal{T}) := \{f \in \mathcal{O}(\mathcal{T}) \,|\, f \text{ is nilpotent}\}$ the ideal of nilpotent sections of $\mathcal{O}(\mathcal{T})$, this yields the canonical projection $\epsilon : \mathcal{O}(\mathcal{T}) \to \mathcal{O}(\mathcal{T})/\mathcal{J}(\mathcal{T}) \cong C^\infty(T_0)$ with T_0 defined via (2.5) which can be extended to the body map

$$\epsilon_{m,n} : (\mathcal{O}(\mathcal{T}) \otimes \mathbb{R}^{m|n})_0 \to C^\infty(T_0)^m \tag{2.20}$$

In the following, we want to restrict to a subclass of supermanifolds $\mathcal{T} \in \mathbf{SMan}_{\mathrm{Alg}}$ for which $C^\infty(T_0) \cong \mathbb{R}$, i.e., for which the underlying topological space $T = \{*\}$ just consists of a single point. Hence, it follows that $\mathcal{T} \cong (\{*\}, \Lambda_N) = \mathbb{R}^{0|N}$ for some $N \in \mathbb{N}_0$.

Definition 2.2.9. An algebro-geometric supermanifold $\mathcal{T}$ is called a *superpoint* if the underlying topological space T only consists of a single point. The subclass of super points form a full subcategory **SPoint** of $\mathbf{SMan}_{\mathrm{Alg}}$ called the *category of superpoints*.

Proposition 2.2.10 (see [98, 99]). *Let* **Gr** *be the* category of (finite-dimensional) Grassmann algebras *whose objects are given by equivalence classes of Grassmann algebras* $\Lambda_N \in \mathbf{Ob}(\mathbf{Gr})$, $N \in \mathbb{N}_0$, *and for* $\Lambda_N, \Lambda_{N'} \in \mathbf{Ob}(\mathbf{Gr})$, $\mathrm{Hom}_{\mathbf{Gr}}(\Lambda_N, \Lambda_{N'})$ *is given by the set of superalgebra morphisms between Grassmann algebras. Then, the assignment*

$$\begin{aligned} \mathbf{Gr}^{\mathrm{op}} \to \mathbf{SPoint},\ \Lambda_N &\mapsto (\{*\}, \Lambda_N) \\ (\phi : \Lambda_N \to \Lambda_{N'}) &\mapsto (\mathrm{id}_{\{*\}}, \phi) \end{aligned} \tag{2.21}$$

yields an equivalence of categories. □

In the following, we will therefore identify superpoints with finite-dimensional Grassmann algebras. From (2.19), it follows for $N \in \mathbb{N}_0$

$$\mathbb{R}^{m|n}(\Lambda_N) \cong (\Lambda_N \otimes \mathbb{R}^{m|n})_0 =: \Lambda_N^{m,n} \tag{2.22}$$

with $\Lambda_N^{m,n}$ the superdomain of dimension (m, n) (see Definition C.1). We equip $\Lambda_N^{m,n}$ with the coarsest topology such that the body map $\epsilon_{m,n} : \Lambda_N^{m,n} \to \mathbb{R}^m$ is continuous, the so-called *DeWitt-topology*. Hence, in this way, it follows that $\mathbb{R}^{m|n}(\Lambda_N)$ can be identified with a trivial supermanifold in the sense of Rogers-DeWitt.

2.2.2. Algebro-geometric and H^∞ supermanifolds: An equivalence of categories

With these preliminaries, in the following, we are ready to describe a concrete link between the algebro-geometric and Rogers-DeWitt approach using the functor of points technique. To this end, we first show that smooth functions on $\mathbb{R}^{m|n}(\Lambda_N) \cong \Lambda_N^{m,n}$ can be described in terms of natural transformations between functor of points.

More precisely, by the Global Chart Theorem 2.2.3, a section $f \in \mathcal{O}(\mathbb{R}^{m|n})$ can be identified with a morphism $f : \mathbb{R}^{m|n} \to \mathbb{R}^{1|1}$. According to (2.11), this in turn induces a natural transformation $f_{\mathcal{T}} : \mathbb{R}^{m|n}(\mathcal{T}) \to \mathbb{R}^{1|1}(\mathcal{T})$ between the respective functor of points. Sticking to Grassmann algebras, we want to find an explicit form of f_{Λ_N}. To this end, let $(x, \xi) \in \Lambda_N^{m,n}$ which we can identify with a morphism $g : \mathbb{R}^{0|N} \to \mathbb{R}^{m|n}$ such that $g^\sharp(t^i) = x^i$ and $g^\sharp(\theta^j) = \xi^j\ \forall i = 1, \dots, m,\ j = 1, \dots, n$. It then follows again from Theorem 2.2.3 that $f_{\Lambda_N}(x, \xi)$ can be identified with an element of $\Lambda_N^{1,1} \equiv \Lambda_N$ whose even and odd part is given by $g^\sharp(f^\sharp(t))$ and $g^\sharp(f^\sharp(\theta))$, respectively, where t and θ denote the global sections of $\mathcal{O}(\mathbb{R}^{1|1})$. Thus, expanding $f = \sum_{\underline{I}} f_{\underline{I}} \theta^{\underline{I}}$, this yields

$$f_{\Lambda_N}(x, \xi) = g^\sharp(f^\sharp(t)) + g^\sharp(f^\sharp(\theta)) = \sum_{\underline{I},\underline{J}} \frac{1}{\underline{J}!} \partial_{\underline{J}} f_{\underline{I}}\big|_{|g|} s(g^\sharp(t))^{\underline{J}} (g^\sharp(\theta))^{\underline{I}}$$

$$= \sum_{\underline{I},\underline{J}} \frac{1}{\underline{J}!} \partial_{\underline{J}} f_{\underline{I}}(\epsilon_{m,n}(x)) s(x)^{\underline{J}} \xi^{\underline{I}} =: \sum_{\underline{I}} \mathbf{G}(f_{\underline{I}})(x) \xi^{\underline{I}} \tag{2.23}$$

where $s(x) := x - \epsilon_{m,n}(x)$ is the soul map and

$$\mathbf{G}(f_{\underline{I}})(x) := \sum_{\underline{J}} \frac{1}{\underline{J}!} \partial_{\underline{J}} f_{\underline{I}}(\epsilon_{m,n}(x)) s(x)^{\underline{J}} \tag{2.24}$$

is called the Grassmann-analytic continuation of $f_{\underline{I}}$ or simply its $\mathbf{G}$-extension. Functions of the form (2.23) are precisely supersmooth functions in the sense of Rogers-DeWitt! In the standard literature, they are also called of class H^∞ (see Appendix C). As a result, $\Lambda_N^{m,n}$ together with functions of the form (2.23) yields a Rogers-DeWitt or H^∞ supermanifold. The assignment

$$\begin{aligned} \mathrm{Hom}_{\mathbf{SMan}_{\mathrm{Alg}}}(\mathbb{R}^{m|n}, \mathbb{R}^{1|1}) &\to H^\infty(\Lambda_N^{m,n}) \\ f &\mapsto f_{\Lambda_N} \end{aligned} \tag{2.25}$$

is clearly surjective but in general not injective unless $N \geq n$. We next want to extend these considerations from superspaces to arbitrary algebro-geometric supermanifolds. To this end, we make the following definition.

Definition 2.2.11. For $N \in \mathbb{N}$, the functor $\mathbf{H}_N : \mathbf{SMan}_{\mathrm{Alg}} \to \mathbf{Sets}$ is defined on objects $\mathcal{M} \in \mathbf{Ob}(\mathbf{SMan}_{\mathrm{Alg}})$ via

$$\mathbf{H}_N(\mathcal{M}) := \mathcal{M}(\Lambda_N) = \mathrm{Hom}(\mathcal{O}(\mathcal{M}), \textstyle\bigwedge \mathbb{R}^N) \tag{2.26}$$

and on morphisms $f : \mathcal{M} \to \mathcal{N}$ according to

$$\mathbf{H}_N(f) : \mathrm{Hom}(\mathcal{O}(\mathcal{M}), \textstyle\bigwedge \mathbb{R}^N) \to \mathrm{Hom}(\mathcal{O}(\mathcal{N}), \textstyle\bigwedge \mathbb{R}^N),\ \phi \mapsto \phi \circ f^\sharp \tag{2.27}$$

The set $\mathcal{M}(\Lambda_N)$ contains the real spectrum $\mathrm{Spec}_{\mathbb{R}}(\mathcal{O}(\mathcal{M})) = \mathrm{Hom}(\mathcal{O}(\mathcal{M}), \mathbb{R}) = \mathcal{M}(\mathbb{R})$ as a proper subset. According to Prop. 2.2.7 (see also (2.13)), this set can be identified with M and thus, in particular, naturally inherits a topology. Using this property, we again introduce the *DeWitt-topology* on $\mathcal{M}(\Lambda_N)$ to be coarsest topology such that the projection[1]

$$\mathbf{B}: \mathcal{M}(\Lambda_N) \to \mathrm{Spec}_{\mathbb{R}}(\mathcal{O}(\mathcal{M})) \cong M,\ \psi \mapsto \epsilon \circ \psi \tag{2.28}$$

is continuous.

Proposition 2.2.12. *Let $U \subseteq M$ be an open subset of the underlying topological space M of an algebro-geometric supermanifold $\mathcal{M}$. Let us identify U via $\Psi: M \to \mathrm{Spec}_{\mathbb{R}}(\mathcal{O}(\mathcal{M}))$, $x \mapsto \mathrm{ev}_x$ with an open subset in the real spectrum. Then, it follows that the open subsets $\mathbf{B}^{-1}(U)$ in $\mathcal{M}(\Lambda_N)$ are given by*

$$\mathbf{B}^{-1}(U) = \{\psi : \mathcal{O}(\mathcal{M}) \to \Lambda_N \,|\, \epsilon \circ \psi = \mathrm{ev}_x \text{ for some } x \in U\} \tag{2.29}$$

In particular, one has $\mathbf{B}^{-1}(U) = \mathcal{M}|_U(\Lambda_N)$ with $\mathcal{M}|_U := (U, \mathcal{O}_M|_U)$.

Proof. The first assertion is immediate, since $\psi \in \mathbf{B}^{-1}(U)$ if and only if $\epsilon \circ \psi \in \Psi(U)$, i.e., $\epsilon \circ \psi = \mathrm{ev}_x$ for some $x \in U$. To prove the last one, note that, by Theorem 2.2.8, one can identify a superalgebra morphism $\psi : \mathcal{O}(\mathcal{M}) \to \Lambda_N$ with the pullback of a supermanifold morphism $\phi := (|\phi|, \phi^\sharp) : \mathbb{R}^{0|N} = (\{*\}, \Lambda_N) \to \mathcal{M}$. For any $f \in \mathcal{O}(\mathcal{M})$, $\epsilon(\phi^\sharp(f))$ is defined as the unique real number such that $\phi^\sharp(f) - \epsilon(\phi^\sharp(f))$ is not invertible. This is precisely the definition of the value of a section of Λ_N at $\{*\}$, i.e., $\epsilon(\phi^\sharp(f)) = \phi^\sharp(f)(\{*\}) = f(|\phi|(\{*\}))$. Since, $\epsilon \circ \phi^\sharp = \mathrm{ev}_x$ for some $x \in M$, this yields $f(|\phi|(\{*\})) = \mathrm{ev}_x(f) = f(x)$ for any $f \in \mathcal{O}(\mathcal{M})$ which implies $|\phi|(\{*\}) = x$. Note that $\phi^\sharp$ is a morphism of sheaves and thus, in particular, commutes with restrictions. Hence, if, for $f \in \mathcal{O}(\mathcal{M})$, there exists an open neighborhood $x \in V$ such that $f|_V = 0$, then $\phi^\sharp(f) = 0$. That is, $\phi^\sharp$ is uniquely determined by the induced stalk morphism $\phi^\sharp_x : \mathcal{O}_{\mathcal{M},x} \to \Lambda_N$. From this, it is immediate to see that any $\psi \in \mathcal{M}|_U(\Lambda_N) = \mathrm{Hom}(\mathcal{O}_M(U), \Lambda_N)$ can trivially be extended to a morphism $\psi : \mathcal{O}(\mathcal{M}) \to \Lambda_N$ satisfying $\epsilon \circ \psi = \mathrm{ev}_x$ for some $x \in U$, i.e., $\psi \in \mathbf{B}^{-1}(U)$. Conversely, it follows that any morphism in $\mathbf{B}^{-1}(U)$ arises in this way. This proves the last assertion. □

By the local property, for any $x \in M$, there exists an open subset $x \in U \subseteq M$ such that $\mathcal{M}|_U$ is isomorphic to a superdomain $\mathcal{U}^{m|n}$ which is a submanifold of the superspace $\mathbb{R}^{m|n}$. Applying the functor (2.26) and using Prop. 2.2.12, we thus obtain an isomorphism

$$\mathbf{B}^{-1}(U) = \mathcal{M}|_U(\Lambda_N) \to \mathcal{U}^{m|n}(\Lambda_N) \subseteq \mathbb{R}^{m|n}(\Lambda_N) \tag{2.30}$$

[1] In case $\mathcal{M}$ is simply given by the superspace $\mathbb{R}^{m|n}$, this coincides with the body map (2.20).

i.e., a local *superchart* of $\mathcal{M}(\Lambda_N)$. By (2.23), it follows immediately that the transition map between two local supercharts defines a H^∞-smooth function. As a consequence, $\mathcal{M}(\Lambda_N)$ indeed carries the structure of a H^∞ supermanifold. Hence, it follows that the Λ-points of an algebro-geometric supermanfold naturally define supermanifolds in the sense of Rogers-DeWitt (or more generally $\mathscr{A}$-manifolds in the sense of Tuynman [97]). Moreover, the corresponding topological space $\mathbf{B}(\mathcal{M}(\Lambda_N)) = \mathrm{Spec}_{\mathbb{R}}(\mathcal{O}(\mathcal{M}))$ has the structure of an ordinary C^∞ manifold.

Remark 2.2.13. Just for sake of completeness, note that each $\mathcal{M} \in \mathbf{Ob}(\mathbf{SMan}_{\mathrm{Alg}})$ gives rise to the obvious functor

$$\mathcal{M} : \mathbf{Gr} \to \mathbf{Top} \tag{2.31}$$

which maps Grassmann algebras Λ to Λ-points $\mathcal{M}(\Lambda)$. This leads to the interpretation of a supermanifold in the sense of Molotkov-Sachse [98–100].

Similar to (2.25), for any $U \subseteq M$ open, one obtains a map

$$\begin{aligned} \mathcal{O}_M(U) \cong \mathrm{Hom}(\mathcal{M}|_U, \mathbb{R}^{1|1}) &\to H^\infty(\mathcal{M}|_U(\Lambda_N)) = \mathbf{B}_* H^\infty_{\mathcal{M}(\Lambda_N)}(U) \\ f &\mapsto f_{\Lambda_N} \end{aligned} \tag{2.32}$$

which is generally surjective but injective iff $N \geq n$. In particular, one can show that it defines a morphism of sheaves, i.e., it commutes with restrictions.

Consider next a H^∞ supermanifold $\mathcal{K} \in \mathbf{Ob}(\mathbf{SMan}_{H^\infty})$. To $\mathcal{K}$, one can associate the *body* $\mathbf{B}(\mathcal{K})$ defined as the subset of $\mathcal{K}$ given by

$$\mathbf{B}(\mathcal{K}) := \{x \in \mathcal{K} | f(x) \in \mathbb{R}, \forall f \in H^\infty(\mathcal{K})\} \tag{2.33}$$

which, by definition, has the structure of an ordinary smooth manifold. This can be extended to morphisms $f : \mathcal{K} \to \mathcal{L}$ between H^∞ supermanifolds setting[2] $\mathbf{B}(f) := f|_{\mathbf{B}(\mathcal{K})} : \mathbf{B}(\mathcal{K}) \to \mathbf{B}(\mathcal{L})$ yielding a functor $\mathbf{B} : \mathbf{SMan}_{H^\infty} \to \mathbf{Man}$ called the *body functor*. In case $\mathcal{K}$ is given by a Λ_N-point $\mathcal{M}(\Lambda_N)$ of an algebro-geometric supermanifold $\mathcal{M} \in \mathbf{Ob}(\mathbf{SMan}_{\mathrm{Alg}})$ with odd dimension bounded by N, one can identify $\mathbf{B}(\mathcal{K})$ with the real spectrum $\mathrm{Spec}_{\mathbb{R}}(\mathcal{O}(\mathcal{M}))$ justifying the notation. To see this, note that, in this case, (2.32) implies that smooth functions on $\mathcal{K}$ are given by natural transformations f_{Λ_N} induced by morphisms $f \in \mathrm{Hom}(\mathcal{M}, \mathbb{R}^{1|1})$. For $\phi \in \mathcal{K} = \mathrm{Hom}(\mathcal{O}(\mathcal{M}), \Lambda_N)$, $f_{\Lambda_N}(\phi)$ can be identified with the element $\phi \circ f^\sharp \in \Lambda_N$. Hence, $\phi \in \mathbf{B}(\mathcal{N}) \Leftrightarrow f_{\Lambda_N}(\phi) \in \mathrm{Hom}(\mathcal{O}(\mathbb{R}^{1|1}), \mathbb{R}) \cong \mathbb{R}\, \forall f \in \mathrm{Hom}(\mathcal{M}, \mathbb{R}^{1|1})$ if and only if $\phi \in \mathrm{Hom}(\mathcal{O}(\mathcal{M}), \mathbb{R})$, that is, iff ϕ is contained in the real spectrum $\mathrm{Spec}_{\mathbb{R}}(\mathcal{O}(\mathcal{M}))$.

---

[2] Note that $f(\mathbf{B}(\mathcal{K})) \subseteq \mathbf{B}(\mathcal{L})$ so that $f|_{\mathbf{B}(\mathcal{K})} : \mathbf{B}(\mathcal{K}) \to \mathbf{B}(\mathcal{L})$ is indeed well-defined. In fact, for any $g \in H^\infty(\mathcal{L})$ and $x \in \mathbf{B}(\mathcal{K})$, it follows $g(f(x)) = (g \circ f)(x) \in \mathbb{R}$ as $g \circ f$ is smooth and therefore $f(x) \in \mathbf{B}(\mathcal{L})$.

To any H^∞ supermanifold $\mathcal{K}$, one can associate the locally ringed space

$$\mathbf{A}(\mathcal{K}) := (\mathbf{B}(\mathcal{K}), \mathbf{B}_* H^\infty_{\mathcal{K}}) \tag{2.34}$$

which has the structure of an algebro-geometric supermanifold. A morphism $f : \mathcal{K} \to \mathcal{L}$ between H^∞ supermanifolds $\mathcal{K}$ and $\mathcal{L}$ canonically induces a morphism

$$\mathbf{A}(f) = (f|_{\mathbf{B}(\mathcal{K})}, f^*) : \mathbf{A}(\mathcal{K}) \to \mathbf{A}(\mathcal{L}) \tag{2.35}$$

between the corresponding algebro-geometric supermanifolds, where f^* denotes the ordinary pullback of smooth functions. Hence, this yields a functor

$$\mathbf{A} : \mathbf{SMan}_{H^\infty} \to \mathbf{SMan}_{\mathrm{Alg}} \tag{2.36}$$

Let us restrict $\mathbf{H}_N$ to the full subcategory $\mathbf{SMan}_{\mathrm{Alg},N} \subset \mathbf{SMan}_{\mathrm{Alg},N}$ of algebro- geometric supermanifolds with odd dimension bounded by N. Then, based on the previous observations, if follows $\mathbf{A}(\mathbf{H}_N(\mathcal{M})) \cong \mathcal{M}$ for any $\mathcal{M} \in \mathbf{Ob}(\mathbf{SMan}_{\mathrm{Alg},N})$. In fact, we have the following.

Theorem 2.2.14. *The functor* $\mathbf{A} \circ \mathbf{H}_N : \mathbf{SMan}_{\mathrm{Alg},N} \to \mathbf{SMan}_{\mathrm{Alg},N}$ *is naturally equivalent to the identity functor* $\mathrm{id} : \mathbf{SMan}_{\mathrm{Alg},N} \to \mathbf{SMan}_{\mathrm{Alg},N}$.

Proof. We have to show that the following diagrams are commutative

$$\begin{array}{ccc} M & \xrightarrow{\ |f|\ } & N \\ {\scriptstyle\Psi}\downarrow & & \downarrow{\scriptstyle\Psi} \\ \mathrm{Spec}_{\mathbb{R}}(\mathcal{O}(\mathcal{M})) & \longrightarrow & \mathrm{Spec}_{\mathbb{R}}(\mathcal{O}(\mathcal{N})) \end{array} \qquad \begin{array}{ccc} \mathcal{O}_M & \xrightarrow{\ f^\sharp\ } & f_*\mathcal{O}_M \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\ \mathbf{B}_* H^\infty_{\mathcal{N}(\Lambda_N)} & \xrightarrow{\ \mathbf{H}_N(f)^*\ } & f_*(\mathbf{B}_* H^\infty_{\mathcal{M}(\Lambda_N)}) \end{array}$$

for any $\mathcal{M}, \mathcal{N} \in \mathbf{Ob}(\mathbf{SMan}_{\mathrm{Alg},N})$ and morphisms $f = (|f|, f^\sharp) : \mathcal{M} \to \mathcal{N}$ where, in the diagram on the left, the lower arrow is given by the restriction of $\mathbf{H}_N(f)$ to the real spectrum $\mathrm{Spec}_{\mathbb{R}}(\mathcal{O}(\mathcal{M}))$. That the left diagram commutes follows immediately, since, by Def. (2.27), we have

$$\mathbf{H}_N(f)(\mathrm{ev_x}) = \mathrm{ev}_x \circ f^\sharp = \mathrm{ev}_{|f|(x)} \tag{2.37}$$

for any $x \in M$. To see the commutativity of the right diagram, note that, by identifying $g \in \mathcal{O}(\mathcal{N})$ with a morphism $g : \mathcal{N} \to \mathbb{R}^{1|1}$, the pullback $f^\sharp(g)$ is given by the morphism $g \circ f : \mathcal{M} \to \mathbb{R}^{1|1}$. Moreover, identifying $\phi \in \mathcal{M}(\Lambda_N)$ with a morphism $\phi : \mathbb{R}^{0|N} \to \mathcal{M}$, we have $\mathbf{H}_N(f)(\phi) = f \circ \phi$. Thus, this yields

$$\begin{aligned}\mathbf{H}_N(f)^*(g_{\Lambda_N})(\phi) &= g_{\Lambda_N}(\mathbf{H}_N(f)(\phi)) \\ &= g \circ f \circ \phi = f^\sharp(g) \circ \phi = (f^\sharp(g))_{\Lambda_N}(\phi)\end{aligned} \tag{2.38}$$

for any $g \in \mathcal{O}(\mathcal{N})$ and $\phi \in \mathcal{M}(\Lambda_N)$. This proves the theorem. □

Conversely, it is immediate to see that $\mathbf{H}_N \circ \mathbf{A}$ is naturally equivalent to the identity functor on the full subcategory $\mathbf{SMan}_{H^\infty,N}$ of H^∞ supermanifolds with odd dimension bounded by N (see also [95, 108]). Thus, the functors $\mathbf{H}_N : \mathbf{SMan}_{\mathrm{Alg},N} \to \mathbf{SMan}_{H^\infty,N}$ and $\mathbf{A} : \mathbf{SMan}_{H^\infty,N} \to \mathbf{SMan}_{\mathrm{Alg},N}$ provide an equivalence of categories.

To summarize, any algebro-geometric supermanfold induces a functor of the form (2.31) assigning Grassmann algebras to the corresponding H^∞ supermanifold. Moreover, in case that the number of odd generators of the Grassmann algebra is large enough, via (2.26), one even obtains an equivalence of categories which allows one to uniquely reconstruct the underlying algebro-geometric supermanifold. For this reason, many constructions on algebro-geometric supermanifolds can equivalently be performed on the corresponding H^∞ supermanifolds (in fact, we will mainly do so in what follows as H^∞ manifolds are often easier to handle for applications in physics). However, the choice of a particular Grassmann algebra is completely arbitrary and therefore tends to introduce superfluous (physical) degrees of freedom. Consequently, any definition made on a H^∞ supermanifold should not depend on a particular choice of a Grassmann algebra but, in the sense of Molotkov-Sachse, behave functorially under the change of Grassmann algebras. In the following, working with a particular H^∞ supermanifold $\mathcal{M}$, we will only assume that the number of odd generators of the Grassmann algebra Λ over which $\mathcal{M}$ is modeled is large enough, i.e., greater than the odd dimension of $\mathcal{M}$.[3]

2.3. Super Lie groups and Lie superalgebras

Super Lie groups and their corresponding algebras play a prominent role in context of supergravity, in particuular, in the framework of the Cartan geometric approach which will be discussed in detail in Chapter 3. In the following, let us recall very briefly the main definition of super Lie groups in the algebraic category and their associated super Lie algebras as well as the relation to their corresponding H^∞ counterparts (see, e.g., [106] for an introduction to super Lie groups in the algebraic category as well as [96]

[3] For this reason, in the standard literature, one typically chooses the infinite-dimensional Grassmann algebra Λ_∞ generated by infinite number of Grassmann-generators which may be obtained as an inductive limit of the finite-dimensional ones. Also in this case, one can show that the category of algebro-geometric and H^∞ supermanifolds modeled over Λ_∞ are indeed equivalent [95, 108].

in the Rogers-DeWitt approach). We will then turn quickly towards important examples which will be of central interest in the context of physical applications.

Definition 2.3.1 (see [106]). An *algebro-geometric super Lie group* $\mathcal{G} := (G, \mathcal{O}_G)$ is an algebro-geometric supermanifold $\mathcal{G} \in \mathbf{Ob}(\mathbf{Man}_{\mathrm{Alg}})$ together with three morphisms

$$\mu : \mathcal{G} \times \mathcal{G} \to \mathcal{G}, \quad i : \mathcal{G} \to \mathcal{G}, \quad e : \mathbb{R}^{0|0} \to \mathcal{G} \tag{2.39}$$

called *multiplication*, *inverse* and *neutral element*, respectively, satisfying the following commutative diagrams

$$\begin{array}{ccc} \mathcal{G}\times\mathcal{G}\times\mathcal{G} & \xrightarrow{\mu\times\mathrm{id}} & \mathcal{G}\times\mathcal{G} \\ {\scriptstyle \mathrm{id}\times\mu}\downarrow & & \downarrow{\scriptstyle\mu} \\ \mathcal{G}\times\mathcal{G} & \xrightarrow{\mu} & \mathcal{G}\end{array} \qquad \begin{array}{ccccc} & & \mathcal{G}\times\mathcal{G} & & \\ & \nearrow^{\langle \mathrm{id}, i\rangle} & & \searrow^{\mu} & \\ \mathcal{G} & & \xrightarrow{\widehat{e}} & & \mathcal{G} \\ & \searrow_{\langle i,\mathrm{id}\rangle} & & \nearrow_{\mu} & \\ & & \mathcal{G}\times\mathcal{G} & & \end{array} \qquad \begin{array}{ccccc} & & \mathcal{G}\times\mathcal{G} & & \\ & \nearrow^{\langle \mathrm{id}, \widehat{e}\rangle} & & \searrow^{\mu} & \\ \mathcal{G} & & \xrightarrow{\mathrm{id}} & & \mathcal{G} \\ & \searrow_{\langle \widehat{e},\mathrm{id}\rangle} & & \nearrow_{\mu} & \\ & & \mathcal{G}\times\mathcal{G} & & \end{array} \tag{2.40}$$

where $\widehat{e}$ denotes the composition of the neutral element $e : \mathbb{R}^{0|0} \to \mathcal{G}$ with the unique morphism $\mathcal{G} \to \mathbb{R}^{0|0}$. Moreover, for two morphisms ϕ and $\psi : \mathcal{G} \to \mathcal{G}$, $\langle \phi, \psi \rangle : \mathcal{G} \to \mathcal{G}$ is defined as the morphism $(\phi \times \psi) \circ d_{\mathcal{G}}$ with $d_{\mathcal{G}}$ the diagonal map $d_{\mathcal{G}} : \mathcal{G} \to \mathcal{G} \times \mathcal{G}$.

Definition 2.3.2. Let $\mathcal{G} = (G, \mathcal{O}_G)$ be an algebro-geometric super Lie group. A smooth *vector field* or *derivation* $X \in \mathrm{Der}(\mathcal{O}_G)$ on the function sheaf $\mathcal{O}_G$ is called *left-* resp. *right-invariant* if

$$\mathbb{1} \otimes X \circ \mu^\sharp = \mu^\sharp \circ X \quad \text{resp.} \quad X \otimes \mathbb{1} \circ \mu^\sharp = \mu^\sharp \circ X \tag{2.41}$$

Definition 2.3.3. The *super Lie algebra* (or *Lie superalgebra*) $\mathfrak{g}$ of an algebro-geometric super Lie group $\mathcal{G} = (G, \mathcal{O}_G)$ is defined as the sub super Lie algebra of left-invariant vector fields on $\mathcal{G}$ where the Lie bracket $[\cdot, \cdot]$ on $\mathfrak{g}$ is defined via the graded commutator of vector fields, i.e.,

$$[X, Y] := X \circ Y - (-1)^{|X||Y|} Y \circ X \tag{2.42}$$

for any homogeneous $X, Y \in \mathfrak{g}$.

Remark 2.3.4. Trivially, any left-invariant vector field $X \in \mathfrak{g}$ on an algebro-geometric super Lie group $\mathcal{G} = (G, \mathcal{O}_G)$ defines a tangent vector at the identity $X_e \in T_e(G, \mathcal{O}_G)$, that is, a linear derivation $X_e : \mathcal{O}_{G,e} \to \mathbb{R}$ on the stalk $\mathcal{O}_{G,e}$ by evaluation $X_e := e^\sharp \circ X$. Conversely, any tangent vector $Y_e \in T_e(G, \mathcal{O}_G)$ canonically induces a left-invariant vector field $Y \in \mathfrak{g}$ via $Y := \mathbb{1} \otimes Y_e \circ \mu^\sharp$. In fact, using associativity of the group multiplication, it follows

$$\begin{aligned} \mathbb{1} \otimes Y \circ \mu^\sharp &= \mathbb{1} \otimes (\mathbb{1} \otimes Y_e \circ \mu^\sharp) \circ \mu^\sharp = \mathbb{1} \otimes \mathbb{1} \otimes Y_e \circ \mathbb{1} \otimes \mu^\sharp \circ \mu^\sharp \\ &= \mathbb{1} \otimes \mathbb{1} \otimes Y_e \circ \mu^\sharp \otimes \mathbb{1} \circ \mu^\sharp = \mu^\sharp \circ (\mathbb{1} \otimes Y_e \circ \mu^\sharp) = \mu^\sharp \circ Y \end{aligned} \tag{2.43}$$

Hence, in this way, this yields an isomorphism of super vector spaces such that we may identify $\mathfrak{g} \cong T_e(G, \mathcal{O}_G)$. For this reason, if not stated otherwise, we will not specify whether a super Lie algebra element $X \in \mathfrak{g}$ is viewed as a left-invariant vector field or as a tangent vector at the identity.

Remark 2.3.5. In Definition 2.3.3, the sub super Lie algebra of left-invariant vector fields was taken for the definition of the super Lie algebra of an algebro-geometric super Lie group $\mathcal{G} = (G, \mathcal{O}_G)$. On the other hand, we could also have taken the right-invariant vector fields as these form a sub super Lie algebra, as well. In the following, we will denote this super Lie algebra by $\mathfrak{g}^R$.

Next, we want to turn to the notion of super Lie groups in the category of Rogers-DeWitt supermanifolds which is more or less in complete analogy to the standard theory of ordinary smooth manifolds.

Definition 2.3.6. A H^∞ super Lie group $\mathcal{G}$ is a H^∞ supermanifold together with two morphisms $\mu : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ and $i : \mathcal{G} \to \mathcal{G}$ called *multiplication* and *inverse*, respectively, as well as an element $e \in \mathcal{G}$ called *neutral element* such that, after applying the forgetful functor $\mathbf{SMan}_{H^\infty} \to \mathbf{Set}$, $(\mathcal{G}, \mu, i, e)$ defines an ordinary group in the category **Set**.

It is clear by functoriality that, given an algebro-geometric super Lie group $\mathcal{G} = (G, \mathcal{O}_G)$, the corresponding Λ_N-points $\mathcal{G}(\Lambda_N) = \mathbf{H}_N(\mathcal{G})$ with N greater than the odd dimension of $\mathcal{G}$ have the structure of H^∞ super Lie groups. Conversely, a H^∞ super Lie group $\mathcal{G}$ naturally induces a corresponding algebro-geometric super Lie group $\mathbf{A}(\mathcal{G}) = (\mathbf{B}(\mathcal{G}), H^\infty_{\mathcal{G}})$. In this context, it is important to note that the neutral element $e \in \mathcal{G}$ of a H^∞ super Lie group $\mathcal{G}$ is an element of the body[4] such that the pullback induces

[4] In fact, since $\mathbf{B}(i) = i|_{\mathbf{B}(\mathcal{G})}$, it follows that, for any $g \in \mathbf{B}(\mathcal{G})$, one has $g^{-1} = i(g) = \mathbf{B}(i)(g) \in \mathbf{B}(\mathcal{G})$. Thus, since $\mathbf{B}(\mu) = \mu|_{\mathbf{B}(\mathcal{G})}$, we have in particular $g \cdot g^{-1} = e \in \mathbf{B}(\mathcal{G})$. This also demonstrates that $\mathbf{B}(\mathcal{G})$ in fact defines an ordinary Lie group.

a morphism $e^* \equiv \mathrm{ev}_e : H^\infty(\mathcal{G}) \to \mathbb{R}$, i.e., an element of the real spectrum $e^* \in \mathrm{Spec}_{\mathbb{R}}(H^\infty(\mathcal{G}))$. Thus, in this way, we obtain an equivalence of categories between algebro-geometric and H^∞ supermanifolds. Again, in what follows, we will mainly work in the H^∞ category as these objects are often easier to handle.

Definition 2.3.7. The *super Lie algebra* $\mathfrak{g}$ of a H^∞ super Lie group $\mathcal{G}$ is defined as the super Lie algebra of left-invariant vector fields on the corresponding algebro-geometric super Lie group $\mathbf{A}(\mathcal{G})$. Furthermore, we define the *super Lie module* $\mathrm{Lie}(\mathcal{G})$ as the tangent space $\mathrm{Lie}(\mathcal{G}) := T_e\mathcal{G}$.

Remark 2.3.8. According to Remark 2.3.4, one can identify $\mathfrak{g} = T_e\mathbf{A}(\mathcal{G})$. Since, left-invariant vector fields induce a homogeneous basis on the tangent spaces and $\mathrm{Lie}(\mathcal{G})$ defines a super Λ-module with Λ the underlying Grassmann algebra over which $\mathcal{G}$ is modeled as a H^∞ supermanifold, one thus has

$$\mathrm{Lie}(\mathcal{G}) = \Lambda \otimes \mathfrak{g} \tag{2.44}$$

Hence, in particular, if follows that $\mathrm{Lie}(\mathcal{G})$ defines a super Λ-vector space with distinguished basis represented by smooth left-invariant vector fields.

Finally, let us briefly mention a very important equivalent characterization of super Lie groups in terms of so-called *super Harish Chandra pairs*. It turns out that super Lie groups $\mathcal{G}$ (concrete or algebraic) have a relatively simple structure: They are completely determined by the data $(G, \mathfrak{g})$ consisting of underlying topological space G and the super Lie algebra $\mathfrak{g}$ and a certain representation of G on $\mathfrak{g}$.

In the algebraic category, this correspondence remains rather implicit (for an abstract proof see for instance [106]). In the H^∞ category, however, a very concrete proof of this correspondence has been given in [97]. As we will see, this theorem will also turn out to be quite useful in constructing invariant measures on super Lie groups to be discussed in Section 5.5.2 in the context of loop quantization of chiral supergravity. It provides a concrete relation between a super Lie group $\mathcal{G}$ and the data $(\mathbf{B}(\mathcal{G}), \mathfrak{g})$. More precisely, one has the following:

Theorem 2.3.9 (Super Harish-Chandra pair (after [97,109])). *Let $\mathcal{G}$ be a H^∞ super Lie group with body $G := \mathbf{B}(\mathcal{G})$. Then, $\mathcal{G}$ is globally split, that is, it is diffeomorphic to the split supermanifold $\mathbf{S}(\mathfrak{g}_1, G) \cong \mathbf{S}(G) \times (\mathfrak{g}_1 \otimes \Lambda)_0$ associated to the trivial vector bundle $G \times \mathfrak{g}_1 \to G$ via the canonical mapping*

$$\begin{aligned}\Phi:\ \mathbf{S}(\mathfrak{g}_1, G) &\to \mathcal{G} \\ (g, X) &\mapsto g \cdot \exp(X)\end{aligned} \tag{2.45}$$

In particular, it follows that there exists a unique super Lie group structure on $\mathbf{S}(\mathfrak{g}_1, G)$ *such that (2.45) turns into a morphism of super Lie groups. Hence, any* H^∞ *super Lie group is uniquely determined via (2.45) by the data* $(G, \mathfrak{g})$ *called a* super Harish Chandra pair *consisting of its body G as well as the super Lie algebra* $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$.

It is interesting to note that, in [110], it has been shown in the Molotkov-Sachse approach that such a correspondence via (2.45) even holds in the case of infinite-dimensional (Fréchet) super Lie groups. With these preparations, let us next turn towards important examples which will play a central role in the geometric approach to supergravity as well as in the context of loop quantum supergravity.

Example 2.3.10 (The super translation group $\mathcal{T}^{1,3|4}$)**.** Let $(\kappa_{\mathbb{R}}, \Delta_{\mathbb{R}})$ be the real Majorana representation of $\mathrm{Spin}^+(1,3)$ (see Section 4.2). Consider the trivial vector bundle bundle $\mathbb{R}^{1,3} \times \Delta_{\mathbb{R}} \to \mathbb{R}^{1,3}$ over Minkowski spacetime $(\mathbb{R}^{1,3}, \eta)$. Applying the split functor, this then yields a split supermanifold

$$\mathbb{R}^{1,3|4} := \mathbf{S}(\Delta_{\mathbb{R}}, \mathbb{R}^{1,3}) \cong \Lambda^{4,4} \tag{2.46}$$

also called *super Minkowski spacetime* with $\Lambda^{4,4}$ the superdomain of dimension (4,4) (Def. C.1). On this supermanifold, we define the map

$$\begin{aligned} \mu :\ & \Lambda^{4,4} \times \Lambda^{4,4} \to \Lambda^{4,4} \\ & ((x,\theta),(y,\eta)) \mapsto (z, \theta+\eta) \end{aligned} \tag{2.47}$$

where $z^I := x^I + y^I - \frac{1}{4}(C\gamma^I)_{\alpha\beta}\theta^\alpha\eta^\beta$ for $I = 0, \ldots, 3$ with C is the charge conjugation matrix and γ^I the gamma matrices of $4D$ Minkowski spacetime satisfying the Clifford algebra relations

$$[\gamma_I, \gamma_J]_+ = 2\eta_{IJ} \tag{2.48}$$

where $\eta \equiv (\eta_{IJ}) = \mathrm{diag}(-+++)$ is the Minkowski metric (see Section 4.2). It follows immediately that μ is smooth and associative. In particular, $\mathbb{R}^{1,3|4}$ equipped with μ defines a super Lie group with neutral element $e = (0,0)$ and inverse $i : \Lambda^{4,4} \to \Lambda^{4,4}$, $(x,\theta) \mapsto (-x, -\theta)$ (note that $C\gamma^I$ is symmetric for $I = 0, \ldots, 3$). From now on, let us denote this super Lie group by $\mathcal{T}^{1,3|4}$ and call it the *super translation group*. To derive the corresponding super Lie algebra, note that the comultiplication is given by

$$\mu^*(x^I) = x^I \otimes 1 + 1 \otimes x^I - \frac{1}{4}(C\gamma^I)_{\alpha\beta}\theta^\alpha \otimes \theta^\beta \tag{2.49}$$

$$\mu^*(\theta^\alpha) = \theta^\alpha \otimes 1 + 1 \otimes \theta^\alpha \tag{2.50}$$

The super Lie module of the super translation group takes the form

$$T_e\mathcal{T}^{1,3|4} =: \Lambda \otimes \mathfrak{t}^{1,3|4} \cong \operatorname{span}_\Lambda \left\{ \frac{\partial}{\partial x^I}\bigg|_e, \frac{\partial}{\partial \theta^\alpha}\bigg|_e \right\} \tag{2.51}$$

so that a homogeneous basis of left-invariant vector fields is given by $P_I := \left(\mathbb{1} \otimes \frac{\partial}{\partial x^I}\big|_e\right) \circ \mu^*$ and $Q_\alpha := \left(\mathbb{1} \otimes \frac{\partial}{\partial \theta^\alpha}\big|_e\right) \circ \mu^*$ for $I = 0, \ldots, 3$ and $\alpha = 1, \ldots, 4$. Hence, their action on the coordinate functions yields

$$\begin{aligned} Q_\alpha(x^I) &= \frac{1}{4}(C\gamma^I)_{\alpha\gamma}\theta^\gamma, \quad Q_\alpha(\theta^\beta) = \delta^\beta_\alpha \\ P_I(x^J) &= \delta^J_I, \quad P_I(\theta^\alpha) = 0 \end{aligned} \tag{2.52}$$

so that the vector fields can explicitly be written in the form

$$Q_\alpha = \frac{\partial}{\partial \theta^\alpha} + \frac{1}{4}(C\gamma^I)_{\alpha\beta}\theta^\beta \frac{\partial}{\partial x^I}, \quad P_I = \frac{\partial}{\partial x^I} \tag{2.53}$$

Using these identities, we can compute the corresponding (graded) commutation relations which yields

$$[Q_\alpha, Q_\beta] = \frac{1}{2}(C\gamma^I)_{\alpha\beta}P_I, \quad [Q_\alpha, P_I] = 0 \text{ and } [P_I, P_J] = 0 \tag{2.54}$$

Example 2.3.11 (Super Poincaré group)**.** The *super Poincaré group* in $D = 4$, $\mathcal{N} = 1$ is defined as the semi-direct product

$$\mathrm{ISO}(\mathbb{R}^{1,3|4}) = \mathcal{T}^{1,3|4} \rtimes_\Phi \mathbf{S}(\mathrm{Spin}^+(1,3)) \tag{2.55}$$

where $\Phi : \mathbf{S}(\mathrm{Spin}^+(1,3)) \to \mathrm{GL}(\mathcal{T}^{1,3|4})$ is the representation of the purely bosonic super Lie group $\mathbf{S}(\mathrm{Spin}^+(1,3))$ on the super translation group $\mathcal{T}^{1,3|4}$ obtained by applying the split functor on the group representation

$$\mathrm{Spin}^+(1,3) \ni g \mapsto \mathrm{diag}(\lambda^+(g), \kappa_\mathbb{R}(g)) \in \mathrm{GL}(\mathbb{R}^{1,3} \oplus \Delta_\mathbb{R}) \tag{2.56}$$

of $\mathrm{Spin}^+(1,3)$ on the super vector space $\mathbb{R}^{1,3} \oplus \Delta_\mathbb{R}$ with $\lambda^+ : \mathrm{Spin}^+(1,3) \to \mathrm{SO}^+(1,3)$ the universal covering map where $\Delta_\mathbb{R}$ is viewed as a purely odd super vector space. The super Lie algebra $\mathfrak{iso}(\mathbb{R}^{1,3|4})$ is generated by the (bosonic) infinitesimal spacetime translations P_I and Lorentz transformations M_{IJ}, $I, J = 0, \ldots, 3$, and four fermionic

Majorana generators Q_α, $\alpha = 1, \dots, 4$. It follows that, in addition to (2.54), the nonvanishing (graded) commutation relations are given by

$$[M_{IJ}, Q_\alpha] = \frac{1}{2} Q_\beta (\gamma_{IJ})^\beta{}_\alpha, \text{ and } [P_I, Q_\alpha] = 0 \tag{2.57}$$

One can equip the super translation group $\mathcal{T}^{1,3|4}$ (or super Minkowski spacetime) with a smooth super metric $\mathscr{S}$ (see Def. 2.3.12 below) setting $\mathscr{S}(P_I, P_J) = \eta_{IJ}$ and $\mathscr{S}(Q_\alpha, Q_\beta) = C_{\alpha\beta}$. By definition, it then follows that $\mathscr{S}$ is invariant under the Adjoint representation of $\mathrm{Spin}^+(1,3)$ on $\mathcal{T}^{1,3|4}$. Moreover, $\mathrm{ISO}(\mathbb{R}^{1,3|4})$ can be identified with the super isometry group of super Minkowski spacetime.

Definition 2.3.12 (after [97, 109]). (i) A (homogeneous) *super bilinear form* $\mathscr{S}$ of parity $|\mathscr{S}| \in \mathbb{Z}_2$ on a free super Λ-module $\mathcal{V}$ is a right bilinear map $\mathscr{S} : \mathcal{V} \times \mathcal{V} \to \Lambda^{\mathbb{C}}$, $\Lambda^{\mathbb{C}} := \Lambda \otimes \mathbb{C}$, which satisfies $\mathscr{S}(\mathcal{V}_i, \mathcal{V}_j) \subseteq (\Lambda^{\mathbb{C}})_{|\mathscr{S}|+i+j}$ and is graded symmetric, i.e.,

$$\mathscr{S}(v, w) = (-1)^{|v||w|} \mathscr{S}(w, v) \tag{2.58}$$

for all homogeneous $v, w \in \mathcal{V}$. Let $V := \mathcal{V}/\mathcal{N}$ with $\mathcal{N} := \{v \in \mathcal{V} | \exists a \in \Lambda : a \neq 0 \text{ and } ax = 0\}$ the subset of nilpotent vectors. A super bilinear form $\mathscr{S}$ is called *smooth*, if $\mathscr{S}(V, V) \subseteq \mathbb{C}$. An even super bilinear form $\mathscr{S} : \mathcal{V} \times \mathcal{V} \to \Lambda^{\mathbb{C}}$ is called a *super metric* if it is non-degenerate, that is, for any $v \in V$ with $v \neq 0$, there exists $w \in V$ such that $\epsilon(\mathscr{S}(v, w)) \neq 0$ where $\epsilon : \Lambda^{\mathbb{C}} \to \mathbb{C}$ is the body map.

(ii) A (homogeneous) *super sesquilinear form* $\mathscr{S}$ of parity $|\mathscr{S}| \in \mathbb{Z}_2$ on a free super Λ-module $\mathcal{V}$ is a right sesquilinear map $\mathscr{S} : \mathcal{V} \times \mathcal{V} \to \Lambda^{\mathbb{C}}$ (i.e., linear in the second and anti-linear in the first argument) satisfying $\mathscr{S}(\mathcal{V}_i, \mathcal{V}_j) \subseteq (\Lambda^{\mathbb{C}})_{|\mathscr{S}|+i+j}$ and is graded Hermitian, i.e.,

$$\mathscr{S}(v, w) = (-1)^{|v||w|} \overline{\mathscr{S}(w, v)} \tag{2.59}$$

for all homogeneous $v, w \in \mathcal{V}$. A super sesquilinear form $\mathscr{S}$ is called *smooth*, if $\mathscr{S}(V, V) \subseteq \mathbb{C}$ with $V := \mathcal{V}/\mathcal{N}$. An even non-degenerate super sesquilinear form $\mathscr{S} : \mathcal{V} \times \mathcal{V} \to \Lambda^{\mathbb{C}}$ is called a *super Hermitian metric* or *super scalar product*.

Remark 2.3.13. From the smoothness requirement of a super bilinear form $\mathscr{S}$, it follows immediately that $\mathscr{S}|_{V \times V}$ defines a graded symmetric bilinear form on the super vector space $V/\mathcal{N}$ in the sense of [111], i.e., $\mathscr{S}(V_i, V_j) = 0$ unless $i + j = |\mathscr{S}|$. Moreover, $\mathscr{S}|_{V_0 \times V_0}$ is symmetric and $\mathscr{S}|_{V_1 \times V_1}$ is antisymmetric. If $\mathscr{S}$ is furthermore even and non-degenerate then so is $\mathscr{S}|_{V \times V}$ which implies that V_1 is necessarily even-dimensional.

Moreover, one can always find a homogeneous basis (e_i, f_j) of V (resp. $\mathcal{V}$) such that, w.r.t. this basis, $\mathscr{S}$ takes the form

$$\begin{pmatrix} \mathbb{1}_m & 0 \\ 0 & J_{2n} \end{pmatrix} \tag{2.60}$$

where $\dim V_0 = m$ and $\dim V_1 = 2n$ and J_{2n} is the standard symplectic structure on $\mathbb{R}^{2n}$ given by

$$J_{2n} := \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix} \tag{2.61}$$

We will call (2.60) the *standard representation* of $\mathscr{S}$.

Example 2.3.14 (The general linear supergroup $\mathrm{GL}(\mathcal{V})$ (see also [97])). For a finite-dimensional super Λ-vector space $\mathcal{V} = V \otimes \Lambda$ with $\dim V = m|n$, the general linear supergroup $\mathrm{GL}(\mathcal{V})$ is defined as the open subset $\mathrm{Aut}(\mathcal{V}) \subset \mathrm{End}_R(\mathcal{V})$ of (right linear) automorphisms of $\mathcal{V}$. Choosing a real homogeneous basis $(e_i)_i$ of $\mathcal{V}$, we may identify $\mathrm{End}_R(\mathcal{V}) \cong \Lambda^{m^2+n^2,2mn}$ yielding $(m+n)^2$ smooth coordinate functions $x^i_{\ j} : \mathrm{End}_R(\mathcal{V}) \to \Lambda$ mapping an endomorphism $A \in \mathrm{End}_R(\mathcal{V})$ to its coordinates $x^i_{\ j}(A) \in \Lambda$ such that $A = x^i_{\ j}(A)\, e_i \otimes e^j$ with $(e_i \otimes e^j)_{i,j}$ the corresponding real homogeneous basis of $\underline{\mathrm{End}}_R(\mathcal{V}) \cong \mathcal{V} \otimes \mathcal{V}^*$ where $(e^i)_i$ denotes the right dual basis of $\mathcal{V}^*$ satisfying $e^i(e_j) = \delta^i_j$. Note that, here and in the following, we will strictly distinguish between the coordinates $x^i_{\ j}(A)$ of an endomorphism $A \in \mathrm{End}_R(\mathcal{V})$ and its matrix coefficients $A^i_{\ j}$ w.r.t. the real homogenous basis $(e_i)_i$ of $\mathcal{V}$ such that $A = e_i \otimes A^i_{\ j} e^j$, with the relation being given by

$$A^i_{\ j} = \mathfrak{C}^{|e_i|}(x^i_{\ j}(A)) \quad \Leftrightarrow \quad x^i_{\ j}(A) = \mathfrak{C}^{|e_i|}(A^i_{\ j}) \tag{2.62}$$

where the involution $\mathfrak{C} : \Lambda \to \Lambda$ is defined as $\mathfrak{C}(\lambda) = (-1)^{|\lambda|}\lambda$ for any homogeneous $\lambda \in \Lambda$. In general, these are equivalent iff A has purely real coordinates. Let $A, B \in \mathrm{End}_R(\mathcal{V})$ be two endomorphisms. The coordinates of the composition $A \circ B$ are then given by

$$\begin{aligned} x^i_{\ j}(A \circ B) &= \mathfrak{C}^{|e_i|}((A \circ B^i_{\ j})) = \mathfrak{C}^{|e_i|}(A^i_{\ k} \cdot B^k_{\ j}) \\ &= \mathfrak{C}^{|e_i|}(A^i_{\ k})\mathfrak{C}^{|e_i|}(B^k_{\ j}) = \mathfrak{C}^{|e_i|}(A^i_{\ k})\mathfrak{C}^{|e_i|+|e_k|}(\mathfrak{C}^{|e_k|}(B^k_{\ j})) \\ &= x^i_{\ k}(A) \cdot \mathfrak{C}^{|e_i|+|e_k|}(x^k_{\ j}(B)) \end{aligned} \tag{2.63}$$

Let us now restrict to the subset $\mathrm{GL}(\mathcal{V}) = \mathrm{Aut}(\mathcal{V}) \subset \mathrm{End}_R(\mathcal{V})$ of (even) automorphisms of $\mathcal{V}$. Since $\mathrm{GL}(\mathcal{V})$ is given by the open subset $\mathbf{B}^{-1}(\mathrm{GL}(V_0) \times \mathrm{GL}(V_1))$

of $\mathrm{End}_R(\mathcal{V})$, this implies that $\mathrm{GL}(\mathcal{V})$ defines a H^∞ supermanifold of dimension $\dim\mathrm{GL}(\mathcal{V})=(m^2+n^2,2mn)$ and body $\mathbf{B}(\mathrm{GL}(\mathcal{V}))=\mathrm{GL}(V_0)\times\mathrm{GL}(V_1)$. Restricting the global derivations $\partial_{x^i{}_j}$ on $\mathrm{End}_R(\mathcal{V})$, defined via $\partial_{x^i{}_j}x^k{}_l=\delta^k_i\delta^j_l$ for $i,j,k,l=1,\ldots,m+n$, to $\mathrm{GL}(\mathcal{V})$, the tangent bundle $T\mathrm{GL}(\mathcal{V})$ can be identified with $\mathrm{GL}(\mathcal{V})\times\underline{\mathrm{End}}_R(\mathcal{V})$, the identification being given by

$$T\mathrm{GL}(\mathcal{V})\ni X_g\mapsto(g,\langle X_g|\mathrm{d}x^i{}_j\rangle\ e_i\otimes e^j)\in\mathrm{GL}(\mathcal{V})\times\underline{\mathrm{End}}_R(\mathcal{V})\tag{2.64}$$

The general linear supergroup $\mathrm{GL}(\mathcal{V})$ forms an abstract group with multiplication μ defined by the composition of endomorphisms. By (2.63), it follows that for $g,h\in\mathrm{GL}(\mathcal{V})$ one has

$$x^i{}_j(g\circ h)=(-1)^{(|e_i|+|e_k|)(|e_k|+|e_j|)}x^i{}_k(g)\cdot x^k{}_j(h)\tag{2.65}$$

since $x^k{}_j(h)$ is homogeneous with parity $|x^k{}_j(h)|=|e_k|+|e_j|$ as h is even. Hence, the coordinates of $\mu(g,h)=g\circ h$ for any $g,h\in\mathrm{GL}(\mathcal{V})$ consists of a sum of products of coordinate functions of g and h and thus μ is of class H^∞ implying that $\mathrm{GL}(\mathcal{V})$ defines a H^∞ super Lie group with super Lie module $\mathrm{Lie}(\mathrm{GL}(\mathcal{V}))=:\Lambda\otimes\mathfrak{gl}(\mathcal{V})$ isomorphic to $\underline{\mathrm{End}}_R(\mathcal{V})$. By (2.65), the comultiplication $\mu^*:H^\infty(\mathrm{GL}(\mathcal{V}))\to H^\infty(\mathrm{GL}(\mathcal{V}))\hat{\otimes}H^\infty(\mathrm{GL}(\mathcal{V}))$ takes the form

$$\mu^*(x^i{}_j)=(-1)^{(|e_i|+|e_k|)(|e_k|+|e_j|)}x^i{}_k\otimes x^k{}_j\tag{2.66}$$

Using (2.66), let us derive the Lie bracket on $\mathrm{Lie}(\mathrm{GL}(\mathcal{V}))$. To do so, for $Y\in\mathfrak{gl}(\mathcal{V})$, we have to compute the corresponding left-invariant vector field $Y^L:=(\mathbb{1}\otimes Y)\circ\mu^*$. Applying it on coordinate functions $x^i{}_j$ yields

$$\begin{aligned}Y^L(x^i{}_j)&=(\mathbb{1}\otimes x^m{}_n(Y)\partial_{x^m{}_n})\circ\mu^*(x^i{}_j)\\&=Y^m_{\ n}\,(\mathbb{1}\otimes\partial_{x^m{}_n})\left((-1)^{(|e_i|+|e_k|)(|e_k|+|e_j|)}x^i{}_k\otimes x^k{}_j\right)\\&=(-1)^{(|e_i|+|e_k|)(|e_k|+|e_j|)}Y^m_{\ n}\,(-1)^{(|e_m|+|e_n|)(|e_i|+|e_k|)}x^i{}_k\delta^k_m\delta^n_j\\&=x^i{}_kY^k_{\ j}\end{aligned}\tag{2.67}$$

where we used that Y has purely real coordinates such that $x^i{}_k(Y)=Y^i_{\ k}\ \forall i,j=1,\ldots,m+n$. Hence,

$$Y^L=(x\circ Y)^i{}_j\,\partial_{x^i{}_j}\tag{2.68}$$

Using (2.68), we find for commutator between two left-invariant vector fields Y^L, Z^L corresponding to $Y, Z \in \mathfrak{gl}(\mathcal{V})$

$$\begin{aligned}[][Y^L, Z^L] &= Y^L Z^L - (-1)^{|Y||Z|} Z^L Y^L \\ &= x^i{}_k Y^k{}_j\, \partial_{x^i{}_j}(x^m{}_l Z^l{}_n \partial_{x^m{}_n}) - (-1)^{|Y||Z|} x^i{}_k Z^k{}_j \partial_{x^i{}_j}(x^m{}_l Y^l{}_n\, \partial_{x^m{}_n}) \\ &= x^i{}_k (Y^k{}_j\, Z^k{}_n - (-1)^{|Y||Z|} Z^k{}_j Y^k{}_n)\partial_{x^i{}_n} \\ &= (x \circ [Y, Z])^i{}_j \partial_{x^i{}_j} = [Y, Z]^L \end{aligned} \tag{2.69}$$

that is, via the identification (2.64), the commutator on $\mathrm{Lie}(\mathrm{GL}(\mathcal{V}))$ coincides with the standard commutator on $\underline{\mathrm{End}}_R(\mathcal{V})$.

Definition 2.3.15. A *super matrix Lie group* is an embedded H^∞ super Lie subgroup $\mathcal{G}$ of the general linear supergroup $\mathrm{GL}(\mathcal{V}) = \mathrm{Aut}(\mathcal{V})$ on some finite-dimensional super Λ-vector space $\mathcal{V}$.

Example 2.3.16 (The super unitary group)**.** On the super Λ-vector space $\mathcal{V} = V \otimes \Lambda$ with $V = \mathbb{C}^{m|n}$, we consider a smooth Hermitian super metric $h : \mathcal{V} \times \mathcal{V} \to \Lambda^{\mathbb{C}}$ which, when restricted to V, takes the form

$$h|_{V\times V} := \begin{pmatrix} \mathbb{1} & 0 \\ 0 & i\mathbb{1} \end{pmatrix} \tag{2.70}$$

The *super unitary group* $\mathrm{U}(m|n)$ is defined as the subgroup of the general group $\mathrm{GL}(\mathcal{V}) \equiv \mathrm{GL}(m|n, \Lambda^{\mathbb{C}})$ consisting of all those group elements preserving h. That is, $g \in \mathrm{GL}(m|n, \Lambda^{\mathbb{C}})$ defines an element of $\mathrm{U}(m|n)$ if and only if

$$h(gv, gw) = h(v, w), \quad \forall v, w \in \mathcal{V} \tag{2.71}$$

It follows that $\mathrm{U}(m|n)$ defines an embedded super Lie subgroup of $\mathrm{GL}(m|n, \Lambda^{\mathbb{C}})$, i.e. super matrix Lie group, with Lie superalgebra $\mathfrak{u}(m|n)$ given by (see e.g. [112])

$$\mathfrak{u}(m|n) = \{X \in \mathfrak{gl}(m|n, \Lambda^{\mathbb{C}})|\, h(Xv, w) + (-1)^{|X||v|} h(v, Xw) = 0\} \tag{2.72}$$

In the literature, for a matrix $X \in \mathfrak{gl}(m|n, \Lambda^{\mathbb{C}})$, one defines its *super adjoint* $X^* \in \mathfrak{gl}(m|n, \Lambda^{\mathbb{C}})$ via

$$h(Xv, w) = (-1)^{|X||v|} h(v, X^* w) \tag{2.73}$$

As can be checked by direct computation, this implies that the super adjoint is given by $X^* = i^{|X|}X^\dagger$ with $X^\dagger$ the ordinary adjoint of X regarded as a morphism between (ungraded) vector spaces. Hence, by (2.73), it follows that $X \in \mathfrak{u}(m|n)$ iff

$$X^* = -X \tag{2.74}$$

From (2.74), one deduces that a generic element $X \in \mathfrak{u}(m|n)$ has to be of the form

$$X = \begin{pmatrix} A & C \\ -iC^\dagger & B \end{pmatrix} \tag{2.75}$$

with $A \in \mathfrak{u}(m)$ and $B \in \mathfrak{u}(n)$ and some arbitrary $n \times m$ matrix C. Thus, $\mathfrak{u}(m|n)$ is a real super vector space of dimension $m^2+n^2|2mn$ and bosonic subalgebra $\mathfrak{u}(m)\oplus\mathfrak{u}(n)$. Since we will come back to this example later in Chapter 5, in what follows, let us focus on the special case $m, n = 1$. According to (2.75), the Lie superalgebra $\mathfrak{u}(1|1)$ of the super unitary group $\mathrm{U}(1|1)$ is generated by the real homogeneous basis

$$X_1 = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix} \quad \Theta_1 = \begin{pmatrix} 0 & 1 \\ -i & 0 \end{pmatrix}, \quad \Theta_2 = \begin{pmatrix} 0 & i \\ -1 & 0 \end{pmatrix} \tag{2.76}$$

which satisfy the following graded commutation relations

$$\begin{aligned} &[X_1, \Theta_1] = \Theta_2, && [X_1, \Theta_2] = -\Theta_1, && [X_i, X_j] = 0 \\ &[X_2, \Theta_1] = -\Theta_2, && [X_2, \Theta_2] = \Theta_1, && [\Theta_i, \Theta_j] = -2\delta_{ij}(X_1 + X_2) \end{aligned} \tag{2.77}$$

for $i, j = 1, 2$. For later purposes, let us discuss the equivalent characterization of the supergroup $\mathrm{U}(1|1)$ in terms of the super Harish-Chandra pair $(\mathrm{U}(1)^2, \mathfrak{u}(1|1))$. By Theorem 2.3.9, it follows that $\mathrm{U}(1|1)$ is diffeomorphic to the split supermanifold $\mathbf{S}(\mathfrak{u}(1|1)_1, \mathrm{U}(1)^2) \cong \mathbf{S}(\mathrm{U}(1))^2 \times (\Lambda \otimes \mathfrak{u}(1|1)_1)_0$ via

$$\Phi : \mathbf{S}(\mathfrak{u}(1|1)_1, \mathrm{U}(1)^2) \to \mathrm{U}(1|1), \ \big(g, Y\big) \mapsto g \cdot \exp(Y) \tag{2.78}$$

The exponential can be computed rather quickly yielding[5]

$$\exp\big(\xi\Theta_1 + \eta\Theta_2\big) = \exp\left(\begin{pmatrix} 0 & \psi \\ i\bar{\psi} & 0 \end{pmatrix}\right) = \begin{pmatrix} 1 + \frac{i}{2}\psi\bar{\psi} & \psi \\ i\bar{\psi} & 1 - \frac{i}{2}\psi\bar{\psi} \end{pmatrix} \tag{2.79}$$

[5] Note that, in contrast to the non-graded case, the matrix representation of an endomorphism $A \in \underline{\mathrm{End}}_R(\mathcal{V})$ on a super Λ-vector space $\mathcal{V}$ is not left linear. More precisely, according to Example 2.3.14, for $\lambda \in \Lambda$, one has $(\lambda A)^i{}_j = \mathfrak{C}^{|e_i|}(\lambda)A^i{}_j$. This explains the additional minus sign involved in the matrix representation of $\xi\Theta_1 + \eta\Theta_2$ in Eq. (2.79).

where we set $\psi := \xi + i\eta$. Hence, the diffeomorphism takes the explicit form

$$\Phi\left(\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, \xi\Theta_1 + \eta\Theta_2\right) = \begin{pmatrix} xA & x\psi \\ iy\bar{\psi} & yA^{-1} \end{pmatrix} \tag{2.80}$$

with $A := 1 + \frac{i}{2}\psi\bar{\psi}$. Moreover, if $x := x^1{}_1$, $y := x^2{}_2$ and $\theta^1 := x^1{}_2$ as well as $\theta^2 := x^2{}_1$ denote the global coordinate functions as defined in Example 2.3.14, this yields

$$\Phi^*(x) = xA, \quad \Phi^*(\theta^1) = x\psi, \quad \Phi^*(\theta^2) = -iy\bar{\psi}, \quad \Phi^*(y) = yA^{-1} \tag{2.81}$$

Example 2.3.17 (The orthosymplectic supergroup)**.** Let $(\mathcal{V}, \mathscr{S}(\cdot,\cdot))$ be a super Λ-vector space $\mathcal{V} = V \otimes \Lambda$ equipped with a smooth super metric $\mathscr{S}: \mathcal{V} \times \mathcal{V} \to \Lambda^{\mathbb{C}}$. The *orthosymplectic supergroup* $\mathrm{OSp}(\mathcal{V})$ is defined as the super Lie subgroup of the general linear supergroup $\mathrm{GL}(\mathcal{V})$ consisting of all those group elements preserving $\mathscr{S}$, i.e., $g \in \mathrm{OSp}(\mathcal{V})$ if and only if

$$\mathscr{S}(gv, gw) = \mathscr{S}(v, w), \quad \forall v, w \in \mathcal{V} \tag{2.82}$$

It follows from the "*Stabilizer Theorem*" (see e.g. Prop. 5.13 in [97] or Prop. 8.4.7. in [106] in the pure algebraic setting) that $\mathrm{OSp}(\mathcal{V})$ defines a super matrix Lie group with corresponding super Lie algebra

$$\mathfrak{osp}(\mathcal{V}) := \{X \in \mathfrak{gl}(\mathcal{V}) |\, \mathscr{S}(Xv, w) + (-1)^{|X||v|}\mathscr{S}(v, Xw)\, \forall v, w \in V\} \tag{2.83}$$

If (e_i, f_j) is homogeneous basis of $\mathcal{V}$ such that $\mathscr{S}$ acquires the standard representation (2.61), the orthosymplectic super Lie group is also simply denoted by $\mathrm{OSp}(m|2n)$. Accordingly. the bosonic sub super Lie algebra takes the form $\mathfrak{osp}(m|2n)_0 = \mathfrak{so}(m) \oplus \mathfrak{sp}(2n)$.

In the following, we want to construct a graded generalization for the isometry group $\mathrm{SO}(2,3)$ of anti-de Sitter spacetime AdS_4.[6] To this end, we consider the following Lie algebra representation of $\mathfrak{so}(2,3)$: Let γ^I, $I = 0, \ldots, 3$, be the gamma matrices of $4D$

[6] Recall from Appendix E, Corollary E.8, that the four-dimensional anti-de Sitter spacetime AdS_4 is defined as the pseudo hyperbolic space $\mathbb{H}^4_1(L)$ defined as an embedded submanifold of the semi-Riemannian manifold $\mathbb{R}^{2,3}$ equipped with the metric $(\eta_{AB}) = \mathrm{diag}(-+++-)$

$$\mathbb{H}^4_1(L) := \{x \in \mathbb{R}^{2,3} |\, \eta_{AB}x^A x^B = -L^2\} \tag{2.84}$$

with L the so-called anti-de Sitter radius

Minkowski spacetime (see Example 2.3.10 and Section 4.2). Similar as in [113, 114], we define totally antisymmetric matrices Ξ^{AB}, $A, B = 0, \ldots, 4$, via

$$\Xi^{IJ} := \frac{1}{2}\gamma^{IJ} := \frac{1}{4}[\gamma^I, \gamma^J]_- \quad \text{as well as} \quad \Xi^{4I} := -\gamma^{I4} := \frac{1}{2}\gamma^I \tag{2.85}$$

where indices are raised and lowered w.r.t. the metric $(\eta_{AB}) = \mathrm{diag}(- + + + -)$. These satisfy the following commutation relations

$$[\Xi_{AB}, \Xi_{CD}] = \eta_{BC}\Xi_{AD} - \eta_{AC}\Xi_{BD} - \eta_{BD}\Xi_{AC} + \eta_{AD}\Xi_{BC} \tag{2.86}$$

and thus provide a representation of $\mathfrak{so}(2,3)$. Indeed, choosing a real representation of the gamma matrices, it follows that the charge conjugation matrix C is of the form $C = -iJ_4$ with J_4 the standard symplectic structure given by (2.61) for $n = 2$ and, by the symmetry properties of the gamma matrices, one has

$$(C\Xi_{AB})^T = C\Xi_{AB} \tag{2.87}$$

Hence, the Ξ_{AB} generate the Lie algebra $\mathfrak{sp}(4)$ of the universal covering group $\mathrm{Sp}(4,\mathbb{R})$ of $\mathrm{SO}(2,3)$. Thus, a candidate for the graded extension of the anti-de Sitter group with $\mathcal{N}$-fermionic generators is given by the orthosymplectic Lie group $\mathrm{OSp}(\mathcal{N}|4)$. We therefore choose $\mathcal{V} = (\Lambda^{\mathbb{C}})^{\mathcal{N}|4}$ as super vector space equipped with the bilinear form

$$\Omega = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & C \end{pmatrix} \tag{2.88}$$

The algebra $\mathfrak{osp}(\mathcal{N}|4)$ is then generated by all $X \in \mathfrak{gl}(\mathcal{V})$ satisfying

$$X^{sT}\Omega + \Omega X = 0 \tag{2.89}$$

where X^{sT} denotes the *super transpose* of X. Writing X in the block form

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} \tag{2.90}$$

(2.89) is equivalent to

$$X_{11}^T = -X, \quad (CX_{22})^T = CX_{22}, \quad X_{12} = -X_{21}^T C \tag{2.91}$$

and therefore, in particular, $X_{11} \in \mathfrak{so}(\mathcal{N})$ and $X_{22} \in \mathfrak{sp}(4)$. Thus, following [115], based on the above observation, we set

$$M_{AB} := \begin{pmatrix} 0 & 0 \\ 0 & \Xi_{AB} \end{pmatrix} \quad \text{and} \quad T^{rs} := \begin{pmatrix} A^{rs} & 0 \\ 0 & 0 \end{pmatrix} \tag{2.92}$$

as generators for the bosonic sub super Lie algebras $\mathfrak{sp}(4)$ and $\mathfrak{so}(\mathcal{N})$, respectively, where $(A^{rs})_{pq} := 2\delta_p^{[r}\delta_q^{s]}$, $p, q, r, s = 1, \ldots, \mathcal{N}$. For the fermionic generators, we set [115]

$$Q_\alpha^r := \begin{pmatrix} 0 & -\bar{e}_\alpha \otimes e_r \\ e_\alpha \otimes e_r^T & 0 \end{pmatrix} \tag{2.93}$$

where $(e_\alpha)_\beta := \delta_{\alpha\beta}$ and $(\bar{e}_\alpha)_\beta := C_{\alpha\beta}$. It then follows by direct computation that

$$[M_{AB}, Q_\alpha^r] = Q_\beta^r(\Xi_{AB})^\beta{}_\alpha \quad \text{and} \quad [T^{pq}, Q_\alpha^r] = \delta^{qr}Q_\alpha^p - \delta^{pr}Q_\alpha^q \tag{2.94}$$

In order to compute the Lie bracket between two fermionic generators, one can use the Fierz identity $2(e_\alpha\bar{e}_\beta + e_\beta\bar{e}_\alpha) = \gamma_I(C\gamma^I)_{\alpha\beta} + \frac{1}{2}\gamma_{JI}(C\gamma^{IJ})_{\alpha\beta}$ (see Eq. (4.8)) where the sum terminates after second order in the gamma matrices as the higher rank gamma matrices are antisymmetric with respect to the charge conjugation C. Thus, one finds

$$[Q_\alpha^r, Q_\beta^s] = \delta^{rs}(C\Xi^{AB})M_{AB} - C_{\alpha\beta}T^{rs} \tag{2.95}$$

Defining $P_I := \Xi_{4I}$ and reintroducing the cosmological constant $\Lambda_{\text{cos}} = -3/L^2$ with L the *anti-de Sitter radius* by rescaling $P_I \to P_I/L$, $Q_\alpha^r \to Q_\alpha^r/\sqrt{2L}$ as well as $T^{rs} \to T^{rs}/2L$, one finally ends up with the following (graded) commutation relations

$$[M_{IJ}, Q_\alpha^r] = \frac{1}{2}Q_\beta^r(\gamma_{IJ})^\beta{}_\alpha \tag{2.96}$$

$$[P_I, Q_\alpha^r] = -\frac{1}{2L}Q_\beta^r(\gamma_I)^\beta{}_\alpha \tag{2.97}$$

$$[T^{pq}, Q_\alpha^r] = \frac{1}{2L}(\delta^{qr}Q_\alpha^p - \delta^{pr}Q_\alpha^q) \tag{2.98}$$

$$[Q_\alpha^r, Q_\beta^s] = \delta^{rs}\frac{1}{2}(C\gamma^I)_{\alpha\beta}P_I + \delta^{rs}\frac{1}{4L}(C\gamma^{IJ})_{\alpha\beta}M_{IJ} - C_{\alpha\beta}T^{rs} \tag{2.99}$$

which is the form we will use in what follows. Performing the Inönü-Wigner contraction, i.e., taking the limit $L \to \infty$, one reobtains the super Poincaré algebra (cf. Def. 2.3.11).

2.4. Super fiber bundles

The content of this section has been reproduced from [1], with slight changes to account for the context of this thesis with the permission of AIP Publishing.

In this section, we want to give a detailed account on super fiber bundles in the category of H^∞ supermanifolds as this will provide us with the necessary mathematical tools needed in the subsequent chapters (see Appendix C for a review on H^∞ supermanifold theory and our choice of conventions; for a relation to algebro-geometric supermanifolds see Section 2.2 as well as Section 2.6 below). If not stated explicitly otherwise, in what follows, we will always work in the category of H^∞ supermanifolds so that smoothness and related notions are always referred to this particular category. Let us start with the basic definition of a super fiber bundle.

Definition 2.4.1 (Super fiber bundle)**.** A *super fiber bundle* $(\mathcal{E},\pi,\mathcal{M},\mathcal{F})$, also simply denoted by $\mathcal{F}\to\mathcal{E}\xrightarrow{\pi}\mathcal{M}$, consists of supermanifolds $\mathcal{E}$, $\mathcal{M}$ and $\mathcal{F}$ called *total space*, *base* and *typical fiber*, respectively, as well as a smooth surjective map $\pi:\,\mathcal{E}\to\mathcal{M}$, called *projection*, satisfying the local triviality property: For any $p\in\mathcal{E}$ there exists an open subset $U\subset\mathcal{M}$ which is an open neighborhood of $\pi(p)$ and a homeomorphism $\phi:\,\pi^{-1}(U)\to U\times\mathcal{F}$ called *local trivialization* such that the following diagram commutes

$$\begin{array}{ccc} \phi:\ \pi^{-1}(U) & \longrightarrow & U\times\mathcal{F} \\ {\scriptstyle\pi}\downarrow & \swarrow{\scriptstyle \mathrm{pr}_1} & \\ U & & \end{array}$$

i.e. $\mathrm{pr}_1\circ\phi=\pi$ where pr_1 denotes the projection onto the first factor.

Proposition 2.4.2. *Let* $\mathcal{F}\to\mathcal{E}\xrightarrow{\pi}\mathcal{M}$ *be a super fiber bundle. Then* $\mathbf{B}(\mathcal{F})\to\mathbf{B}(\mathcal{E})\xrightarrow{\bar{\pi}}\mathbf{B}(\mathcal{M})$*, with* $\bar{\pi}:=\mathbf{B}(\pi)$ *and* $\mathbf{B}:\,\mathbf{SMan}_{H^\infty}\to\mathbf{Man}$ *the body functor, defines a smooth fiber bundle in the category* **Man** *of ordinary* C^∞*-smooth manifolds.*

Proof. This is an immediate consequence of Prop. C.10 as well as the fact that $\mathbf{B}:\,\mathbf{SMan}_{H^\infty}\to\mathbf{Man}$ is a functor. □

Definition 2.4.3. Let $\mathcal{M}$ and $\mathcal{F}$ be supermanifolds, $\mathcal{E}$ an abstract set and $\pi:\,\mathcal{E}\to\mathcal{M}$ a surjective map.

(i) Let $U\subset\mathcal{M}$ be open and $\phi:\,\pi^{-1}(U)\to U\times\mathcal{F}$ a bijective map such that $\mathrm{pr}_1\circ\phi=\pi|_{\pi^{-1}(U)}$, then (U,ϕ_U) is called a *formal bundle chart*.

(ii) A family $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in\Upsilon}$ of formal bundle charts is called a *(smooth) formal bundle atlas* of $\mathcal{E}$ (w.r.t. π) iff $\{U_\alpha\}_{\alpha\in\Upsilon}$ is an open covering of $\mathcal{M}$ and for any $\alpha, \beta \in \Upsilon$ with $U_\alpha \cap U_\beta \neq \emptyset$, the transition functions

$$\phi_\beta \circ \phi_\alpha^{-1} : (U_\alpha \cap U_\beta) \times \mathcal{F} \to (U_\alpha \cap U_\beta) \times \mathcal{F} \tag{2.100}$$

are smooth.

Theorem 2.4.4. *Let $\mathcal{M}$ and $\mathcal{F}$ be supermanifolds of dimensions* $\dim \mathcal{M} = (m, n)$ *and* $\dim \mathcal{F} = (p, q)$*, respectively, $\mathcal{E}$ an abstract set and $\pi : \mathcal{E} \to \mathcal{M}$ a surjective map. Let furthermore $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in\Upsilon}$ be a smooth formal bundle atlas of $\mathcal{E}$ (w.r.t. π). Then, there exists a unique topology and smooth structure on $\mathcal{E}$ such that $\mathcal{E}$ becomes a supermanifold of dimension* $\dim \mathcal{E} = (m + p, n + q)$ *and $(\mathcal{E}, \pi, \mathcal{M}, \mathcal{F})$ a super fiber bundle which is locally trivial w.r.t. the bundle atlas $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in\Upsilon}$.*

Proof. We define a topology on $\mathcal{E}$ by declaring a subset $O \subseteq \mathcal{E}$ to be open if and only if for any $\alpha \in \Upsilon$ the image

$$\phi_\alpha(O \cap \pi^{-1}(U_\alpha)) \subseteq U_\alpha \times \mathcal{F} \tag{2.101}$$

is an open subset in $\mathcal{M} \times \mathcal{F}$ (note that this condition is mandatory in order for ϕ_α to define a homeomorphism). Since the formal bundle charts are bijective, it follows immediately that arbitrary unions and finite intersections of open sets are open. Moreover, the empty set and $\mathcal{E}$ are open as well, so that this indeed defines a topology on $\mathcal{E}$.

By intersection, one may assume that the $\{U_\alpha\}_\alpha$ are coordinate neighborhoods of $\mathcal{M}$. Let then $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\Upsilon}$ and $\{(V_\beta, \psi_\beta)\}_{\beta\in\Sigma}$ be smooth atlases of $\mathcal{M}$ and $\mathcal{F}$, respectively. For $(\alpha, \beta) \in \Upsilon \times \Sigma$ we define open sets $W_{\alpha\beta} := \phi_\alpha^{-1}(U_\alpha \times V_\beta) \subseteq \mathcal{E}$ as well as bijective maps

$$\theta_{\alpha\beta} := (\varphi_\alpha \times \psi_\beta) \circ \phi_\alpha : W_{\alpha\beta} \to \varphi_\alpha(U_\alpha) \times \psi_\alpha(V_\beta) \subseteq \Lambda^{m+p,n+q} \tag{2.102}$$

For $(\alpha', \beta') \in \Upsilon \times \Sigma$ such that $W_{\alpha\beta} \cap W_{\alpha'\beta'} \neq \emptyset$, it then follows

$$\theta_{\alpha'\beta'} \circ \theta_{\alpha\beta}^{-1} = (\varphi_{\alpha'} \times \psi_{\beta'}) \circ \phi_{\alpha'} \circ \phi_\alpha^{-1} \circ (\varphi_\alpha^{-1} \times \psi_\beta^{-1}) \tag{2.103}$$

on $\theta_{\alpha\beta}(W_{\alpha\beta} \cap W_{\alpha'\beta'})$, which is smooth by definition of a formal bundle atlas. Hence, $\{(W_{\alpha\beta}, \theta_{\alpha\beta})\}$ defines a smooth atlas of $\mathcal{E}$ turning it into a proto H^∞ supermanifold (see Def. C.4) of dimension $\dim \mathcal{E} = (m + p, n + q)$.

Let $U \subseteq \mathcal{M}$ be open. Then, for any $\alpha \in \Upsilon$, $\phi_\alpha(\pi^{-1}(U) \cap \pi^{-1}(U_\alpha)) = \phi_\alpha(\pi^{-1}(U \cap U_\alpha)) = (U \cap U_\alpha) \times \mathcal{F}$ is open in $U_\alpha \times \mathcal{F}$ and thus $\pi^{-1}(U) \subseteq \mathcal{E}$ is open proving that

$\pi : \mathcal{E} \longrightarrow \mathcal{M}$ is continuous. To see that is also smooth, let $(\alpha, \beta) \in \Upsilon \times \Sigma$ and $\alpha' \in \Upsilon$ such that $U_{\alpha'} \cap U_\alpha \neq \emptyset$. It follows

$$\varphi_{\alpha'} \circ \pi \circ \theta_{\alpha\beta}^{-1} = \varphi_{\alpha'} \circ \mathrm{pr}_1 \circ (\varphi_\alpha^{-1} \times \psi_\beta^{-1}) : \varphi_\alpha(U_{\alpha'} \cap U_\alpha) \times \psi_\beta(V_\beta) \to \varphi_\alpha(U_{\alpha'} \cap U_\alpha) \tag{2.104}$$

which is obviously smooth. It remains to show that $\mathcal{E}$ is in fact a H^∞ supermanifold, i.e., $\mathbf{B}(\mathcal{E})$ is a second countable Hausdorff topological space.

To prove that it is Hausdorff, consider first $p, q \in \mathbf{B}(\mathcal{E})$ with $\mathbf{B}(\pi)(p) \neq \mathbf{B}(\pi)(q)$. Since $\mathbf{B}(\mathcal{M}) \times \mathbf{B}(\mathcal{F})$ is Hausdorff by assumption, there are open subsets $\mathbf{B}(\pi)(p) \in U_p$ and $\mathbf{B}(\pi)(q) \in U_q$ in $\mathbf{B}(\mathcal{M}) \times \mathbf{B}(\mathcal{F})$ with $U_p \cap U_q \neq \emptyset$. Since, $\mathbf{B}(\pi) : \mathbf{B}(\mathcal{E}) \to \mathbf{B}(\mathcal{M})$ is smooth, these yield disjoint open subsets $p \in \mathbf{B}(\pi)^{-1}(U_p)$ and $q \in \mathbf{B}(\pi)^{-1}(U_q)$ in $\mathbf{B}(\mathcal{E})$ separating p and q. If $\mathbf{B}(\pi)(p) = \mathbf{B}(\pi)(q)$, consider $\alpha \in \Upsilon$ with $p, q \in \mathbf{B}(\pi)^{-1}(\mathbf{B}(U_\alpha))$, where $\mathbf{B}(\phi) : \mathbf{B}(\pi)^{-1}(\mathbf{B}(U_\alpha)) \to \mathbf{B}(U_\alpha) \times \mathbf{B}(\mathcal{F})$ is the corresponding bundle chart on the body which, in particular, defines a homeomorphism. Let then $V_p, V_q \subset \mathbf{B}(\mathcal{F})$ be disjoint open subsets with $\mathbf{B}(\phi)(p) \in V_p$ and $\mathbf{B}(\phi)(q) \in V_q$. This then finally yields disjoint open subsets $p \in \mathbf{B}(\phi)^{-1}(\mathbf{B}(U_\alpha) \times V_p)$ and $q \in \mathbf{B}(\phi)^{-1}(\mathbf{B}(U_\alpha) \times V_q)$ in $\mathbf{B}(\mathcal{E})$ seperating p and q. This shows that $\mathbf{B}(\mathcal{E})$ is indeed Hausdorff. That $\mathbf{B}(\mathcal{E})$ is also second countable follows similarly using that $\mathbf{B}(\mathcal{M}) \times \mathbf{B}(\mathcal{F})$ is second countable. Hence, this proves that $\mathcal{E}$ indeed defines a H^∞ supermanifold and that $(\mathcal{E}, \pi, \mathcal{M}, \mathcal{F})$ is a super fiber bundle. □

Example 2.4.5 (Pullback bundle)**.** Given a smooth map $f : \mathcal{N} \to \mathcal{M}$ between supermanifolds and a super fiber bundle $(\mathcal{E}, \pi, \mathcal{M}, \mathcal{F})$, one can construct a new bundle by setting

$$f^*\mathcal{E} := \{(x, p) \in \mathcal{N} \times \mathcal{E} | f(x) = \pi(p)\} \subseteq \mathcal{N} \times \mathcal{E} \tag{2.105}$$

as total space together with the projection

$$\pi_f : f^*\mathcal{E} \to \mathcal{N},\ (x, p) \mapsto x \tag{2.106}$$

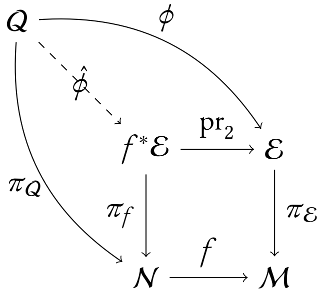
that is, fibers over $\mathcal{M}$ are pulled back w.r.t. f to fibers over $\mathcal{N}$. Let $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in\Upsilon}$ be a smooth bundle atlas on $\mathcal{E}$. For $\alpha \in \Upsilon$, define the map $\psi_\alpha : f^*\mathcal{E} \supseteq \pi_f^{-1}(U_\alpha) \to U_\alpha \times \mathcal{F}$ via

$$\psi_\alpha(x, p) := (x, \mathrm{pr}_2 \circ \phi_\alpha(p)) \tag{2.107}$$

It is clear by definition that ψ_α is bijective and fiber-preserving. Moreover, for $\alpha, \beta \in \Upsilon$ with $U_\alpha \cap U_\beta \neq \emptyset$, we compute

$$\psi_\beta \circ \psi_\alpha^{-1}(x, p) = \psi_\beta(x, \phi_\alpha^{-1}(f(x), p)) = (x, \mathrm{pr}_2 \circ \phi_\beta \circ \phi_\alpha^{-1}(f(x), p)) \tag{2.108}$$

$\forall (x, p) \in (U_\alpha \cap U_\beta) \times \mathcal{F}$ and thus is smooth. It follows that $\{(U_\alpha, \psi_\alpha)\}_{\alpha\in\Upsilon}$ satisfies the properties of a formal bundle atlas so that, by Theorem 2.4.4, $(f^*\mathcal{E}, \pi_f, \mathcal{N}, \mathcal{F})$ has the structure of a super fiber bundle called the *pullback bundle* of $\mathcal{E}$ w.r.t. f. The pullback bundle defines the *pullback* in the category of super fiber bundles. More precisely, it satisfies the following universal property: Given super fiber bundles $\mathcal{E} \overset{\pi_{\mathcal{E}}}{\to} \mathcal{M}$ and $\mathcal{Q} \overset{\pi_{\mathcal{Q}}}{\to} \mathcal{N}$ as well as a smooth bundle morphism $(\phi, f) : (\mathcal{Q}, \mathcal{N}) \to (\mathcal{E}, \mathcal{M})$, there exists a unique amooth bundle morphism $(\hat{\phi}, \mathrm{id}_{\mathcal{N}}) : (\mathcal{Q}, \mathcal{N}) \to (f^*\mathcal{E}, \mathcal{N})$ such that the following diagram commutes



(2.109)

In fact, $\hat{\phi}$ has to be of the form $\hat{\phi} : \mathcal{Q} \ni q \mapsto (\pi_{\mathcal{Q}}(q), \phi(q))$ and therefore is smooth.

We next consider a particular subclass of super fiber bundles whose typical fiber carries the structure of a super Λ-vector space (see also [97]).

Definition 2.4.6 (Super vector bundle)**.** A *super vector bundle of rank $m|n$* is a super fiber bundle $\mathcal{V} \to \mathcal{E} \overset{\pi}{\to} \mathcal{M}$ such that

(i) the typical fiber is a super Λ-vector space $\mathcal{V}$ of dimension $\dim \mathcal{V} = m|n$.

(ii) for each $x \in \mathcal{M}$, the fibers $\mathcal{E}_x := \pi^{-1}(\{x\})$ have the structure of free super Λ-modules.

(iii) there exists a smooth bundle atlas $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in \Upsilon}$ of $\mathcal{E}$ such that, for each $\alpha \in \Upsilon$, the induced map

$$\begin{aligned} \phi_{\alpha,x} : \mathcal{E}_x &\to \mathcal{V} \\ v_x &\mapsto \mathrm{pr}_2 \circ \phi_\alpha(v_x) \end{aligned} \tag{2.110}$$

$\forall x \in U_\alpha$ is a (right linear) isomorphism of super Λ-modules.

A super vector bundle of rank $1|1$ is also called a *super line bundle*.

Lemma 2.4.7 (after [97])**.** *Let $\mathcal{V}$ and $\mathcal{W}$ be super Λ-vector spaces, $\mathcal{M}$ a supermanifold and $\phi : \mathcal{M} \times \mathcal{V} \to \mathcal{W}$ a smooth map such that $\phi(p, \cdot) : \mathcal{V} \to \mathcal{W} \in \underline{\mathrm{End}}_R(\mathcal{V}, \mathcal{W})$*

is a right linear map for any $p \in \mathcal{M}$. Then, there exists a unique smooth map $\psi : \mathcal{M} \to \underline{\mathrm{End}}_R(\mathcal{V}, \mathcal{W})$ such that $\psi(p)(v) = \phi(p, v)$ for any $p \in \mathcal{M}$, $v \in \mathcal{V}$.

Proof. Let $(e_i)_{i=1,\dots,n}$ and $(f_j)_{j=1,\dots,m}$ be real homogeneous bases of $\mathcal{V}$ and $\mathcal{W}$, respectively, and $({}^j f)_{j=1,\dots,m}$ the corresponding right dual basis of $\mathcal{W}^* := \underline{\mathrm{Hom}}_R(\mathcal{W}, \Lambda)$. Since e_i are body points and ${}^j f : \mathcal{W} \to \Lambda$ are smooth, the maps ${\phi^j}_i := {}^j f \circ \phi(\,\cdot\,, e_i) : \mathcal{M} \to \Lambda$ are of class H^∞ for any $i = 1, \dots, n$ and $j = 1, \dots, m$. Hence, we can define a smooth map $\psi : \mathcal{M} \to \underline{\mathrm{End}}_R(\mathcal{V}, \mathcal{W})$ via $\psi(p) := f_j \otimes {\phi^j}_i(p) e^i \; \forall p \in \mathcal{M}$ which, by construction, satisfies $\psi(p)(v) = \phi(p, v)$ for any $v \in \mathcal{V}$. That ψ is unique is immediate. □

Remark 2.4.8. Given a super vector bundle $\mathcal{V} \to \mathcal{E} \to \mathcal{M}$ and local trivializations (U_α, ϕ_α) and (U_β, ϕ_β), this yields the map

$$\mathrm{pr}_2 \circ (\phi_\alpha \circ \phi_\beta^{-1}) : (U_\alpha \cap U_\beta) \times \mathcal{V} \to \mathcal{V} \tag{2.111}$$

which, in particular, is linear in the second argument. Thus, using Lemma 2.4.7, this yields a smooth map

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(\mathcal{V}) \tag{2.112}$$

satisfying $g_{\alpha\beta}(x)v = \mathrm{pr}_2 \circ (\phi_\alpha \circ \phi_\beta^{-1})(x, v)$, that is, $g_{\alpha\beta}(x) = \phi_{\alpha,x} \circ \phi_{\beta,x}^{-1}$, which we will call *transition maps*.

Proposition 2.4.9 (after [97]). *There is a one-to-one correspondence between local trivializations of a super vector bundle $\mathcal{V} \to \mathcal{E} \xrightarrow{\pi} \mathcal{M}$ and families $(X_i)_i$ of smooth local sections $X_i : U \to \mathcal{E}$ of $\mathcal{E}$, $U \subset \mathcal{M}$ open, such that $(X_{ix})_i$ defines a homogeneous basis of the super Λ-module $\mathcal{E}_x$ $\forall x \in U$.*

Proof. Let (U, ϕ_U) be a local trivialization of $\mathcal{E}$. If $(e_i)_i$ is a real homogeneous of the super Λ-vector space $\mathcal{V}$, define $X_{ix} \equiv X_i(x) := \phi_U^{-1}(x, e_i)$ $\forall x \in U$. Since the e_i are body points, it follows that the X_i define smooth local sections of $\mathcal{E}$ and $(X_{ix})_i$ is a homogeneous basis of $\mathcal{E}_x$ $\forall x \in U$. Conversely, if $(X_i)_i$ is a family of smooth local sections $X_i : U \to \mathcal{E}$ of $\mathcal{E}$ such that, for any $x \in \mathcal{M}$, $(X_{ix})_i$ defines a homogeneous basis of $\mathcal{E}_x$ $\forall x \in U$, consider the map

$$\psi : U \times \mathcal{V} \to \pi^{-1}(U) \subseteq \mathcal{E}, \; (x, v^i e_i) \mapsto v^i X_{ix} \tag{2.113}$$

By definition, ψ is bijective, smooth and an isomorphism of super Λ-modules on each fiber. Hence, it remains to show that has a smooth inverse. To see this, following

[97], let (W, ϕ_W) be a local trivialization of $\mathcal{E}$ with $U \cap W \neq \emptyset$. Then, $\phi_W \circ \psi$: $(U \cap W) \times \mathcal{V} \to (U \cap W) \times \mathcal{V}$ is of the form

$$\phi_W \circ \psi(x, v) = (x, v^i \widetilde{\phi}_i^{\ j}(x) e_j) \tag{2.114}$$

with smooth coefficients $\widetilde{\phi}_i^{\ j} := \langle \mathrm{pr}_2 \circ \phi_W(\,\cdot\,, e_i)|\,{}^i e\rangle$ defining an invertible matrix which has a smooth inverse yielding a smooth inverse of $\phi_W \circ \psi$. But, as ϕ_W is a diffeomorphism, this implies that ψ is a local diffeomorphism and hence admits a smooth inverse $\psi^{-1} : \pi^{-1}(U) \to U \times \mathcal{V}$ providing a local trivialization of $\mathcal{E}$. □

Corollary 2.4.10 (after [97]). *A super vector bundle $\mathcal{V} \to \mathcal{E} \xrightarrow{\pi} \mathcal{M}$ is trivial if and only if there exists a family $(X_i)_i$ of smooth global sections $X_i \in \Gamma(\mathcal{E})$ of $\mathcal{E}$ such that $(X_{ix})_i$ defines a homogeneous basis of the super Λ-module $\mathcal{E}_x$ $\forall x \in \mathcal{M}$.* □

Example 2.4.11 (The dual vector bundle). Recall that, to a super Λ-module $\mathcal{V}$, one can associate its corresponding left dual ${}^*\mathcal{V}$ defined as the super Λ-module ${}^*\mathcal{V} := \underline{\mathrm{Hom}}_L(\mathcal{V}, \Lambda)$. Analogously, one defines the right dual $\mathcal{V}^* := \underline{\mathrm{Hom}}_R(\mathcal{V}, \Lambda)$. Thus, given a super vector bundle $\mathcal{V} \to \mathcal{E} \xrightarrow{\pi} \mathcal{M}$, we can construct the corresponding *left dual bundle* as follows. As total space, we set

$${}^*\mathcal{E} := \coprod_{x \in \mathcal{M}} {}^*\mathcal{E}_x \tag{2.115}$$

together with the surjective map

$$\pi_{{}^*\mathcal{E}} : {}^*\mathcal{E} \to \mathcal{M}, {}^*\mathcal{E}_x \ni T_x \mapsto x \tag{2.116}$$

Let $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in \Upsilon}$ be a smooth bundle atlas on $\mathcal{E}$. Then, for any $\alpha \in \Upsilon$, define the map

$$\begin{aligned} \phi_\alpha^* : {}^*\mathcal{E} \supseteq \pi_{{}^*\mathcal{E}}^{-1}(U_\alpha) &\to U_\alpha \times {}^*\mathcal{V} \\ T_x &\mapsto (x, \phi_\alpha^*(T_x)) \end{aligned} \tag{2.117}$$

where $\phi_\alpha^*(T_x) \in {}^*\mathcal{V}$ is defined as $\langle v|\phi_\alpha^*(T_x)\rangle := \langle\langle v|\phi_{\alpha,x}^{-1}\rangle\,|T_x\rangle$ $\forall v \in \mathcal{V}$. It follows that ϕ_α^* is bijective, fiber-preserving and an isomorphism of super Λ-modules on each fiber. The inverse is given by

$$\phi_\alpha^{*-1} : U_\alpha \times {}^*\mathcal{V} \to \pi_{{}^*\mathcal{E}}^{-1}(U_\alpha),\ (x, \ell) \mapsto (\phi_\alpha^{-1})^* \ell_x \in {}^*\mathcal{E}_x \tag{2.118}$$

where $\langle v_x|(\phi_\alpha^{-1})^*\ell_x\rangle := \langle\langle v_x|\phi_{\alpha,x}\rangle\,|\ell\rangle\ \forall v_x \in \mathcal{E}_x,\ x \in \mathcal{M}$. For $\alpha, \beta \in \Upsilon$ with $U_\alpha \cap U_\beta \neq \emptyset$, the transition function $\phi_\beta^* \circ \phi_\alpha^{*-1} : (U_\alpha \cap U_\beta) \times {}^*\mathcal{V} \to (U_\alpha \cap U_\beta) \times {}^*\mathcal{V}$ takes the form

$$\phi_\beta^* \circ \phi_\alpha^{*-1}(x,\ell) = \phi_\beta^*((\phi_\alpha^{-1})^*\ell_x) = (x, \phi_\beta^*((\phi_\alpha^{-1})^*\ell_x)) \tag{2.119}$$

Choosing a real homogeneous basis $(e_i)_i$ of $\mathcal{V}$, the r.h.s. of (2.119) becomes

$$\begin{aligned}\langle e_i|\phi_\beta^*((\phi_\alpha^{-1})^*\ell_x)\rangle &= \langle\langle e_i|\phi_{\beta,x}^{-1}\rangle\,|(\phi_\alpha^{-1})^*\ell_x)\rangle = \langle\langle e_i|\phi_{\beta,x}^{-1} \diamond \phi_{\alpha,x}\rangle\,|\ell\rangle \\ &= (g_{\alpha\beta})^{sT}{}_i^{\ j}(x)\ell_j\end{aligned} \tag{2.120}$$

with "sT" denoting super transposition, for any $x \in U_\alpha$ and thus is smooth. Hence, it follows that $\{(U_\alpha, \phi_\alpha^*)\}_{\alpha\in\Upsilon}$ defines a formal bundle atlas of ${}^*\mathcal{E}$ so that, by Theorem 2.4.3, ${}^*\mathcal{V} \to {}^*\mathcal{E} \to \mathcal{M}$ has the structure of a super vector bundle called the *left dual super vector bundle* of $\mathcal{E}$. Analogously, one defines the *right dual super vector bundle* $\mathcal{V}^* \to \mathcal{E}^* \to \mathcal{M}$ of $\mathcal{E}$.

Suppose $\mathcal{V} \to \mathcal{E} \xrightarrow{\pi} \mathcal{M}$ is a super vector bundle and $(X_i)_i$ a family of smooth local sections $X_i : U \to \mathcal{E}$ of $\mathcal{E}$ over $U \subseteq \mathcal{M}$ open such that, for any $x \in U$, $(X_{ix})_i$ defines a homogeneous basis of the super Λ-module $\mathcal{E}_x$. According to Prop. 2.4.9, this yields a local trivialization (U, ϕ_U) of $\mathcal{E}$ with the inverse given by

$$\phi_U^{-1} : U \times \mathcal{V} \to \pi^{-1}(U) \subseteq \mathcal{E},\ (x, v^i e_i) \mapsto v^i X_{ix} \tag{2.121}$$

with $(e_i)_i$ a real homogeneous basis of $\mathcal{V}$. By (2.118), this in turn induces a local local trivialization (U, ϕ_U^*) of the left dual bundle ${}^*\mathcal{E}$ with inverse

$$\phi_U^{*-1} : U \times {}^*\mathcal{V} \to \pi_{{}^*\mathcal{E}}^{-1}(U),\ (x,\ell) \mapsto (\phi^{-1})^*\ell_x \in {}^*\mathcal{E}_x \tag{2.122}$$

Hence, if $({}^ie)_i$ denotes the corresponding left dual basis of ${}^*\mathcal{V}$ satisfying $\langle e_i|{}^je\rangle = \delta_i^j$, again by Prop. 2.4.9, this yields a family $({}^i\omega)_i$ of smooth local sections ${}^i\omega \in \Gamma_U({}^*\mathcal{E})$ of ${}^*\mathcal{E}$ given by

$${}^i\omega := \phi_U^{*-1}(\,\cdot\,, {}^ie) \tag{2.123}$$

such that, for any $x \in U$, $({}^i\omega_x)_i$ defines a homogeneous basis of the corresponding left dual super Λ-module ${}^*\mathcal{E}_x$. Evaluation on the X_i then yields

$$\begin{aligned}\langle X_{ix}|{}^j\omega_x\rangle &= \langle X_{ix}|(\phi_U^{-1})^{*\,j}e_x\rangle = \langle\langle X_{ix}|\phi_{U,x}\rangle\,|{}^je\rangle = \langle\langle\langle e_i|\phi_{U,x}^{-1}\rangle\,|\phi_{U,x}\rangle\,|{}^je\rangle \\ &= \langle\langle e_i|\phi_{U,x}^{-1} \diamond \phi_{U,x}\rangle\,|{}^je\rangle = \langle e_i|{}^je\rangle = \delta_i^j\end{aligned} \tag{2.124}$$

$\forall x \in U$, that is, $\langle X_i|{}^j\omega\rangle = \delta_i^j\ \forall i, j$. Hence, we have established the following.

Proposition 2.4.12. *Let $\mathcal{V} \to \mathcal{E} \xrightarrow{\pi} \mathcal{M}$ be a super vector bundle, $(X_i)_{i=1,\ldots,n}$ a family of local sections $X_i : U \to \mathcal{E}$ of $\mathcal{E}$ over $U \subseteq \mathcal{M}$ open such that, for any $x \in U$, $(X_{ix})_i$ defines a homogeneous basis of the super Λ-module $\mathcal{E}_x$. Then, there exists a family $({}^i\omega)_{i,\ldots,n}$ of smooth local sections ${}^i\omega \in \Gamma_U({}^*\mathcal{E})$ of the corresponding left dual super vector bundle ${}^*\mathcal{V} \to {}^*\mathcal{E} \to \mathcal{M}$ such that $({}^i\omega_x)_i$ defines a homogeneous basis of ${}^*\mathcal{E}_x$ $\forall x \in U$ and are dual in the sense that*

$$\langle X_i | {}^j\omega\rangle = \delta_i^j \tag{2.125}$$

$\forall i, j = 1, \ldots, n$ □

Example 2.4.13 (Maurer-Cartan form)**.** Given a super Lie group $\mathcal{G}$, one can choose a real homogeneous basis $(X_{ie})_i$ of the super Lie module $\mathrm{Lie}(\mathcal{G}) \equiv T_e\mathcal{G} = \Lambda \otimes \mathfrak{g}$. It then follows that the corresponding left-invariant vector fields $(X_i)_i$ induce a homogeneous basis of the tangent module $T_g\mathcal{G}$ at any $g \in \mathcal{G}$. Thus, via Prop. 2.4.12, this in turn induces a basis $({}^i\omega)_i$ of smooth 1-forms ${}^i\omega \in \Omega^1(\mathcal{M}) := \Gamma({}^*T\mathcal{M})$, that is, smooth sections of the *left dual cotangent bundle* ${}^*T\mathcal{M}$. It follows immediately from the left-invariance of the X_i that the 1-forms ${}^i\omega$ are also left-invariant.

The *Maurer-Cartan form* on $\mathcal{G}$ is defined as the $\mathrm{Lie}(\mathcal{G})$-valued 1-form $\theta_{\mathrm{MC}} \in \Omega^1(\mathcal{G}, \mathfrak{g}) := \Omega^1(\mathcal{G}) \otimes \mathfrak{g}$ given by

$$\theta_{\mathrm{MC}} := {}^i\omega \otimes X_i \tag{2.126}$$

By definition, the Maurer-Cartan form is left-invariant. Moreover, using the generalized tangent map (see Definition 2.5.9 in Section 2.5), it follows that one can also write $(\theta_{\mathrm{MC}})_g = L_{g^{-1}*}$ $\forall g \in \mathcal{G}$ where $L_g := \mu_{\mathcal{G}}(g, \cdot)$ (resp. $R_g := \mu_{\mathcal{G}}(\cdot, g)$) denotes the *left (resp. right) translation* on $\mathcal{G}$ w.r.t. $g \in \mathcal{G}$.

Example 2.4.14 (Tensor product of super vector bundles)**.** Let $\mathcal{V} \to \mathcal{E} \to \mathcal{M}$ and $\widetilde{\mathcal{V}} \to \widetilde{\mathcal{E}} \to \mathcal{M}$ be super vector bundles. Set

$$\mathcal{E} \otimes \widetilde{\mathcal{E}} := \coprod_{x \in \mathcal{M}} \mathcal{E}_x \otimes_\Lambda \widetilde{\mathcal{E}_x} \tag{2.127}$$

together with the surjective map

$$\pi_\otimes : \mathcal{E} \otimes \widetilde{\mathcal{E}} \to \mathcal{M},\ \mathcal{E}_x \otimes \widetilde{\mathcal{E}}_x \ni v_x \mapsto x \tag{2.128}$$

Let $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in \Upsilon}$ and $\{(V_{\alpha'}, \widetilde{\phi}_{\alpha'})\}_{\alpha' \in \Sigma}$ be smooth bundle atlases of $\mathcal{E}$ and $\widetilde{\mathcal{E}}$, respectively. For $(\alpha, \alpha') \in \Upsilon \times \Sigma$ with $U_{\alpha,\alpha'} := U_\alpha \cap V_{\alpha'} \neq \emptyset$, define

$$\begin{aligned} \psi_{\alpha,\alpha'} : \mathcal{E} \otimes \widetilde{\mathcal{E}} \supseteq \pi_\otimes^{-1}(U_{\alpha,\alpha'}) &\to U_{\alpha,\alpha'} \times (\mathcal{V} \otimes_\Lambda \widetilde{\mathcal{V}}) \\ v_x \otimes w_x &\mapsto (x, (\phi_\alpha \otimes \widetilde{\phi}_{\alpha'})(v_x \otimes w_x)) \end{aligned} \tag{2.129}$$

Then, $\{(U_{\alpha,\alpha'},\psi_{\alpha,\alpha'})\}_{(\alpha,\alpha')\in\Upsilon\times\Sigma}$ defines a formal bundle atlas of $\mathcal{E}\otimes\widetilde{\mathcal{E}}$ turning $\mathcal{V}\otimes\widetilde{\mathcal{V}}\to\mathcal{E}\otimes\widetilde{\mathcal{E}}\stackrel{\pi_{\otimes}}{\to}\mathcal{M}$ into a super vector bundle called the *tensor product super vector bundle*.

Definition 2.4.15 (Principal super fiber bundle)**.** A *principal super fiber bundle* is a super fiber bundle $\mathcal{G}\to\mathcal{P}\stackrel{\pi}{\to}\mathcal{M}$ such that

(i) the typical fiber is a super Lie group $\mathcal{G}$.

(ii) the total space $\mathcal{P}$ is equipped with a smooth $\mathcal{G}$-right action $\Phi:\mathcal{P}\times\mathcal{G}\to\mathcal{P}$ preserving the fibers, that is, $\pi\circ\Phi=\pi\circ\mathrm{pr}_1$, or explicitly, $p\cdot g:=\Phi(p,g)\in\mathcal{P}_x$ $\forall g\in\mathcal{G}$ and $p\in\mathcal{P}$ with $\pi(p)=x\in\mathcal{M}$.

(iii) there exists a smooth bundle atlas $\{(U_\alpha,\phi_\alpha)\}_{\alpha\in\Upsilon}$ of $\mathcal{P}$ such that, for each $\alpha\in\Upsilon$, the bundle chart $\phi_\alpha:\pi^{-1}(U_\alpha)\to U_\alpha\times\mathcal{G}$ is $\mathcal{G}$-equivariant, i.e.

$$\phi_\alpha\circ\Phi=(\mathrm{id}\times\mu_{\mathcal{G}})\circ(\phi_\alpha\times\mathrm{id}_{\mathcal{G}}) \tag{2.130}$$

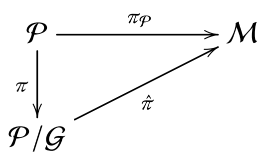
where $U_\alpha\times\mathcal{G}$ is equipped with the standard $\mathcal{G}$-right action $\mathrm{id}\times\mu_{\mathcal{G}}:(U_\alpha\times\mathcal{G})\times\mathcal{G}\to U_\alpha\times\mathcal{G},\ ((x,g),g')\mapsto(x,gg')$.

Proposition 2.4.16. *Let $\mathcal{G}\to\mathcal{P}\stackrel{\pi_{\mathcal{P}}}{\to}\mathcal{M}$ be a principal super fiber bundle, then the orbit space $\mathcal{P}/\mathcal{G}$ equipped with the quotient topology can be given the structure a supermanifold such that $\mathcal{P}/\mathcal{G}$ is canonically isomorphic to $\mathcal{M}$ and the body $\mathbf{B}(\mathcal{P}/\mathcal{G})$ is isomorphic to $\mathbf{B}(\mathcal{P})/\mathbf{B}(\mathcal{G})$ in the sense of ordinary smooth manifolds. Moreover, the sheaf $H^\infty_{\mathcal{P}/\mathcal{G}}$ of smooth functions on $\mathcal{P}/\mathcal{G}$ is canonically isomorphic to the quotient sheaf*

$$H^\infty_{\mathcal{P}}/H^\infty_{\mathcal{G}}:\mathcal{P}/\mathcal{G}\supset U\to(H^\infty_{\mathcal{P}}/H^\infty_{\mathcal{G}})(U):=\{f\in H^\infty(\pi^{-1}(U))|\,\Phi^*(f)=f\otimes 1_{\mathcal{G}}\} \tag{2.131}$$

where $\pi:\mathcal{P}\to\mathcal{P}/\mathcal{G}$ is the canonical projection.

Proof. Since $\pi_{\mathcal{P}}\circ\Phi=\pi_{\mathcal{P}}\circ\mathrm{pr}_1$, $\pi_{\mathcal{P}}$ is constant on $\mathcal{G}$-orbits. Hence, by universal property of the quotient, there exists a unique continuous map $\hat{\pi}:\mathcal{P}/\mathcal{G}\to\mathcal{M}$ such that the following diagram commutes



It follows immediately that $\hat{\pi}$ is bijective. Moreover, since $\pi_{\mathcal{P}}$ is open as a bundle map, it follows, by definition of the quotient topology, that $\hat{\pi}$ is a homeomorphism. Choosing an atlas $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in\Upsilon}$ on $\mathcal{M}$, this yields a corresponding atlas $\{(V_\alpha, \psi_\alpha)\}_{\alpha\in\Upsilon}$ of $\mathcal{P}/\mathcal{G}$ by setting $V_\alpha := \hat{\pi}^{-1}(U_\alpha)$ and $\psi_\alpha := \phi_\alpha \circ \hat{\pi} : V_\alpha \to \phi_\alpha(U_\alpha) \subset \Lambda^{m,n}\ \forall\alpha \in \Upsilon$, where $\dim\mathcal{M} = (m,n)$. In this way, $\mathcal{P}/\mathcal{G}$ becomes a supermanifold diffeomorphic to $\mathcal{M}$ via $\hat{\pi}$. By Prop. 2.4.2, $\mathbf{B}(\mathcal{G}) \to \mathbf{B}(\mathcal{P}) \to \mathbf{B}(\mathcal{M})$ defines a ordinary smooth principal fiber bundle over $\mathbf{B}(\mathcal{M})$ with structure group $\mathbf{B}(\mathcal{G})$. Hence, arguing as above, one concludes that $\mathbf{B}(\mathcal{P})/\mathbf{B}(\mathcal{G})$ is canonically diffeomorphic to $\mathbf{B}(\mathcal{M})$. To summarize, we have the isomorphism

$$\mathbf{B}(\mathcal{P}/\mathcal{G}) \xrightarrow{\mathbf{B}(\hat{\pi})} \mathbf{B}(\mathcal{M}) \xrightarrow{\cong} \mathbf{B}(\mathcal{P})/\mathbf{B}(\mathcal{G}) \tag{2.132}$$

Finally, let $U \subseteq \mathcal{P}/\mathcal{G}$ be an open subset and $f \in H^\infty(\pi^{-1}(U))$ a smooth map with $\Phi^*(f) = f \otimes 1_{\mathcal{G}}$. Then, f is constant on $\mathcal{G}$-orbits such that, by the universal property of the quotient, there exists a unique continuous map $\tilde{f} : U \to \Lambda$ with $\tilde{f} \circ \pi = f$. By definition of the differential structure on $\mathcal{P}/\mathcal{G}$, it follows immediately that $\tilde{f}$ is smooth, i.e., $\tilde{f} \in H^\infty(U)$. Conversely, if $g \in H^\infty(U)$, then $g' := \pi^* g = g \circ \pi|_{\pi^{-1}(U)} \in H^\infty(\pi^{-1}(U))$ satisfying $\Phi^*(g') = \Phi^* \circ \pi^*(g) = (\pi \circ \Phi)^*(g) = (\pi \circ \mathrm{pr}_1)^*(g) = \mathrm{pr}_1^* \circ \pi^*(g) = g' \otimes 1_{\mathcal{G}}$, i.e., $g' \in (H^\infty_{\mathcal{P}}/H^\infty_{\mathcal{G}})(U)$. This closes the prove of the above proposition. □

Proposition 2.4.17 (after [97]). *There is a one-to-one correspondence between local trivializations of a principal super fiber bundle $\mathcal{G} \to \mathcal{P} \xrightarrow{\pi} \mathcal{M}$ and smooth local sections $s \in \Gamma_U(\mathcal{P})$ of $\mathcal{P}$ for any $U \subseteq \mathcal{M}$ open.*

Proof. If (U, ϕ_U) is a local trivialization of $\mathcal{P}$, the map $s : \pi^{-1}(U) \ni p \mapsto s(x) := \phi_U^{-1}(x, e) \in \mathcal{P}_x$ defines, as $e \in \mathbf{B}(\mathcal{G})$, a smooth local section of $\mathcal{P}$ over U. Conversely, if $s \in \Gamma_U(\mathcal{P})$ is a smooth local section of $\mathcal{P}$ over $U \subseteq \mathcal{M}$, consider the map

$$\psi : U \times \mathcal{G} \to \pi^{-1}(U),\ (x, g) \mapsto \Phi(s(x), g) \tag{2.133}$$

Then, ψ defines a smooth fiber-preserving and $\mathcal{G}$-equivariant map which, in particular, is bijective, as the $\mathcal{G}$-right action $\Phi : \mathcal{P} \times \mathcal{G} \to \mathcal{P}$ on $\mathcal{P}$ is free and transitive. That the inverse is also smooth follows as in proof of Prop. 2.4.9. □

Example 2.4.18 (The frame bundle $\mathscr{F}(\mathcal{E})$). Given a super vector bundle $\mathcal{V} \to \mathcal{E} \xrightarrow{\pi} \mathcal{M}$, one can construct a new bundle as follows. For any $x \in \mathcal{M}$ we define a *frame at x* as an isomorphism of super Λ-modules $p_x : \mathcal{V} \to \mathcal{E}_x$. Let $\mathscr{F}(\mathcal{E})_x$ denote the set of all frames at $x \in \mathcal{M}$. We set

$$\mathscr{F}(\mathcal{E}) := \coprod_{x\in\mathcal{M}} \mathscr{F}(\mathcal{E})_x \tag{2.134}$$

together with the surjective map

$$\pi_{\mathscr{F}}: \mathscr{F}(\mathcal{E}) \to \mathcal{M}, \; \mathscr{F}(\mathcal{E})_x \ni p_x \mapsto x \in \mathcal{M} \tag{2.135}$$

Furthermore, we introduce a $\mathrm{GL}(\mathcal{V})$-right action on $\mathscr{F}(\mathcal{E})$ via

$$\begin{aligned} \Phi: \mathscr{F}(\mathcal{E}) \times \mathrm{GL}(\mathcal{V}) &\mapsto \mathscr{F}(\mathcal{E}) \\ (p_x, g) &\mapsto (p_x \circ g)_x \end{aligned} \tag{2.136}$$

It follows that Φ is fiber-preserving, i.e., $\Phi(p_x, g) \in \mathscr{F}(\mathcal{E})_x \; \forall p_x \in \mathscr{F}(\mathcal{E})_x, g \in \mathrm{GL}(\mathcal{V})$ and $x \in \mathcal{M}$, and therefore is also free and transitive on each fiber. Let $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in \Upsilon}$ be a smooth bundle atlas on $\mathcal{E}$. Then, for any $\alpha \in \Upsilon$ and $x \in U_\alpha$, the induced map $\phi_{\alpha,x}^{-1}: \mathcal{V} \to \mathcal{E}_x$ canonically yields a frame at x. Hence, let us define the new map

$$\begin{aligned} \psi_\alpha: U_\alpha \times \mathrm{GL}(\mathcal{V}) &\to \pi_{\mathscr{F}}^{-1}(U_\alpha) \subseteq \mathscr{F}(\mathcal{E}) \\ (x, g) &\mapsto \Phi(\phi_{\alpha,x}^{-1}, g) \end{aligned} \tag{2.137}$$

This map is bijective with inverse $\psi_\alpha^{-1}(p_x) = (x, \phi_{\alpha,x} \circ p_x) \; \forall p_x \in \mathscr{F}(\mathcal{E})_x, x \in U_\alpha$ (note that indeed $\phi_{\alpha,x} \circ p_x \in \mathrm{GL}(\mathcal{V})$). Let $\alpha, \beta \in \Upsilon$ with $U_\alpha \cap U_\beta \neq \emptyset$, the transition function $\psi_\beta^{-1} \circ \psi_\alpha: (U_\alpha \cap U_\beta) \times \mathrm{GL}(\mathcal{V}) \to (U_\alpha \cap U_\beta) \times \mathrm{GL}(\mathcal{V})$ then takes the form

$$\begin{aligned} \psi_\beta^{-1} \circ \psi_\alpha(x, g) = \psi_\beta^{-1}(\Phi(\phi_{\alpha,x}^{-1}, g)) &= (x, \phi_{\beta,x} \circ \phi_{\alpha,x}^{-1} \circ g) \\ &= (x, g_{\beta\alpha}(x) \circ g) \end{aligned} \tag{2.138}$$

$\forall (x, g) \in (U_\alpha \cap U_\beta) \times \mathrm{GL}(\mathcal{V})$ and thus is smooth. Hence, we have constructed an appropriate formal bundle of $\mathscr{F}(\mathcal{E})$ such that $\mathrm{GL}(\mathcal{V}) \to \mathscr{F}(\mathcal{E}) \xrightarrow{\pi} \mathcal{M}$ turns into a principal super fiber bundle with structure group $\mathrm{GL}(\mathcal{V})$ and $\mathrm{GL}(\mathcal{V})$-right action $\Phi: \mathscr{F}(\mathcal{E}) \times \mathrm{GL}(\mathcal{V}) \mapsto \mathscr{F}(\mathcal{E})$, called the *frame bundle* of $\mathcal{E}$.

Definition 2.4.19. Given a supermanifold, the *frame bundle* $\mathscr{F}(\mathcal{M})$ of $\mathcal{M}$ is defined as the frame bundle $\mathscr{F}(\mathcal{M}) \equiv \mathscr{F}(T\mathcal{M})$ of the associated tangent bundle $T\mathcal{M}$.

Proposition 2.4.20 (Associated fiber bundle)**.** *Let $\mathcal{G} \to \mathcal{P} \xrightarrow{\pi_{\mathcal{P}}} \mathcal{M}$ be a principal super fiber bundle with $\mathcal{G}$-right action $\Phi: \mathcal{P} \times \mathcal{G} \to \mathcal{P}$. Let furthermore $\rho: \mathcal{G} \times \mathcal{F} \to \mathcal{F}$ be a smooth left action of $\mathcal{G}$ on a supermanifold $\mathcal{F}$. On $\mathcal{P} \times \mathcal{F}$ consider the map*

$$\Phi^\times: (\mathcal{P} \times \mathcal{F}) \times \mathcal{G} \to \mathcal{P} \times \mathcal{F}, \; ((p, v), g) \mapsto (\Phi(p, g), \rho(g^{-1}, v)) \tag{2.139}$$

Then, $\Phi^\times$ defines an effective smooth $\mathcal{G}$-right action on $\mathcal{P}\times\mathcal{F}$. Let $\mathcal{E} := \mathcal{P}\times_\rho \mathcal{F} := (\mathcal{P}\times\mathcal{F})/\mathcal{G}$ be the corresponding coset space and $\pi_{\mathcal{E}} : \mathcal{E}\to\mathcal{M}$ be defined as

$$\begin{aligned}\pi_{\mathcal{E}} : \mathcal{E} &\to \mathcal{M} \\ [p,v] &\mapsto \pi_{\mathcal{P}}(p)\end{aligned} \tag{2.140}$$

Then, $\mathcal{E}$ can be equipped with the structure of a supermanifold such that $\pi_{\mathcal{E}}$ is a smooth surjective map and $(\mathcal{E},\pi_{\mathcal{E}},\mathcal{M},\mathcal{F})$ turns into a super fiber bundle, called the super fiber bundle associated to $\mathcal{P}$ *w.r.t. the $\mathcal{G}$-left action ρ on $\mathcal{F}$.*

Proof. It is immediate that $\Phi^\times : (\mathcal{P}\times\mathcal{F})\times\mathcal{G}\to\mathcal{P}\times\mathcal{F}$ defines a smooth $\mathcal{G}$-right action. That it also is effective follows from the fact that $\Phi : \mathcal{P}\times\mathcal{G}\to\mathcal{P}$ is effective. Let $\{(U_\alpha,\phi_\alpha)\}_{\alpha\in\Upsilon}$ be a smooth bundle atlas of $\mathcal{G}\to\mathcal{P}\xrightarrow{\pi_{\mathcal{P}}}\mathcal{M}$. For $\alpha\in\Upsilon$ define

$$\begin{aligned}\phi_\alpha^\times : \pi_{\mathcal{E}}^{-1}(U_\alpha) &\to U_\alpha\times\mathcal{F} \\ [p,v] &\to (\pi_{\mathcal{P}}(p),\rho(\mathrm{pr}_2\circ\phi_\alpha(p),v))\end{aligned} \tag{2.141}$$

which is well-defined as ρ is a $\mathcal{G}$-left action on $\mathcal{F}$. Moreover, $\phi_\alpha^\times$ is a bijection with inverse $(\phi_\alpha^\times)^{-1} : U_\alpha\times\mathcal{F}\to\pi_{\mathcal{E}}^{-1}(U_\alpha),\ (x,v)\mapsto[\phi_\alpha^{-1}(x,e),v]$. For $\alpha,\alpha'\in\Upsilon$ with $U_\alpha\cap U_{\alpha'}\neq\emptyset$, it follows

$$\phi_\beta^\times\circ(\phi_\alpha^\times)^{-1}(x,v) = (x,\rho(\mathrm{pr}_2\circ\phi_\beta\circ\phi_\alpha^{-1}(x,e),v)) \tag{2.142}$$

$\forall(x,v)\in(U_\alpha\cap U_\beta)\times\mathcal{F}$, and thus $\phi_\beta^\times\circ(\phi_\alpha^\times)^{-1}$ is smooth as $\mathrm{pr}_2\circ\phi_\beta\circ\phi_\alpha^{-1}$ is smooth and $e\in\mathbf{B}(\mathcal{G})$ is a body point. Thus, the family $\{(U_\alpha,\phi_\alpha^\times)\}_{\alpha\in\Upsilon}$ defines a formal bundle atlas of $\mathcal{E}$ (w.r.t. $\pi_{\mathcal{E}}$) and hence induces a topology and smooth supermanifold structure on $\mathcal{E}$ such that $(\mathcal{E},\pi_{\mathcal{E}},\mathcal{M},\mathcal{F})$ becomes a super fiber bundle. □

Proposition 2.4.21. *Under the conditions of Prop. 2.4.20, the canonical projection*

$$\pi : \mathcal{P}\times\mathcal{F}\to\mathcal{E}=\mathcal{P}\times_\rho\mathcal{F} \tag{2.143}$$

is a smooth bundle map, i.e., $\mathcal{P}\times\mathcal{F}\xrightarrow{\pi}\mathcal{P}\times_\rho\mathcal{F}$ carries the structure of a super fiber bundle with typical fiber $\mathcal{G}$. As a consequence, π is an open submersion and the topology on $\mathcal{E}$ defined via the construction in the proof of Theorem 2.4.4 coincides with the quotient topology.

Proof. First, let us show that π is continuous. Therefore, if $U \subseteq \mathcal{E}$ is open, by definition of the topology on $\mathcal{E}$, for any $\alpha \in \Upsilon$, $U \cap \pi_{\mathcal{E}}^{-1}(U_\alpha)$ is of the form $U \cap \pi_{\mathcal{E}}^{-1}(U_\alpha) = (\phi_\alpha^\times)^{-1}(\widetilde{U}_\alpha)$ with $\widetilde{U}_\alpha \subseteq U_\alpha \times \mathcal{F}$ open. Then,

$$\begin{aligned}\pi^{-1}(U) = \pi^{-1}\left(\bigcup_{\alpha\in\Upsilon} U \cap \pi_{\mathcal{E}}^{-1}(U_\alpha)\right) &= \bigcup_{\alpha\in\Upsilon} \pi^{-1}((\phi_\alpha^\times)^{-1}(\widetilde{U}_\alpha)) \\ &= \bigcup_{\alpha\in\Upsilon} (\phi_\alpha^\times \circ \pi)^{-1}(\widetilde{U}_\alpha) \end{aligned} \tag{2.144}$$

and hence $\pi^{-1}(U) \subseteq \mathcal{P} \times \mathcal{F}$ is open proving that π is continuous. That it is smooth can be checked by direct computation.

Next, to see that is in fact a bundle map, i.e., $\mathcal{P} \times \mathcal{F} \xrightarrow{\pi} \mathcal{P} \times_\rho \mathcal{F}$ carries the structure of a super fiber bundle, let us define an atlas $\{(\pi_{\mathcal{E}}^{-1}(U_\alpha), \widetilde{\phi}_\alpha)\}_{\alpha\in\Upsilon}$ of locally trivializing bundle charts as follows. For $\alpha \in \Upsilon$, the map $\widetilde{\phi}_\alpha : \pi^{-1}(\pi_{\mathcal{E}}^{-1}(U_\alpha)) = \pi_{\mathcal{P}}^{-1}(U_\alpha) \times \mathcal{F} \to \pi_{\mathcal{E}}^{-1}(U_\alpha) \times \mathcal{G}$ is obtained via the following commutative diagram

$$\begin{array}{ccc} \pi_{\mathcal{P}}^{-1}(U_\alpha) \times \mathcal{F} & \xrightarrow[\cong]{\phi_\alpha \times \mathrm{id}_{\mathcal{F}}} & U_\alpha \times \mathcal{G} \times \mathcal{F} \\ {\scriptstyle \widetilde{\phi}_\alpha}\big\downarrow & & \big\downarrow {\scriptstyle \cong \,|\, \mathrm{id}_{U_\alpha} \times \Theta} \\ \pi_{\mathcal{E}}^{-1}(U_\alpha) \times \mathcal{G} & \xleftarrow[(\phi^\times)^{-1} \times \mathrm{id}_{\mathcal{G}}]{\cong} & U_\alpha \times \mathcal{F} \times \mathcal{G} \end{array}$$

where $\Theta : \mathcal{G} \times \mathcal{F} \to \mathcal{F} \times \mathcal{G}$ is the diffeomorphism given by $\Theta(g, v) := (\rho(g, v), g)$ $\forall (g, v) \in \mathcal{G} \times \mathcal{F}$ and thus $\widetilde{\phi}_\alpha(p, v) = ([p, v], \mathrm{pr}_2 \circ \phi_\alpha(p))\ \forall (p, v) \in \pi^{-1}(\pi_{\mathcal{E}}^{-1}(U_\alpha))$. It follows that $\widetilde{\phi}_\alpha$ is a diffeomorphism preserving the fibers, i.e., $\mathrm{pr}_1 \circ \widetilde{\phi}_\alpha = \pi$ and thus indeed defines a local trivialization.

That $\pi : \mathcal{P} \times \mathcal{F} \to \mathcal{E} = \mathcal{P} \times_\rho \mathcal{F}$ is an open map follows by a standard argument using the fact that it is a bundle map and thus locally coincides with the projection pr_1 which is open. It remains to show that the topology on $\mathcal{E}$ coincides with the quotient topology. Obviously, any open subset $U \subseteq \mathcal{E}$ is also open w.r.t. the quotient topology as π is continuous. Conversely, if $U \subseteq \mathcal{E}$ is a subset of $\mathcal{E}$ such that $\pi^{-1}(U) \subseteq \mathcal{P} \times \mathcal{F}$ is open, it follows from $\pi(\pi^{-1}(U)) = U$ that U is open, as well. Hence, this proves the proposition. ☐

Corollary 2.4.22. *Let $\mathcal{G} \to \mathcal{P} \xrightarrow{\pi_{\mathcal{P}}} \mathcal{M}$ be a principal super fiber bundle with structure group $\mathcal{G}$ and $\rho : \mathcal{G} \to \mathrm{GL}(\mathcal{V})$ be a representation of $\mathcal{G}$ on a finite-dimensional super Λ-vector space $\mathcal{V}$ which induces the smooth $\mathcal{G}$-left action $\rho : \mathcal{G} \times \mathcal{V} \to \mathcal{V}$, $(g, v) \mapsto \rho(g, v) \equiv \rho(g)v$ on $\mathcal{V}$. Then, the associated fiber bundle $\mathcal{E} := \mathcal{P} \times_\rho \mathcal{V}$ can be given the* *structure of a super vector bundle where, for each $x \in \mathcal{M}$, the fiber $\mathcal{E}_x = \mathcal{P}_x \times_\rho \mathcal{V}$ carries the structure of a free super Λ-module via*

$$a[p,v] + b[p,w] := [p, av + bw] \tag{2.145}$$

$\forall [p,v], [p,w] \in \mathcal{E}_x$, $a, b \in \Lambda$ *and* $\mathbb{Z}_2$*-grading*

$$(\mathcal{E}_x)_0 := \{[p,v] \in \mathcal{E}_x | \, v \in \mathcal{V}_0\}, \quad (\mathcal{E}_x)_1 := \{[p,v] \in \mathcal{E}_x | \, v \in \mathcal{V}_1\} \tag{2.146}$$

The bundle $\mathcal{V} \to \mathcal{E} \to \mathcal{M}$ *is called the* super vector bundle *associated to* $\mathcal{P}$ *w.r.t. the representation* ρ. □

Corollary 2.4.23. *Let* $\mathcal{G} \to \mathcal{P} \stackrel{\pi_\mathcal{P}}{\to} \mathcal{M}$ *be a principal super fiber bundle with* $\mathcal{G}$*-right action* $\Phi : \mathcal{P} \times \mathcal{G} \to \mathcal{P}$ *and let* $\rho : \mathcal{G} \to \mathrm{GL}(\mathcal{V})$ *be a representation of* $\mathcal{G}$ *on a finite-dimensional super* Λ*-vector space* $\mathcal{V}$. *Let* $H^\infty(\mathcal{P},\mathcal{V})^\mathcal{G}$ *be the subspace of smooth* $\mathcal{G}$*-equivariant functions on* $\mathcal{P}$ *with values in* $\mathcal{V}$ *defined as*

$$H^\infty(\mathcal{P},\mathcal{V})^\mathcal{G} := \{f \in H^\infty(\mathcal{P},\mathcal{V}) | \, f(p \cdot g) = \rho(g^{-1})(f(p)), \, \forall p \in \mathcal{P}, g \in \mathcal{G}\} \tag{2.147}$$

Let $\mathcal{E} := \mathcal{P} \times_\rho \mathcal{V}$ *be the associated super vector bundle. Then, there exists an isomorphism of super vector spaces given by*

$$\Psi : H^\infty(\mathcal{P},\mathcal{V})^\mathcal{G} \to \Gamma(\mathcal{E}), \, f \mapsto (\Psi(f) : \pi_\mathcal{P}(p) \mapsto [p, f(p)]) \tag{2.148}$$

Proof. For $f \in H^\infty(\mathcal{P},\mathcal{V})^\mathcal{G}$ define the smooth map $\hat{f} : \mathcal{P} \to \mathcal{E}$ via $\hat{f}(p) := [p, f(p)], \forall p \in \mathcal{P}$. By construction, this yields $\Phi^*(\hat{f}) = \hat{f} \otimes 1_\mathcal{G}$, that is, $\hat{f}$ is constant on $\mathcal{G}$-orbits. Thus, according to Prop. 2.4.16, this induces a smooth map $\Psi(f) : \mathcal{M} \to \mathcal{E}$ satisfying $\Psi(f) \circ \pi_\mathcal{P} = \hat{f}$ and, in particular, is fiber-preserving. Hence, $\Psi(f) \in \Gamma(\mathcal{E})$. Conversely, suppose one has given a smooth section $X \in \Gamma(\mathcal{E})$. For any $p \in \mathcal{P}$, we define the bijective map $[p] : \mathcal{V} \to \mathcal{E}_x$, $v \mapsto [p,v]$ with $x := \pi_\mathcal{P}(p)$. Hence, let us define a map $f : \mathcal{P} \to \mathcal{V}$ via $f(p) = [p]^{-1}(X(x)), \forall p \in \mathcal{P}_x$. By construction, it then follows $f(p \cdot g) = \rho(g^{-1})(f(p))$ for any $g \in \mathcal{G}$. To see that it is smooth, let us choose a local section $s : U \to \mathcal{P}$ of the principal super fiber bundle $\mathcal{P}$ with $U \subseteq \mathcal{M}$ open which, in turn, induces a local trivialization $\psi : \mathcal{P} \supseteq \pi_\mathcal{P}^{-1}(U) \to U \times \mathcal{G}$ of $\mathcal{P}$ as well as smooth sections $\hat{s}_i : U \to \mathcal{E}$ of the associated super vector bundle $\mathcal{E}$ via $\hat{s}(x) := [s(x), e_i] \, \forall x \in U$ where $(e_i)_i$ is a real homogeneous basis of $\mathcal{V}$. By Prop. 2.4.9, this induces a local trivialization $\phi : \mathcal{E} \supseteq \pi_\mathcal{E}^{-1}(U) \to U \times \mathcal{V}$ of $\mathcal{E}$ over U. It is then easy to see that there exists a smooth map $v : U \to \mathcal{V}$ such that $X(x) = [s(x), v(x)]$ $\forall x \in U$. Then, it follows that $f \circ \psi^{-1}(x,g) = \rho(g^{-1}, v(x)) \, \forall (x,g) \in U \times \mathcal{G}$ which is obviously smooth proving that indeed $f \in H^\infty(\mathcal{P},\mathcal{V})^\mathcal{G}$. This shows that Ψ is bijective. That Ψ is linear and preserves the grading is immediate. □

Example 2.4.24 (Vector bundles as associated bundles)**.** Let $\mathcal{V} \to \mathcal{E} \stackrel{\pi}{\to} \mathcal{M}$ be a super vector bundle. The general linear supergroup $\mathrm{GL}(\mathcal{V})$ acts in a trivial way on $\mathcal{V}$ via the so-called *fundamental representation*

$$\rho \equiv \mathrm{id} : \mathrm{GL}(\mathcal{V}) \to \mathrm{GL}(\mathcal{V}),\ g \mapsto g \tag{2.149}$$

This yields the associated super vector bundle $\mathscr{F}(\mathcal{E}) \times_\rho \mathcal{V}$. Consider the map

$$\begin{aligned} \mathscr{F}(\mathcal{E}) \times_\rho \mathcal{V} &\to \mathcal{E} \\ [p_x, v] &\to (v^i X_{ix})_x \end{aligned} \tag{2.150}$$

where we have chosen a real homogeneous basis $(e_i)_i$ of $\mathcal{V}$ and set $X_{ix} := p_x(e_i)\ \forall i$. It is immediate that (2.150) is smooth and well-defined, fiber-preserving as well as an isomorphism of super Λ-modules on each fiber. Hence, it indeed defines an isomorphism of super vector bundles. This shows that each super vector bundle is associated to a principal super fiber bundle.

This construction also yields a new characterization of the (left) dual super vector bundle ${}^*\mathcal{V} \to {}^*\mathcal{E} \to \mathcal{M}$ as follows. A representation $\rho : \mathcal{G} \to \mathrm{GL}(\mathcal{V})$ of a super Lie group on a super Λ-vector space yields a representation ${}^*\rho : \mathcal{G} \to \mathrm{GL}({}^*\mathcal{V})$ on the corresponding left dual super Λ-vector space ${}^*\mathcal{V}$ via

$${}^*\rho(g)\ell : v \mapsto \langle v|{}^*\rho(g)\ell\rangle := \langle \langle v|\rho(g^{-1})\rangle\, |\ell\rangle \tag{2.151}$$

$\forall g \in \mathcal{G}$, $\ell \in {}^*\mathcal{V}$ and $v \in \mathcal{V}$, called the *left dual representation* of $\mathcal{G}$. In our case, i.e., $\mathcal{G} = \mathrm{GL}(\mathcal{V})$ and $\rho \equiv \mathrm{id}$, this yields the associated super vector bundle $\mathscr{F}(\mathcal{E}) \times_{{}^*\rho} {}^*\mathcal{V}$. In fact, in turns out that this bundle is isomorphic to the left dual super vector bundle via

$$\begin{aligned} \mathscr{F}(\mathcal{E}) \times_{{}^*\rho} {}^*\mathcal{V} &\to {}^*\mathcal{E} \\ [p_x, \ell] &\to ({}^iX_x\ \ell_i)_x \end{aligned} \tag{2.152}$$

where, for any $x \in \mathcal{M}$, $({}^iX_x)_i$ denotes the left dual basis of $(X_{ix})_i$ in ${}^*\mathcal{E}_x$.

Corollary 2.4.25. *Let $\mathcal{H} \to \mathcal{P} \stackrel{\pi_\mathcal{P}}{\to} \mathcal{M}$ be a principal super fiber bundle with structure group $\mathcal{H}$ and $\mathcal{H}$-right action $\Phi : \mathcal{P} \times \mathcal{H} \to \mathcal{H}$. Let $\lambda : \mathcal{H} \to \mathcal{G}$ be a morphism of super Lie groups and $\rho_\lambda := \mu_\mathcal{G} \circ (\lambda \times \mathrm{id}_\mathcal{G}) : \mathcal{H} \times \mathcal{G} \to \mathcal{G}$ the induced smooth left* *action of $\mathcal{H}$ on $\mathcal{G}$. Then, the associated fiber bundle $\mathcal{P}\times_{\mathcal{H}}\mathcal{G}:=\mathcal{P}\times_{\rho_\lambda}\mathcal{G}$ can be given the structure of a principal super fiber bundle with structure group $\mathcal{G}$ and $\mathcal{G}$-right action*

$$\begin{aligned}\widetilde{\Phi}:\ (\mathcal{P}\times_{\mathcal{H}}\mathcal{G})\times\mathcal{G}&\to\mathcal{P}\times_{\mathcal{H}}\mathcal{G}\\ ([p,g],h)&\mapsto[p,\mu_{\mathcal{G}}(g,h)]\end{aligned}\tag{2.153}$$

Furthermore, let $\iota:\ \mathcal{P}\to\mathcal{P}\times_{\mathcal{H}}\mathcal{G}$ be defined as $\iota(p):=[p,e]\,\forall p\in\mathcal{P}$, then ι is smooth, fiber-preserving and $\mathcal{H}$-equivariant in the sense that $\iota\circ\Phi=\widetilde{\Phi}\circ(\iota\times\lambda)$. Moreover, if $\lambda:\ \mathcal{H}\hookrightarrow\mathcal{G}$ is an embedding, then ι is an embedding.

Proof. First, let us show that $\widetilde{\Phi}:\ (\mathcal{P}\times_{\mathcal{H}}\mathcal{G})\times\mathcal{G}\to\mathcal{P}\times_{\mathcal{H}}\mathcal{G}$ is a smooth $\mathcal{G}$-right action with respect to which the local trivializations $\phi_\alpha^\times$ are $\mathcal{G}$-equivariant. To see that is continuous, consider the smooth map $\Phi':\ (\mathcal{P}\times\mathcal{G})\times\mathcal{G}\to\mathcal{P}\times_{\mathcal{H}}\mathcal{G}$, $((p,g),h)\mapsto[p,\mu_{\mathcal{G}}(g,h)]$. Since Φ' is constant on $\mathcal{G}$-orbits it follows from Prop. (2.4.21), i.e., the topology on $\mathcal{E}$ coincides with the quotient topology, that this induces a continuous map $\widetilde{\Phi}:\ (\mathcal{P}\times_{\mathcal{H}}\mathcal{G})\times\mathcal{G}\to\mathcal{P}\times_{\mathcal{H}}\mathcal{G}$ such that the following diagram commutes

$$\begin{array}{ccc}(\mathcal{P}\times\mathcal{G})\times\mathcal{G} & \xrightarrow{\ \Phi'\ } & \mathcal{P}\times_{\mathcal{H}}\mathcal{G}\\ {\scriptstyle\pi\times\mathrm{id}_{\mathcal{G}}}\big\downarrow & \nearrow_{\widetilde{\Phi}} & \\ (\mathcal{P}\times_{\mathcal{H}}\mathcal{G})\times\mathcal{G} & & \end{array}$$

That $\widetilde{\Phi}$ is smooth can be checked by direct computation. Furthermore, for a locally trivializing bundle chart $\phi_\alpha^\times:\ \pi_{\mathcal{E}}^{-1}(U_\alpha)\to U_\alpha\times\mathcal{G}$, we compute

$$\begin{aligned}\phi_\alpha^\times\circ\widetilde{\Phi}([p,g],h)&=(\pi_{\mathcal{P}}(p),\rho(\mathrm{pr}_2\circ\phi_\alpha(p),\mu_{\mathcal{G}}(g,h)))\\&=(\mathrm{id}\times\mu_{\mathcal{G}})(\pi_{\mathcal{P}}(p),\rho(\mathrm{pr}_2\circ\phi_\alpha(p),g)),h)\\&=(\mathrm{id}\times\mu_{\mathcal{G}})\circ\phi_\alpha^\times([p,g],h)\end{aligned}\tag{2.154}$$

and hence $\phi_\alpha^\times\circ\widetilde{\Phi}=(\mathrm{id}\times\mu_{\mathcal{G}})\circ\phi_\alpha^\times$ as required.

Finally, let us consider the map $\iota:\ \mathcal{P}\to\mathcal{P}\times_{\mathcal{H}}\mathcal{G}$, $p\mapsto[p,e]$. Since $\iota=\pi\circ\iota_{\mathcal{P}}$ is a composition of the canonical projection π as well as the embedding $\iota_{\mathcal{P}}:\ \mathcal{P}\to\mathcal{P}\times\{e\}\subset\mathcal{P}\times\mathcal{G}$, it follows that ι is smooth. That ι is fiber-preserving is immediate and for $(p,h)\in\mathcal{P}\times\mathcal{H}$ it follows $\iota\circ\Phi(p,h)=[\Phi(p,h),e]=[p,\rho_\lambda(h,e)]=[p,\mu_{\mathcal{G}}(e,\lambda(h))]=\widetilde{\Phi}\circ(\iota\times\lambda)(p,h)$, that is, ι is $\mathcal{H}$-equivariant. To prove the last assertion, if $\lambda:\ \mathcal{H}\hookrightarrow\mathcal{G}$ is an embedding, it follows that ι is injective. Moreover, since π is an open submersion, ι is a homeomorphism onto its image. To see that it is an immersion and thus an embedding, let $\{(U_\alpha, \phi^\times_\alpha)\}_{\alpha\in\Upsilon}$ be a smooth bundle atlas of $\mathcal{P}\times_{\rho_\lambda}\mathcal{G}$ induced by a smooth bundle atlas of $\mathcal{P}$. Then, for $\alpha\in\Upsilon$, this yields

$$\phi^\times_\alpha\circ\iota(p)=(\pi_{\mathcal{P}}(p),\lambda(\mathrm{pr}_2\circ\phi_\alpha(p)))=(\mathrm{id}\times\lambda)\circ\phi_\alpha(p) \tag{2.155}$$

$\forall p\in\mathcal{P}$. But, since ϕ_α is a diffeomorphism and $\mathrm{id}\times\lambda:\mathcal{P}\times\mathcal{H}\to\mathcal{P}\times\mathcal{G}$ an embedding the claim follows. □

Definition 2.4.26. Under the assumptions of Corollary 2.4.25, the associated principal super fiber bundle $\mathcal{P}\times_{\mathcal{H}}\mathcal{G}:=\mathcal{P}\times_{\rho_\lambda}\mathcal{G}$ is called the *λ-extension* of $\mathcal{P}$. If $\lambda:\mathcal{H}\hookrightarrow\mathcal{G}$ is an embedding, $\mathcal{P}\times_{\mathcal{H}}\mathcal{G}$ will also simply be called the *$\mathcal{G}$-extension* of $\mathcal{P}$. In the latter case, we also simply write $\mathcal{P}[\mathcal{G}]:=\mathcal{P}\times_{\mathcal{H}}\mathcal{G}$.

Definition 2.4.27. Let $\mathcal{G}\to\mathcal{P}\xrightarrow{\pi_{\mathcal{P}}}\mathcal{M}$ be a principal super fiber bundle with structure group $\mathcal{G}$ and $\lambda:\mathcal{H}\longrightarrow\mathcal{G}$ a morphism of super Lie groups. A $\mathcal{H}$-bundle $\mathcal{H}\to\mathcal{Q}\xrightarrow{\pi_{\mathcal{Q}}}\mathcal{M}$ over $\mathcal{M}$ is called a *λ-reduction* of $\mathcal{P}$, if there exists a smooth map $\Lambda:\mathcal{Q}\longrightarrow\mathcal{P}$ such that the following diagram commutes

$$\begin{array}{ccccc} \mathcal{Q}\times\mathcal{H} & \longrightarrow & \mathcal{Q} & & \\ \downarrow{\scriptstyle\Lambda\times\lambda} & & \downarrow{\scriptstyle\Lambda} & \searrow^{\pi_{\mathcal{Q}}} & \mathcal{M} \\ \mathcal{P}\times\mathcal{G} & \longrightarrow & \mathcal{P} & \nearrow_{\pi_{\mathcal{P}}} & \end{array} \tag{2.156}$$

i.e. Λ is fiber-preserving and $\mathcal{H}$-equivariant in the sense that $\Lambda\circ\Phi_{\mathcal{Q}}=\Phi_{\mathcal{P}}\circ(\Lambda\times\lambda)$ with $\Phi_{\mathcal{Q}}:\mathcal{Q}\times\mathcal{H}\to\mathcal{Q}$ and $\Phi_{\mathcal{P}}:\mathcal{P}\times\mathcal{G}\to\mathcal{P}$ the super Lie group actions on $\mathcal{Q}$ and $\mathcal{P}$, respectively.

Corollary 2.4.28. *Under the assumptions of Corollary 2.4.25, the $\mathcal{H}$-bundle $\mathcal{P}$ is a λ-reduction of the $\mathcal{G}$-bundle $\mathcal{P}\times_{\mathcal{H}}\mathcal{G}$.* □

2.5. $\mathcal{S}$-relative super connection forms

The content of this section has been reproduced from [1], with slight changes to account for the context of this thesis with the permission of AIP Publishing.

One of the main issues when working in the standard category of supermanifolds, both in the H^∞ or the algebraic category, is that superfields on the body of a supermanifold only contain commuting (bosonic) degrees of freedom, that is, there are no anticommuting (fermionic) field configurations on the body. This, however, turns out to be incompatible with various constructions in physics, such as in the geometric approach to supergravity. In the Castellani-D'Auria-Fré approach to supergravity [71, 72] (see Section 3.4), for instance, one has the so-called rheonomy principle stating that physical fields are completely fixed by their pullback to the body manifold.

This can be cured by factorizing a given supermanifold $\mathcal{M}$ by an additional parametrizing supermanifold $\mathcal{S}$ and studying superfields on $\mathcal{S}\times\mathcal{M}$. One is then interested in a certain subclass of such superfields which depend as little as possible on this additional supermanifold making them covariant in a specific sense under a change of parametrization. This idea is based on a proposal formulated already by Schmitt in [73] which is motivated by the functorial approach to supermanifold theory according to Molotkov [98] and Sachse [99, 100]. This approach also recently found application in context of superconformal field theories on super Riemannian surfaces [75, 116] as well as in context of the local approach to super quantum field theories (QFT) [101]. Moreover, as will be explained in more detail later in Section 3.6, the description of fermionic fields turns out to be quite similar to considerations in perturbative algebraic QFT [76, 77].

In the following, we will adopt the terminology of [101] introducing the notion of a relative supermanifold. However, unlike [101], in order to study fermionic fields, we will not restrict to superpoints as parametrizing supermanifolds. We will then define principal connections and connection 1-forms on parametrized supermanifolds. These results will then be applied in Section 2.7.2 for the construction of the parallel tansport map.

Definition 2.5.1. Let $\mathcal{S}$ and $\mathcal{M}$ be supermanifolds. The pair $(\mathcal{S}\times\mathcal{M},\mathrm{pr}_{\mathcal{S}})$ with $\mathrm{pr}_{\mathcal{S}}:\mathcal{S}\times\mathcal{M}\to\mathcal{S}$ the projection onto the first factor is called a *$\mathcal{S}$-relative supermanifold* also denoted by $\mathcal{M}_{/\mathcal{S}}$. The supermanifold $\mathcal{S}$ is called *parametrizing supermanifold* or simply *parametrization*. A morphism $\phi:\mathcal{M}_{/\mathcal{S}}\to\mathcal{N}_{/\mathcal{S}}$ between $\mathcal{S}$-relative supermanifolds is a morphism $\phi:\mathcal{S}\times\mathcal{M}\to\mathcal{S}\times\mathcal{N}$ of supermanifolds preserving the projections, i.e., the following diagram is commutative

$$\begin{array}{ccc} \mathcal{S}\times\mathcal{M} & \xrightarrow{\quad\phi\quad} & \mathcal{S}\times\mathcal{N} \\ & {\scriptstyle \mathrm{pr}_{\mathcal{S}}}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_{\mathcal{S}}} & \\ & \mathcal{S} & \end{array}$$

Hence, $\phi(s,p)=(s,\tilde{\phi}(s,p))\;\forall(s,p)\in\mathcal{S}\times\mathcal{M}$ with $\tilde{\phi}:=\mathrm{pr}_{\mathcal{N}}\circ\phi:\mathcal{S}\times\mathcal{M}\to\mathcal{N}$. This yields a category $\mathbf{SMan}_{/\mathcal{S}}$ called the *category of $\mathcal{S}$-relative supermanifolds*.

The following proposition gives a different characterization of morphism between $\mathcal{S}$-relative supermanifolds.

Proposition 2.5.2 (after [101]). *Let $\mathcal{M}_{/\mathcal{S}}, \mathcal{N}_{/\mathcal{S}} \in \mathbf{Ob}(\mathbf{SMan}_{/\mathcal{S}})$ be $\mathcal{S}$-relative supermanifolds. Then, the map*

$$\begin{aligned} \alpha_{\mathcal{S}}: \mathrm{Hom}_{\mathbf{SMan}_{/\mathcal{S}}}(\mathcal{M}_{/\mathcal{S}}, \mathcal{N}_{/\mathcal{S}}) &\to \mathrm{Hom}_{\mathbf{SMan}_{H^\infty}}(\mathcal{S}\times\mathcal{M}, \mathcal{N}) \\ (\phi: \mathcal{S}\times\mathcal{M}\to\mathcal{S}\times\mathcal{N}) &\mapsto (\mathrm{pr}_{\mathcal{N}}\circ\phi: \mathcal{S}\times\mathcal{M}\to\mathcal{N}) \end{aligned} \tag{2.157}$$

is a bijection with the inverse given by

$$\begin{aligned} \alpha_{\mathcal{S}}^{-1}: \mathrm{Hom}_{\mathbf{SMan}_{H^\infty}}(\mathcal{S}\times\mathcal{M}, \mathcal{N}) &\to \mathrm{Hom}_{\mathbf{SMan}_{/\mathcal{S}}}(\mathcal{M}_{/\mathcal{S}}, \mathcal{N}_{/\mathcal{S}}) \\ (\psi: \mathcal{S}\times\mathcal{M}\to\mathcal{N}) &\mapsto ((\mathrm{id}_{\mathcal{S}}\times\psi)\circ(d_{\mathcal{S}}\times\mathrm{id}_{\mathcal{M}}): \mathcal{S}\times\mathcal{M}\to\mathcal{S}\times\mathcal{N}) \end{aligned} \tag{2.158}$$

with $d_{\mathcal{S}}: \mathcal{S}\to\mathcal{S}\times\mathcal{S}$ the diagonal map. □

Let $\lambda: \mathcal{S}\to\mathcal{S}'$ be a morphism between parametrizing supermanifolds, we will also call such a morphism a *change of parametrization*. Then, any smooth map $\phi: \mathcal{S}'\times\mathcal{M}\to\mathcal{N}$ can be pulled back via λ to a morphism $\lambda^*\phi := \phi\circ(\lambda\times\mathrm{id}_{\mathcal{M}}): \mathcal{S}\times\mathcal{M}\to\mathcal{N}$. Using 2.158, this yields the map [101]

$$\begin{aligned} \lambda^*: \mathrm{Hom}_{\mathbf{SMan}_{/\mathcal{S}'}}(\mathcal{M}_{/\mathcal{S}'}, \mathcal{N}_{/\mathcal{S}'}) &\to \mathrm{Hom}_{\mathbf{SMan}_{/\mathcal{S}}}(\mathcal{M}_{/\mathcal{S}}, \mathcal{N}_{/\mathcal{S}}) \\ \phi &\mapsto \alpha_{\mathcal{S}}^{-1}(\alpha_{\mathcal{S}'}(\phi)\circ(\lambda\times\mathrm{id}_{\mathcal{M}})) \end{aligned} \tag{2.159}$$

Hence, explicitly, for $\phi: \mathcal{M}_{/\mathcal{S}'}\to\mathcal{N}_{/\mathcal{S}'}$, $\lambda^*(\phi)$ reads

$$\lambda^*(\phi)(s,p) = (s, \mathrm{pr}_{\mathcal{N}}\circ\phi(\lambda(s),p)) \tag{2.160}$$

$\forall (s,p)\in\mathcal{S}\times\mathcal{M}$. The following proposition demonstrates that the set of morphisms between relative supermanifolds is functorial in the parametrizing supermanifold and thus indeed has the required properties under change of parametrization.

Proposition 2.5.3 (after [101]). *The assignment*

$$\begin{aligned} \mathbf{SMan}\to\mathbf{Set}: \mathbf{Ob}(\mathbf{SMan})\ni\mathcal{S} &\mapsto \mathrm{Hom}_{\mathbf{SMan}_{/\mathcal{S}}}(\mathcal{M}_{/\mathcal{S}}, \mathcal{N}_{/\mathcal{S}}) \in \mathbf{Ob}(\mathbf{Set}) \\ (\lambda: \mathcal{S}\to\mathcal{S}') &\mapsto \lambda^* \end{aligned} \tag{2.161}$$

defines a contravariant functor on the category $\mathbf{SMan}_{H^\infty}$ of H^∞ supermanifolds. Moreover, the map λ^* associated to the morphism $\lambda: \mathcal{S}\to\mathcal{S}'$ preserves compositions, i.e., $\lambda^*(\phi\circ\psi) = \lambda^*(\phi)\circ\lambda^*(\psi)$ for any $\psi: \mathcal{M}\to\mathcal{N}$ and $\phi: \mathcal{N}\to\mathcal{L}$

Proof. This is an immediate consequence of the identities (2.157), (2.158) and (2.159). Alternatively, one may directly prove this proposition using the explicit formula (2.160) valid in the H^∞ category. □

Definition 2.5.4. (i) Let $\mathcal{M}_{/\mathcal{S}} \in \mathbf{Ob}(\mathbf{SMan}_{/\mathcal{S}})$ be a $\mathcal{S}$-relative supermanifold and $p \in \mathcal{M}_{/\mathcal{S}}$ a point, that is, a tuple $p \equiv (s, x)$ in $\mathcal{S} \times \mathcal{M}$. A *tangent vector* X_p at p is defined as a tangent vector $X_p \in T_p(\mathcal{S} \times \mathcal{M})$ satisfying

$$X_p(f \otimes 1) = 0, \; \forall f \in H^\infty(\mathcal{S}) \tag{2.162}$$

The collection of tangent vectors at p defines a super Λ-sub module of $T_p(\mathcal{S}\times\mathcal{M})$ denoted by $T_p(\mathcal{M}_{/\mathcal{S}})$ which we call the *tangent module* of $\mathcal{M}_{/\mathcal{S}}$ at p. A *smooth vector field* $X \in \mathfrak{X}(\mathcal{M}_{/\mathcal{S}})$ on $\mathcal{M}_{/\mathcal{S}}$ is a smooth section $X \in \mathfrak{X}(\mathcal{S} \times \mathcal{M})$ of the tangent bundle of $\mathcal{S} \times \mathcal{M}$ such that $X_p \in T_p(\mathcal{M}_{/\mathcal{S}})$ for any $p \in \mathcal{S} \times \mathcal{M}$. The vector fields on $\mathcal{M}_{/\mathcal{S}}$ form a super $H^\infty(\mathcal{S} \times \mathcal{M})$-sub module $\mathfrak{X}(\mathcal{M}_{/\mathcal{S}})$ of $\mathfrak{X}(\mathcal{S} \times \mathcal{M})$ isomorphic to $H^\infty(\mathcal{S}) \otimes \mathfrak{X}(\mathcal{M})$.

(ii) A *cotangent vector* ω_p at $p \in \mathcal{M}_{/\mathcal{S}}$ is defined a left linear morphism $\omega_p : T_p(\mathcal{M}_{/\mathcal{S}}) \rightarrow \Lambda$. A 1*-form* ω on $\mathcal{M}_{/\mathcal{S}}$ is a left linear morphism of super $H^\infty(\mathcal{S}\times \mathcal{M})$-modules $\omega \in \underline{\mathrm{Hom}}_L(\mathfrak{X}(\mathcal{M}_{/\mathcal{S}}), H^\infty(\mathcal{S} \times \mathcal{M}))$. The set $\Omega^1(\mathcal{M}_{/\mathcal{S}})$ of $\mathcal{S}$-relative 1-forms on $\mathcal{M}_{/\mathcal{S}}$ defines a super $H^\infty(\mathcal{S}\times\mathcal{M})$-sub module of $\Omega^1(\mathcal{S}\times\mathcal{M})$ isomorphic to $\Omega^1(\mathcal{M}) \otimes H^\infty(\mathcal{S})$.

(iii) Analogously, one defines k-forms $\omega \in \Omega^k(\mathcal{M}_{/\mathcal{S}})$ on $\mathcal{M}_{/\mathcal{S}}$, $k \in \mathbb{N}$, as skew-symmetric k-left linear morphisms $\omega \in \underline{\mathrm{Hom}}_{a,L}(\mathfrak{X}(\mathcal{M}_{/\mathcal{S}})^k, H^\infty(\mathcal{S} \times \mathcal{M}))$ of super $H^\infty(\mathcal{S} \times \mathcal{M})$-modules. For $k = 0$, we set $\Omega^0(\mathcal{M}_{/\mathcal{S}}) \equiv H^\infty(\mathcal{S} \times \mathcal{M})$. Operations on forms, such as the *exterior* and *interior derivative* as well as *Lie derivative* can be defined as in the non-relative setting (see e.g. [74, 97, 102, 117]). For instance, given a 1-form $\omega \in \Omega^1(\mathcal{M}_{/\mathcal{S}})$ and homogeneous $X \in \mathfrak{X}(\mathcal{M}_{/\mathcal{S}})$, the interior derivative is given by $\iota_X(\omega) := \langle X|\omega\rangle$ and similarly for arbitrary k-forms. The Lie derivative is then defined via the *(graded) Cartan formula* $L_X := \mathrm{d} \circ \iota_X + \iota_X \circ \mathrm{d}$. Moreover, one has the important identity

$$[L_X, \iota_Y] \equiv L_X \circ \iota_Y - (-1)^{|X||Y|}\iota_Y \circ L_X = \iota_{[X,Y]} \tag{2.163}$$

for any homogeneous $X, Y \in \mathfrak{X}(\mathcal{M}_{/\mathcal{S}})$.

(iv) For a super Λ-module $\mathcal{V}$, we denote by $\Omega^k(\mathcal{M}_{/\mathcal{S}}, \mathcal{V}) := \Omega^k(\mathcal{M}_{/\mathcal{S}}) \otimes \mathcal{V}$ the super $H^\infty(\mathcal{S} \times \mathcal{M})$-module of *$\mathcal{V}$-valued k-forms on* $\mathcal{M}_{/\mathcal{S}}$. In case, $\mathcal{V}$ is given by the super Lie module $\mathrm{Lie}(\mathcal{G}) = \Lambda \otimes \mathfrak{g}$ of a Lie group $\mathcal{G}$, we simply write $\Omega^k(\mathcal{M}_{/\mathcal{S}}, \mathfrak{g}) := \Omega^k(\mathcal{M}_{/\mathcal{S}}) \otimes \mathfrak{g}$. Operations on $\Omega^k(\mathcal{M}_{/\mathcal{S}})$ such as exterior, interior or Lie derivative are extended in a straightforward way to $\Omega^k(\mathcal{M}_{/\mathcal{S}}, \mathcal{V})$.

Definition 2.5.5. (i) Consider a principal super fiber bundle $\mathcal{G} \to \mathcal{P} \xrightarrow{\pi} \mathcal{M}$ with $\mathcal{G}$-right action $\Phi : \mathcal{P} \times \mathcal{G} \to \mathcal{P}$ as well as a parametrizing supermanifold $\mathcal{S}$. Taking products, this yields a fiber bundle

$$\begin{array}{ccc} \mathcal{S} \times \mathcal{P} & \longleftarrow & \mathcal{G} \\ \Big\downarrow{\scriptstyle \pi_{\mathcal{S}}} & & \\ \mathcal{S} \times \mathcal{M} & & \end{array}$$

with projection $\pi_{\mathcal{S}} := \mathrm{id}_{\mathcal{S}} \times \pi$ and $\mathcal{G}$-right action $\Phi_{\mathcal{S}} := \mathrm{id}_{\mathcal{S}} \times \Phi : (\mathcal{S} \times \mathcal{P}) \times \mathcal{G} \to \mathcal{S} \times \mathcal{P}$. By construction, $\pi_{\mathcal{S}}$ defines a morphism of $\mathcal{S}$-relative supermanifolds. Moreover, $\Phi_{\mathcal{S}}$ satisfies $\pi_{\mathcal{S}} \circ \Phi_{\mathcal{S}} = \pi_{\mathcal{S}} \circ \mathrm{pr}_{\mathcal{S}\times\mathcal{P}}$ as well as

$$\Phi_{\mathcal{S}} \circ (\Phi_{\mathcal{S}} \times \mathrm{id}) = \Phi_{\mathcal{S}} \circ (\mathrm{id} \times \mu) \tag{2.164}$$

We will call such a group action a $\mathcal{G}$-right action of $\mathcal{S}$-relative supermanifolds. Hence, this yields a fiber bundle $\mathcal{G} \to \mathcal{P}_{/\mathcal{S}} \xrightarrow{\pi_{\mathcal{S}}} \mathcal{M}_{/\mathcal{S}}$ in the category $\mathbf{SMan}_{/\mathcal{S}}$ of $\mathcal{S}$-relative supermanifolds which will be called a *$\mathcal{S}$-relative principal super fiber bundle*.

(ii) Let $\mathcal{V} \to \mathcal{E} \xrightarrow{\pi} \mathcal{M}$ be a super vector bundle. Similarly as above, taking products, this yields a fiber bundle $\mathcal{V} \to \mathcal{E}_{/\mathcal{S}} \xrightarrow{\pi_{\mathcal{S}}} \mathcal{M}_{/\mathcal{S}}$ in the category $\mathbf{SMan}_{/\mathcal{S}}$ with typical fiber given by a super Λ-vector space $\mathcal{V}$ which will be called a *$\mathcal{S}$-relative super vector bundle*. A smooth section $X \in \Gamma(\mathcal{E}_{/\mathcal{S}})$ of the $\mathcal{S}$-relative super vector bundle $\mathcal{E}_{/\mathcal{S}}$ is given by morphisms $X : \mathcal{M}_{/\mathcal{S}} \to \mathcal{E}_{/\mathcal{S}}$ of $\mathcal{S}$-relative supermanifolds satisfying $\pi_{\mathcal{S}} \circ X = \mathrm{id}$.

Remark 2.5.6. Let $\mathcal{M}_{/\mathcal{S}} \in \mathbf{Ob}(\mathbf{SMan}_{/\mathcal{S}})$ be a $\mathcal{S}$-relative supermanifold. Choosing a local coordinate neighborhood of $\mathcal{M}$, it is immediate to see that for any $(s,p) \in \mathcal{M}_{/\mathcal{S}}$, the tangent module $T_{(s,p)}(\mathcal{M}_{/\mathcal{S}})$ is isomorphic to $T_p\mathcal{M}$ via $T_p\mathcal{M} \ni X_p \mapsto \mathbb{1} \otimes X_p \in T_{(s,p)}(\mathcal{M}_{/\mathcal{S}})$. On $\mathcal{S} \times \mathcal{M}$ consider the assignment $T(\mathcal{M}_{/\mathcal{S}}) : \mathcal{S} \times \mathcal{M} \ni (s,p) \mapsto T_{(s,p)}(\mathcal{M}_{/\mathcal{S}})$ which defines a subbundle of $T(\mathcal{S} \times \mathcal{M})$ and which we call the *tangent bundle* of $\mathcal{M}_{/\mathcal{S}}$. On the other hand, according to Definition 2.5.5 (ii), one can also consider the $\mathcal{S}$-relative super vector bundle $(T\mathcal{M})_{/\mathcal{S}}$. Together with the previous observation, we obtain an isomorphism

$$(T\mathcal{M})_{/\mathcal{S}} \xrightarrow{\sim} T(\mathcal{M}_{/\mathcal{S}}),\ (s, X_p) \mapsto (\mathbb{1} \otimes X)_{(s,p)} \in T_{(s,p)}(\mathcal{M}_{/\mathcal{S}}) \tag{2.165}$$

Moreover, via this identification, it follows that smooth vector fields on $\mathcal{M}_{/\mathcal{S}}$ can be identified with smooth sections $X \in \Gamma((T\mathcal{M})_{/\mathcal{S}})$ of the $\mathcal{S}$-relative super vector bundle $(T\mathcal{M})_{/\mathcal{S}}$. In a similar way, it follows that $\mathcal{S}$-relative 1-forms on $\mathcal{M}_{/\mathcal{S}}$ can be identified with smooth sections $\omega \in \Gamma((^*T\mathcal{M})_{/\mathcal{S}})$ of the $\mathcal{S}$-relative super vector bundle $(^*T\mathcal{M})_{/\mathcal{S}}$.

As in the ordinary theory of principal fiber bundles and gauge theory in physics, connections and connection 1-forms play a very central role. To implement these notions in the category of relative supermanifolds, we first have to introduce the notion of a geometric distribution.

Definition 2.5.7. A *smooth geometric distribution $\mathcal{E}$ of rank $k|l$* on a $\mathcal{S}$-relative supermanifold $\mathcal{M}_{/\mathcal{S}}$ is an assignment

$$\mathcal{E} : \mathcal{M}_{/\mathcal{S}} \ni p \mapsto \mathcal{E}_p \subseteq T_p(\mathcal{M}_{/\mathcal{S}}) \tag{2.166}$$

mapping each point $p \in \mathcal{M}_{/\mathcal{S}}$ to a super Λ-sub module $\mathcal{E}_p$ of $T_p(\mathcal{M}_{/\mathcal{S}})$ of dimension $k|l$ such that, for any $p \in \mathcal{M}_{/\mathcal{S}}$, there exists $p \in U \subseteq \mathcal{S} \times \mathcal{M}$ open as well as a family $(X_i)_{i=1,\ldots,k+l}$ of smooth vector fields $X \in \mathfrak{X}(U)$ on $\mathcal{S} \times \mathcal{M}$ such that $X_i(q) \in T_q(\mathcal{M}_{/\mathcal{S}})$ $\forall q \in U$ and $(X_i(q))_i$ defines a homogeneous basis of $\mathcal{E}_q$.

Lemma 2.5.8 (a generalization of [97])**.** *Let $\mathcal{M}_{/\mathcal{S}}$ be a $\mathcal{S}$-relative supermanifold of dimension (m,n) and $\mathcal{W} \to \mathcal{F}_{/\mathcal{S}} \to \mathcal{N}_{/\mathcal{S}}$ be a $\mathcal{S}$-relative super vector bundle of rank $k|l$. Let $(\phi, g) : T\mathcal{M}_{/\mathcal{S}} \to \mathcal{F}_{/\mathcal{S}}$ a (even, right linear) morphism of $\mathcal{S}$-relative super vector bundles such that $\phi_p : T_p\mathcal{M}_{/\mathcal{S}} \to (\mathcal{F}_{/\mathcal{S}})_{g(p)}$ is surjective $\forall p \in \mathcal{M}_{/\mathcal{S}}$. Then*

$$\ker(\phi) := \coprod_{p \in \mathcal{M}_{/\mathcal{S}}} \ker(\phi_p : T_p\mathcal{M}_{/\mathcal{S}} \to (\mathcal{F}_{/\mathcal{S}})_{g(p)}) \tag{2.167}$$

defines a smooth geometric distribution of rank $(m-k)|(n-l)$ on $\mathcal{M}_{/\mathcal{S}}$.

Proof. The following proof is a generalization of the proof in the ordinary H^∞ category as given in [97]. For an arbitrary but fixed $p_0 \in \mathcal{M}_{/\mathcal{S}}$, let $(U_{/\mathcal{S}}, \psi_U)$ and $(V_{/\mathcal{S}}, \psi'_V)$ be local trivializations of $T\mathcal{M}_{/\mathcal{S}}$ and $\mathcal{F}_{/\mathcal{S}}$, respectively, such that $p_0 \in U_{/\mathcal{S}}$ and $\mathcal{S} \times U \subset g^{-1}(\mathcal{S} \times V)$. Consider the map $\phi_{VU} := \psi'_V \circ \phi \circ \psi_U^{-1} : U_{/\mathcal{S}} \times \Lambda^{m|n} \to V_{/\mathcal{S}} \times \mathcal{W}$. Since ϕ is a right linear super vector bundle morphism, $\mathrm{pr}_2 \circ \phi_{VU}(p, \cdot) : \Lambda^{m|n} \to \mathcal{W} \in \mathrm{End}_R(\Lambda^{m|n}, \mathcal{W})$ is a right linear map $\forall p \in U_{/\mathcal{S}}$. Hence, there exists a (unique) smooth map $\widetilde{\phi}_{VU} : \mathcal{S} \times U \to \mathrm{End}_R(\Lambda^{m|n}, \mathcal{W})$ such that $\widetilde{\phi}_{VU}(p)v = \mathrm{pr}_2 \circ \phi_{VU}(p,v)$ $\forall (p,v) \in (\mathcal{S} \times U) \times \Lambda^{m|n}$. Thus, w.r.t. a real homogeneous basis $(e_i)_{i=1,\ldots,m+n}$ and $(f_j)_{j=1,\ldots,k+l}$ of $\Lambda^{m|n}$ and $\mathcal{W}$, respectively, this yields

$$\phi_{VU}(p,v) = (g(p), f_j \widetilde{\phi}_{VU}{}^j{}_i(p) v^i) \tag{2.168}$$

Since $\phi_p : T_p\mathcal{M}_{/\mathcal{S}} \to (\mathcal{F}_{/\mathcal{S}})_{g(p)}$ is surjective, the rank of the super matrix $\widetilde{\phi}_{VU}{}^j{}_i(p)$ has to be $k|l$ $\forall p \in \mathcal{S} \times U$. Hence, after reordering the real homogeneous basis $(e_i)_i$, we may assume that the sub super matrix $(\widetilde{\phi}_{VU}{}^r{}_i(p_0))_{i,r=1,\ldots,k+l}$ is invertible. Hence, there exists $U' \subset \mathcal{S} \times U$ open such that $(\widetilde{\phi}_{VU}{}^r{}_i(p))_{i,r=1,\ldots,k+l}$ is invertible $\forall p \in U'$. As taking the inverse of super matrices is smooth, the inverse matrix $(h^i{}_r(p))_{i,r=1,\ldots,k+l}$ of $(\widetilde{\phi}_{VU}{}^r{}_i(p))$ $\forall p \in U'$ defines a smooth right linear map from $\mathcal{W}$ to $\Lambda^{m|n}$.
For $i' = k + l, \ldots, m + n$, set

$$s_{i'}(p) := \psi_U^{-1}\left(p, e_{i'} - e_i h^i{}_r(p)\widetilde{\phi}_{VU}{}^r{}_{i'}(p)\right) \tag{2.169}$$

$\forall p \in U'$. Then, since ϕ is homogeneous, it follows that $s_{i'} \in \mathfrak{X}(U)$, $i' = k+l, \ldots, m+n$ define smooth vector fields on U' such that $s_{i'}(p) \in T_p\mathcal{M}_{/\mathcal{S}}$ $\forall p \in U'$ and $(s_{i'}(p))_{i'}$ is a homogeneous basis of $\ker(\phi)_p$. Hence, this proves that $\ker(\phi)$ defines a smooth geometric distribution of rank $(m-k)|(n-l)$ on $\mathcal{M}_{/\mathcal{S}}$. □

Before we proceed with the definition of vertical and horizontal distributions as well as connection forms on principal super fiber bundles, let us first note a very important fact concerning partial evaluation of smooth maps defined on supermanifolds. More precisely, note that, in the H^∞ category, the space of smooth functions defines a $\mathbb{R}$-vector space. Thus, it follows that, given a smooth map $\phi : \mathcal{M} \times \mathcal{N} \to \mathcal{L}$ between H^∞ supermanifolds as well as a point $q \in \mathcal{N}$, the map

$$\phi_q := \phi(\cdot, q) : \mathcal{M} \to \mathcal{L} \tag{2.170}$$

in general, will not be of class H^∞, unless $g \in \mathbf{B}(\mathcal{N})$.[7] However, following [97], one can still associate a tangent map to ϕ_q even if q is not an element of the body. More precisely, one makes the following definition.

Definition 2.5.9. Let $\phi : \mathcal{M} \times \mathcal{N} \to \mathcal{L}$ be a smooth map between supermanifolds and $q \in \mathcal{N}$ be an arbitrary but fixed point. Using the identification $T(\mathcal{M} \times \mathcal{N}) \cong T(\mathcal{M}) \times T\mathcal{N}$, the *generalized tangent map* $\phi_{q*} : T_p\mathcal{M} \to T_{\phi(p,q)}\mathcal{L}$ of $\phi_q \equiv \phi(\cdot, q)$ at $p \in \mathcal{M}$ is defined as

$$\phi_{q*}(X_p) \equiv D_p\phi_q(X_p) := D_{(p,g)}\phi(X_p, 0_q) \tag{2.171}$$

---

7 Note that the set smooth functions on a H^∞ supermanifold is a $\mathbb{R}$-vector space. For a general product supermanifold $\mathcal{M} \times \mathcal{N}$, one has $H^\infty(\mathcal{M} \times \mathcal{N}) \cong H^\infty(\mathcal{M}) \otimes H^\infty(\mathcal{N})$. If then $f \otimes g$ is a smooth function on $\mathcal{M} \times \mathcal{N}$, it follows that $f \otimes g(\cdot, p) = f \cdot g(p) \in H^\infty(\mathcal{M}) \Leftrightarrow g(p) \in \mathbb{R} \Leftrightarrow p \in \mathbf{B}(\mathcal{N})$. In fact, this has its explanation in the algebraic category since, due to super Milnor's exercise (Prop. 2.2.7), the real spectrum $\mathrm{Hom}(\mathcal{O}(\mathcal{M}), \mathbb{R})$ is given by the set of morphisms $\mathrm{ev}_p : \mathcal{O}(\mathcal{M}) \to \mathbb{R}$ associated to points p on the underlying topological space of an algebro-geometric supermanifold.

for any $X_p \in T_p\mathcal{M}$. In a similar way, one defines the generalized tangent map ϕ_{p*} for $p \in \mathcal{M}$.

Definition 2.5.10. Let $\psi : \mathcal{M} \times \mathcal{N} \to \mathcal{L}$ and $\phi : \mathcal{K} \times \mathcal{L} \to \mathcal{P}$ be smooth maps between supermanifolds and $p \in \mathcal{K}$ and $q \in \mathcal{M}$ arbitrary fixed points. Then, the generalized tangent map $(\phi_p \circ \psi_q)_*$ of the map $\phi_p \circ \psi_q = \phi(p, \psi(q, \cdot)) : \mathcal{N} \to \mathcal{P}$ is defined, according to 2.5.9, as the generalized tangent map $(\phi \times (\mathrm{id} \times \psi))_{(p,q)*}$ associated to $(\phi \times (\mathrm{id} \times \psi))_{(p,q)} : \mathcal{N} \to \mathcal{P}$.

Proposition 2.5.11. *Let $\psi : \mathcal{M} \times \mathcal{N} \to \mathcal{L}$ and $\phi : \mathcal{K} \times \mathcal{L} \to \mathcal{P}$ be smooth maps between supermanifolds and $p \in \mathcal{K}$ and $q \in \mathcal{M}$ arbitrary fixed points. Then,*

$$(\phi_p \circ \psi_q)_* = \phi_{p*} \circ \psi_{q*} \tag{2.172}$$

Proof. For $X_g \in T_g\mathcal{N}$, $g \in \mathcal{N}$, we compute

$$\begin{aligned}(\phi_p \circ \psi_q)_*(X_g) &:= (\phi \times (\mathrm{id} \times \psi))_{(p,q)*}(X_g) = D(\phi \times (\mathrm{id} \times \psi))(0_{(p,q)}, X_g) \\ &= D\phi(D(\mathrm{id} \times \psi)(0_{(p,q)}, X_g)) \\ &= D\phi(0_p, D\psi(0_q, X_g)) \\ &= \phi_{p*} \circ \psi_{q*}(X_g)\end{aligned} \tag{2.173}$$

□

Definition 2.5.12. Let $\Phi_{\mathcal{S}} : \mathcal{M}_{/\mathcal{S}} \times \mathcal{G} \to \mathcal{M}_{/\mathcal{S}}$ be a smooth right action of a super Lie group $\mathcal{G}$ on a $\mathcal{S}$-relative supermanifold $\mathcal{M}_{/\mathcal{S}}$. A *fundamental tangent vector* $\widetilde{X}_p \in T_p\mathcal{M}_{/\mathcal{S}}$ at $p \in \mathcal{M}_{/\mathcal{S}}$ associated to $X \in \mathrm{Lie}(\mathcal{G}) = T_e\mathcal{G}$ is defined as

$$\widetilde{X}_p := (\Phi_{\mathcal{S}})_{p*}(X) \in T_p\mathcal{M}_{/\mathcal{S}} \tag{2.174}$$

where we made use of the generalized tangent map. In case $X \in \mathfrak{g}$, this yields a smooth vector field $\widetilde{X} \in \mathfrak{X}(\mathcal{M}_{/\mathcal{S}})$ which can equivalently be written as

$$\widetilde{X} = (\mathbb{1} \otimes X_e) \circ \Phi_{\mathcal{S}}^* \tag{2.175}$$

and which is called the *fundamental vector field* generated by X.

Definition 2.5.13. Let $\mathcal{G} \to \mathcal{P}_{/\mathcal{S}} \stackrel{\pi_{\mathcal{S}}}{\to} \mathcal{M}_{/\mathcal{S}}$ be a $\mathcal{S}$-relative principal super fiber bundle. The *vertical tangent module* $\mathscr{V}_p$ of $\mathcal{P}_{/\mathcal{S}}$ at a point $p \in \mathcal{P}_{/\mathcal{S}}$ is a super Λ-sub module of the tangent module $T_p\mathcal{P}_{/\mathcal{S}}$ defined as

$$\mathscr{V}_p := \ker(D_p\pi_{\mathcal{S}}) \tag{2.176}$$

Lemma 2.5.14. *Let $\Phi_{\mathcal{S}} : \mathcal{M}_{/\mathcal{S}} \times \mathcal{G} \to \mathcal{M}_{/\mathcal{S}}$ be a smooth right action of a super Lie group $\mathcal{G}$ on a $\mathcal{S}$-relative supermanifold $\mathcal{M}_{/\mathcal{S}}$. Then, for $X \in \mathrm{Lie}(\mathcal{G}) = T_e\mathcal{G}$, one has*

$$(\Phi_{\mathcal{S}})_{g*}\widetilde{X}_p = \widetilde{\mathrm{Ad}_{g^{-1}}X}_{p\cdot g}, \quad \forall p \in \mathcal{P}_{/\mathcal{S}},\ g \in \mathcal{G} \tag{2.177}$$

with $\widetilde{\mathrm{Ad}_{g^{-1}}X}_{p\cdot g}$ the fundamental tangent vector associated to $\mathrm{Ad}_{g^{-1}}X \in \mathrm{Lie}(\mathcal{G})$ at $p \cdot g \in \mathcal{M}_{/\mathcal{S}}$. Here, $\mathrm{Ad} : \mathcal{G} \to \mathrm{GL}(\mathrm{Lie}(\mathcal{G}))$ *denotes the* Adjoint representation *of $\mathcal{G}$ with pushforward* $\mathrm{ad} := \mathrm{Ad}_* : \mathrm{Lie}(\mathcal{G}) \to \underline{\mathrm{End}}_R(\mathrm{Lie}(\mathcal{G})),\ X \mapsto [X, \cdot]$ *given by the* adjoint representation *of* $\mathrm{Lie}(\mathcal{G})$.

Proof. By Prop. 2.5.11, we find

$$\begin{aligned}(\Phi_{\mathcal{S}})_{g*}\widetilde{X}_p &= (\Phi_{\mathcal{S}})_{g*} \circ (\Phi_{\mathcal{S}})_{p*}(X) = ((\Phi_{\mathcal{S}})_g \circ (\Phi_{\mathcal{S}})_p)_*(X) \\ &= (\Phi_{\mathcal{S}} \circ (\Phi_{\mathcal{S}} \times \mathrm{id}))_{(p,\cdot,g)*}(X) \\ &= (\Phi_{\mathcal{S}} \circ (\mathrm{id} \times \mu))_{(p,\cdot,g)*}(X) \\ &= (\Phi_{\mathcal{S}})_{p*} \circ R_{g*}(X)\end{aligned} \tag{2.178}$$

Since $R_{g*} = L_{g^*} \circ L_{g^{-1}*} \circ R_{g*} = L_{g^*} \circ \mathrm{Ad}_{g^{-1}}$, it thus follows

$$\begin{aligned}(\Phi_{\mathcal{S}})_{g*}\widetilde{X}_p &= (\Phi_{\mathcal{S}})_{p*} \circ L_{g^*} \circ L_{g^{-1}*} \circ R_{g*}(X) \\ &= (\Phi_{\mathcal{S}})_{p\cdot g*} \circ \mathrm{Ad}_{g^{-1}}(X) = \widetilde{\mathrm{Ad}_{g^{-1}}X}_{p\cdot g}\end{aligned} \tag{2.179}$$

as claimed. □

Proposition 2.5.15. *Let $\mathcal{G} \to \mathcal{P}_{/\mathcal{S}} \to \mathcal{M}_{/\mathcal{S}}$ be a $\mathcal{S}$-relative principal super fiber bundle with right-action $\Phi_{\mathcal{S}} : \mathcal{P}_{/\mathcal{S}} \times \mathcal{G} \to \mathcal{P}_{/\mathcal{S}}$. For any $p \in \mathcal{P}_{/\mathcal{S}}$, one has*

$$\mathscr{V}_p = \{\widetilde{X}_p |\, X \in \mathrm{Lie}(\mathcal{G})\} \tag{2.180}$$

i.e., the vertical tangent module $\mathscr{V}_p$ is generated by the fundamental tangent vectors at p. In particular, the assignment $\mathscr{V} : \mathcal{M}_{/\mathcal{S}} \ni p \mapsto \mathscr{V}_p$ defines a smooth geometric distribution of rank $\dim \mathfrak{g}$ *called the* vertical tangent bundle *which is right-invariant in the sense that $(\Phi_{\mathcal{S}})_{g*}\mathscr{V}_p = \mathscr{V}_{p\cdot g}\ \forall g \in \mathcal{G}$.*

Proof. For any $p \in \mathcal{P}_{/\mathcal{S}}$, let $s : U_{/\mathcal{S}} \to \mathcal{P}_{/\mathcal{S}}$ be a smooth local section of $\mathcal{P}_{/\mathcal{S}}$ with $\pi_{\mathcal{S}}(p) \in \mathcal{S} \times U \subseteq \mathcal{S} \times \mathcal{M}$ open. Then, $\pi_{\mathcal{S}} \circ s = \mathrm{id}$ implies $\pi_{\mathcal{S}*} \circ s_* = \mathrm{id}$ and thus $D_p\pi_{\mathcal{S}} : T_p\mathcal{M}_{/\mathcal{S}} \to T_{\pi_{\mathcal{S}}(p)}\mathcal{P}_{/\mathcal{S}}$ is surjective. Since, $D_p\pi_{\mathcal{S}}$ is homogeneous, it follows that $\mathscr{V}_p = \ker D_p\pi_{\mathcal{S}}$ is a super Λ-sub module of $T_p(\mathcal{P}_{/\mathcal{S}})$ of dimension $\dim \mathscr{V}_p = \dim T_p\mathcal{P}_{/\mathcal{S}} - \dim T_{\pi_{\mathcal{S}}(p)}\mathcal{M}_{/\mathcal{S}} = \dim \mathrm{Lie}(\mathcal{G})\ \forall p \in \mathcal{P}_{/\mathcal{S}}$.

For $X \in T_e\mathcal{G} = \mathrm{Lie}(\mathcal{G})$, the associated fundamental tangent vector $\widetilde{X}_p$ at $p \in \mathcal{P}_{/\mathcal{S}}$ is given by $\widetilde{X}_p = (\Phi_{\mathcal{S}})_{p*}X$. By definition of the generalized tangent map, this yields

$$
\begin{aligned}
D_p\pi_{\mathcal{S}}(\widetilde{X}_p) &= D_p\pi_{\mathcal{S}} \circ D_{(p,e)}\Phi_{\mathcal{S}}(0_p, X) \\
&= D_{(p,e)}(\pi_{\mathcal{S}} \circ \Phi_{\mathcal{S}})(0_p, X) \\
&= D_{(p,e)}(\pi_{\mathcal{S}} \circ \mathrm{pr}_1)(0_p, X) = 0
\end{aligned} \tag{2.181}
$$

i.e., $\widetilde{X}_p \in \mathscr{V}_p$. As $\Phi_{p*} : T_e\mathcal{G} \mapsto T_p\mathcal{P}_{/\mathcal{S}}$ is an even and injective map of super Λ-modules, it follows from the observation above that it is an isomorphism onto $\mathscr{V}_p$ proving (2.180).

To prove the last assertion, let $(X_i)_i$ be a real homogeneous basis of $T_e\mathcal{G}$ and $\widetilde{X} = \mathbb{1} \otimes X \circ \Phi^*_{\mathcal{S}}$ the associated smooth fundamental vector fields on $\mathcal{P}_{/\mathcal{S}}$. Then, $(\widetilde{X}_i(p))_i$ is a homogeneous basis of $\mathscr{V}_p\ \forall p \in \mathcal{P}_{/\mathcal{S}}$ and thus $\mathscr{V} : \mathcal{M}_{/\mathcal{S}} \ni p \mapsto \mathscr{V}_p$ defines a smooth geometric distribution of rank $\dim\mathfrak{g}$. It remains to show that $\mathscr{V}$ is indeed right-invariant, that is, $(\Phi_{\mathcal{S}})_{g*}\mathscr{V}_p = \mathscr{V}_{p\cdot g}\ \forall g \in \mathcal{G}$. Therefore, if $\widetilde{X}_p \in \mathscr{V}_p$ with $X \in \mathrm{Lie}(\mathcal{G})$, it follows from Lemma 2.5.14

$$
(\Phi_{\mathcal{S}})_{g*}\widetilde{X}_p = \widetilde{\mathrm{Ad}_{g^{-1}}X}_{p\cdot g} \in \mathscr{V}_{p\cdot g}. \tag{2.182}
$$

Since $(\Phi_{\mathcal{S}})_{g*} \circ (\Phi_{\mathcal{S}})_{g^{-1}*} = \mathrm{id}$, the claim follows. □

Definition 2.5.16. Let $\mathcal{G} \to \mathcal{P}_{/\mathcal{S}} \xrightarrow{\pi} \mathcal{M}_{/\mathcal{S}}$ be a $\mathcal{S}$-relative principal super fiber bundle with right-action $\Phi_{\mathcal{S}} : \mathcal{P}_{/\mathcal{S}} \times \mathcal{G} \to \mathcal{P}_{/\mathcal{S}}$. A *principal connection* (à la Ehresmann) $\mathscr{H}$ on $\mathcal{P}_{/\mathcal{S}}$ is a smooth geometric distribution $\mathscr{H} : \mathcal{P}_{/\mathcal{S}} \ni p \mapsto \mathscr{H}_p \subset T_p(\mathcal{P}_{/\mathcal{S}})$ of horizontal tangent modules on $\mathcal{P}_{/\mathcal{S}}$ of rank $\dim\mathcal{M}$ such that $\mathscr{H}_p \oplus \mathscr{V}_p = T_p(\mathcal{P}_{/\mathcal{S}})$ and $\mathscr{H}$ is right-invariant in the sense that

$$
(\Phi_{\mathcal{S}})_{g*}\mathscr{H}_p = \mathscr{H}_{p\cdot g} \tag{2.183}
$$

$\forall p \in \mathcal{P}_{/\mathcal{S}}, g \in \mathcal{G}$.

Definition 2.5.17. Let $\mathcal{G} \to \mathcal{P}_{/\mathcal{S}} \to \mathcal{M}_{/\mathcal{S}}$ be a $\mathcal{S}$-relative principal super fiber bundle and $\mathscr{H} : \mathcal{P}_{/\mathcal{S}} \to T(\mathcal{P}_{/\mathcal{S}})$ a principal connection on $\mathcal{P}_{/\mathcal{S}}$. A tangent vector $X_p \in T_p(\mathcal{P}_{/\mathcal{S}})$ at $p \in \mathcal{P}_{/\mathcal{S}}$ is called *horizontal* or *vertical* if $X_p \in \mathscr{H}_p$ or $X_p \in \mathscr{V}_p$, respectively. Analogously, one defines horizontal and vertical vector fields.

Remark 2.5.18. Since, $\mathscr{H}_p \oplus \mathscr{V}_p = T_p(\mathcal{P}_{/\mathcal{S}})\ \forall p \in \mathcal{P}_{/\mathcal{S}}$, this induces projections pr_h and pr_v on $T(\mathcal{P}_{/\mathcal{S}})$ onto the horizontal and vertical tangent modules. As $\mathscr{H}$ and $\mathscr{V}$ define smooth geometric distributions, it clear that, for any smooth vector field

$X \in \Gamma(T\mathcal{P}_{/\mathcal{S}})$, the projections $\mathrm{pr}_h \circ X$ and $\mathrm{pr}_v \circ X$ define smooth horizontal and vertical vector fields, respectively.

We finally come to an equivalent characterization of principal connections in terms of kernels of particular 1-forms defined on (relative) principal super fiber bundles. These so-called super connection 1-forms yield a generalization of the well-known gauge fields playing a prominent role in ordinary gauge theory in physics.

Definition 2.5.19. A *super connection* 1*-form* $\mathcal{A}$ on the $\mathcal{S}$-relative principal super fiber bundle $\mathcal{G} \to \mathcal{P}_{/\mathcal{S}} \xrightarrow{\pi_{\mathcal{S}}} \mathcal{M}_{/\mathcal{S}}$ is an even Lie($\mathcal{G}$)-valued 1-form $\mathcal{A} \in \Omega^1(\mathcal{P}_{/\mathcal{S}}, \mathfrak{g})$ such that

(i) $\langle \widetilde{X} | \mathcal{A} \rangle = X \; \forall X \in \mathfrak{g}$

(ii) $(\Phi_{\mathcal{S}})^*_g \mathcal{A} = \mathrm{Ad}_{g^{-1}} \circ \mathcal{A} \; \forall g \in \mathcal{G}$.

where, in condition (ii), the generalized tangent map was used (see Def. 2.5.9).

Theorem 2.5.20 (a generalization of [97]). *For a $\mathcal{S}$-relative principal super fiber bundle $\mathcal{G} \to \mathcal{P}_{/\mathcal{S}} \to \mathcal{M}_{/\mathcal{S}}$, there is a one-to-one correspondence between principal connections and connection* 1*-forms on $\mathcal{P}_{/\mathcal{S}}$. More precisely,*

(i) if $\mathscr{H} : \mathcal{P}_{/\mathcal{S}} \ni p \mapsto \mathscr{H}_p$ is a principal connection on $\mathcal{P}_{/\mathcal{S}}$, then $\mathcal{A} \in \Omega^1(\mathcal{P}_{/\mathcal{S}}, \mathfrak{g})_0$ defined via

$$\langle (\widetilde{X}_p, Y_p) | \mathcal{A}_p \rangle := X \tag{2.184}$$

$\forall (\widetilde{X}_p, Y_p) \in \mathscr{H}_p \oplus \mathscr{V}_p = T_p\mathcal{P}$, $p \in \mathcal{P}_{/\mathcal{S}}$ and $X \in$ Lie($\mathcal{G}$), defines a connection 1*-form on $\mathcal{P}_{/\mathcal{S}}$.*

(ii) if $\mathcal{A} \in \Omega^1(\mathcal{P}_{/\mathcal{S}}, \mathfrak{g})_0$ is a connection 1*-form on $\mathcal{P}_{/\mathcal{S}}$, then the assignment*

$$\mathscr{H} : \mathcal{P}_{/\mathcal{S}} \ni p \mapsto \ker(\mathcal{A}_p) \subset T_p(\mathcal{P}_{/\mathcal{S}}) \tag{2.185}$$

defines a principal connection on $\mathcal{P}_{/\mathcal{S}}$

Proof. (i) We have to show that $\mathcal{A}$ as defined via (2.185) satisfies the conditions (i) and (ii) in 2.5.19 of a super connection 1-form on $\mathcal{P}_{/\mathcal{S}}$. First, to see that $\mathcal{A}$ indeed defines a smooth even Lie($\mathcal{G}$)-valued 1-form on $\mathcal{P}_{/\mathcal{S}}$, i.e., $\mathcal{A} \in \Omega^1(\mathcal{P}_{/\mathcal{S}}, \mathfrak{g})_0$, let $(X_i)_i$ be a homogeneous basis of $\mathfrak{g} \subset \mathrm{Lie}(\mathcal{G}) = T_e\mathcal{G}$. Consider the components ${}^i\mathcal{A} := \mathcal{A} \diamond {}^iX \; \forall i$ with $({}^iX)_i$ the corresponding left dual basis of ${}^*\mathrm{Lie}(\mathcal{G})$. According to Remark 2.5.18, if $X \in \mathfrak{X}(\mathcal{P}_{/\mathcal{S}})$ is a smooth vector field, we can decompose X into vertical and horizontal parts via $X = \mathrm{pr}_v \circ X + \mathrm{pr}_h \circ X =: X_v + X_h$. Since X_v and X_h are smooth, it follows that ${}^i\mathcal{A}$ is smooth iff $\langle X_v | {}^i\mathcal{A} \rangle$

and $\langle X_h | {}^i\mathcal{A} \rangle$ are smooth for any X and thus iff ${}^i\mathcal{A}$ is smooth when restricted to smooth vertical and horizontal vector fields. The fundamental vector fields $\widetilde{X}_i$ generated by X_i define global smooth vertical vector fields such that $(\widetilde{X}_i(p))_i$ is a homogeneous basis of $\mathscr{V}_p$ $\forall p \in \mathcal{P}_{/\mathcal{S}}$. By definition of $\mathcal{A}$, we have

$$\langle \widetilde{X}_i | {}^j\mathcal{A} \rangle = \delta_i^j \quad \forall i, j \tag{2.186}$$

which is smooth and thus $\mathcal{A}$ is smooth on vertical vector fields. Finally, since $\mathscr{H}$ is a smooth geometric distribution, for any $p \in \mathcal{P}_{/\mathcal{S}}$, there exists a $p \in U_p \subseteq \mathcal{S}\times\mathcal{P}$ open as well as a family $(Y_i)_i$ of smooth horizontal vector fields on U_p such that $(Y_i(q))_i$ is a homogeneous basis of $\mathscr{H}_q$ $\forall q \in U_p$. Since $\langle Y_i | \mathcal{A} \rangle = 0$ and thus is smooth, it follows that $\mathcal{A}$ is also smooth on horizontal vector fields. Hence, indeed $\mathcal{A} \in \Omega^1(\mathcal{P}_{/\mathcal{S}}, \mathfrak{g})$. That $\mathcal{A}$ has to be even is immediate.

It remains to show that $\mathcal{A}$ in fact satisfies the conditions (i) and (ii). By definition, (i) is immediate. Moreover, by right-invariance of $\mathscr{H}$, it suffices to show (ii) for vertical tangent vectors. Hence, let $\widetilde{X}_p \in \mathscr{V}_p \subset T_p(\mathcal{P}_\mathcal{S})$ with $X \in \mathrm{Lie}(\mathcal{G})$. Using Lemma 2.5.14, we compute

$$\langle (\Phi_\mathcal{S})_{g*}\widetilde{X}_p | \mathcal{A}_{pg} \rangle = \langle \widetilde{\mathrm{Ad}_{g^{-1}}X}_{p\cdot g} | \mathcal{A}_{pg} \rangle = \mathrm{Ad}_{g^{-1}}X = \mathrm{Ad}_{g^{-1}} \langle \widetilde{X}_p | \mathcal{A}_p \rangle \tag{2.187}$$

This shows that $\mathcal{A}$ defines a principal connection 1-form on $\mathcal{P}_{/\mathcal{S}}$.

(ii) Conversely, for $\mathcal{A} \in \Omega^1(\mathcal{P}_{/\mathcal{S}}, \mathfrak{g})_0$ a principal connection 1-form on $\mathcal{P}_{/\mathcal{S}}$, we have to show that $\mathscr{H} : \mathcal{P}_{/\mathcal{S}} \ni p \mapsto \ker(\mathcal{A}_p) \subset T_p(\mathcal{P}_{/\mathcal{S}})$ defines a principal connection on $\mathcal{P}_{/\mathcal{S}}$. To this end, similar as in [97], consider the map

$$T(\mathcal{P}_{/\mathcal{S}}) \to (\mathcal{P} \times \mathrm{Lie}(\mathcal{G}))_{/\mathcal{S}},\ T_p\mathcal{P} \ni X_p \mapsto (p, \langle X_p | \mathcal{A}_p \rangle) \tag{2.188}$$

from $T(\mathcal{P}_{/\mathcal{S}})$ to the trivial $\mathcal{S}$-relative super vector bundle $\mathrm{Lie}(\mathcal{G}) \to (\mathcal{P} \times \mathrm{Lie}(\mathcal{G}))_{/\mathcal{S}} \to \mathcal{P}_{/\mathcal{S}}$. By Remark 2.5.6, we can identify $T(\mathcal{P}_{/\mathcal{S}})$ with the $\mathcal{S}$-relative super vector bundle $(T\mathcal{P})_{/\mathcal{S}}$. Hence, it follows that (2.188) defines a smooth even and surjective, as $\mathcal{A}$ is even and surjective, morphism of $\mathcal{S}$-relative super vector bundles. Hence, by Lemma 2.5.8, the kernel of (2.188), which coincides with $\mathscr{H}$, defines a smooth geometric distribution. To see that it is right-invariant, note that, by condition (ii), for $p \in \mathcal{P}_{/\mathcal{S}}$ and $X_p \in \mathscr{H}_p = \ker(\mathcal{A}_p)$, we have

$$\langle (\Phi_\mathcal{S})_{g*}X_p | \mathcal{A}_{p\cdot g} \rangle = \mathrm{Ad}_{g^{-1}} \langle X_p | \mathcal{A}_p \rangle = 0 \tag{2.189}$$

and thus $(\Phi_\mathcal{S})_{g*}X_p \in \mathscr{H}_{p\cdot g}$.

□

We finally want to define the notion of a covariant derivative and curvature 2-forms corresponding to super connection forms defined on relative principal super fiber bundles.

Definition 2.5.21. Let $\mathcal{G} \to \mathcal{P}_{/\mathcal{S}} \to \mathcal{M}_{/\mathcal{S}}$ be a $\mathcal{S}$-relative principal super fiber bundle and $\mathcal{A} \in \Omega^1(\mathcal{P}_{/\mathcal{S}}, \mathfrak{g})_0$ a super connection 1-form on $\mathcal{P}_{/\mathcal{S}}$. The linear map $D^{(\mathcal{A})} : \Omega^k(\mathcal{P}_{/\mathcal{S}}, \mathcal{V}) \to \Omega^{k+1}(\mathcal{P}_{/\mathcal{S}}, \mathcal{V})$ defined as

$$\langle X_0, \ldots, X_k | D^{(\mathcal{A})}\omega \rangle := \langle \mathrm{pr}_h \circ X_0, \ldots, \mathrm{pr}_h \circ X_k | \mathrm{d}\omega \rangle \tag{2.190}$$

for smooth vector fields $X_i \in \mathfrak{X}(\mathcal{P}_{/\mathcal{S}})$, $i = 0, \ldots, k$, is called the *covariant derivative induced by* $\mathcal{A}$ where $\mathrm{pr}_h : T\mathcal{P}_{/\mathcal{S}} \to \mathscr{H} := \ker \mathcal{A}$ denotes the projection onto the horizontal tangent modules induced by $\mathcal{A}$.

An important subclass of vector-valued forms on (relative) principal super fiber bundles is provided by forms that transform covariantly in a specific sense under gauge transformaions. This is the content of the following definition. One then asks the question, whether one can define a derivative on such forms so that the transformation property is preserved. As we will see, it follows that the covariant derivative induced by super connections forms indeed has the right properties.

Definition 2.5.22. Let $\mathcal{G} \to \mathcal{P}_{/\mathcal{S}} \to \mathcal{M}_{/\mathcal{S}}$ be a $\mathcal{S}$-relative principal super fiber bundle with $\mathcal{G}$-right action $\Phi : \mathcal{P}_{/\mathcal{S}} \times \mathcal{G} \to \mathcal{P}_{/\mathcal{S}}$ and $\rho : \mathcal{G} \to \mathrm{GL}(\mathcal{V})$ be a representation of $\mathcal{G}$ on a super Λ-vector space $\mathcal{V}$. A k-form ω on $\mathcal{P}_{/\mathcal{S}}$ with values in $\mathcal{V}$ is called *horizontal of type* $(\mathcal{G}, \rho)$, symbolically $\omega \in \Omega^k_{hor}(\mathcal{P}_{/\mathcal{S}}, \mathcal{V})^{(\mathcal{G},\rho)}$, if ω vanishes on vertical tangent vectors and

$$\Phi_g^*\omega = \rho(g)^{-1} \circ \omega, \ \forall g \in \mathcal{G} \tag{2.191}$$

Proposition 2.5.23 (a generalization of [97]). *Let $\mathcal{G} \to \mathcal{P}_{/\mathcal{S}} \to \mathcal{M}_{/\mathcal{S}}$ be a $\mathcal{S}$-relative principal super fiber bundle and $\mathcal{A} \in \Omega^1(\mathcal{P}_{/\mathcal{S}}, \mathfrak{g})_0$ a super connection 1-form on $\mathcal{P}_{/\mathcal{S}}$. Let $\omega \in \Omega^k_{hor}(\mathcal{P}_{/\mathcal{S}}, \mathcal{V})^{(\mathcal{G},\rho)}$ be a horizontal k-form on $\mathcal{P}_{/\mathcal{S}}$ of type $(\mathcal{G}, \rho)$. Then, the induced covariant derivative $D^{(\mathcal{A})}$ takes the form*

$$D^{(\mathcal{A})}\omega = \mathrm{d}\omega + \rho_*(\mathcal{A}) \wedge \omega \tag{2.192}$$

where the $(k+1)$-form $\rho_*(\mathcal{A}) \wedge \omega$ on $\mathcal{P}_{/\mathcal{S}}$ is defined as

$$\begin{aligned}&\langle X_0, \ldots, X_k | \rho_*(\mathcal{A}) \wedge \omega \rangle \\ &:= \sum_{i=0}^{k} (-1)^{k-i+\sum_{l=0}^{i-1}|X_i||X_l|} \rho_*(\iota_{X_i}\mathcal{A})(\langle X_0, \ldots, \widehat{X_i}, \ldots, X_k | \omega \rangle)\end{aligned} \tag{2.193}$$

In particular, $D^{(\mathcal{A})}\omega \in \Omega^{k+1}_{hor}(\mathcal{P}_{/\mathcal{S}}, \mathcal{V})^{(\mathcal{G},\rho)}$*, i.e.,* $D^{(\mathcal{A})}$ *induces a covariant derivative on the subspace of horizontal forms of type* $(\mathcal{G}, \rho)$*.*

Proof. That $D^{(\mathcal{A})}\omega$ defines a horizontal form of type $(\mathcal{G}, \rho)$ if ω does, follows immediately from the right-invariance of the horizontal distribution which implies $\mathrm{pr}_h \circ (\Phi_{\mathcal{S}})_g = (\Phi_{\mathcal{S}})_g \circ \mathrm{pr}_h\, \forall g \in \mathcal{G}$ where $\Phi_{\mathcal{S}} : \mathcal{P}_{/\mathcal{S}} \times \mathcal{G} \to \mathcal{P}_{/\mathcal{S}}$ denotes the $\mathcal{G}$-right action on $\mathcal{P}_{/\mathcal{S}}$.

To prove (2.192), it is sufficient to show equality when evaluating (2.192) on smooth horizontal and vertical vector fields X_i, $i = 0, \ldots, k$. Moreover, if X is vertical, it suffices to assume that is a fundamental vector field $X = \widetilde{Y}$ generated by some $Y \in \mathfrak{g}$. In case all X_i are vertical, then it is clear that both sides in (2.192) trivially vanish by horizontality. If all X_i are horizontal, then (2.193) simply vanishes as $\mathcal{A}$ vanishes on horizontal vector fields proving equality also in this case. Next, assume that at least two vector fields are vertical. Then, note that if $\widetilde{X}$ for some $X \in \mathfrak{g}$ is a fundamental vector field and Y^* is horizontal then $[\widetilde{X}, Y^*]$ is also horizontal. In fact, according to Prop. 2.6.10, we have $L_{\widetilde{X}}\mathcal{A} = -\mathrm{ad}_X \circ \mathcal{A}$ such that, following [97], this yields

$$\begin{aligned}\langle[\widetilde{X}, Y^*]|\mathcal{A}\rangle &= \widetilde{X}\,\langle Y^*|\mathcal{A}\rangle - (-1)^{|X||Y|}\,\langle Y^*|L_{\widetilde{X}}\mathcal{A}\rangle \\ &= \mathrm{ad}_X \circ \langle Y^*|\mathcal{A}\rangle = 0 \end{aligned} \tag{2.194}$$

Hence, again, both sides in (2.192) vanish as ω is horizontal and $\mathcal{A}$ vanishes on horizontal vector fields. It thus remains to consider the case where at least one vector field is vertical. Thus, suppose $\widetilde{X}$ is a fundamental vector field generated by $X \in \mathfrak{g}$ and Y_i are horizontal $\forall i = 1, \ldots, k$. By definition, the left-hand side of (2.192) simply vanishes. On the other hand, following precisely the same steps as in the proof of Prop. 2.6.10 to be discussed in Appendix F, it follows that $L_{\widetilde{X}}\omega = -\rho_*(X) \circ \omega$. This yields

$$\begin{aligned}(-1)^k\,\langle\widetilde{X}, Y_1 \ldots, Y_k|d\omega\rangle &= (-1)^{\sum_{i=1}^k |X||Y_i|}\,\langle Y_1, \ldots, Y_k|L_{\widetilde{X}}\omega\rangle \\ &= -\rho_*(X)(\langle Y_1, \ldots, Y_k|\omega\rangle) \end{aligned} \tag{2.195}$$

where identity (2.163) was used. Moreover, using Definition (2.193), we get

$$\begin{aligned}&\langle\widetilde{X}, Y_1 \ldots, Y_k|\rho_*(\mathcal{A}) \wedge \omega\rangle \\ &= (-1)^k \rho_*(\iota_{\widetilde{X}}\mathcal{A})(\langle Y_1, \ldots, Y_k|\omega\rangle) = \rho_*(X)(\langle Y_1, \ldots, Y_k|\omega\rangle) \end{aligned} \tag{2.196}$$

proving that the right-hand side of (2.192) vanishes, as well. □

Definition 2.5.24. Let $\alpha \in \Omega^k(\mathcal{M}_{/\mathcal{S}}, \mathfrak{g})$ and $\beta \in \Omega^l(\mathcal{M}_{/\mathcal{S}}, \mathfrak{g})$ be a k- resp. l-form on a $\mathcal{S}$-relative supermanifold $\mathcal{M}_{/\mathcal{S}}$ with values in a super Lie module $\mathrm{Lie}(\mathcal{G})$

corresponding to a super Lie group $\mathcal{G}$. The $k+l$-form $[\alpha \wedge \beta] \in \Omega^{k+l}(\mathcal{M}_{/\mathcal{S}}, \mathfrak{g})$ is then defined via

$$[\alpha \wedge \beta] := \alpha^i \wedge \mathfrak{C}^{|e_i|}(\beta^j) \otimes [e_i, e_j] \tag{2.197}$$

where we have expanded $\alpha = \alpha^i \otimes e_i$ and $\beta = \beta^i \otimes e_i$ w.r.t. a homogeneous basis $(e_i)_i$ of $\mathfrak{g}$. Here, the involution $\mathfrak{C} : \Lambda \to \Lambda$ is defined as $\mathfrak{C}(\lambda) = (-1)^{|\lambda|}\lambda$ for any homogeneous $\lambda \in \Lambda$.

Remark 2.5.25. For $\omega \in \Omega^k_{hor}(\mathcal{P}_{/\mathcal{S}}, \mathfrak{g})^{(\mathcal{G},\mathrm{Ad})}$, it follows that

$$D^{(\mathcal{A})}\omega = \mathrm{d}\omega + [\mathcal{A} \wedge \omega] \tag{2.198}$$

To see this, let us consider the case $k = 1$. By direct computation, it then follows

$$\begin{aligned}\langle X, Y | [\mathcal{A} \wedge \omega] \rangle =& \iota_X \circ \iota_Y \left(\mathcal{A}^i \wedge \mathfrak{C}^{|e_i|}(\omega^j) \right) \otimes [e_i, e_j] \\ =& \Big((-1)^{(|e_i|+|Y|)|X|} \langle Y | \mathcal{A}^i \rangle \langle X | \mathfrak{C}^{|e_i|}(\omega^j) \rangle \\ & -(-1)^{|e_i||Y|} \langle X | \mathcal{A}^i \rangle \langle Y | \mathfrak{C}^{|e_i|}(\omega^j) \rangle \Big) \otimes [e_i, e_j] \\ =& - [\langle X | \mathcal{A} \rangle, \langle Y | \omega \rangle] + (-1)^{|X||Y|} [\langle Y | \mathcal{A} \rangle, \langle X | \omega \rangle] \\ =& \langle X, Y | \mathrm{ad}(\mathcal{A}) \wedge \omega \rangle \end{aligned} \tag{2.199}$$

Hence, indeed, $[\mathcal{A} \wedge \omega] = \mathrm{ad}(\mathcal{A}) \wedge \omega$.

Definition 2.5.26. Let $\mathcal{A} \in \Omega^1(\mathcal{P}_{/\mathcal{S}}, \mathfrak{g})_0$ be a super connection 1-form on a $\mathcal{S}$-relative principal super fiber bundle $\mathcal{G} \to \mathcal{P}_{/\mathcal{S}} \to \mathcal{M}_{/\mathcal{S}}$. The horizontal 2-form

$$F(\mathcal{A}) := D^{(\mathcal{A})}\mathcal{A} \in \Omega^2_{hor}(\mathcal{P}_{/\mathcal{S}}, \mathfrak{g})^{(\mathcal{G},\mathrm{Ad})} \tag{2.200}$$

is called the *curvature* of $\mathcal{A}$.

Proposition 2.5.27. *Let $\mathcal{A} \in \Omega^1(\mathcal{P}_{/\mathcal{S}}, \mathfrak{g})_0$ be a super connection 1-form on a $\mathcal{S}$-relative principal super fiber bundle $\mathcal{G} \to \mathcal{P}_{/\mathcal{S}} \to \mathcal{M}_{/\mathcal{S}}$. Then,*

(i) $F(\mathcal{A}) = \mathrm{d}\mathcal{A} + [\mathcal{A} \wedge \mathcal{A}]$

(ii) $D^{(\mathcal{A})}F(\mathcal{A}) = 0$ *(Bianchi identity)*

Proof. The first identity can be shown similarly as in the proof of Prop. 2.5.23 by applying both sides separately on horizontal and vertical vector fields.

To show the Bianchi identity, let us decompose $\mathcal{A} = \mathcal{A}^i \otimes e_i$ with $(e_i)_i$ a real homogeneous basis of $\mathfrak{g}$. Then,

$$\begin{aligned} \mathrm{d}F(\mathcal{A}) &= \frac{1}{2}\mathrm{d}[\mathcal{A} \wedge \mathcal{A}] = (-1)^{|e_i||e_j|}\frac{1}{2}\left(\mathrm{d}\mathcal{A}^i \wedge \mathcal{A}^j - \mathcal{A}^i \wedge \mathrm{d}\mathcal{A}^j\right) \otimes [e_i, e_j] \\ &= \frac{1}{2}\left(\mathcal{A}^j \wedge \mathrm{d}\mathcal{A}^i - (-1)^{|e_i||e_j|}\mathcal{A}^i \wedge \mathrm{d}\mathcal{A}^j\right) \otimes [e_i, e_j] = -[\mathcal{A} \wedge \mathrm{d}\mathcal{A}] \end{aligned} \tag{2.201}$$

Thus, it follows that $\mathrm{d}F(\mathcal{A})$ vanishes when evaluating it on horizontal vector fields. But, since $D^{(\mathcal{A})}F(\mathcal{A}) = \mathrm{pr}_h \diamond \mathrm{d}F(\mathcal{A})$ (see Eq. (A.8)), the claim follows. □

2.6. Graded principal bundles and graded connections

The content of this section has been reproduced from [1], with slight changes to account for the context of this thesis with the permission of AIP Publishing.

In Section 2.2, a precise link between the algebro-geometric approach and Rogers-DeWitt approach has been discussed (see also [95, 108] as well as [98, 99] in the context of the categorial approach). In the following, we want to show in which sense super principal fiber bundles and connection forms in the category of algebro-geometric supermanifolds $\mathbf{SMan}_{\mathrm{Alg}}$ as given in [102] can be related to the respective formulation in the H^∞ category. Just for notational simplication, only in this section, we will call objects and morphisms in the category $\mathbf{SMan}_{\mathrm{Alg}}$ with the addition *graded* to distinguish them from their respective H^∞ counterparts. However, we have to emphasize that the definition of graded manifolds as chosen here is different from the original definition as given by Berezin-Kostant-Leites [92, 93] using the notion of *finite duals*, the latter being much more general (see also [106] for a comparison). Also, in this section, we are considering trivial parametrizing supermanifolds $\mathcal{S} = \{*\}$. The generalization to nontrivial parametrizing supermanifolds is straightforward, though with the addition graded is a bit cumbersome.

Definition 2.6.1. A *right action* of a graded Lie group $\mathcal{G} = (G, \mathcal{O}_G)$ on a graded manifold $\mathcal{M} = (M, \mathcal{O}_M)$ is a morphism $\Phi : (M, \mathcal{O}_M) \times (G, \mathcal{O}_G) \to (M, \mathcal{O}_M)$ of graded manifolds such that[8]

$$\Phi^\sharp \circ (\Phi^\sharp \otimes \mathbb{1}) = \Phi^\sharp \circ (\mathbb{1} \otimes \mu_{\mathcal{G}}^\sharp), \quad (\mathbb{1} \otimes e_{\mathcal{G}}^\sharp) \circ \Phi^\sharp = \mathbb{1} \tag{2.202}$$

[8] By the "Global Chart Theorem", Theorem 2.2.3, it follows that a morphism between graded manifolds is uniquely determined by its pullback. Hence, in the following, we will often state certain properties of morphisms that only involve the corresponding pullback.

where $\mu_{\mathcal{G}}^{\sharp}$ and $e_{\mathcal{G}}^{\sharp}$ denote the pullback of the group multiplication $\mu_{\mathcal{G}} : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ on $\mathcal{G}$ as well as its *neutral element* $e_{\mathcal{G}} : \mathbb{R}^{0|0} \to \mathcal{G}$. Analogously, one defines a *left action* of $(G, \mathcal{O}_G)$ on $(M, \mathcal{O}_M)$ as a morphism $\Phi : (G, \mathcal{O}_G) \times (M, \mathcal{O}_M) \to (M, \mathcal{O}_M)$ of graded manifolds such that

$$\Phi^{\sharp} \circ (\mathbb{1} \otimes \Phi^{\sharp}) = \Phi^{\sharp} \circ (\mu_{\mathcal{G}}^{\sharp} \otimes \mathbb{1}), \quad (e_{\mathcal{G}}^{\sharp} \otimes \mathbb{1}) \circ \Phi^{\sharp} = \mathbb{1} \tag{2.203}$$

The following definition is a particular variant of the definition of graded principal bundles given in [102].

Definition 2.6.2 (after [102])**.** A *graded principal bundle* over a graded manifold $(M, \mathcal{O}_M)$ consists of a graded manifold $(P, \mathcal{O}_P)$ as well as a right action $\Phi : (P, \mathcal{O}_P) \times (G, \mathcal{O}_G) \to (P, \mathcal{O}_P)$ of a graded Lie group $(G, \mathcal{O}_G)$ on $(P, \mathcal{O}_P)$ such that

(i) the quotient $(P/G, \mathcal{O}_P/\mathcal{O}_G)$ exists as a graded manifold isomorphic to $(M, \mathcal{O}_M)$ and the canonical projection $\pi : (P, \mathcal{O}_P) \to (P/G, \mathcal{O}_P/\mathcal{O}_G)$ is a submersion

(ii) $(P, \mathcal{O}_P)$ satisfies the local triviality property: For any $p \in M$, there exists an open neighborhood $U \subseteq M$ of p as well as an isomorphism $\phi : (V, \mathcal{O}_M|_V) \to (U \times G, \mathcal{O}_M|_U \hat{\otimes}_\pi \mathcal{O}_G)$ of graded manifolds, where $V := |\iota \circ \pi|^{-1}(U) \subseteq P$ with ι the isomorphism $\iota : (P/G, \mathcal{O}_P/\mathcal{O}_G) \xrightarrow{\sim} (M, \mathcal{O}_M)$, such that ϕ is $(G, \mathcal{O}_G)$-equivariant, that is,

$$\Phi^{\sharp} \circ \phi^{\sharp} = (\phi^{\sharp} \otimes \mathbb{1}) \circ (\mathbb{1} \otimes \mu_{\mathcal{G}}^{\sharp}) \tag{2.204}$$

Remark 2.6.3. As demonstrated in Section 2.2, using the functor of points technique, there exists an equivalence of categories given by two functors $\mathbf{A} : \mathbf{SMan}_{H^\infty, N} \to \mathbf{SMan}_{\mathrm{Alg}, N}$ and $\mathbf{H}_N : \mathbf{SMan}_{\mathrm{Alg}, N} \to \mathbf{SMan}_{H^\infty, N}$ where, for a given Grassmann algebra $\Lambda \equiv \Lambda_N$, both categories have to be restricted to the subcategories of supermanifolds with odd dimensions bounded by the number N of generators in Λ (this can be avoided choosing instead the infinite-dimensional Grassmann algebra Λ_∞, see e.g. [95, 108]). If $\mathcal{M}$ is a H^∞ supermanifold, the corresponding graded manifold $\mathbf{A}(\mathcal{M})$ is defined as

$$\mathbf{A}(\mathcal{M}) := (\mathbf{B}(\mathcal{M}), \mathbf{B}_* H^\infty_{\mathcal{M}}) \tag{2.205}$$

with $\mathbf{B}_* H^\infty_{\mathcal{M}}$ the pushforward sheaf $\mathbf{B}(\mathcal{M}) \supseteq U \mapsto \mathbf{B}_* H^\infty_{\mathcal{M}}(U) := H^\infty(\mathbf{B}^{-1}(U))$. On the other hand, for any graded manifold $\mathcal{K} = (K, \mathcal{O}_K)$, the corresponding H^∞ supermanifold is defined as the Λ-point

$$\mathbf{H}_N(\mathcal{K}) := \mathrm{Hom}_{\mathbf{SAlg}}(\mathcal{O}(\mathcal{K}), \Lambda) \tag{2.206}$$

As it turns out, these functors, in particular, preserve products. In fact, for instance, consider two H^∞ supermanifolds $\mathcal{M}$ and $\mathcal{N}$, then $H^\infty(\mathcal{M}\times\mathcal{N})\cong H^\infty(\mathcal{M})\hat{\otimes}_\pi H^\infty(\mathcal{N})$ where the completion is taken w.r.t. the *Grothendiek's π-topology*. Moreover, one has $\mathbf{B}(\mathcal{M}\times\mathcal{N})=\mathbf{B}(\mathcal{M})\times\mathbf{B}(\mathcal{N})$ which yields $\mathbf{A}(\mathcal{M}\times\mathcal{N})\cong\mathbf{A}(\mathcal{M})\times\mathbf{A}(\mathcal{N})$. On the other hand, for given graded manifolds $\mathcal{K}$ and $\mathcal{L}$, the corresponding Λ-point is given by $\mathbf{H}_N(\mathcal{K}\times\mathcal{L})=\mathrm{Hom}_{\mathbf{SAlg}}(\mathcal{O}(\mathcal{K})\hat{\otimes}_\pi\mathcal{O}(\mathcal{L}),\Lambda)$. Given morphisms $\phi:\mathcal{O}(\mathcal{K})\to\Lambda$ and $\psi:\mathcal{O}(\mathcal{L})\to\Lambda$, this yields a morphism $\langle\phi\otimes\psi\rangle:\mathcal{O}(\mathcal{K})\hat{\otimes}_\pi\mathcal{O}(\mathcal{L})\to\Lambda$ which on elementary tensors is defined as

$$\langle\phi\otimes\psi\rangle(f\otimes g):=\phi(f)\psi(g) \tag{2.207}$$

Conversely, given a morphism $\Psi:\mathcal{O}(\mathcal{K})\hat{\otimes}_\pi\mathcal{O}(\mathcal{L})\to\Lambda$, we can define morphims $\phi:\mathcal{O}(\mathcal{K})\to\Lambda$ and $\psi:\mathcal{O}(\mathcal{L})\to\Lambda$ setting $\phi(f):=\Psi(f\otimes 1)$ and $\psi(g):=\Psi(1\otimes g)$. Hence, this yields an isomorphism between Λ-points $\mathbf{H}_N(\mathcal{K}\times\mathcal{L})\cong\mathbf{H}_N(\mathcal{K})\times\mathbf{H}_N(\mathcal{L})$. To summarize, it follows that $\mathbf{A}$ and $\mathbf{H}_N$ can be extended to *monoidal functors* between *monoidal categories*.

Proposition 2.6.4. *There is a bijective correspondence between graded principal bundles and H^∞-principal super fiber bundles. More precisely,*

(i) if $\mathcal{G}\to\mathcal{P}\stackrel{\pi_{\mathcal{P}}}{\to}\mathcal{M}$ is a H^∞-principal super fiber bundle with structure group $\mathcal{G}$, then $\mathbf{A}(\mathcal{P})=(\mathbf{B}(\mathcal{P}),\mathbf{B}_*H^\infty_{\mathcal{P}})$ has the structure of a graded principal bundle over the graded manifold $\mathbf{A}(\mathcal{M})=(\mathbf{B}(\mathcal{M}),\mathbf{B}_*H^\infty_{\mathcal{M}})$.

(ii) if $(P,\mathcal{O}_P)$ is a graded principal bundle over a graded manifold $(M,\mathcal{O}_M)$, then $\mathbf{H}_N(G,\mathcal{O}_G)\to\mathbf{H}_N(P,\mathcal{O}_P)\to\mathbf{H}_N(M,\mathcal{O}_M)$ for a suitably large $N\in\mathbb{N}$ has the structure of a H^∞-principal super fiber bundle with bundle map $\tilde{\pi}:=\mathbf{H}_N(\iota\circ\pi)=\mathbf{H}_N(\iota)\circ\mathbf{H}_N(\pi):\mathbf{H}_N(P,\mathcal{O}_P)\to\mathbf{H}_N(M,\mathcal{O}_M)$, where $\pi:(P,\mathcal{O}_P)\to(P/G,\mathcal{O}_P/\mathcal{O}_G)$ is the canonical projection and ι the isomorphism $\iota:(P/G,\mathcal{O}_P/\mathcal{O}_G)\stackrel{\sim}{\to}(M,\mathcal{O}_M)$.

Proof. This is an immediate consequence of Remark 2.6.3 as well as Prop. 2.4.16. □

Hence, this proposition demonstrates that the categorical equivalence between graded and H^∞ supermanifolds even carries over to principal super fiber bundles. Next, we want to show that connection 1-forms defined on these bundles are in fact in one-to-one correspondence.

Definition 2.6.5. Let $(M,\mathcal{O}_M)$ be a graded manifold and $\mathfrak{g}$ be a super Lie algebra. A $\mathfrak{g}$-valued differential form ω on $(M,\mathcal{O}_M)$ is an element of $\Omega^\bullet(M,\mathcal{O}_M,\mathfrak{g})\equiv\Omega^\bullet(M,\mathcal{O}_M)\otimes\mathfrak{g}$.

Definition 2.6.6. Let $(G, \mathcal{O}_G)$ be a graded Lie group with super Lie algebra $\mathfrak{g}$. The morphism

$$\mathrm{Ad}: G \to \mathrm{GL}(\mathfrak{g}),\ \mathrm{Ad}_g(X) := (\mathrm{ev}_g \otimes X \otimes \mathrm{ev}_{g^{-1}}) \circ (\mathbb{1} \otimes \mu^\sharp) \circ \mu^\sharp \tag{2.208}$$

$\forall X \in \mathfrak{g}$, is called the Adjoint representation of G on $\mathfrak{g}$.

Definition 2.6.7. Let $(P, \mathcal{O}_P)$ be a graded principal bundle over a graded manifold $(M, \mathcal{O}_M)$ and right action $\Phi: (P, \mathcal{O}_P) \times (G, \mathcal{O}_G) \to (P, \mathcal{O}_{\mathcal{P}})$ of a graded Lie group $(G, \mathcal{O}_G)$ on $(P, \mathcal{O}_P)$. A *graded connection* 1*-form* ω on $(P, \mathcal{O}_P)$ is an even $\mathfrak{g}$-valued 1-form $\omega \in \Omega^1(P, \mathcal{O}_P, \mathfrak{g})_0$ such that

(i) $\langle \widetilde{X} | \omega \rangle = X\ \forall X \in \mathfrak{g}$ and $\widetilde{X} := \mathbb{1} \otimes X \circ \Phi^* \in \mathrm{Der}(\mathcal{O}(\mathcal{P}))$ the associated fundamental vector field

(ii) $\Phi_g^* \omega = \mathrm{Ad}_{g^{-1}} \circ \omega\ \forall g \in G$ and $L_{\widetilde{X}} \omega = -\mathrm{ad}_X \circ \omega\ \forall X \in \mathfrak{g}$

Here, for $g \in G$, $\Phi_g: (P, \mathcal{O}_P) \to (P, \mathcal{O}_P)$ is the isomorphism of graded manifolds induced by the pullback morphism $\Phi_g^\sharp := (\mathbb{1} \otimes \mathrm{ev}_g) \circ \Phi^\sharp: \mathcal{O}_P \to \mathcal{O}_P$. Moreover, for homogeneous $X \in \mathfrak{g}$, $\mathrm{ad}_X \circ \omega$ is defined as

$$\langle Z | \mathrm{ad}_X \circ \omega \rangle = (-1)^{|Z||X|} \mathrm{ad}_X \langle Z | \omega \rangle \tag{2.209}$$

for any homogeneous $Z \in \mathrm{Der}(\mathcal{O}(\mathcal{P}))$.

In order to provide a link between graded and H^∞-super connection 1-forms, the following lemma will play a central role. It is based on the equivalent characterization of super Lie groups in terms of the corresponding super Harish-Chandra pair $(\mathbf{B}(\mathcal{G}), \mathfrak{g})$, Theorem 2.3.9.

Lemma 2.6.8. *Given a H^∞-super Lie group as a well as a smooth map $F \in H^\infty(\mathcal{G})$. Then, F vanishes identically on $\mathcal{G}$ if and only if $XF|_{\mathbf{B}(\mathcal{G})} \equiv 0$ for all $X \in \mathcal{U}(\mathfrak{g})$ (and thus in particular $F|_{\mathbf{B}(\mathcal{G})} \equiv 0$) with $\mathcal{U}(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$.*

Proof. One direction is clear, so suppose that for some smooth function $F \in H^\infty(\mathcal{G})$, $XF|_{\mathbf{B}(\mathcal{G})} \equiv 0$ for all $X \in \mathcal{U}(\mathfrak{g})$. By the super Harish Chandra theorem, $\mathcal{G}$ can be identified with the globally split super Lie group $\mathbf{S}(\mathfrak{g}, \mathbf{B}(\mathcal{G})) \cong \mathcal{G}_0 \times (\mathfrak{g}_1 \otimes \Lambda_1)$ associated to the trivial vector bundle $\mathbf{B}(\mathcal{G}) \times \mathfrak{g} \to \mathbf{B}(\mathcal{G})$. Hence, there exist odd functions $\theta^\alpha \in H^\infty(\mathcal{G})$, $\alpha = 1, \ldots, n$, $n = \dim \mathfrak{g}_1$, such that any $f \in H^\infty(\mathcal{G})$ is of the form

$$f = \sum_{\underline{I}} \mathbf{S}(f_{\underline{I}}) \theta^{\underline{I}} \tag{2.210}$$

with $\mathbf{S}(f_{\underline{I}})$ the (generalized) Grassmann extension of smooth functions $f_{\underline{I}} \in C^\infty(\mathbf{B}(\mathcal{G}))$ for any ordered multi-index $\underline{I}$ of length $\leq n$. Hence, F can be written in the form

$$F = \sum_{\underline{I}} \mathbf{S}(F_{\underline{I}})\theta^{\underline{I}} \tag{2.211}$$

for some $F_{\underline{I}} \in C^\infty(\mathbf{B}(\mathcal{G}))$. It then follows from the assumptions that for $1 \in \mathcal{U}(\mathfrak{g})$, $F(g) = 0 = F_\emptyset(g)$ for any body point $g \in \mathbf{B}(\mathcal{G})$, that is, $F_\emptyset \equiv 0$.

Let ∂_α be the derivations on $H^\infty(\mathcal{G})$ satisfying $\partial_\alpha\theta^\beta = \delta^\beta_\alpha$, $(X_i)_i$ be a homogeneous basis of smooth left-invariant vector fields $X_i \in \mathfrak{g}\,\forall i$ on $\mathcal{G}$ and, according to Prop. 2.4.12, $({}^i\omega)_i$ the corresponding smooth left-invariant 1-forms satisfying $\langle X_j|{}^i\omega\rangle = \delta^i_j$. Then, ∂_α can be written in the form $\partial_\alpha = \langle\partial_\alpha|{}^i\omega\rangle\, X_i$ with $\langle\partial_\alpha|{}^i\omega\rangle \in H^\infty(\mathcal{G})\,\forall\alpha = 1,\ldots,n$ and i. As a consequence, for each multi-index $\underline{I}$, $\partial_{\underline{I}} := \partial_{\alpha_1}\cdots\partial_{\alpha_k}$ with $k = |\underline{I}|$ is a $H^\infty(\mathcal{G})$-linear expansion of elements in $\mathcal{U}(\mathfrak{g})$. Hence, by hypothesis, this implies $0 = \partial_{\underline{I}}F(g) = (-1)^{k(k-1)/2}F_{\underline{I}}(g)$ for any body point $g \in \mathbf{B}(\mathcal{G})$, i.e., $F_{\underline{I}} \equiv 0$, and therefore $F \equiv 0$ as claimed. □

Remark 2.6.9. It is clear that Lemma 2.6.8 equally holds if one replaces $\mathcal{U}(\mathfrak{g})$ by the respective right-invariant counterpart $\mathcal{U}(\mathfrak{g}^R)$, i.e., the universal enveloping algebra of the super Lie algebra $\mathfrak{g}^R$ of smooth right-invariant vector fields on $\mathcal{G}$.

Proposition 2.6.10. *Let $\mathcal{G} \to \mathcal{P} \stackrel{\pi_\mathcal{P}}{\to} \mathcal{M}$ be a H^∞ principal super fiber bundle with $\mathcal{G}$-right action $\Phi : \mathcal{P}\times\mathcal{G} \to \mathcal{P}$. A smooth even* Lie$(\mathcal{G})$*-valued* 1*-form $\mathcal{A} \in \Omega^1(\mathcal{P},\mathfrak{g})_0$ is a connection* 1*-form on $\mathcal{P}$ if and only if*

(i) $\langle\widetilde{X}|\mathcal{A}\rangle = X\,\forall X \in \mathfrak{g}$ *and* $\widetilde{X} := \mathbb{1}\otimes X\circ\Phi^* \in \Gamma(T\mathcal{P})$ *the associated smooth fundamental vector field*

(ii) $\Phi_g^*\mathcal{A} = \mathrm{Ad}_{g^{-1}}\circ\mathcal{A}\,\forall g \in \mathbf{B}(\mathcal{G})$ *and* $L_{\widetilde{X}}\mathcal{A} = -\mathrm{ad}_X\circ\mathcal{A}\,\forall X \in \mathfrak{g}$.

Proof. The proof of this proposition is a bit lengthy and technical as one always has to care about smoothness in the various construction since H^∞-smoothness is not preserved under partial evaluation. Therefore, we have moved it to Appendix F. □

Proposition 2.6.11. *There is a bijective correspondence between graded connection* 1*-forms on graded principal bundles and H^∞-smooth super connection* 1*-forms on H^∞ principal super fiber bundles.*

Proof. This is an immediate consequence of Prop. 2.6.4 and 2.6.10 as well as Remark 2.6.3. □

2.7. Parallel transport map

The content of this section has been reproduced from [1], with slight changes to account for the context of this thesis with the permission of AIP Publishing.

2.7.1. Preliminaries and first construction

In this section, we want to derive the parallel transport map corresponding to super connection forms. To this end, at the beginning, we want restrict to trivial parametrizing supermanifolds $\mathcal{S} = \{*\}$. This will provide us already with the main ideas behind the construction and, at the same time, points out the necessity of the parametrization. The generalization to the relative category will then be considered in the subsequent section. The following proposition plays a central role.

Proposition 2.7.1. *Let* $\Phi : \mathcal{M} \times \mathcal{G} \to \mathcal{M}$ *be a smooth right action of a super Lie group* $\mathcal{G}$ *on a supermanifold* $\mathcal{M}$*. Then,*

$$D_{(p,g)}\Phi(X_p, Y_g) = \Phi_{g*}(X_p) + \widetilde{\theta_{\mathrm{MC}}(Y_g)}_{p\cdot g} \tag{2.212}$$

at any $(p,q) \in \mathcal{M} \times \mathcal{G}$ *and tangent vectors* $X_p \in T_p\mathcal{M}$*,* $Y_g \in T_g\mathcal{G}$*, where* $\theta_{\mathrm{MC}} \in \Omega^1(\mathcal{G}, \mathfrak{g})$ *is the* Maurer-Cartan form *on* $\mathcal{G}$ *(see Example 2.4.13).*

Proof. The proof is very similar to the classical theory of ordinary smooth manifolds. One only has to care about smoothness. We will therefore employ the notion of the generalized tangent map. By linearity, we have

$$D_{(p,g)}\Phi(X_p, Y_g) = D_{(p,g)}\Phi(X_p, 0_g) + D_{(p,g)}\Phi(0_p, Y_g) = \Phi_{g*}(X_p) + \Phi_{p*}(Y_g) \tag{2.213}$$

Since $Y_g = L_{g*} \circ L_{g^{-1}*}(Y_g) = L_{g*}(\theta_{\mathrm{MC}}(Y_g))$ by definition of the Maurer-Cartan form, this yields, using Prop. 2.5.11 and that Φ defines a right action,

$$\begin{aligned}\Phi_{p*}(Y_g) &= \Phi_{p*} \circ L_{g*}(\theta_{\mathrm{MC}}(Y_g)) = (\Phi_p \circ L_g)_*(\theta_{\mathrm{MC}}(Y_g)) \\ &= (\Phi \circ (\mathrm{id} \times \mu))_{(p,g)*}(\theta_{\mathrm{MC}}(Y_g)) = (\Phi \circ (\Phi \times \mathrm{id}))_{(p,g)*}(\theta_{\mathrm{MC}}(Y_g)) \\ &= \Phi_{p\cdot g*}(\theta_{\mathrm{MC}}(Y_g))\end{aligned} \tag{2.214}$$

which implies $\Phi_{p*}(Y_g) = \widetilde{\theta_{\mathrm{MC}}(Y_g)}_{p\cdot g}$ by definition of the fundamental tangent vector. $\square$

Before we state the main definition of this section concerning the horizontal lift of smooth paths on supermanifolds, let us briefly recall the notion of a local flow of a smooth vector field. In the super category, one needs to distinguish between even and odd vector fields whose corresponding local flows turn out to possess different properties. In what follows, we want to focus on odd vector fields as these seem to be rarely discussed in the literature. In contrast to the classical theory, it follows that the corresponding local flow depends on two parameters (t, θ) given by both an even and odd parameter t and θ, respectively. These define elements of the superdomain $\Lambda^{1,1}$ which can be given the structure of a super Lie group also called the *super translation group* with multiplication defined via $(t, \theta) \cdot (s, \eta) = (t + s + \theta\eta, \theta + \eta)\ \forall (t, \theta), (s, \eta) \in \Lambda^{1,1}$.

Definition 2.7.2. Let $X \in \Gamma(T\mathcal{M})_1$ be an odd smooth vector field on a supermanifold $\mathcal{M}$, $f : \mathcal{M} \to \mathcal{M}$ a smooth map and $t_0 \in \Lambda^{1,1}$ a body point. A smooth map $\phi^X : \mathcal{I} \times U \to V$, with $U, V \subset \mathcal{M}$ open and $t_0 \in \mathcal{I} \subset \Lambda^{1,1}$ an open (connected) *super interval*, is called a *local flow* of X around t_0 with initial condition f, if ϕ^X satisfies

$$(\mathcal{D} \otimes \mathbb{1}) \circ (\phi^X)^* = X \circ \phi^X \tag{2.215}$$

as well as $\phi^X_{(t_0,0)} := \phi^X(t_0, 0, \cdot\,) = f$ on U, where $\mathcal{D}$ denotes the right-invariant vector field $\mathcal{D} := \partial_\theta + \theta\partial_t$ on $\Lambda^{1,1}$. If, in particular, $\mathcal{I} = \Lambda^{1,1}$ and $U = V = \mathcal{M}$, ϕ^X is called a *global flow*.

Proposition 2.7.3. *Let $X \in \Gamma(T\mathcal{M})_1$ be an odd smooth vector field on a supermanifold $\mathcal{M}$ and $f : \mathcal{M} \to \mathcal{M}$ a smooth map, then, for any body point $t_0 \in \Lambda^{1,1}$, X admits a local flow ϕ^X around t_0 with initial condition f.*

Proof. See for instance [78] for a proof in the pure algebraic setting using the concept of *functor of points*. □

Corollary 2.7.4. *If ϕ^X is a local flow of an odd smooth vector field X, then*

$$\phi^X_{(t,\theta)} \circ \phi^X_{(s,\eta)} = \phi^X_{(t+s+\theta\eta,\theta+\eta)} \tag{2.216}$$

whenever both sides are defined.

Proof. We give an algebraic proof of this proposition. To this end, consider the smooth maps

$$\Phi_1 : (t, \theta, s, \eta, p) \mapsto (t + s + \theta\eta, \theta + \eta, \phi^X_{(t+s+\theta\eta,\theta+\eta)}(p)) \tag{2.217}$$

$$\Phi_2 : (t, \theta, s, \eta, p) \mapsto (t + s + \theta\eta, \phi^X_{(t,\theta)}(\phi_{(s,\eta)}(p))) \tag{2.218}$$

defined on some open subsets of $\Lambda^{1,1}\times\Lambda^{1,1}\times\mathcal{M}$. It then follows that both maps satisfy

$$(\mathcal{D}\otimes\mathbb{1})\circ\Phi_i^*=\Phi_i^*\circ(\mathcal{D}\otimes\mathbb{1}+\mathbb{1}\otimes X),\quad\forall i=1,2\tag{2.219}$$

For instance, since $\Phi_1=(\mathrm{id}\times\phi^X)\circ(d\times\mathrm{id})\circ(\mu\times\mathrm{id})$, with d the diagonal map on $\Lambda^{1,1}$, it follows from the right-invariance of $\mathcal{D}$ as well as $\mathcal{D}\circ d^*=d^*\circ(\mathcal{D}\otimes\mathbb{1}+\mathbb{1}\otimes\mathcal{D})$

$$\begin{aligned}(\mathcal{D}\otimes\mathbb{1})\circ\Phi_i^*&=(\mathcal{D}\otimes\mathbb{1})\circ(\mu^*\otimes\mathbb{1})\circ(d^*\otimes\mathbb{1})\circ(\mathbb{1}\otimes\phi^{X*})\\&=(\mu^*\circ\mathcal{D}\otimes\mathbb{1})\circ(d^*\otimes\mathbb{1})\circ(\mathbb{1}\otimes\phi^{X*})\\&=(\mu^*\otimes\mathbb{1})\circ(d^*\otimes\mathbb{1})\circ(\mathcal{D}\otimes\mathbb{1}+\mathbb{1}\otimes\mathcal{D})\circ(\mathbb{1}\otimes\phi^{X*})\\&=(\mu^*\otimes\mathbb{1})\circ(d^*\otimes\mathbb{1})\circ(\mathcal{D}\otimes\phi^{X*}+\mathbb{1}\otimes(\phi^{X*}\circ X))\\&=\Phi_1^*\circ(\mathcal{D}\otimes\mathbb{1}+\mathbb{1}\otimes X)\end{aligned}\tag{2.220}$$

and similarly for Φ_2. Hence, Φ_i for $i=1,2$ both define local flows of the odd vector field $\mathcal{D}\otimes\mathbb{1}+\mathbb{1}\otimes X$ with the initial condition $\Phi_i(0,0,\cdot)=(\mathrm{id}\times\phi^X)\circ(d\times\mathrm{id})$. By uniqueness, it thus follows that they have to coincide on the intersection of their domains. □

Definition 2.7.5. Let $\mathcal{P}\xrightarrow{\pi}\mathcal{M}$ be a principal super fiber bundle and $\mathscr{H}\subset T\mathcal{P}$ a principal connection on $\mathcal{P}$. Let $\mathcal{I}\subseteq\Lambda^{1,1}$ be a super interval and $\gamma:\mathcal{I}\rightarrow\mathcal{M}$ be a smooth map also called a *path* on $\mathcal{M}$. Then, a smooth path $\gamma^{hor}:\Lambda^{1,1}\supseteq\mathcal{I}\rightarrow\mathcal{P}$ on $\mathcal{P}$ is called a *horizontal lift* of γ, if $\pi\circ\gamma^{hor}=\gamma$ and $(\gamma^{hor})_*\mathcal{D}\subset(\gamma^{hor})^*\mathscr{H}$.

The following theorem provides the existence of horizontal lifts of paths defined on supermanifolds. Since, the zero element $0\in\mathcal{I}$ of a super interval defines a body point and paths are supposed to be smooth, it is important to note that the initial values of both the path and its horizontal lift need to be body points, as well. Moreover, as will be proven below, in order to obtain a nontrivial parallel transport map, one necessarily needs to consider smooth paths depending on both even and odd parameters. However, as will be discussed in the subsequent Section 2.7.2, this can be remedied considering instead relative supermanifolds.

Theorem 2.7.6. *Let $\mathcal{A}\in\Omega^1(\mathcal{P},\mathfrak{g})_0$ be a super connection 1-form on a principal super fiber bundle $\mathcal{G}\rightarrow\mathcal{P}\xrightarrow{\pi}\mathcal{M}$ over $\mathcal{M}$ defining a principal connection $\mathscr{H}\subset T\mathcal{P}$ on $\mathcal{P}$. Then, for any smooth path $\gamma:\Lambda^{1,1}\supseteq\epsilon_{1,1}^{-1}([0,1])\rightarrow\mathcal{M}$ on the supermanifold $\mathcal{M}$ and any body point $p\in\mathbf{B}(\mathcal{P})$, there exists a unique horizontal lift $\gamma_p^{hor}:\epsilon_{1,1}^{-1}([0,1])\rightarrow\mathcal{P}$ of γ such that $\gamma_p^{hor}(0)=p$.*

If, in particular, the path is bosonic, i.e., $\gamma : \Lambda_0 \supseteq \epsilon_{1,0}^{-1}([0,1]) \rightarrow \mathcal{M}$ the horizontal lift $\gamma_p^{hor} : \epsilon_{1,0}^{-1}([0,1]) \rightarrow \mathcal{P}$ of γ is given by

$$\gamma_p^{hor} = \mathbf{S}(\mathbf{B}(\gamma)_p^{hor}) \tag{2.221}$$

where $\mathbf{B}(\gamma)_p^{hor}$ is the (unique) horizontal lift of $\mathbf{B}(\gamma)$ through $p \in P := \mathbf{B}(\mathcal{P})$ to the ordinary principal bundle P where the principal connection $H \subset TP$ on P is induced by the ordinary connection 1-form $A := \mathbf{B}(\mathcal{A}) \in \Omega^1(P, \mathfrak{g}_0)$.

Proof. Using a gluing argument, it suffices to show that, for any local trivialization neighborhood $\pi^{-1}(U) \subset \mathcal{P}$ of $\mathcal{P}$ with $U \subset \mathcal{M}$ open and $\mathrm{im}\,\gamma \cap U \neq \emptyset$, there exists a horizontal lift $\gamma^{hor} : \mathcal{I} \longrightarrow \mathcal{P}$ of γ with $\mathcal{I} = \gamma^{-1}(U)$. Hence, in the following, let us assume that γ is contained within a local trivialization neighborhood of $\mathcal{P}$, i.e., there is a local section $s : U \rightarrow \mathcal{P}$ of $\mathcal{P}$ such that $\mathrm{im}\,\gamma \subset U$. Let $\mathcal{I} = \epsilon_{1,1}^{-1}([0,1])$ and $\delta := s \circ \gamma$. A horizontal lift then has to be of the form $\gamma^{hor} := \Phi \circ (\delta \times g)$ for some smooth function $g : \tilde{\mathcal{I}} \rightarrow \mathcal{G}$ defined on some open subset $\tilde{\mathcal{I}} \subseteq \mathcal{I}$. By Prop. 2.7.1, this yields

$$\mathcal{D}\gamma^{hor}(t,\theta) = D_{(\delta(t,\theta),g(t,\theta))}\Phi(\mathcal{D}\delta, \mathcal{D}g) = \Phi_{g(t,\theta)*}(\mathcal{D}\delta) + \widetilde{\theta_{\mathrm{MC}}(\mathcal{D}g)}_{\gamma^{hor}(t,\theta)} \tag{2.222}$$

Hence, γ^{hor} defines a horizontal lift of γ if and only if

$$\begin{aligned} 0 = \langle \mathcal{D}\gamma^{hor}(t,\theta) | \mathcal{A} \rangle &= \mathrm{Ad}_{g(t,\theta)^{-1}} \langle \mathcal{D}\delta | \mathcal{A} \rangle + \langle \mathcal{D}g | \theta_{\mathrm{MC}} \rangle \\ &= L_{g(t,\theta)^{-1}*}(R_{g(t,\theta)*} \langle \mathcal{D}\delta | \mathcal{A} \rangle + \mathcal{D}g) \end{aligned} \tag{2.223}$$

But, since the pushforward of the left translation is an isomorphism of tangent modules, this equivalent to

$$\mathcal{D}g(t,\theta) = -R_{g(t,\theta)*} \langle \mathcal{D}\delta(t,\theta) | \mathcal{A} \rangle = -R_{g(t,\theta)*}\mathcal{A}^{\gamma}(t,\theta) \tag{2.224}$$

where we set $\mathcal{A}^{\gamma}(t,\theta) := \langle \mathcal{D}\delta(t,\theta) | \mathcal{A} \rangle = \langle \mathcal{D}\gamma(t,\theta) | s^*\mathcal{A} \rangle$. Hence, the claim follows if we can show that (2.224) admits a smooth solution on all of $\mathcal{I}$, i.e., $\tilde{\mathcal{I}} = \mathcal{I}$. To see this, let us define the vector field

$$\begin{aligned} Z : \mathcal{I} \times \mathcal{G} &\rightarrow T(\mathcal{I} \times \mathcal{G}) \cong \Lambda^2 \times T\mathcal{G} \\ (s,\eta,g) &\mapsto (\eta, 1, -D_{(e,g)}\mu_{\mathcal{G}}(\mathcal{A}^{\gamma}(s,\eta), 0_g)) \end{aligned} \tag{2.225}$$

Since $\mathcal{A}^{\gamma}$ and the zero section $\mathbf{0} : \mathcal{G} \rightarrow T\mathcal{G},\ g \mapsto 0_g$ are both of class H^{∞}, it follows that Z defines a smooth section of the tangent bundle of the supermanifold $\mathcal{I} \times \mathcal{G}$. In particular, as $\mathcal{A}$ is even and $\mathcal{D}$ is odd, $\mathcal{A}^{\gamma}(t,\theta)$ defines an odd derivation $\forall (t,\theta) \in \mathcal{I}$ and therefore Z is an odd vector field. Hence, by Prop. (2.7.3), there exists a smooth function $F : \Lambda^{1,1} \supseteq \epsilon_{1,1}^{-1}(\tilde{I}) \to \Lambda^{1,1} \times \mathcal{G}$, $(t,\theta) \mapsto (h(t,\theta), g(t,\theta))$, with smooth functions h and g defined on some interval $\epsilon_{1,1}^{-1}(\tilde{I})$ with $\tilde{I} = [0,\delta')$, $0 < \delta' < 1$, such that

$$\mathcal{D}F(t,\theta) = Z_{F(t,\theta)} \tag{2.226}$$

with the initial condition $F(0,0) = (0,0,e)$. Since $h : (t,\theta) \mapsto h(t,\theta) \in \Lambda^{1,1}$ has to be of the form $h(t,\theta) = (a(t), \theta b(t))$ with smooth functions $a, b : \epsilon_{1,0}^{-1}(\tilde{I}) \to \Lambda_0$, it follows that $\mathcal{D}h(t,\theta) = (\theta\partial_t a(t), b(t))$. Hence, by definition of the vector field Z, the differential equation (2.226) yields $b \equiv 1$ and $\partial_t a = b \equiv 1$ such that, by the initial condition, it follows $h(t,\theta) = (t,\theta)$. Thus, (2.226) becomes

$$\mathcal{D}F(t,\theta) = (\theta, 1, \mathcal{D}g(t,\theta)) = (\theta, 1, -D_{(t,\theta,g(t,\theta))}\mu_{\mathcal{G}}(\mathcal{A}^{\gamma}(t,\theta), 0_{g(t,\theta)})) \tag{2.227}$$

That is, $g : \epsilon_{1,1}^{-1}(\tilde{I}) \to \mathcal{G}$ defines a smooth solution of (2.224) with initial condition $g(0,0) = e$.

It remains to show that g can be extended to all of $\mathcal{I}$. To this end, since $\mathcal{I} \times \{e\} \subset \mathcal{I} \times \mathcal{G}$ is compact, there exists finitely many open subsets $U_i \subset \mathcal{I}$ and $V_i \subset \mathcal{G}$, $i = 1, \ldots, n$, such that $\bigcup_{i=1}^{n} U_i = \mathcal{I}$ and $e \in V_i\ \forall i = 1, \ldots, n$, as well as smooth maps $\phi_i^Z : \epsilon_{1,1}^{-1}((-\delta_i, \delta_i)) \times U_i \times V_i \to \mathcal{I} \times \mathcal{G}$, $0 < \delta_i < 1\ \forall i = 1, \ldots, n$, such that ϕ_i^Z defines a local flow of Z. If we set $\delta := \min\{\delta', \delta_i\}_{i=1,\ldots,n} > 0$ and $V := \bigcap_{i=1}^{n} V_i \ni e$, we may glue the ϕ_i^Z to get a local flow $\phi^Z : \epsilon_{1,1}^{-1}((-\delta, \delta)) \times \mathcal{I} \times V \to \mathcal{I} \times \mathcal{G}$. Hence, let us choose $t_i \in [0,1]$, $i = 0, \ldots, m$, such that $0 =: t_0 < t_1 < \ldots < t_m := 1$ and $|t_i - t_{i-1}| < \delta\ \forall i = 1, \ldots, m$. Set $\mathcal{I}_i := \epsilon_{1,1}^{-1}([t_i, t_{i-1}])$ for $i = 1, \ldots, m$.

By assumption, we know that g is well-defined on $\mathcal{I}_1$. Hence, let us next consider the path $G : \mathcal{I}_2 \to \mathcal{I} \times \mathcal{G}$ defined as $G(s,\theta) = \phi_{(s,\theta)}^Z(t_1, 0, e)$ which is well-defined by definition of $\mathcal{I}_2$. Similarly as above, it follows that G has to be of the form $G(s,\theta) = (s + t_1, \theta, b(s,\theta))$ with $\mathcal{D}b(s,\theta) = -R_{b(s,\theta)*}\mathcal{A}^{\gamma}(s + t_1, \theta)$. If we then define $G'(s,\theta) := (s + t_1, \theta, b(s,\theta) \cdot g(t_1, 0))$ on $\mathcal{I}_2$, it follows (note that G' is smooth as $g(t_1, 0)$ is a body point)

$$\begin{aligned}\mathcal{D}G'(s,\theta) &= (\theta, 1, R_{g(t_1,0)*}\mathcal{D}b(s,\theta)) = (\theta, 1, -R_{g(t_1,0)*}R_{b(s,\theta)*}\mathcal{A}^{\gamma}(s + t_1, \theta)) \\ &= (\theta, 1, -R_{b(s,\theta)\cdot g(t_1,0)*}\mathcal{A}^{\gamma}(s + t_1, \theta)) = Z_{G'(s,\theta)}\end{aligned} \tag{2.228}$$

That is, G' defines an integral curve of Z through $G'(0,0) = (t_1, 0, g(t_1, 0))$. Thus, let us define the smooth path $\tilde{g} : \mathcal{I}_1 \cup \mathcal{I}_2 \to \mathcal{G}$ via

$$\tilde{g}(t,\theta) := \begin{cases} g(t,\theta), & \text{if } (t,\theta) \in \mathcal{I}_1 \\ b(t - t_1, \theta) \cdot g(t_1, 0), & \text{if } (t,\theta) \in \mathcal{I}_2 \end{cases} \tag{2.229}$$

It then follows that $\tilde{g}$ is smooth and defines a solution of (2.224). We have thus found an extension of g to $\mathcal{I}_1 \cup \mathcal{I}_2$. Hence, by induction, it follows that g can be extended to all of $\mathcal{I}$.

To prove the last claim note that, by pulling back $\mathcal{A}$ to the bosonic sub supermanifold $\mathcal{P}_0 := \mathbf{S} \circ \mathbf{B}(\mathcal{P})$, $\mathcal{A}$ only takes values in the even super Lie sub module $\Lambda \otimes \mathfrak{g}_0 = \mathrm{Lie}(\mathcal{G}_0)$. Hence, on $\mathcal{P}_0$, $\mathcal{A}$ can be reduced to a super connection 1-form on the principal super fiber bundle $\mathcal{G}_0 \to \mathcal{P}_0|_{\mathcal{M}_0} \to \mathcal{M}_0$. The claim now follows immediately. □

2.7.2. Parallel transport map revisited

As we have seen in the last section, in the ordinary theory of principal super fiber bundles, in order to obtain a nontrivial parallel transport map, one necessarily has to consider smooth paths depending on both even and odd parameters. But, still, due to smoothness, the endpoints at which the parallel transport map is constructed have to be points on the body of a supermanifold. Hence, it is important to emphasize that, in the ordinary category of supermanifolds, *one cannot use the parallel transport map to compare points on different fibers of a super fiber bundle!*

As we will see in what follows, a resolution is given considering instead super connections forms on parametrized super fiber bundles. At the same time, this also allows us to include (anticommutative) fermionic degrees of freedom on the body of a supermanifold which is of utmost importance in context of the geometric approach to supergravity to be discussed in Chapter 3. In this framework, it moreover suffices to consider (parametrized) paths depending solely on an even time parameter. The generalization to both even and odd parameters can be obtained along the lines of the previous section.

Definition 2.7.7. Let $\mathcal{M}_{/\mathcal{S}}$ be a $\mathcal{S}$-relative supermanifold. A *(smooth) path* γ on $\mathcal{M}_{/\mathcal{S}}$ is a smooth map $\gamma : \mathcal{S} \times \mathcal{I} \to \mathcal{M}_{/\mathcal{S}}$ with $\mathcal{I} \subseteq \Lambda^{1,0}$ a *super interval* which will mostly be assumed to be of the form $\mathcal{I} \equiv \epsilon_{1,0}^{-1}([0,1])$. Let $f, g : \mathcal{S} \to \mathcal{M}$ be smooth functions. A smooth path $\gamma : f \to g$ between f and g is a smooth path $\gamma : \mathcal{S} \times \mathcal{I} \to \mathcal{M}_{/\mathcal{S}}$ on $\mathcal{M}_{/\mathcal{S}}$ such that $\gamma_0 := \gamma(\cdot, 0) = f$ and $\gamma_1 := \gamma(\cdot, 1) = g$.

Definition 2.7.8. Let $\mathcal{G} \to \mathcal{P}_{/\mathcal{S}} \stackrel{\pi_{\mathcal{S}}}{\to} \mathcal{M}_{/\mathcal{S}}$ be a $\mathcal{S}$-relative principal super fiber bundle and $\gamma : \mathcal{S} \times \mathcal{I} \to \mathcal{M}_{/\mathcal{S}}$ a smooth path. Given a super connection 1-form $\mathcal{A}$ on $\mathcal{P}_{/\mathcal{S}}$, a smooth path $\gamma^{hor} : \mathcal{S} \times \mathcal{I} \to \mathcal{P}$ on $\mathcal{P}_{/\mathcal{S}}$ is called a horizontal lift of γ w.r.t. $\mathcal{A}$ if $\pi \circ \gamma^{hor} = \gamma$ and $\langle (\mathbb{1} \otimes \partial_t) \alpha_{\mathcal{S}}^{-1}(\gamma^{hor})(s,t) | \mathcal{A} \rangle = 0 \, \forall (s,t) \in \mathcal{S} \times \mathcal{I}$.

Proposition 2.7.9. *Let $\mathcal{A} \in \Omega^1(\mathcal{P}_{/\mathcal{S}}, \mathfrak{g})_0$ be a super connection 1-form on the $\mathcal{S}$-relative principal super fiber bundle $\mathcal{G} \to \mathcal{P}_{/\mathcal{S}} \stackrel{\pi_{\mathcal{S}}}{\to} \mathcal{M}_{/\mathcal{S}}$ as well as $\gamma : \mathcal{S} \times \mathcal{I} \to \mathcal{M}$ a smooth path. Let furthermore $f : \mathcal{S} \to \mathcal{P}$ be a smooth map. Then, there exists a unique horizontal lift $\gamma^{hor} : \mathcal{S} \times \mathcal{I} \to \mathcal{M}$ of γ w.r.t. $\mathcal{A}$ such that $\gamma^{hor}(\,\cdot\,, 0) = f$.*

Proof. The proof of this proposition is similar as in Theorem 2.7.6. Let us therefore only sketch the most important steps. Again, it suffices to assume that γ is contained within a local trivialization neighborhood of $\mathcal{P}_{/\mathcal{S}}$, i.e., there exists an open subset $U \subseteq \mathcal{M}$ and a smooth morphism $\tilde{s}: U_{/\mathcal{S}} := \mathcal{M}_{/\mathcal{S}}|_{\mathcal{S}\times U} \to \mathcal{P}_{/\mathcal{S}}$ of $\mathcal{S}$-relative supermanifolds such that $\pi_{\mathcal{S}} \circ \tilde{s} = \mathrm{id}_{U_{/\mathcal{S}}}$ and $\mathrm{im}\,\gamma \subseteq \pi_{\mathcal{S}}^{-1}(\mathcal{S}\times U)$. Furthermore, w.l.o.g., we can assume that $\tilde{s}(\,\cdot\,,\gamma(\,\cdot\,,0)) = f$. In fact, suppose this would not be the case. Then, since the super Lie group $\mathcal{G}$ acts transitively on each fiber of the underlying principal super fiber bundle $\mathcal{P}$, there exists a unique map $g': \mathcal{S}\times U \to \mathcal{G}$ with $\Phi_{\mathcal{S}}(\tilde{s}(\,\cdot\,,\gamma(\,\cdot\,,0)), g') = f$. Since $\tilde{s}(\,\cdot\,,\gamma(\,\cdot\,,0)), f$ as well as the inverse operation in the super Lie group $\mathcal{G}$ are of class H^∞, it follows immediately that g' is also smooth. Hence, replacing $\tilde{s}$ by $\Phi_{\mathcal{S}}(\tilde{s}, g')$ the map thus obtained will have the required properties.

Set $\delta := \tilde{s}\circ(\mathrm{id}\times\gamma): \mathcal{I}_{/\mathcal{S}} \to \mathcal{P}_{/\mathcal{S}}$. It follows that a horizontal lift has to be of the form $\gamma^{hor} := \Phi_{\mathcal{S}}\circ(\delta\times g)$ for some smooth function $g: \mathcal{S}\times\tilde{\mathcal{I}} \to \mathcal{G}$ defined on some open subset $\tilde{\mathcal{I}} \subseteq \mathcal{I}$. Using Prop. 2.7.1, one finds that $\langle(\mathbb{1}\otimes\partial_t)\alpha_{\mathcal{S}}^{-1}(\gamma^{hor})(s,t)|\mathcal{A}\rangle = 0$ if and only if

$$(\mathbb{1}\otimes\partial_t)g(s,t) = -R_{g(s,t)*}\langle(\mathbb{1}\otimes\partial_t)\delta(s,t)|\mathcal{A}\rangle = -R_{g(s,t)*}\mathcal{A}^\gamma(s,t) \tag{2.230}$$

where $\mathcal{A}^\gamma(s,t) := \langle(\mathbb{1}\otimes\partial_t)\delta(s,t)|\mathcal{A}\rangle = \langle(\mathbb{1}\otimes\partial_t)\alpha_{\mathcal{S}}^{-1}(\gamma)(s,t)|\tilde{s}^*\mathcal{A}\rangle$ with the initial condition $g(\,\cdot\,,0) = e$. Hence, the claim follows if one can show that (2.230) admits a smooth solution with $\tilde{\mathcal{I}} = \mathcal{I}$. Therefore, consider the even smooth vector field

$$\begin{aligned}\tilde{Z}:\ (\mathcal{S}\times\mathcal{I})\times\mathcal{G} &\to T(\mathcal{S}\times\mathcal{I})\times T\mathcal{G}\\ (s',t',g) &\mapsto (0_{s'},1,-D_{(e,g)}\mu_{\mathcal{G}}(\mathcal{A}^\gamma(s',t'),0_g))\end{aligned} \tag{2.231}$$

It follows that there exists a smooth local solution $F: U' \to (\mathcal{S}\times\mathcal{I})\times\mathcal{G}, U' \subset \mathcal{S}\times\mathcal{I}$ open, of the equation

$$\partial_t F(s,t) = \tilde{Z}_{F(s,t)} \tag{2.232}$$

with the initial condition $F(\,\cdot\,,0) = (\,\cdot\,,0,e)$. Moreover, F has to be of the form $F(s,t) = (s,t,g(s,t))$ for some smooth function $g: U' \to \mathcal{G}$ such that $g(\,\cdot\,,0) = e$ and

$$\partial_t F(s,t) = (0_s,1,(\mathbb{1}\otimes\partial_t)g(s,t)) = (0_s,1,-D_{(e,g)}\mu_{\mathcal{G}}(\mathcal{A}^\gamma(s,t),0_g)) \tag{2.233}$$

that is, g is a solution of (2.230) proving that a local smooth solution indeed exists. Remains to show that g can be extended to all of $\mathcal{S}\times\mathcal{I}$.

Therefore, one can proceed as in the proof of Theorem 2.7.6. In fact, restricting on compact subsets and gluing local solutions together, it follows that, for any $s_0 \in \mathcal{S}$, there exists an open neighborhood $s_0 \in V \subset \mathcal{S}$ as well as a smooth map $g_{s_0}: V\times\mathcal{I} \to \mathcal{G}$ such that V is contained in a compact subset and g_{s_0} is a solution of (2.230) with

$g(\,\cdot\,,0) = e$. Thus, by uniqueness of solutions of differential equations, these maps can be glued together yielding a global solution $g : \mathcal{S} \times \mathcal{I} \to \mathcal{G}$ of (2.230). □

Remark 2.7.10. For a smooth map $f : \mathcal{S} \to \mathcal{M}$, one can consider the pullback super fiber bundle

$$f^*\mathcal{P} = \{(s,p)|\, f(s) = \pi(p)\} \subset \mathcal{S} \times \mathcal{P} \tag{2.234}$$

over $\mathcal{S}$. A smooth section $\tilde{\phi} : \mathcal{S} \to f^*\mathcal{P}$ of the pullback bundle is then of the form $\tilde{\phi}(s) = (s, \phi(s))\ \forall s \in \mathcal{S}$ with $\phi : \mathcal{S} \to \mathcal{P}$ a smooth map satisfying $\pi \circ \phi = f$. Hence, we can identify

$$\Gamma(f^*\mathcal{P}) = \{\phi : \mathcal{S} \to \mathcal{P}|\, \pi \circ \phi = f\} \tag{2.235}$$

Definition 2.7.11. Under the conditions of Prop. 2.7.9, the *parallel transport map* in $\mathcal{P}_{/\mathcal{S}}$ along γ w.r.t. the connection $\mathcal{A}$ is defined as

$$\begin{aligned} \mathscr{P}^{\mathcal{A}}_{\mathcal{S},\gamma} : \Gamma(\gamma_0^*\mathcal{P}) &\to \Gamma(\gamma_1^*\mathcal{P}) \\ \phi &\mapsto \gamma_\phi^{hor}(\,\cdot\,,1) \end{aligned} \tag{2.236}$$

where, for $\Gamma(\gamma_0^*\mathcal{P}) \ni \phi : \mathcal{S} \to \mathcal{P}$, γ_ϕ^{hor} is the unique horizontal lift of γ with respect to $\mathcal{A}$ such that $\gamma_\phi^{hor}(\,\cdot\,,0) = \phi$.

Given a change of parametrization $\lambda : \mathcal{S} \to \mathcal{S}'$, this induces the pullback $\lambda^* : H^\infty(\mathcal{S}') \to H^\infty(\mathcal{S})$, $f \mapsto \lambda^* f = f \circ \lambda$ on the respective function sheaves. Since the super $H^\infty(\mathcal{S} \times \mathcal{M})$-module $\mathfrak{X}(\mathcal{M}_{/\mathcal{S}})$ of smooth vector fields on $\mathcal{M}_{/\mathcal{S}}$ is isomorphic to $H^\infty(\mathcal{S}) \otimes \mathfrak{X}(\mathcal{M})$, this yields the morphism

$$\begin{aligned} \lambda^* \equiv \lambda^* \otimes \mathbb{1} : \mathfrak{X}(\mathcal{M}_{/\mathcal{S}'}) &\to \mathfrak{X}(\mathcal{M}_{/\mathcal{S}}) \\ f \otimes X &\mapsto \lambda^* f \otimes X \end{aligned} \tag{2.237}$$

Moreover, from $\Omega^1(\mathcal{M}_{/\mathcal{S}}) \cong \Omega^1(\mathcal{M}) \otimes H^\infty(\mathcal{S})$ we obtain the morphism

$$\begin{aligned} \lambda^* \equiv \mathbb{1} \otimes \lambda^* : \Omega^1(\mathcal{M}_{/\mathcal{S}'}) &\to \Omega^1(\mathcal{M}_{/\mathcal{S}}) \\ \omega \otimes f &\mapsto \omega \otimes \lambda^* f \end{aligned} \tag{2.238}$$

By definition, it then follows

$$\langle \lambda^* X | \lambda^* \mathcal{A} \rangle = \lambda^* \langle X | \mathcal{A} \rangle \tag{2.239}$$

In fact, since (2.239) is a local property, let us choose a local coordinate neighborhood such that X and ω can be locally expanded in the form $X = f^i \otimes X_i$ and $\omega = \omega_j \otimes g^j$ with X_i and ω_j smooth vector fields and 1-forms on $\mathcal{M}$, respectively. We then compute

$$\begin{aligned}\langle \lambda^* X | \lambda^* \mathcal{A} \rangle &= \langle \lambda^* f^i \otimes X_i | \omega_j \otimes \lambda^* g^j \rangle = \lambda^* f^i \, \langle X_i | \omega_j \rangle \, \lambda^* g^j \\ &= \lambda^* \, \langle f^i \otimes X_i | \omega_j \otimes g^j \rangle = \lambda^* \, \langle X | \mathcal{A} \rangle \end{aligned} \tag{2.240}$$

The following proposition summarizes some important properties of the parallel transport map such as the functoriality under composition of paths as well as covariance under change of parametrization demonstrating the independence of the choice of a particular parametrizing supermanifold.

Proposition 2.7.12. *The parallel tansport map enjoys the following properties:*

(i) $\mathscr{P}^{\mathcal{A}}_{\mathcal{S}}$ *is functorial under compositions of paths, that is, for smooth paths* $\gamma : \mathcal{S} \times \mathcal{I} \to \mathcal{M}$ *and* $\delta : \mathcal{S} \times \mathcal{I} \to \mathcal{M}$ *on* $\mathcal{M}_{/\mathcal{S}}$*, one has*

$$\mathscr{P}^{\mathcal{A}}_{\mathcal{S},\gamma\circ\delta} = \mathscr{P}^{\mathcal{A}}_{\mathcal{S},\gamma} \circ \mathscr{P}^{\mathcal{A}}_{\mathcal{S},\delta} \tag{2.241}$$

(ii) $\mathscr{P}^{\mathcal{A}}_{\mathcal{S},\gamma}$ *is covariant under change of parametrization in the sense that if* $\lambda : \mathcal{S} \to \mathcal{S}'$ *is a morphism of supermanifolds, then the diagram*

$$\begin{array}{ccc} \Gamma(f^*\mathcal{P}) & \xrightarrow{\ \mathscr{P}^{\mathcal{A}}_{\mathcal{S}',\gamma}\ } & \Gamma(g^*\mathcal{P}) \\ \Big\downarrow{\scriptstyle \lambda^*} & & \Big\downarrow{\scriptstyle \lambda^*} \\ \Gamma((f\circ\lambda)^*\mathcal{P}) & \xrightarrow{\ \mathscr{P}^{\lambda^*\mathcal{A}}_{\mathcal{S},\lambda^*\gamma}\ } & \Gamma((g\circ\lambda)^*\mathcal{P}) \end{array} \tag{2.242}$$

is commutative for any smooth path $\gamma : f \to g$ *on* $\mathcal{M}_{/\mathcal{S}'}$*.*

Proof. The functoriality property of the parallel transport map under the composition of paths is an immediate consequence of Eq. (2.230) or (2.232) and the uniqueness of solutions of differential equations once fixing the inital conditions. In fact, this implies $(\gamma\circ\delta)^{hor} = \gamma^{hor}\circ\delta^{hor}$ yielding (2.241) by Definition (2.236). To prove covariance under change of parametrization, notice that for a supermanifold morphism $\lambda : \mathcal{S} \to \mathcal{S}'$, one has $(\mathbb{1} \otimes \partial_t)\lambda^*\gamma^{hor} = \lambda^*((\mathbb{1} \otimes \partial_t)\gamma^{hor})$ so that, by Definition (2.159), it follows

$$(\mathbb{1} \otimes \partial_t)\alpha_{\mathcal{S}}^{-1}(\lambda^*\gamma^{hor}) = \lambda^*((\mathbb{1} \otimes \partial_t)\alpha_{\mathcal{S}'}^{-1}(\gamma^{hor})) \tag{2.243}$$

and thus

$$\begin{aligned}\langle(\mathbb{1}\otimes\partial_t)\alpha_{\mathcal{S}}^{-1}(\lambda^*\gamma^{hor})|\lambda^*\mathcal{A}\rangle &= \langle\lambda^*((\mathbb{1}\otimes\partial_t)\alpha_{\mathcal{S}'}^{-1}(\gamma^{hor}))|\lambda^*\mathcal{A}\rangle\\ &= \langle(\mathbb{1}\otimes\partial_t)\alpha_{\mathcal{S}'}^{-1}(\gamma^{hor})|\mathcal{A}\rangle = 0 \end{aligned}\tag{2.244}$$

according to (2.239). Since $\lambda^*\gamma_{\phi}^{hor}(\,\cdot\,,0) = \phi\circ\lambda = \lambda^*\phi$ and $\pi\circ\lambda^*\gamma_{\phi}^{hor} = \lambda^*\gamma$, by uniqueness, this yields $\lambda^*\gamma_{\phi}^{hor} = (\lambda^*\gamma)_{\lambda^*\phi}^{hor}$ and therefore

$$\mathscr{P}_{\mathcal{S},\lambda^*\gamma}^{\lambda^*\mathcal{A}}(\lambda^*\phi) = (\lambda^*\gamma)_{\lambda^*\phi}^{hor}(\,\cdot\,,1) = \lambda^*\gamma_{\phi}^{hor}(\,\cdot\,,1) = \lambda^*(\mathscr{P}_{\mathcal{S}',\gamma}^{\mathcal{A}}(\phi)) \tag{2.245}$$

$\forall\phi\in\Gamma(f^*\mathcal{P})$ proving the commutativity of the diagram (2.242). □

Definition 2.7.13. A *global gauge transformation* f on the $\mathcal{S}$-relative principal super fiber bundle $\mathcal{G}\to\mathcal{P}_{/\mathcal{S}}\to\mathcal{M}_{/\mathcal{S}}$ is a morphism $f:\mathcal{P}_{/\mathcal{S}}\to\mathcal{P}_{/\mathcal{S}}$ of $\mathcal{S}$-relative supermanifolds which is fiber-preserving and $\mathcal{G}$-equivariant, i.e., $\pi_{\mathcal{S}}\circ f = \pi_{\mathcal{S}}$ and $f\circ\Phi_{\mathcal{S}} = \Phi_{\mathcal{S}}\circ(f\times\mathrm{id})$. The set of global gauge transformations on $\mathcal{P}_{/\mathcal{S}}$ will be denoted by $\mathscr{G}(\mathcal{P}_{/\mathcal{S}})$.

Proposition 2.7.14. *There exists a bijective correspondence between the set $\mathscr{G}(\mathcal{P}_{/\mathcal{S}})$ of global gauge transformations on the $\mathcal{S}$-relative principal super fiber bundle $\mathcal{G}\to\mathcal{P}_{/\mathcal{S}}\to\mathcal{M}_{/\mathcal{S}}$ and the set*

$$H^\infty(\mathcal{S}\times\mathcal{P},\mathcal{G})^{\mathcal{G}} := \{\sigma:\mathcal{S}\times\mathcal{P}\to\mathcal{G}|\,\sigma\circ\Phi_{\mathcal{S}} = \alpha_{g^{-1}}\circ\sigma\} \tag{2.246}$$

via

$$H^\infty(\mathcal{S}\times\mathcal{P},\mathcal{G})^{\mathcal{G}}\ni\sigma\mapsto\Phi_{\mathcal{S}}\circ(\mathrm{id}\times\sigma)\circ d_{\mathcal{S}\times\mathcal{P}}\in\mathscr{G}(\mathcal{P}_{\mathcal{S}}) \tag{2.247}$$

In particular, global gauge transformations are super diffeomorphisms on $\mathcal{P}_{/\mathcal{S}}$ and $\mathscr{G}(\mathcal{P}_{/\mathcal{S}})$ forms an abstract group under composition of smooth maps.

Proof. The proof of this proposition is almost the same as in the classical theory. Hence, let us only show that the map $\sigma_f\in H^\infty(\mathcal{S}\times\mathcal{P},\mathcal{G})^{\mathcal{G}}$ corresponding to a global gauge transformation $f\in\mathscr{G}(\mathcal{P}_{/\mathcal{S}})$ such that $f(s,p) = (s,p)\cdot\sigma_f(s,p)\ \forall(s,p)\in\mathcal{S}\times\mathcal{P}$ is indeed of class H^∞. To this end, choose a local trivialization (U,ϕ_U) of $\mathcal{P}$ and set $\tilde{\phi}_U := \mathrm{id}\times\phi_U:\pi_{\mathcal{S}}^{-1}(\mathcal{S}\times U)\to(\mathcal{S}\times U)\times\mathcal{G}$. On the local trivialization neighborhood, f is then of the form

$$\tilde{\phi}_U\circ f\circ\tilde{\phi}_U^{-1}((s,x),g) = ((s,x),\sigma(s,x,g)) \tag{2.248}$$

for some smooth function $\sigma:(\mathcal{S}\times U)\times G\to\mathcal{G}$. Hence,

$$\sigma_f\circ\phi_U^{-1}((s,x),g) = \mu_{\mathcal{G}}(g^{-1},\sigma(s,x,g)) \tag{2.249}$$

on $(\mathcal{S} \times U) \times \mathcal{G}$ proving that σ_f is smooth. That global gauge transformations are diffeomorphisms and $\mathscr{G}(\mathcal{P}_{/\mathcal{S}})$ forms an abstract group now follows immediately from the respective properties of $H^\infty(\mathcal{S} \times \mathcal{P}, \mathcal{G})^{\mathcal{G}}$. $\square$

Proposition 2.7.15. *Let $\mathcal{A} \in \Omega^1(\mathcal{P}_{/\mathcal{S}}, \mathfrak{g})_0$ be a $\mathcal{S}$-relative super connection 1-form and $f \in \mathscr{G}(\mathcal{P}_{/\mathcal{S}})$ a global gauge transformation on the $\mathcal{S}$-relative principal super fiber bundle $\mathcal{G} \to \mathcal{P}_{/\mathcal{S}} \stackrel{\pi_{\mathcal{S}}}{\to} \mathcal{M}_{/\mathcal{S}}$. Then,*

(i) $f^*\mathcal{A} \in \Omega^1(\mathcal{P}_{/\mathcal{S}}, \mathfrak{g})_0$ is a connection 1-form and, in particular,

$$f^*\mathcal{A} = \mathrm{Ad}_{\sigma_f^{-1}} \circ \mathcal{A} + \sigma_f^* \theta_{\mathrm{MC}} \tag{2.250}$$

(ii) the diagram

$$\begin{array}{ccc} \Gamma(g^*\mathcal{P}) & \xrightarrow{\mathscr{P}^{\mathcal{A}}_{\mathcal{S},\gamma}} & \Gamma(h^*\mathcal{P}) \\ {\scriptstyle \alpha_{\mathcal{S}} \circ f \circ \alpha_{\mathcal{S}}^{-1}} \big\downarrow & & \big\downarrow {\scriptstyle \alpha_{\mathcal{S}} \circ f \circ \alpha_{\mathcal{S}}^{-1}} \\ \Gamma(g^*\mathcal{P}) & \xrightarrow{\mathscr{P}^{f^*\mathcal{A}}_{\mathcal{S},\gamma}} & \Gamma(h^*\mathcal{P}) \end{array}$$

is commutative for any smooth path $\gamma : g \to h$ on $\mathcal{M}_{/\mathcal{S}}$.

Proof. First, let us show that $f^*\mathcal{A}$ is a $\mathcal{S}$-relative connection 1-form on $\mathcal{P}_{/\mathcal{S}}$. To this end, since f is $\mathcal{G}$-equivariant, it follows that fundamental vector fields $\widetilde{X}$ on $\mathcal{P}_{/\mathcal{S}}$ associated to $X \in \mathfrak{g}$ satisfy

$$\begin{aligned} f_*\widetilde{X} &= \mathbb{1} \otimes X \circ \Phi_{\mathcal{S}}^* \circ f^* = \mathbb{1} \otimes X \circ f^* \otimes \mathbb{1} \circ \Phi_{\mathcal{S}} \\ &= f^* \circ \mathbb{1} \otimes X \circ \Phi_{\mathcal{S}}^* = f^* \circ \widetilde{X} \end{aligned} \tag{2.251}$$

which yields

$$\langle \widetilde{X} | f^*\mathcal{A} \rangle = \langle f_*\widetilde{X} | \mathcal{A}_{f(\,\cdot\,)} \rangle = f^* \langle \widetilde{X} | \mathcal{A} \rangle = X \tag{2.252}$$

$\forall X \in \mathfrak{g}$. Moreover,

$$(\Phi_{\mathcal{S}})_g^*(f^*\mathcal{A}) = (f \circ \Phi_{\mathcal{S}})^*\mathcal{A} = f^*((\Phi_{\mathcal{S}})_g^*\mathcal{A}) = \mathrm{Ad}_{g^{-1}} \circ f^*\mathcal{A} \tag{2.253}$$

This proves that $f^*\mathcal{A} \in \Omega^1(\mathcal{P}_{/\mathcal{S}}, \mathfrak{g})_0$ indeed defines a connection 1-form on $\mathcal{P}_{/\mathcal{S}}$. Next, applying Prop. 2.7.1, we find

$$\begin{aligned} f_*(X_p) &= D_p(\Phi_{\mathcal{S}} \circ (\mathrm{id} \times \sigma_f))(X_p, X_p) = D_{(p, \sigma_f(p))}\Phi_{\mathcal{S}}(X_p, D_p\sigma_f(X_p)) \\ &= (\Phi_{\mathcal{S}})_{\sigma_f(p)*}(X_p) + [\theta_{\mathrm{MC}}(D_p\sigma_f(X_p))]^{\sim} \end{aligned} \tag{2.254}$$

$\forall X_p \in T_p(\mathcal{P}_{/\mathcal{S}})$, $p \in \mathcal{P}_{/\mathcal{S}}$, and thus

$$\begin{aligned}\langle X_p|f^*\mathcal{A}_p\rangle &= \langle D_pf(X_p)|\mathcal{A}_{f(p)}\rangle = \langle (\Phi_{\mathcal{S}})_{\sigma_f(p)^*}(X_p)|\mathcal{A}\rangle + \langle D_p\sigma_f(X_p)|\theta_{\mathrm{MC}}\rangle \\ &= \mathrm{Ad}_{\sigma_f(p)^{-1}}\langle X_p|\mathcal{A}_p\rangle + \langle X_p|\sigma_f^*\theta_{\mathrm{MC}}\rangle \end{aligned} \tag{2.255}$$

which yields (2.250). To prove the last assertion, let $\gamma^{hor}_{f,\phi}:\ \mathcal{S}\times\mathcal{P}\to\mathcal{P}$ for any $\phi\in\Gamma(g^*\mathcal{P})$ be the unique horizontal lift of the smooth path $\gamma:\ g\to h$ w.r.t. $\mathcal{A}$ through $\alpha_{\mathcal{S}}(f\circ\alpha_{\mathcal{S}}^{-1}(\phi))\in\Gamma(g^*\mathcal{P})$. Set

$$\tilde{\gamma}_\phi := \alpha_{\mathcal{S}}(f^{-1}\circ\alpha_{\mathcal{S}}^{-1}(\gamma^{hor}_{f,\phi})):\ \mathcal{S}\times\mathcal{P}\to\mathcal{P} \tag{2.256}$$

Then, $\tilde{\gamma}_\phi$ is a smooth path on $\mathcal{P}_{/\mathcal{S}}$ with $\tilde{\gamma}_\phi(\,\cdot\,,0)=\phi$ and

$$\langle(\mathbb{1}\otimes\partial_t)\tilde{\gamma}_\phi|f^*\mathcal{A}\rangle = \langle(\mathbb{1}\otimes\partial_t)\alpha_{\mathcal{S}}^{-1}(\gamma^{hor}_{f,\phi})|\mathcal{A}\rangle = 0 \tag{2.257}$$

so that $\tilde{\gamma}_\phi$ coincides with the unique horizontal lift of γ w.r.t. $f^*\mathcal{A}$ through ϕ. □

Example 2.7.16. We want to give an explicit local expression of the parallel transport map as derived above making it more accessible for concrete applications in the context of quantum supergravity to be discussed in Chapter 5 and 6. To this end, let us assume that $\mathcal{G}$ is a super matrix Lie group, i.e., an embedded super Lie subgroup of the general linear supergroup $\mathrm{GL}(\mathcal{V})$ on a super Λ-vector space $\mathcal{V}$ (Def. 2.3.15). In this case, the pushforward R_{g*} of the right translation for any $g\in\mathcal{G}$ then just coincides with the right multiplication by the super matrix g. Hence, let $\gamma:\ \mathcal{S}\times\mathcal{I}\to\mathcal{M}$ be a smooth path which is contained within a local trivialization neighborhood of $\mathcal{P}_{/\mathcal{S}}$ and $\tilde{s}:\ U_{/\mathcal{S}} := \mathcal{M}_{/\mathcal{S}}|_{\mathcal{S}\times U}\to\mathcal{P}_{/\mathcal{S}}$ the corresponding smooth section. Then, Eq. (2.230) in the proof of Prop. 2.7.9 reads

$$(\mathbb{1}\otimes\partial_t)g(s,t) = -\mathcal{A}^\gamma(s,t)\cdot g(s,t) \tag{2.258}$$

with $\mathcal{A}^\gamma(s,t) := \langle(\mathbb{1}\otimes\partial_t)\alpha_{\mathcal{S}}^{-1}(\gamma)(s,t)|\tilde{s}^*\mathcal{A}\rangle$. Furthermore, suppose that U defines a local coordinate neighborhood of $\mathcal{M}$. The 1-form $\tilde{s}^*\mathcal{A}$ on $\mathcal{S}\times U$ can then be expanded in the form

$$\tilde{s}^*\mathcal{A} = \mathrm{d}x^\mu\mathcal{A}^{(\tilde{s})}_\mu + \mathrm{d}\theta^\alpha\mathcal{A}^{(\tilde{s})}_\alpha \tag{2.259}$$

with smooth even and odd functions $\mathcal{A}^{(\tilde{s})}_\mu$ and $\mathcal{A}^{(\tilde{s})}_\alpha$ on $\mathcal{S}\times U$, respectively. This yields

$$\mathcal{A}^\gamma(s,t) =: \dot{x}^\mu\mathcal{A}^{(\tilde{s})}_\mu(s,t) + \dot{\theta}^\alpha\mathcal{A}^{(\tilde{s})}_\alpha(s,t) \tag{2.260}$$

Hence, the solution of Eq. (2.258) with the initial condition $g(\,\cdot\,,0)=\mathbb{1}$ takes the form

$$g(s,t)=\mathcal{P}\exp\left(-\int_0^t \mathrm{d}t'\,\dot{x}^\mu\mathcal{A}^{(\tilde{s})}_\mu(s,t')+\dot{\theta}^\alpha\mathcal{A}^{(\tilde{s})}_\alpha(s,t')\right) \tag{2.261}$$

where $\mathcal{P}\exp(\ldots)$ denotes the usual *path-ordered exponential*. This is the most general local expression of the parallel transport map corresponding to a $\mathcal{S}$-relative super connection 1-form. This form is used for instance in [118] in the discussion about the relation between super twistor theory and $\mathcal{N}=4$ super Yang-Mills theory (see also [79]). Note that in case $\mathcal{S}=\{*\}$ is a single point, the odd coefficients in (2.261) become zero so that this expression just reduces to the parallel transport map of an ordinary connection 1-form on a principal fiber bundle in accordance with Theorem 2.7.6.

By definition, $g[\mathcal{A}]:=g(\cdot,1)$ defines a smooth map $g[\mathcal{A}]:\mathcal{S}\rightarrow\mathcal{G}$ from the parametrizing supermanifold $\mathcal{S}$ to the gauge group $\mathcal{G}$. As explained in detail in Section 2.2 (see also Section 2.6), there exists an equivalence of categories $\mathbf{A}:\mathbf{Man}_{H^\infty}\rightarrow\mathbf{Man}_{\mathrm{Alg}}$ between the category $\mathbf{Man}_{H^\infty}$ of H^∞ supermanifolds and the category $\mathbf{Man}_{\mathrm{Alg}}$ of algebro-geometric supermanifolds. Using this equivalence, it thus follows

$$H^\infty(\mathcal{S},\mathcal{G})\cong\mathrm{Hom}_{\mathbf{SMan}_{\mathrm{Alg}}}(\mathbf{A}(\mathcal{S}),\mathbf{A}(\mathcal{G})) \tag{2.262}$$

Hence, $g[\mathcal{A}]$ can be identified with a $\mathbf{A}(\mathcal{S})$-point of $\mathbf{A}(\mathcal{G})$. This coincides with the results of [78] and [79] where the parallel transport induced by covariant derivatives on super vector bundles in the pure algebraic setting has been considered. It was found that the parallel transport map has the interpretation in terms of $\mathcal{T}$-points of a general linear supergroup.

Example 2.7.17. Finally, let us restrict to a subclass of smooth paths on $\mathcal{M}_{/\mathcal{S}}$ obtained via the lift of smooth paths $\gamma:\mathcal{I}\rightarrow\mathcal{M}$ on the bosonic sub supermanifold[9] $\mathcal{M}_0$ of $\mathcal{M}$ defined as the split supermanifold $\mathcal{M}_0:=\mathbf{S}(\mathbf{B}(\mathcal{M}))$. A $\mathcal{S}$-relative connection 1-form $\mathcal{A}\in\Omega^1(\mathcal{P}_{/\mathcal{S}},\mathfrak{g})$ induces via pullback along the inclusion $\iota:\mathcal{S}\times\mathcal{M}_0\hookrightarrow\mathcal{S}\times\mathcal{M}$ a $\mathcal{S}$-relative super connection 1-form $\iota^*\mathcal{A}$ on the pullback bundle $\mathcal{G}\rightarrow\iota^*\mathcal{P}_{/\mathcal{S}}\rightarrow(\mathcal{M}_0)_{/\mathcal{S}}$. Let

$$\iota^*\mathcal{A}=\mathrm{pr}_{\mathfrak{g}_0}\circ\iota^*\mathcal{A}+\mathrm{pr}_{\mathfrak{g}_1}\circ\iota^*\mathcal{A}=:\omega+\psi \tag{2.263}$$

be the decomposition of $\iota^*\mathcal{A}$ according to the even and odd part of the super Lie algebra $\mathfrak{g}=\mathfrak{g}_0\oplus\mathfrak{g}_1$. Since $\omega\in\Omega^1(\iota^*\mathcal{P}_{/\mathcal{S}},\mathfrak{g}_0)_0\cong\Omega^1(\mathcal{P}|_{\mathcal{M}_0},\mathfrak{g}_0)_0\otimes H^\infty(\mathcal{S})_0$, it follows that ω can be reduced to a $\mathcal{S}$-relative super connection 1-form on the $\mathcal{S}$-relative principal super fiber bundle $\mathcal{G}_0\rightarrow(\mathcal{P}_0)_{/\mathcal{S}}\rightarrow(\mathcal{M}_0)_{/\mathcal{S}}$ which will be denoted by the same symbol. Hence, ω gives rise to a parallel transport map $\mathscr{P}^{\omega}_{\mathcal{S},\gamma}$ along $\alpha_{\mathcal{S}}(\mathrm{id}\times\gamma):\mathcal{S}\times\mathcal{I}\rightarrow\mathcal{M}_0$.

[9] In [97, 109] this is also called the $\mathbf{G}$-extension of $\mathbf{B}(\mathcal{M})$ as this can be viewed as a generalization of the ordinary $\mathbf{G}$-extension of smooth functions (Eq. (C.2))

Suppose that γ is contained within a local tirvialization neighborhood of $\mathcal{P}_0$ and let $\tilde{s} : U_{/\mathcal{S}} \to (\mathcal{P}_0)_{/\mathcal{S}}$ be the corresponding local section with $U \subset \mathcal{M}_0$ open. Let $g[\mathcal{A}] : \mathcal{S} \times \mathcal{I} \to \mathcal{G}$ be the solution of the parallel transport equation (2.230) of $\mathcal{A}$

$$\partial_t g[\mathcal{A}](s,t) = -R_{g[\mathcal{A}](s,t)*}\mathcal{A}^\gamma(s,t) \tag{2.264}$$

with the initial condition $g(\,\cdot\,, 0) = e$, where

$$\mathcal{A}^\gamma := \langle \mathbb{1} \otimes \partial_t\gamma | \tilde{s}^*\mathcal{A}\rangle = \langle \mathbb{1} \otimes \partial_t\gamma | \tilde{s}^*\omega\rangle + \langle \mathbb{1} \otimes \partial_t\gamma | \tilde{s}^*\psi\rangle =: \omega^\gamma + \psi^\gamma \tag{2.265}$$

Furthermore, let $g[\omega] : \mathcal{S} \times \mathcal{I} \to \mathcal{G}_0$ be the solution of the corresponding parallel transport equation of ω. Set $g[\psi] := g[\omega]^{-1} \cdot g[\mathcal{A}] : \mathcal{S} \times \mathcal{I} \to \mathcal{G}$. Using $\partial_t(g[\omega]^{-1}) = -L_{g[\omega]^{-1}*}R_{g[\omega]^{-1}*}(\partial_t g[\omega]) = L_{g[\omega]^{-1}*}\omega^\gamma$, it then follows

$$\begin{aligned}\partial_t g[\psi] &= D\mu_{\mathcal{G}}(\partial_t(g[\omega]^{-1}), \partial_t g[\mathcal{A}]) = R_{g[\mathcal{A}]*}L_{g[\omega]^{-1}*}\omega^\gamma - L_{g[\omega]^{-1}*}R_{g[\mathcal{A}]*}\mathcal{A}^\gamma \\ &= -R_{g[\mathcal{A}]*}L_{g[\omega]^{-1}*}\psi^\gamma = -R_{g[\psi]*}R_{g[\omega]*}L_{g[\omega]^{-1}*}\psi^\gamma \\ &= -R_{g[\psi]*}\mathrm{Ad}_{g[\omega]^{-1}}(\psi^\gamma)\end{aligned} \tag{2.266}$$

that is, $g[\psi]$ is the solution of the equation

$$\partial_t g[\psi] = -R_{g[\psi]*}\mathrm{Ad}_{g[\omega]^{-1}}(\psi^\gamma) \tag{2.267}$$

For a super matrix Lie group $\mathcal{G}$, the solution of (2.267) can be explicitly written as

$$g[\psi](s,t) = \mathcal{P}\exp\left(-\int_0^t \mathrm{d}\tau\, (\mathrm{Ad}_{g[\omega]^{-1}}\psi^\gamma)(s,\tau)\right) \tag{2.268}$$

such that

$$g[\mathcal{A}](s,t) = g[\omega](s,t) \cdot \mathcal{P}\exp\left(-\int_0^t \mathrm{d}\tau\, (\mathrm{Ad}_{g[\omega]^{-1}}\psi^\gamma)(s,\tau)\right) \tag{2.269}$$

As a consequence, if γ is closed loop on $\mathcal{M}_0$, in this gauge, the *super Wilson loop* takes the form

$$W_\gamma[\mathcal{A}] = \mathrm{str}\left(g_\gamma[\omega] \cdot \mathcal{P}\exp\left(-\oint_\gamma \mathrm{Ad}_{g_\gamma[\omega]^{-1}}\psi^{(\tilde{s})}\right)\right) : \mathcal{S} \to \mathcal{G} \tag{2.270}$$

where $\psi^{(\tilde{s})} := \tilde{s}^*\psi$. It follows from Prop. 2.7.15 that $W_\gamma[\mathcal{A}]$ is invariant under local gauge transformations. In fact, $g_\gamma[\mathcal{A}]$ transforms as

$$g_\gamma[\mathcal{A}](s) \to \phi(s) \cdot g_\gamma[\mathcal{A}](s) \cdot \phi(s)^{-1}, \quad \forall s \in \mathcal{S} \tag{2.271}$$

for some smooth function $\phi : \mathcal{S} \to \mathcal{G}$. Hence, due to cyclicity of the supertrace, (2.270) is indeed invariant. Finally, by Prop. 2.7.12 (ii) the super Wilson loop is also invariant under change of parametrizations. That is, if $\lambda : \mathcal{S}' \to \mathcal{S}$ is a supermanifold morphism, then

$$\lambda^* W_\gamma[\mathcal{A}] = W_\gamma[\lambda^*\mathcal{A}] : \mathcal{S}' \to \mathcal{G} \tag{2.272}$$

Thus, due these properties, $W_\gamma[\mathcal{A}]$ can be regarded as a fundamental physical quantity according to [73].

As explained in Example 2.7.16, the parallel transport map corresponding to super connection 1-forms on relative principal super fiber bundles shares many properties with the parallel transport map as studied in the pure algebraic setting in [78, 79] in the context of covariant derivatives on super vector bundles. To make this link even more precise, let us start with an equivalent characterization of horizontal forms in terms of forms with values in the associated bundle.

Proposition 2.7.18. *Let $\mathcal{G} \to \mathcal{P}_{/\mathcal{S}} \stackrel{\pi_\mathcal{S}}{\to} \mathcal{M}_{/\mathcal{S}}$ be a $\mathcal{S}$-relative principal super fiber bundle and $\rho : \mathcal{G} \to \mathrm{GL}(\mathcal{V})$ be a representation of $\mathcal{G}$ on a super Λ-vector space $\mathcal{V}$. Then, there exists an isomorphism between $\Omega^k_{hor}(\mathcal{P}_{/\mathcal{S}}, \mathcal{V})^{(\mathcal{G},\rho)}$ and $\Omega^k(\mathcal{M}_{/\mathcal{S}}, \mathcal{E}_{/\mathcal{S}}) \cong \Omega^k(\mathcal{M}_{/\mathcal{S}}) \otimes \mathcal{E}_{/\mathcal{S}}$, i.e., k-forms on $\mathcal{M}_{/\mathcal{S}}$ with values in the associated $\mathcal{S}$-relative super vector bundle $\mathcal{E}_{/\mathcal{S}} := (\mathcal{P} \times_\rho \mathcal{V})_{/\mathcal{S}}$.*

Proof. For $k = 0$, this is straightforward generalization of Corollary 2.4.23. For general $k \in \mathbb{N}$, suppose ω is a horizontal k-form on $\mathcal{P}_{/\mathcal{S}}$ of type $(\mathcal{G}, \rho)$. Choose a local trivialization $s : U_{/\mathcal{S}} \to \mathcal{P}_{/\mathcal{S}}$ with $U \subseteq \mathcal{M}$ open. On $U_{/\mathcal{S}}$, we then define a k-form $\overline{\omega} \in \Omega^k(\mathcal{M}_{/\mathcal{S}}, \mathcal{E}_{/\mathcal{S}})$ as follows

$$\langle X_1, \ldots, X_k | \overline{\omega} \rangle := [s, \langle X_1, \ldots, X_k | s^*\omega \rangle] \tag{2.273}$$

for any smooth vector fields X_i on $\mathcal{M}_{/\mathcal{S}}$, $i = 1, \ldots, k$. By horizontality and $\mathcal{G}$-equivariance of ω, it is then immediate to see that $\overline{\omega}$ is indeed well-defined and independent of the choice of a local section. The inverse direction follows similarly. □

Definition 2.7.19. Under the assumptions of Prop. 2.7.18, on $\Omega(\mathcal{M}_{/\mathcal{S}}, \mathcal{E}_{/\mathcal{S}})$, the *exterior covariant derivative* $\mathrm{d}_\mathcal{A} : \Omega^k(\mathcal{M}_{/\mathcal{S}}, \mathcal{E}_{/\mathcal{S}}) \to \Omega^{k+1}(\mathcal{M}_{/\mathcal{S}}, \mathcal{E}_{/\mathcal{S}})$ *induced by* $\mathcal{A}$ is defined via

$$\mathrm{d}_\mathcal{A}\overline{\omega} := \overline{D^{(\mathcal{A})}\omega} \tag{2.274}$$

for any $\omega \in \Omega^k_{hor}(\mathcal{P}_{/\mathcal{S}}, \mathcal{V})^{(\mathcal{G},\rho)}$. For $k = 0$, we also write $\mathrm{d}_\mathcal{A} \equiv \nabla^{(\mathcal{A})}$.

Definition 2.7.20. Under the assumptions of Prop. 2.7.18, let $f : \mathcal{S} \to \mathcal{M}$ be a smooth map. As in Remark 2.7.10, for the pullback bundle $f^*\mathcal{E}$, we have

$$\Gamma(f^*\mathcal{E}) = \{\phi : \mathcal{S} \to \mathcal{E} | \, \pi_{\mathcal{E}} \circ \phi = f\} \tag{2.275}$$

By definition, any $\phi \in \Gamma(f^*\mathcal{E})$ is of the form $\phi = [\phi_0, v]$ with $\phi_0 \in \Gamma(f^*\mathcal{P})$. Hence, let $\gamma : \mathcal{S} \times \mathcal{I} \to \mathcal{M}$ be a smooth path on $\mathcal{M}_{/\mathcal{S}}$. The connection 1-form $\mathcal{A}$ on $\mathcal{P}_{/\mathcal{S}}$ then induces a parallel transport map $\mathscr{P}^{\mathcal{E},\mathcal{A}}_{\mathcal{S},\gamma}$ on $\mathcal{E}_{/\mathcal{S}}$ along γ via

$$\mathscr{P}^{\mathcal{E},\mathcal{A}}_{\mathcal{S},\gamma} : \Gamma(\gamma_0^*\mathcal{E}) \to \Gamma(\gamma_1^*\mathcal{E}), \; \phi = [\phi_0, v] \mapsto [\mathscr{P}^{\mathcal{A}}_{\mathcal{S},\gamma}(\phi_0), v] \tag{2.276}$$

with $\mathscr{P}^{\mathcal{A}}_{\mathcal{S}}$ the parallel transport map induced by $\mathcal{A}$ as defined via Def. 2.7.11.

The following proposition, together with Prop. 2.5.23 giving an explicit form of the exterior covariant derivative on horizontal forms, provides a link between the parallel transport on associated $\mathcal{S}$-relative super vector bundles and the parallel transport on algebraic super vector bundles as constructed in [78, 79].

Proposition 2.7.21. *Under the assumptions of Prop. 2.7.18, let $\mathcal{A} \in \Omega^1(\mathcal{P}_{/\mathcal{S}}, \mathfrak{g})_0$ be a super connection 1-form on the $\mathcal{S}$-relative principal super fiber bundle $\mathcal{G} \to \mathcal{P}_{/\mathcal{S}} \to \mathcal{M}_{/\mathcal{S}}$ and $\mathscr{P}^{\mathcal{E},\mathcal{A}}_{\mathcal{S}}$ the induced parallel transport map on the associated $\mathcal{S}$-relative super vector bundle. Let furthermore $\gamma : \mathcal{S} \times \mathcal{I} \to \mathcal{M}$ be a smooth path on $\mathcal{M}_{/\mathcal{S}}$ and $e \in \Gamma(\mathcal{E}_{/\mathcal{S}})$ a smooth section which is covariantly constant along γ w.r.t. $\mathcal{A}$, i.e.,*

$$\langle (\mathbb{1} \otimes \partial_t)\hat{\gamma} | \nabla^{(\mathcal{A})} e \rangle = 0 \tag{2.277}$$

$\forall (s,t) \in \mathcal{S} \times \mathcal{I}$ with $\hat{\gamma} := \alpha_{\mathcal{S}}^{-1}(\gamma)$. Then, the pullback of e along the path γ is given by $\hat{\gamma}^* e = [\gamma^{hor}_{\phi}, v]$ with $[\phi, v] =: e \circ \hat{\gamma}(\cdot, 0) \in \Gamma(\gamma_0^*\mathcal{E})$ and γ^{hor}_{ϕ} is the unique horizontal lift through ϕ. In particular,

$$\hat{\gamma}_1^* e = \mathscr{P}^{\mathcal{E},\mathcal{A}}_{\mathcal{S},\gamma}(\hat{\gamma}_0^* e) \tag{2.278}$$

Proof. By locality, it suffices to assume that the claim holds on a local trivilization neighborhood. Hence, w.l.o.g. suppose that γ is contained within a local trivialization neighborhood of $\mathcal{P}_{/\mathcal{S}}$ induced by a local section $\tilde{s} : U_{/\mathcal{S}} \to \mathcal{P}_{/\mathcal{S}}$. With respect to this trivialization, the section e is then of the form $e = [\tilde{s}, v]$ with $v : \mathcal{S} \times U \to \mathcal{V}$ a smooth map. Using (2.273), it then immediately follows by definition of the covariant derivative that

$$\langle (\mathbb{1} \otimes \partial_t)\hat{\gamma} | \nabla^{(\mathcal{A})} e \rangle = [\tilde{s}, (\mathbb{1} \otimes \partial_t)\hat{v} + \rho_*(\mathcal{A}^{\gamma})\hat{v}] \tag{2.279}$$

where $\mathcal{A}^{\gamma} := \langle(\mathbb{1}\otimes\partial_t)\hat{\gamma}|\tilde{s}^*\mathcal{A}\rangle$, $\hat{\delta} := \tilde{s}\circ\hat{\gamma} : \mathcal{I}_{/\mathcal{S}} \to \mathcal{P}_{/\mathcal{S}}$ and $\hat{v} = v\circ\hat{\gamma}$ such that $\hat{v}(\cdot,0) = v_0$. Hence, e is covariantly constant along γ iff

$$(\mathbb{1}\otimes\partial_t)\hat{v} + \rho_*(\mathcal{A}^{\gamma})\hat{v} = 0 \tag{2.280}$$

On the other hand, consider the smooth path $\tilde{e}(s,t) := [\gamma_{\phi}^{hor}(s,t), v_0]\,\forall(s,t)\in\mathcal{S}\times\mathcal{I}$. By the proof of Prop. 2.7.9, w.r.t. to the chosen local trivialization, the horizontal lift takes the form $\gamma_{\phi} = \Phi_{\mathcal{S}}(\hat{\delta}, g)$ with $g : \mathcal{S}\times\mathcal{I}\to\mathcal{G}$ a smooth map satisfying

$$(\mathbb{1}\otimes\partial_t)g(s,t) = -R_{g(s,t)*}\mathcal{A}^{\gamma}(s,t) \tag{2.281}$$

together with the initial condition $g(\cdot,0) = e$. Hence, this yields $\tilde{e} = [\Phi_{\mathcal{S}}(\hat{\delta}, g), v_0] = [\hat{\delta}, \tilde{v}]$ with $\tilde{v} := \rho(g)v_0 : \mathcal{S}\times\mathcal{I}\to\mathcal{V}$. Taking the partial time derivative of $\tilde{v}$, this then yields, together with $\rho\circ R_g = R_{\rho(g)}\circ\rho\,\forall g\in\mathcal{G}$,

$$\begin{aligned}(\mathbb{1}\otimes\partial_t)\tilde{v}(s,t) &= D_{g(s,t)}\rho((\mathbb{1}\otimes\partial_t)g(s,t))v_0 = -D_e(\rho\circ R_{g(s,t)})(\mathcal{A}^{\gamma})v_0\\ &= -R_{\rho(g(s,t))*}(\rho_*(\mathcal{A}^{\gamma}))v_0 = -\rho_*(\mathcal{A}^{\gamma})\rho(g(s,t))v_0\\ &= -\rho_*(\mathcal{A}^{\gamma})\tilde{v}(s,t)\end{aligned} \tag{2.282}$$

where, in the second line, we used that the pushforward of the right translation on $\mathrm{GL}(\mathcal{V})$ can be identified with the ordinary right group multiplication. Since, $\tilde{v}(\cdot,0) = v_0$ it thus follows from the uniqueness of solutions of differential equations once fixing the initial conditions that $\tilde{v} = \hat{v}$ on $\mathcal{S}\times\mathcal{I}$. This proves the proposition. □

2.8. Discussion

In this chapter, we have studied the theory of super fiber bundles, in particular, principal super fiber bundles and super connection forms defined on them. We studied these objects mainly in the Rogers-DeWitt category extending the seminal work of Tuynman [97] to relative supermanifolds defining objects in enriched categories. To this end, at the beginning, we discussed some important aspects of supermanifold theory and established a concrete link between various different approaches to this subject via the functor of points prescription. We also used this technique in order to provide a link to the theory of principal super fiber bundles and super connection forms in the algebro-geometric approach [102] and showed that both approaches are in fact equivalent.

We then studied the parallel transport map induced by super connection 1-forms. A generic issue in both the algebro-geometric and concrete approach is the lack of (anti-commutative) fermionic degrees of freedom on the body of a supermanifold. From a mathematical point of view, this implies that the parallel transport map cannot be used to compare points on different fibers of the bundle in contrast to the classical theory. A resolution is given by considering relative supermanifolds as studied, e.g., in [74,101] and which is rooted in the Molotkov-Sachse approach to supermanifold theory [98–100]. In this chapter, a rigorous mathematical account on this subject was given. In particular, we defined and analyzed super connection 1-forms on relative principal super fiber bundles. Finally, the parallel transport map was constructed in this enriched category. It follows that the parallel transport map indeed has the right properties as it provides an isomorphism between the fibers of the underlying relative principal super fiber bundle. Moreover, it behaves functorially under composition of (parametrized) paths and, in particular, transforms covariantly under change of parametrization.

Finally, the induced parallel transport map on associated super vector bundles was considered. In this context, among other things, we established a link to similar constructions in the algebraic approach [78,79] studying the parallel transport map induced by covariant derivatives on super vector bundles.

We will use these results in the following chapters. On the one hand, in Chapter 3, a mathematically rigorous approach towards geometric supergravity will be established. In this context, we will work in the category of relative supermanifolds. Again, this turns out to be mandatory in order to resolve the fermionic degrees of freedom of the theory. On the other hand, we will need the parallel transport map in Chapter 5 as well as Chapter 6 in a symmetry reduced setting in order to construct the graded holonomy-flux algebra of chiral supergravity. To this end, in Example 2.7.17, for a particular choice of a gauge, we derived an explicit expression of the parallel transport map in which bosonic and fermionic degrees of freedom are separated as far as possible.

There are many possible and interesting extensions of the present formalism. For instance, it would be desirable to generalize it to include higher gauge theories which typically arise in context of higher dimensional supergravity theories. To this end, one needs to generalize the theory of higher principal super fiber bundles and super connection forms defined on them as studied for instance in [119,120] to the relative category. Finally, there is quite some recent interest in the description of boundary charges [121–124]. This has been addressed for instance in context of AdS supergravity in [125]. There, among other things, it was found that a consistent treatment of (supersymmetric) boundary charges may be possible in the context of the geometric approach to supergravity (see Section 3.4). It would therefore be very interesting to generalize the Iyer-Wald's Noether charge formalism [126] to supergravity which, in particular, explicitly takes into account the underlying supersymmetry of the theory. This may be achieved generalizing the work of Prahbu [127] to field theories defined on principal super fiber bundles.

3. Supergravity and super Cartan geometry

3.1. Introduction

Soon after the first discovery of supergravity in 1976 by Freedman, Ferrara and van Nieuwenhuizen [128], Ne'eman and Regge studied a new geometric approach based on the ideas of Cartan of a purely geometric interpretation of gravity [70]. In this theory, now commonly known as Cartan geometry, gravity arises by considering the underlying symmetry groups of flat Minkowski spacetime, i.e., a Klein geometry consisting of the isometry group given by the Poincaré group and the Lorentz group as stabilizer subgroup of a particular spacetime event. Gravity is then obtained by deforming this flat initial data in a particular way by studying a certain kind of connection forms, called Cartan connections, taking values in the Lie algebra of the isometry group of the flat model. This Cartan geometric approach gives gravity a very clear geometric interpretation and even allows the inclusion of matter fields via Kaluza-Klein reduction of higher dimensional pure gravity theories leading for instance to Einstein-Yang-Mills theories. However, it still has some limitations as, for instance, it does not include fermionic fields. This changes in case of supersymmetry as graded Lie algebras, by definition, naturally include fermionic generators. It was then realized extending Cartan geometry to the super category that this in fact leads to supergravity. The fermion field, given by the superpartner of the graviton field, then arises from the odd components of a super Cartan connection taking values in the graded extension of the Poincaré algebra. Besides, this description also yields a geometric interpretation of supersymmetry transformations in terms of infinitesimal superdiffeomorphisms.

These ideas were studied more systematically and developed even further by Castellani-D'Auria-Fré [71, 72] to include extended and higher dimensional supergravity theories. Moreover, generalizing the Maurer-Cartan equations to include higher p-form gauge fields which naturally appear in higher dimensions, such as the supergravity C-field in the unique maximal $D = 11$, $\mathcal{N} = 1$ supergravity theory, then lead to the concept of free graded differential algebras (FDA). These type of algebras then turned out to have a rigorous geometric interpretation in higher category theory describing the higher gauge fields as components of a higher Cartan connection [129, 130].

In this chapter, we want to provide a mathematically rigorous approach towards geometric supergravity introducing the notion of a super Cartan geometry. However, the problem of modeling anticommuting classical fermion fields, which is crucial in the context of supersymmetry, turns out to be by far non-straightforward. This seems to be usually ignored in the physical literature. Again, motivated from algebraic geometry, this problem has an intriguing resolution using the concept of enriched categories as studied in detail in Chapter 2 and used for instance for the construction of the parallel transport map. We will then see that pure $D=4$, $\mathcal{N}=1$ Poincaré supergravity arises naturally in this framework. For an interesting approach which is different from the present one, using the notion of *integral forms* see [131, 132, 220].

Later, in Chapter 5, these considerations will be extended, though with slightly less mathematical rigor, to include pure $\mathcal{N}$-extended (Holst-)AdS SUGRA in $D=4$ for $\mathcal{N}=1,2$ as well as a discussion about the appropriate description of boundary theories compatible with SUSY. Moreover, we will use this geometric formulation in Section 5.4.1 in context of chiral supergravity in order to give a geometric interpretation of the super Asthekar connection in terms of a generalized super Cartan connection. This provides a conceptual explanation for the observation of Fülöp [63]. As we will see, this connection appears quite naturally when studying the chiral structure of the underlying supersymmetry algebra corresponding to the super Klein geometry and is rooted in the special properties of the (bosonic) self-dual variables and even survives in case of extended supersymmetry.

The structure of this chapter is as follows: First, in Section 3.2, we will review the geometrical interpretation of gravity in terms of a Cartan geometry. In Section 3.3 We will then introduce the notion of a (metric reductive) super Cartan geometry in the framework of enriched categories and discuss some of its properties such as their strong relation to Yang-Mills gauge theories. This formulation will subsequently be used in Section 3.4 in order to give a geometric interpretation of $\mathcal{N}=1$, $D=4$ Poincaré supergravity.

As an interesting application, we will use this geometric approach in Section 3.5 to discuss global symmetries in supergravity and to describe Killing spinors in terms of odd Killing vector fields on super Riemannian manifolds induced by metric reductive super Cartan geometries. These play a prominent role in context of supersymmetric black holes. Finally, in Section 3.6, we will sketch a concrete link between the description of anticommuting fermionic fields in context of enriched categories as well as in the framework of pAQFT [76, 77] demonstrating the strong relation between these two approaches.

A list of important symbols as well as an overview of our choice of conventions concerning indices, physical constants etc. can be found in the List of symbols, notations and conventions.

3.2. Review: Gravity as Cartan geometry

In this section, mostly following [133], we want to review the interpretation of gravity in terms of a Cartan geometry as this will serve a starting point for a very elegant approach to supergravity as described in detail in Section 3.4 and a derivation of a super analog of Asthekar's connection discussed in Section 5.4. For a more detailed introduction to Cartan geometry see, e.g., [134–136]. For more details on the relation between Cartan geometry and general relativity we refer to [133] (see also [137] for a nice exposition).

In his famous Erlangen program, Klein studied the idea of classifying the geometry of space via the underlying group of symmetries. For instance, one can consider Minkowski spacetime $(\mathbb{R}^{1,3}, \eta)$ and study the corresponding Lie group $\mathrm{ISO}(\mathbb{R}^{1,3})$ of isometries which is isomorphic to the Poincaré group $\mathbb{R}^{1,3} \rtimes \mathrm{SO}^+(1,3)$. If one then chooses a specific spacetime event $p \in \mathbb{R}^{1,3}$, one can consider the corresponding stabilizer subgroup $\mathrm{SO}^+(1,3)$ which preserves that point. Since the isometry group acts transitively on $\mathbb{R}^{1,3}$, it follows that Minkowski spacetime can be described in terms of the coset space

$$\mathbb{R}^{1,3} \cong \mathrm{ISO}(\mathbb{R}^{1,3})/\mathrm{SO}^+(1,3) \tag{3.1}$$

Hence, the collection of spacetime events can equivalently be described in terms of the underlying symmetry groups. A similar kind of reasoning applies in case of the other maximally symmetric homogeneous spacetimes such as de Sitter of anti-de Sitter spacetime (see Appendix E, Corollary E.8) playing a central role in general relativity and cosmology. Hence, one makes the following definition (see also [133, 134]):

Definition 3.2.1. A *Klein geometry* is a pair (G, H) consisting of a Lie Group G and an embedded Lie subgroup $H \hookrightarrow G$ such that G/H is connected.

Given a Klein geometry (G, H), the coset space G/H has the structure of principal H-bundle

$$\begin{array}{ccc} G & \longleftarrow & H \\ {\scriptstyle \pi}\big\downarrow & & \\ G/H & & \end{array}$$

Moreover, on G, there exists a canonical $\mathfrak{g}$-valued 1-form given by the Maurer-Cartan form $\theta_{\mathrm{MC}} \in \Omega^1(G, \mathfrak{g})$ (cf. Example 2.4.13) which, choosing a basis of left-invariant vector fields $X_i \in \mathfrak{g}$, $i = 1, \ldots, \dim \mathfrak{g}$, is defined as

$$\theta_{\mathrm{MC}} = X_i \otimes \omega^i \tag{3.2}$$

where $\omega^i \in \Omega^1(G)$ is the corresponding dual basis of left-invariant one-forms on G satisfying $\omega^i(X_j) = \delta^i_j$. It follows by definition that the Maurer-Cartan form is G-equivariant, i.e.[1]

$$R_g^* \theta_{\mathrm{MC}} = \mathrm{Ad}_{g^{-1}} \circ \theta_{\mathrm{MC}} \tag{3.3}$$

[1] This can be seen directly using the equivalent definition in terms of the left-translation $\theta_{\mathrm{MC}}(X_g) = L_{g^{-1}*} X_g$.

$\forall g \in G$ with $R_g : G \to G$ denoting the right translation on G. By definition, θ_{MC} maps left-invariant vector fields to themselves, i.e., $\theta_{\mathrm{MC}}(X) = X_e \; \forall X \in \mathfrak{g}$ and, as a consequence, yields an isomorphism $\theta_{\mathrm{MC}} : T_g G \to \mathfrak{g}$ of vector spaces at any $g \in G$. Moreover, it satisfies the *Maurer-Cartan structure equation*

$$\mathrm{d}\theta_{\mathrm{MC}} + \frac{1}{2}[\theta_{\mathrm{MC}} \wedge \theta_{\mathrm{MC}}] = 0 \tag{3.4}$$

As seen above, standard examples of Klein geometries (G, H) arising in physics are given by the Minkowski spacetime $(\mathrm{ISO}(\mathbb{R}^{1,3}), \mathrm{SO}^+(1,3))$, de Sitter $(\mathrm{SO}(1,4), \mathrm{SO}^+(1,3))$ or anti-de Sitter spacetime $(\mathrm{SO}(2,3), \mathrm{SO}^+(1,3))$, respectively. These have in common that the Lie algebra $\mathfrak{g}$ of G can be split into $\mathrm{Ad}(H)$-invariant subspaces $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}/\mathfrak{h}$ with $\mathfrak{h}$ the Lie algebra of H. Moreover, on the moduli space $\mathfrak{g}/\mathfrak{h}$ there exists a canonical $\mathrm{Ad}(H)$-invariant bilinear form. In this case, the Klein geometry is called *metric* and *reductive* [133]. Hence, we see that flat spacetime can equivalently be described in terms of a Klein geometry. Based on this observation, Cartan formulated a theory now known as Cartan geometry which can be interpreted as a deformed Klein geometry such as gravity is a deformed version of flat Minkowski spacetime (see also [133, 134]):

Definition 3.2.2. A *metric reductive Cartan geometry* $(\pi : P \to M, A; \eta)$ modeled on a metric reductive Klein geometry $(H, G; \eta)$ is a principal fiber bundle $H \to P \to M$ with structure group H together with a $\mathfrak{g}$-valued 1-form $A \in \Omega^1(P, \mathfrak{g})$ on P called *Cartan connection* such that

(i) $A_p(X_p) = X \; \forall X \in \mathfrak{h} = T_e H, \; p \in P$

(ii) $\Phi_h^* A = \mathrm{Ad}_{h^{-1}} \circ A \; \forall h \in H$

(iii) the map $A_p : T_p P \to \mathfrak{g}$ defines an isomorphism of vector spaces for any $p \in P$

where the last condition is also called the *Cartan condition*.

Given a metric reductive Cartan geometry $(\pi : P \to M, A; \eta)$, one can split the Cartan connection A by projecting it according to the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}/\mathfrak{h}$ of the Lie algebra of G yielding

$$A = \mathrm{pr}_{\mathfrak{g}/\mathfrak{h}} \circ A + \mathrm{pr}_{\mathfrak{h}} \circ A =: e + \omega \tag{3.5}$$

with 1-forms $\omega \in \Omega^1(P, \mathfrak{h})$ and $e \in \Omega^1(P, \mathfrak{g}/\mathfrak{h})$ with the latter also referred to as the *soldering form*. Due to the conditions (i) and (ii) of the Cartan connection, it follows immediately that ω defines an ordinary principal connection 1-form in the sense of Ehresmann. Let $\mathscr{H} := \ker(\omega)$ be the induced horizontal distribution on the tangent bundle TP. If $\widetilde{X} \in \mathscr{V}$ is a (vertical) fundamental vector field generated by $X \in \mathfrak{h}$, one has $A(\widetilde{X}) = X = \omega(\widetilde{X})$ and thus $e(\widetilde{X}) = 0$. Hence, since $\mathfrak{g}/\mathfrak{h}$ defines a H-invariant

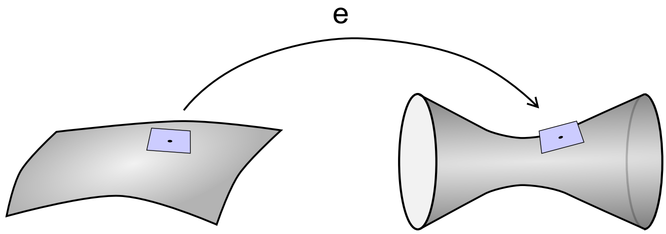


Figure 1.: Pictorial representation of a Cartan geometry. The soldering form provides a local identification of tangent spaces on the spacetime manifold (left) with tangent spaces on the flat model (right) corresponding to the Klein geometry.

subspace, together with condition (ii), this immediately implies that the soldering form is horizontal of type (H, Ad), i.e. $e \in \Omega^1_{hor}(P, \mathfrak{g}/\mathfrak{h})^{(H,\mathrm{Ad})}$. In fact, the soldering form even provides an identification of the principal bundle P as a H-reduction of the frame bundle $\mathscr{F}(M)$ explaining its name (see also Figure 1). To see this, following [136], note that $e_p^{-1} := (e_p|_{\mathscr{H}_p})^{-1}$ for any $p \in P$ defines an isomorphism on $\mathfrak{g}/\mathfrak{h}$ and $\omega_p|_{\mathscr{H}_p}$ is an isomorphism onto $T_{\pi(p)}M$ so that $D_p\pi \circ e_p^{-1} : \mathfrak{g}/\mathfrak{h} \xrightarrow{\sim} T_{\pi(p)}M$ is a linear frame at $\pi(p)$. Hence, this yields a map

$$\iota : P \to \mathscr{F}(M), \; p \mapsto D_p\pi \circ e_p^{-1} \tag{3.6}$$

By condition (ii), we have $\Phi_h^* e_p(Y_p) = e_{ph}(D_p\Phi_h(Y_p)) = \mathrm{Ad}_{h^{-1}}(e_p(Y_p)) \; \forall Y_p \in T_pP$ and $h \in H$ and therefore

$$e_{ph}^{-1} = D_p\Phi_h \circ e_p^{-1} \circ \mathrm{Ad}_g \tag{3.7}$$

from which we obtain

$$\iota(p \cdot h) = D_{ph}\pi \circ e_{ph}^{-1} = D_{ph}\pi \circ D_p\Phi_h \circ e_p^{-1} \circ \mathrm{Ad}_h = \iota(p) \circ \mathrm{Ad}_h \tag{3.8}$$

$\forall p \in P, \, h \in H$. That is, $\iota : P \to \mathscr{F}(M)$ is H-equivariant and fiber-preserving so that P defines a H-reduction of the frame bundle w.r.t. the group morphism $\mathrm{Ad} : H \to \mathrm{GL}(\mathfrak{g}/\mathfrak{h})$. Moreover, it follows that ι induces an isomorphism [136] (denoted by the same symbol)

$$\begin{aligned} \iota : P \times_{\mathrm{Ad}} \mathfrak{g}/\mathfrak{h} &\xrightarrow{\sim} TM \\ [(p, X)] &\longmapsto D_p\pi(e_p^{-1}(X)) \end{aligned} \tag{3.9}$$

between the associated vector bundle $P \times_{\mathrm{Ad}} \mathfrak{g}/\mathfrak{h}$ and the tangent bundle of M.

To a Cartan connection A one associates the *Cartan curvature* $F(A) \in \Omega^2(P,\mathfrak{g})$ according to

$$F(A) := \mathrm{d}A + \frac{1}{2}[A \wedge A] \tag{3.10}$$

In case of a "flat" Klein geometry, the Cartan connection is given by the Maurer-Cartan form (3.2) which satisfies the structure equation (3.4), i.e., the Cartan curvature is identically zero. Thus $F(A)$ indicates the deviation of a Cartan geometry from a flat Klein geometry [136]. In fact, one can prove that a Cartan geometry modeled on the Klein geometry (G,H) is locally isomorphic to the homogeneous model $(G \to G/H, \theta_{\mathrm{MC}})$ if and only if the associated Cartan curvature vanishes (for a proof see, e.g., [134, 135]).

Decomposing $F(A)$ according to the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}/\mathfrak{h}$ of the Lie algebra, one obtains

$$F(A) = \mathrm{pr}_{\mathfrak{h}} \circ F(A) + \mathrm{pr}_{\mathfrak{g}/\mathfrak{h}} \circ F(A) = F(\omega) + \Theta^{(\omega)} + \frac{1}{2}[e \wedge e] \tag{3.11}$$

where

$$F(\omega) = D^{(\omega)}\omega = \mathrm{d}\omega + \frac{1}{2}[\omega \wedge \omega] \tag{3.12}$$

is the curvature of the connection 1-form ω and

$$\Theta^{(\omega)} := D^{(\omega)}e = \mathrm{d}e + [\omega \wedge e] \tag{3.13}$$

is the corresponding *torsion* 2*-form*. To see that (3.13) in fact encodes the torsion of the connection, one may proceed similar as in [136] and note that ω induces a connection on the associated vector bundle $P \times_{\mathrm{Ad}} \mathfrak{g}/\mathfrak{h}$ and thus, via (3.9), an affine connection $\nabla \equiv \nabla^{(\omega)} : \Gamma(TM) \to \Gamma(T^*M \otimes TM)$ on the tangent bundle. For vector fields $X, Y \in \Gamma(TM)$, it is given by

$$(\nabla_X Y)_x = \iota([p, X^{hor}e(Y^{hor})]) \tag{3.14}$$

for any $x \in M$ and $p \in P$ with $\pi(p) = x$ where X^{hor}, Y^{hor} denote the horizontal lifts of X and Y, respectively.[2] Moreover, in general, given a representation $\rho : H \to \mathrm{GL}(V)$ of H on a vector space V, there exists an isomorphism

$$\Omega^k_{hor}(P,V)^{(H,\rho)} \xrightarrow{\sim} \Omega^k(M, P \times_\rho V),\ \omega \mapsto \overline{\omega} \tag{3.15}$$

[2] Recall from Def. 2.5.17 in the ungraded case, for a vector field $X \in \Gamma(TM)$, the corresponding horizontal lift $X^{hor} \in \Gamma(TP)$ is a vector field on P which is horizontal, i.e., $X^{hor}_p \in \mathscr{H}_p\,\forall p \in P$, and satisfies $D_p\pi(X^{hor}) = X_{\pi(p)}$.

between V-valued k-forms of type (H, ρ) and k-forms with values in the associated bundle $P \times_\rho V$. Hence, we can associate to $\Theta^{(\omega)}$ a 2-form $\overline{\Theta^{(\omega)}} \in \Omega^2(M, P \times_{\mathrm{Ad}} \mathfrak{g}/\mathfrak{h})$, which, applying (3.9), yields another form $\iota \circ \overline{\Theta^{(\omega)}} \in \Omega^2(M) \otimes \Gamma(TM)$. For vector fields $X, Y \in \Gamma(TM)$, we then compute

$$\begin{aligned}\iota \circ \overline{\Theta^{(\omega)}}(X_x, Y_x) &= \iota \circ [p, \mathrm{d}e(X^{hor}, Y^{hor})] \\ &= \iota \circ [p, X^{hor}e(Y^{hor}) - Y^{hor}e(X^{hor}) - e([X^{hor}, Y^{hor}])] \\ &= \nabla_X Y - \nabla_Y X - D_p\pi([X,Y]^{hor}) = T^\nabla(X_x, Y_x) \end{aligned} \tag{3.16}$$

for any $x \in M$ and $p \in P$ such that $\pi(p) = x$. Hence, $\Theta^{(\omega)}$ indeed encodes the torsion of the associated affine connection ∇ on the tangent bundle of M.

With all these observations, let us now make contact to general relativity. As seen already at the beginning, flat Minkowski spacetime can be described in terms of the metric Klein geometry $(\mathrm{ISO}(\mathbb{R}^{1,3}), \mathrm{SO}^+(1,3); \eta)$. Hence, we consider gravity as a metric reductive Cartan geometry $(P \to M, A; \eta)$ modeled on the metric reductive Klein geometry $(\mathrm{ISO}(\mathbb{R}^{1,3}), \mathrm{SO}^+(1,3); \eta)$ where $\mathrm{SO}^+(1,3) \to P \xrightarrow{\pi} M$ is a principal bundle with structure group $\mathrm{SO}^+(1,3)$ and $A \in \Omega^1(P, \mathfrak{iso}(\mathbb{R}^{1,3}))$ is a Cartan connection.

By (3.6), we know that P defines a $\mathrm{SO}^+(1,3)$-reduction of the frame bundle $\mathscr{F}(M)$ of M. As such, it induces a Lorentzian metric $g \in \Gamma(T^*M \otimes T^*M)$ on M which, for vector fields $X, Y \in \Gamma(TM)$, is defined as [133, 137]

$$g(X_x, Y_x) := \eta(\iota_p^{-1}(X_x), \iota_p^{-1}(Y_x)) \tag{3.17}$$

$\forall p \in P_x,\ x \in M$. Note that g is in fact well-defined, i.e., independent of the choice of $p \in P_x$, since ι is equivariant and η, by definition, is a bilinear form invariant under the Adjoint representation of $\mathrm{SO}^+(1,3)$ on $\mathbb{R}^{1,3}$. Hence, M is in fact a Lorentzian manifold and P can be identified with the bundle $\mathscr{F}_{\mathrm{SO}}(M)$ of Lorentz frames on M.

Let e^I for $I = 0, \dots, 3$ be defined via $e =: e^I P_I$. Given a local section $s: M \supset U \to P$ of the bundle, the corresponding pullback then induces 1-forms (denoted by the same symbol for convenience) $e^I \equiv s^*e^I \in \Omega^1(U)$ which satisfy

$$g_{\mu\nu} = \eta(s^*e_\mu, s^*e_\nu) = e^I_\mu e^J_\nu \eta_{IJ} \tag{3.18}$$

i.e. $(e^I)_I$ defines a local co-frame on M with the corresponding frame fields being given by $e_I := s^*\iota(P_I)$. With these ingredients, we can define an action on M via

$$S(A) = \frac{1}{4\kappa} \int_M s^*(F(\omega)^{IJ} \wedge e^K \wedge e^L)\epsilon_{IJKL} \tag{3.19}$$

where $\kappa = 8\pi G$. This action precisely coincides with the first-order Palatini action of pure Einstein gravity. As we see, the whole theory including the underlying geometrical structure of the spacetime is completely encoded in the Cartan connection.

Following [133], there also exists another version of action (3.19) which depends on the Cartan connection in a more explicit way. This requires a nonvanishing cosmological constant which we take as negative for convenience (for a positive cosmological constant this in fact completely analogous (see, e.g., [133])). To this end, let us consider a Cartan geometry modeled on the Klein gravity $(\mathrm{SO}(2,3), \mathrm{SO}^+(1,3))$ corresponding to anti-de Sitter space. Since then $[P_I, P_J] = \frac{1}{L^2}M_{IJ}$ with L the anti-de Sitter radius (see Example 2.3.17), it follows that the Lorentzian part of the Cartan curvature acquires an additional contribution depending on the soldering form yielding

$$F(A)^{IJ} = F(\omega)^{IJ} + \frac{1}{L^2}e^I \wedge e^J \tag{3.20}$$

One can then define the so-called *MacDowell-Mansouri action* as follows [133, 138]

$$S_{\mathrm{MM}}(A) = \frac{L^2}{8\kappa}\int_M s^*(F(A)^{IJ} \wedge F(A)^{KL})\epsilon_{IJKL} \tag{3.21}$$

which, in particular, solely depends on the curvature of the Cartan connection and thus has the structure of a Yang-Mills-type action. Expanding (3.21) using (3.20), it follows that the term quadratic in $F(\omega)$ is given by the well-known *Gauss-Bonnet term* an thus is purely topological. Hence, it follows that, up to boundary terms, (3.21) indeed leads back to first-order Einstein gravity with a nontrivial cosmological constant.

3.3. Super Cartan geometry

The content of this section has been reproduced from [1], with slight changes to account for the context of this thesis with the permission of AIP Publishing.

As discussed in the previous section, gravity has a very elegant geometrical interpretation in terms of a Cartan geometry modeled on the Klein geometry corresponing to flat Minkowski, de Sitter or anti-de Sitter spacetime. As it turns out, this description also carries over to the super category providing a geometrical foundation of supergravity. This is the starting point of the Castellani-D'Auria-Fré approach to supergravity [71, 72]. However, in order to obtain nontrivial fermionic degrees of freedom as well as supersymmetry transformations on the body of a supermanifold, in the following, we will define the notion of super Cartan geometry using the concept of enriched categories. In [101], *super Cartan structures* on supermanifolds were introduced and also lifted trivially to Cartan structures in the relative category. However, a precise definition of super Cartan geometries on (nontrivial) relative principal super fiber bundles in the framework of enriched categories has not been given so far in the mathematical literature. For a motivation of super Cartan geometry, let us consider first the "flat" case given by a super Klein geometry[3].

Definition 3.3.1. A *super Klein geometry* is a pair $(\mathcal{G},\mathcal{H})$ consisting of a super Lie Group $\mathcal{G}$ and an embedded super Lie subgroup $\mathcal{H}\hookrightarrow\mathcal{G}$.

Remark 3.3.2. Suppose one has given a pair $(\mathcal{G},\mathcal{H})$ of super Lie groups with $\mathcal{H}\hookrightarrow\mathcal{G}$ an embedded super Lie subgroup. By definition of the DeWitt topology, $\mathcal{G}/\mathcal{H}$ is connected iff $\mathbf{B}(\mathcal{G}/\mathcal{H})\cong\mathbf{B}(\mathcal{G})/\mathbf{B}(\mathcal{H})$ is connected, that is, iff $(\mathbf{B}(\mathcal{G}),\mathbf{B}(\mathcal{H}))$ is a Klein geometry.

As shown in [109], as in the classical theory, a super Klein geometry $(\mathcal{G},\mathcal{H})$ canonically induces super fiber bundle with typical fiber $\mathcal{H}$ via

$$\begin{array}{ccc} \mathcal{G} & \longleftarrow & \mathcal{H} \\ {\scriptstyle\pi}\big\downarrow & & \\ \mathcal{G}/\mathcal{H} & & \end{array}$$

together with the natural $\mathcal{H}$-right action $\Phi:\mathcal{G}\times\mathcal{H}\rightarrow\mathcal{G}$ on $\mathcal{G}$. Hence $\mathcal{H}\rightarrow\mathcal{G}\stackrel{\pi}{\rightarrow}\mathcal{G}/\mathcal{H}$ has the structure of a principal $\mathcal{H}$-bundle. Let $(X_i)_i$ be a homogeneous basis of $\mathfrak{g}$ and $({}^i\omega)_i$ the associated left dual basis (see Prop. 2.4.12) of left-invariant 1-forms ${}^i\omega\in\Omega^1(\mathcal{G})$ on $\mathcal{G}$ satisfying $\langle X_i|{}^j\omega\rangle=\delta_i^j,\forall i,j=1,\ldots,n$. By Example 2.4.13, the Maurer-Cartan form $\theta_{\mathrm{MC}}\in\Omega^1(\mathcal{G},\mathfrak{g})$ on $\mathcal{G}$ is then given by

$$\theta_{\mathrm{MC}}={}^i\omega\otimes X_i \tag{3.22}$$

By definition, the fundamental vector fields on $\mathcal{G}$ correspond to the subspace of left-invariant vector fields $X\in\mathrm{Lie}(\mathcal{H})$. Hence, it follows immediately from (3.22) that $\langle X|\theta_{\mathrm{MC}}\rangle=X_e\;\forall X\in\mathrm{Lie}(\mathcal{H})$. Moreover, this also implies that the map $(\theta_{\mathrm{MC}})_g: T_g\mathcal{G}\rightarrow T_e\mathcal{G}$ is an isomorphism of super Λ-modules for any $g\in\mathcal{G}$ (and even an isomorphism of super Λ-vector spaces if $g\in\mathbf{B}(\mathcal{G})$). Finally, since the right action on $\mathcal{G}$ essentially coincides with the restriction of the group multiplication, this yields

$$R_h^*\theta_{\mathrm{MC}}=\mathrm{Ad}_{h^{-1}}\circ\theta_{\mathrm{MC}} \tag{3.23}$$

[3] All the following definitions will be formulated in the H^∞ category. However, they can also be extended to the algebraic category without major changes (cf. Section 2.6)

$\forall h \in \mathcal{H}$, where R_h^* denotes the generalized pullback w.r.t. the right translation $R_h := \Phi(\cdot, h)$, on $\mathcal{G}$ w.r.t. $h \in \mathcal{H}$ (cf. Definition 2.5.9). This motivates the following definition.

Definition 3.3.3 (Super Cartan geometry)**.** A *super Cartan geometry* $(\pi_{\mathcal{S}} : \mathcal{P}_{/\mathcal{S}} \to \mathcal{M}_{/\mathcal{S}}, \mathcal{A})$ modeled on a super Klein geometry $(\mathcal{G}, \mathcal{H})$ is a $\mathcal{S}$-relative principal super fiber bundle $\mathcal{H} \to \mathcal{P}_{/\mathcal{S}} \to \mathcal{M}_{/\mathcal{S}}$ with structure group $\mathcal{H}$ together with a smooth even $\mathrm{Lie}(\mathcal{G})$-valued 1-form $\mathcal{A} \in \Omega^1(\mathcal{P}_{/\mathcal{S}}, \mathfrak{g})_0$ on $\mathcal{P}_{/\mathcal{S}}$ called *super Cartan connection* such that

(i) $\langle \widetilde{X} | \mathcal{A} \rangle = X, \quad \forall X \in \mathfrak{h}$

(ii) $(\Phi_{\mathcal{S}})_h^* \mathcal{A} = \mathrm{Ad}_{h^{-1}} \circ \mathcal{A}, \quad \forall h \in \mathcal{H}$

(iii) for any $s \in \mathbf{B}(\mathcal{S})$, the pullback of $\mathcal{A}$ w.r.t. the induced embedding $\iota_{\mathcal{P}} : \mathcal{P} \hookrightarrow \mathcal{S} \times \mathcal{P},\ p \mapsto (s, p)$ yields an isomorphism $\iota_{\mathcal{P}}^* \mathcal{A}_p : T_p\mathcal{P} \to \mathrm{Lie}(\mathcal{G})$ of free super Λ-modules for any $p \in \mathcal{P}$

where the last condition will be called the *super Cartan condition*. If (iii) is not satisfied, $\mathcal{A}$ will be called a *generalized super Cartan connection*.

Remark 3.3.4. Note that, by definition, it follows that condition (iii) for a super Cartan connection $\mathcal{A}$ is preserved under change of parametrization. In fact, let $\lambda : \mathcal{S}' \to \mathcal{S}$ be a smooth map. Then, since $\lambda|_{\mathbf{B}(\mathcal{S}')} \subseteq \mathbf{B}(\mathcal{S})$, it follows that the pullback $\lambda^* \mathcal{A}$ also satisfies (iii).

Definition 3.3.5. A super Cartan geometry $(\pi_{\mathcal{S}} : \mathcal{P}_{/\mathcal{S}} \to \mathcal{M}_{/\mathcal{S}}, \mathcal{A})$ modeled on a super Klein geometry $(\mathcal{G}, \mathcal{H})$ is called

(i) *reductive* if the super Lie algebra $\mathfrak{g}$ of $\mathcal{G}$ admits a decomposition of the form $\mathfrak{g} = \mathfrak{g}/\mathfrak{h} \oplus \mathfrak{h}$ with $\mathfrak{h}$ the super Lie algebra of $\mathcal{H}$ and $\mathfrak{g}/\mathfrak{h}$ a super vector space such that the corresponding super Λ-vector space $\Lambda \otimes \mathfrak{g}/\mathfrak{h}$ is invariant w.r.t. the Adjoint action of $\mathcal{H}$ on $\mathrm{Lie}(\mathcal{G})$.

(ii) *metric* if it is reductive and if the super Λ-vector space $\Lambda \otimes \mathfrak{g}/\mathfrak{h}$ admits a smooth super metric (Def. 2.3.12) that is invariant w.r.t. the Adjoint action of $\mathcal{H}$.

Definition 3.3.6. For $\mathcal{M}$ a supermanifold, consider the right dual tensor product super vector bundle $(T\mathcal{M} \otimes T\mathcal{M})^*$. A smooth section $g \in \Gamma^{\mathbb{C}}((T\mathcal{M} \otimes T\mathcal{M})^*) \cong \mathrm{Hom}_R(\Gamma(T\mathcal{M})^2, H^\infty(\mathcal{M}) \otimes \mathbb{C})$ is called a *super metric* on $\mathcal{M}$, if g_x for any $x \in \mathcal{M}$ defines a super metric on the tangent module $T_x\mathcal{M}$. Thus, for any homogeneous smooth vector fields $X, Y \in \Gamma(T\mathcal{M})$, $g(X, Y) = (-1)^{|X||Y|} g(Y, X)$ and the map $\Gamma(T\mathcal{M}) \ni X \mapsto g(X, \cdot) \in \Gamma(T\mathcal{M}^*)$ is an isomorphism. Thus, g defines a super metric in the sense of [139].

Remark 3.3.7. Let g be a super metric on a supermanifold $\mathcal{M}$. Similar as in Remark 2.3.8, it follows that, if $p \in \mathbf{B}(\mathcal{M})$ is a body point, the tangent module $\mathcal{V}_p := T_p\mathcal{M}$ has the structure of a super Λ-vector space $\mathcal{V}_p = \Lambda \otimes V_p$ with the super vector space V_p consisting of derivations $X_p \in T_p\mathcal{M}$ satisfying

$$X_p(f) \in \mathbb{R}, \forall f \in H^\infty(\mathcal{M}) \tag{3.24}$$

Hence, it follows that V_p can be identified with the tangent space $T_p\mathbf{A}(\mathcal{M})$ of the corresponding algebro-geometric supermanifold $\mathbf{A}(\mathcal{M})$ (cf. Remark 2.3.4). By definition, any $X_p \in V_p$ arises from the restriction of a local smooth vector field on $\mathcal{M}$ to p. Consequently, as g is smooth and p is a body point, it follows that $g_p(V_p, V_p) \subseteq \mathbb{C}$, that is, g_p is smooth according to Definition 2.3.12.

Proposition 3.3.8. *Let $(\pi_{\mathcal{S}} : \mathcal{P}_{/\mathcal{S}} \rightarrow \mathcal{M}_{/\mathcal{S}}, \mathcal{A})$ be a super Cartan geometry modeled on a super Klein geometry $(\mathcal{G}, \mathcal{H})$. Let $\mathrm{pr}_{\mathfrak{h}} : \mathrm{Lie}(\mathcal{G}) \rightarrow \mathrm{Lie}(\mathcal{H})$ denote the projection of $\mathrm{Lie}(\mathcal{G})$ onto the super Lie sub module $\mathrm{Lie}(\mathcal{H})$. Then, $\mathrm{pr}_{\mathfrak{h}} \circ \mathcal{A} \in \Omega^1(\mathcal{P}_{/\mathcal{S}}, \mathfrak{h})_0$ defines a super connection 1-form on $\mathcal{P}_{/\mathcal{S}}$. Let the super Cartan geometry, in addition, be reductive and $\mathrm{pr}_{\mathfrak{g}/\mathfrak{h}}$ denote the projection of $\mathrm{Lie}(\mathcal{G})$ onto the super Λ-vector space $\Lambda \otimes \mathfrak{g}/\mathfrak{h}$. Then,*

$$E := \mathrm{pr}_{\mathfrak{g}/\mathfrak{h}} \circ \mathcal{A} \tag{3.25}$$

called super soldering form *or* supervielbein*, defines an even horizontal $\Lambda \otimes \mathfrak{g}/\mathfrak{h}$-valued 1-form on $\mathcal{P}_{/\mathcal{S}}$ of type $(\mathcal{H}, \mathrm{Ad})$.*

Proof. That $\omega := \mathrm{pr}_{\mathfrak{h}} \circ \mathcal{A}$ defines a super connection 1-form in the sense of Ehresmann is immediate by condition (i) and (ii) of a super Cartan connection. Furthermore, if the Cartan geometry is reductive, $\Lambda \otimes \mathfrak{g}/\mathfrak{h}$ defines a $\mathrm{Ad}(\mathcal{H})$-invariant super Λ-module. Hence, by condition (ii) of a super Cartan connection, the super soldering form E yields a well-defined $\mathcal{H}$-equivariant 1-form on $\mathcal{P}_{/\mathcal{S}}$. To see that is horizontal, let $\widetilde{X}$ be a fundamental vector field on $\mathcal{P}_{/\mathcal{S}}$ generated by $X \in \mathfrak{h}$. Then, by condition (i), it follows

$$X = \langle\widetilde{X}|\mathcal{A}\rangle = \langle\widetilde{X}|\omega\rangle + \langle\widetilde{X}|E\rangle = X + \langle\widetilde{X}|E\rangle \tag{3.26}$$

Hence, $\langle\widetilde{X}|E\rangle = 0$ proving that E is horizontal. □

Proposition 3.3.9. *Let $(\pi_{\mathcal{S}} : \mathcal{P}_{/\mathcal{S}} \rightarrow \mathcal{M}_{/\mathcal{S}}, \mathcal{A})$ be a reductive super Cartan geometry modeled on a super Klein geometry $(\mathcal{G}, \mathcal{H})$ with super Cartan connection $\mathcal{A}$. Let $E := \mathrm{pr}_{\mathfrak{g}/\mathfrak{h}} \circ \mathcal{A}$ be the super soldering form as defined in Prop. 3.3.8. Let $\iota_{\mathcal{P}} : \mathcal{P} \hookrightarrow \mathcal{S} \times \mathcal{P}$*

furthermore be an embedding. Then, the pullback $\theta := \iota_{\mathcal{P}}^* E \in \Omega^1_{hor}(\mathcal{P}, \Lambda \otimes \mathfrak{g}/\mathfrak{h})^{(\mathcal{H},\mathrm{Ad})}$ *defines a non-degenerate* 1*-form and induces a smooth map*[4]

$$\iota : \mathcal{P} \to \mathscr{F}(\mathcal{M}),\ p \mapsto D_p\pi \circ \theta_p^{-1} \tag{3.27}$$

which is fiber-preserving and $\mathcal{H}$*-equivariant in the sense that* $\iota \circ \Phi = \Upsilon \circ (\iota \times \mathrm{Ad})$ *with* $\mathrm{Ad} : \mathcal{H} \to \mathrm{GL}(\Lambda \otimes \mathfrak{g}/\mathfrak{h})$ *the Adjoint action and* Φ *and* Υ *the group right actions on* $\mathcal{P}$ *and* $\mathscr{F}(\mathcal{M})$*, respectively. In particular,* $\mathcal{P}$ *defines a* $\mathcal{H}$*-reduction of the frame bundle* $\mathscr{F}(\mathcal{M})$*.*

Proof. By the previous proposition, it is clear that θ defines a horizontal 1-form on $\mathcal{P}$ of type $(\mathcal{H}, \mathrm{Ad})$. Let $\mathscr{H} := \ker(\iota_{\mathcal{P}}^*\omega)$ be the horizontal distribution induced by the pullback of the super connection 1-form $\omega := \mathrm{pr}_{\mathfrak{h}} \circ \mathcal{A}$. Then, by the super Cartan condition (iii) in Def. 3.3.3, it follows that $\theta_p : \mathscr{H}_p \to \Lambda \otimes \mathfrak{g}/\mathfrak{h}$, for any $p \in \mathcal{P}$, yields an isomorphism of free super Λ-modules. Moreover, the pushforward of the bundle projection induces an isomorphism $D_p\pi : \mathscr{H}_p \xrightarrow{\sim} T_p\mathcal{M}$. Hence, this in turn induces an isomorphism

$$D_p\pi \circ \theta_p^{-1} : \Lambda \otimes \mathfrak{g}/\mathfrak{h} \to T_p\mathcal{M} \tag{3.28}$$

that is, a linear frame at p. It thus remains to show that (3.27) indeed defines a $\mathcal{H}$-reduction of $\mathscr{F}(\mathcal{M})$. To this end, note that, by condition (ii) for a super Cartan connection, we have $\theta(\Phi_{h*}(Y_p)) = \Phi_h^*\theta(Y_p) = \mathrm{Ad}_{h^{-1}}(\theta(Y_p))\ \forall Y_p \in T_p\mathcal{P}$ and $h \in \mathcal{H}$. Hence, this yields

$$\iota(p \cdot h) = D_{ph}\pi \circ \theta_{ph}^{-1} = D_{ph}\pi \circ D_p\Phi_h \circ \theta_p^{-1} \circ \mathrm{Ad}_h = \iota(p) \circ \mathrm{Ad}_h \tag{3.29}$$

which proves that ι is $\mathcal{H}$-equivariant. □

Remark 3.3.10. Under the assumptions of Prop. 3.3.9, let $\iota : \mathcal{P} \to \mathcal{S} \times \mathcal{P}$ be an embedding and suppose $\mathcal{P}$ is trivial, i.e., $\mathcal{P} \cong \mathcal{M} \times \mathcal{G}$ with respect to a global trivialization $s : \mathcal{M} \to \mathcal{P}$. Let $(e_i)_i$ be a homogeneous basis of the quotient super vector space $\mathfrak{m} := \mathfrak{g}/\mathfrak{h}$. This in turn induces a (homogeneous) basis $(s_i)_i$ of global sections $s_i := [s, e_i]$ of the associated super vector bundle $\mathcal{E} := \mathcal{P} \times_{\mathcal{H}} \mathcal{V}$ with $\mathcal{V} := \Lambda \otimes \mathfrak{m}$. This yields an isomorphism

$$\begin{aligned} \Omega^1(\mathcal{M}, \mathcal{V}^*) \to \Omega^1(\mathcal{M}, \mathcal{E}) &\cong \Omega^1_{hor}(\mathcal{P}, \mathcal{V})^{(\mathcal{H},\mathrm{Ad})} \\ \omega \mapsto \omega^i s_i & \end{aligned} \tag{3.30}$$

4 Note that, in this definition, the soldering form θ is regarded as right linear morphism which is possible as θ is even.

It thus follows from condition (iii) of a super Cartan connection that, via (3.30), the pullback $\iota^*E \in \Omega^1_{hor}(\mathcal{P},\mathcal{V})$ induces a non-degenerate 1-form $\tilde{E} \in \Omega^1(\mathcal{M},\mathcal{V}^*)$. Consequently, the pair $(\mathcal{M},\tilde{E})$ defines super Cartan structure in the sense of [101]. Conversely, if $(\mathcal{M},\tilde{E})$ is a super Cartan structure with $\tilde{E} \in \Omega^1(\mathcal{M},\mathcal{V})$ being non-degenerate, one can use (3.30) to get a non-degenerate 1-form $E \in \Omega^1_{hor}(\mathcal{P},\mathcal{V})$ which can be lifted trivially to a $\mathcal{S}$-relative 1-form $\mathbb{1}\otimes E \in \Omega^1_{hor}(\mathcal{P}_{/\mathcal{S}},\mathcal{V})$ satisfying the super Cartan condition (iii). Hence, definition (3.3.3) provides a generalization of super Cartan structures in the sense of [101] to a generalized notion of super Cartan connections on nontrivial $\mathcal{S}$-relative principal super fiber bundles.

Corollary 3.3.11. *Let $(\pi_{\mathcal{S}}:\mathcal{P}_{/\mathcal{S}}\to\mathcal{M}_{/\mathcal{S}},\mathcal{A})$ be a metric reductive super Cartan geometry modeled on a super Klein geometry $(\mathcal{G},\mathcal{H})$ with a super Cartan connection $\mathcal{A}$ and smooth super metric $\mathscr{S}$ on $\Lambda\otimes\mathfrak{g}/\mathfrak{h}$. Let $\theta := \iota_{\mathcal{P}}^*E \in \Omega^1(\mathcal{P},\Lambda\otimes\mathfrak{g}/\mathfrak{h})^{(\mathcal{H},\mathrm{Ad})}$ be the pullback of the super soldering form to the principal super fiber bundle $\mathcal{P}$ w.r.t. an embedding $\iota_{\mathcal{P}}:\mathcal{P}\hookrightarrow\mathcal{S}\times\mathcal{P}$. For any $x\in\mathcal{M}$, consider the map $g_x: T_x\mathcal{M}\times T_x\mathcal{M}\to\Lambda^{\mathbb{C}}$ defined as*

$$g(X_x,Y_x) := \mathscr{S}(\theta_p(X^*_p),\theta(Y^*_p)) \tag{3.31}$$

for any $p\in\mathcal{P}_x$ and $X^*_p,Y^*_p\in T_p\mathcal{P}$ the unique horizontal tangent vectors such that $D_p\pi(X^*_p)=X_x$ and $D_p\pi(Y^*_p)=Y_x$. Then, g_x is a well-defined super metric on $T_x\mathcal{M}$ for any $x\in\mathcal{M}$. In particular, the assignment $g:\mathcal{M}\ni x\mapsto g_x$ defines a smooth super metric on $\mathcal{M}$.

Proof. Since $D_p\pi:\mathscr{H}_p\xrightarrow{\sim}T_p\mathcal{M}$ is an isomorphism of super Λ-modules for any $p\in\mathcal{P}$, where $\mathscr{H}:=\ker(\iota_{\mathcal{P}}^*\omega)$ is the horizontal distribution induced by the pullback of the super connection 1-form $\omega:=\mathrm{pr}_{\mathfrak{h}}\circ\mathcal{A}$., it is clear that, for any tangent vectors $X_x,Y_x\in T_x\mathcal{M}$, the horizontal lifts $X^*_p,Y^*_p\in T_p\mathcal{P}$ exist and are unique. Moreover, as θ_p is a right linear isomorphism of super Λ-modules, it is clear that g_x defines a super metric on $T_x\mathcal{M}$ once we have shown that g_x is well-defined. To this end, let $p'\in\mathcal{P}_x$ be another point on the fiber over x. Then, there exists $g\in\mathcal{P}$ such that $p'=p\cdot g$. By uniqueness, it follows $X^*_{p\cdot g}=\Phi_{g*}X_p$ and $Y^*_{p\cdot g}=\Phi_{g*}Y_p$. Thus,

$$\theta_{p'}(X^*_{p'})=\theta_{p\cdot g}(\Phi_{g*}X_p)=\mathrm{Ad}_{g^{-1}}(\theta_p(X^*_p)) \tag{3.32}$$

and similarly for Y^*. Thus, as $\mathscr{S}$ is $\mathrm{Ad}(\mathcal{H})$-invariant, it follows that g_x is indeed well-defined. To see that $g:\mathcal{M}\ni x\mapsto g_x$ is smooth, let $s:\mathcal{M}\supseteq U\to\mathcal{P}$ be a local section and $(e_i)_i$ be a homogeneous basis of $\mathfrak{g}/\mathfrak{h}$. Then, on U, the super metric is given by

$$g(X,Y)=\mathscr{S}(s^*\theta(X),s^*\theta(Y))=(-1)^{(|e_i|+|X|)|e_j|}\mathscr{S}_{ij}(s^*\theta)(X)^i(s^*\theta)(Y)^j \tag{3.33}$$

where $\mathscr{S}_{ij} := \mathscr{S}(e_i, e_j) \in \mathbb{C}$ as $\mathscr{S}$ is smooth and we have made the expansion $s^*\theta(X) = e_i \otimes (s^*\theta)(X)^i$ with $(s^*\theta)(X)^i$ smooth functions on U and similarly for $s^*\theta(Y)$. Thus, it follows that $g(X,Y) \in H^\infty(U) \otimes \mathbb{C}$ proving that g is smooth. □

The following proposition demonstrates the strong link between super Cartan connections and Ehresmann connections defined on associated ($\mathcal{S}$-relative) principal super fiber bundles (for a discussion in the category of ordinary smooth manifolds see, e.g., [140]). In physics, this thus provides a concrete relation between Cartan geometries and Yang-Mills gauge theories. This is due to the fact that, by definition, both type of connections turn out to be already fixed uniquely on vertical vector fields, i.e., vector fields tangent to the fibers of the underlying principal super fiber bundles. Indeed, on vertical fields, both connections are related to the Maurer-Cartan forms on the respective structure groups. Using this observation, one then arrives at the following.

Proposition 3.3.12. *Let $\mathcal{H} \to \mathcal{P}_{/\mathcal{S}} \to \mathcal{M}_{/\mathcal{S}}$ be a $\mathcal{S}$-relative principal super fiber bundle with structure group $\mathcal{H}$ as well as $(\mathcal{G}, \mathcal{H})$ a super Klein geometry. Then, there is a bijective correspondence between generalized super Cartan connections in $\Omega^1(\mathcal{P}_{/\mathcal{S}}, \mathfrak{g})_0$ and super connection 1-forms in $\Omega^1(\mathcal{P}_{/\mathcal{S}} \times_{\mathcal{H}} \mathcal{G}, \mathfrak{g})_0$ with $\mathcal{P}_{/\mathcal{S}} \times_{\mathcal{H}} \mathcal{G} := (\mathcal{P} \times_{\mathcal{H}} \mathcal{G})_{/\mathcal{S}}$ the $\mathcal{G}$-extension of $\mathcal{P}_{/\mathcal{S}}$.*

Proof. The following proof is a generalization of the proof given in [140] to the super category. One direction is immediate, i.e., given a $\mathcal{S}$-relative super connection 1-form $\mathcal{A}$ on $\mathcal{P}_{/\mathcal{S}} \times_{\mathcal{H}} \mathcal{G}$, the pullback $\hat{\iota}^*\mathcal{A}$ w.r.t. the embedding $\hat{\iota} := \mathrm{id} \times \iota : \mathcal{P}_{/\mathcal{S}} \to \mathcal{P}_{/\mathcal{S}} \times_{\mathcal{H}} \mathcal{G}$, with ι as defined in Corollary 2.4.25, yields a generalized super Cartan connection on $\mathcal{P}_{/\mathcal{S}}$ according to Def. 3.3.3. Conversely, suppose $\mathcal{A} \in \Omega^1(\mathcal{P}_{/\mathcal{S}}, \mathfrak{g})_0$ is a generalized super Cartan connection. Let $\hat{\pi} : \mathcal{P}_{/\mathcal{S}} \times \mathcal{G} \to \mathcal{P}_{/\mathcal{S}} \times_{\mathcal{H}} \mathcal{G}$ be the canonical projection. If $\hat{\Phi}_{\mathcal{S}}$ denotes the $\mathcal{G}$-right action on $\mathcal{P}_{/\mathcal{S}} \times_{\mathcal{H}} \mathcal{G}$, it follows that the fundamental vector fields are given by

$$\begin{aligned}\widetilde{Y}_{[p,g]} &= (\hat{\Phi}_{\mathcal{S}})_{[p,g]*}(Y_e) = D_{([p,g],e)}\hat{\Phi}_{\mathcal{S}}(0_{[p,g]}, Y)\\ &= D_{([p,g],e)}\hat{\Phi}_{\mathcal{S}}(D_{(p,g)}\hat{\pi}(0_p, 0_g), Y) = D_{(p,g,e)}(\hat{\Phi}_{\mathcal{S}} \circ (\hat{\pi} \times \mathrm{id}))(0_p, 0_g, Y)\\ &= D_{(p,g,e)}(\hat{\pi} \circ (\mathrm{id} \times \mu_{\mathcal{G}}))(0_p, 0_g, Y) = D_{(p,g)}\hat{\pi}(0_p, \mu_{\mathcal{G}}(0_g, Y))\\ &= D_{(p,g)}\hat{\pi}(0_p, L_{g*}Y) = D_{(p,g)}\hat{\pi}(0_p, L_{g*}Y)\end{aligned} \tag{3.34}$$

for any $Y \in \mathrm{Lie}(\mathcal{G})$ and $p \in \mathcal{P}_{/\mathcal{S}}$, $g \in \mathcal{G}$, where the generalized tangent map was used at various stages. Furthermore, for any $X_p \in T_p(\mathcal{P}_{/\mathcal{S}})$, one has

$$\begin{aligned}D_{(p,g)}\hat{\pi}(X_p, 0_g) &= D_{(p,g)}\hat{\pi}(X_p, D_{(g,e)}\mu_{\mathcal{G}}(0_g, 0_e))\\ &= D_{(p,g,e)}(\hat{\pi} \circ (\mathrm{id} \times \mu_{\mathcal{G}}))(X_p, 0_g, 0_e)\end{aligned}$$

$$
\begin{aligned}
&= D_{(p,g,e)}(\hat{\pi} \circ (\hat{\Phi}_{/\mathcal{S}} \times \mathrm{id}))(X_p, 0_g, 0_e) \\
&= D_{(p\cdot g,e)}\hat{\pi}((\hat{\Phi}_{/\mathcal{S}})_{g*}X_p, 0_e) \\
&= \hat{\iota}_*((\Phi_{\mathcal{S}})_{g*}X_p) \qquad (3.35)
\end{aligned}
$$

$\forall(p, g) \in \mathcal{P}_{/\mathcal{S}} \times \mathcal{G}$. Hence, this yields

$$
\begin{aligned}
D_{(p,g)}\hat{\pi}(X_p, Y_g) &= D_{(p,g)}\hat{\pi}(X_p, 0_g) + D_{(p,g)}\hat{\pi}(0_p, Y_g) \\
&= \hat{\iota}_*((\Phi_{\mathcal{S}})_{g*}X_p) + D_{(p,g)}\hat{\pi}(0_p, L_{g*} \circ L_{g^{-1}*}(Y_g)) \\
&= \iota_*((\Phi_{\mathcal{S}})_{g*}X_p) + \widetilde{\langle Y_g|\theta_{\mathrm{MC}}\rangle}_{[p,g]} \qquad (3.36)
\end{aligned}
$$

Therefore, if there exists a super connection 1-form $\hat{\iota}_*\mathcal{A}$ whose pullback under $\hat{\iota}$ is given by $\mathcal{A}$, then it necessarily has to be of the form

$$
\langle D_{(p,g)}\hat{\pi}(X_p, Y_g)|\hat{\iota}_*\mathcal{A}_{[p,g]}\rangle = \mathrm{Ad}_{g^{-1}}\langle X_p|\mathcal{A}_p\rangle + \langle Y_g|\theta_{\mathrm{MC}}\rangle \qquad (3.37)
$$

In particular, as $\hat{\pi}$ is a submersion, it is uniquely determined by (3.37). Hence, it remains to show that $\iota_*\mathcal{A}$ as given by via (3.37) is in fact well-defined and provides a super connection 1-form on $\mathcal{P}_{/\mathcal{S}} \times_{\mathcal{H}} \mathcal{G}$.

To see that it is well-defined, note that, for any $(p, g) \in \mathcal{P}_{/\mathcal{S}} \times \mathcal{G}$, the kernel of $D_{(p,g)}\hat{\pi}$ is given by $\{(\widetilde{Y}_p, -R_{g*}Y)|\, Y \in \mathrm{Lie}(\mathcal{H})\} \subset T_p(\mathcal{P}_{/\mathcal{S}}) \times T_g\mathcal{G}$[5]. This yields

$$
\begin{aligned}
\langle(\widetilde{Y}_p, -R_{g*}Y)|\iota_*\mathcal{A}_{[p,g]}\rangle &= \mathrm{Ad}_{g^{-1}}\langle\widetilde{Y}_p|\mathcal{A}_p\rangle - \langle R_{g*}Y|\theta_{\mathrm{MC}}\rangle \\
&= \mathrm{Ad}_{g^{-1}}(Y) - L_{g^{-1}*} \circ R_{g*}(Y) = 0 \qquad (3.38)
\end{aligned}
$$

$\forall Y \in \mathrm{Lie}(\mathcal{H})$. Finally, to see that is independent of the choice of a representative of $[p, g] \in \mathcal{P}_{/\mathcal{S}} \times_{\mathcal{H}} \mathcal{G}$, we compute $\forall h \in \mathcal{H}$

$$
\begin{aligned}
&\langle D_{(ph,h^{-1}g)}\hat{\pi}((\hat{\Phi}_{\mathcal{S}})_{h*}X_p, L_{h^{-1}*}Y_g)|\iota_*\mathcal{A}_{[ph,h^{-1}g]}\rangle \\
&= \mathrm{Ad}_{(h^{-1}g)^{-1}}\langle(\hat{\Phi}_{\mathcal{S}})_{h*}X_p|\mathcal{A}_{ph}\rangle + \langle L_{h^{-1}*}Y_g|\theta_{\mathrm{MC}}\rangle \\
&= \mathrm{Ad}_{g^{-1}}\langle X_p|\mathcal{A}_p\rangle + L_{(h^{-1}g)^{-1}*} \circ L_{h^{-1}*}(Y_g) \\
&= \mathrm{Ad}_{g^{-1}}\langle X_p|\mathcal{A}_p\rangle + \langle Y_g|\theta_{\mathrm{MC}}\rangle \\
&= \langle D_{(p,g)}\hat{\pi}(X_p, Y_g)|\iota_*\mathcal{A}_{[p,g]}\rangle \qquad (3.39)
\end{aligned}
$$

[5] This may be checked by direct computation using the local trivializations $\{(\pi_{\mathcal{E}}^{-1}(U_\alpha), \widetilde{\phi}_\alpha)\}_{\alpha\in\Upsilon}$ of the bundle $\pi : \mathcal{P} \times \mathcal{G} \to \mathcal{P} \times_{\mathcal{H}} \mathcal{G}$ as defined in the proof of Prop. 2.4.21.

This shows that $\hat{\iota}_*\mathcal{A}$ is in fact well-defined. To see that it is also $\mathcal{G}$-equivariant, we compute

$$\begin{aligned}
&\langle(\hat{\Phi}_{\mathcal{S}})_{h*}D_{(p,g)}\pi(X_p,Y_g)|\hat{\iota}_*\mathcal{A}_{[p,gh]}\rangle \\
&=\langle D_{(p,g,h)}(\hat{\Phi}_{\mathcal{S}}\circ(\hat{\pi}\times\mathrm{id}))(X_p,Y_g,0_h)|\iota_*\mathcal{A}_{[p,g]}\rangle \\
&=\langle D_{(p,g,h)}(\hat{\pi}\circ(\mathrm{id}\times\mu_{\mathcal{G}}))(X_p,Y_g,0_h)|\iota_*\mathcal{A}_{[p,gh]}\rangle \\
&=\langle D_{(p,gh)}\hat{\pi}(X_p,R_{h*}Y_g)|\iota_*\mathcal{A}_{[p,gh]}\rangle \\
&=\mathrm{Ad}_{(gh)^{-1}}\langle X_p|\mathcal{A}_p\rangle+\langle R_{h*}Y_g|\theta_{\mathrm{MC}}\rangle \\
&=\mathrm{Ad}_{h^{-1}}(\mathrm{Ad}_{g^{-1}}\langle X_p|\mathcal{A}_p\rangle)+L_{(gh)^{-1}*}\circ R_{h*}(Y_g) \\
&=\mathrm{Ad}_{h^{-1}}(\mathrm{Ad}_{g^{-1}}\langle X_p|\mathcal{A}_p\rangle+\langle Y_g|\theta_{\mathrm{MC}}\rangle) \\
&=\mathrm{Ad}_{h^{-1}}\langle D_{(p,g)}\hat{\pi}(X_p,Y_g)|\hat{\iota}_*\mathcal{A}_{[p,g]}\rangle
\end{aligned}\tag{3.40}$$

Finally, for $Y\in\mathrm{Lie}(\mathcal{G})$, it follows

$$\langle\widetilde{Y}_{[p,g]}|\hat{\iota}_*\mathcal{A}_{[p,g]}\rangle=\langle D_{(p,g)}\hat{\pi}(X_p,L_{g*}Y)|\iota_*\mathcal{A}_{[p,g]}\rangle=\langle L_{g*}Y|\theta_{\mathrm{MC}}\rangle=Y\tag{3.41}$$

Hence, this proves that $\iota_*\mathcal{A}\in\Omega^1(\mathcal{P}_{/\mathcal{S}}\times_{\mathcal{H}}\mathcal{G},\mathfrak{g})_0$ is a well-defined super connection 1-form on $\mathcal{P}_{/\mathcal{S}}\times_{\mathcal{H}}\mathcal{G}$. □

3.4. Supergravity as super Cartan geometry and the Castellani-D'Auria-Fré approach

The content of this section has been reproduced from [1], with slight changes to account for the context of this thesis with the permission of AIP Publishing.

With these preparations, let us turn next to supergravity. We want to describe $D=4$, $\mathcal{N}=1$ Poincaré supergravity as a metric reductive super Cartan geometry $(\pi_{\mathcal{S}}:\mathcal{P}_{/\mathcal{S}}\to\mathcal{M}_{/\mathcal{S}},\mathcal{A})$ modeled on the super Klein geometry $(\mathcal{G},\mathcal{H})=(\mathrm{ISO}(\mathbb{R}^{1,3|4}),\mathrm{Spin}^+(1,3))$ corresponding to super Minkowski spacetime (see Example 2.3.10 and 2.3.11). Here and in the following, for notational simplification, we will often identify $\mathrm{Spin}^+(1,3)$ with the corresponding bosonic split super Lie group $\mathbf{S}(\mathrm{Spin}^+(1,3))$. Let us split the super Cartan connection according to the decomposition $\mathfrak{g}=\mathfrak{g}/\mathfrak{h}\oplus\mathfrak{h}$ of the super Lie algebra of $\mathcal{G}$ yielding

$$\mathcal{A}=\mathrm{pr}_{\mathfrak{g}/\mathfrak{h}}\circ\mathcal{A}+\mathrm{pr}_{\mathfrak{h}}\circ\mathcal{A}=:E+\omega\tag{3.42}$$

with E the supervielbein. According to Prop. 3.3.8, ω defines a super connection 1-form in the sense of Ehresmann (Definition 2.5.19) whereas the supervielbein provides an even horizontal 1-form of type $(\mathcal{H},\mathrm{Ad})$, i.e., $E\in\Omega^1_{hor}(\mathcal{P}_{/\mathcal{S}},\Lambda\otimes\mathfrak{g}/\mathfrak{h})_0^{(\mathcal{H},\mathrm{Ad})}$.

Let $s \in \mathbf{B}(\mathcal{S})$ and $\iota_{\mathcal{P}} : \mathcal{P} \to \mathcal{S} \times \mathcal{P} : p \mapsto (s, p)$ be a smooth embedding. This induces a smooth horizontal 1-form $\iota_{\mathcal{P}}^* E$ on $\mathcal{P}$ which, by the super Cartan condition, is non-degenerate. Furthermore, it follows that $\iota_{\mathcal{P}}^*\omega \in \Omega^1(\mathcal{P}, \mathfrak{h})_0$ defines an ordinary super connection 1-form on $\mathcal{P}$. According to Prop. 3.3.9, this induces a $\mathcal{H}$-equivariant morphism between $\mathcal{P}$ and the frame bundle $\mathscr{F}(\mathcal{M})$ via

$$\mathcal{P} \to \mathscr{F}(\mathcal{M}),\ p \mapsto D_p\pi \circ E_p^{-1} \tag{3.43}$$

where, for any $p \in \mathcal{P}$, $D_p\pi \circ E_p^{-1} : (\mathfrak{g}/\mathfrak{h}) \otimes \Lambda \xrightarrow{\sim} T_{\pi(p)}\mathcal{M}$ is an isomorphism of free super Λ-modules. Hence, $\mathcal{P}$ defines a $\mathcal{H}$-reduction of $\mathscr{F}(\mathcal{M})$. Applying the body functor, this in turn induces a $\mathrm{Spin}^+(1,3)$-reduction $P := \mathbf{B}(\mathcal{P}) \to \mathscr{F}(M)$ of the frame bundle of the body $M := \mathbf{B}(\mathcal{M})$. That is, the body carries a spin structure. Moreover, it follows that $\mathcal{M}$ has the same dimension as the super vector space $\mathfrak{g}/\mathfrak{h} =: \mathfrak{t}$ where

$$\mathfrak{t} \equiv \mathfrak{t}^{1,3|4} := \mathbb{R}^{1,3} \oplus \Delta_{\mathbb{R}} \tag{3.44}$$

denotes the super Lie algebra of the super translation group $\mathcal{T}^{1,3|4}$ (Example 2.3.10). Hence, the supervielbein can be further split in the following way

$$E = \mathrm{pr}_{\mathfrak{t}} \circ \mathcal{A} =: e + \psi =: e^I P_I + \psi^\alpha Q_\alpha \tag{3.45}$$

with $\psi \in \Omega^1(\mathcal{P}_{/\mathcal{S}}, \Delta_{\mathbb{R}})$ and $e \in \Omega^1(\mathcal{P}_{/\mathcal{S}}, \mathbb{R}^{1,3})$ defining even horizontal 1-forms of type $(\mathrm{Spin}^+(1,3), \mathrm{Ad})$ called the *Rarita-Schwinger field* and *co-frame*, respectively. As a consequence, the super Cartan connection takes the form

$$\mathcal{A} = e^I P_I + \frac{1}{2}\omega^{IJ} M_{IJ} + \psi^\alpha Q_\alpha \tag{3.46}$$

The *super Cartan curvature* of the super Cartan connection $\mathcal{A}$ is defined as

$$F(\mathcal{A}) = \mathrm{d}\mathcal{A} + \frac{1}{2}[\mathcal{A} \wedge \mathcal{A}] = \mathrm{d}\mathcal{A} + \frac{1}{2}(-1)^{|T_{\underline{A}}||T_{\underline{B}}|}\mathcal{A}^{\underline{A}} \wedge \mathcal{A}^{\underline{B}} \otimes [T_{\underline{A}}, T_{\underline{B}}] \tag{3.47}$$

w.r.t. a homogeneous basis $(T_{\underline{A}})_{\underline{A}}$, $\underline{A} \in (I, IJ, \alpha)$, of the super Porincaré algebra $\mathfrak{iso}(\mathbb{R}^{1,3|4})$ (Example 2.3.11) where the minus sign in (3.47) appears due to the (anti) commutation of $T_{\underline{A}}$ and $\mathcal{A}^{\underline{B}}$. It then follows from $[M_{IJ}, P_K] = \eta_{IK}P_J - \eta_{JK}P_I$ as well as (2.96)-(2.99) that the components of $F(\mathcal{A})$ in the translational part of the super Lie algebra, also called the *supertorsion*, take the form

$$\begin{aligned} F(\mathcal{A})^I &= \mathrm{d}e^I + \omega^I{}_J \wedge e^J + \frac{1}{4}((-1)^{|Q_\alpha||Q_\beta|}\psi^\alpha \wedge \psi^\beta \otimes [Q_\alpha, Q_\beta])^I \\ &= \Theta^{(\omega)I} - \frac{1}{4}\bar{\psi} \wedge \gamma^I \psi \end{aligned} \tag{3.48}$$

since $(-1)^{|Q_\alpha||Q_\beta|} = -1$, with $\Theta^{(\omega)}$ is the torsion 2-form associated to the spin connection ω. For the spinorial components, we find

$$F(\mathcal{A})^{IJ} = \mathrm{d}\omega^{IJ} + \omega^I{}_K \wedge \omega^{KJ} =: F(\omega)^{IJ} \tag{3.49}$$

with $F(\omega)$ the curvature of ω. Finally, for the odd part, we immediately obtain

$$F(\mathcal{A})^\alpha = \mathrm{d}\psi^\alpha + \frac{1}{4}\omega^{IJ}(\gamma_{IJ})^\alpha{}_\beta \wedge \psi^\beta = D^{(\omega)}\psi^\alpha \tag{3.50}$$

with $D^{(\omega)}\psi = \mathrm{d}\psi + \kappa_{\mathbb{R}*}(\omega) \wedge \psi$ the exterior covariant derivative in the Majorana representation.

With these preliminary considerations, let us finally state the action of $D = 4$, $\mathcal{N} = 1$ Poincaré supergravity. It can be derived from the MacDowell-Mansouri action of anti-de Sitter supergravity to be discussed in a more general context in Section 5.2 (see Eq. (5.3)) performing the Inönü-Wigner contraction, i.e., considering the limit $L \to \infty$ with L the anti-de Sitter radius corresponding to a vanishing cosmological constant (see also [138, 141]). Let $P \to M$ be the underlying ordinary $\mathrm{Spin}^+(1,3)$-bundle obtained after applying the body functor. Choosing a local section $s: M \supset U \to P \subset \mathcal{P}$, it follows that the action takes the form

$$S^{\mathcal{N}=1}(\mathcal{A}) = \frac{1}{2\kappa}\int_M s^*\mathscr{L} \tag{3.51}$$

with Langrangian $\mathscr{L} \in \Omega^4_{hor}(\mathcal{P}_{/\mathcal{S}})$ defining a horizontal form on $\mathcal{P}_{/\mathcal{S}}$ given by

$$\mathscr{L} := \frac{1}{2}F(\omega)^{IJ} \wedge e^K \wedge e^L \epsilon_{IJKL} + i\bar{\psi} \wedge \gamma_*\gamma_I D^{(\omega)}\psi \wedge e^I \tag{3.52}$$

In what follows, we want to study the local symmetries of the Lagrangian of Poincaré supergravity. To this end, we will adapt the "group-geometric" approach of Castellani-D'Auria-Fré. Before we proceed, however, we need to make some preparations.

Definition 3.4.1. Let $\mathcal{H} \to \mathcal{P}_{/\mathcal{S}} \to \mathcal{M}_{/\mathcal{S}}$ be a $\mathcal{S}$-relative principal super fiber bundle. On $\mathcal{P}_{/\mathcal{S}}$, the set of *infinitesimal automorphisms* is defined as

$$\mathfrak{aut}(\mathcal{P}_{/\mathcal{S}}) = \{X \in \Gamma(T\mathcal{P}_{/\mathcal{S}}) |\, (\Phi_\mathcal{S})_{h*}X = X,\ \forall h \in \mathcal{H}\} \tag{3.53}$$

Since the generalized tangent map commutes with the commutator between smooth vector fields, it follows that $\mathfrak{aut}(\mathcal{P}_{/\mathcal{S}})$ defines a proper super Lie subalgebra of the super Lie algebra $\Gamma(T\mathcal{P}_{/\mathcal{S}})$ of smooth sections of the tangent bundle of $\mathcal{P}_{/\mathcal{S}}$.

Suppose $X \in \Gamma(T\mathcal{M}_{/\mathcal{S}})$ is a smooth vector field. As the restriction of the bundle projection to the horizontal distribution $\mathscr{H} := \ker(\omega)$ induced by the Ehresmann connection ω yields an isomorphism to the tangent bundle of $\mathcal{M}_{/\mathcal{S}}$, it follows that there exists a unique horizontal lift $X^* \in \Gamma(T\mathcal{P}_{/\mathcal{S}})$ such that $X^*_p \in \mathscr{H}_p$ and $\pi_{\mathcal{S}*}X^* = X$. Then, by uniqueness, it follows $(\Phi_{\mathcal{S}})_{g*}X^* = X^*$ since $\pi_{\mathcal{S}} \circ \Phi_{\mathcal{S}} = \pi_{\mathcal{S}} \circ \mathrm{pr}_1$. The horizontal lift thus defines an infinitesimal automorphism on $\mathcal{P}_{/\mathcal{S}}$. Moreover, it follows that we can identify $\Gamma(T\mathcal{P}_{/\mathcal{S}}) \subset \mathfrak{aut}(\mathcal{P}_{/\mathcal{S}})$ with smooth horizontal vector fields on $\mathcal{P}_{/\mathcal{S}}$. Let us next consider the vertical counterpart.

Definition 3.4.2. The set $\mathfrak{gau}(\mathcal{P}_{/\mathcal{S}})$ of *vertical infinitesimal automorphisms* or *infinitesimal gauge transformations* is defined as the subset of $\mathfrak{aut}(\mathcal{P}_{/\mathcal{S}})$ given by

$$\mathfrak{gau}(\mathcal{P}_{/\mathcal{S}}) = \{X \in \mathfrak{aut}(\mathcal{P}_{/\mathcal{S}}) |\, X_p \in \mathscr{V}_p,\, \forall p \in \mathcal{P}_{/\mathcal{S}}\} \tag{3.54}$$

Again, since the generalized tangent map preserves the commutator, it follows that the commutator between infinitesimal gauge transformations is again an infinitesimal gauge transformation. Thus, $\mathfrak{gau}(\mathcal{P}_{/\mathcal{S}})$ defines a proper super Lie subalgebra of $\mathfrak{aut}(\mathcal{P}_{/\mathcal{S}})$.

Together with Proposition 2.7.18, we obtain the following.

Proposition 3.4.3. *There exists an isomorphism between infinitesimal gauge transformations* $\mathfrak{gau}(\mathcal{P}_{/\mathcal{S}})$ *and* $\mathcal{H}$*-equivariant smooth functions on* $\mathcal{S} \times \mathcal{P}$ *with values in* $\mathrm{Lie}(\mathcal{H})$ *via*

$$\Gamma(\mathrm{Ad}(\mathcal{P}_{/\mathcal{S}})) \cong H^\infty(\mathcal{S} \times \mathcal{P}, \mathrm{Lie}(\mathcal{H}))^{\mathcal{H}} \xrightarrow{\sim} \mathfrak{gau}(\mathcal{P}_{/\mathcal{S}}) \tag{3.55}$$

$$f \mapsto X : \, p \mapsto D_{(p,e)}\Phi_{\mathcal{S}}(0_p, f(p)) \tag{3.56}$$

where $\mathrm{Ad}(\mathcal{P}_{/\mathcal{S}}) := (\mathcal{P} \times_{\mathrm{Ad}} \mathrm{Lie}(\mathcal{H}))_{/\mathcal{S}}$ *is the* Adjoint bundle.

Remark 3.4.4. Note that, according to Prop. 2.7.14, one can identify global gauge transformations on $\mathscr{G}(\mathcal{P}_{/\mathcal{S}})$ with the set $H^\infty(\mathcal{S} \times \mathcal{P}, \mathcal{H})^{\mathcal{H}}$ which forms an abstract group via pointwise multiplication. Taking pointwise derivatives, this suggests that one may thus interpret $\mathfrak{gau}(\mathcal{P}_{/\mathcal{S}})$ as the super Lie algebra of the group of global gauge transformations $\mathscr{G}(\mathcal{P}_{/\mathcal{S}})$ if one may be able to equip it with a smooth supermanifold structure. An explicit proof, however, requires the study of infinite-dimensional supermanifolds (see for instance [99, 100, 110] for recent results in this direction).

With these preparations, let us discuss the local symmetries of Poincaré supergravity. Since, the Lagrangian $\mathscr{L}$ is pulled back to the body $M = \mathbf{B}(\mathcal{M})$ of the base supermanifold $\mathcal{M}$, we only have to require that $\mathscr{L}$ is invariant after restriction to the bosonic subbundle $\mathcal{P}_0$ of $\mathcal{P}$ defined as $\mathcal{P}_0 := \mathbf{S}(\mathbf{B}(\mathcal{P}))$. Hence, let $X \in \mathfrak{aut}(\mathcal{P}_{/\mathcal{S}})$ be an infinitesimal automorphism. We say that X defines a *local symmetry* iff

$$\delta_X\mathscr{L}|_{\mathcal{P}_0} := \iota^*_{\mathcal{P}_0}(L_X\mathscr{L}) = \mathrm{d}\alpha \tag{3.57}$$

for some smooth 3-form $\alpha \in \Omega^3((\mathcal{P}_0)_{/\mathcal{S}})$. In order to compute the variation of the Lagrangian, one has to determine the variations of the individual field components. To this end, note that the super Cartan connection induces an isomorphism

$$\mathcal{A}: \mathfrak{aut}(\mathcal{P}_{/\mathcal{S}}) \to H^\infty(\mathcal{S}\times\mathcal{P}, \mathrm{Lie}(\mathcal{G}))^{\mathcal{H}},\ X \mapsto \langle X|\mathcal{A}\rangle \tag{3.58}$$

from the super Lie algebra of infinitesimal automorphisms to the super Lie algebra of $\mathcal{H}$-equivariant smooth functions with values in $\mathrm{Lie}(\mathcal{G})$. In fact, for any $X \in \mathfrak{aut}(\mathcal{P}_{/\mathcal{S}})$, we have $(\Phi_{\mathcal{S}})_{h*}X = X\ \forall h \in \mathcal{H}$ which implies

$$\langle X|\mathcal{A}\rangle\,(p\cdot h) = \langle X_p|(\Phi_{\mathcal{S}})^*_h\mathcal{A}\rangle = \mathrm{Ad}_{h^{-1}}\,\langle X|\mathcal{A}\rangle\,(p) \tag{3.59}$$

Hence, it follows that the variation of the super Cartan connection takes the form

$$\delta_X\mathcal{A} := L_X\mathcal{A} = \iota_X F(\mathcal{A}) + D^{(\mathcal{A})}(\iota_X\mathcal{A}) \tag{3.60}$$

where, according to Remark 2.5.25, $D^{(\mathcal{A})}(\iota_X\mathcal{A}) = \mathrm{d}(\iota_X\mathcal{A}) + [\mathcal{A}\wedge\iota_X\mathcal{A}]$. To see this, choosing a homogeneous basis $(T_{\underline{A}})_{\underline{A}}$ of $\mathfrak{g}$, a direct calculation yields (setting $|\underline{A}| \equiv |T_{\underline{A}}|$)

$$\begin{aligned}\iota_X[\mathcal{A}\wedge\mathcal{A}] &= (-1)^{|\underline{A}||\underline{B}|}\iota_X(\mathcal{A}^{\underline{A}}\wedge\mathcal{A}^{\underline{B}})\otimes[T_{\underline{A}},T_{\underline{B}}]\\ &= (-1)^{|\underline{A}||\underline{B}|}(\iota_X\mathcal{A}^{\underline{A}}\wedge\mathcal{A}^{\underline{B}} - (-1)^{|\underline{A}||X|}\mathcal{A}^{\underline{A}}\wedge\iota_X\mathcal{A}^{\underline{B}})\otimes[T_{\underline{A}},T_{\underline{B}}]\\ &= (-1)^{(|\underline{A}|+|X|)|\underline{B}|}\mathcal{A}^{\underline{B}}\wedge\iota_X\mathcal{A}^{\underline{A}}\otimes[T_{\underline{B}},T_{\underline{A}}]\\ &\quad - (-1)^{(|\underline{B}|+|X|)|\underline{A}|}\mathcal{A}^{\underline{A}}\wedge\iota_X\mathcal{A}^{\underline{B}}\\ &= -2[\mathcal{A}\wedge\iota_X\mathcal{A}]\end{aligned} \tag{3.61}$$

which immediately gives (3.60). Since $\mathscr{L}$ is obviously invariant under global $\mathrm{Spin}^+(1,3)$-gauge transformations by definition, i.e., it is horizontal, it follows that any $X \in \mathfrak{gau}(\mathcal{P}_{/\mathcal{S}})$, in particular, defines a local symmetry of the Lagrangian without even pulling it back to $\mathcal{P}_0$. That is, we have

$$\delta_X\mathscr{L} = 0, \quad \forall X \in \mathfrak{gau}(\mathcal{P}_{/\mathcal{S}}) \tag{3.62}$$

Since the Cartan curvature $F(\mathcal{A})$ is horizontal, this furthermore implies that the curvature contribution to the variations (3.60) of the Cartan connection vanish in general iff $X \in \mathfrak{gau}(\mathcal{P}_{/\mathcal{S}})$. In this case, the variation of the connection takes the form of an ordinary infinitesimal gauge transformation.

On the other hand, this is not immediately the case if X is horizontal and the curvature contributions do not vanish a priori. Let us briefly explain how these are treated in the Castellani-D'Auria-Fré approach to supergravity and how this approach may be related to the present formalism (for a detailed introduction to this fascinating subject see for instance [71, 72, 142] as well as [131, 132, 220] using the concept of integral forms).

The horizontal vector fields can be subdivided into two different categories. In fact, note that the super Poincaré algebra splits into three $\mathcal{H}$-equivariant subspaces $\mathfrak{iso}(\mathbb{R}^{1,3|4}) = \mathbb{R}^{1,3} \oplus \mathfrak{spin}^+(1,3) \oplus \Delta_{\mathbb{R}}$ so that one can decompose $H^\infty(\mathcal{S}\times\mathcal{P}, \mathrm{Lie}(\mathcal{G}))^{\mathcal{H}} \cong \Gamma(\mathcal{E}_{/\mathcal{S}})$ according to

$$\mathcal{E} := \mathcal{E}^{\mathbb{R}^{1,3}} \oplus \mathcal{E}^{\mathfrak{spin}} \oplus \mathcal{E}^{\Delta_{\mathbb{R}}} \tag{3.63}$$

where, for instance, $\mathcal{E}^{\Delta_{\mathbb{R}}} := \mathcal{P} \times_{\mathrm{Ad}} (\Lambda \otimes \Delta_{\mathbb{R}})$ denotes the associated super vector bundle corresponding to the real Majorana representation on $\Delta_{\mathbb{R}}$. Horizontal vector fields then correspond to sections of the bundle $\mathcal{E}^{\mathbb{R}^{1,3}} \oplus \mathcal{E}^{\Delta_{\mathbb{R}}}$. In the Castellani-D'Auria-Fré approach, horizontal vector fields corresponding to $\mathcal{E}^{\mathbb{R}^{1,3}}$ are typically referred to as infinitesimal spacetime translations while those corresponding to $\mathcal{E}^{\Delta_{\mathbb{R}}}$ are associated to supersymmetry transformations. In general, they do not provide local symmetries of the Lagrangian. In the Castellani-D'Auria-Fré approach, one then tries to resolve this by appropriately fixing the curvature contributions (or rather their pullback to $\mathcal{P}_0$) to the variations of the connection. These are typically referred to as the *horizontality* and *rheonomy conditions*. However, this cannot be done arbitrarily as, for instance, one has to ensure consistency with the Bianchi identity $D^{(\mathcal{A})}F(\mathcal{A}) = 0$.

To explain this in a bit more detail, following [142], note that the Lie derivative of the Lagrangian $L_X\mathscr{L} = \mathrm{d}(\iota_X\mathscr{L}) + \iota_X\mathrm{d}\mathscr{L}$ picks up an additional exact form so that, after pulling back to $\mathcal{P}_0$, condition (3.57) can be written in the equivalent form

$$(\iota_X \mathrm{d}\mathscr{L})|_{\mathcal{P}_0} = \mathrm{d}\alpha' \tag{3.64}$$

for some $\alpha' \in \Omega^3((\mathcal{P}_0)_{/\mathcal{S}})$. In order to compute the variation of the Lagrangian, one observes that its exterior derivative can be completely re-expressed in terms of the components (3.48) of the super Cartan curvature. In fact, exploiting the manifest $\mathrm{Spin}^+(1,3)$-invariance of the Lagrangian, it follows that

$$\begin{aligned}\mathrm{d}\mathscr{L} =& F(\omega)^{IJ} \wedge D^{(\omega)}e^K \wedge e^L \epsilon_{IJKL} + iD^{(\omega)}\bar{\psi} \wedge \gamma_*\gamma_I D^{(\omega)}\psi \wedge e^I \\ &- i\bar{\psi} \wedge \gamma_*\gamma_I D^{(\omega)}D^{(\omega)}\psi \wedge e^I - i\bar{\psi} \wedge \gamma_*\gamma_I D^{(\omega)}\psi \wedge D^{(\omega)}e^I\end{aligned}$$

$$
\begin{aligned}
=&F(\mathcal{A})^{IJ}\wedge F(\mathcal{A})^{K}\wedge e^{L}\epsilon_{IJKL}+i\bar{\rho}\wedge\gamma_{*}\gamma_{I}\rho\wedge e^{I}\\
&+\frac{1}{4}F(\omega)^{IJ}\wedge\bar{\psi}\wedge\gamma^{K}\psi\wedge e^{L}\epsilon_{IJKL}-\frac{1}{4}F(\omega)^{JK}\wedge\bar{\psi}\wedge\gamma_{*}\gamma_{JK}\psi\wedge e^{I}\\
&-i\bar{\psi}\wedge\gamma_{*}\gamma_{I}\rho\wedge F(\mathcal{A})^{I}-i\bar{\rho}\wedge\gamma_{*}\gamma_{I}\psi\wedge\bar{\psi}\wedge\gamma^{I}\psi
\end{aligned}
\tag{3.65}
$$

where we set $\rho^{\alpha}:=F(\mathcal{A})^{\alpha}$ and used the relation $\bar{\psi}\wedge\gamma_{*}\gamma_{I}\rho=\bar{\rho}\wedge\gamma_{*}\gamma_{I}\psi$. Moreover, in the second equality, we made use of the Bianchi identity

$$
D^{(\omega)}D^{(\omega)}\psi=\kappa_{\mathbb{R}*}(F(\omega))\wedge\psi=\frac{1}{4}F(\omega)^{IJ}\wedge\gamma_{IJ}\psi
\tag{3.66}
$$

Using the relation $\bar{\psi}\gamma_{*}\gamma_{I}\gamma_{JK}\psi=-\bar{\psi}\gamma_{*}\gamma_{JK}\gamma_{I}\psi$ which gives

$$
\bar{\psi}\wedge\gamma_{*}\gamma_{I}\gamma_{JK}\psi=\frac{1}{2}\bar{\psi}\wedge\gamma_{*}[\gamma_{I},\gamma_{JK}]_{+}\psi=i\epsilon_{IJKL}\bar{\psi}\wedge\gamma^{L}\psi
\tag{3.67}
$$

it follows that the third and forth term in the second equality of (3.65) exactly cancel. Furthermore, due to the *Fierz identity* $\gamma_{I}\psi\wedge\bar{\psi}\wedge\gamma^{I}\psi=0$, it follows that (3.64) reduces to

$$
\mathrm{d}\mathscr{L}=F(\mathcal{A})^{IJ}\wedge F(\mathcal{A})^{K}\wedge e^{L}\epsilon_{IJKL}+i\bar{\rho}\wedge\gamma_{*}\gamma_{I}\rho\wedge e^{I}-i\bar{\psi}\wedge\gamma_{*}\gamma_{I}\rho\wedge F(\mathcal{A})^{I}
\tag{3.68}
$$

Thus, indeed, the exterior derivative of the Lagrangian can be solely expressed in terms of the curvature of the super Cartan connection.

Let us consider variations of the Lagrangian which correspond to supersymmetry transformations. Thus, let $X\in\Gamma(\mathcal{E}^{\Delta_{\mathbb{R}}}_{/S})_{0}$ be an even smooth vector field and $\epsilon:=\langle X|\mathcal{A}\rangle$ be the corresponding $\mathrm{Spin}^{+}(1,3)$-equivariant function. The contraction of (3.68) with X then takes the form

$$
\begin{aligned}
\iota_{X}\mathrm{d}\mathscr{L}=&\iota_{X}F(\mathcal{A})^{IJ}\wedge F(\mathcal{A})^{K}\wedge e^{L}\epsilon_{IJKL}+F(\mathcal{A})^{IJ}\wedge\iota_{X}F(\mathcal{A})^{K}\wedge e^{L}\epsilon_{IJKL}\\
&+2i\bar{\rho}\wedge\gamma_{*}\gamma_{I}(\iota_{X}\rho)\wedge e^{I}-i\bar{\epsilon}\wedge\gamma_{*}\gamma_{I}\rho\wedge F(\mathcal{A})^{I}\\
&+i\bar{\epsilon}\wedge\gamma_{*}\gamma_{I}(\iota_{X}\rho)\wedge F(\mathcal{A})^{I}+i\bar{\psi}\wedge\gamma_{*}\gamma_{I}\rho\wedge\iota_{X}F(\mathcal{A})^{I}
\end{aligned}
\tag{3.69}
$$

Thus, when pulled back to the bosonic subbundle $\mathcal{P}_{0}$, we see that condition (3.64) for a local symmetry is satisfied if

$$
\iota_{X}F(\mathcal{A})^{I}=0,\quad \iota_{X}F(\mathcal{A})^{\alpha}=0,\quad \iota_{X}F(\mathcal{A})^{IJ}\wedge e^{K}\epsilon_{IJKL}=-i\bar{\rho}\gamma_{*}\gamma_{L}\epsilon
\tag{3.70}
$$

where, here and in the following, the pullback to $\mathcal{P}_0$ is always implicitly assumed. As can be checked by direct computation, the last condition in (3.70) is solved by

$$\iota_X F(\mathcal{A})^{IJ} = -\frac{i}{4}\left(\epsilon^{IJKL}\bar{\rho}_{KL}\gamma_*\gamma_M\epsilon\, e^M + \epsilon^{KLM[I}\bar{\rho}_{KL}\gamma_*\gamma_M\epsilon\, e^{J]}\right) =: \bar{\theta}^{IJ}_K\,\epsilon\, e^K \tag{3.71}$$

where we set $\rho =: \frac{1}{2}\rho_{IJ}e^I \wedge e^J$. Thus, to summarize, it follows that even smooth vector fields $X \in \Gamma(\mathcal{E}^{\Delta_{\mathbb{R}}}_{/\mathcal{S}})_0$ define local symmetries of the Lagrangian provided that their contraction with the super Cartan curvature satisfy the conditions (3.70) and (3.71) also called the *rheonomy conditions*. Inserting these conditions into the general formula (3.60), it follows that the supersymmetry transformations of the individual components of the super Cartan connection take the form

$$\delta_X e^I = \frac{1}{2}\bar{\epsilon}\gamma^I\psi, \quad \delta_X\psi = D^{(\omega)}\epsilon \quad \text{and} \quad \delta_X\omega^{IJ} = \bar{\theta}^{IJ}_K\,\epsilon\, e^K \tag{3.72}$$

Note again that, by definition, $X \in \Gamma(\mathcal{E}^{\Delta_{\mathbb{R}}}_{/\mathcal{S}})_0$ implies $\langle X|\omega\rangle = 0$, that is, X is horizontal. Hence, in this framework, supersymmetry transformations have the interpretation in terms of superdiffeomorphisms on the base supermanifold $\mathcal{M}$. Of course, one still needs to check whether the rheonomy conditions are in fact compatible with the Bianchi identity. As it turns out, this is indeed the case provided that the basic fields satisfy their equations of motion. Thus, in this framework, it follows that supersymmetry transformations can be interpreted in terms of superdiffeomorphisms only when applied on solutions of the field equations [142]. For general field configurations, this will no longer be the case. In this case, one needs to add additional fields, so-called *auxilary fields*, to the theory. For more details on this subject, the interested reader may be referred to [142] and references therein.

Remark 3.4.5. Let us emphasize that, technically, the existence of a nonvanishing $\epsilon := \langle X|\mathcal{A}\rangle \in H^\infty(\mathcal{S}\times\mathcal{P},\Lambda\otimes\Delta_{\mathbb{R}})^{\mathcal{H}}_0$ for $X \in \Gamma(\mathcal{E}^{\Delta_{\mathbb{R}}}_{/\mathcal{S}})_0$ when pulled back to the body of $\mathcal{P}$ relies crucially on the additional parametrizing supermanifold $\mathcal{S}$. Hence, working in the relative category resolves both, nontrivial anticommuting fermionic fields as well as supersymmetry transformations on the body of a supermanifold.

In the Cartan geometric framework, there also exists another kind of variation of the Lagrangian which arises from the lift of the Cartan connection to an Ehresmann connection on the associated bundle. More precisely, one can consider the $\mathcal{G}$-extension $\mathcal{P}[\mathcal{G}]_{/\mathcal{S}} := (\mathcal{P}\times_{\mathcal{H}}\mathcal{G})_{/\mathcal{S}}$ with respect to which $\mathcal{P}_{/\mathcal{S}}$ defines a $\mathcal{H}$-reduction via the embedding

$$\hat{\iota}: \mathcal{P}_{/\mathcal{S}} \to \mathcal{P}[\mathcal{G}]_{/\mathcal{S}},\ p \mapsto [p,e] \tag{3.73}$$

By Prop. 3.3.12, the Cartan connection can be lifted to an Ehresmann connection $\hat{\mathcal{A}}$ on the associated bundle in a unique way such that $\hat{\imath}^*\hat{\mathcal{A}} = \mathcal{A}$. Again, this connection can be decomposed in the following way

$$\hat{\mathcal{A}} =: \hat{e}^I P_I + \frac{1}{2}\hat{\omega}^{IJ} M_{IJ} + \hat{\psi}^\alpha Q_\alpha \tag{3.74}$$

In fact, it follows that the Lagrangian $\mathscr{L}$ can also be lifted uniquely to a horizontal form $\hat{\mathscr{L}} \in \Omega^4_{hor}(\mathcal{P}[\mathcal{G}]_{/\mathcal{S}})$ on the associated bundle which, using (3.74), is explicitly given by

$$\hat{\mathscr{L}} = \frac{1}{2}F(\hat{\omega})^{IJ} \wedge \hat{e}^K \wedge \hat{e}^L \epsilon_{IJKL} + i\hat{\bar{\psi}} \wedge \gamma_* \gamma_I D^{(\hat{\omega})}\hat{\psi} \wedge \hat{e}^I \tag{3.75}$$

It is clear that, via the embedding (3.73), the bosonic subbundle $\mathcal{P}_0$ (or body P) can be identified with the bosonic subbundle (or body) of the $\mathcal{G}$-extension $\mathcal{P}[\mathcal{G}]$. Thus, given a local section $s : M \to P \subset \mathcal{P}[\mathcal{G}]$, the action $S^{\mathcal{N}=1}(\mathcal{A})$ as defined via (3.51) turns out to be equivalent to

$$S^{\mathcal{N}=1}(\hat{\mathcal{A}}) := \int_M s^*\hat{\mathscr{L}} \tag{3.76}$$

Consider then an infinitesimal automorphism of the form $X \in \mathfrak{gau}(\mathcal{P}[\mathcal{G}]_{/\mathcal{S}})_0$ such that $\epsilon := \langle X|\hat{\mathcal{A}}\rangle \in H^\infty(\mathcal{S}\times\mathcal{P}[\mathcal{G}], \Lambda\otimes\Delta_{\mathbb{R}})^{\mathcal{G}}_0$. We say that X defines a local symmetry of the Lagrangian (3.75) provided that

$$(\iota_X \mathrm{d}\hat{\mathscr{L}})|_{\mathcal{P}_0} = \mathrm{d}\alpha' \tag{3.77}$$

for some 3-form $\alpha' \in \Omega^3(\mathcal{P}[\mathcal{G}]_{/\mathcal{S}})$. The variations $\delta_X\hat{\mathcal{A}} := L_X\hat{\mathcal{A}}$ of the individual components of the connection take the form

$$\delta_X\hat{e}^I = \frac{1}{2}\bar{\epsilon}\gamma^I\hat{\psi}, \quad \delta_X\hat{\psi} = D^{(\hat{\omega})}\epsilon \quad \text{and} \quad \delta_X\hat{\omega}^{IJ} = 0 \tag{3.78}$$

Thus, in contrast to the previous considerations, it follows that the curvature now does not enter to the field variations as the infinitesimal automorphism X is vertical, i.e., it is a local gauge transformation, not a superdiffeomorphism. Comparing with (3.72), when pulled back to $\mathcal{P}_0$, one observes that this yields precisely the supersymmetry transformation for the supervielbein E as found in the previous prescription while the variation of the spin connection is altered.

The variation of the Lagrangian can be computed following the same steps as before which, when pulled back to $\mathcal{P}_0$, leads to a similar expression for $\iota_X\mathrm{d}\hat{\mathscr{L}}$ as in (3.69) just replacing all the fields by their respective hatted counterparts. However, as X is vertical, this implies that the curvature contributions of the form $\iota_X F(\hat{\mathcal{A}})$ are identically zero. Thus, in this case, we end up with

$$(\iota_X \mathrm{d}\hat{\mathscr{L}})|_{\mathcal{P}_0} = (-i\bar{\epsilon} \wedge \gamma_* \gamma_I \rho \wedge F(\mathcal{A})^I)|_{\mathcal{P}_0} \tag{3.79}$$

Hence, as we see, it follows that X indeed provides a local symmetry of the theory provided that the *supertorsion constraint* $F(\mathcal{A})^I = 0$ is satisfied which is equivalent to requiring that ω satisfies its field equations (see also [128, 143]). This observation will in fact play a crucial role in the context of chiral supergravity to be discussed in Section 5.4. There, it turns out that a certain subclass of local gauge transformations $X \in \mathfrak{gau}(\mathcal{P}[\mathcal{G}]_{/\mathcal{S}})_0$ provides a local symmetry of the theory, even without pulling back the Lagrangian to $\mathcal{P}_0$ and, in particular, without requiring ω to satisfy its field equations. Thus, in this framework, in follows that (at least a certain sublcass of) supersymmetry transformations have the interpretation in terms of true gauge transformations.

3.5. Application: Killing vector fields and Killing spinors

The content of this section has been reproduced from [1], with slight changes to account for the context of this thesis with the permission of AIP Publishing.

In this section, we want to discuss an interesting application of the Cartan geometric interpretation of supergravity. More precisely, We want to describe global symmetries of supergravity theories in terms of Killing vector fields on the underlying super Riemannian manifold which arises from a metric reductive super Cartan geometry. This has important applications in the description of supersymmetric black holes in supergravity.

Let $(\pi_{\mathcal{S}} : \mathcal{P}_{/\mathcal{S}} \to \mathcal{M}_{/\mathcal{S}}, \mathcal{A})$ be a metric and reductive super Cartan geometry modeled on a super Klein geometry $(\mathcal{G}, \mathcal{H})$ with super Cartan connection $\mathcal{A}$ and smooth $\mathrm{Ad}(\mathcal{H})$-invariant super metric $\mathscr{S}$ defined on $\Lambda \otimes \mathfrak{g}/\mathfrak{h}$. For sake of simplicity, let us assume that $\mathcal{S}$ is a *superpoint*, i.e., the body just consists of a single point $\mathbf{B}(\mathcal{S}) = \{*\}$. From a physical perspective, the necessity of the choice of a nontrivial parametrizing supermanifold $\mathcal{S}$ is based on the requirement of nonvanishing (anticommuting) fermionic degrees of freedom on the body of a supermanifold. The structure of the underlying base supermanifold $\mathcal{M}$ in the Cartan geometric framework, however, is encoded in the super soldering form $E = \mathrm{pr}_{\mathfrak{g}/\mathfrak{h}} \circ \mathcal{A}$ when restricted on bosonic configurations (see also the discussion in Section 3.6 below). In fact, let $\mathscr{H} = \ker(\omega|_{\mathbf{B}(\mathcal{S})})$ be the horizontal distribution induced by pullback of the Ehresmann connection $\omega := \mathrm{pr}_{\mathfrak{h}} \circ \mathcal{A}$ to $\mathbf{B}(\mathcal{S})$. By the super Cartan condition, it then follows that the restriction $\theta := E|_{\mathbf{B}(\mathcal{S})} \in \Omega^1(\mathcal{P}, \Lambda \otimes \mathfrak{g}/\mathfrak{h})$ yields an isomorphism

$$\theta_p : \mathscr{H}_p \to \Lambda \otimes \mathfrak{g}/\mathfrak{h} \tag{3.80}$$

of free super Λ-modules at any $p \in \mathcal{P}$. Horizontal vector fields on $\mathcal{P}$ are in one-to-one correspondence with smooth vector fields on the base supermanifold $\mathcal{M}$. If $X \in \Gamma(T\mathcal{M})$ is a smooth vector field and X^* its horizontal lift, then X^*, in particular, defines an infinitesimal bundle automorphism. By $\mathcal{H}$-equivariance, it follows that $\langle X^*|\theta\rangle$ defines a $\mathcal{H}$-equivariant smooth function on $\mathcal{M}$ with values in $\Lambda \otimes \mathfrak{g}/\mathfrak{h}$. Thus, to summarize, we have an isomorphism

$$\theta : \Gamma(T\mathcal{M}) \xrightarrow{\sim} H^\infty(\mathcal{P}, \Lambda \otimes \mathfrak{g}/\mathfrak{h})^{\mathcal{H}},\ X \to \langle X^*|\theta\rangle \tag{3.81}$$

Since the super Cartan geometry is metric, according to Corollary 3.3.11, the super soldering form θ induces a super metric g on $\mathcal{M}$ which, w.r.t. any local section $s \in \Gamma(\mathcal{P})$ of the principal super fiber bundle, takes the form

$$g = (-1)^{|e_i||e_j|}\mathscr{S}_{ij}(s^*\theta)^i \otimes (s^*\theta)^j \tag{3.82}$$

where we have chosen a homogeneous basis $(e_i)_i$ of $\mathfrak{g}/\mathfrak{h}$ and set $\mathscr{S}_{ij} := \mathscr{S}(e_i, e_j) \in \mathbb{C}$. With respect to this metric, the base $(\mathcal{M}, g)$ has the structure of a *super Riemannian manifold*. We now want to introduce the super analog of an infinitesimal isometry on a super Riemannian manifold.

Definition 3.5.1. Let $(\mathcal{M}, g)$ be a super Riemannian manifold. A smooth vector field $X \in \Gamma(T\mathcal{M})$ is called a *Killing vector field* if g is constant along the flow generated by X, that is,

$$L_X g = 0 \tag{3.83}$$

Since $L_{[X,Y]} = [L_X, L_Y]$ for any smooth vector fields X and Y, Killing vector fields form a super Lie subalgebra $\mathfrak{k}(\mathcal{M}, g)$ of $\Gamma(T\mathcal{M})$.

In the supergeometric framework, Killing vector fields $X \in \mathfrak{k}(\mathcal{M}, g)$ fall into two categories depending on their grading. In context of the super Cartan geometry, we will call an odd Killing vector field $X \in \mathfrak{k}(\mathcal{M})_1$ a *Killing spinor*. To explain its name, let us consider for example the super Cartan geometry corresponding to $\mathcal{N} = 1$ Poincaré supergravity and let $X \in \mathfrak{k}(\mathcal{M})_1$ be a odd Killing vector field. Via the super soldering form, this then corresponds to an odd smooth function

$$\epsilon := H^\infty(\mathcal{P}, \Lambda \otimes \mathfrak{t})_1^{\mathrm{Spin}^+(1,3)} \cong \Gamma(\mathcal{P} \times_{\mathrm{Spin}^+(1,3)} \Lambda \otimes \mathfrak{t})_1 \tag{3.84}$$

with $\mathfrak{t} = \mathbb{R}^{1,3} \oplus \Delta_{\mathbb{R}}$ the super Lie algebra of the super translation group $\mathcal{T}^{1,3|4}$. If one restricts to the bosonic sub supermanifold $\mathcal{M}_0$ this then implies that ϵ defines a section of the associated spinor bundle, that is, it defines a Majorana spinor. Thus, Killing spinors of the super Riemannian geometry, induced by the metric reductive Cartan geometry, are associated to Majorana spinors.

Note that odd Killing vector fields on a supermanifold vanish when pulled back to the underlying bosonic supermanifold $\mathcal{M}_0 := \mathbf{S}(\mathbf{B}(\mathcal{M}))$. Thus, they define trivial infinitesimal isometries of the bosonic Riemannian manifold $(\mathcal{M}_0, g_0)$ with g_0 the even part of the supermetric g. Neverthless, existence of Killing spinors, in general, may still impose strong restrictions on the structure of the underlying bosonic geometry. In fact, given two Killing spinors $X, Y \in \mathfrak{k}(\mathcal{M}, g)_1$ their (graded) commutator $[X, Y] \in \mathfrak{k}(\mathcal{M}, g)_0$ defines an even Killing vector field which can be nonvanishing when pulled back to the bosonic submanifold.

To illustrate this, let us consider the homogeneous super Cartan geometry $(\pi : \mathcal{G} \to \mathcal{G}/\mathcal{H}, \theta_{\mathrm{MC}})$ given by the super Klein geometry $(\mathcal{G}, \mathcal{H}) = (\mathrm{OSp}(1|4), \mathrm{Spin}^+(1,3))$ corresponding to super anti-de Sitter space $\mathcal{M} := \mathcal{G}/\mathcal{H}$ (see Example 2.3.17) where θ_{MC} denotes the Maurer-Cartan form on $\mathcal{G}$. On $\mathfrak{osp}(1|4)$, we can define a smooth Ad-invariant super metric $\mathscr{S}$ induced by the *supertrace* str on $\mathrm{Mat}(1|4, \Lambda^{\mathbb{C}})$ setting

$$\mathscr{S}(X, Y) := -\mathrm{str}(X \cdot Y), \quad \forall X, Y \in \mathrm{Lie}(\mathcal{G}) \tag{3.85}$$

This yields an orthogonal decomposition of $\mathfrak{osp}(1|4) = \mathbb{R}^{1,3} \oplus \mathfrak{spin}^+(1,3) \oplus \Delta_{\mathbb{R}}$ into $\mathrm{Ad}(\mathcal{H})$-invariant subspaces generated by momenta and Lorentz transformations P_I and M_{IJ}, respectively, as well as four Majorana charges Q_α such that $\mathscr{S}(P_I, P_J) = L^2\eta_{IJ}$ and $\mathscr{S}(Q_\alpha, Q_\beta) = \frac{1}{L}C_{\alpha\beta}$. Thus, on $\mathfrak{g}/\mathfrak{h} = \mathbb{R}^{1,3} \oplus \Delta_{\mathbb{R}}$, $\mathscr{S}$ takes the form

$$\mathscr{S} = \begin{pmatrix} L^2\eta & 0 \\ 0 & \frac{1}{L}C \end{pmatrix} \tag{3.86}$$

Let furthermore,

$$E := \mathrm{pr}_{\mathfrak{g}/\mathfrak{h}} \circ \theta_{\mathrm{MC}} =: e^I P_I + \psi^\alpha Q_\alpha \tag{3.87}$$

be the super soldering form which induces a super metric g on $\mathcal{G}/\mathcal{H}$ via (3.82). According to Definition 3.4.1, an infinitesimal automorphism $X \in \mathfrak{aut}(\mathcal{G})$ needs to satisfy

$$R_{h*}X = X, \quad \forall h \in \mathcal{H} \tag{3.88}$$

Thus, in particular, it follows that infinitesimal automorphisms are provided by right-invariant vector fields on $\mathcal{G}$. In fact, it follows that right-invariant vector fields are Killing vector fields of the induced super Riemannian geometry $(\mathcal{M}, g)$. To see this, note that for right-invariant and left-invariant vector fields X and Y on the super Lie group $\mathcal{G}$, respectively, it follows that

$$\begin{aligned} \langle Y | L_X E\rangle &= (-1)^{|X||Y|}(X\langle Y|E\rangle - \langle [X,Y]|E\rangle) \\ &= (-1)^{|X||Y|}X\langle Y|E\rangle = 0 \end{aligned} \tag{3.89}$$

since left- and right-invariant vector fields commute[6] and $\langle Y|E\rangle$ is constant by left-invariance. But, since, at any $g \in \mathcal{G}$, the left-invariant vector fields yield a homogeneous basis of $T_g\mathcal{G}$, this implies $L_X E = 0$ so that, by (3.82), X is indeed a Killing vector field. According to our definition above, odd right-invariant vector fields define Killing spinors which can be identified with $\mathfrak{g}_1 = \Delta_{\mathbb{R}}$, that is, they are Majorana spinors. Let $\epsilon := \epsilon^\alpha Q_\alpha \in \mathfrak{g}_1$ be such a spinor. Using (3.60) and $L_X E = 0$ for any right-invariant vector field X, it then follows in particular (note that the super Cartan curvature vanishes by the Maurer-Cartan equation (3.4) which equally holds also in the super category)

$$L_{\epsilon^R}\psi = D^{(\omega)}\epsilon' - \frac{1}{2L}e^I\gamma_I\epsilon' = 0 \tag{3.91}$$

where we set $\epsilon' := \iota_{\epsilon^R}E$. Spinors satisfying an equation of the form (3.91) are called *twistor spinors* [144]. Thus, we see that odd Killing vector fields of the super Riemannian manifold $(\mathcal{M}, g)$ correspond to twistor spinors.

Next, let $\epsilon, \eta \in \mathfrak{g}_1$ be two Majorana spinors and ϵ^R, η^R be the corresponding right-invariant vector fields. Since the commutator of two right-invariant vector fields is again right-invariant, the bilinear $K^* := [\epsilon^R, \eta^R]$ again defines a Killing vector field. Using, $[\epsilon^R, \eta^R] = -[\epsilon, \eta]^R$, it follows

$$K^* = -\bar{\epsilon}\gamma^I\eta P_I^R - \frac{1}{4L}\bar{\epsilon}\gamma^{IJ}\eta M_{IJ}^R \tag{3.92}$$

As K^* is purely bosonic, its pushforward $K := \pi_* K^*$ defines, in general, a nonvanishing Killing vector field on the bosonic semi-Riemannian manifold $(\mathcal{M}_0, g_0)$, i.e., ordinary $D = 4$ AdS spacetime.

Remark 3.5.2. Much more generally, in the algebraic framework, in [144], it was shown that for the split supermanifold $\mathbf{S}(E_{\mathbb{R}}, M)$ associated to a Majorana bundle $E_{\mathbb{R}} \times M \to M$ with M spin, odd Killing vector fields of the corresponding super Riemannian geometry precisely correspond to twistor spinors. Its highly suggestive that any super Cartan geometry corresponding to Poincaré supergravity is of this form. More precisely, the principal super fiber bundle may be of the form $\mathcal{P} \to \mathbf{S}(E_{\mathbb{R}}, M)$ with $\mathcal{P}$ a spin-reduction of the frame bundle $\mathscr{F}(\mathbf{S}(E_{\mathbb{R}}, M))$. This is supported by the fact that the super soldering form induces frame fields on the base supermanifold which are of the form as stated in [144].

---

6 A direct proof is given using the algebraic Definition 2.3.2. Let X and Y be homogeneous right- and left-invariant vector fields on $\mathcal{G}$, respectively. Then,

$$Y\circ X = \mathbb{1}\otimes Y_e\circ\mu^*\circ X = \mathbb{1}\otimes Y_e\circ X\otimes\mathbb{1}\circ\mu^* = (-1)^{|X||Y|}X\circ\mathbb{1}\otimes Y_e\circ\mu^* = (-1)^{|X||Y|}X\circ Y \tag{3.90}$$

Remark 3.5.3. In physics, Killing spinors appear by studying consistency conditions for the bosonic background of solutions of supergravity theories. For instance, let us consider a super Cartan geometry corresponding to $\mathcal{N}=1$, $D=4$ AdS supergravity. As will be discussed in detail in Chapter 5, the super Cartan connection is again of the form (3.46) and the supersymmetry transformations of the individual field components are given by

$$\delta_\epsilon e^I = \frac{1}{2}\bar{\epsilon}\gamma^I\psi, \quad \delta_\epsilon\psi = D^{(\omega)}\epsilon - \frac{1}{2L}e^I\gamma_I\epsilon \text{ and } \delta_\epsilon\omega^{IJ} = \frac{1}{2L}\bar{\epsilon}\gamma^{IJ}\psi \tag{3.93}$$

One may then be interested in the structure of the bosonic background of a general solution of the SUGRA field equations. As will be argued in Section 3.6 below, based on the considerations in algebraic QFT [76, 77], the parametrizing supermanifold $\mathcal{S}$ may be interpreted as the field configuration space. Bosonic (c-valued) field configurations taking values in the ordinary complex numbers are then encoded in the body $\mathbf{B}(\mathcal{S})$. But, if one then pulls back the field variations (3.93) to the underlying spacetime manifold $\mathbf{B}(\mathcal{M})$, the fermionic fields, as being anticommutative, simply vanish. Hence, if the bosonic background solutions are required to be consistent with supersymmetry, there has to exist a nontrivial spinor field ϵ such that

$$D^{(\omega)}\epsilon - \frac{1}{2L}e^I\gamma_I\epsilon = 0 \tag{3.94}$$

Thus, again, this leads back to the Killing spinor equation (3.91).

3.6. On the role of the parametrizing supermanifold

The content of this section has been reproduced from [1], with slight changes to account for the context of this thesis with the permission of AIP Publishing.

As we have frequently observed in the previous sections as well as Chapter 2, among other things, studying parametrized supermanifolds is mandatory in order to incorporate nontrivial fermionic degrees of freedom on the body of a supermanifold. For instance, for the action $S^{\mathcal{N}=1}(\mathcal{A})$ corresponding to Poincaré supergravity (Eq. (3.51) and (3.52)), one has

$$S^{\mathcal{N}=1}(\mathcal{A}) = \int_M s^*\mathscr{L} \in H^\infty(\mathcal{S})_0 \tag{3.95}$$

that is, the action defines an even smooth map on the underlying parametrizing supermanifold $\mathcal{S}$. If $\mathcal{S}$ would be trivial, this then implies that the fermionic fields contained in the action would simply drop off, that is, the action reduces the standard action of ordinary Einstein gravity.

It follows from the definition of super connection forms on parametrized super fiber bundles that all physical quantities transform covariantly under change of parametrization. More precisely, let $\lambda : \mathcal{S}' \to \mathcal{S}$ be a change of parametrization. Then, the super Cartan connection $\mathcal{A}$ transforms via $\mathcal{A} \to \lambda^*\mathcal{A}$ (see Eq. (2.238)) so that for the action of the theory it follows

$$S^{\mathcal{N}=1}(\mathcal{A}) \to S^{\mathcal{N}=1}(\lambda^*\mathcal{A}) = \lambda^* S^{\mathcal{N}=1}(\mathcal{A}) \in H^\infty(\mathcal{S}')_0 \tag{3.96}$$

This may be regarded as the mathematical realization of the physical requirement that the physical theory should not depend on a particular choice of a parametrizing supermanifold.

In what follows, we want to further analyze the structure of field configuration space of Poincaré supergravity and work out explicitly the relation to pAQFT [76, 77]. However, let us emphasize that a full consistent treatment requires the study of infinite-dimensional supermanifolds. Hence, in the following, we will only sketch the main ideas behind such a link.

Choosing a reference connection, the configuration space of the theory can be identified with $\Omega^1(\mathcal{M}_{/\mathcal{S}}, \mathrm{Ad}(\mathcal{P}_{/\mathcal{S}}))_0$. Since, by the *rheonomy principle* [72], we are actually only dealing with the pullback of super connections on the body of the supermanifold, in what follows, it suffices restrict on forms defined on the body $M := \mathbf{B}(\mathcal{M})$ so that, for a specific choice of parametrization $\mathcal{S}$, the configuration space C of the theory can be taken to be

$$C := (H^\infty(\mathcal{S}) \otimes \Omega^1(M, P \times_{\mathrm{Ad}} \mathfrak{g}))_0 = H^\infty(\mathcal{S})_0 \otimes F_0 \oplus H^\infty(\mathcal{S})_1 \otimes F_1 \tag{3.97}$$

where F denotes the infinite-dimensional $\mathbb{Z}_2$-graded vector space given by

$$F := \Omega^1(M, E) = \Omega^1(M, E_0) \oplus \Omega^1(M, E_1) \tag{3.98}$$

Here, $E := P \times_{\mathrm{Ad}} \mathfrak{g}$ is the associated bundle which itself carries the structure of a super vector space with even and odd part respectively given by $E_0 = P \times_{\mathrm{Ad}} \mathfrak{g}_0$ and $E_1 = P \times_{\mathrm{Ad}} \mathfrak{g}_1$ with E_1 the spinor bundle of Majorana fermions. For $\Phi \in C$ it follows that, for any $s \in \mathcal{S}$, $\Phi(s)$ defines an element of the superspace

$$\overline{F}(\Lambda) := (F \otimes \Lambda)_0 = F_0 \otimes \Lambda_0 \oplus F_1 \otimes \Lambda_1 \tag{3.99}$$

where Λ is the Grassmann algebra over which $\mathcal{S}$ is modeled as a H^∞ supermanifold. Hence, in this sense, it follows that one can identify the configuration space C with the space of H^∞-smooth functions on $\mathcal{S}$ with values in the superspace $\overline{F}(\Lambda)$.

So far, our consideration was based on the choice of a particular parametrizing supermanifold $\mathcal{S}$. However, as explained at the beginning of this section, by definition of super connection forms defined on parametrized principal super fiber bundles, all the above constructions behave covariantly under change of parametrization. Due to this property, one may ask the question, whether there exists a particular choice of a parametrization $\mathbb{S}$, possibly infinite-dimensional, such that any field configuration associated to some finite $\mathcal{S}$ can be obtained via pullback. Hence, $\mathbb{S}$ should be suitably large enough to encode all the configurations associated to $\mathcal{S}$-parametrized field theories with $\mathcal{S}$ finite.

In fact, this idea has been first studied by Schmitt in [73] where, in this context, a theory of infinite-dimensional *analytic supermanifolds* has been developed. In the following, we want to sketch this idea using the Molotkov-Sachse approach to supermanifold theory [98, 99], more precisely locally convex supermanifolds as considered in [110], as this seems to be the more established approach to this subject. In fact, as already outlined in the introduction, the Moltokov-Sachse approach can be regarded as a generalization of the correspondence between finite-dimensional algebro-geometric and Rogers-De Witt supermanifolds via the functor of points prescription to the case of infinite dimensions. This is actually one of the reasons why we have focused on the Rogers-DeWitt approach in this work.

To find a suitable candidate for $\mathbb{S}$, note that, for any $\Lambda \in \mathbf{Ob}(\mathbf{Gr})$, the superspace $\overline{F}(\Lambda)$ can be endowed with the structure of a locally convex space by choosing a particular locally convex topology on F and extending it to $\overline{F}(\Lambda)$ using the product topology. In this way, the assignment $\mathbf{Ob}(\mathbf{Gr}) \ni \Lambda \mapsto \overline{F}(\Lambda)$ induces a functor

$$(\overline{F} : \mathbf{Gr} \to \mathbf{Top}) \in \mathbf{Top}^{\mathbf{Gr}} \tag{3.100}$$

from the category of superpoints to the category **Top** of topological spaces. Hence, $\overline{F}$ defines a supermanifold (more precisely, a superdomain) in the sense of Molotkov-Sachse [98, 99, 110].

Next, note that the parametrizing supermanifold $\mathcal{S}$, as a H^∞ supermanifold, can be regarded as a Λ-point $\mathcal{S} \equiv \overline{\mathcal{S}}(\Lambda)$ of a particular algebro-geometric supermanifold $\overline{\mathcal{S}}$. On the other hand, via the functor of points prescription, $\overline{\mathcal{S}}$ itself induces a functor $\overline{\mathcal{S}} : \mathbf{Gr} \to \mathbf{Top}$ and thus yields a Molotkov-Sachse supermanifold (for a proof see, e.g., [99]). Hence, if we do not focus on a particular Grassmann algebra, according to the discussion above, we may identify the configuration space C with $(\mathcal{S}C^\infty(\overline{\mathcal{S}})\hat{\otimes}F)_0$, i.e., smooth functions (in the sense of Molotkov-Sachse) on $\overline{\mathcal{S}}$ with values in $\overline{F}$. We now make the important assumption that this space can be identified with $\mathcal{S}C^\infty(\overline{\mathcal{S}}, \overline{F})$, that is, the space smooth maps between the infinite-dimensional supermanifolds $\overline{\mathcal{S}}$ and $\overline{F}$. Note that this would be trivially the case, if F were finite-dimensional.

Hence, in this case, it follows that any $\Phi \in C$ can be identified with a morphism $\mu_\Phi : \overline{S} \to \overline{F}$. But, note that $\mu_\Phi \equiv \mu_\Phi^*(x_{\overline{F}})$ with $x_{\overline{F}} := \mathrm{id} : \overline{F} \to \overline{F}$ the identity morphism. Hence, any field configuration Φ associated to a particular parametrizing supermanifold $\overline{S}$ can be obtained via pullback of $x_{\overline{F}}$ w.r.t. a (unique) morphism $\mu_\Phi : \overline{S} \to \overline{F}$. As in [73], we may therefore call $x_{\overline{F}}$ a *fundamental coordinate* and the morphism μ_Φ the *classifying morphism* of the field configuration Φ. Hence, to summarize, based on these observations, this suggests setting $\mathcal{S} := \overline{F}$. With slight abuse of terminology, in what follows, we will also often refer to $\mathcal{S}$ as a configuration space.

As shown in [110], similar to the finite-dimensional case, it turns out that any smooth function $f \in \mathcal{SC}^\infty(\overline{F}) \equiv \mathcal{SC}^\infty(\overline{F}, \overline{\mathbb{R}^{1|1}})$ on $\overline{F}$ can be uniquely described in terms of a so-called *skeleton* $(f_n)_n$ consisting of smooth maps between locally convex spaces $f_n : F_0 \to \mathcal{A}\mathrm{lt}^n(F_1, \mathbb{R}^{1|1})$, $n \in \mathbb{N}_0$, from F_0 to $\mathcal{A}\mathrm{lt}^n(F_1, \mathbb{R}^{1|1})$, i.e., the space of graded symmetric n-multilinear smooth functionals on F_1. Thus, we can make the identification

$$\mathcal{SC}^\infty(\overline{F}, \overline{\mathbb{R}^{1|1}}) \cong C^\infty(F_0, \mathcal{A}\mathrm{lt}(F_1, \mathbb{R}^{1|1})) \tag{3.101}$$

It is interesting to note that (3.101) is precisely the field configuration space as considered in [76, 77] in context of pAQFT. There, among other things, this space has been considered in order to (classically) consistently incorporate the anticommutative nature of fermionic fields. As we see, here, it arises quite naturally studying relative supermanifolds.

To make this link to the description of fermionic fields in pAQFT even more precise, let us choose a mutual local trivialization neighborhood of M and the vector bundle E. Let $(e_{\underline{A}})_{\underline{A}}$ with $\underline{A} \in \{I, \alpha\}$ be a corresponding homogeneous basis of local sections of E. Decomposing the fundamental coordinate $x_{\overline{F}}$ w.r.t. the homogeneous basis $(e_{\underline{A}})_{\underline{A}}$ and evaluating on coordinate differentials $\partial_\mu|_p$ at any point $p \in M$, it follows that the odd components induce smooth functionals $\Psi^\alpha_\mu(p) \in (F_1)'$, with $(F_1)' = \Gamma'(T^*M \otimes E_1)$ the topological dual of F_1, via $\Psi^\alpha_\mu(p) := \mathrm{pr}_\alpha \circ \langle \partial_\mu|_p | x_{\overline{F}}(\cdot) \rangle$ such that

$$\Psi^\alpha_\mu(p) : F_1 \to \mathbb{R}^{0|1} \cong \mathbb{R},\ \psi \mapsto \psi^\alpha_\mu(p) \tag{3.102}$$

Thus, it follows that, in this framework, fermionic fields are described in terms of odd evaluation functionals on the configuration space. This is exactly the interpretation of (classical) anticommutative fermionic fields in pAQFT [76, 77]. Given two fermionic fields $\Psi^\alpha_\mu(p)$ and $\Psi^\beta_\nu(p)$ at the same point $p \in M$, their product is defined via the ordinary wedge product yielding the bilinear map

$$\Psi^\alpha_\mu(p)\Psi^\beta_\nu(p) \equiv \Psi^\alpha_\mu(p) \wedge \Psi^\beta_\nu(p) = -\Psi^\beta_\nu(p) \wedge \Psi^\alpha_\mu(p) \tag{3.103}$$

so that, in this sense, the fermionic fields are indeed anticommutative.

3.7. Discussion

In this chapter, we have studied the geometric approach to supergravity. To this end, we provided the mathematical foundations for the formulation of super Cartan geometries. A crucial ingredient for supersymmetry is the anticommutative nature of fermionic fields. However, as we have seen, modeling anticommuting classical fermion fields turns out to be by far non-straightforward. Again, a resolution is given considering enriched categories as studied in detail in Chapter 2 based on standard techniques in algebraic geometry. This procedure requires the choice of an additional parametrizing supermanifold which encodes the fermionic degrees of freedom. Since the choice is arbitrary, one needs to ensure that physical quantities behave functorially under a change of parametrization. This property follows naturally, if one works in the category of relative supermanifolds. This also reflects the interpretation of supermanifolds in the sense of Molotkov-Sachse [98,99] in terms of a functor $\mathbf{Gr} \rightarrow \mathbf{Top}$ assigning Grassmann algebras to Rogers-DeWitt supermanifolds (see Remark 2.2.13).

Having formulated the notion of a super Cartan geometry in the framework of enriched categories, we then turned towards applications in context of supergravity. More precisely, we considered $D = 4$, $\mathcal{N} = 1$ Poincaré supergravity as a metric reductive super Cartan geometry and analyzed local symmetries of this model. In this context, we also discussed a possible embedding of the Castellani-D'Auria-Fré approach to supergravity [71, 72, 131, 132, 142] into the present formalism. In this framework, it follows that, under certain conditions on the fields as well as the on the individual components on super Cartan curvature associated to the super Cartan connection, supersymmetry transformations have the interpretation in terms of infinitesimal superdiffeomorphisms along the odd directions of the underlying base supermanifold.

Alternatively, using the Cartan geometric description as well as the strong link between super Cartan connections and Ehresmann connections as provided in detail in Section 3.3, it follows that supersymmetry transformations can also be interpreted in terms of gauge transformations on an associated principal super fiber bundle. This interpretation will in fact play an important role in explaining the manifest enlarged gauge symmetry of the chiral theory to be discussed in Section 5.4.

Finally, Killing vector fields on super Riemannian manifolds arising from metric reductive super Cartan geometries were discussed. In this context, odd Killing vector fields were identified with Killing spinors typically arising as consistency conditions for the bosonic background of solutions of supergravity with supersymmetry.

Furthermore, using the functorial dependence of the supergravity action as well as the resulting configuration space of the theory on the underlying parametrization supermanifold, we sketched a concrete link to the description of anticommutative (classical) fermionic fields in pAQFT. More precisely, by adapting the idea of [73] to the

Molotkov-Sachse approach to supermanifold theory a kind of infinite-dimensional universal parametrization supermanifold was constructed such that any field configuration defined on a finite-dimensional parametrization can be obtained via pullback. Fermionic fields on this universal parametrization supermanifold then turn out to be described in terms of evaluation functionals on configuration space in complete analogy to pAQFT.

There are many possible and interesting extensions of the present formalism, both from the mathematical and physical perspective. On the one hand, It would be interesting to see how extended supergravity theories can be described via super Cartan geometries in the framework of enriched categories as presented here. The case of pure $\mathcal{N} = 2$, $D = 4$ AdS SUGRA will be discussed in Section 5.3. In this context, it would also be very interesting to compare the present formalism to other approaches towards geometric formulations of supergravity theories [131, 132, 145].

On the other hand, it would be also interesting to generalize the formalism to include higher dimensional supergravity theories. Higher dimensional supergravity theories typically involve higher gauge fields. Connection forms on higher principal bundles are studied for instance in [119, 120]. It would be very interesting to see how these approaches can be related. As will be discussed in Chapter 5.4, the Cartan geometric approach to supergravity leads to an intriguing geometric structure of the corresponding chiral theory. Hence, generalizing this formalism to supergravity theories with extended SUSY or even higher spacetime dimensions may also have important applications in LQG. Among other things, the geometric approach may lead to a very natural quantization scheme of higher gauge fields in the framework of LQG. For an interesting treatment of higher gauge fields in a complementary approach that does not keep a part of the supersymmetry manifest but can handle higher dimensional SUGRA theories see [68].

4. Loop quantum supergravity and the quantum SUSY constraint

4.1. Introduction

The content of this chapter has been reproduced with permission from [2], with slight changes to account for the context of this thesis.

About ten years after the discovery of supergravity, Jacobson [62] introduced a chiral variant of the real $\mathcal{N} = 1$ Poincaré supergravity action using Ashtekar's self-dual connection variables. Soon after, Fülöp [63] extended this theory to anti-de Sitter supergravity including a cosmological constant where he also pointed out some interesting remnant supersymmetric structure in the resulting Poisson algebra between the Gauss and left SUSY constraint. This paved the way towards a new approach to non-perturbative supergravity in which parts of SUSY were kept manifest.[1] In particular, this was more intensively studied by Gambini and Pullin et al. [84] as well as Ling and Smolin [85, 86], where the notion of super spin networks first appeared. Later it was also considered by Livine and Oeckl [147] in the spinfoam approach to quantum gravity.

Canonical supergravity with *real* Asthekar-Barbero variables was for the first time considered by Tsuda [65] where a generalization of the chiral $\mathcal{N} = 1$ supergravity action to arbitrary real Barbero-Immirzi parameters was found. In parallel, Sawaguchi [64] constructed the phase space in terms of real Ashtekar-Barbero variables by performing a canonical transformation of the ADM phase space. However, as mentioned already in the main introduction, these considerations did not include a fully consistent treatment of half-densitized fermionic fields as proposed by Thiemann in [80] in order to solve the reality conditions to be satisfied by the Rarita-Schwinger field. Generalizations in the classical setting have been studied for instance in [148], where Holst actions for extended $D = 4$ supergravity theories have been constructed.

Finally, these considerations have been extended to higher spacetime dimensions by Bodendorfer et al. [67, 68] based on a new method discovered by the same authors in [69] allowing them to construct Ashekar-Barbero type variables in case of more general spacetime dimensions going beyond the limitations of the variables usually applied in LQG. Since, we are not working in higher dimensions, we use the standard Ashtekar connection, shifted by some torsion terms. These are slightly different variables for the gravitational field than [67, 68]. However, [67] uses half-densitized variables for

[1] For an earlier approach to the canonical quantization of supergravity using ADM variables (in which this manifest part of SUSY is absent) see [146].

the Rarita-Schwinger field, and it introduces an ingenious technique for dealing with its Majorana-nature, which we will also employ.

In this work, we will be mainly interested in the $\mathcal{N} = 1$, $D = 4$ case, in particular, in the implementation of the SUSY constraint in the quantum theory. In the chiral approach, Jacobson studied the classical Poisson algebra generated by the left and right SUSY constraints which maintain the right balance between fermionic and bosonic degrees of freedom. In particular, it was shown that the Poisson bracket among the SUSY constraints generates the Hamiltonian constraint which is in fact a generic feature in canonical supergravity theories. Similar results obtained in [64] using real Ashtekar variables supported this hypothesis showing that, on the constrained surface of gauge- and diffeomorphism-invariant states, the Poisson bracket between the SUSY constraints is indeed proportional to the Hamiltonian constraint.

This has interesting consequences implying that the SUSY constraint is superior to the Hamiltonian constraint in the sense that the solutions of the SUSY constraint immediately are solution of the latter. Hence, in case of presence local supersymmetry, the SUSY constraint plays a similar role as the Hamiltonian constraint in ordinary field theories. In fact, it has been conjectured early on that the SUSY constraint could be understood as the "square root" of the Hamilton constraint, in the same sense and with the same resulting simplifications as the relation between Dirac and Klein-Gordon operator [149–151]. This is precisely what makes its study in LQG particularly interesting. However, an explicit implementation of the SUSY constraint in the quantum theory has not been considered so far in the literature. In fact, the SUSY constraint turns out to have a different structure than the Hamiltonian constraint which also requires special care for its regularization. As a result, its implementation in the quantum theory leads to an operator which has a different structure than the Hamiltonian constraint operator. It would be interesting to check by computing the commutators, in which sense these operators can be related to each other. This may also fix some of the quantization ambiguities. In fact, for a certain subclass of symmetry reduced models, we will explicitly show in Chapter 6 that such a strong relationship can indeed be maintained in the quantum theory. It would be of great interest to see whether these results can be extended to the full theory.

The structure of this chapter is as follows: At the beginning, in Section 4.2, we will review very briefly some important aspects about Clifford algebras and Majorana spinors. We will use this opportunity to fix our notation and conventions as well as to collect important identities used in the main part of this chapter. In Section 4.3, we will subsequently discuss the canonical analysis of the Holst action of $D = 4$, $\mathcal{N} = 1$ Poincaré supergravity as introduced in [65] filling in some details concerning half-densitized fermion fields. We will finally derive a compact expression of the supersymmetry constraint that will be used for the implementation in the quantum theory. The quantization of the

Rarita-Schwinger field will be discussed in detail in Section 4.5.1 following the proposal of [67] performing an appropriate extension of the canonical phase space. We will also use this occasion to point out some interesting mathematical structure underlying the usual quantization scheme of fermion fields in LQG also discussed in more detail in Section 5.5.4 in the context of the manifestly supersymmetric approach to quantum supergravity.

Finally, in Section 4.5.2, we will turn to the quantization of the SUSY constraint in the quantum theory. In particular, an explicit expression of the quantum SUSY constraint will be derived using a specific adapted regularization scheme. In this way, we will also find some explicit formulas for its action on spin network states which may be of particular interest in order to find relations to the standard quantization scheme of Hamiltonian constraint. In Section 4.5.3, possible solutions of the SUSY constraint will be discussed on a qualitative level showing that general solutions may indeed be supersymmetric in the sense that they need to contain both fermionic and bosonic degrees of freedom.

As already explained at the end of the main introduction of this thesis in Chapter 1, in the following, we will drop many mathematical details such as the underlying parametrization supermanifold in order to simplify the notation and to make the following discussion easier accessible for the reader.

A list of important symbols as well as an overview of our choice of conventions concerning indices, physical constants etc. can be found in the List of symbols, notations and conventions.

4.2. Some notes on Clifford algebras and Majorana spinors

In this section, we will only recall some essential aspects of Clifford algebras and Majorana spinors. To this end, we will mainly follow the mathematical exposition in [104], although our conventions are those in [152].

Let $(\mathbb{R}^{s,t}, \eta)$ be the inner product space where η is a symmetric bilinear form of signature (s, t), i.e., with respect to the standard basis $\{e_I\}$ of $\mathbb{R}^{s,t}$, $I = 0, \ldots, D-1$ with $D := s + t$, one has

$$\eta(e_I, e_I) = \begin{cases} -1, & \text{for } I = 0, \ldots, s-1 \\ +1, & \text{for } I = s, \ldots, D-1 \end{cases} \tag{4.1}$$

and $\eta(e_I, e_J) = 0$ for $I \neq J$. In case $s = 1$, η is also called the *D-dimensional Minkowski metric*. The *Clifford algebra* $\mathrm{Cl}(\mathbb{R}^{s,t}, \eta)$ is an associative algebra over the reals with unit

$\mathbb{1}$ generated defined via D elements $\gamma_I \in \mathrm{Cl}(\mathbb{R}^{s,t},\eta)$ called *gamma matrices* satisfying the anticommutation relations

$$[\gamma_I,\gamma_J]_+ = 2\eta_{IJ} \tag{4.2}$$

It follows that $\mathrm{Cl}(\mathbb{R}^{s,t},\eta)$ is real vector space of dimension $\dim \mathrm{Cl}(\mathbb{R}^{s,t},\eta) = 2^D$ spanned by the unit $\mathbb{1}$ together with elements of the form

$$\gamma_{I_1 I_2\cdots I_k} := \gamma_{[I_1}\gamma_{I_2}\cdot\ldots\cdot\gamma_{I_k]} \tag{4.3}$$

for $k = 1,\ldots,D$, where the bracket denotes antisymmetrization. A basis of the Clifford algebra is provided by the set

$$\{\mathbb{1},\gamma^I,\gamma^{I_1 I_2},\gamma^{I_1\cdots I_D}\} \tag{4.4}$$

where $I_1 < I_2 < \ldots < I_r$ with $0 \leq r \leq D$ also referred to as the *rank* of $\gamma^{I_1\cdots I_r}$. It follows that the $\gamma^{I_1\cdots I_r}$ can be subdivided into two sub classes of symmmetric and antisymmetric elements. More precisely, there exists a unitary matrix C called *charge conjugation matrix* such that, w.r.t. that matrix, the basis elements satisfy

$$(C\gamma^{I_1\cdots I_r})^T = -t_r C\gamma^{I_1\cdots I_r} \tag{4.5}$$

for certain $t_r \in \{\pm 1\}$. The coefficients t_r are fixed by the choice of t_0 and t_1 via $t_2 = -t_0$, $t_3 = -t_1$ and $t_{r+4} = t_r$.

In concrete applications, we will usually work in Lorentzian signature, i.e., $s = 1$ and $t = D-1$. In this case, for even spacetime dimensions D, a useful formula which will often be used in the main text interrelating elements of the form (4.3) of different rank r is given by the following

$$\gamma^{I_1 I_2\cdots I_r}\gamma_* = (-i)^{\frac{D}{2}+1}\frac{1}{(D-r)!}\epsilon^{I_r I_{r-1}\cdots I_1 J_1\cdots J_{D-r}}\gamma_{J_1\cdots J_{D-r}} \tag{4.6}$$

for $0 \leq r \leq D$ where γ_* denotes the unique highest rank Clifford algebra element also commonly denoted by $\gamma_* \equiv \gamma_{D+1}$ defined as

$$\gamma_* := (-i)^{\frac{D}{2}+1}\gamma_0\gamma_1\ldots\gamma_{D-1} \tag{4.7}$$

Moreover, $\epsilon^{I_1\cdots I_D} = -\epsilon_{I_1\cdots I_D}$ denotes the completely antisymmetric symbol in D spacetime dimensions with the convention $\epsilon^{01\cdots D-1} = 1$. Finally, another important identity that we will frequently use is given by the *Fierz rearrangement formula* which, in case of even spacetime dimensions D, states that

$$M = \frac{1}{2^{D/2}} \sum_{r=0}^{D} \frac{1}{r!} \gamma_{I_1 \cdots I_r} \mathrm{tr}(\gamma^{I_r \cdots I_1} M) \tag{4.8}$$

for an arbitrary $D \times D$-matrix M. The Clifford algebra has the structure of a graded algebra via the decomposition $\mathrm{Cl}(\mathbb{R}^{s,t}, \eta) = \mathrm{Cl}(\mathbb{R}^{s,t}, \eta)_0 \oplus \mathrm{Cl}(\mathbb{R}^{s,t}, \eta)_1$ where $\mathrm{Cl}(\mathbb{R}^{s,t}, \eta)_i$ for $i = 0$ or 1 is the subalgebra generated by elements of the form (4.3) containing an even resp. odd number of elements γ_I. The even part $\mathrm{Cl}(\mathbb{R}^{s,t}, \eta)_0$ contains a subset $\mathrm{Spin}^+(s, t)$ which turns out to have the structure of a Lie group. In particular, it follows that this Lie group defines a universal covering of the orthochronous pseudo-orthogonal group $\mathrm{SO}^+(s, t)$ together with a covering map

$$\lambda^+ : \mathrm{Spin}^+(s, t) \to \mathrm{SO}^+(s, t) \tag{4.9}$$

In case of Minkowski spacetime in $D = 4$, $\mathrm{Spin}^+(1, 3)$ is isomorphic to $\mathrm{SL}(2, \mathbb{C})$. The Lie algebra $\mathfrak{spin}^+(s, t)$ of $\mathrm{Spin}^+(s, t)$ is generated by the elements

$$M_{IJ} := \frac{1}{2} \gamma_{IJ} \tag{4.10}$$

In this work, we are mainly concerned about four spacetime dimensions. In fact, most of the computations do not require a specific representation of the Clifford algebra. However, in Section 4.5.1, it will be worthwhile to choose a representation in which the gamma matrices are explicitly real which is also referred to as the *real* or *Majorana representation* of the gamma matrices. For instance, in case $D = 4$, a concrete realization of such a representation is provided by (for a discussion in case of arbitrary even spacetime dimensions see, e.g., [152])

$$\gamma_0 = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix} \text{ and } \gamma_3 = \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix} \tag{4.11}$$

where σ_i, $i = 1, 2, 3$ denote the ordinary Pauli matrices satisfying the product relation

$$\sigma_i \sigma_j = \delta_{ij} \mathbb{1} + i \epsilon_{ij}{}^{k} \sigma_k \tag{4.12}$$

On the other hand, in context of the quantization of the SUSY constraint to be discussed in Section 4.5.2, it will prove particularly beneficial to work instead in a *chiral representation* or *Weyl representation*. This will also play a prominent role in the context of self-dual variables as discussed in detail in Chapter 5 and 6. In this representation, the gamma matrices take the form

$$\gamma_I = \begin{pmatrix} 0 & \sigma_I \\ \bar{\sigma}_I & 0 \end{pmatrix} \quad \text{and} \quad \gamma_* = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \tag{4.13}$$

where $\sigma_I := (-\mathbb{1}, \sigma_i)$ and $\bar{\sigma}_I := (\mathbb{1}, \sigma_i)$. It follows that, in this representation, the generators (4.10) of $\mathfrak{spin}^+(1,3)$ then take the form

$$M_{IJ} = \frac{1}{2}\gamma_{IJ} = \frac{1}{4}\begin{pmatrix} \sigma_I\bar{\sigma}_J - \sigma_J\bar{\sigma}_I & 0 \\ 0 & \bar{\sigma}_I\sigma_J - \bar{\sigma}_J\sigma_I \end{pmatrix} \tag{4.14}$$

Moreover, they satisfy well-known Lie algebra relations

$$[M_{IJ}, M_{KL}] = \eta_{JK}M_{IL} - \eta_{IK}M_{JL} - \eta_{JL}M_{IK} + \eta_{IL}M_{JK} \tag{4.15}$$

The charge conjugation matrix C is given by $C = i\gamma^3\gamma^1$ and, according to (4.5) with $t_0 = 1$ and $t_2 = -1$, satisfies the symmetry relations

$$C^T = -C, \quad (C\gamma_I)^T = C\gamma_I \Leftrightarrow \gamma_I^T = -C\gamma_I C^{-1} \text{ and } (C\gamma_{IJ})^T = C\gamma_{IJ} \tag{4.16}$$

Next, let us briefly say something about Majorana representations and Majorana spinors. Let $\kappa : \mathrm{Spin}^+(s,t) \to \mathrm{GL}(\Delta_D)$ be the complex Dirac representation (for a detailed account on complex Dirac representations in arbitrary spacetime dimensions see for instance [104] and references therein). A Majorana representation is then defined as an induced representation on a real subspace of the complex vector space Δ_D. More precisely:

Definition 4.2.1. The complex spinor representation κ is called *Majorana* if it admits a *real structure* σ, i.e. a complex antilinear map $\sigma : \Delta_D \to \Delta_D$ such that σ is $\mathrm{Spin}^+(s,t)$-equivariant

$$\sigma \circ \kappa(g) = \kappa(g) \circ \sigma \tag{4.17}$$

$\forall g \in \mathrm{Spin}^+(s,t)$ and σ is involutive $\sigma^2 = \mathrm{id}_{\Delta_D}$.
The real structure defines a proper real $\mathrm{Spin}^+(s,t)$-invariant subspace

$$\Delta_{\mathbb{R}} := \{\psi \in \Delta_D | \, \sigma(\psi) = \psi\} \tag{4.18}$$

of Δ_D of real dimension $\dim_{\mathbb{R}} \Delta_{\mathbb{R}} = \dim_{\mathbb{C}} \Delta_D$. Moreover, due to $\mathrm{Spin}^+(s,t)$-equivariance, it induces a real sub representation

$$\kappa_{\mathbb{R}} : \mathrm{Spin}^+(s,t) \to \mathrm{GL}(\Delta_{\mathbb{R}}) \tag{4.19}$$

of the complex Dirac representation of $\mathrm{Spin}^+(s,t)$ on $\Delta_{\mathbb{R}}$ called the *Majorana representation* of $\mathrm{Spin}^+(s,t)$.

Choosing a basis of Δ_D, one can write the condition $\psi = \sigma(\psi)$ equivalently in the form

$$\psi^* = B\psi \tag{4.20}$$

which is also often referred to as the *Majorana condition* in the literature where B is a complex matrix satisfying $B^*B = \mathbb{1}$. This matrix is related to the charge conjugation matrix C via $B = it_0C\gamma^0$ where $t_0 \in \{\pm 1\}$ depends on the signature and the dimension of the spacetime. In the case of Minkowski spacetime in four spacetime dimensions, one usually sets $t_0 = 1$ in which case the charge conjugation matrix is given by $C = i\gamma^3\gamma^1$ and therefore, in the chiral representation,

$$B = \gamma^0\gamma^1\gamma^3 = \begin{pmatrix} 0 & -i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix} \tag{4.21}$$

For a Dirac fermion $\psi = (\chi, \phi)^T$, the Majorana condition (4.20) then reads

$$\psi^* = B\psi \quad \Leftrightarrow \quad \chi = -i\sigma_2\phi^* \text{ or } \phi = i\sigma_2\chi^* \tag{4.22}$$

In the Majorana representation (4.11), one has $C = i\gamma_0$ so that, in this case, the matrix B reduces to the identity matrix $B = \mathbb{1}$. Hence, it follows that the Majorana condition (4.20) is equivalent to $\psi^* = \psi$, that is, in the Majorana representation the Majorana spinor ψ is explicitly real.

Finally, by convention, the gamma matrices γ_I are defined to have the natural index position $(\gamma_I)^\alpha{}_\beta$ whereas spinors are denoted by ψ^α. On the other hand, the conjugate spinor $\bar{\psi} := \psi^T C$, by definition, has the natural index position $\bar{\psi}_\alpha \equiv \psi_\alpha$. Indices are raised and lowered w.r.t. $C^{\alpha\beta} := (C^{-1T})^{\alpha\beta}$ and $C_{\alpha\beta} := C_{\alpha\beta}$ with the convention

$$\psi^\alpha = C^{\alpha\beta}\psi_\beta \quad \text{and} \quad \psi_\alpha = \psi^\beta C_{\beta\alpha} \tag{4.23}$$

In the Weyl representation for $D = 4$, the individual 2-component Weyl spinors contained in the Dirac (resp. Majorana) spinor ψ^α are denoted by ψ^A and $\bar{\psi}_{A'}$, respectively, such that $\psi^\alpha = (\psi^A, \bar{\psi}_{A'})^T$. Since $C = \mathrm{diag}(i\epsilon, i\epsilon)$ with $\epsilon := i\sigma_2$ the completely antisymmetric symbol which itself carries the index structure ϵ^{AB} and $\epsilon^{A'B'}$, respectively, in accordance with (4.23) up to global factor of $\pm i$, primed and unprimed Weyl spinor indices are raised and lowered via

$$\psi_A = \psi^B\epsilon_{BA} \quad \text{and} \quad \psi^A = \epsilon^{AB}\psi_B \tag{4.24}$$

and analogously for primed indices. With respect to chiral spinorial indices, σ_I and $\bar{\sigma}_I$ are written as $\sigma_I^{AA'}$ and $\bar{\sigma}_{I\,A'A}$. These can be used to map the internal indices I of the co-frame e^I to spinorial indices setting

$$e_\mu^{AA'} = e_\mu^I \sigma_I^{AA'} \tag{4.25}$$

Due to $\epsilon\sigma_i\epsilon = \sigma_i^T$, one has the useful formula

$$\sigma_{I\,AA'} = \sigma_I^{BB'}\epsilon_{BA}\epsilon_{B'A'} = -\bar{\sigma}_{I\,A'A} \tag{4.26}$$

Using (4.26), it is easy to see that

$$\sigma^I_{AA'}\sigma_J^{AA'} = -2\delta^I_J \tag{4.27}$$

Finally, in view of the canonical analysis of chiral supergravity to be discussed in detail in Chapter 6, let us mention that in a 3+1-decomposition $\mathcal{M} \cong \mathbb{R}\times\Sigma$ of the four dimensional spacetime $\mathcal{M}$, one considers the spinor-valued one-forms $e_a^{AA'}$ which are related to the spatial metric q on Σ according to

$$2q_{ab} = -e_{aAA'}e_b^{AA'} \tag{4.28}$$

with $a = 1, 2, 3$. These, together with the future-directed unit normal vector field $n^{AA'}$ which is normal to the time slices Σ_t and satisfies

$$n_{AA'}e_a^{AA'} = 0 \quad \text{and} \quad n_{AA'}n^{AA'} = 2 \tag{4.29}$$

form a basis of spinors with one primed and one unprimed index. On then has the following important identities

$$n_{AA'}n^{AB'} = \delta^{B'}_{A'} \tag{4.30}$$
$$n_{AA'}n^{BA'} = \delta^B_A \tag{4.31}$$
$$\sigma_{iAA'}\sigma_j^{AB'} = -\delta_{ij}\delta^{B'}_{A'} - i\epsilon_{ij}{}^k n_{AA'}\sigma_k^{AB'} \tag{4.32}$$
$$\sigma_{iAA'}\sigma_j^{BA'} = -\delta_{ij}\delta^B_A + i\epsilon_{ij}{}^k n_{AA'}\sigma_k^{BA'} \tag{4.33}$$

4.3. Holst action for Supergravity in $D = 4$ and its $3+1$ decomposition

Recall from Section 3.4 that Poincaré supergravity in $D = 4$ with $\mathcal{N} = 1$ supersymmetry can be described geometrically as a super Cartan geometry modeled on the super

Klein geometry $(\mathrm{ISO}(\mathbb{R}^{1,3|4}), \mathrm{Spin}^+(1,3))$ with $\mathrm{ISO}(\mathbb{R}^{1,3|4})$ the super Poincaré group (Example 2.3.11) with super Lie algebra

$$\mathfrak{iso}(\mathbb{R}^{1,3|4}) = \mathbb{R}^{1,3} \oplus \mathfrak{spin}^+(1,3) \oplus \Delta_{\mathbb{R}} \tag{4.34}$$

The super Cartan connection $\mathcal{A} = e^I P_I + \frac{1}{2}\omega^{IJ} M_{IJ} + \psi^\alpha Q_\alpha$ splits into the spin connection $\omega \in \Omega^1(P, \mathfrak{spin}^+(1,3))$, the soldering form $e \in \Omega^1_{hor}(P, \mathbb{R}^{1,3})$ as well as the Rarita-Schwinger field $\psi \in \Omega^1_{hor}(P, \Delta_{\mathbb{R}})$ with $P \xrightarrow{\pi} M$ the underlying spin structure.[2]

For the purpose of describing supergravity in the context of LQG, we take the Holst action of $\mathcal{N} = 1$ Poincaré supergravity as stated in [65] which, adapted to our conventions and written in a coordinate-free form, reads[3] (see also Section 5.2.1)

$$\begin{aligned} S_{\mathrm{H}}^{\mathcal{N}=1}(e, \omega, \psi) = \frac{1}{2\kappa} \int_M & \Sigma^{IJ} \wedge (P_\beta \circ F(\omega))^{KL} \epsilon_{IJKL} \\ & + \kappa e^I \wedge \bar{\psi} \wedge \gamma_I \frac{\mathbb{1} + i\beta\gamma_*}{\beta} D^{(\omega)}\psi \end{aligned} \tag{4.35}$$

where $\kappa = 8\pi G$ and $D^{(\omega)}\psi := \mathrm{d}\psi + \kappa_{\mathbb{R}*}(\omega) \wedge \psi$ denotes the exterior covariant derivative of ψ and

$$(P_\beta \circ F(\omega))^{IJ} := {P_\beta}^{IJ}{}_{KL} F(\omega)^{KL} \quad \text{with} \quad {P_\beta}^{IJ}{}_{KL} := \frac{1}{2}\left(\delta^I_{[K}\delta^J_{L]} - \frac{1}{2\beta}\epsilon^{IJ}{}_{KL}\right) \tag{4.36}$$

with β the *Barbero-Immirzi parameter* which is either assumed to be real, i.e., $\beta \in \mathbb{R}^\times = \mathbb{R}\backslash\{0\}$, in case of real variables, or purely imaginary, i.e., $\beta = \pm i$ in case of the chiral theory. In this chapter, we are mostly interested in the case of real variables. The chiral theory will be discussed in detail in the following Chapter 5. In (4.35), $F(\omega) = \mathrm{d}\omega + \omega \wedge \omega$ is the associated curvature of ω and

$$\Sigma := e \wedge e \in \Omega^2_{hor}(P, \mathfrak{spin}^+(1,3)) \tag{4.37}$$

Note that, in the action (4.35), we have implicitly chosen a local section $s : M \supset U \to P$ of the bundle so that the differential forms appearing the action are implicitly assumed to be pulled back to respective differential forms on M. One needs to ensure that the equations of motion resulting from (4.35) are independent on the choice of the Barbero-Immirzi parameter and, at second order, are equivalent to those of ordinary

[2] Recall that the spin structure arises as the body of the principal super fiber bundle corresponding to the super Cartan geometry.

[3] For convenience, the factor $1/\sqrt{\kappa}$ will be absorbed in the Rarita-Schwinger field.

$\mathcal{N}=1$ Poincaré supergravity. To this end, one has to the vary (4.35) with respect to the spin connection ω. As this is rarely done explicitly in the literature, let us perform the variation for a general matter contribution. That is, we consider an action S of the form $S=S_{\mathrm{H}}+S_{\text{H-matter}}$, where S_{H} is the standard Holst action of pure first-order Einstein gravity and $S_{\text{H-matter}}$ is some Holst-like modification of the matter contribution such that the resulting equations of motion remain unchanged.

First, let us consider the Holst term of pure gravity

$$S_{\mathrm{H}}=\frac{1}{2\kappa}\int_M \Sigma^{IJ}\wedge(P_\beta\circ F(\omega))^{KL}\epsilon_{IJKL}=:\frac{1}{2\kappa}\int_M\langle\Sigma\wedge P_\beta\circ F(\omega)\rangle \tag{4.38}$$

where $\langle\cdot\wedge\cdot\rangle\,:\,\Omega^2(M,\mathfrak{spin}^+(1,3))\times\Omega^2(M,\mathfrak{spin}^+(1,3))\to\mathbb{R}$ is the extension of the Adjoint invariant bilinear form on $\mathfrak{spin}^+(1,3)$ to $\mathfrak{spin}^+(1,3)$-valued forms on M. Let us then consider a variation of connection $\omega+\delta\omega$. The variation of $F(\omega)$ is then given by $\delta F(\omega)=D^{(\omega)}\delta\omega$. Since $P_\beta\circ D^{(\omega)}\delta\omega=D^{(\omega)}(P_\beta\circ\delta\omega)$ and $\langle\Sigma\wedge D^{(\omega)}(P_\beta\circ\delta\omega)\rangle=-\langle D^{(\omega)}\Sigma\wedge P_\beta\circ\delta\omega\rangle$ up to a total derivative [105], this yields

$$\delta S_{\mathrm{H}}=\frac{1}{2\kappa}\int_M\langle D^{(\omega)}\Sigma\wedge P_\beta\circ\delta\omega\rangle=-\frac{1}{2\kappa}\int_M D^{(\omega)}\Sigma^{IJ}\wedge(P_\beta\circ\delta\omega)^{KL}\epsilon_{IJKL} \tag{4.39}$$

Using (4.15), it follows

$$\begin{aligned}D^{(\omega)}\Sigma^{IJ}&=\mathrm{d}(e^I\wedge e^J)+\frac{1}{4}\omega^{IJ}\wedge\Sigma^{KL}\otimes[M_{IJ},M_{KL}]^{IJ}\\&=\mathrm{d}e^I\wedge e^J-e^I\wedge\mathrm{d}e^J+\omega^I{}_K\wedge\Sigma^{KJ}+\omega^J{}_K\wedge\Sigma^{IK}\\&=\Theta^I\wedge e^J-e^I\wedge\Theta^J\end{aligned} \tag{4.40}$$

with $\Theta^I\equiv\Theta^{(\omega)I}=\mathrm{d}e^I+\omega^I{}_K\wedge e^K$ the components of the associated torsion 2-form $\Theta^{(\omega)}$. Inserting (4.40) into (4.39), this yields

$$\begin{aligned}\delta S_{\mathrm{H}}&=-\frac{1}{\kappa}\int_M\Theta^I\wedge e^J\wedge(P_\beta\circ\delta\omega)^{KL}\epsilon_{IJKL}\\&=-\frac{1}{2\kappa}\int_M\epsilon^{MNJO}\epsilon_{IJKL}\Theta^I_{\mu\nu}e^\mu_M e^\nu_N(P_\beta\circ\delta\omega_\rho)^{KL}e^\rho_O\,\mathrm{dvol}_M\\&=-\frac{1}{2\kappa}\int_M 3!\delta^{[M}_I\delta^N_K\delta^{O]}_L\Theta^I_{\mu\nu}e^\mu_M e^\nu_N(P_\beta\circ\delta\omega_\rho)^{KL}e^\rho_O\,\mathrm{dvol}_M\\&=-\frac{1}{\kappa}\int_M P_\beta{}^{KL}{}_{IJ}(2\Theta^\rho_{\rho\mu}e^\mu_K e^\nu_L+\Theta^\nu_{\mu\rho}e^\mu_K e^\rho_L)\delta\omega^{IJ}_\nu\,\mathrm{dvol}_M\end{aligned} \tag{4.41}$$

Hence, including the matter contribution, we find for the variation of the total action

$$\delta S = \int_M -\frac{1}{\kappa} P_\beta{}^{KL}{}_{IJ}(2\Theta^\rho_{\rho\mu} e^\mu_K e^\nu_L + \Theta^\nu_{\mu\rho} e^\mu_K e^\rho_L)\delta\omega^{IJ}_\nu + \frac{\delta S_{\text{H-matter}}}{\delta \omega^{IJ}_\nu}\delta\omega^{IJ}_\nu \, \mathrm{dvol}_M \tag{4.42}$$

which vanishes if and only if

$$P_\beta{}^{KL}{}_{IJ}(2\Theta^\rho_{\rho K} e^\nu_L + \Theta^\nu_{KL}) = \kappa e^{-1}\frac{\delta S_{\text{H-matter}}}{\delta \omega^{IJ}_\nu} \tag{4.43}$$

with $e := \det(e^I_\mu)$. Applying the inverse

$$(P^{-1}_\beta)_{IJ}{}^{KL} = \frac{2\beta^2}{1+\beta^2}\left(\delta^K_{[I}\delta^L_{J]} + \frac{1}{2\beta}\epsilon_{IJ}{}^{KL}\right) \tag{4.44}$$

on both sides of (4.43), this gives

$$2\Theta^\rho_{\rho I} e^\nu_J + \Theta^\nu_{IJ} = \kappa e^{-1}(P^{-1}_\beta)_{IJ}{}^{KL}\frac{\delta S_{\text{H-matter}}}{\delta \omega^{KL}_\nu} \tag{4.45}$$

This is the most general formula for the equations of motion of the spin connection for arbitrary matter contributions resulting from the variation of the Holst action. In case of $\mathcal{N} = 1$ supergravity, we have

$$\frac{\delta S_{\text{H-matter}}}{\delta \omega^{KL}_\nu} = -\frac{1}{4}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_\sigma\frac{\mathbb{1} + i\beta\gamma_*}{2\beta}\gamma_{KL}\psi_\rho \tag{4.46}$$

so that

$$(P^{-1}_\beta)_{IJ}{}^{KL}\frac{\delta S_{\text{H-matter}}}{\delta \omega^{KL}_\nu} = -\frac{\beta^2}{2(1+\beta^2)}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_\sigma\frac{\mathbb{1} + i\beta\gamma_*}{2\beta}\left(\gamma_{IJ} + \frac{1}{2\beta}\epsilon_{IJ}{}^{KL}\gamma_{KL}\right)\psi_\rho \tag{4.47}$$

Since $\epsilon_{IJ}{}^{KL}\gamma_{KL} = 2i\gamma_{IJ}\gamma_*$ by (4.6), this implies

$$\begin{aligned}(P^{-1}_\beta)_{IJ}{}^{KL}\frac{\delta S_{\text{H-matter}}}{\delta \omega^{KL}_\nu} &= -\frac{\beta^2}{2(1+\beta^2)}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_\sigma\frac{\mathbb{1} + i\beta\gamma_*}{2\beta}\left(\gamma_{IJ} + \frac{i}{\beta}\gamma_{IJ}\gamma_*\right)\psi_\rho \\ &= -\frac{\beta^2}{2(1+\beta^2)}i\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_\sigma\gamma_{IJ}\gamma_*\frac{\mathbb{1} + i\beta\gamma_*}{2\beta}\frac{\mathbb{1} - i\beta\gamma_*}{\beta}\psi_\rho \\ &= -\frac{i}{4}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_\sigma\gamma_{IJ}\gamma_*\psi_\rho \end{aligned} \tag{4.48}$$

Finally, using $\epsilon^{\mu\nu\rho\sigma}\gamma_\sigma = ie\gamma^{\mu\nu\rho}\gamma_*$, we find

$$2\Theta^{\rho}_{\ \rho I}e^{\nu}_{J} + \Theta^{\nu}_{IJ} = \frac{\kappa}{4}\bar{\psi}_\mu\gamma^{\mu\nu\rho}\gamma_{IJ}\psi_\rho \tag{4.49}$$

which are exactly the equations of motion of ω of $\mathcal{N} = 1$ supergravity, in particular, completely independent of the Barbero-Immirzi parameter. These can equivalently be written in the form [152]

$$\Theta^{\rho}_{\mu\nu} = \frac{\kappa}{2}\bar{\psi}_\mu\gamma^\rho\psi_\nu \tag{4.50}$$

In view of the canonical decomposition of the action, let us rewrite (4.35) in a coordinate-dependent form which gives

$$\begin{aligned} S_{\mathrm{H}}^{\mathcal{N}=1} = \int_M \mathrm{d}^4x\, \frac{e}{2\kappa}e^{\mu}_{I}e^{\nu}_{J}\left(F(\omega)^{IJ}_{\mu\nu} - \frac{1}{2\beta}\epsilon^{IJ}_{\ \ KL}F(\omega)^{KL}_{\mu\nu}\right) \\ + \epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_\sigma\frac{\mathbb{1}+i\beta\gamma_*}{2\beta}D^{(\omega)}_\nu\psi_\rho \end{aligned} \tag{4.51}$$

As shown above, variation of (4.51) yields the same equation of motion as the standard action (3.51) of $\mathcal{N} = 1$ Poincaré supergravity. As will be demonstrated explicitly in Section 5.3.1 in the context of AdS supergravity with $\mathcal{N} = 2$, reinserting the EOM of ω into the Holst action, the terms proportional to β^{-1} together become purely topological (see also [148]). Hence, the Holst action coincides with the ordinary one provided ω satisfies its field equations.

The $3+1$-split of the action (4.51) follows the standard procedure (see for instance [18] and references therein for a nice review on the canonical analysis of ordinary Einstein gravity). Since M is supposed to be globally hyperbolic, it is diffeomorphic to a foliation of the form $\mathbb{R}\times\Sigma$, where Σ is a spacelike Cauchy hypersurface. Let $\phi : \mathbb{R}\times\Sigma \to M$ denote such a diffeomorphism. Then, for a specific time $t \in \mathbb{R}$, we define the time slice Σ_t via $\Sigma_t = \phi_t(\Sigma)$, where $\phi_t := \phi(t,\cdot)$ describing the evolution of Σ in M. Furthermore, the flow of the time slices induces a global timelike vector field ∂_t which, on smooth functions $f \in C^\infty(M)$, acts via

$$\partial_t(f) = \frac{\mathrm{d}}{\mathrm{d}t}(f\circ\phi_t) \tag{4.52}$$

We choose a unit normal vector field n which is normal to the time slices such that there exists a *lapse function* N as well as a *shift vector field* $\vec{N}$ tangential to the foliation, such that

$$\partial_t = Nn + \vec{N} \tag{4.53}$$

In order to perform the 3+1-split the action (4.51), we have to decompose the covariant tensors according to the foliation. To this end, following [153], let us define the smooth geometric distribution $M \ni p \mapsto T_p^{\|}M := \{v \in T_pM \,|\, g(n,v) = 0\} \subset T_pM$ together with the projection $P^{\|}$ defined via [153]

$$\begin{aligned} P^{\|} : T_pM &\longrightarrow T_p^{\|}M \\ v &\longmapsto v + g(n,v)n \end{aligned} \tag{4.54}$$

$\forall p \in M$ where g denotes the metric on M induced by the soldering form, i.e., $g_{\mu\nu} = \eta_{IJ}e^I_\mu e^J_\nu$. By duality, this induces a corresponding projection $P_{\|}$ on the space of covariant tensor fields which for any $T \in \Gamma((T^*M)^{\otimes k})$, $k \in \mathbb{N}$, is defined as [153]

$$P_{\|}T := T \circ P^{\|} \tag{4.55}$$

where the projection on the right-hand side acts on each slot such that the contraction of any index of $(P_{\|}T)_{\mu\nu\dots}$ with n^ρ yields zero. Thus, for instance, in case of the curvature tensor $F(\omega)$, this yields

$$P_{\|}F(\omega) = F(\omega) - \iota_n F(\omega) \wedge n^\flat \tag{4.56}$$

where we set $n^\flat := g(n,\cdot)$. As another example, if L_{∂_t} denotes the Lie derivative with respect to the global timelike vector field ∂_t, one finds

$$P_{\|}L_{\partial_t}\omega = P_{\|}\left(i_{\partial_t}\mathrm{d}\omega + \mathrm{d}(\omega(\partial_t))\right) = NP_{\|}\left(i_n\mathrm{d}\omega\right) + i_{\vec{N}}\mathrm{d}(P_{\|}\omega) + P_{\|}\mathrm{d}\omega_t) \tag{4.57}$$

with $\omega_t := \omega(\partial_t)$ which yields the important identity

$$P_{\|}\left(i_n\mathrm{d}\omega\right) = \frac{1}{N}\left(P_{\|}L_{\partial_t}\omega - i_{\vec{N}}\mathrm{d}(P_{\|}\omega) - P_{\|}\mathrm{d}\omega_t\right) \tag{4.58}$$

In local coordinates, this reads

$$n^\rho\partial_{[\rho}\omega_{a]} = \frac{1}{2N}\left(L_{\partial_t}\omega_a - 2N^b\partial_{[b}\omega_{a]} - \partial_a\omega_t\right) \tag{4.59}$$

where $a, b \ldots = 1,2,3$ are local coordinate indices on Σ. With these preparations, we are ready to perform the 3+1-split of the action functional (4.51). As the canonical analysis of the purely bosonic term in (4.51) is very well-known [23] (see also [154] for a nice treatment), let us only comment on some main steps. By (4.56), it follows that the decomposition of the curvature tensor w.r.t. the unit normal (co)vector field yields

$$\frac{e}{2}e^\mu_I e^\nu_J P^{IJ}{}_{KL}\,F(\omega)^{KL}_{\mu\nu} = \frac{e}{2}e^a_i e^b_j P^{ij}{}_{KL}\,F^{KL}_{ab} + ee^\mu_I e^\nu_J n^\rho P^{IJ}{}_{KL}\,F^{KL}_{\rho[\mu}n_{\nu]} \tag{4.60}$$

with $F^{KL}_{\mu\nu} = 2\partial_{[\mu}\omega^{KL}_{\nu]} + 2\omega^{K}_{[\mu|M|}\omega^{ML}_{\nu]}$. Using identity (4.59), the last term in (4.60) becomes

$$
\begin{aligned}
&ee^{\mu}_{I}e^{\nu}_{J}n^{\rho}P^{IJ}{}_{KL}F^{KL}_{\rho[\mu}n_{\nu]}\\
=&-N\sqrt{q}n^{\rho}e^{\mu}_{i}\left(F^{i0}_{\rho\mu}-\frac{1}{2\beta}\epsilon^{i0}{}_{kl}F^{kl}_{\rho\mu}\right)=N\sqrt{q}e^{a}_{i}P^{0i}{}_{KL}n^{\rho}F^{KL}_{\rho a}\\
=&N\sqrt{q}e^{a}_{i}P^{0i}{}_{KL}\left(2n^{\rho}\partial_{[\rho}\omega^{KL}_{a]}+2n^{\rho}\omega^{K}_{[\rho|M|}\omega^{ML}_{a]}\right)\\
=&\frac{1}{\beta}\sqrt{q}e^{a}_{i}L_{\partial_t}\left(\beta\omega^{0i}_{a}-\frac{1}{2}\epsilon^{i}{}_{kl}\omega^{kl}_{a}\right)-\sqrt{q}e^{a}_{i}P^{0i}{}_{KL}\partial_{a}\omega^{KL}_{t}\\
&-2N^{b}\sqrt{q}e^{a}_{i}P^{0i}{}_{KL}\partial_{[b}\omega^{KL}{}_{a]}+2\sqrt{q}e^{a}_{i}P^{0i}{}_{KL}n^{\rho}\omega^{K}_{[\rho|M|}\omega^{ML}_{a]}\\
=&\frac{1}{\beta}E^{a}_{i}L_{\partial_t}{}^{\beta}\!A^{i}_{a}-\frac{1}{\beta}E^{a}_{i}\partial_{a}A^{i}_{t}+2E^{a}_{i}P^{0i}{}_{KL}\omega^{K}_{t\ M}\omega^{ML}_{a}\\
&-N^{b}E^{a}_{i}P^{0i}{}_{KL}\left(2\partial_{[b}\omega^{KL}{}_{a]}+2\omega^{K}_{[b|M|}\omega^{ML}_{a]}\right)
\end{aligned}
\tag{4.61}
$$

where

$$
{}^{\beta}\!A^{i}_{a}=\Gamma^{i}_{a}+\beta K^{i}_{a}\quad\text{and}\quad E^{a}_{i}=\sqrt{q}e^{a}_{i}
\tag{4.62}
$$

are the usual (real) *Ashtekar-Barbero connection* and the canonically conjugate *(gravitational) electric field*, respectively. Here, we set $\Gamma^{i}_{a}:=-\frac{1}{2}\epsilon^{i}{}_{kl}\omega^{kl}_{a}$ and $K^{i}_{a}:=\omega^{0i}_{a}$ for the $3D$ spin connection on Σ and extrinsic curvature, respectively. Moreover, $q_{ab}=\delta_{ij}e^{i}_{a}e^{j}_{b}$ denotes the induced metric on Σ. The canonically conjugate variables (4.62) satisfy the nonvanishing Poisson brackets

$$
\{{}^{\beta}\!A^{i}_{a}(x),E^{b}_{j}(y)\}=\kappa\beta\delta^{i}_{j}\delta^{b}_{a}\delta^{(3)}(x,y)
\tag{4.63}
$$

Furthermore, in (4.61), we introduced the Lagrange multiplier $A^{i}_{t}:=-\frac{1}{2}\epsilon^{i}{}_{kl}\omega^{kl}_{t}+\beta\omega^{0i}_{t}=:\Gamma^{i}_{t}+\beta K^{i}_{t}$. Since

$$
2P^{0i}{}_{KL}\omega^{K}_{t\ M}\omega^{ML}_{a}=\frac{1}{\beta}A^{m}_{t}\epsilon_{mn}{}^{i}\,{}^{\beta}\!A^{n}_{a}-\frac{1+\beta^{2}}{\beta}K^{m}_{t}\epsilon_{mn}{}^{i}K^{n}_{a}
\tag{4.64}
$$

the two mid terms in (4.61) can be combined to give, after integration by parts and dropping a boundary term,

$$
\begin{aligned}
&\frac{1}{\beta}A^{i}_{t}\partial_{a}E^{a}_{i}+2E^{a}_{i}P^{0i}{}_{KL}\omega^{K}_{t\ M}\omega^{ML}_{a}\\
=&A^{i}_{t}\frac{1}{\beta}\left(\partial_{a}E^{a}_{i}+\epsilon_{ik}{}^{l}\,{}^{\beta}\!A^{k}_{a}E^{a}_{l}\right)-\frac{1+\beta^{2}}{\beta}K^{m}_{t}\epsilon_{mn}{}^{i}K^{n}_{a}E^{a}_{i}
\end{aligned}
$$

$$=A^i_t \frac{1}{\beta} D^{(^\beta A)}_a E^a_i - \frac{1+\beta^2}{\beta} K^m_t \epsilon_{mn}{}^i K^n_a E^a_i \tag{4.65}$$

For the last term in (4.61) proportional to the shift vector field, it follows [154]

$$\begin{aligned}&- N^a E^b_i P^{0i}{}_{KL} \left(2\partial_{[a}\omega^{KL}{}_{b]} + 2\omega^K_{[a|M|}\omega^{ML}_{b]}\right)\\ &= N^a \frac{1}{\beta} E^b_i \left(F(^\beta A)^i_{ab} + (1+\beta^2)\epsilon^i{}_{kl} K^k_a K^l_b\right)\end{aligned} \tag{4.66}$$

with $F(^\beta A)^i = \mathrm{d}^\beta A^i + \frac{1}{2}\epsilon^i{}_{jk}{}^\beta A^j \wedge {}^\beta A^k$ the curvature of the Ashtekar-Barbero connection. Finally, following [154], let us comment on the first term appearing in the decomposition (4.60). Since $e = N\sqrt{q}$, this can be written in the form

$$\begin{aligned}\frac{e}{2} e^a_i e^b_j P^{ij}{}_{KL} F(\omega)^{KL}_{ab} &= \frac{N\sqrt{q}}{2} e^a_i e^b_j (F(\omega)^{ij}_{ab} + \frac{1}{\beta}\epsilon^{ij}{}_k F(\omega)^{0k}_{ab})\\ &= \frac{NE^a_i E^b_j}{2\sqrt{q}}(F(\Gamma)^{ij}_{ab} + 2\omega^{0i}_{[a}\omega^{0j}_{b]} + \frac{1}{\beta}\epsilon^{ij}{}_k F(\omega)^{0k}_{ab})\end{aligned} \tag{4.67}$$

with $F(\Gamma)$ the curvature of the $3D$ spin connection Γ. Using

$$F(\Gamma)^i_{ab} = F(^\beta A)^i_{ab} - 2\beta D^{(\Gamma)}_{[a} K^i_{b]} - \beta^2 \epsilon^i{}_{jk} K^j_a K^k_b \tag{4.68}$$

it follows that (4.67) can be written as

$$\begin{aligned}\frac{e}{2} e^a_i e^b_j P^{ij}{}_{KL} F(\omega)^{KL}_{ab} =& - \frac{NE^a_i E^b_j}{2\sqrt{q}} \epsilon^{ij}{}_k \left(F(^\beta A)^k_{ab} - (1+\beta^2)\epsilon^k{}_{mn} K^m_a K^n_b\right.\\ &\left. - \frac{2(1+\beta^2)}{\beta} D^{(\Gamma)}_{[a} K^k_{b]}\right)\end{aligned} \tag{4.69}$$

Next, let us decompose the fermionic part of the supergravity action (4.51). Following [65], since $e^0_t = -n^\flat(\partial_t) = N$ and $e^i_t = N^a e^i_a$, we find

$$\begin{aligned}\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_\sigma \frac{\mathbb{1} + i\beta\gamma_*}{2\beta} D^{(\omega)}_\nu \psi_\rho =& \epsilon^{abc}\bar{\psi}_t\gamma_a \frac{\mathbb{1} + i\beta\gamma_*}{2\beta} D^{(\omega)}_b \psi_c\\ &- N\epsilon^{abc}\bar{\psi}_a\gamma_0 \frac{\mathbb{1} + i\beta\gamma_*}{2\beta} D^{(\omega)}_b \psi_c\\ &- N^d\epsilon^{abc}\bar{\psi}_a\gamma_d \frac{\mathbb{1} + i\beta\gamma_*}{2\beta} D^{(\omega)}_b \psi_c +\end{aligned}$$

$$+\epsilon^{abc}\bar{\psi}_a\gamma_b\frac{\mathbb{1}+i\beta\gamma_*}{2\beta}\left(L_{\partial_t}\psi_c+\frac{1}{4}\omega_t^{IJ}\gamma_{IJ}\psi_c\right)$$
$$-\epsilon^{abc}\bar{\psi}_a\gamma_b\frac{\mathbb{1}+i\beta\gamma_*}{2\beta}\left(\partial_c\psi_t+\frac{1}{4}\omega_c^{IJ}\gamma_{IJ}\psi_t\right) \quad (4.70)$$

Hence, taking the left-derivative of kinematical term apearing in (4.70) with respect to ψ_t and noticing that fermionic fields are anticommuting, it follows that the momentum conjugate to ψ_a is given by

$$\pi^a=-\epsilon^{abc}\bar{\psi}_b\gamma_c\frac{\mathbb{1}+i\beta\gamma_*}{2\beta} \quad (4.71)$$

These satisfy the nonvanishing Poisson brackets

$$\{\psi_a^\alpha(x),\pi_\beta^b(y)\}=-\delta_b^a\delta_\beta^\alpha\delta^{(3)}(x,y) \quad (4.72)$$

In particular, according to (4.71), the canonically conjugate momentum π^a is related to ψ_a via the reality condition

$$\Omega^a:=\pi^a+\epsilon^{abc}\bar{\psi}_b\gamma_c\mathcal{P}_\beta=0 \quad (4.73)$$

where we set

$$\mathcal{P}_\beta:=\frac{\mathbb{1}+i\beta\gamma_*}{2\beta} \quad (4.74)$$

If we consider the last term in (4.70), it again follows after integration by parts and dropping a boundary term

$$\begin{aligned}-\epsilon^{abc}\bar{\psi}_a\gamma_b\frac{\mathbb{1}+i\beta\gamma_*}{2\beta}\left(\partial_c\psi_t+\frac{1}{4}\omega_c^{IJ}\gamma_{IJ}\psi_t\right)=&\epsilon^{abc}\partial_c\bar{\psi}_t\frac{\mathbb{1}+i\beta\gamma_*}{2\beta}\gamma_b\psi_a\\&-\frac{1}{4}\epsilon^{abc}\bar{\psi}_t\omega_c^{IJ}\gamma_{IJ}\frac{\mathbb{1}+i\beta\gamma_*}{2\beta}\gamma_b\psi_a\\=&\bar{\psi}_t\frac{\mathbb{1}+i\beta\gamma_*}{2\beta}\partial_a\left(\epsilon^{abc}\gamma_b\psi_c\right)\\&+\bar{\psi}_t\frac{\mathbb{1}+i\beta\gamma_*}{2\beta}\frac{1}{4}\epsilon^{abc}\omega_a^{IJ}\gamma_{IJ}\gamma_b\psi_c\\=&\bar{\psi}_t\frac{\mathbb{1}+i\beta\gamma_*}{2\beta}D_a^{(\omega)}\left(\epsilon^{abc}\gamma_b\psi_c\right)\end{aligned} \quad (4.75)$$

Let us rewrite (4.75) in terms of the covariant derivative of the Ashtekar connection. Since

$$\begin{aligned}\omega_a^{IJ}\gamma_{IJ} &= \omega_a^{ij}\gamma_{ij} + 2\omega_a^{0i}\gamma_{0i}\\ &= 2i\Gamma^i_a\gamma_*\gamma_{0i} + 2K^i_a\gamma_{0i}\end{aligned} \tag{4.76}$$

we find

$$\begin{aligned}\frac{\mathbb{1}+i\beta\gamma_*}{2\beta}\omega_a^{IJ}\gamma_{IJ} &= -\frac{\mathbb{1}+i\beta\gamma_*}{i\beta}\left(\Gamma^i_a\gamma_*\gamma_{0i} - iK^i_a\gamma_{0i}\right)\\ &= -\frac{1}{i\beta}\left(\Gamma^i_a\gamma_*\gamma_{0i} - iK^i_a\gamma_{0i} + i\beta\Gamma^i_a\gamma_{0i} + \beta K^i_a\gamma_*\gamma_{0i}\right)\\ &= -\frac{1}{i\beta}\left({}^{\beta}\!A^i_a - iK^i_a\gamma_* + i\beta\Gamma^i_a\gamma_*\right)\gamma_*\gamma_{0i}\\ &= -\frac{1}{i\beta}\left({}^{\beta}\!A^i_a + i\beta\,{}^{\beta}\!A^i_a\gamma_* - i(1+\beta^2)K^i_a\gamma_*\right)\gamma_*\gamma_{0i}\\ &= \frac{\mathbb{1}+i\beta\gamma_*}{2\beta}2i\,{}^{\beta}\!A^i_a\gamma_*\gamma_{0i} + \frac{1+\beta^2}{\beta}K^i_a\gamma_{0i}\end{aligned} \tag{4.77}$$

Hence, this yields

$$\frac{\mathbb{1}+i\beta\gamma_*}{2\beta}D_a^{(\omega)}\psi_b = \frac{\mathbb{1}+i\beta\gamma_*}{2\beta}D_a^{({}^{\beta}\!A)}\psi_b + \frac{1+\beta^2}{4\beta}K^i_a\gamma_{0i}\psi_b \tag{4.78}$$

with

$$D_a^{({}^{\beta}\!A)}\psi_b := \partial_a\psi_b + \frac{i}{2}{}^{\beta}\!A^i_a\gamma_*\gamma_{0i}\psi_b \tag{4.79}$$

With respect to the chiral representation of the gamma matrices, one has

$$\frac{i}{2}\gamma_*\gamma_{0i} = \begin{pmatrix}\tau_i & 0\\ 0 & \tau_i\end{pmatrix} \tag{4.80}$$

where $\tau_i := \frac{1}{2i}\sigma_i$ for $i = 1, 2, 3$ is a basis of generators of $\mathfrak{su}(2)$. Hence, in particular, in the chiral representation the covariant derivative acts separately on the respective chiral sub components of the Rarita-Schwinger field. We will use this property later in Section 4.5.2, when we will study the action of SUSY constraint on spin network states. Note that the appearance of the term $\frac{i}{2}\gamma_*\gamma_{0i}$ in the covariant derivative in (4.79) is not a coincidence, but follows from the identification of $\mathfrak{su}(2)$ as a Lie subalgebra of $\mathfrak{spin}^+(1,3)$ generated by $M_{jk}=\frac{1}{2}\gamma_{jk}$ such that ${}^{\beta}\!A=-\frac{1}{2}\epsilon_i^{\ jk}\,{}^{\beta}\!A^i M_{jk}$ which implies

$$\kappa_{\mathbb{R}*}({}^{\beta}\!A)=-\frac{1}{2}{}^{\beta}\!A^i\epsilon_i^{\ jk}\kappa_{\mathbb{R}*}(M_{jk})=-\frac{1}{4}{}^{\beta}\!A^i\epsilon_i^{\ jk}\gamma_{jk}=\frac{i}{2}\gamma_*\gamma_{0i}{}^{\beta}\!A^i \tag{4.81}$$

For the derivation of the SUSY constraint, we need to collect the terms in (4.75) proportional to ψ_t. Using (4.78), one finds again by integration by parts and eventually dropping boundary terms

$$\begin{aligned}
&\epsilon^{abc}\bar{\psi}_t\gamma_a\frac{\mathbb{1}+i\beta\gamma_*}{2\beta}D_b^{(\omega)}\psi_c-\epsilon^{abc}\bar{\psi}_a\gamma_b\frac{\mathbb{1}+i\beta\gamma_*}{2\beta}\left(\partial_c\psi_t+\frac{1}{4}\omega_c^{IJ}\gamma_{IJ}\psi_t\right)\\
=&\bar{\psi}_t\left(\epsilon^{abc}\gamma_a\frac{\mathbb{1}+i\beta\gamma_*}{2\beta}D_b^{(\omega)}\psi_c+\frac{\mathbb{1}+i\beta\gamma_*}{2\beta}D_a^{(\omega)}\left(\epsilon^{abc}\gamma_b\psi_c\right)\right)\\
=&\bar{\psi}_t\left(\epsilon^{abc}\gamma_a\frac{\mathbb{1}+i\beta\gamma_*}{2\beta}D_b^{({}^{\beta}\!A)}\psi_c+\frac{\mathbb{1}+i\beta\gamma_*}{2\beta}D_a^{({}^{\beta}\!A)}\left(\epsilon^{abc}\gamma_b\psi_c\right)\right.\\
&\left.-\frac{1+\beta^2}{4\beta}\epsilon^{abc}K_b^i e_a^j\gamma_0[\gamma_i,\gamma_j]_+\psi_c\right)\\
=&\bar{\psi}_t\left(\epsilon^{abc}\gamma_a\frac{\mathbb{1}+i\beta\gamma_*}{2\beta}D_b^{({}^{\beta}\!A)}\psi_c+\frac{\mathbb{1}+i\beta\gamma_*}{2\beta}D_a^{({}^{\beta}\!A)}\left(\epsilon^{abc}\gamma_b\psi_c\right)-\frac{1+\beta^2}{2\beta}\epsilon^{abc}K_{ba}\gamma_0\psi_c\right)
\end{aligned} \tag{4.82}$$

Hence, the SUSY constraint of the theory takes the form

$$\begin{aligned}
S=&\epsilon^{abc}\gamma_a\frac{\mathbb{1}+i\beta\gamma_*}{2\beta}D_b^{({}^{\beta}\!A)}\psi_c+\frac{\mathbb{1}+i\beta\gamma_*}{2\beta}D_a^{({}^{\beta}\!A)}\left(\epsilon^{abc}\gamma_b\psi_c\right)\\
&-\frac{1+\beta^2}{2\beta}\epsilon^{abc}\gamma_0\psi_c K_{ba}
\end{aligned} \tag{4.83}$$

For the term proportional to ω_t in (4.70) we compute, using (4.77),

$$\begin{aligned}
\frac{1}{4}\epsilon^{abc}\bar{\psi}_a\gamma_b\frac{\mathbb{1}+i\beta\gamma_*}{2\beta}\omega_t^{IJ}\gamma_{IJ}\psi_c=&A_t^i\left(-\frac{1}{4}\epsilon^{abc}\bar{\psi}_a\gamma_b\frac{\mathbb{1}+i\beta\gamma_*}{i\beta}\gamma_*\gamma_{0i}\psi_c\right)\\
&+\frac{1+\beta^2}{4\beta}K_t^i\epsilon^{abc}\bar{\psi}_a\gamma_b\gamma_{0i}\psi_c\\
=&A_t^i\left(-\frac{i}{2}\pi^a\gamma_*\gamma_{0i}\psi_a\right)+\frac{1+\beta^2}{4\beta}K_t^i\epsilon^{abc}\bar{\psi}_a\gamma_b\gamma_{0i}\psi_c
\end{aligned} \tag{4.84}$$

so that, combining with (4.65), this yields

$$A_t^i G_i = A_t^i \left(\frac{1}{\kappa\beta} D_a^{(\beta A)} E_i^a - \frac{i}{2} \pi^a \gamma_* \gamma_{0i} \psi_a \right) \tag{4.85}$$

Hence, the Gauss constraint takes the form

$$\begin{aligned} G_i &= \frac{1}{\kappa\beta} D_a^{(\beta A)} E_i^a - \frac{i}{2} \pi^a \gamma_* \gamma_{0i} \psi_a \\ &= \frac{1}{\kappa\beta} D_a^{(\beta A)} E_i^a + \frac{i}{2} \epsilon^{abc} \bar{\psi}_a \gamma_* \gamma_0 \gamma_b \gamma_i \frac{\mathbb{1} + i\beta\gamma_*}{2\beta} \psi_c \end{aligned} \tag{4.86}$$

As fermion fields anticommute, it follows that

$$\epsilon^{abc} \bar{\psi}_a \gamma_0 \gamma_{de} \psi_c = \epsilon^{abc} \bar{\psi}_c \gamma_{de} \gamma_0 \psi_a = -\epsilon^{abc} \bar{\psi}_a \gamma_0 \gamma_{de} \psi_c = 0 \tag{4.87}$$

Therefore, combining the last term in (4.84) with the last term in (4.65), this gives

$$\begin{aligned} & - \frac{1+\beta^2}{\beta} K_t^i \left(\frac{1}{\kappa} \epsilon_{ik}{}^l K_a^k E_l^a - \frac{1}{4} \epsilon^{abc} \bar{\psi}_a \gamma_b \gamma_{0i} \psi_c \right) \\ =& - \frac{1+\beta^2}{\beta} K_t^i \left(\frac{1}{\kappa} \epsilon_{ik}{}^l K_a^k E_l^a + \frac{1}{4} \epsilon^{abc} e_{bi} \bar{\psi}_a \gamma_0 \psi_c \right) \end{aligned} \tag{4.88}$$

yielding the second class constraint

$$\epsilon_{ik}{}^l K_a^k E_l^a + \frac{\kappa}{4} \epsilon^{abc} e_{bi} \bar{\psi}_a \gamma_0 \psi_c = 0 \tag{4.89}$$

For the vector constraint, we need to collect terms proportional to the shift vector field N^a. From (4.66), we deduce, using (4.84),

$$\begin{aligned} & N^d \frac{1}{\kappa\beta} E_i^b \left(F(^\beta A)^i_{db} + (1+\beta^2) \epsilon^i{}_{kl} K_d^k K_b^l \right) \\ =& N^d \frac{1}{\kappa\beta} E_i^b F(^\beta A)^i_{db} + \frac{1+\beta^2}{4\kappa\beta} N^d K_d^k \epsilon_{kl}{}^i K_b^l E_i^b \\ =& N^d \frac{1}{\kappa\beta} E_i^b F(^\beta A)^i_{db} - N^d \frac{1+\beta^2}{4\beta} \epsilon^{abc} K_{db} \bar{\psi}_a \gamma_0 \psi_c \end{aligned} \tag{4.90}$$

On the other hand, (4.70) yields together with (4.78)

$$
\begin{aligned}
&-N^d\epsilon^{abc}\bar{\psi}_a\gamma_d\frac{\mathbb{1}+i\beta\gamma_*}{2\beta}D_b^{(\omega)}\psi_c\\
=&-N^d\epsilon^{abc}\bar{\psi}_a\gamma_d\frac{\mathbb{1}+i\beta\gamma_*}{2\beta}D_b^{({}^\beta\!A)}\psi_c-N^d\frac{1+\beta^2}{4\beta}\epsilon^{abc}K_b^i\bar{\psi}_a\gamma_d\gamma_{0i}\psi_c\\
=&-N^d\epsilon^{abc}\bar{\psi}_a\gamma_d\frac{\mathbb{1}+i\beta\gamma_*}{2\beta}D_b^{({}^\beta\!A)}\psi_c+N^d\frac{1+\beta^2}{4\beta}\epsilon^{abc}K_{bd}\bar{\psi}_a\gamma_0\psi_c
\end{aligned}
\tag{4.91}
$$

Therefore, the vector constraint is given by

$$
H_d:=\frac{1}{\kappa\beta}E_i^bF({}^\beta\!A)_{db}^i-\epsilon^{abc}\bar{\psi}_a\gamma_d\frac{\mathbb{1}+i\beta\gamma_*}{2\beta}D_b^{({}^\beta\!A)}\psi_c+\frac{1+\beta^2}{2\beta}\epsilon^{abc}K_{[bd]}\bar{\psi}_a\gamma_0\psi_c
\tag{4.92}
$$

Finally, using (4.69), we find for the Hamilton constraint of the theory, modulo the second class constraint,

$$
\begin{aligned}
H=&\frac{E_i^aE_j^b}{2\kappa\sqrt{q}}\epsilon^{ij}{}_k\left(F({}^\beta\!A)_{ab}^k-(1+\beta^2)\epsilon^k{}_{mn}K_a^mK_b^n\right)\\
&+\epsilon^{abc}\bar{\psi}_a\gamma_0\frac{\mathbb{1}+i\beta\gamma_*}{2\beta}D_b^{({}^\beta\!A)}\psi_c+\frac{1+\beta^2}{4\beta}\epsilon^{abc}K_b^i\bar{\psi}_a\gamma_{0i}\psi_c
\end{aligned}
\tag{4.93}
$$

The form of the constraints as derived in this section are consistent with those found in [65]. At this point, we have expressed them so far in terms of ${}^\beta\!A$, E, ψ, π, Γ and K. However, while we can further express K as $K({}^\beta\!A,\Gamma)$, Γ is undetermined as of yet. At the same time we have a further second class constraint, coming from the variation of the action with respect to

$$
{}^-A_a^i=\Gamma_a^i-\beta K_a^i.
\tag{4.94}
$$

The 9 components of this constraint, together with the 3 components of (4.89) should allow us to solve for Γ and K_t, thus solving the second class constraints. The calculation is tedious already for Dirac fermions coupled to gravity [155], so we take a shortcut. The precise expression for K_t is not relevant for our purposes and the gravitational contribution to Γ, the torsion-free spin connection, is well known. The fermionic contribution is simply the spatial component of the *contortion tensor* $C_{\rho IJ}$ which, using (4.50), is given by

$$
C_a^i:=-\epsilon^{ijk}C_{ajk}=-\frac{\kappa}{8\sqrt{q}}\epsilon^{bcd}e_d^i(\bar{\psi}_b\gamma_a\psi_c+2\bar{\psi}_b\gamma_c\psi_a)
\tag{4.95}
$$

This is a function of E, ψ, π. From now on, we always assume that Γ and K are determined by the canonical variables in this way.

4.3.1. Introducing half-densitized fermion fields

As proposed in [80], in order to solve the reality conditions of fermion fields in canonical quantum gravity, it is worthwhile to go over to half-densitized fermion fields. In the case of the Rarita-Schwinger field, this amounts to introducing the new fields [64]

$$\phi_i = \sqrt[4]{q} e^a_i \psi_a \quad \text{and} \quad \pi^i_\phi = \frac{1}{\sqrt[4]{q}} e^i_{\ a} \pi^a \tag{4.96}$$

As both sides have been rescaled by the spatial metric, it is clear that this, a priori, does not define a canonical transformation. In fact, as we will see in the following, this requires a redefinition of the Ashtekar connection. Therefore, following the same steps as in [67], we substitute the transformed fields (4.96) in the symplectic potential which yields

$$\begin{aligned}
&\int_{\mathbb{R}} \mathrm{d}t \int_\Sigma \mathrm{d}^3x \, \frac{1}{\kappa\beta} E^a_i {}^\beta\dot{A}^i_a - \pi^a \dot{\psi}_a \\
&= \int_{\mathbb{R}} \mathrm{d}t \int_\Sigma \mathrm{d}^3x \, \frac{1}{\kappa\beta} E^a_i {}^\beta\dot{A}^i_a - \frac{1}{\sqrt[4]{q}} E^a_i \pi^i_\phi L_{\partial_t} \left(\sqrt[4]{q} E^j_a \phi_j\right) \\
&= \int_{\mathbb{R}} \mathrm{d}t \int_\Sigma \mathrm{d}^3x \, \frac{1}{\kappa\beta} E^a_i {}^\beta\dot{A}^i_a - \pi^i_\phi \dot{\phi}_i - \pi^i_\phi E^a_i \dot{E}^j_a \phi_j \\
&= \int_{\mathbb{R}} \mathrm{d}t \int_\Sigma \mathrm{d}^3x \, \frac{1}{\kappa\beta} E^a_i {}^\beta\dot{A}^i_a - \pi^i_\phi \dot{\phi}_i + \pi^i_\phi \dot{E}^a_i E^j_a \phi_j \\
&= \int_{\mathbb{R}} \mathrm{d}t \int_\Sigma \mathrm{d}^3x \, \frac{1}{\kappa\beta} E^a_i {}^\beta\dot{A}^i_a - \pi^i_\phi \dot{\phi}_i - E^a_i L_{\partial_t} \left(\pi^i_\phi E^j_a \phi_j\right) \\
&= \int_{\mathbb{R}} \mathrm{d}t \int_\Sigma \mathrm{d}^3x \, \frac{1}{\kappa} E^a_i L_{\partial_t} \left({}^\beta A^i_a - \kappa\beta \pi^i_\phi E^j_a \phi_j\right) - \pi^i_\phi \dot{\phi}_i
\end{aligned} \tag{4.97}$$

where we have dropped a boundary term from the third to the fourth line. Hence, transforming the Ashtekar-Barbero connection via

$${}^\beta A^i_a \to {}^\beta A'^i_a = \Gamma^i_a + \beta K'^i_a \tag{4.98}$$

with

$$\begin{aligned}
K'^i_a &= K^i_a - \kappa \pi^i_\phi E^l_a \phi_l = K^i_a + \frac{\kappa}{q} \epsilon^{dbc} e^i_{\ d} e^j_{\ b} e^k_{\ c} e^l_{\ a} \bar{\phi}_j \gamma_k \frac{\mathbb{1} + i\beta\gamma_*}{2\beta} \phi_l \\
&= K^i_a + \frac{i\kappa}{2\sqrt{q}} \epsilon^{ijk} e^l_{\ a} \bar{\phi}_j \gamma_k \frac{\mathbb{1} + i\beta\gamma_*}{i\beta} \phi_l
\end{aligned} \tag{4.99}$$

this yields a canonical transformation with the new canonically conjugate pairs $({A'}^i_a, E^a_i)$ and (ϕ_i, π^i_ϕ) and the nonvanishing Poisson brackets

$$\{{}^\beta {A'}^i_a(x), E^b_j(y)\} = \kappa\beta\delta^{(3)}(x,y) \quad \text{and} \quad \{\phi^\alpha_i(x), \pi^j_{\phi\,\beta}(y)\} = -\delta^j_i \delta^\alpha_\beta \delta^{(3)}(x,y) \tag{4.100}$$

In the new variables, the reality condition (4.73) takes the form

$$\Omega^i := \pi^i_\phi + \epsilon^{ijk}\bar{\phi}_j \gamma_k \mathcal{P}_\beta = 0 \tag{4.101}$$

which now, in particular, neither depends on the internal triad nor on the spatial metric simplifying significantly the further canonical analysis. As a next step, we have to reformulate the constraints in the new variables. Since we will mainly be interested in the explicit form of the SUSY constraint, we will only derive the transformed expressions of the Gauss and SUSY constraint in what follows. The remaining constraints can be treated in complete analogy.

4.3.1.1. Gauss constraint

By (4.86), the Gauss constraint takes the form

$$\begin{aligned} G_i &= \frac{1}{\kappa\beta} D^{({}^\beta A)}_a E^a_i + \frac{i}{2}\epsilon^{abc}\bar{\psi}_a \gamma_* \gamma_0 \gamma_b \gamma_i \frac{\mathbb{1} + i\beta\gamma_*}{2\beta}\psi_c \\ &= \frac{1}{\kappa\beta} D^{({}^\beta A)}_a E^a_i + \frac{i}{2}\epsilon^{jmk}\bar{\phi}_j \gamma_* \gamma_0 \gamma_m \gamma_i \frac{\mathbb{1} + i\beta\gamma_*}{2\beta}\phi_k \end{aligned} \tag{4.102}$$

Considering the first part in (4.102), we find

$$\begin{aligned} D^{({}^\beta A)}_a E^a_i =& \partial_a E^a_i + \epsilon_{im}{}^n ({}^\beta {A'}^m_a + \kappa\beta\pi^m_\phi E^l_a \phi_l) E^a_n \\ =& D^{({}^\beta A')}_a E^a_i + \frac{i\kappa\beta}{2}\epsilon_{mi}{}^l \epsilon^{mjk}\bar{\phi}_j \gamma_k \frac{\mathbb{1} + i\beta\gamma_*}{i\beta}\phi_l \\ =& D^{({}^\beta A')}_a E^a_i + \frac{i\kappa\beta}{2}\bar{\phi}_i \gamma_k \frac{\mathbb{1} + i\beta\gamma_*}{i\beta}\phi^k - \frac{i\kappa\beta}{2}\bar{\phi}_l \gamma_i \frac{\mathbb{1} + i\beta\gamma_*}{i\beta}\phi^l \\ =& D^{({}^\beta A')}_a E^a_i + \frac{\kappa}{2}\bar{\phi}_i \gamma_k \phi^k - \frac{i\kappa\beta}{2}\bar{\phi}_i \gamma_* \gamma_k \phi^k + \frac{i\kappa\beta}{2}\bar{\phi}_l \gamma_* \gamma_i \phi^l \end{aligned} \tag{4.103}$$

Since $\gamma_i \gamma_j = \delta_{ij} + \gamma_{ij}$, one has

$$\begin{aligned}\frac{i}{2}\epsilon^{jmk}\bar{\phi}_j\gamma_*\gamma_0\gamma_m\gamma_i\frac{\mathbb{1}+i\beta\gamma_*}{2\beta}\phi_k =& \frac{1}{4}\epsilon^{jmk}\bar{\phi}_j\gamma_0\gamma_m\gamma_i\phi_k + \frac{i}{4\beta}\epsilon^{jmk}\bar{\phi}_j\gamma_*\gamma_0\gamma_m\gamma_i\phi_k \\ =& \frac{1}{4}\epsilon^{jmk}\bar{\phi}_j\gamma_0\gamma_m\gamma_i\phi_k - \frac{i}{4\beta}\epsilon^{ijk}\bar{\phi}_j\gamma_*\gamma_0\phi_k \\ &+ \frac{i}{4\beta}\epsilon^{jmk}\bar{\phi}_j\gamma_*\gamma_0\gamma_{mi}\phi_k \end{aligned} \tag{4.104}$$

By antisymmetry of the fermion fields, it follows

$$\epsilon^{ijk}\bar{\phi}_j\gamma_*\gamma_0\phi_k = \epsilon^{ijk}\bar{\phi}_k\gamma_*\gamma_0\phi_j = -\epsilon^{ijk}\bar{\phi}_j\gamma_*\gamma_0\phi_k = 0 \tag{4.105}$$

so that, using $\gamma_*\gamma_{ij} = -i\epsilon_{ij}{}^k\gamma_{0k}$, we find

$$\begin{aligned}\frac{i}{2}\epsilon^{jmk}\bar{\phi}_j\gamma_*\gamma_0\gamma_m\gamma_i\frac{\mathbb{1}+i\beta\gamma_*}{2\beta}\phi_k =& \frac{1}{4}\epsilon^{jmk}\bar{\phi}_j\gamma_0\gamma_m\gamma_i\phi_k - \frac{1}{4\beta}\bar{\phi}_i\gamma_k\phi_k + \frac{1}{4\beta}\bar{\phi}_k\gamma^k\phi_i \\ =& \frac{1}{4}\epsilon^{jmk}\bar{\phi}_j\gamma_0\gamma_m\gamma_i\phi_k - \frac{1}{2\beta}\bar{\phi}_i\gamma_k\phi_k \\ =& -\frac{1}{4}\epsilon_i{}^{jk}\bar{\phi}_j\gamma_0\phi_k - \frac{1}{2\beta}\bar{\phi}_i\gamma_k\phi_k \end{aligned} \tag{4.106}$$

where from the second to the last line we again used (4.105). Hence, the Gauss constraint can be written as

$$G_i = D_a^{(^\beta A')}E^a_i - \frac{1}{4}\epsilon_i{}^{jk}\bar{\phi}_j\gamma_0\phi_k - \frac{i}{2}\bar{\phi}_i\gamma_*\gamma_k\phi^k + \frac{i}{2}\bar{\phi}_k\gamma_*\gamma_i\phi^k \tag{4.107}$$

In fact, this can be simplified even further. To see this, note that

$$\begin{aligned}\bar{\phi}_j\gamma_*\gamma_k\gamma_i\gamma^{(j}\phi^{k)} =& \frac{1}{2}\bar{\phi}_j\gamma_*\gamma_k\gamma_i\gamma^j\phi^k + \frac{1}{2}\bar{\phi}_j\gamma_*\gamma_k\gamma_i\gamma^k\phi^j \\ =& \frac{1}{2}\bar{\phi}_j\gamma_*\gamma_k\gamma_i\gamma^j\phi^k - \frac{1}{2}\bar{\phi}_j\gamma_*\gamma_i\phi^j \end{aligned} \tag{4.108}$$

which, due to $\gamma_i\gamma^j = 2\delta_i^j - \gamma^j\gamma_i$ yields

$$\begin{aligned}\bar{\phi}_j\gamma_*\gamma_k\gamma_i\gamma^{(j}\phi^{k)} =& \bar{\phi}_i\gamma_*\gamma_k\phi^k - \frac{1}{2}\bar{\phi}^j\gamma_*\gamma_{kj}\gamma_i\phi^k - \bar{\phi}_k\gamma_*\gamma_i\phi^k \\ =& \frac{i}{2}\epsilon^{klj}\bar{\phi}_k\gamma_0\gamma_l\gamma_i\phi_j + \bar{\phi}_i\gamma_*\gamma_k\phi^k - \bar{\phi}_k\gamma_*\gamma_i\phi^k \\ =& -\frac{i}{2}\epsilon_i{}^{kj}\bar{\phi}_k\gamma_0\phi_j + \bar{\phi}_i\gamma_*\gamma_k\phi^k - \bar{\phi}_k\gamma_*\gamma_i\phi^k \end{aligned} \tag{4.109}$$

Thus, to summarize, in the new variables, we find that the Gauss constraint can be written in the following compact form

$$G_i = D_a^{(\beta A')} E_i^a - \frac{i}{2}\bar{\phi}_j \gamma_* \gamma_k \gamma_i \gamma^{(j}\phi^{k)} \tag{4.110}$$

4.3.1.2. Supersymmetry constraint

Finally, we want to express the supersymmetry constraint S in the new variables. To this end, inserting (4.96) as well as (4.98) and (4.99) into (4.83), the first two terms in (4.83) become

$$\begin{aligned}
&\epsilon^{abc} e^i_a \gamma_i \frac{\mathbb{1} + i\beta\gamma_*}{2\beta} D_b^{(\beta A')}\left(\frac{1}{\sqrt[4]{q}} e^j_c \phi_j\right) + \frac{\mathbb{1} + i\beta\gamma_*}{2\beta} D_a^{(\beta A')}\left(\frac{1}{\sqrt[4]{q}}\epsilon^{ijk} E^a_i \gamma_j \phi_k\right) \\
&+ \frac{i\kappa\beta}{2\sqrt[4]{q}}\epsilon^{lmn}\epsilon^{ijk}\frac{\mathbb{1} - i\beta\gamma_*}{2\beta}\gamma_*\gamma_0\gamma_m\gamma_i\phi_n\left(\bar{\phi}_j\gamma_k\frac{\mathbb{1} + i\beta\gamma_*}{2\beta}\phi_l\right) \\
&- \frac{i\kappa\beta}{2\sqrt[4]{q}}\epsilon^{lmn}\epsilon^{ijk}\frac{\mathbb{1} + i\beta\gamma_*}{2\beta}\gamma_*\gamma_0\gamma_i\gamma_m\phi_n\left(\bar{\phi}_j\gamma_k\frac{\mathbb{1} + i\beta\gamma_*}{2\beta}\phi_l\right)
\end{aligned} \tag{4.111}$$

where the second and last line in (4.111) can be summarized as

$$\begin{aligned}
&\frac{i\kappa\beta}{2\sqrt[4]{q}}\epsilon^{lmn}\epsilon^{ijk}\left(\bar{\phi}_j\gamma_k\frac{\mathbb{1} + i\beta\gamma_*}{2\beta}\phi_l\right)\left[\frac{1}{2\beta}\gamma_*\gamma_0[\gamma_m,\gamma_i]_- - \frac{i}{2}\gamma_0[\gamma_m,\gamma_i]_+\right]\phi_n \\
=&\frac{i\kappa}{2\sqrt[4]{q}}\epsilon^{lmn}\epsilon^{ijk}\gamma_*\gamma_{0mi}\phi_n\left(\bar{\phi}_j\gamma_k\frac{\mathbb{1} + i\beta\gamma_*}{2\beta}\phi_l\right) \\
&- \frac{\kappa\beta}{2\sqrt[4]{q}}\epsilon_i{}^{ln}\epsilon^{ijk}\gamma_0\phi_n\left(\bar{\phi}_j\gamma_k\frac{\mathbb{1} + i\beta\gamma_*}{2\beta}\phi_l\right)
\end{aligned} \tag{4.112}$$

Since $\gamma_*\gamma_{0mi} = -i\epsilon_{mi}{}^p\gamma_p$, the first term in the second line of (4.112) takes the form

$$\frac{\kappa}{2\sqrt[4]{q}}\epsilon^{lmn}\epsilon_{mi}{}^p\epsilon^{ijk}\gamma_p\phi_n\left(\bar{\phi}_j\gamma_k\frac{\mathbb{1} + i\beta\gamma_*}{2\beta}\phi_l\right) = \frac{\kappa}{\sqrt[4]{q}}\epsilon^{ijk}\gamma^l\phi_{[l}\left(\bar{\phi}_{i]}\frac{\mathbb{1} + i\beta\gamma_*}{2\beta}\gamma_k\phi_j\right) \tag{4.113}$$

Next, let us rewrite the "K-term" of the supersymmetry constraint (4.83) as

$$\begin{aligned}
\epsilon^{abc}\gamma_0\psi_c K_{ba} =&\frac{1}{\sqrt[4]{q}}\epsilon^{adc}e^n_c e_{ai}\gamma_0\phi_n K^i_b e^j_d e^b_j = \frac{1}{\sqrt[4]{q}}\epsilon_i{}^{jn}\gamma_0\phi_n K^i_b E^b_j \\
=& -\frac{\kappa}{4\sqrt[4]{q}}\epsilon^{abc}e^n_b\gamma_0\phi_n(\bar{\psi}_a\gamma_0\psi_c) = \frac{\kappa}{4\sqrt[4]{q}}\epsilon^{njk}\gamma_0\phi_n(\bar{\phi}_j\gamma_0\phi_k)
\end{aligned} \tag{4.114}$$

Hence, combining (4.114) with second term in the second line of (4.112), this yields

$$
\begin{aligned}
&-\frac{\kappa\beta}{2\sqrt[4]{q}}\epsilon_i{}^{ln}\epsilon^{ijk}\gamma_0\phi_n\left(\bar{\phi}_j\gamma_k\frac{\mathbb{1}+i\beta\gamma_*}{2\beta}\phi_l\right)-\frac{\kappa}{4\sqrt[4]{q}}\frac{1+\beta^2}{2\beta}\epsilon^{njk}\gamma_0\phi_n(\bar{\phi}_j\gamma_0\phi_k)\\
=&\frac{i\kappa\beta}{4\sqrt[4]{q}}\gamma_0\phi^k\left(\bar{\phi}_l\gamma_*\gamma_k\phi^l\right)+\frac{\kappa\beta}{2\sqrt[4]{q}}\gamma_0\phi^j\left(\bar{\phi}_j\gamma^l\frac{\mathbb{1}+i\beta\gamma_*}{2\beta}\phi_l\right)\\
&-\frac{\kappa}{4\sqrt[4]{q}}\frac{1+\beta^2}{2\beta}\epsilon^{njk}\gamma_0\phi_n\bar{\phi}_j\gamma_0\phi_k\\
=&\frac{i\kappa\beta}{4\sqrt[4]{q}}\gamma_0\phi^i\left(\bar{\phi}_l\gamma_*\gamma_i\phi^l-\bar{\phi}_i\gamma_*\gamma_l\phi^l+\frac{i}{2}\epsilon_i{}^{jk}\bar{\phi}_j\gamma_0\phi_k\right)+\frac{\kappa}{4\sqrt[4]{q}}\gamma_0\phi^i\left(\bar{\phi}_i\gamma^l\phi_l\right)\\
&-\frac{\kappa}{8\beta\sqrt[4]{q}}\gamma_0\phi_i\epsilon^{ijk}(\bar{\phi}_j\gamma_0\phi_k)\\
=&-\frac{i\kappa\beta}{4\sqrt[4]{q}}\gamma_0\phi^i\left(\bar{\phi}_j\gamma_*\gamma_k\gamma_i\gamma^{(j}\phi^{k)}\right)+\frac{\kappa}{4\sqrt[4]{q}}\gamma_0\phi^i\left(\bar{\phi}_i\gamma^l\phi_l\right)-\frac{\kappa}{8\beta\sqrt[4]{q}}\gamma_0\phi_i\epsilon^{ijk}(\bar{\phi}_j\gamma_0\phi_k)
\end{aligned}
\tag{4.115}
$$

where, from the third to the last line, identity (4.109) was used. Since (this can be shown along the lines of Eq. (4.108) and (4.109))

$$
\bar{\phi}_i\gamma^l\phi_l=-\frac{i}{2}\epsilon^{jkl}\bar{\phi}_j\gamma_*\gamma_0\gamma_k\gamma_i\phi_l+\bar{\phi}_j\gamma_k\gamma_i\gamma^{(j}\phi^{k)}
\tag{4.116}
$$

and $\epsilon^{ijk}\bar{\phi}_j\gamma_0\phi_k=-\epsilon^{jkl}\bar{\phi}_j\gamma_0\gamma_k\gamma_i\phi_l$, the last line of (4.115) finally takes the form

$$
\begin{aligned}
&-\frac{i\kappa\beta}{4\sqrt[4]{q}}\gamma_0\phi^i\left(\bar{\phi}_j\gamma_*\gamma_k\gamma_i\gamma^{(j}\phi^{k)}\right)+\frac{\kappa}{4\sqrt[4]{q}}\gamma_0\phi^i\left(\bar{\phi}_i\gamma^l\phi_l\right)-\frac{\kappa}{8\beta\sqrt[4]{q}}\gamma_0\phi_i\epsilon^{ijk}(\bar{\phi}_j\gamma_0\phi_k)\\
=&\frac{\kappa\beta}{2\sqrt[4]{q}}\gamma_0\phi^i\left(\bar{\phi}_j\gamma_k\frac{\mathbb{1}+i\beta\gamma_*}{2\beta}\gamma_i\gamma^{(j}\phi^{k)}\right)+\frac{\kappa}{4\sqrt[4]{q}}\gamma_0\phi^i\left(\epsilon^{jkl}\bar{\phi}_j\gamma_0\frac{\mathbb{1}+i\beta\gamma_*}{2\beta}\gamma_k\gamma_i\phi_l\right)
\end{aligned}
\tag{4.117}
$$

To summarize, we have found the following form of the supersymmetry constraint in the new variables

$$
\begin{aligned}
S=&\epsilon^{abc}e^i_a\gamma_i\frac{\mathbb{1}+i\beta\gamma_*}{2\beta}D^{(^\beta\! A')}_b\left(\frac{1}{\sqrt[4]{q}}e^j_c\phi_j\right)+\frac{\mathbb{1}+i\beta\gamma_*}{2\beta}D^{(^\beta\! A')}_a\left(\frac{1}{\sqrt[4]{q}}\epsilon^{ijk}E^a_i\gamma_j\phi_k\right)\\
&+\frac{\kappa}{\sqrt[4]{q}}\epsilon^{ijk}\gamma^l\phi_{[l}\left(\bar{\phi}_{i]}\frac{\mathbb{1}+i\beta\gamma_*}{2\beta}\gamma_k\phi_j\right)+\frac{\kappa\beta}{2\sqrt[4]{q}}\gamma_0\phi^i\left(\bar{\phi}_j\gamma_k\frac{\mathbb{1}+i\beta\gamma_*}{2\beta}\gamma_i\gamma^{(j}\phi^{k)}\right)\\
&+\frac{\kappa}{4\sqrt[4]{q}}\gamma_0\phi^i\left(\epsilon^{jkl}\bar{\phi}_j\gamma_0\frac{\mathbb{1}+i\beta\gamma_*}{2\beta}\gamma_k\gamma_i\phi_l\right)
\end{aligned}
\tag{4.118}
$$

With an eye towards quantization of this expression, it is useful to rewrite the second term in (4.118) depending on the covariant derivative of the fermion field. In fact, using $\gamma_*\gamma_{0i}\gamma_k = -2i\epsilon_{ik}{}^{l}\gamma_l + \gamma_k\gamma_*\gamma_{0i}$, we find

$$\begin{aligned} D_a^{(^\beta\! A')}\left(\frac{1}{\sqrt[4]{q}}\epsilon^{abc}e_b^k\gamma_k e_c^l\phi_l\right) &= \partial_a\left(\frac{1}{\sqrt[4]{q}}\epsilon^{abc}e_b^k\gamma_k e_c^l\phi_l\right) + \frac{1}{\sqrt[4]{q}}\epsilon^{abc}e_b^k e_c^l\,{}^\beta\! A'^i_a\frac{i}{2}\gamma_*\gamma_{0i}\gamma_k\phi_l \\ &= (D_a^{(^\beta\! A')}e_b^k)\frac{1}{\sqrt[4]{q}}\epsilon^{abc}\gamma_k e_c^l\phi_l + \epsilon^{abc}e_b^k\gamma_k D_a^{(^\beta\! A')}\left(\frac{1}{\sqrt[4]{q}}e_c^l\phi_l\right) \end{aligned} \tag{4.119}$$

so that we can equivalently write (4.118) as follows

$$\begin{aligned} S =& i\epsilon^{abc}e_a^i\gamma_i\gamma_* D_b^{(^\beta\! A')}\left(\frac{1}{\sqrt[4]{q}}e_c^j\phi_j\right) + \frac{1}{\sqrt[4]{q}}\epsilon^{abc}e_c^l\frac{\mathbb{1}+i\beta\gamma_*}{2\beta}\gamma_k(D_a^{(^\beta\! A')}e_b^k)\phi_l \\ &+ \frac{\kappa}{\sqrt[4]{q}}\epsilon^{ijk}\gamma^l\phi_{[l}\left(\bar{\phi}_{i]}\frac{\mathbb{1}+i\beta\gamma_*}{2\beta}\gamma_k\phi_j\right) + \frac{\kappa\beta}{2\sqrt[4]{q}}\gamma_0\phi^i\left(\bar{\phi}_j\gamma_k\frac{\mathbb{1}+i\beta\gamma_*}{2\beta}\gamma_i\gamma^{(j}\phi^{k)}\right) \\ &+ \frac{\kappa}{4\sqrt[4]{q}}\gamma_0\phi^i\left(\epsilon^{jkl}\bar{\phi}_j\gamma_0\frac{\mathbb{1}+i\beta\gamma_*}{2\beta}\gamma_k\gamma_i\phi_l\right) \end{aligned} \tag{4.120}$$

This is the most compact form of the supersymmetry constraint that we will use for quantization of the theory.

4.4. Anti-de Sitter Supergravity

The canonical analysis of $\mathcal{N} = 1$ anti-de Sitter supergravity in context of the chiral theory has been studied, for instance, in [63, 84, 85] (see also Section 6.4). For sake of completeness, let us briefly discuss it in case of arbitrary real Barbero-Immirzi parameters. As seen in Example 2.3.17, it follows that the isometry group $SO(2,3)$ of anti-de Sitter space AdS_4 can be extended to a super Lie group with $\mathcal{N}$ fermionic generators given by the orthosymplectic Lie group $OSp(\mathcal{N}|4)$. This leads to a supergravity theory with negative cosmological constant $\Lambda_{\text{cos}} = -\frac{3}{L^2}$ where L is the anti-de Sitter radius. As will be discussed in detail in Section 5.2.1, for $\mathcal{N} = 1$, the Holst action then takes the form

$$S^{\mathcal{N}=1}_{\text{H-AdS}} = S^{\mathcal{N}=1}_{\text{H}} + \int_M d^4x\left(-e\frac{1}{2L}\bar{\psi}_\mu\gamma^{\mu\nu}\psi_\nu + \frac{3}{\kappa L^2}e\right) \tag{4.121}$$

with $S^{\mathcal{N}=1}_{\text{H}}$ the Holst action (4.35) (or (4.51)) of $\mathcal{N} = 1$ Poincaré supergravity. Since these additional terms do not depend on the spin connection, it follows immediately that the variation of (4.121) w.r.t. ω yields the same equations of motion as in the $\Lambda_{\text{cos}} = 0$ case and thus, in particular, are again independent of the Barbero-Immirzi parameter. The 3+1-split of the additional terms is straightforward and yields

$$-e\frac{1}{2L}\bar{\psi}_\mu\gamma^{\mu\nu}\psi_\nu + \frac{3}{\kappa L^2}e = -\frac{1}{2L}N\sqrt{q}\left(2\bar{\psi}_t\gamma^{ta}\psi_a + \bar{\psi}_a\gamma^{ab}\psi_b\right) + N\frac{3}{\kappa L^2}\sqrt{q} \tag{4.122}$$

as for anticommuting fermionic fields one has $\bar{\psi}_a\gamma^{at}\psi_t = \bar{\psi}_t\gamma^{ta}\psi_a$. Since $e^t_i = 0$ and $e^t_0 = \frac{1}{N}$, we find

$$-e\frac{1}{2L}\bar{\psi}_\mu\gamma^{\mu\nu}\psi_\nu + \frac{3}{\kappa L^2}e = -\frac{1}{L}E^a_i\bar{\psi}_t\gamma^{0i}\psi_a + N\left(\frac{1}{2L}\sqrt{q}\bar{\psi}_a\gamma^{ab}\psi_b + \frac{3}{\kappa L^2}\sqrt{q}\right) \tag{4.123}$$

The first term in (4.123) yields an additional contribution to the SUSY constraint whereas the second term contributes to the Hamiltonian constraint. Hence, it follows that the SUSY constraint in AdS supergravity takes the form

$$\begin{aligned} S =& \epsilon^{abc}\gamma_a\frac{\mathbb{1} + i\beta\gamma_*}{2\beta}D^{(^\beta A)}_b\psi_c + \frac{\mathbb{1} + i\beta\gamma_*}{2\beta}D^{(^\beta A)}_a\left(\epsilon^{abc}\gamma_b\psi_c\right) \\ &- \frac{1+\beta^2}{2\beta}\epsilon^{abc}\gamma_0\psi_c K_{ba} - \frac{1}{L}E^a_i\gamma^{0i}\psi_a \end{aligned} \tag{4.124}$$

which again can be re-expressed in terms of half-densitized fermionic variables.

4.5. Quantum theory

4.5.1. Quantization of the Rarita-Schwinger field

In what follows, we want to discuss the quantization of the fermionic sector of canonical supergravity expressed in terms of real Asthekar-Barbero variables. The quantization of the gravitational sector of the theory, including the proper definition of *(super) holonomies* and *electric fluxes*, follows along the lines of Section 5.5.1-5.5.3 by identifying the underlying super gauge group $\mathcal{G}$ of the theory with the purely bosonic super Lie group $\mathcal{G} = \mathbf{S}(\mathrm{SU}(2))$ (see also Section 5.5.4).

The quantization of the Rarita-Schwinger field is more subtle than for ordinary Dirac fermions due to the form (4.101) of the reality condition Ω^i_α which, however, has already been drastically simplified using half-densitized fermionic fields since then (4.101) no longer depends on the triads and the spatial metric. In order to solve this second class constraint, we follow the standard procedure and compute the corresponding Dirac brackets for which we have to compute Poisson brackets of the form $\{\Omega^i_\alpha, \Omega^j_\beta\}$. Using

(4.101) as well as (4.100), this yields (occasionally omitting the delta distribution $\delta^{(3)}$ for notational convenience)

$$\begin{aligned}\{\Omega^i_\alpha, \Omega^j_\beta\} &= \epsilon^{ikl}\{\bar{\phi}_{k\delta}, \pi^j_\beta\}(\gamma_l\mathcal{P}_\beta)^\delta{}_\alpha + \epsilon^{jmn}\{\pi^i_\alpha, \bar{\phi}_{m\delta}\}(\gamma_n\mathcal{P}_\beta)^\delta{}_\beta \\ &= -\epsilon^{ijk}C_{\beta\delta}(\gamma_k\mathcal{P}_\beta)^\delta{}_\alpha + \epsilon^{ijk}C_{\alpha\delta}(\gamma_k\mathcal{P}_\beta)^\delta{}_\beta \\ &= -\epsilon^{ijk}\left[(C\gamma_k\mathcal{P}_\beta)^T_{\alpha\beta} - (C\gamma_k\mathcal{P}_\beta)_{\alpha\beta}\right] \\ &= -\epsilon^{ijk}(C\gamma_k[\mathcal{P}_\beta + \mathcal{P}_{-\beta}])_{\alpha\beta} = i\epsilon^{ijk}(C\gamma_*\gamma_k)_{\alpha\beta} =: \mathbf{C}^{ij}_{\alpha\beta}\end{aligned} \tag{4.125}$$

As we see, the operator $\mathcal{P}_\beta$ has dropped out completely so that, in particular, (4.125) is independent of the Barbero-Immirzi parameter. Finally, since

$$\{\phi^\alpha_i, \Omega^j_\beta\} = -\delta^j_i\,\delta^\alpha_\beta \quad \text{and} \quad \{\Omega^i_\alpha, \bar{\phi}_{j\beta}\} = -\delta^i_j C_{\alpha\beta} \tag{4.126}$$

it follows that the graded Dirac brackets for the Rarita-Schwinger field take the form

$$\{\phi^\alpha_i, \bar{\phi}_{j\beta}\}_{\mathrm{DB}} = -\{\phi^\alpha_i, \Omega^k_\gamma\}(\mathbf{C}^{-1})^{\gamma\delta}_{kl}\{\Omega^l_\delta, \bar{\phi}_{j\beta}\} = -((\mathbf{C}^{-1})_{ij}C)^\alpha{}_\beta \tag{4.127}$$

with $\mathbf{C}^{-1}$ the inverse of (4.125) which satisfies $(\mathbf{C}^{-1})_{ij}\mathbf{C}^{jk} = \delta^k_i\mathbb{1}$. As can be checked by direct computation, this matrix takes the form

$$(\mathbf{C}^{-1})_{ij} = -\gamma_0\left(\mathbb{1}\delta_{ij} - \frac{1}{2}\gamma_i\gamma_j\right)C^{-1} \tag{4.128}$$

so that the resulting Dirac brackets can be written as

$$\{\phi^\alpha_i(x), \bar{\phi}_{j\beta}(y)\}_{\mathrm{DB}} = \left(\left(\mathbb{1}\delta_{ij} - \frac{1}{2}\gamma_i\gamma_j\right)\gamma_0\right)^\alpha{}_\beta\,\delta^{(3)}(x,y) \tag{4.129}$$

Note that, since (4.101) does not depend on the internal triads, the Dirac brackets of the bosonic degrees of freedom (A^i_a, E^i_a) coincide with the original Poisson brackets. In particular, the mixed Dirac brackets between bosonic and fermionic degrees of freedom are still vanishing. For further simplification, we will work in the real representation (4.11) of the gamma matrices such that Majorana fermions are explicitly real. In this representation, the charge conjugation matrix is given by $C = i\gamma^0$ and (4.129) yields

$$\{\phi^\alpha_i(x), \phi^\beta_j(y)\}_{\mathrm{DB}} = \frac{i}{2}\left(\mathbb{1}\delta_{ij} - \frac{1}{2}\gamma_i\gamma_j\right)^{\alpha\beta}\delta^{(3)}(x,y) \tag{4.130}$$

together with the Majorana condition $\phi^*_i = \phi_i$. Due to the complicated form of the Dirac bracket (4.130), the implementation of the Rarita-Schwinger field which simultaneously also allows a direct solution of the Gauss constraint in the quantum theory is by far not straightforward. However, in [67], a clever way was found to solve all these issues simultaneously by appropriately enlarging the phase space. More precisely, the idea in [67] is to decompose ϕ_i in its trace part $\sigma := \gamma^i\phi_i$ and its trace-free part $\rho_i := \phi_i - \frac{1}{3}\gamma_i\sigma$ w.r.t. to the gamma matrices γ_i such that $\phi_i = \rho_i + \frac{1}{3}\gamma_i\sigma$. On the enlarged phase space, we then impose the Poisson brackets

$$\{\rho_i^\alpha(x), \rho_j^\beta(y)\} = i\delta_{ij}\delta^{\alpha\beta}\delta^{(3)}(x,y) \text{ and } \{\sigma^\alpha(x), \sigma^\beta(y)\} = -\frac{9i}{2}\delta_{ij}\delta^{\alpha\beta}\delta^{(3)}(x,y) \tag{4.131}$$

with the remaining brackets being zero such that the Dirac bracket (4.130) is recovered. Moreover, in order to account for the superfluous degrees of freedom, i.e. the trace-freeness of ρ_i, one has to add the additional secondary constraint $\Lambda := \gamma^i\rho_i = 0$ [67]. Using $\{\Lambda^\alpha, \Lambda^\beta\} = 3i\delta^{\alpha\beta}$, this yields the Dirac brackets

$$\{\rho_i^\alpha(x), \rho_j^\beta(y)\}_{\mathrm{DB}} = i\left(\delta_{ij}\delta^{\alpha\beta} - \frac{1}{3}(\gamma_i\gamma_j)^{\alpha\beta}\right)\delta^{(3)}(x,y) =: i\mathbf{P}_{ij}^{\alpha\beta}\delta^{(3)}(x,y) \tag{4.132}$$

where $\mathbf{P}_{ij}^{\alpha\beta}$ is the projection operator onto the subspace of trace-free Rarita-Schwinger fields, i.e., $\rho_i = \mathbf{P}_{ij}\phi^j$. Due to the fact that, in contrast to (4.130), this indeed defines a projection now allows for a direct implementation in the quantum theory.

Before we do so, following [80], we first exploit the fact that the ϕ_i (resp. ρ_i and σ) are half densities and introduce new Grassmann-valued variables. For later purposes, in contrast to [80], in view of the regularization of the supersymmetry constraint, we therefore triangulate the spatial slice Σ by disjoint (again up to common faces, edges and vertices) tetrahedra Δ_i instead of boxes at countably infinite discrete points $x_i \in \Sigma$, $i \in \mathcal{I}$ ($|\mathcal{I}| = \aleph_0$), and coordinate volume $\delta_i^3/6$ such that $\Sigma = \bigcup_{i\in\mathcal{I}}\Delta_i$. Here, $\delta_i > 0$ $\forall i \in \mathcal{I}$ are small positive numbers determining the fineness of the triangulation. Then, for each $i \in \mathcal{I}$, we define [80]

$$\theta^{(\delta_i)}(x_i) := \int_\Sigma \mathrm{d}^3y\, \frac{\chi_{\delta_i}(x_i - y)}{\sqrt{\frac{\delta_i^3}{6}}}\phi(y) \tag{4.133}$$

where $\chi_{\delta_i}(x_i - y)$ is the characteristic function of the tetrahedron Δ_i centered at x_i. These satisfy the bracket relations

$$\begin{aligned}\{\theta_i^{(\delta_k)}(x_k), \theta_j^{(\delta_l)}(x_l)\} &= \int_\Sigma \mathrm{d}^3x\, \frac{\chi_{\delta_k}(x_k - x)}{\sqrt{\frac{\delta_k^3}{6}}}\int_\Sigma \mathrm{d}^3y\, \frac{\chi_{\delta_l}(x_l - y)}{\sqrt{\frac{\delta_l^3}{6}}}\{\phi_i(x), \phi_j(y)\}_{\mathrm{DB}} \\ &= \frac{i}{2}\left(\mathbb{1}\delta_{ij} - \frac{1}{2}\gamma_i\gamma_j\right)\delta_{kl}\int_\Sigma \mathrm{d}^3x\, \frac{\chi_{\delta_k}(x_k - x)}{\delta_k^3/6}\end{aligned}$$

$$= \frac{i}{2}\left(\mathbb{1}\delta_{ij} - \frac{1}{2}\gamma_i\gamma_j\right)\delta_{kl} \tag{4.134}$$

We then take the continuum limit $\sup_{i\in I}\{\delta_i\} \to 0$ and set $\theta_i(x) := \lim_{\delta_x\to 0}\,\theta_i^{(\delta_x)}(x)$ $\forall x \in \Sigma$. Furthermore, setting $\theta_i^{(\rho)}(x) := \mathbf{P}_{ij}\theta^j(x)$ as well as $\theta^{(\sigma)} := \gamma^i\theta_j(x)$, this finally yields

$$\{\theta_i^{(\rho)}(x), \theta_j^{(\rho)}(y)\} = i\mathbf{P}_{ij}\delta_{x,y} \quad \text{and} \quad \{\theta^{(\sigma)}(x), \theta^{(\sigma)}(y)\} = -\frac{9i}{2}\mathbb{1}\delta_{x,y} \tag{4.135}$$

together with the Majorana conditions $\theta_i^{(\rho)}(x)^* = \theta_i^{(\rho)}(x)$ and $\theta^{(\sigma)}(x)^* = \theta^{(\sigma)}(x)$ $\forall x, y \in \Sigma$. Hence, one ends up with an abstract CAR *-algebra at any point $x \in \Sigma$. The quantization of the theory can be performed following [67]. In what follows, let us sketch some basic ideas lying behind this quantization scheme and also point out some further mathematical structures which have a natural interpretation in the framework of supergeometry and even naturally arise in the chiral approach. For more details, from both a mathematical and physical point of view, we refer to Section 5.5.4.

For any point $x \in \Sigma$ we choose the supermanifold $\mathbb{R}^{0|N}_x := (\{x\}, \Lambda_N)$, also called a superpoint (Def. 2.2.9), with N fermionic generators $\theta^{\mathscr{A}}$, with $\mathscr{A}$ an index $\mathscr{A} \in \{1, \ldots, N\}$, whose sections $f \in \Lambda_N^{\mathbb{C}} := \Lambda_N \otimes \mathbb{C}$ of the complexified function sheaf take the form

$$f = \sum_{\underline{I}} f_{\underline{I}}\theta^{\underline{I}} \tag{4.136}$$

with $f_{\underline{I}} \in \mathbb{C}$ for all ordered multi-indices $\underline{I}$ of length $0 \le |I| \le N$. On the superspace one has the standard translation-invariant super scalar product $\mathscr{S} : \Lambda_N^{\mathbb{C}} \times \Lambda_N^{\mathbb{C}} \to \mathbb{C}$ given by the *Berezin integral*[4]

$$\mathscr{S}(f|g) := \int_B \mathrm{d}\theta^1 \cdots \mathrm{d}\theta^N\, \bar{f} g, \quad \forall f, g \in \Lambda_N^{\mathbb{C}} \tag{4.137}$$

This gives the space $(\Lambda_N^{\mathbb{C}}, \mathscr{S})$ the structure of an indefinite inner product space for which there exists an endomorphism $J \in \mathrm{End}(\Lambda_N^{\mathbb{C}})$ such that $\mathscr{S}(\cdot|J\cdot)$ defines a positive definite scalar product on $\Lambda_N^{\mathbb{C}}$. The choice of such an endomorphism J is not unique but is strongly restricted by the implementation of the reality conditions. A standard choice of a scalar product is given by identifying $\Lambda_N^{\mathbb{C}} \cong \mathbb{C}^{2^N}$ and setting

$$\langle f|g\rangle := \sum_{\underline{I}} \bar{f}_{\underline{I}} g_{\underline{I}} \tag{4.138}$$

[4] For a generic section $f = \sum_{\underline{I}} f_{\underline{I}}\theta^{\underline{I}} \in \Lambda_N^{\mathbb{C}}$, the Berezin integral is defined via $\int_B \mathrm{d}\theta^1 \cdots \mathrm{d}\theta^N f := f_{12\cdots N}$, i.e., the Berezin integral selects the coefficient of the component of highest degree in θ

It follows, even for general super Lie groups, that there always exists an endomorphism J on $\Lambda_N^{\mathbb{C}}$ such that[5] [109]

$$\langle\cdot|\cdot\rangle = \mathscr{S}(\cdot|J\cdot) \tag{4.139}$$

Hence, this yields a Hilbert space or, more prescisely, a standard super Hilbert space $\mathfrak{H}_x^N := (\Lambda_N^{\mathbb{C}}, \langle\cdot|\cdot\rangle)$ (see Def. 5.5.9 and Remark 5.5.10). On $\mathfrak{H}_x^N$ we define the multiplication operators $\widehat{\theta}^{\mathscr{A}}$ as well as odd derivations $\partial_{\mathscr{A}} \equiv \frac{\partial}{\partial\theta^{\mathscr{A}}}$ for $\mathscr{A} = 1, \ldots, N$ via

$$\widehat{\theta}^{\mathscr{A}} f := \theta^{\mathscr{A}} f \quad \text{and} \quad \partial_{\mathscr{A}}\theta^{\mathscr{B}} := \delta_{\mathscr{A}}^{\mathscr{B}} \tag{4.140}$$

$\forall f \in \Lambda_N^{\mathbb{C}}$. As shown in [67], due to the choice of the scalar product (4.138), these operators are indeed self-adjoint on $\mathfrak{H}_x^N$. With these ingredients, one can then construct a faithful representation of the CAR *-algebra (4.135). To this end, one takes the tensor product Hilbert space $\mathfrak{H}_x := \mathfrak{H}_x^N \otimes \mathfrak{H}_x^M$ with $N = 12$ and $M = 4$ and defines

$$\widehat{\theta}_i^{(\rho)\alpha}(x) := \mathbf{P}_{ij}^{\alpha\beta}\left[\sqrt{\frac{\hbar}{2}}(\theta_\beta^j + \partial_\beta^j)\right] \quad \text{and} \quad \widehat{\theta}^{(\sigma)\alpha}(x) := \frac{3\sqrt{\hbar}}{2}(\theta^\alpha + \partial^\alpha) \tag{4.141}$$

on $\mathfrak{H}_x^N$ and $\mathfrak{H}_x^M$, respectively. By construction, these operators are then self-adjoint as required by the Majorana conditions and moreover satisfy the anticommutation relations

$$[\widehat{\theta}_i^{(\rho)}(x), \widehat{\theta}_j^{(\rho)}(x)] = \hbar\mathbf{P}_{ij} \quad \text{and} \quad [\widehat{\theta}_i^{(\sigma)}(x), \widehat{\theta}^{(\sigma)}(x)] = \frac{9\hbar}{2}\mathbb{1} \tag{4.142}$$

The quantized Rarita-Schwinger field on $\mathfrak{H}_x$ is then given by

$$\widehat{\theta}_i(x) := \widehat{\theta}_i^{(\rho)}(x) + \frac{1}{3}\gamma_i\widehat{\theta}^{(\sigma)}(x) \tag{4.143}$$

This construction then takes over to a family of points $\{x_1, \ldots, x_k\}$ yielding the tensor product Hilbert space $\mathfrak{H}_{\{x_1,\ldots,x_k\}} := \bigotimes_{i=1}^k \mathfrak{H}_{x_i}$. The fermionic Hilbert space $\mathfrak{H}_f$ is then obtained as the inductive limit over the corresponding family of Hilbert spaces $\mathfrak{H}_{\{x_1,\ldots,x_k\}}$. As result, the total super Hilbert space $\mathfrak{H}^{\text{LQSG}}$ of the theory is given by

$$\mathfrak{H}^{\text{LQSG}} = \mathfrak{H}_{\text{grav}} \otimes \mathfrak{H}_f \tag{4.144}$$

with $\mathfrak{H}_{\text{grav}}$ the Hilbert space of the quantized bosonic degrees of freedom generated by SU(2) spin network states (see also Section 5.5).

[5] For this situation, such an endomorphism has in fact been constructed explicitly in [67], although it is important to emphasize that their definition of the super scalar product differs from the definition chosen here.

4.5.2. Quantization of the SUSY constraint

4.5.2.1. Part I

Having derived the compact expression (4.120) of the classical supersymmetry constraint with half-densitized fermionic fields, we next want to find an implementation in the quantum theory. As stated in [64], the Poisson bracket of the SUSY constraint with itself should be proportional to the Hamiltonian constraint modulo Gauss and diffeomorphism constraint. Hence, in the quantum theory, on the subspace of gauge- and diffeomorphism-invariant states, it is expected that the commutator of the SUSY constraint operator reproduces the Hamiltonian constraint operator. This is a very interesting and important feature in canonical supergravity theories as this provides a very strong relationship between both operators and thus serves as a consistency condition in the quantum theory. This may also fix some of the quantization ambiguities. In fact, in the framework of self-dual loop quantum cosmology, for a certain subclass of symmetry reduced models, it will explicitly be shown in Chapter 6 that this strong relationship even holds exactly in the quantum theory. More precisely, we will show that the (graded) commutator between the SUSY constraints exactly reproduces the classical Poisson relation.

Another point of view is that the SUSY constraint is superior to the Hamiltonian constraint in the sense that once the SUSY constraint is quantized (or even solved) this immediately yields the quantization (or solution) of the Hamiltonian constraint by computing the commutator. For this reason, it is desirable to quantize the SUSY constraint in a way that does not involve the Hamiltonian constraint. For instance, it should not depend on the extrinsic curvature as this, via Thiemann's proposal, would involve commutators with the Euclidean part of the Hamiltonian. On the other hand, in order to be able to compare it with the Hamiltonian constraint, it is desirable to find an as compact expression as possible. In the following, we will propose a specific quantization scheme of the SUSY constraint that does not involve the Hamiltonian constraint.

As a first step, let us therefore consider the first part in the classical expression (4.120) depending on the covariant derivative of the fermionic fields

$$S^{(1)}[\eta] := \int_{\Sigma} \mathrm{d}^3x\, \bar{\eta} i \epsilon^{abc} e^i_a \gamma_i \gamma_* D_b^{(^{\beta}\!A)} \left(\frac{1}{\sqrt[4]{q}} e^j_c \phi_j \right) \tag{4.145}$$

Here and in what follows, in order to simplify the notation, the prime indicating the transformed Ashtekar connection in case of half-densitized fermionic variables will be dropped. The expression (4.145) looks quite similar to the Dirac Hamiltonian studied for instance in [156] with the crucial difference that the conjugate spinor $\bar{\eta}$ in (4.145) now plays the role a smearing function and thus is not a dynamical variable. Hence, in

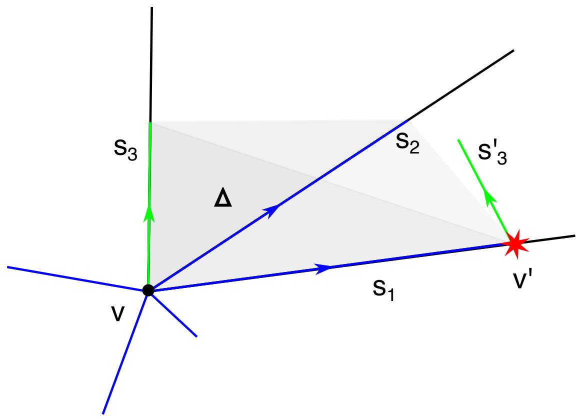


Figure 2.: A tetrahedron Δ with the edges used for the regularization. The star marks the location of the fermion operator (source: [2]).

contrast to [156], we cannot change its density weight going over to half-densities for the regularization as this will change the density weight of the constraint operator as a whole. Moreover, changing the density weight of the smearing function may change the constraint algebra which should be avoided. Hence, particular attention is required for its regularization.

We will proceed in analogy with [157], i.e., we will consider triangulations adapted to a graph γ. First, we describe triangulations of the neighborhood of a vertex v of γ that are labeled by a triplet of edges (e_I, e_J, e_K) at v. We will keep track of the fineness of these triangulations, measured in a fixed fiducial metric around the vertex, in terms of a parameter $\delta > 0$.

(i) All edges of the graph are assumed be outgoing in the sense that if e is an edge with vertices v, v' as endpoints, subdivide it into two new edges e_1 and e_2 such that $e = e_1 \circ e_2$ and e_1 and e_2 are outgoing at v and v', respectively.

(ii) Given an edge e_I incident at a vertex v, choose a segment $s_I : [0,1] \to \Sigma$ of e_I such that s_I is also incident and outgoing at v and such that it does not include any other endpoint of the edge e_I.

(iii) In order to treat all edges of the graph equally, at each vertex v, let (e_I, e_J, e_K) be an arbitrary triple of mutually distinct edges incident at the common vertex v.[6] For each triple, we chose corresponding segments (s_I, s_J, s_K) shorter than δ. They span a tetrahedron Δ with basepoint $v(\Delta) = v$ (see Figure 2), where

[6] If the vertex is two-valent, one can adjoin a third edge in an arbitrary manner. However, it will become clear below that the action of the operator on such vertices is trivial.

the missing three edges of Δ are chosen in a diffeomorphism covariant way [157]. Furthermore, we assume that the triple is ordered in such a way that the tangents of the segments are positively oriented, i.e., $\det(\dot{s}_I, \dot{s}_J, \dot{s}_K) > 0$.

(iv) Let (e_I, e_J, e_K) be a positively oriented triple of edges as in (iii) with corresponding segments (s_I, s_J, s_K). For any $\delta > 0$, we introduce another segment $s'_K : [0,1] \rightarrow \Sigma$ which is incident and at outgoing at $s_I(1)$ in such a way, that in the limit $\delta \rightarrow 0$, s'_K converges to the segment s_K (see Figure 2). As it will become clear in what follows, the end result will not depend on the specific choice of such an additional edge provided it satisfies the requirements just mentioned.

(v) To obtain a triangulation $T(\gamma, v, \delta, IJK)$ of a neighborhood of v, we proceed as in [157] and construct seven additional ("mirror") tetrahedra.

We will now write down a regularization of the classical expression (4.145), using some triangulation $T(\delta)$ of fineness δ. Let Δ_i be a tetrahedron from this triangulation spanned by some triplet (s_I, s_J, s_K) of edges. We will additionally assume that edges s'_I have been chosen according to (iv) above. As usual, we apply Thiemann's trick and replace the co-frame fields e^i_a by the Poisson bracket of the connection with the volume

$$2e^i_a = \frac{1}{\kappa}\{{}^{\beta}\!A^i_a, V\} = \frac{1}{\kappa}\{{}^{\beta}\!A^i_a, V(x,\delta)\} \tag{4.146}$$

where

$$V(x,\delta) := \int_\Sigma \mathrm{d}^3y\, \chi_\delta(x,y)\sqrt{q(y)} \tag{4.147}$$

is the volume of the tetrahedron Δ containing $x \in \Sigma$, with χ_δ its *characteristic function*, such that, in the limit $\delta \rightarrow 0$, one has $\lim_{\delta\rightarrow 0}\frac{6}{\delta^3}V(x,\delta) = \sqrt{q(x)}$. Let $h_s[{}^{\beta}\!A]$ denote the holonomy induced by ${}^{\beta}\!A$ along an arbitrary segment s in the triple (see Equation (5.161) for the case of the purely bosonic super Lie group $\mathbf{S}(\mathrm{SU}(2))$). For $\delta > 0$ small enough, it follows that $h_s[{}^{\beta}\!A]$ can approximately be written as $h_s[{}^{\beta}\!A] = \mathbb{1} + \delta\dot{s}^a\,{}^{\beta}\!A^i_a\tau_i + O(\delta^2)$ such that, using $\mathrm{tr}(\tau_i\tau_j) = -\frac{1}{2}\delta_{ij}$, it follows that

$$2\mathrm{tr}(\tau_i h_s[{}^{\beta}\!A]\{h_s[{}^{\beta}\!A]^{-1}, V(x,\delta)\}) = \delta_{ij}\delta\dot{s}^a\{{}^{\beta}\!A^j_a(x), V(x,\delta)\} \tag{4.148}$$

This enables one to express (4.146) in terms of holonomies and fluxes with the latter implicitly contained in the definition of the volume.

Finally, in order regulate the covariant derivative in (4.168), for any segment s, let

$$H_s[{}^{\beta}\!A] := \mathcal{P}\exp\left(\int_s \kappa_{\mathbb{R}^*}({}^{\beta}\!A)\right) \tag{4.149}$$

be the holonomy induced by ${}^{\beta}\!A$ in the $\mathfrak{su}(2)$-sub representation of the real Majorana representation $\kappa_{\mathbb{R}*}$ which, according to (4.80), in the chiral representation of the gamma matrices consists of a direct sum of two spin-$\frac{1}{2}$ representations. Hence, w.r.t. this representation, $H_s[{}^{\beta}\!A] = \mathrm{diag}(h_s[{}^{\beta}\!A], h_s[{}^{\beta}\!A])$ is in fact block diagonal. Again, in the limit of small $\delta > 0$, the holonomy can approximately be written in the form $H_s[{}^{\beta}\!A] = \mathbb{1} + \delta\dot{s}^a \frac{i}{2}\gamma_*\gamma_{0i}{}^{\beta}\!A^i_a + O(\delta^2)$ which yields

$$H_s[{}^{\beta}\!A](0,\delta)\Psi(s(\delta)) - \Psi(s(0)) = \delta\dot{s}^a(0)(D_a^{({}^{\beta}\!A)}\Psi)(s(0)) \tag{4.150}$$

where Ψ stands for an arbitrary spinor-valued field defined on Σ. With these preparations, we are now ready to write down a regularization of (4.145). Given the triangulation $T(\delta)$ of fineness $\delta > 0$, we set

$$\begin{aligned} S^{(1)}_{\delta}[\eta] := \frac{1}{6\kappa^2}\sum_{\Delta_i\in T(\gamma,\delta)} \bar{\eta}(x_i)i\epsilon^{IJK}\mathrm{tr}(\tau_j h_{s_I(\Delta_i)}\{h^{-1}_{s_I(\Delta_i)}, V(x_i,\delta)\})\times \\ \times\gamma_j\gamma_*[\mathscr{X}_K(s_J(\Delta_i)) - \mathscr{X}_K(x_i)] \end{aligned} \tag{4.151}$$

with

$$\mathscr{X}_K(s_J(\Delta_i)) := \frac{\mathrm{tr}(\tau_k h_{s'_K(\Delta_i)}\{h^{-1}_{s'_K(\Delta_i)}, V(s_J(\Delta_i),\delta)\})}{\sqrt{V(s_J(\Delta_i),\delta)}} H_{s_J(\Delta_i)}\theta^{\delta}_k(s_J(\Delta_i)(\delta)) \tag{4.152}$$

and

$$\mathscr{X}_K(x_i) := \frac{\mathrm{tr}(\tau_k h_{s_K(\Delta_i)}\{h^{-1}_{s_K(\Delta_i)}, V(x_i,\delta)\})}{\sqrt{V(x_i,\delta)}}\theta^{\delta}_k(x_i) \tag{4.153}$$

where in (4.151), for any basepoint $x_i \equiv v(\Delta_i)$, we have chosen a particular triple of segments $(s_I(\Delta_i), s_J(\Delta_i), s_K(\Delta_i))$ incident at x_i and an additional segment s'_K such that the above requirements are satisfied. First, let us show that (4.151) indeed provides a regularization of (4.145). To this end, we use the fact that, by property (iv), s'_K converges to s_K in the limit $\delta \to 0$ such that for small δ, due to (4.150), we can approximately write

$$\mathscr{X}_K(s_J(\Delta_i)) - \mathscr{X}_K(x_i) \approx \delta^2\dot{s}^b_J(\Delta_i)\dot{s}^c_K(\Delta_i)D_b^{({}^{\beta}\!A)}\left(\frac{\{{}^{\beta}\!A^k_c, V(x_i,\delta)\}}{\sqrt{V(x_i,\delta)}}\theta^{\delta}_k(x_i)\right) \tag{4.154}$$

Recall that, by (4.133), θ_i^δ is defined as

$$\theta_i^\delta(x) = \int \mathrm{d}^3y\, \frac{\chi_\delta(x-y)}{\sqrt{\frac{\delta^3}{6}}}\phi_i(y) \tag{4.155}$$

so that, using $\partial_{x^a}\chi_\delta(x-y) = -\partial_{y^a}\chi_\delta(x-y)$ [156], it follows that

$$\partial_{x^a}\theta_i^\delta(x) = -\int_\Sigma \mathrm{d}^3y\, \frac{\partial_{y^a}\chi_\delta(x-y)}{\sqrt{\frac{\delta^3}{6}}}\phi_i(y) = \int_\Sigma \mathrm{d}^3y\, \frac{\chi_\delta(x-y)}{\sqrt{\frac{\delta^3}{6}}}\partial_{y^a}\phi_i(y) \tag{4.156}$$

Hence, if $\mathcal{B}^k(x_i)$ denotes the term inside the covariant derivative of (4.154) depending on the volume $V(x_i,\delta)$, we can rewrite (4.154) as

$$\begin{aligned}
&D_a^{(^\beta A)}\left(\mathcal{B}^k(x_i)\theta_k^\delta(x_i)\right)\\
=&(\partial_{x^a}\mathcal{B}^k)(x_i)\theta_k^\delta(x_i) + \mathcal{B}^k(x_i)\partial_{x^a}\theta_k^\delta(x_i) + \mathcal{B}^k(x_i)\frac{i}{2}\gamma_*\gamma_{0i}{}^\beta A_a^i(x_i)\theta_k^\delta(x_i)\\
=&\int \mathrm{d}^3y\, \frac{\chi_\Delta(x_i-y)}{\sqrt{\frac{\delta^3}{6}}}\Big((\partial_{x^a}\mathcal{B}^k)(x_i)\theta_k^\delta(y) + \mathcal{B}^k(x_i)\partial_{x^a}\theta_k^\delta(y)\\
&\qquad +\mathcal{B}^k(x_i)\frac{i}{2}\gamma_*\gamma_{0i}{}^\beta A_a^i(x_i)\partial_{y^a}\phi_i(y)\Big)
\end{aligned} \tag{4.157}$$

By definition, for small δ we have $V(x_i,\delta) \approx \frac{\delta^3}{6}\sqrt{q(x_i)}$. Hence, approximating the denominator in $\mathcal{B}^k(x_i)$ by $\sqrt{\delta^3/6}\sqrt[4]{q(x_i)}$ and inserting it into Eq. (4.157) and finally using the fact that in the limit $\delta \to 0$ one has $\chi_\delta(x_i-y)/\frac{\delta^3}{6} \to \delta(x_i-y)$, (4.151) becomes

$$\begin{aligned}
&\frac{1}{24\kappa^2}\lim_{\delta\to 0}\sum_{\Delta_i\in T(\gamma,\delta)} \bar{\eta}(x_i)i\{^\beta A_a^j(x_i), V(x_i,\delta)\})\gamma_j\gamma_* D_b^{(^\beta A)}\left(\frac{\{^\beta A_c^k, V(x_i,\delta)\}}{\sqrt[4]{q(x_i)}}\phi_k(x_i)\right)\times\\
&\qquad\qquad \times\epsilon^{IJK}\delta^3\dot{s}_I^a(\Delta_i)\dot{s}_J^b(\Delta_i)\dot{s}_K^c(\Delta_i)
\end{aligned} \tag{4.158}$$

Hence, if we finally use

$$\epsilon^{IJK}\delta^3\dot{s}_I^a(\Delta_i)\dot{s}_J^b(\Delta_i)\dot{s}_K^c(\Delta_i) = \epsilon^{abc}\delta^3\det(\dot{s}_I,\dot{s}_J,\dot{s}_K)(\Delta_i) = 6\epsilon^{abc}\mathrm{vol}(\Delta_i) \tag{4.159}$$

Equation (4.158) takes the form of a Riemann sum which in the limit $\delta \to 0$ converges to a Riemann integral which precisely coincides with expression (4.145). That is, we found

$$\lim_{\delta\to 0} S_\delta^{(1)}[\eta] = S^{(1)}[\eta] \tag{4.160}$$

Hence, we can use (4.151) as a staring point for the quantization. To this end, we apply the identity

$$\{{}^{\beta}\!A^i_a, \sqrt{V(x,\delta)}\} = \frac{1}{2\sqrt{V(x,\delta)}}\{{}^{\beta}\!A^i_a, V(x,\delta)\} \tag{4.161}$$

in order to express (4.151) resp. (4.152) purely in terms of Poisson brackets between holonomies and volume. The corresponding quantum operator is then obtained by replacing the classical phase space variables by their respective quantum counterparts and replacing the Poisson bracket by the commutator $\{\cdot,\cdot\} \to \frac{1}{i\hbar}[\cdot,\cdot]$.

At this point we have to pause, however, since we have to specify the triangulation $T(\delta)$ in adaptation to the graph γ. To do this, We follow precisely the procedure from [157]: Triangulations around the vertices are chosen as $T(\gamma, v, \delta, IJK)$, and the rest of the space triangulated arbitrarily. Finally, an averaging over I, J, K at each vertex is carried out. To write out this averaging, we denote by $E(v)$ the number of triples at the given vertex. With this procedure, we end up with

$$\begin{aligned}\widehat{S}^{(1)}_{\delta}[\eta] := -\frac{1}{3\hbar^2\kappa^2}\sum_{v\in V(\gamma)}\frac{8}{E(v)}\bar{\eta}(x_i)i\epsilon^{IJK}\gamma_j\gamma_*[\widehat{\mathscr{X}_K}(s_J(\Delta)) - \widehat{\mathscr{X}_K}(x)]\times \\ \times\operatorname{tr}(\tau_j h_{s_K(\Delta)}[h^{-1}_{s_I(\Delta)}, \widehat{V}_v])\end{aligned} \tag{4.162}$$

with

$$\widehat{\mathscr{X}_K}(s_J(\Delta)) := \operatorname{tr}(\tau_k h_{s'_K(\Delta)}[h^{-1}_{s'_K(\Delta)}, \sqrt{\widehat{V}}_{s_J(\Delta)}])H_{s_J(\Delta)}\widehat{\theta}_k(s_J(\Delta)) \tag{4.163}$$

and

$$\widehat{\mathscr{X}_K}(x) := \operatorname{tr}(\tau_k h_{s_K(\Delta)}[h^{-1}_{s_K(\Delta)}, \sqrt{\widehat{V}}_v])\widehat{\theta}_k(v) \tag{4.164}$$

where, for reasons that will become clear below, the first factor in the classical expression (4.151) depending on the volume has been ordered to the right. Here, $\widehat{V}_v$ denotes the volume operator at a vertex $v \in V(\gamma)$ (see the following discussion below).

Note that in (4.162) we have implicitly assumed that the discrete sum over all tetrahedra in the triangulation collapses to a sum over the vertices of the underlying spin network graph γ. This is permissible in case of the *Ashtekar-Lewandowski volume operator* $\widehat{V} \equiv \widehat{V}^{\mathrm{AL}}$ [31, 32] as this operator acts trivially on planar vertices. However, this also implies that the operator $\widehat{\mathscr{X}_K}(s_J(\Delta))$ in (4.163) becomes trivial as $\sqrt{\widehat{V}}_{s_J(\Delta)}$ acts on a vertex with coplanar tangent vectors. But then $\widehat{\mathscr{X}_K}(s_J(\Delta)) - \widehat{\mathscr{X}_K}(x)$ is not a difference operator and therefore this would not resemble a quantization of a regularized covariant derivative. A resolution would be to quantize a different classical quantity in which the covariant derivative operator acts directly on the Rarita-Schwinger field. The regularization can then be performed as described above. However, we would like to keep the SUSY constraint operator as simple as possible. For this reason, we consider another possibility ensuring nontriviality of the action of $\widehat{\mathscr{X}_K}(s_J(\Delta))$. To this end, let us choose instead the *Rovelli-Smolin* variant of the *volume operator* $\widehat{V} \equiv \widehat{V}^{\mathrm{RS}}$ [29, 31, 158]. This operator is defined according to [31, 32]

$$\widehat{V}_v T_{\gamma,\vec{\pi},\vec{m},\vec{n}} := \sum_{v \in V(\gamma)} \sqrt{|\widehat{q}_v|} T_{\gamma,\vec{\pi},\vec{m},\vec{n}} \tag{4.165}$$

for any SU(2) spin network state $T_{\gamma,\vec{\pi},\vec{m},\vec{n}}$ w.r.t. a finite graph γ with edges $e \in E(\gamma)$ labeled by irreps $j_e \equiv \pi^{j_e}$ (see Section 5.5.3.3, in particular, Eq. (5.258)). In (4.165), the operator $|\widehat{q}_v|$ is defined as

$$|\widehat{q}_v| := \frac{1}{48} \sum_{I \neq J \neq K \neq I} |\widehat{q}_{IJK}| := \frac{1}{48} \sum_{I \neq J \neq K \neq I} |\epsilon_{ijk} J_I^i J_J^j J_K^k| \tag{4.166}$$

where the sum is taken over all possible triples $(e_I, e_J.e_K)$ of mutually distinct edges at v. Moreover, J_I^i for $i = 1, 2, 3$ denote the components of the angular momentum operator J_I at the edge e_I (see also Equation 4.212 below). The operator $\widehat{q}_{IJK}$ can also be written in the form

$$\widehat{q}_{IJK} = \epsilon_{ijk} J_I^i J_J^j J_K^k = \frac{i}{4}[(J_{IJ})^2, (J_{JK})^2] \tag{4.167}$$

with $(J_{IJ})^2 := (J_I + J_J)^2$ the Casimir operator corresponding to the total angular momentum $J_{IJ} := J_I + J_J$. Note that the modulus appears inside the sum. For this reason, the action of the Rovelli-Smolin volume operator on vertices with coplanar tangent vectors is in general nontrivial. At first sight, this seems to be a problem as then the sum in (4.162) would also include basepoints of tetrahedra located inside a given edge of a spin network graph, i.e., the sum would be a priori infinite. However, due to our choice of the factor ordering, we will see that this indeed not the case. To this end, let us consider the operator

$$\widehat{O} := \mathrm{tr}\left(\tau_i h_e [h_e^{-1}, \sqrt{\widehat{V}}]\right) \tag{4.168}$$

appearing for instance to the right in (4.162) where the holonomy h_e is taken along an edge e incident at a vertex sitting inside a spin network edge and which is transversal

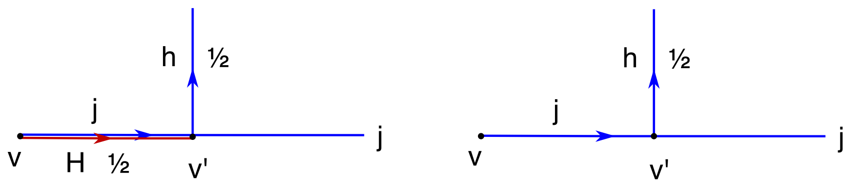


Figure 3.: Illustrations of the action of $\widehat{S}^{(1)}[\eta]$ on spin network states. The picture on the right shows the action of the trace operator $\widehat{O}$ defined in (4.168) creating a new vertex v' by adding a new edge labeled with spin-1/2. The picture on the left illustrates the action of $\widehat{\mathscr{X}_K}(s_J(\Delta))$ in (4.162) which, in contrast to $\widehat{O}$, creates two new spin-1/2 edges at v', one parallel and one transversal to the spin network edge j (source: [2]).

to that particular edge (see Figure 3). Given a spin network state $\Psi_\gamma \equiv T_{\gamma,\vec{\pi},\vec{m},\vec{n}}$, this operator takes the form

$$\widehat{O}\Psi_\gamma = \mathrm{tr}\left(\tau_i h_e[h_e^{-1}, \sqrt{\widehat{V}}]\right)\Psi_\gamma = \mathrm{tr}(\tau_i)\sqrt{\widehat{V}}\Psi_\gamma - \mathrm{tr}\left(\tau_i h_e\sqrt{\widehat{V}}h_e^{-1}\right)\Psi_\gamma$$
$$= -{\tau_i}^A{}_B {h_e}^B{}_C \sqrt{\widehat{V}}\, {h_e^{-1}}^C{}_A \Psi_\gamma \tag{4.169}$$

where the first term in second equation vanishes due the trace-freeness of the Pauli matrices. Since, the matrix components of a holonomy $h_e[{}^\beta\!A]^A{}_B = \pi_{\frac{1}{2}}(h_e[{}^\beta\!A])^A{}_B$ can be identified with the matrix components of the spin-$\frac{1}{2}$ representation, it follows that

$$(\widehat{O}\Psi_\gamma)[{}^\beta\!A] = -{\tau_i}^A{}_B \pi_{\frac{1}{2}}(h_e^{-1}[{}^\beta\!A])^B{}_C \sqrt{\widehat{V}}\left(\pi_{\frac{1}{2}}(h_e[{}^\beta\!A])^C{}_A \Psi_\gamma[{}^\beta\!A]\right) \tag{4.170}$$

Hence, according to (4.170), the holonomy h_e adds a new edge to the spin network graph γ with spin quantum number $j = \frac{1}{2}$ (see Figure 3). To evaluate the action of the volume operator, note that, effectively, the state located at the new created vertex can symbolically be written in the form

$$\Psi_{j_{12}} := |(j_1 j_2) j_{12}, \frac{1}{2}; \underline{j}\,\underline{m}\rangle \tag{4.171}$$

where $j_1 = j_2 = j$ denote the spin quantum numbers of the original spin network edge coupling to $j_{12} = 0$ (for divalent spin network vertices), $j_3 = \frac{1}{2}$ is the spin quantum number of the new created edge and $\underline{j}$ (resp. $\underline{m}$) denotes the total spin (resp. magnetic) quantum number. For later purposes, it is worthwhile to keep the computation a bit more general and assume that j_1 and j_2 are not necessarily equal (therefore j_{12} does not have be zero). For the vertex under consideration, the operator (4.167) takes the form

$$\widehat{q}_{123} =: \widehat{q} = \frac{i}{4}[(J_{12})^2, (J_{23})^2] \tag{4.172}$$

Hence, in order to determine its action on (4.171), we have to perform a recoupling of angular momenta by coupling j_2 and j_3. This can be done using the Wigner 6-j symbols which yields

$$\Psi_{j_{12}} = \sum_{j_{23}} (-1)^{j_1+j_2+\frac{1}{2}+\underline{j}} \sqrt{(2j_{12}+1)}\sqrt{(2j_{23}+1)} \begin{Bmatrix} \frac{1}{2} & j_{12} & \underline{j} \\ j_1 & j_{23} & j_2 \end{Bmatrix} |j_1, (j_2 \tfrac{1}{2}) j_{23}; \underline{jm}\rangle \tag{4.173}$$

In this form, it is particularly easy to compute the action of $(J_{12})^2$ which gives

$$\begin{aligned}
&(J_{23})^2\Psi_{j_{12}} = \\
&= \sum_{j_{23}} (-1)^{j_1+j_2+\frac{1}{2}+\underline{j}} j_{23}(j_{23}+1)\sqrt{(2j_{12}+1)}\sqrt{(2j_{23}+1)} \begin{Bmatrix} \frac{1}{2} & j_{12} & \underline{j} \\ j_1 & j_{23} & j_2 \end{Bmatrix} \times \\
&\quad \times |j_1, (j_2 \tfrac{1}{2}) j_{23}; \underline{jm}\rangle \\
&= \sum_{j_{23}} (-1)^{j_1+j_2+\frac{1}{2}+\underline{j}} j_{23}(j_{23}+1)\sqrt{(2j_{12}+1)}\sqrt{(2j_{23}+1)} \begin{Bmatrix} \frac{1}{2} & j_{12} & \underline{j} \\ j_1 & j_{23} & j_2 \end{Bmatrix} \times \\
&\quad \times \sum_{j'_{12}} (-1)^{j_1+j_2+\frac{1}{2}+\underline{j}} \sqrt{(2j'_{12}+1)}\sqrt{(2j_{23}+1)} \begin{Bmatrix} \frac{1}{2} & j'_{12} & \underline{j} \\ j_1 & j_{23} & j_2 \end{Bmatrix} |(j_1 j_2) j'_{12}, \tfrac{1}{2}; \underline{jm}\rangle \\
&= \sqrt{(2j_{12}+1)} \sum_{j_{23}, j'_{12}} j_{23}(j_{23}+1)(2j_{23}+1)\sqrt{(2j'_{12}+1)} \begin{Bmatrix} \frac{1}{2} & j_{12} & \underline{j} \\ j_1 & j_{23} & j_2 \end{Bmatrix} \times \\
&\quad \times \begin{Bmatrix} \frac{1}{2} & j'_{12} & \underline{j} \\ j_1 & j_{23} & j_2 \end{Bmatrix} \Psi_{j'_{12}}
\end{aligned} \tag{4.174}$$

where in the last line we have again performed a recoupling by coupling j_1 with j_2. This immediately yields

$$(J_{12})^2[(J_{23})^2\Psi_{j_{12}}] = \sqrt{(2j_{12}+1)} \sum_{j'_{12}} j'_{12}(j'_{12}+1)\sqrt{(2j'_{12}+1)} \times$$

$$\times \sum_{j_{23}} j_{23}(j_{23}+1)(2j_{23}+1) \begin{Bmatrix} \frac{1}{2} & j_{12} & j \\ j_1 & j_{23} & j_2 \end{Bmatrix} \begin{Bmatrix} \frac{1}{2} & j'_{12} & j \\ j_1 & j_{23} & j_2 \end{Bmatrix} \Psi_{j'_{12}} \quad (4.175)$$

Remains to evaluate the last term in the commutator of (4.172). In a similar way as above, one finds

$$\begin{aligned} &(J_{23})^2[(J_{12})^2\Psi_{j_{12}}] = j_{12}(j_{12}+1)(J_{23})^2\Psi_{j_{12}} \\ =& j_{12}(j_{12}+1)\sqrt{(2j_{12}+1)} \sum_{j'_{12}} \sqrt{(2j'_{12}+1)} \times \\ &\times \sum_{j_{23}} j_{23}(j_{23}+1)(2j_{23}+1) \begin{Bmatrix} \frac{1}{2} & j_{12} & j \\ j_1 & j_{23} & j_2 \end{Bmatrix} \begin{Bmatrix} \frac{1}{2} & j'_{12} & j \\ j_1 & j_{23} & j_2 \end{Bmatrix} \Psi_{j'_{12}} \end{aligned} \quad (4.176)$$

Hence, we found

$$\begin{aligned} \widehat{q}\Psi_{j_{12}} =& \frac{i}{4}[(J_{12})^2, (J_{23})^2]\Psi_{j_{12}} \\ =& \frac{i}{4}\sqrt{(2j_{12}+1)} \sum_{j'_{12}} \sqrt{(2j'_{12}+1)} \left(j'_{12}(j'_{12}+1) - j_{12}(j_{12}+1)\right) \times \\ &\times \sum_{j_{23}} j_{23}(j_{23}+1)(2j_{23}+1) \begin{Bmatrix} \frac{1}{2} & j_{12} & j \\ j_1 & j_{23} & j_2 \end{Bmatrix} \begin{Bmatrix} \frac{1}{2} & j'_{12} & j \\ j_1 & j_{23} & j_2 \end{Bmatrix} \Psi_{j'_{12}} \end{aligned} \quad (4.177)$$

In fact, this expression can be further simplified using the identity [159]

$$\begin{aligned} &\sum_{j_{23}} (2j_{23}+1) j_{23}(j_{23}+1) \begin{Bmatrix} j_1 & j_{12} & j_2 \\ j_3 & j_{23} & j_4 \end{Bmatrix} \begin{Bmatrix} j_1 & j'_{12} & j_2 \\ j_3 & j_{23} & j_4 \end{Bmatrix} \\ =& \frac{1}{2}(-1)^{j_1+j_2+j_3+j_4+j_{12}+j'_{12}+1} X(j_1, j_4)^{\frac{1}{2}} \begin{Bmatrix} j_2 & j_1 & j_{12} \\ 1 & j'_{12} & j_1 \end{Bmatrix} \begin{Bmatrix} j_3 & j_4 & j_{12} \\ 1 & j'_{12} & j_4 \end{Bmatrix} \\ &+ \frac{j_1(j_1+1) + j_4(j_4+1)}{2j_{12}+1} \delta_{j_{12}j'_{12}} \end{aligned} \quad (4.178)$$

with $X(j_1, j_4) := 2j_1(2j_1+1)(2j_1+2)2j_4(2j_4+1)(2j_4+2)$. Due to the difference appearing in (4.177), it is immediate that the matrix representation of $\widehat{q}$ is purely off-diagonal, i.e., only entries with $j_{12} \neq j'_{12}$ are nonzero. In this case, (4.178) becomes

$$\sum_{j_{23}} (2j_{23}+1) j_{23}(j_{23}+1) \begin{Bmatrix} \frac{1}{2} & j_{12} & j \\ j_1 & j_{23} & j_2 \end{Bmatrix} \begin{Bmatrix} \frac{1}{2} & j'_{12} & j \\ j_1 & j_{23} & j_2 \end{Bmatrix} =$$

$$= \frac{1}{2}(-1)^{j_1+j_2+\underline{j}+j_{12}+j'_{12}+\frac{3}{2}} X(\frac{1}{2}, j_2)^{\frac{1}{2}} \begin{Bmatrix} \underline{j} & \frac{1}{2} & j_{12} \\ 1 & j'_{12} & \frac{1}{2} \end{Bmatrix} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ 1 & j'_{12} & j_2 \end{Bmatrix} \tag{4.179}$$

with $X(\frac{1}{2}, j_2)^{\frac{1}{2}} = 2\sqrt{6}\sqrt{j_2(j_2+1)}\sqrt{(2j_2+1)}$. Furthermore, by the properties of the 6-j symbols, in order for (4.179) to be nonzero j'_{12} has to appear in the decomposition of the tensor product representation $j_{12} \otimes 1 \cong (j_{12}-1) \otimes j_{12} \otimes (j_{12}+1)$, that is $j'_{12} \in \{j_{12}-1, j_{12}+1\}$. Thus, inserting (4.179) into (4.177), we finally obtain

$$\begin{aligned} \widehat{q}\Upsilon_{j_{12}} = &-\frac{i\sqrt{6}}{4}(-1)^{j_1+j_2+2j_{12}+\underline{j}+\frac{3}{2}}\sqrt{(2j_{12}+1)}\sqrt{j_2(j_2+1)}\sqrt{(2j_2+1)}\times \\ &\times \sum_{k\in\{\pm 1\}} k(2j_{12}+k+1)\sqrt{2j_{12}+2k+1}\begin{Bmatrix} \underline{j} & \frac{1}{2} & j_{12} \\ 1 & j_{12}+k & \frac{1}{2} \end{Bmatrix}\times \\ &\times \begin{Bmatrix} j_1 & j_2 & j_{12} \\ 1 & j_{12}+k & j_2 \end{Bmatrix}\Upsilon_{j_{12}+k} \end{aligned} \tag{4.180}$$

This is the most general form for the action of $\widehat{q}$ on a planar vertex with an additional decoupled edge labeled by spin-$\frac{1}{2}$. Applying (4.180) to our situation, i.e., $j_1 = j_2 =: j$ and $j_{12} = 0$, this yields

$$\begin{aligned} \widehat{q}\Upsilon_0 = &\frac{3i\sqrt{2}}{2}(-1)^{2j+1}\sqrt{(2j+1)}\sqrt{j(j+1)}\begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & 1 & \frac{1}{2} \end{Bmatrix}\begin{Bmatrix} j & j & 1 \\ 0 & 1 & j \end{Bmatrix}\Upsilon_1 \\ = &\frac{3i\sqrt{2}}{2}(-1)^{2j+1}\sqrt{(2j+1)}\sqrt{j(j+1)}\frac{1}{\sqrt{6}}\frac{(-1)^{2j+1}}{\sqrt{2j+1}\sqrt{3}}\Upsilon_1 \\ = &\frac{i}{2}\sqrt{j(j+1)}\Upsilon_1 \end{aligned} \tag{4.181}$$

where we used that [160]

$$\begin{Bmatrix} a & b & c \\ 0 & c & b \end{Bmatrix} = \frac{(-1)^{a+b+c}}{\sqrt{(2b+1)}\sqrt{(2c+1)}} \tag{4.182}$$

Similarly, for $j_{12} = 1$, one obtains

$$\widehat{q}\Upsilon_1 = -\frac{i}{2}\sqrt{j(j+1)}\Upsilon_0 \tag{4.183}$$

Hence, w.r.t. the subspace spanned by the orthonormal basis Υ_0 and Υ_1, the operator $\widehat{q}$ has the following matrix representation

$$\widehat{q} = \frac{i}{2}\sqrt{j(j+1)}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4.184}$$

from which we can directly deduce that

$$|\widehat{q}| = \sqrt{\widehat{q}^{\dagger}\widehat{q}} = \frac{1}{2}\sqrt{j(j+1)}\mathbb{1} =: \tilde{C}\mathbb{1} \tag{4.185}$$

Hence, the Rovelli-Smolin volume operator (4.165) acts via multiplication with the constant factor $\tilde{C}^{\frac{1}{2}}$ on the subspace spanned by Υ_0 and Υ_1. This immediately implies that the action of (4.168) is given by

$$\begin{aligned}(\widehat{O}\Upsilon_{\gamma})[{}^{\beta}\!A] =& -(\tau_i)^A{}_B\,\pi_{\frac{1}{2}}(h_e[{}^{\beta}\!A])^B{}_C\sqrt{\widehat{V}}\left(\pi_{\frac{1}{2}}(h_e^{-1}[{}^{\beta}\!A])^C{}_A\Upsilon_{\gamma}[{}^{\beta}\!A]\right)\\ =& -\tilde{C}^{\frac{1}{4}}\mathrm{tr}(\tau_i h_e[{}^{\beta}\!A]h_e^{-1}[{}^{\beta}\!A])\Upsilon_{\gamma}[{}^{\beta}\!A] = -\tilde{C}^{\frac{1}{4}}\mathrm{tr}(\tau_i)\Upsilon_{\gamma}[{}^{\beta}\!A] = 0\end{aligned} \tag{4.186}$$

that is, $\widehat{O}$ simply vanishes on these type of edges and therefore is only nonzero in case of spin network vertices proving that (4.162) is indeed finite also justifying or choice of the factor ordering. This is in fact different to the situation of the standard regularization of the Hamiltonian constraint [157] as, e.g, the Euclidean part contains a term of the form $\mathrm{tr}(h_{\alpha}h_e[h_e^{-1},\widehat{V}])$ where α is a closed loop. In contrast to (4.168), the action of this operator will then, in general, be nonzero (in fact, as observed in (4.186), the triviality of the action of $\widehat{O}$ mainly arose due to the appearence of the Pauli matrix inside the trace). At first sight, this may look like a contradiction, as the the commutator of the SUSY constraint should reproduce the Hamiltonian constraint. However, as already explained at the beginning of this section, the SUSY constraint is superior to the Hamiltonian constraint, i.e., once the SUSY constraint is quantized, this yields a quantization of the Hamiltonian constraint by computing its commutator. Hence, our proposal of the quantum SUSY constraint provides, at least in principle, another possibility for the quantization of the Hamiltonian constraint.

It finally remains to the check that the action of the operator $\widehat{\mathscr{X}_K}(s_J(\Delta))$ in (4.163) is nontrivial such that $\widehat{\mathscr{X}_K}(s_J(\Delta)) - \widehat{\mathscr{X}_K}(x)$ can indeed be viewed as a quantization of a regularized covariant derivative. To this end, we have to study the action of $\widehat{q}$ on decoupled product states of the form

$$|(j\,j)0\rangle \otimes |\frac{1}{2}, m\rangle \otimes |\frac{1}{2}, m'\rangle \tag{4.187}$$

where $|(j\ j)0\rangle$ is again the gauge-invariant divalent vertex located inside a spin network edge and $|\frac{1}{2}, m\rangle$ resp. $|\frac{1}{2}, m'\rangle$ are the additional edges with spin-$\frac{1}{2}$ arising from the holonomies $h_{s'_K(\Delta)}$ resp. $H_{s_J(\Delta)}$ contained in (4.163) (see Figure 3). Note that, for the ansatz (4.187), we have implicitly chosen the chiral represenation of the gamma matrices so that the holonomy H_e is indeed block diagonal according to the decomposition of the restricted Majorana representation into a direct sum of two spin-$\frac{1}{2}$ representations. Hence, this operator does not mix between the two chiral sub representations so that it suffices to restrict to one particular chiral sector. However, note that for the quantization of the Rarita-Schwinger field in Section 4.5.1 a representation was chosen in which the gamma matrices are explicitly real. But, since both representations are related via a similarity transformations, one can map from one representation to the other.

In order to compute the action of (4.172) on the state (4.187), we first need to couple the angular momentum j corresponding to the one part of the spin network edge e that is incident at the vertex $v \in V(\gamma)$ under consideration with the spin-$\frac{1}{2}$ quantum number corresponding to the segment $s'_K(\Delta)$ that is parallel to that edge. Using again Wigner 6-j symbols, we find

$$
\begin{aligned}
&|(j\ j)0\rangle \otimes |\frac{1}{2}, m\rangle \otimes |\frac{1}{2}, m'\rangle = |(j\ j)0, \frac{1}{2}; \frac{1}{2}\ m\rangle \otimes |\frac{1}{2}, m'\rangle \\
=&\left(\sum_{j_{23}}(-1)^{2j+1}\sqrt{2j_{23}+1}\begin{Bmatrix} j_{23} & j & \frac{1}{2} \\ 0 & \frac{1}{2} & j \end{Bmatrix} |j, (j\ \frac{1}{2})j_{23}, \frac{1}{2}\ m\rangle\right) \otimes |\frac{1}{2}, m'\rangle \\
=&\frac{(-1)^{2j+1}}{\sqrt{2}\sqrt{2j+1}}\sum_{j_{23}}(-1)^{j+\frac{1}{2}+j_{23}}\sqrt{2j_{23}+1}\,|j, (j\ \frac{1}{2})j_{23}, \frac{1}{2}\ m\rangle \otimes |\frac{1}{2}, m'\rangle \\
=&\sqrt{\frac{j+1}{2j+1}}\,|j, (j\ \frac{1}{2})j+\frac{1}{2}, \frac{1}{2}\ m\rangle \otimes |\frac{1}{2}, m'\rangle \\
&-\sqrt{\frac{j}{2j+1}}\,|j, (j\ \frac{1}{2})j-\frac{1}{2}, \frac{1}{2}\ m\rangle \otimes |\frac{1}{2}, m'\rangle
\end{aligned}
\tag{4.188}
$$

This can then be coupled with the remaining spin-$\frac{1}{2}$ quantum number using the well-known identities

$$
|\frac{1}{2}, \frac{1}{2}\rangle \otimes |\frac{1}{2}, \frac{1}{2}\rangle = |1, 1\rangle\,, \quad |\frac{1}{2}, -\frac{1}{2}\rangle \otimes |\frac{1}{2}, -\frac{1}{2}\rangle = |1, -1\rangle \tag{4.189}
$$

and

$$
|\frac{1}{2}, \pm\frac{1}{2}\rangle \otimes |\frac{1}{2}, \mp\frac{1}{2}\rangle = \frac{1}{\sqrt{2}}\,|1, 0\rangle \pm \frac{1}{\sqrt{2}}\,|0, 0\rangle \tag{4.190}
$$

Hence, we have to determine the action of (4.172) on states of the form

$$\Psi^{\pm}_{\frac{1}{2},\frac{1}{2},\underline{j}} := |(j \pm \frac{1}{2}\ j)\frac{1}{2}, \frac{1}{2}, \underline{j}\ \underline{m}\rangle\,, \quad \text{with} \quad \underline{j} \in \{0,1\} \tag{4.191}$$

The action of $\widehat{q}$ on (4.191) now follows directly from the general formula (4.180) setting $j_1 = j \pm \frac{1}{2}$ and $j_2 = j$. Since $j_{12} = \frac{1}{2}$ in this case, only the $k = +1$-term in the sum of (4.180) remains yielding

$$\widehat{q}\Psi^{\pm}_{\frac{1}{2},\frac{1}{2},\underline{j}} = -3i\sqrt{3}(-1)^{2j+\frac{1}{2}\pm\frac{1}{2}+\underline{j}}\sqrt{j(j+1)}\sqrt{(2j+1)}\begin{Bmatrix}\underline{j} & \frac{1}{2} & \frac{1}{2}\\ 1 & \frac{3}{2} & \frac{1}{2}\end{Bmatrix}\times \tag{4.192}$$

$$\times\begin{Bmatrix}j\pm\frac{1}{2} & j & \frac{1}{2}\\ 1 & \frac{3}{2} & j\end{Bmatrix}\Psi_{\frac{3}{2},\frac{1}{2},\underline{j}} \tag{4.193}$$

which, according to the first $6j$-symbol appearing in (4.192), will be nonzero if and only if $\underline{j} \in \frac{3}{2}\otimes\frac{1}{2} \cong 1 \oplus 2$. Hence, in particular, for $\underline{j} = 0$ this immediately implies

$$\widehat{q}\Psi^{\pm}_{\frac{1}{2},\frac{1}{2},0} = 0 \tag{4.194}$$

On the other hand, for $\underline{j} = 1$, one obtains

$$\begin{aligned}\widehat{q}\Psi^{\pm}_{\frac{1}{2},\frac{1}{2},1} =&3i\sqrt{3}(-1)^{2j+\frac{1}{2}\pm\frac{1}{2}}\sqrt{j(j+1)}\sqrt{(2j+1)}\begin{Bmatrix}1 & \frac{1}{2} & \frac{1}{2}\\ 1 & \frac{3}{2} & \frac{1}{2}\end{Bmatrix}\times\\ &\times\begin{Bmatrix}j\pm\frac{1}{2} & j & \frac{1}{2}\\ 1 & \frac{3}{2} & j\end{Bmatrix}\Psi_{\frac{3}{2},\frac{1}{2},1}\end{aligned} \tag{4.195}$$

Using the general formula [160]

$$\begin{aligned}&\begin{Bmatrix}a & j & \frac{1}{2}\\ 1 & \frac{3}{2} & j\end{Bmatrix} = \begin{Bmatrix}a & j & \frac{3}{2}\\ 1 & \frac{1}{2} & j\end{Bmatrix}\\ =&\frac{(-1)^{a+\frac{3}{2}+j}}{4\sqrt{3}\sqrt{2j+1}\sqrt{j(j+1)}}\left(\left(a+j+\frac{5}{2}\right)\left(\frac{3}{2}+j-a\right)\left(\frac{3}{2}+a-j\right)\left(a-\frac{1}{2}+j\right)\right)^{\frac{1}{2}}\end{aligned} \tag{4.196}$$

it follows for $a=1$ and $j=\frac{1}{2}$

$$\begin{Bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 1 & \frac{3}{2} & \frac{1}{2} \end{Bmatrix} = -\frac{1}{3} \tag{4.197}$$

For $a=j+\frac{1}{2}$ one finds

$$\begin{Bmatrix} j+\frac{1}{2} & j & \frac{1}{2} \\ 1 & \frac{3}{2} & j \end{Bmatrix} = \frac{(-1)^{2j}}{2\sqrt{3}}\frac{\sqrt{j(2j+3)}}{\sqrt{2j+1}\sqrt{j(j+1)}} \tag{4.198}$$

and finally for $a=j-\frac{1}{2}$

$$\begin{Bmatrix} j-\frac{1}{2} & j & \frac{1}{2} \\ 1 & \frac{3}{2} & j \end{Bmatrix} = \frac{(-1)^{2j+1}}{2\sqrt{3}}\frac{\sqrt{(j+1)(2j-1)}}{\sqrt{2j+1}\sqrt{j(j+1)}} \tag{4.199}$$

Thus, inserting (4.197), (4.198) and (4.199) into (4.195) this yields

$$\widehat{q}\Psi^{\pm}_{\frac{1}{2},\frac{1}{2},1} = \frac{ia_{\pm}}{2}\Psi_{\frac{3}{2},\frac{1}{2},1} \tag{4.200}$$

with $a_+ := \sqrt{j(2j+3)}$ and $a_- := \sqrt{(j+1)(2j-1)}$. Since $\widehat{q}$ is Hermitian, its matrix representation in the subspace spanned by the orthonormal basis $\Psi^{\pm}_{\frac{1}{2},\frac{1}{2},1}$ and $\Psi^{\pm}_{\frac{3}{2},\frac{1}{2},1}$ thus takes the form

$$\widehat{q} = \frac{ia_{\pm}}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4.201}$$

As a consequence, the Rovelli-Smolin volume operator is diagonal on this subspace so that, in particular,

$$\sqrt{\widehat{V}} = \sqrt[4]{|\widehat{q}|} = \sqrt[4]{\frac{a_{\pm}}{2}}\mathbb{1} =: C_{\pm}\mathbb{1} \tag{4.202}$$

i.e. $\sqrt{\widehat{V}}$ acts as a multiplication operator with the constant factor $C_{\pm}$. Let us define

$$|(j\,j)0,\frac{1}{2};\frac{1}{2}\,m\rangle \otimes |\frac{1}{2},m'\rangle = |(j\,j)0\rangle \otimes |\frac{1}{2},m\rangle \otimes |\frac{1}{2},m'\rangle$$
$$:= \begin{cases} |0,\uparrow\uparrow\rangle, & \text{for } m=m'=\frac{1}{2} \\ |0,\uparrow\downarrow\rangle, & \text{for } m=\frac{1}{2},\ m'=-\frac{1}{2} \\ |0,\downarrow\uparrow\rangle, & \text{for } m=-\frac{1}{2},\ m'=\frac{1}{2} \\ |0,\downarrow\downarrow\rangle, & \text{for } m=m'=-\frac{1}{2} \end{cases}$$

in order to simply our notation. Using then (4.188), (4.189) and (4.190) as well as (4.202), we find

$$\begin{aligned}
\sqrt{\widehat{V}}\,|0,\uparrow\uparrow\rangle =&\sqrt{\frac{j+1}{2j+1}}\sqrt{\widehat{V}}\,|j,(j\,\tfrac{1}{2})j+\tfrac{1}{2};\tfrac{1}{2},\tfrac{1}{2}\rangle\otimes|\tfrac{1}{2},\tfrac{1}{2}\rangle\\
&-\sqrt{\frac{j}{2j+1}}\sqrt{\widehat{V}}\,|j,(j\,\tfrac{1}{2})j-\tfrac{1}{2};\tfrac{1}{2},\tfrac{1}{2}\rangle\otimes|\tfrac{1}{2},\tfrac{1}{2}\rangle\\
=&\sqrt{\frac{j+1}{2j+1}}C_+\,|(j\,j+\tfrac{1}{2})\tfrac{1}{2},\tfrac{1}{2};1,1\rangle-\sqrt{\frac{j}{2j+1}}C_-\,|(j\,j-\tfrac{1}{2})\tfrac{1}{2},\tfrac{1}{2};1,1\rangle\\
=&\sqrt{\frac{j+1}{2j+1}}C_+\,|j,(j\,\tfrac{1}{2})j+\tfrac{1}{2};\tfrac{1}{2},\tfrac{1}{2}\rangle\otimes|\tfrac{1}{2},\tfrac{1}{2}\rangle\\
&-\sqrt{\frac{j}{2j+1}}C_-\,|j,(j\,\tfrac{1}{2})j-\tfrac{1}{2};\tfrac{1}{2},\tfrac{1}{2}\rangle\otimes|\tfrac{1}{2},\tfrac{1}{2}\rangle\\
=:&A_1\,|+,\uparrow\rangle\otimes|\uparrow\rangle-A_2\,|-,\uparrow\rangle\otimes|\uparrow\rangle
\end{aligned}\tag{4.203}$$

and similarly

$$\begin{aligned}
\sqrt{\widehat{V}}\,|0,\downarrow\downarrow\rangle =&\sqrt{\frac{j+1}{2j+1}}C_+\,|j,(j\,\tfrac{1}{2})j+\tfrac{1}{2};\tfrac{1}{2},-\tfrac{1}{2}\rangle\otimes|\tfrac{1}{2},-\tfrac{1}{2}\rangle\\
&-\sqrt{\frac{j}{2j+1}}C_-\,|j,(j\,\tfrac{1}{2})j-\tfrac{1}{2};\tfrac{1}{2},-\tfrac{1}{2}\rangle\otimes|\tfrac{1}{2},-\tfrac{1}{2}\rangle\\
=&A_1\,|+,\downarrow\rangle\otimes|\downarrow\rangle-A_2\,|-,\downarrow\rangle\otimes|\downarrow\rangle
\end{aligned}\tag{4.204}$$

Finally, using (4.190) and the fact that the action of the volume operator on states with vanishing total angular momentum $\underline{j}=0$ is zero (see Eq. (4.194)), we find for the mixed spin-components

$$\begin{aligned}
&\sqrt{\widehat{V}}\,|0,\downarrow\uparrow\rangle =\\
&=\sqrt{\frac{j+1}{2j+1}}\sqrt{\widehat{V}}\,|j,(j\,\tfrac{1}{2})j+\tfrac{1}{2};\tfrac{1}{2},-\tfrac{1}{2}\rangle\otimes|\tfrac{1}{2},\tfrac{1}{2}\rangle\\
&\quad-\sqrt{\frac{j}{2j+1}}\sqrt{\widehat{V}}\,|j,(j\,\tfrac{1}{2})j-\tfrac{1}{2};\tfrac{1}{2},-\tfrac{1}{2}\rangle\otimes|\tfrac{1}{2},\tfrac{1}{2}\rangle\\
&=\frac{1}{\sqrt{2}}\sqrt{\frac{j+1}{2j+1}}C_+\,|(j\,j+\tfrac{1}{2})\tfrac{1}{2},\tfrac{1}{2};1,0\rangle-\frac{1}{\sqrt{2}}\sqrt{\frac{j}{2j+1}}C_-\,|(j\,j-\tfrac{1}{2})\tfrac{1}{2},\tfrac{1}{2};1,0\rangle
\end{aligned}$$

$$= \sqrt{\frac{j+1}{2j+1}}\frac{C_+}{2}\left(|+,\uparrow\rangle\otimes|\downarrow\rangle + |+,\downarrow\rangle\otimes|\uparrow\rangle\right)$$
$$- \sqrt{\frac{j}{2j+1}}\frac{C_-}{2}\left(|-,\uparrow\rangle\otimes|\downarrow\rangle + |-,\downarrow\rangle\otimes|\uparrow\rangle\right)$$
$$= \frac{A_1}{2}|+,\uparrow\rangle\otimes|\downarrow\rangle + \frac{A_1}{2}|+,\downarrow\rangle\otimes|\uparrow\rangle - \frac{A_2}{2}|-,\uparrow\rangle\otimes|\downarrow\rangle - \frac{A_2}{2}|-,\downarrow\rangle\otimes|\uparrow\rangle \tag{4.205}$$

and analogously

$$\sqrt{\widehat{V}}\,|0,\uparrow\downarrow\rangle = \frac{A_1}{2}|+,\uparrow\rangle\otimes|\downarrow\rangle + \frac{A_1}{2}|+,\downarrow\rangle\otimes|\uparrow\rangle - \frac{A_2}{2}|-,\uparrow\rangle\otimes|\downarrow\rangle - \frac{A_2}{2}|-,\downarrow\rangle\otimes|\uparrow\rangle \tag{4.206}$$

Recall that we want to the determine the action of (4.163) on the spin network state Υ_γ. We therefore have already derived all necessary ingredients. It only remains to evaluate the trace appearing in (4.163). For this, in the following remark, let us recall some basic facts concerning the action of flux operators appearing e.g. in the volume operator (see also Section 5.5.1).

Remark 4.5.1. In the *Asthekar-Lewandowski representation*, the quantized (bosonic) electric flux operator $\widehat{X}_n(S)$ smeared over two-dimensional surfaces S with smearing function n acts on holonomies $h_e[{}^\beta\!A]$ via [18] (see Eq. (5.180) with coupling constant $g = -\kappa\beta$)

$$\widehat{X}_n(S)h_e[{}^\beta\!A] = -\frac{i\hbar\kappa\beta}{4}\epsilon(e,S)n(b(e))h_e[{}^\beta\!A] \tag{4.207}$$

Since $\{E_n(S), h_e[{}^\beta\!A]^{-1}\} = -h_e[{}^\beta\!A]^{-1}\{E_n(S), h_e[{}^\beta\!A]\}h_e[{}^\beta\!A]^{-1}$, this yields in case of a single edge e ingoing at $S\cap e$

$$\widehat{X}_n(S)h_e[{}^\beta\!A]^{-1} = -h_e[{}^\beta\!A]^{-1}(X_n(S)h_e[{}^\beta\!A])h_e[{}^\beta\!A]^{-1} = \frac{i\hbar\kappa\beta}{4}h_e[{}^\beta\!A]^{-1}n(b(e)) \tag{4.208}$$

Hence, in case that $f \equiv f_e$ is a cylindrical function w.r.t. a graph consisting of the single edge e, this yields

$$\begin{aligned} X_n(S)f(h_e[{}^\beta\!A]^{-1}) &= \frac{\partial f}{\partial\left(h_e[{}^\beta\!A]^{-1}\right)^A{}_B}(h_e[{}^\beta\!A]^{-1})\left(\frac{i\hbar\kappa\beta}{4}h_e[{}^\beta\!A]^{-1}n(b(e))\right)^A{}_B \\ &= \frac{i\hbar\kappa\beta}{4}n(b(e))^j\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}f(h_e[{}^\beta\!A]^{-1}e^{t\tau_j}) \\ &= \frac{\kappa\beta}{4}n(b(e))^j(i\hbar L_jf)(h_e[{}^\beta\!A]^{-1}) \end{aligned} \tag{4.209}$$

with L_j the left-invariant vector field generated by $\tau_j \in \mathfrak{su}(2)$, $j \in 1,2,3$, which is related to the pushforward representation of the right regular representation

$$\rho_R : \mathrm{SU}(2) \to \mathcal{B}(L^2(\mathrm{SU}(2))), \quad g \mapsto (\rho_R(g) : f \mapsto f(\cdot\, g)) \tag{4.210}$$

according to

$$(L_j f)(h) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(he^{t\tau_j}) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \rho_R(e^{t\tau_j})(f)(h) = \rho_{R*}(\tau_j)f(h) \tag{4.211}$$

$\forall f \in C^\infty(\mathrm{SU}(2))$ and $h \in \mathrm{SU}(2)$ and extended uniquely to a (unbounded) self-adjoint operator on $L^2(\mathrm{SU}(2))$, that is,

$$J^j := i\hbar\rho_{R*}(\tau_j) = i\hbar L_j \tag{4.212}$$

For the concrete situation considered here, we are interested in the action on cylindrical function f corresponding to the matrix components of the spin-$\frac{1}{2}$ representation of $\mathrm{SU}(2)$, i.e.

$$f = \pi_{\frac{1}{2}}(h_e[{}^{\beta}\!A]^{-1})^A{}_B \tag{4.213}$$

for any $A, B \in \{\pm\}$. As it is very well-known, these matrix components generate a proper invariant subrepresentation of the right regular representation on $L^2(\mathrm{SU}(2))$. In fact, since for general spin-j

$$\rho_R(g)(\pi_j)^A{}_B(h) = \pi_j(hg)^A{}_B = \pi_j(h)^A{}_C\pi_j(g)^C{}_B \tag{4.214}$$

for any group element $g \in \mathrm{SU}(2)$, it follows that $\rho_R(g)V_A \subseteq V_A$ with $V_A := \mathrm{span}_{\mathbb{C}}\left\{(\pi_j)^A{}_B|\, B \in \{\pm\}\right\}$ and thus, in particular,

$$J^j V_A \subseteq V_A, \quad \forall A \in \{\pm\} \tag{4.215}$$

Moreover, for $j = \frac{1}{2}$, it follows

$$\begin{aligned} J^3(\pi_{\frac{1}{2}})^A{}_B(h) &= i\hbar \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \pi_{\frac{1}{2}}(he^{t\tau_3})^A{}_B = i\hbar\pi_{\frac{1}{2}}(h)^A{}_C \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(e^{t\tau_3}\right)^C{}_B \\ &= i\hbar\pi_{\frac{1}{2}}(h)^A{}_C\tau_3{}^C{}_B \end{aligned} \tag{4.216}$$

so that

$$J^3(\pi_{\frac{1}{2}})^A{}_+ = \frac{\hbar}{2}(\pi_{\frac{1}{2}})^A{}_+ \quad \text{and} \quad J^3(\pi_{\frac{1}{2}})^A{}_- = -\frac{\hbar}{2}(\pi_{\frac{1}{2}})^A{}_- \tag{4.217}$$

To summarize, we have

$$\pi_{\frac{1}{2}}=\begin{pmatrix}|\frac{1}{2},\frac{1}{2}\rangle & |\frac{1}{2},-\frac{1}{2}\rangle\\ |\frac{1}{2},\frac{1}{2}\rangle & |\frac{1}{2},-\frac{1}{2}\rangle\end{pmatrix}\tag{4.218}$$

and, due to (4.214), the rows in (4.218) define 2-dimensional invariant subspaces w.r.t. the angular momentum operator J^j and thus, in particular, w.r.t. the quantized electric fluxes $\widehat{\mathcal{X}}_n(S)$.

Using the observations of Remark 4.5.1, let us now compute the action of (4.163) on the spin network state Ψ_γ which, for sake of simplicity, we assume to be a product state of the form $\Psi_\gamma=\psi_b\otimes\psi_f$ with ψ_b a proper spin network function and ψ_f an element of the fermionic part of the Hilbert space. Using (4.203) as well as (4.218) and (4.214), we then immediately find

$$\begin{aligned}\sqrt{\widehat{V}}h[{}^{\beta}\!A]^{-1A}{}_{+}H^{+}{}_{+}\,\widehat{\theta}_i^{+}\Psi_\gamma[{}^{\beta}\!A]&\equiv\sqrt{\widehat{V}}\,|0,\uparrow\uparrow\rangle\otimes\widehat{\theta}_i^{+}\psi_f\\&=\left(A_1h^{-1A}{}_{+}\,|+,\uparrow\rangle-A_2h^{-1A}{}_{+}\,|-,\uparrow\rangle\right)\otimes\widehat{\theta}_i^{+}\psi_f\end{aligned}\tag{4.219}$$

On the other hand, we have

$$\begin{aligned}\sqrt{\widehat{V}}h[{}^{\beta}\!A]^{-1A}{}_{-}H^{+}{}_{+}\,\widehat{\theta}_i^{+}\Psi_\gamma[{}^{\beta}\!A]\equiv&\sqrt{\widehat{V}}\,|0,\downarrow\uparrow\rangle\otimes\widehat{\theta}_i^{+}\psi_f\\=&\left(\frac{A_1}{2}h^{-1A}{}_{+}\,|+,\downarrow\rangle+\frac{A_1}{2}h^{-1A}{}_{-}\,|+,\uparrow\rangle\right.\\&\left.-\frac{A_2}{2}h^{-1A}{}_{+}\,|-,\downarrow\rangle-\frac{A_2}{2}h^{-1A}{}_{-}\,|-,\uparrow\rangle\right)\otimes\widehat{\theta}_i^{+}\psi_f\end{aligned}\tag{4.220}$$

as well as

$$\begin{aligned}\sqrt{\widehat{V}}h[{}^{\beta}\!A]^{-1A}{}_{+}H^{+}{}_{-}\,\widehat{\theta}_i^{-}\Psi_\gamma[{}^{\beta}\!A]\equiv&\sqrt{\widehat{V}}\,|0,\uparrow\downarrow\rangle\otimes\widehat{\theta}_i^{-}\psi_f\\=&\left(\frac{A_1}{2}h^{-1A}{}_{+}\,|+,\downarrow\rangle+\frac{A_1}{2}h^{-1A}{}_{-}\,|+,\uparrow\rangle\right.\\&\left.-\frac{A_2}{2}h^{-1A}{}_{+}\,|-,\downarrow\rangle-\frac{A_2}{2}h^{-1A}{}_{-}\,|-,\uparrow\rangle\right)\otimes\widehat{\theta}_i^{-}\psi_f\end{aligned}\tag{4.221}$$

and finally

$$\sqrt{\widehat{V}}\,h[{}^{\beta}\!A]^{-1A}{}_{-}H^{+}{}_{-}\,\widehat{\theta}_i^{-}\Psi_\gamma[{}^{\beta}\!A] \equiv \sqrt{\widehat{V}}\,|0,\downarrow\downarrow\rangle\otimes\widehat{\theta}_i^{-}\psi_f$$
$$= \left(A_1 h^{-1A}{}_{-}\,|+,\downarrow\rangle - A_2 h^{-1A}{}_{-}\,|-,\downarrow\rangle\right)\otimes\widehat{\theta}_i^{-}\psi_f \tag{4.222}$$

If we write for the holonomy

$$h^{-1} := \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \tag{4.223}$$

for some complex coefficients $\alpha,\beta,\gamma,\delta\in\mathbb{C}$, this yields for the action of (4.163)

$$\begin{aligned}
&(\widehat{\mathscr{X}}\Psi_\gamma)[{}^{\beta}\!A] =\\
&= \mathrm{tr}(\tau_i h[{}^{\beta}\!A]\sqrt{\widehat{V}}\,h[{}^{\beta}\!A]^{-1})H^{+}{}_{B}\,\widehat{\theta}^{B}\psi[{}^{\beta}\!A]\\
&= \tau_i{}^{A}{}_{B}h^{B}{}_{C}\sqrt{\widehat{V}}\,h^{-1C}{}_{A}H^{+}{}_{B}\,\psi_b\otimes\widehat{\theta}_i^{B}\psi_f\\
&= \mathrm{tr}(\tau_i h\begin{pmatrix}A_1\alpha & \frac{A_1}{2}\beta\\ A_1\gamma & \frac{A_1}{2}\delta\end{pmatrix})\,|+,\uparrow\rangle\otimes\widehat{\theta}_i^{+}\psi_f + \mathrm{tr}(\tau_i h\begin{pmatrix}0 & \frac{A_1}{2}\alpha\\ 0 & \frac{A_1}{2}\gamma\end{pmatrix})\,|+,\downarrow\rangle\otimes\widehat{\theta}_i^{+}\psi_f\\
&- \mathrm{tr}(\tau_i h\begin{pmatrix}A_2\alpha & \frac{A_2}{2}\beta\\ A_2\gamma & \frac{A_2}{2}\delta\end{pmatrix})\,|-,\uparrow\rangle\otimes\widehat{\theta}_i^{+}\psi_f - \mathrm{tr}(\tau_i h\begin{pmatrix}0 & \frac{A_2}{2}\alpha\\ 0 & \frac{A_2}{2}\gamma\end{pmatrix})\,|-,\downarrow\rangle\otimes\widehat{\theta}_i^{+}\psi_f\\
&+ \mathrm{tr}(\tau_i h\begin{pmatrix}\frac{A_1}{2}\beta & 0\\ \frac{A_1}{2}\delta & 0\end{pmatrix})\,|+,\uparrow\rangle\otimes\widehat{\theta}_i^{-}\psi_f + \mathrm{tr}(\tau_i h\begin{pmatrix}\frac{A_1}{2}\alpha & A_1\beta\\ \frac{A_1}{2}\gamma & A_1\delta\end{pmatrix})\,|+,\downarrow\rangle\otimes\widehat{\theta}_i^{-}\psi_f\\
&- \mathrm{tr}(\tau_i h\begin{pmatrix}\frac{A_2}{2}\beta & 0\\ \frac{A_2}{2}\delta & 0\end{pmatrix})\,|-,\uparrow\rangle\otimes\widehat{\theta}_i^{-}\psi_f - \mathrm{tr}(\tau_i h\begin{pmatrix}\frac{A_2}{2}\alpha & A_2\beta\\ \frac{A_2}{2}\gamma & A_2\delta\end{pmatrix})\,|-,\downarrow\rangle\otimes\widehat{\theta}_i^{-}\psi_f
\end{aligned} \tag{4.224}$$

This can be further simplified using that

$$\begin{pmatrix}A_1\alpha & \frac{A_1}{2}\beta\\ A_1\gamma & \frac{A_1}{2}\delta\end{pmatrix} = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}A_1 & 0\\ 0 & \frac{A_1}{2}\end{pmatrix} = h^{-1}\begin{pmatrix}A_1 & 0\\ 0 & \frac{A_1}{2}\end{pmatrix} \tag{4.225}$$

and

$$\begin{pmatrix}0 & \frac{A_1}{2}\alpha\\ 0 & \frac{A_1}{2}\gamma\end{pmatrix} = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}0 & \frac{A_1}{2}\\ 0 & 0\end{pmatrix} = h^{-1}\begin{pmatrix}0 & \frac{A_1}{2}\\ 0 & 0\end{pmatrix} \tag{4.226}$$

as well as

$$\begin{pmatrix} \frac{A_1}{2}\beta & 0 \\ \frac{A_1}{2}\delta & 0 \end{pmatrix} = h^{-1}\begin{pmatrix} 0 & 0 \\ \frac{A_1}{2} & 0 \end{pmatrix} \tag{4.227}$$

such that, for instance,

$$\mathrm{tr}(\tau_i \begin{pmatrix} A_1\alpha & \frac{A_1}{2}\beta \\ A_1\gamma & \frac{A_1}{2}\delta \end{pmatrix}) = \begin{cases} 0, & \text{for } i = 1 \\ 0, & \text{for } i = 2 \\ \frac{A_1}{4i}, & \text{for } i = 3 \end{cases} \tag{4.228}$$

and similar for the other traces. Hence, we finally end up with

$$\begin{aligned}(\widehat{\mathscr{X}}\Upsilon_\gamma)[{}^\beta A] =& \frac{A_1}{4i}\,|+,\uparrow\rangle \otimes \widehat{\theta}_3^+\psi_f + \frac{A_1}{4i}\,|+,\downarrow\rangle \otimes (\widehat{\theta}_1^+ + i\widehat{\theta}_2^+)\psi_f \\ &- \frac{A_2}{4i}\,|-,\uparrow\rangle \otimes \widehat{\theta}_3^+\psi_f - \frac{A_2}{4i}\,|-,\downarrow\rangle \otimes (\widehat{\theta}_1^+ + i\widehat{\theta}_2^+)\psi_f \\ &+ \frac{A_1}{4i}\,|+,\uparrow\rangle \otimes (\widehat{\theta}_1^- - i\widehat{\theta}_2^-)\psi_f - \frac{A_1}{4i}\,|+,\downarrow\rangle \otimes \widehat{\theta}_3^-\psi_f \\ &- \frac{A_2}{4i}\,|-,\uparrow\rangle \otimes (\widehat{\theta}_1^- - i\widehat{\theta}_2^-)\psi_f + \frac{A_2}{4i}\,|-,\downarrow\rangle \otimes \widehat{\theta}_3^-\psi_f \end{aligned} \tag{4.229}$$

and thus

$$\begin{aligned}(\widehat{\mathscr{X}}\Upsilon_\gamma)[{}^\beta A] =& \frac{A_1}{4i}\,|+,\uparrow\rangle \otimes (\widehat{\theta}_3^+ + \widehat{\theta}_1^- - i\widehat{\theta}_2^-)\psi_f + \frac{A_1}{4i}\,|+,\downarrow\rangle \otimes (\widehat{\theta}_1^+ + i\widehat{\theta}_2^+ - \widehat{\theta}_3^-)\psi_f \\ &- \frac{A_2}{4i}\,|-,\uparrow\rangle \otimes (\widehat{\theta}_3^+ + \widehat{\theta}_1^- - i\widehat{\theta}_2^-)\psi_f - \frac{A_2}{4i}\,|-,\downarrow\rangle \otimes (\widehat{\theta}_1^+ + i\widehat{\theta}_2^+ - \widehat{\theta}_3^-)\psi_f \end{aligned} \tag{4.230}$$

where

$$A_1 = \sqrt{\frac{j+1}{2(2j+1)}}\,\big(j(2j+3)\big)^{\frac{1}{4}} \quad \text{and} \quad A_2 = \sqrt{\frac{j}{2(2j+1)}}\,\big((j+1)(2j-1)\big)^{\frac{1}{4}} \tag{4.231}$$

As we see, the action of (4.163) is indeed nontrivial as required and, moreover, creates a new vertex coupled to a fermion. In particular, we see that (4.230) is completely independent of the additional segment $s'_K(\Delta)$ which was needed for the regularization. This is indeed a good thing as the choice of such an additional segment would be completely arbitrary and not based on any fundamental principles justifying the assumption made in (iv) above. Let us make two final remarks about the quantization chosen here.

Remark 4.5.2. We have seen that the properties of the additional edge added at the new vertex, in the definition of (4.230) are irrelevant for the end result. This property can have some side effects, however. Consider the situation depicted in Figure 2, and additionally consider a second tetrahedron spanned by the edge segments s_1, s_2 and a third segment t_3 along an edge different from s_1, s_2, s_3. Depending on the orientation of the tangent vectors, the triplet (s_1, s_2, t_3) may be either positively or negatively oriented. However, the action of (4.230) will otherwise be exactly the same in both cases. The relative orientation of the two triplets enters through the ϵ tensor and gives a relative minus sign in one of the cases. If the orientations differ, the two contributions to the operator $\widehat{S}^{(1)}$ cancel after all. This runs counter to the intuition from the classical theory. Thus one might consider defining a variant of this operator in which an additional sign depending on the orientation is introduced in (4.230).

Remark 4.5.3. Another possibility in quantizing the first term in the SUSY constraint (4.120) would be to choose a different variant in which the covariant derivative acts directly on the Rarita-Schwinger field involving of course additional contributions due to the derivation property. That is, one could instead consider an expression of the form

$$S'^{(1)}[\eta] := \int_\Sigma \mathrm{d}^3x\, \bar{\eta}\frac{i}{\sqrt[4]{q}}\epsilon^{abc}e^i_a\gamma_i\gamma_* e^j_c D^{({}^\beta\!A)}_b\phi_j \tag{4.232}$$

Following the standard procedure, it is then immediate to see that a regularization of (4.232) is given by (see also Part II below)

$$\begin{aligned}
&S'^{(1)}_\delta[\eta] = \\
&= \sum_{\Delta_i\in T(\gamma,\delta)} \bar{\eta}(x_i)\frac{1}{\kappa^2\sqrt{V(x_i,\delta)}}\epsilon^{IJK}\mathrm{tr}(\tau_l h_{s_I(\Delta_i)}[{}^\beta\!A]\{h_{s_I(\Delta_i)}[{}^\beta\!A]^{-1}, V(x_i,\delta)\})\times \\
&\times\gamma_l\gamma_*\mathrm{tr}(\tau_j h_{s_J(\Delta_i)}[{}^\beta\!A]\{h_{s_J(\Delta_i)}[{}^\beta\!A]^{-1}, V(x_i,\delta)\})\times \\
&\times\left(H({}^\beta\!A)(s_K(\Delta_i)(\delta))\theta^\delta_j(s_K(\Delta_i)(\delta)) - \theta^\delta_j(x_i)\right)
\end{aligned} \tag{4.233}$$

For the quantization of (4.233), one can now use either the Ashtekar-Lewandowski or Rovelli-Smolin volume operator. In both cases, based on our observations above, the resulting operator will be finite, i.e., only terms involving spin network vertices contribute. Moreover, one obtains a nontrivial action for the difference operator resulting from the last term in (4.233) which is consistent for a regularization of a covariant derivative.

4.5.2.2. Part II

Next, let us turn to the quantization of the second term in the SUSY constraint (4.120) depending on the covariant derivative of the frame field

$$S^{(2)}[\eta] := \int_{\Sigma} \mathrm{d}^3x\, \bar{\eta} \frac{1}{\sqrt[4]{q}} \epsilon^{abc} e^l_c \frac{\mathbb{1} + i\beta\gamma_*}{2\beta} \gamma_k (D_a^{(^\beta A)} e^k_b) \phi_l \tag{4.234}$$

We want to quantize this expression by similar means as in the foregoing section. As we have recently observed, the implementation of the regularized covariant derivative in (4.151) yields an operator that creates new vertices. However, according to (4.230), this new vertex is strongly coupled to the fermion. Hence, in order for this additional contribution to be nonzero, the presence of a fermion is crucial. One may therefore expect that the quantization of the covariant derivative in (4.234) by similar means will lead to vanishing contributions of the operator acting apart from the spin network vertex which seems to be inconsistent for the regularization of a covariant derivative. For this reason, let us introduce the total covariant derivative $\nabla^{(^\beta A)}$ which acts on both internal indices and spinor indices. With respect to this covariant derivative, we can write

$$(D_a^{(^\beta A)} e^k_b)\phi_l = \nabla_a^{(^\beta A)}(e^k_b \phi_l) - e^k_b \nabla_a^{(^\beta A)} \phi_l \tag{4.235}$$

In the quantum theory, this then has the advantage of creating vertices coupled to fermion fields and therefore, based on our previous observations, yields nontrivial contributions. Inserting (4.235) into (4.234) yields two terms, one which is very similar to expression (4.145) replacing the covariant derivative acting on purely spinor indices with the new total covariant derivative which also acts on internal indices. The implementation of this quantity can be performed in analogy to the foregoing section. For this reason, we will not explain the steps in detail. Concerning the second contribution, one arrives at an expression of the form

$$S'^{(2)}[\eta] := \int_{\Sigma} \mathrm{d}^3x\, \bar{\eta} \frac{1}{\sqrt[4]{q}} \epsilon^{abc} e^l_c \frac{\mathbb{1} + i\beta\gamma_*}{2\beta} \gamma_k e^k_b \nabla_a^{(^\beta A)} \phi_l \tag{4.236}$$

We make the following ansatz for a regularization of (4.236)

$$\begin{aligned} &S'^{(2)}_\delta[\eta] = \\ &= \sum_{\Delta_i \in T(\gamma,\delta)} \bar{\eta}(x_i) \frac{-1}{6\kappa^2 \sqrt{V(x_i,\delta)}} \epsilon^{IJK} \mathrm{tr}(\tau_l h_{s_K(\Delta_i)}[^\beta A]\{h_{s_K(\Delta_i)}[^\beta A]^{-1}, V(x,\delta)\}) \times \\ &\times \frac{\mathbb{1} + i\beta\gamma_*}{2\beta} \gamma_k \mathrm{tr}(\tau_k h_{s_J(\Delta_i)}[^\beta A]\{h_{s_J(\Delta_i)}[^\beta A]^{-1}, V(x,\delta)\}) \left(\mathscr{Y}^\delta_l(s_I(\Delta)) - \mathscr{Y}^\delta_l(x_i)\right) \end{aligned} \tag{4.237}$$

where

$$\mathscr{Y}_l^{\delta}(s_I(\Delta)) := \underline{H}(^{\beta}\!A)(s_I(\Delta_i)(\delta))\theta_l^{\delta}(s_I(\Delta_i)(\delta)) \tag{4.238}$$

and

$$\mathscr{Y}_l(x_i) := \theta_l^{\delta}(x_i) \tag{4.239}$$

Here, $\underline{H}(^{\beta}\!A)$ denotes the holonomy induced by the total covariant derivative $\nabla^{(^{\beta}\!A)}$ which, in the limit of small δ, satisfies

$$\underline{H}(^{\beta}\!A)(s_I(\Delta_i)(\delta))\Psi_l(s_I(\Delta_i)(\delta)) - \Psi_l(x_i) = \delta \dot{s}_I(\Delta_i)^a \nabla_a^{(^{\beta}\!A)}\Psi_l(x_i) \tag{4.240}$$

where Ψ is some spinor-valued co-vector field (w.r.t. internal indices) defined on Σ. Following the same steps as in the previous section, it can be shown immediately that for $\delta \to 0$, one obtains

$$\begin{aligned}\lim_{\delta\to 0} S_\delta'^{(2)}[\eta] = \lim_{\delta\to 0} \sum_{\Delta_i\in T(\gamma,\delta)} \frac{-1}{32\kappa^2\sqrt[4]{q(x_i)}}\{A_c^l, V(x_i,\delta)\}\frac{\mathbb{1}+i\beta\gamma_*}{2\beta}\gamma_k\times \\ \times\{A_b^k, V(x_i,\delta)\}\nabla_a^{(\beta A)}\phi_l(x_i)\epsilon^{IJK}\delta^3\dot{s}_I^a(\Delta_i)\dot{s}_J^b(\Delta_i)\dot{s}_K^c(\Delta_i)\end{aligned} \tag{4.241}$$

so that, together with (4.159) and (4.146), this yields a Riemann sum so that in the limit $\delta \to 0$ one finally arrives at

$$\lim_{\delta\to 0} S_\delta'^{(2)}[\eta] = S'^{(2)}[\eta] \tag{4.242}$$

For the quantization of the regularized expression (4.237), we use

$$\frac{1}{\sqrt{V(x,\delta)}}\{A_c^l, V(x,\delta)\}\{A_b^k, V(x,\delta)\} = \frac{16}{9}\{A_c^l, V(x,\delta)^{\frac{3}{4}}\}\{A_b^k, V(x,\delta)^{\frac{3}{4}}\} \tag{4.243}$$

and replace Poisson brackets by the respective commutators yielding

$$\begin{aligned}\widehat{S}_\delta'^{(2)}[\eta] &:= \\ &= \frac{8}{27\hbar^2\kappa^2}\sum_{v\in\gamma}\frac{8}{E(v)}\bar{\eta}(v)\epsilon^{IJK}\frac{\mathbb{1}+i\beta\gamma_*}{2\beta}\gamma_k \mathrm{tr}(\tau_k h_{s_J(\Delta)}[^{\beta}\!A][h_{s_J(\Delta)}[^{\beta}\!A]^{-1}, \widehat{V}_v^{\frac{3}{4}}])\times \\ &\times\left(\widehat{\mathscr{Y}_l}(s_I(\Delta)) - \widehat{\mathscr{Y}_l}(x_i)\right)\mathrm{tr}(\tau_l h_{s_K(\Delta)}[^{\beta}\!A][h_{s_K(\Delta)}[^{\beta}\!A]^{-1}, \widehat{V}_v^{\frac{3}{4}}])\end{aligned} \tag{4.244}$$

with

$$\widehat{\mathscr{Y}_l}(s_I(\Delta)) := \underline{H}({}^{\beta}\!A)(s_I(\Delta)(\delta))\widehat{\theta}_l(s_I(\Delta)(\delta)) \quad \text{and} \quad \widehat{\mathscr{Y}_l}(v) := \widehat{\theta}_l(v) \tag{4.245}$$

In the infinite sum of (4.244) we were again allowed to restrict to the sum over the vertices of the underlying spin network graph since one of the trace terms was ordered to the right. By (4.186), this yields vanishing contributions in case that the Rovelli-Smolin volume operator does not act on a spin network vertex.

4.5.2.3. Part III

Finally, we need to quantize the last three terms in the SUSY constraint (4.120). These terms are all of very similar structure and, in particular, do not contain any covariant derivative. Hence, it suffices for instance to consider the last one which we write in the form

$$S^{(3)}[\eta] := \int_{\Sigma} \mathrm{d}^3x\, \bar{\eta}\frac{\kappa}{4\sqrt[4]{q}}\gamma_0\phi^i\left(\epsilon^{jkl}\bar{\phi}_j\gamma_0\frac{\mathbb{1}+i\beta\gamma_*}{2\beta}\gamma_k\gamma_i\phi_l\right) \tag{4.246}$$

For its regularization, we make the ansatz

$$S^{(3)}_{\delta}[\eta] := \sum_{\Delta_i\in T(\gamma,\delta)} \bar{\eta}(x_i)\frac{\kappa}{4\sqrt{V(x_i,\delta)}}\gamma_0\theta^{\delta}_i(x_i)\left(\epsilon^{jkl}\bar{\theta}^{\delta}_j(x_i)\gamma_0\frac{\mathbb{1}+i\beta\gamma_*}{2\beta}\gamma_k\gamma^i\theta^{\delta}_l(x_i)\right) \tag{4.247}$$

Due to (4.133), we have

$$\begin{aligned}&\epsilon^{jkl}\bar{\theta}^{\delta}_j(x_i)\gamma_0\frac{\mathbb{1}+i\beta\gamma_*}{2\beta}\gamma_k\gamma^i\theta^{\delta}_l(x_i) \\ &= \int \mathrm{d}^3y\int \mathrm{d}^3z\,\frac{\chi_{\delta}(x_i-y)\chi_{\delta}(x_i-z)}{\delta^3/6}\epsilon^{jkl}\bar{\phi}_j(y)\gamma_0\frac{\mathbb{1}+i\beta\gamma_*}{2\beta}\gamma_k\gamma^i\phi_l(z)\end{aligned} \tag{4.248}$$

and on the other hand

$$\frac{\kappa}{4\sqrt{V(x_i,\delta)}}\gamma_0\theta^{\delta}_i(x_i) = \int \mathrm{d}^3x\,\frac{\chi_{\delta}(x_i-x)}{\delta^3/6}\frac{\kappa}{4\sqrt[4]{q(x_i)}}\gamma_0\phi_i(x) \tag{4.249}$$

In the limit $\delta \to 0$, it follows $\frac{6}{\delta^3}\chi_{\delta}(x_i-x) \to \delta(x_i-x)$ and moreover $\frac{6}{\delta^3}\chi_{\delta}(x_i-z) \to \delta(x_i-z)$ and $\chi_{\delta}(x_i-y)$ can be replaced by the Kronecker delta $\delta_{x_i,y}$.

Therefore, in this limit, (4.247) finally becomes

$$\begin{aligned}\lim_{\delta\to 0} S^{(3)}_\delta[\eta] &= \\ &= \lim_{\delta\to 0}\sum_{\Delta_i\in T(\gamma,\delta)}\bar{\eta}(x_i)\frac{\kappa}{4\sqrt[4]{q(x_i)}}\gamma_0\phi_i(x_i)\left(\epsilon^{jkl}\bar{\phi}_j\gamma_0\frac{\mathbb{1}+i\beta\gamma_*}{2\beta}\gamma_k\gamma^i\phi_l(x_i)\right)\mathrm{vol}(\Delta_i)\\ &= \int_\Sigma \mathrm{d}^3x\,\bar{\eta}\frac{\kappa}{4\sqrt[4]{q}}\gamma_0\phi^i\left(\epsilon^{jkl}\bar{\phi}_j\gamma_0\frac{\mathbb{1}+i\beta\gamma_*}{2\beta}\gamma_k\gamma_i\phi_l\right) = S^{(3)}[\eta] \qquad (4.250)\end{aligned}$$

and therefore (4.247) indeed provides an appropriate regularization of (4.246). Its implementation in the quantum theory is now straightforward yielding

$$\widehat{S}^{(3)}[\eta] := \frac{\kappa}{4}\sum_{v\in V(\gamma)}\frac{8}{E(v)}\bar{\eta}(v)\sqrt{\widehat{V_v^{-1}}}\gamma_0\widehat{\theta}_i(v)\left(i\epsilon^{jkl}\widehat{\theta}^T_j(v)\frac{\mathbb{1}+i\beta\gamma_*}{2\beta}\gamma_k\gamma^i\widehat{\theta}_l(v)\right) \qquad (4.251)$$

where, in the real representation of the gamma matrices, we used that the charge conjugation matrix is given by $C = i\gamma^0$. There exist various possibilities for the implementation of the inverse volume operator $\widehat{V^{-1}}$ such that this operator is well-defined and non-singular. For instance, one can re-express it in terms of a product of Poisson brackets of the form (4.146). However, for sake of simplicity, let us choose a quantization as proposed in [161]. There, one quantizes the inverse volume via

$$\widehat{V^{-1}} := \lim_{t\to 0}(\widehat{V}^2 + t^2 l_p^6)^{-1}\widehat{V} \qquad (4.252)$$

with l_p the Planck length. This operator then simply vanishes while acting on vertices with zero volume and therefore provides a suitable regularization.

4.5.3. Solutions of the quantum SUSY constraint

In this last section, we would like to sketch possible solutions of the quantum SUSY constraint. Going over to the sector of diffeomorphism-invariant states, we are thus looking for vectors $\Upsilon_{\mathrm{phys}} \in \mathcal{D}^*_{\mathrm{diff}}$ (see [18]) such that[7]

$$(\Upsilon_{\mathrm{phys}}|\widehat{S}[\eta]\psi\rangle = 0, \quad \forall\psi\in\mathfrak{H}^{\mathrm{LQSG}} = \mathfrak{H}_{\mathrm{grav}}\otimes\mathfrak{H}_f,\ \eta\in\Gamma(E_{\mathbb{R}}) \qquad (4.253)$$

[7] Actually, working on the dual requires an antilinear representation of the constraint algebra involving rather the adjoint $\widehat{S}[\eta]^\dagger$ of the SUSY constraint. However, since the classical theory, the SUSY constraint is a real function and thus we could equally quantize the complex conjugate $\bar{S}[\eta]$ which then yields $\widehat{S}[\eta]^\dagger$.

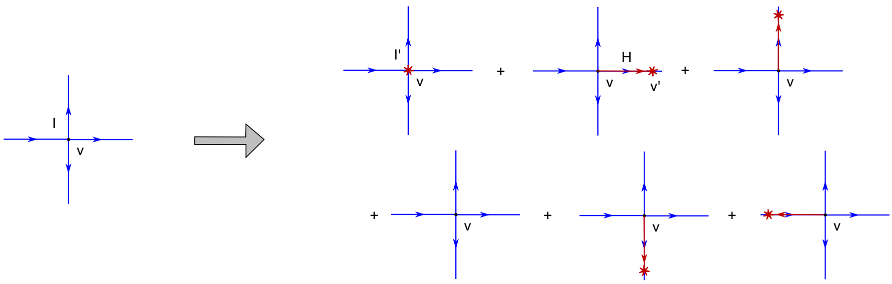


Figure 4.: Schematic depiction of the action of the supersymmetry constraint on a 4-valent vertex v with intertwiner I. Each sub diagram on the right side of the arrow represents a type of term that is appearing in the result. The star symbol represents a vertex containing a fermion, and H is the new holonomy that connects a new vertex v' to the intertwiner at v (source: [2]).

where $\Gamma(E_{\mathbb{R}})$ denotes the space of smooth sections of the spinor bundle $E_{\mathbb{R}} := P \times_{\kappa_{\mathbb{R}}} \Delta_{\mathbb{R}}$ induced by the Majorana representation on $\Delta_{\mathbb{R}}$.

Considering the first part (4.162) of the quantum SUSY constraint studied in Section 4.5.1, this operator creates new vertices coupled to a fermion. A qualitative description of the action is depicted in Figure 4. Each diagram on the right side of the arrow represents a type of term that is appearing in the result. Fermions are created both, at the original vertex v and at new vertices v' that lie on the edges incident at v. The creation of fermions is a generic feature of the quantum SUSY constraint because the conjugate spinor plays the role of smearing function. In case of an ordinary Dirac fermion, this would mean that even if, on the right-hand side of (4.253), one initially started with a state ψ in the pure gravitational sector of the Hilbert state, i.e., an ordinary spin network state without any fermions, this operator would always create states with nontrivial fermionic degrees of freedom. But then, any pure gravitational state Ψ_{phys} would be a solution of (4.253) as the inner product between a pure bosonic and fermionic state is always zero by (4.138) (or (4.139)). This is, however, no longer true in case of Majorana fermions. In fact, as seen in Section 4.5.1 (see formula (4.141)), due to the Majorana condition, it follows that the quantization of the Rarita-Schwinger field necessarily involves both multiplication operators and derivations, i.e., creation and annihilation operators. Therefore, the quantum SUSY constraint generically both creates and annihilates fermionic degrees of freedom. As a consequence, pure gravitational states cannot be a solution of (4.253). For purely fermionic states, the situation is less clear, we can not immediately rule out their existence. In any case, such solutions of (4.253) would seem to be unphysical.

4.6. Discussion

In this chapter, we have studied the canonical theory of $\mathcal{N} = 1$ supergravity in four spacetime dimensions based on the Holst action of supergravity as first introduced by Tsuda [65] as well as its extension to the case of a nonvanishing cosmological constant. In this framework, we considered half-densitized fermion fields as suggested by Thiemann [80] in order to simplify the reality conditions for the Rarita-Schwinger field. We then derived a compact expression for the classical SUSY constraint which then served as a starting point for its implementation in the quantum theory. To this end, following [67], we quantitzed the Rarita-Schwinger field by appropriately extending the classical phase space.

With these prerequisites, we turned to the quantitzation of the SUSY constraint which so far has not been considered in the literature. This is important because the quantum SUSY constraint in canonical supergravity theories plays a similar role as the quantum Hamiltonian constraint in quantum gravity theories without local supersymmetry. For this purpose, we first had to derive a suitable regularization of the continuum expression guided by the principle that the resulting operator should be as compact as possible. For the regularization, special care was required. This is mainly due to the fact that, although the SUSY constraint looks similar to the Dirac Hamiltonian constraint, there is a crucial difference: The conjugate spinor plays the role of a Lagrangian multiplier. As a result, one cannot simply follow the standard regularization procedure as the density weight of the smearing function should be kept fixed in order to not the change the density weight of the SUSY constraint as a whole. Changing its density weight may change the resulting quantum algebra and thus its strong relationship to the Hamiltonian constraint as indicated in the classical regime in [64] in case of real Ashtekar-Barbero variables. We succeeded in finding an appropriate regularization such the density weight is maintained.

The resulting operator consists of various different terms one of which arose from the quantization of the covariant derivative on the fermion field considered in Section 4.5.2. Requiring consistency with the classical theory forced us to choose the Rovelli-Smolin variant of the volume operator for the quantization of the triads via Thiemann's trick. Based on an explicit calculation, it was shown, choosing an appropriate factor ordering, that the resulting operator was still finite as the sum over the tetrahedra in the triangulation again restricts to the sum over vertices of the underlying graph. Different implementations in the quantum theory involving the Ashtekar-Lewandowski volume operator have also been discussed. For this, a different but equivalent form of the classical SUSY constraint has to be considered.

As it turns out, the operator thus obtained has an interesting feature as it creates new vertices strongly coupled to fermions. This was shown via explicit computation evaluating its action on generic spin network states. Due to this fact, it is expected that solutions of the quantum SUSY constraint need to contain both gravity and matter degrees of freedom as required for supersymmetry. We have seen that the reality condition enforced on Majorana spinors is important. Whether these solutions indeed contain the same number of bosons and fermions, however, is still unclear so far and remains a question for the future. Also it would be highly desirable to study the commutator algebra of the quantum SUSY constraint. In particular, it would be very interesting to see in which sense the commutator on diffeomorphism- and gauge-invariant states is related to the Hamiltonian constraint. In context of the chiral theory, we will study this question explicitly in Chapter 6 considering a class of symmetry reduced models. In the full theory, as a first step, one could try to evaluate the commutator of the terms involving the quantization of the covariant derivative and investigate whether this can be related to the quantization of the curvature of the connection along loops.

5. Holst-MacDowell-Mansouri action for (extended) supergravity with boundaries, and chiral LQSG

5.1. Introduction

The physics of boundaries, in particular, the interaction between degrees of freedom on the boundary and those in the bulk play an important role in diverse areas of physics, from solid state physics to gravity. In the latter area, this is particularly the case for the horizons of black holes. Bekenstein-Hawking entropy [162, 163] assigns the black hole an entropy as if there was one bit of information encoded in each Planck unit of its horizon area, and Hawking radiation looks as if it was perfectly thermal at its surface [164]. The holographic principle as advocated by 't Hooft, Susskind and others states that the entire state of the black hole is represented on its surface [165]. In loop quantum gravity, a picture that is consistent with these holographic ideas emerges partially from an observation about the classical theory and its boundary at the horizon: If a spacetime with an inner boundary is considered, and boundary condition are imposed at the inner boundary consistent with it being an apparent horizon, the symplectic structure attains a contribution corresponding to a Chern-Simons theory on the horizon [35–39]. In the quantum theory, the excitations of the gravitational field create defects in the horizon Chern-Simons theory, thereby changing the size of the state space and account for black hole entropy [40–44].

Boundary theories in supergravity also play a crucial role in string theory such as in context of the celebrated $\mathrm{AdS}_{d+1}/\mathrm{CFT}_d$ conjecture [14–16], a far reaching duality which attained a lot of interest since its discovery by Maldacena. It describes a duality between string theory on a $d+1$ asymptotically anti-de Sitter spacetime and a d-dimensional conformal field theory on the boundary such as, most prominently, between type IIB superstring theory on an $\mathrm{AdS}_5 \times S^5$ background and $\mathcal{N}=4$ super Yang-Mills theory living on the boundary. In the low-energy limit of string theory aka supergravity, this holographic correspondence has been studied very intensively. There, one observes a one-to-one correspondence between fields of the bulk supergravity theory satisfying certain boundary conditions and quantum operators associated to the boundary conformal field theory.

On the other hand, boundaries in string theory have also recently been explored in [166] where a specific brane configuration in the framework of type IIB superstring theory has been considered consisting of a stack of D3 branes on two sides of a NS5 brane where the worldvolume theory on the D3 branes corresponds to a maximally supersymmetric Yang-Mills theory with $\mathrm{U}(n)$ gauge group. There, it has been observed that the boundary theory is described by a super Chern-Simons theory with gauge group given by the super unitary group $\mathrm{U}(m|n)$ and complex Chern-Simons level. But also other configurations have been considered leading to super Chern-Simons theories with gauge supergroups $\mathrm{OSp}(m|n)$.

In the context of supergravity, there exist various different approaches on the proper description of boundaries (see e.g. [167–170]). More recently, boundaries in supergravity have been considered in [81, 83, 171–173] in the framework of the Castellani-D'Auria-Fré approach [71, 72] (see Section 3.4). There, a systematic approach for $D = 4$ pure supergravity theories both with and without a cosmological constant has been developed, by studying the most general class of possible boundary terms that are compatible with the symmetry of the bulk Lagrangian. By demanding supersymmetry invariance at the boundary, these boundary terms then turned out to be determined even uniquely. Moreover, within this formalism, one finds in both cases, i.e., with and without a cosmological constant, that the associated boundary conditions are not of Dirichlet-type but require the vanishing of the super curvatures on the boundary. Finally, it follows that the resulting action of the theory including bulk and boundary degrees of freedom takes a very intriguing form which, for $\mathcal{N} = 1$ and nontrivial cosmological constant, exactly reproduces the well-known MacDowell-Mansouri action [138]. In particular, in this way, a similar structure has been found for $\mathcal{N} = 1$, $D = 4$ Poincaré supergravity [171] as well as $\mathcal{N} = 2$, $D = 4$ pure AdS supergravity [81].

In this chapter, we want to study the classical theory of boundaries in supergravity in $D = 4$ using Asthtekar-Barbero variables. There are several reasons why this is an interesting topic to study. Among other things, it may shed further light on the quantum description of black holes in loop quantum gravity (LQG) and string theory. We will not use the formalism of isolated or dynamical horizons ([174–176] and [177] for an overview and further literature) as it has not yet been thoroughly studied in the context of supergravity, and because its boundary conditions seem to not be well-adapted to the requirement of local supersymmetry at the boundary. Rather, following [81, 83], we make the condition of local supersymmetry extending to the boundary the guiding principle for finding appropriate boundary terms and boundary conditions.

As already mentioned in the main introduction in Chapter 1, in the context of LQG, $\mathcal{N} = 1$ supergravity in terms of self-dual variables has been studied e.g. in [62, 63]. In particular, in [63], on the kinematical level, a hidden $\mathfrak{osp}(1|2)$ gauge symmetry in the constraint algebra has been observed which subsequently has been used to formulate a quantum theory à la LQG by Gambini, Pullin et al. [84] and Ling and Smolin [85] and in the context of spin foam models in $D = 3$ for instance in [147, 178]. Extended $\mathcal{N} = 2$, $D = 4$ chiral supergravity has been studied e.g. in [179, 180], and in terms of a constrained super BF-theory in [181]. Boundaries in supergravity in the framework of LQG have been discussed using self-dual variables already a long time ago in [86, 182].

Interestingly, there the authors already seem to suggest that topological terms contained in the (chiral) MacDowell-Mansouri action may play a role in (quantum) description of boundaries in supergravity.

In what follows, we want to study this question from a more general perspective following newer developments in the geometric approach [81, 83] and pointing out the importance of supersymmetry invariance at the boundary and also explicitly including real Ashtekar-Barbero variables. Moreover, we will start from the original supergravity Lagrangians instead from constrained field theories. To this end, using the interpretation of supergravity in terms of a super Cartan geometry, we will derive the Holst modification of the MacDowell-Mansouri action for arbitrary Barbero-Immirzi parameter β for $\mathcal{N}=1$ and $\mathcal{N}=2$ pure AdS supergravity as derived in [81] which as mentioned above, by construction, already contains the most general class of boundary terms maintaining supersymmetry invariance at the boundary. To do so, inspired by [183–185] in the context of ordinary gravity, we will introduce some kind of a β-deformed inner product induced by a β-dependent operator $\mathbf{P}_\beta$ defined on super Lie algebra-valued differential forms. As we will see, this approach then also allows for a very elegant and unified discussion of the chiral limit of the theory. There, it follows that $\mathbf{P}_{\pm i}$ leads to a projection operator onto a proper subalgebra of the super anti-de Sitter algebra corresponding to the (complex) orthosymplectic group $\mathrm{OSp}(\mathcal{N}|2)_{\mathbb{C}}$. As a consequence, the resulting action becomes manifestly invariant under $\mathrm{OSp}(\mathcal{N}|2)_{\mathbb{C}}$ leading to the notion of the super Ashtekar connection. This reveals the underlying enlarged gauge symmetry of the chiral theory for both cases in a very clear way.

In particular, it follows that the resulting boundary terms correspond to a super Chern-Simons action with gauge group given by the supergroups $\mathrm{OSp}(\mathcal{N}|2)_{\mathbb{C}}$ and complex Chern-Simons level. For $\mathcal{N}=1$, we will also prove explicitly that the full action is indeed invariant under left- and right-handed supersymmetry transformations on the boundary and turns out to be even fixed uniquely by this requirement. Moreover, we will derive the boundary conditions of the full theory describing the coupling between the bulk and boundary degrees of freedom. As we will see, these turn out to be in strong analogy to the standard boundary conditions as usually applied in LQG and, in particular, transform covariantly under the enlarged gauge symmetry of the chiral theory.

Finally, using the gauge-theoretic structure of the canonical phase space of $\mathcal{N}$-extended chiral SUGRA, we will derive a graded analog of the holonomy-flux algebra as well-known in standard LQG with real variables. This will be done rigorously explicitly taking into account the parametrization supermanifold required in order to resolve the fermionic degrees of freedom of the theory and using the parallel transport map as derived in Chapter 2 induced by the super Ashtekar connection. It follows that the configuration space of generalized super connections carries an intriguing structure similar to a Moltokov-Sachse supermanifold which, in case of compact super Lie groups, becomes projectively Hausdorff. Based on these observations, we will then sketch the quantization of the theory choosing a Ashtekar-Lewandowski-type representation of the superalgebra and introduce the notion of super spin networks as first considered in [84–86]. Finally, we will compare this quantization scheme with the standard quantization techniques of LQG coupled to fermions as proposed in [67, 80, 87] and will encounter many conceptual similarities.

The structure of this chapter is as follows: At the beginning, we recall very briefly some basic elements of the Cartan geometric approach to $\mathcal{N}=1$ pure AdS supergravity and discuss the most general class of possible boundary terms following [81, 83]. We then define in Section 5.2.1 the Holst-MacDowell-Mansouri action by introducing a β-dependent operator and corresponding inner product on the underlying superalgebra. We then repeat this procedure for the $\mathcal{N}=2$ extended case in the subsequent Sections 5.3 and 5.3.1. In particular, we will extend the $\mathcal{N}=2$ pure supergravity action as found in [81] to arbitrary β including the most general class of boundary terms compatible with local supersymmetry. We will then discuss the chiral limit as well as the boundary theory in Section 5.4 and compare our results with those found in [81, 83] using standard variables. In Section 5.5, we will derive the graded holonomy-flux algebra and quantize the theory adapting quantization techniques of standard LQG. In Section 5.6, we will give an outlook on the application of these results to the quantum description of supersymmetric black holes in chiral LQSG. Finally, in the last Section 5.7, we will discuss some results on super Peter-Weyl theory considering the supergroups $\mathrm{U}(1|1)$ and $\mathrm{SU}(1|1)$.

As already explained in the previous chapter as well as at the end of the main introduction of this thesis in Chapter 1, in the following, we will drop many mathematical details in order to simplify the notation and to make the following discussion easier accessible for the reader. In particular, we will not explicitly mention the underlying parametrization supermanifold except in Section 5.5 in the context of the construction of the graded holonomy-flux algebra and the quantization of chiral supergravity where the parametrization turns out be essential.

A list of important symbols as well as an overview of our choice of conventions concerning indices, physical constants etc. can be found in the List of symbols, notations and conventions.

5.2. Geometric $\mathcal{N}=1$ supergravity with boundaries

The content of this section has been reproduced from [3], with slight changes to account for the context of this thesis with the permission of Springer-Nature.

In this section, we want to briefly recall the geometric interpretation of $\mathcal{N} = 1$ AdS supergravity in terms of a super Cartan geometry. In particular, following [81], we discuss the extension of the theory in the presence of boundaries and the implementation of supersymmetric boundary conditions.

Pure $D = 4$, $\mathcal{N} = 1$ AdS supergravity can be described in terms of a super Cartan geometry modeled on the flat super Klein geometry (OSp$(1|4)$,Spin$^+(1,3)$) corresponding to super anti-de Sitter space (see Example 2.3.17). Using the decomposition $\mathfrak{osp}(1|4) = \mathbb{R}^{1,3} \oplus \mathfrak{spin}^+(1,3) \oplus \Delta_{\mathbb{R}}$ of the super Lie algebra, with odd part $\Delta_{\mathbb{R}}$ corresponding to a real Majorana representation of Spin$^+(1,3)$, the super Cartan connection $\mathcal{A}$ of the theory takes the form

$$\mathcal{A} = e^I P_I + \frac{1}{2}\omega^{IJ} M_{IJ} + \psi^\alpha Q_\alpha \tag{5.1}$$

with e^I the co-frame, ω^{IJ} the spin connection and ψ^α the Rarita-Schwinger field. Moreover, the horizontal forms contained in the Cartan connection build up the supervielbein or super soldering form

$$E = e^I P_I + \psi^\alpha Q_\alpha \tag{5.2}$$

which provides a local identification of the curved supermanifold with the flat model given by super AdS$_4$. This is a direct consequence the super Cartan condition (condition (iii) in Def. 3.3.3). The action of the theory takes the form [141, 142, 152]

$$\begin{aligned} S^{\mathcal{N}=1}_{\text{AdS}}(\mathcal{A}) =&: \frac{L^2}{\kappa}\int_M \mathscr{L} \\ :=& \frac{1}{2\kappa}\int_M \Big(\frac{1}{2}F(\omega)^{IJ}\wedge e^K \wedge e^L \epsilon_{IJKL} + i\bar{\psi}\wedge\gamma_*\gamma_I D^{(\omega)}\psi\wedge e^I \\ &- \frac{1}{4L}\bar{\psi}\wedge\gamma^{IJ}\psi\wedge e^K\wedge e^L\epsilon_{IJKL} + \frac{1}{4L^2}e^I\wedge e^J\wedge e^K\wedge e^L\epsilon_{IJKL}\Big) \end{aligned} \tag{5.3}$$

with $F(\omega)^{IJ} = \mathrm{d}\omega^{IJ} + \omega^I{}_K \wedge \omega^{KJ}$ the curvature of the spin connection ω and $D^{(\omega)}\psi = \mathrm{d}\psi + \frac{1}{4}\omega^{IJ}\gamma_{IJ}\wedge\psi$ the induced exterior covariant derivative. Similar as in the discussion at the end of Section 3.4, it follows that the underlying supersymmetry of the theory can be described using the bijective correspondence between super Cartan connections and Ehresmann connections (Prop. 3.3.12), i.e. ordinary gauge fields playing a role for instance in Yang-Mills gauge theories, on the associated OSp$(1|4)$-bundle. One can then interpret local supersymmetry in terms of local gauge transformations in the odd direction of the supergroup. In fact, using the (graded) commutation relations

(2.96)-(2.99), it follows that, under such kind of gauge transformations, the individual fields transform as

$$\delta_\epsilon e^I = \frac{1}{2}\bar{\epsilon}\gamma^I\psi, \quad \delta_\epsilon\psi^\alpha = D^{(\omega)}\epsilon^\alpha - \frac{1}{2L}e^I(\gamma_I)^\alpha{}_\beta \wedge \epsilon^\beta, \quad \delta_\epsilon\omega^{IJ} = \frac{1}{2L}\bar{\epsilon}\gamma^{IJ}\psi \tag{5.4}$$

for some Grassmann-odd Majorana spinor ϵ^α. It then turns out that (5.3) is indeed invariant under (5.4) provided the spin connection satisfies its field equations.

So far, we have excluded the possibility of boundaries in our discussion. In the presence of boundaries, it follows that one needs to add additional (topological) boundary terms in order to maintain functional differentiability of the action functional (5.3). Moreover, in case of supergravity, one is particularly interested in boundary terms which also ensure invariance of the full action under supersymmetry transformations at the boundary. It turns out that this requirement strongly restricts the structure of possible boundary terms to be added to the theory. To this end, following [81], one notes that the only possible topological terms which are consistent with the symmetries of the bulk Lagrangian $\mathscr{L} \equiv \mathscr{L}_{\text{bulk}}$ in (5.3) are of the form

$$\begin{aligned}\mathscr{L}_{\text{bdy}} =& C_1 F(\omega)^{IJ} \wedge F(\omega)^{KL}\epsilon_{IJKL} + C_2\Big(D^{(\omega)}\bar{\psi} \wedge \gamma_* D^{(\omega)}\psi \\ &+ \frac{i}{8}\epsilon_{IJKL}F(\omega)^{IJ} \wedge \bar{\psi} \wedge \gamma^{KL}\psi\Big)\end{aligned} \tag{5.5}$$

for any constant coefficients C_1, C_2. The first term in (5.5) is given by the Gauss-Bonnet term which is indeed topological. The second term can equivalently be written as

$$D^{(\omega)}\bar{\psi} \wedge \gamma_* D^{(\omega)}\psi + \frac{i}{8}\epsilon_{IJKL}F(\omega)^{IJ} \wedge \bar{\psi} \wedge \gamma^{KL}\psi = \mathrm{d}(\bar{\psi} \wedge \gamma_* D^{(\omega)}\psi) \tag{5.6}$$

and therefore is also a total derivative. As shown in [81] (see also the discussion in Section 5.3), if one requires invariance of the full Lagrangian $\mathscr{L}_{\text{full}} = \mathscr{L}_{\text{bulk}} + \mathscr{L}_{\text{bdy}}$ under supersymmetry transformations, then the coefficients C_1, C_2 are uniquely fixed to particular values given by $C_1 = 1/8$ and $C_2 = i/(2L)$, respectively. We will in fact show this explicitly in the context of the chiral theory in Section 5.4.2 below. Moreover, using (5.6) it follows that the full Lagrangian $\mathscr{L}_{\text{full}}$ is that of the MacDowell-Mansouri action [81, 138, 141], i.e. quadratic in the super Cartan curvature $F(\mathcal{A})$ to be defined in the next section such that

$$\mathscr{L}_{\text{full}} = \frac{1}{4}F(\mathcal{A})^{IJ} \wedge F(\mathcal{A})^{KL}\epsilon_{IJKL} + \frac{i}{L}F(\mathcal{A})^\alpha \wedge F(\mathcal{A})^\delta (C\gamma_*)_{\alpha\delta} \tag{5.7}$$

To summarize, the boundary terms for $D = 4$, $\mathcal{N} = 1$ AdS supergravity in the presence of boundaries are uniquely fixed by requirement of supersymmetry invariance at the boundary, and they are neatly contained in the MacDowell-Mansouri action.

5.2.1. Holst-MacDowell-Mansouri action of $\mathcal{N} = 1$ SUGRA

The content of this section has been reproduced from [3], with slight changes to account for the context of this thesis with the permission of Springer-Nature.

In this section, we want to discuss $D = 4$, $\mathcal{N} = 1$ AdS supergravity in the context of LQG. As in the previous section, we want to explicitly include the possibility of boundaries in the theory. We therefore need to derive a Holst variant of the MacDowell-Mansouri action for arbitrary Barbero-Immirzi parameter β. A derivation of the Holst action of $D = 4$, $\mathcal{N} = 1$ supergravity via a MacDowell-Mansouri action, by adding a suitable topological term and treating β as kind of a θ-ambiguity similar to Yang-Mills theory, has been given in [186] and for the special case of the chiral theory in [187]. Here, we want to follow the ideas of [183, 184] in the context of classical first-order Einstein gravity and its reformulation in terms of a constrained BF-theory [185]. As we will show, these ideas can naturally be extended to supergravity by introducing a *β-deformed inner product* on the superalgebra.

To this end, note that, using the explicit representation of $\mathfrak{osp}(1|4)$ as derived in Example 2.3.17, the generators of $\mathfrak{spin}^+(1,3)$ take the form $M_{IJ} = \frac{1}{2}\gamma_{IJ}$. One can then define an operator $\mathcal{P}_\beta$ on $\mathfrak{spin}^+(1,3)$ via

$$\mathcal{P}_\beta := \frac{\mathbb{1} + i\beta\gamma_*}{2\beta} : \mathfrak{spin}^+(1,3) \to \mathfrak{spin}^+(1,3) \tag{5.8}$$

That this operator indeed leaves the Lie algebra $\mathfrak{spin}^+(1,3)$ invariant follows from $i\gamma_*\gamma_{IJ} = \frac{1}{2}\epsilon_{IJ}{}^{KL}\gamma_{KL}$ which, moreover, yields the important identity

$$\mathcal{P}_\beta\gamma^{IJ} = i\gamma_* P_\beta{}^{IJ}{}_{KL}\gamma^{KL}, \text{ with } \quad P_\beta{}^{IJ}{}_{KL} = \frac{1}{2}\left(\delta^I_{[K}\delta^J_{L]} - \frac{1}{2\beta}\epsilon^{IJ}{}_{KL}\right) \tag{5.9}$$

Since the odd part of $\mathfrak{osp}(1|4)$, in particular, defines a Clifford module, we can naturally extend $\mathcal{P}_\beta$ to an operator on $\Delta_\mathbb{R}$ via $\mathcal{P}_\beta Q_\alpha = Q_\delta(\mathcal{P}_\beta)^\delta{}_\alpha$. Hence, using the identification $\mathfrak{osp}(1|4) \cong \mathbb{R}^{1,3} \oplus \mathfrak{spin}^+(1,3) \oplus \Delta_\mathbb{R}$, one can introduce an operator $\mathbf{P}_\beta$ on the super Lie algebra (or rather its complexification), by setting

$$\mathbf{P}_\beta := \underline{\mathbf{0}} \oplus \mathcal{P}_\beta \oplus \mathcal{P}_\beta : \mathfrak{osp}(1|4) \to \mathfrak{osp}(1|4) \tag{5.10}$$

Using this operator, we can define an inner product on the super Lie algebra. Note first that a standard Adjoint-invariant inner product on $\mathfrak{osp}(1|4)$ is given by the supertrace

$\langle\cdot,\cdot\rangle$:= str. When combined with (5.10), this yields a the corresponding β-deformed inner product setting

$$\langle X, Y\rangle_\beta := \mathrm{str}(X \cdot \mathbf{P}_\beta Y), \quad \forall X, Y \in \mathfrak{osp}(1|4) \tag{5.11}$$

which is invariant under $\mathrm{Spin}^+(1,3)$ but not under the full supergoup $\mathrm{OSp}(1|4)$. Extending this inner product to an inner product on $\Omega^2(\mathcal{M}, \mathfrak{g})$ with $\mathcal{M}$ the underyling supermanifold (see Section 5.6.1), we can now formulate the Holst-MacDowell-Mansouri action of $\mathcal{N} = 1$, $D = 4$ AdS supergravity. It is given by

$$S^{\mathcal{N}=1}_{\text{H-MM}}(\mathcal{A}) = \frac{L^2}{\kappa}\int_M \langle F(\mathcal{A}) \wedge F(\mathcal{A})\rangle_\beta \tag{5.12}$$

where $F(\mathcal{A})$ is the Cartan curvature of $\mathcal{A}$ defined as

$$F(\mathcal{A}) = \mathrm{d}\mathcal{A} + \frac{1}{2}[\mathcal{A} \wedge \mathcal{A}] = \mathrm{d}\mathcal{A} + \frac{1}{2}(-1)^{|T_{\underline{A}}||T_{\underline{B}}|}\mathcal{A}^{\underline{A}} \wedge \mathcal{A}^{\underline{B}} \otimes [T_{\underline{A}}, T_{\underline{B}}] \tag{5.13}$$

w.r.t. the homogeneous basis $(T_{\underline{A}})_{\underline{A}}$ of $\mathfrak{osp}(1|4)$, $\underline{A} \in (I, IJ, \alpha)$, where the minus sign in (5.12) appears due to the (anti)commutation of $T_{\underline{A}}$ and $\mathcal{A}^{\underline{B}}$. Using the graded commutation relations (2.96)-(2.99) in case $\mathcal{N} = 1$ as well as $[M_{IJ}, P_K] = \eta_{IK}P_J - \eta_{JK}P_I$, it follows that the translational and Lorentzian sub components of $F(\mathcal{A})$ take the form

$$F(\mathcal{A})^I = \Theta^{(\omega)I} - \frac{1}{4}\bar{\psi} \wedge \gamma^I\psi, \quad F(\mathcal{A})^{IJ} = F(\omega)^{IJ} + \frac{1}{L^2}\Sigma^{IJ} - \frac{1}{4L}\bar{\psi} \wedge \gamma^{IJ}\psi \tag{5.14}$$

respectively, with $\Sigma^{IJ} = e^I \wedge e^J$ and $\Theta^{(\omega)I} = \mathrm{d}e^I + \omega^I{}_J \wedge e^J$ the torsion 2-form associated to ω. For the odd part of the curvature, we find

$$F(\mathcal{A})^\alpha = D^{(\omega)}\psi^\alpha - \frac{1}{2L}e^I \wedge (\gamma_I)^\alpha{}_\beta\psi^\beta \tag{5.15}$$

To see that this in fact leads to the Holst action $S^{\mathcal{N}=1}_{\text{H-AdS}}$ (Eq. (4.121)) of $\mathcal{N} = 1$ AdS supergravity, let us expand the action (5.12). If we use (5.14) and (5.15), we find[1]

$$\begin{aligned}\langle F(\mathcal{A}) \wedge F(\mathcal{A})\rangle_\beta =& \frac{1}{4}F(\mathcal{A})^{IJ} \wedge F(\mathcal{A})^{KL}\langle M_{IJ}, M_{KL}\rangle_\beta \\ & - F(\mathcal{A})^\alpha \wedge F(\mathcal{A})^\delta\langle Q_\alpha, Q_\delta\rangle_\beta\end{aligned}$$

[1] To symplify our notation, we write $\bar{\psi} \wedge \gamma^{\bullet\bullet}\psi$ for the $\mathfrak{spin}^+(1,3)$-valued 2-form with components $\bar{\psi} \wedge \gamma^{IJ}\psi$.

$$
\begin{aligned}
&= \langle F(\omega) \wedge F(\omega) \rangle_\beta + \frac{2}{L^2} \langle \Sigma \wedge F(\omega) \rangle_\beta \\
&\quad - \frac{1}{2L} \langle F(\omega) \wedge \bar{\psi} \wedge \gamma^{\bullet\bullet} \psi \rangle_\beta - \frac{1}{2L^3} \langle \Sigma \wedge \bar{\psi} \wedge \gamma^{\bullet\bullet} \psi \rangle_\beta \\
&\quad + \frac{1}{L^4} \langle \Sigma \wedge \Sigma \rangle_\beta + \frac{1}{L} F(\mathcal{A})^\alpha \wedge F(\mathcal{A})^\delta (C\mathcal{P}_\beta)_{\alpha\delta}
\end{aligned} \tag{5.16}
$$

where we used that $\langle Q_\alpha, Q_\delta \rangle_\beta = -\frac{1}{L}(C\mathcal{P}_\beta)_{\alpha\delta}$ which can be checked by direct computation using the explicit representation given in Example 2.3.17. Using

$$
\langle M_{IJ}, M_{KL} \rangle_\beta = -\frac{i}{4} P_\beta{}^{MN}{}_{KL} \mathrm{tr}(\gamma_{IJ}\gamma_{MN}\gamma_*) = P_\beta{}^{MN}{}_{KL} \epsilon_{IJMN} \tag{5.17}
$$

this yields

$$
\begin{aligned}
&\langle F(\omega) \wedge F(\omega) \rangle_\beta + \frac{2}{L^2} \langle \Sigma \wedge F(\omega) \rangle_\beta - \frac{1}{2L} \langle F(\omega) \wedge \bar{\psi} \wedge \gamma^{\bullet\bullet} \psi \rangle_\beta \\
&- \frac{1}{2L^3} \langle \Sigma \wedge \bar{\psi} \wedge \gamma^{\bullet\bullet} \psi \rangle_\beta + \frac{1}{L^4} \langle \Sigma \wedge \Sigma \rangle_\beta \\
&= \langle F(\omega) \wedge F(\omega) \rangle_\beta + \frac{2}{L^2} \langle \Sigma \wedge F(\omega) \rangle_\beta - \frac{1}{8L} F(\omega)^{IJ} \wedge P_\beta{}^{KL}{}_{MN} \bar{\psi} \wedge \gamma^{MN} \psi \epsilon_{IJKL} \\
&- \frac{1}{8L^3} \Sigma^{IJ} \wedge P_\beta{}^{KL}{}_{MN} \bar{\psi} \wedge \gamma^{MN} \psi \epsilon_{IJKL} + \frac{1}{4L^4} \Sigma^{IJ} \wedge P_\beta{}^{KL}{}_{MN} \Sigma^{MN} \epsilon_{IJKL} \\
&= \langle F(\omega) \wedge F(\omega) \rangle_\beta + \frac{2}{L^2} \langle \Sigma \wedge F(\omega) \rangle_\beta + \frac{i}{8L} F(\omega)^{IJ} \wedge \bar{\psi} \wedge \gamma_* \gamma^{KL} \mathcal{P}_\beta \psi \epsilon_{IJKL} \\
&+ \frac{i}{8L^3} \Sigma^{IJ} \wedge \bar{\psi} \wedge \gamma_* \gamma^{KL} \mathcal{P}_\beta \psi \epsilon_{IJKL} + \frac{1}{8L^4} \Sigma^{IJ} \wedge \Sigma^{KL} \epsilon_{IJKL}
\end{aligned} \tag{5.18}
$$

On the other hand, we have

$$
\begin{aligned}
\frac{1}{L} F(\mathcal{A})^\alpha \wedge F(\mathcal{A})^\delta (C\mathcal{P}_\beta)_{\alpha\delta} &= \langle D^{(\omega)}\psi \wedge D^{(\omega)}\psi \rangle_\beta - \frac{1}{L^2} \bar{\psi} \wedge \boldsymbol{\gamma} \wedge \mathcal{P}_\beta D^{(\omega)} \psi \\
&\quad - \frac{1}{4L^3} \bar{\psi} \wedge \gamma_{IJ} \mathcal{P}_{-\beta} \psi \wedge \Sigma^{IJ}
\end{aligned} \tag{5.19}
$$

where we set $\boldsymbol{\gamma} := e^I \gamma_I$. Adding the Equations (5.18) and (5.19) and again using the identity $\epsilon_{IJ}{}^{KL}\gamma_{KL} = 2i\gamma_{IJ}\gamma_*$ yielding

$$
\begin{aligned}
&\frac{i}{8L^3} \Sigma^{IJ} \wedge \bar{\psi} \wedge \gamma_* \gamma^{KL} \mathcal{P}_\beta \psi \epsilon_{IJKL} - \frac{1}{4L^3} \bar{\psi} \wedge \gamma_{IJ} \mathcal{P}_\beta \psi \wedge \Sigma^{IJ} \\
=& \frac{i}{8L^3} \bar{\psi} \wedge \gamma_* \gamma^{KL} \left[\mathcal{P}_\beta + \mathcal{P}_{-\beta} \right] \psi \wedge \Sigma^{IJ} \epsilon_{IJKL} = -\frac{1}{8L^3} \bar{\psi} \wedge \gamma^{KL} \psi \wedge \Sigma^{IJ} \epsilon_{IJKL}
\end{aligned} \tag{5.20}
$$

it follows that the Holst-MacDowell-Mansouri action takes the form

$$\begin{aligned}S^{\mathcal{N}=1}_{\text{H-MM}}(\mathcal{A}) = \frac{1}{2\kappa}\int_M \Sigma^{IJ}\wedge(P_\beta\circ F(\omega))^{KL}\epsilon_{IJKL} - \bar{\psi}\wedge\gamma\wedge\frac{\mathbb{1}+i\beta\gamma_*}{\beta}D^{(\omega)}\psi \\ -\frac{1}{4L}\bar{\psi}\wedge\gamma^{KL}\psi\wedge\Sigma^{IJ}\epsilon_{IJKL} + \frac{1}{4L^2}\Sigma^{IJ}\wedge\Sigma^{KL}\epsilon_{IJKL} + S_{\text{bdy}}\end{aligned} \tag{5.21}$$

where we wrote $(P_\beta\circ F(\omega))^{IJ} = P_\beta{}^{IJ}{}_{KL}F(\omega)^{KL}$. Moreover, S_{bdy} denotes a boundary term given by

$$\begin{aligned}S_{\text{bdy}}(\mathcal{A}) = \frac{L^2}{\kappa}\int_M \langle F(\omega)\wedge F(\omega)\rangle_\beta + \langle D^{(\omega)}\psi\wedge D^{(\omega)}\psi\rangle_\beta \\ -\frac{1}{4L}F(\omega)^{IJ}\wedge\bar{\psi}\wedge\gamma_{IJ}\mathcal{P}_\beta\psi\end{aligned} \tag{5.22}$$

Thus, we see, up to a topological term, (5.12) indeed reduces to the Holst action (4.121) of $\mathcal{N}=1$, $D=4$ AdS supergravity and which in the limit of a vanishing cosmological constant, i.e. $L\to\infty$, yields the respective action (4.35) of Poincaré supergravity. To see that (5.22) is in fact purely topological, note that, by the Bianchi identity, we have

$$D^{(\omega)}D^{(\omega)}\psi = \kappa_{\mathbb{R}*}(F(\omega))\wedge\psi = \frac{1}{4}F(\omega)^{IJ}\gamma_{IJ}\wedge\psi \tag{5.23}$$

such that by the $\text{Spin}^+(1,3)$-invariance of the inner product, this yields

$$\begin{aligned}\langle D^{(\omega)}\psi\wedge D^{(\omega)}\psi\rangle_\beta &= \mathrm{d}\langle\psi\wedge D^{(\omega)}\psi\rangle_\beta + \langle\psi\wedge D^{(\omega)}D^{(\omega)}\psi\rangle_\beta \\ &= \mathrm{d}\langle\psi\wedge D^{(\omega)}\psi\rangle_\beta + \frac{1}{4L}F(\omega)^{IJ}\wedge\bar{\psi}\wedge\gamma_{IJ}\psi\end{aligned} \tag{5.24}$$

Moreover, according to the general discussion in Section 5.6.1 in case of arbitrary (super) connections, we have

$$\langle F(\omega)\wedge F(\omega)\rangle_\beta = \mathrm{d}\left\langle\omega\wedge\mathrm{d}\omega + \frac{1}{3}\omega\wedge[\omega\wedge\omega]\right\rangle_\beta \tag{5.25}$$

Thus, to summarize, it follows that (5.22) can be equivalently be written in the form

$$S_{\text{bdy}}(\mathcal{A}) = \frac{L^2}{\kappa}\int_{\partial M}\left\langle\omega\wedge\mathrm{d}\omega + \frac{1}{3}\omega\wedge[\omega\wedge\omega]\right\rangle_\beta + \langle\psi\wedge D^{(\omega)}\psi\rangle_\beta \tag{5.26}$$

and hence, in particular, is nonvanishing in the presence of boundaries. According to the general discussion in the previous section, this is the most general boundary term one can have in the context of $\mathcal{N} = 1$, $D = 4$ anti-de Sitter supergravity in the framework of LQG if one requires invariance of the full theory under local supersymmetry transformations.

In this context, note that the deformed action (5.12) is invariant under the same SUSY transformations (5.4) as in the standard theory. In fact, since in the $\mathcal{N} = 1$ case the SUSY transformations can be regarded as super gauge transformations, it follows that the transformation of the super Cartan curvature takes the form

$$\delta_\epsilon F(\mathcal{A}) = -[\epsilon, F(\mathcal{A})] \tag{5.27}$$

Thus, if we set $\rho^\alpha = F(\mathcal{A})^\alpha$ this implies that the variation of the Lagrangian in (5.12) yields

$$\begin{aligned}\delta_\epsilon \mathscr{L} =& - \langle [\epsilon, F(\mathcal{A})] \wedge F(\mathcal{A})\rangle_\beta - \langle F(\mathcal{A}) \wedge [\epsilon, F(\mathcal{A})]\rangle_\beta \\ =& \frac{1}{4L} F(\mathcal{A})^{IJ} \wedge \bar{\epsilon}\gamma^{KL}\rho\, P_\beta{}^{MN}{}_{KL}\epsilon_{IJMN} - \frac{1}{L^2} F(\mathcal{A})^I \wedge \bar{\rho}\mathcal{P}_\beta\gamma_I\epsilon \\ & - \frac{1}{2L} F(\mathcal{A})^{IJ} \wedge \bar{\epsilon}\mathcal{P}_\beta\gamma_{IJ}\rho = -\frac{1}{L^2} F(\mathcal{A})^I \wedge \bar{\rho}\mathcal{P}_\beta\gamma_I\epsilon\end{aligned} \tag{5.28}$$

Hence, it follows that the Lagrangian of the full β-deformed action is invariant under the SUSY transformations, both in the bulk and at the boundary, provided that the supertorsion constraint $F(\mathcal{A})^I = 0$ is satisfied which is equivalent to requiring that the spin connection ω satisfies its equations of motion. As far as the bulk theory is concerned, this was actually to be expected since, as will be proven explicitly for the case $\mathcal{N} = 2$ in Section 5.3.1 below, at second order, the deformed action coincides with the standard action up to topological terms, so that the SUSY variations are indeed unaltered.

Due to the deformed inner product appearing in (5.26) the boundary action contains additional topological terms compared to the standard theory. For instance, writing out the bosonic contribution in (5.26), this yields

$$\langle F(\omega) \wedge F(\omega)\rangle_\beta = \frac{1}{8} F(\omega)^{IJ} \wedge F(\omega)^{KL}\epsilon_{IJKL} + \frac{1}{4\beta} F(\omega)^{IJ} \wedge F(\omega)_{IJ} \tag{5.29}$$

so that the bosonic part of the boundary action splits into the ordinary Gauss-Bonnet term as in the standard theory as well as an additional topological *Pontryagin term*. As discussed in [188], these are in fact the most general boundary terms one can expect in the pure bosonic theory in case of a finite Barbero-Immirzi parameter which are compatible with the symmetries of the bulk Lagrangian. The fermionic contribution in (5.26) takes the form $\langle \psi \wedge D^{(\omega)}\psi\rangle_\beta = \frac{1}{L}\bar{\psi} \wedge \mathcal{P}_\beta D^{(\omega)}\psi$. Similar to the bulk theory as discussed in Section 4.3, in the canonical description of the boundary theory, the operator $\mathcal{P}_\beta$ implies that the covariant derivative can be re-rexpressed in terms of the covariant derivative associated to the (real) Ashtekar-Barbero connection ${}^{\beta}\!A$.

Finally, comparing with Eq. (5.296) in Section 5.6.1, one may suspect that the boundary action (5.26) almost looks like a super Chern-Simons action. Note, however, that the deformed inner product is not invariant under the full supergroup $\mathrm{OSp}(1|4)$ but only under the action of its bosonic subgroup so that (5.26), at least in general, will not correspond to a Chern-Simons action with a supergroup as a gauge group. This is of course in contrast e.g. to the IH formalism [177], where the IH boundary conditions imply that the boundary theory is generically described in terms of a Chern-Simons theory. As we will see however in Section 5.4, this changes drastically in case of the chiral theory.

5.3. $\mathcal{N} = 2$ pure SUGRA with boundaries

The content of this section has been reproduced from [3], with slight changes to account for the context of this thesis with the permission of Springer-Nature.

In the ungauged theory, for $\mathcal{N} = 2$, the full R-symmetry group is given by the unitary group $\mathrm{U}(2)$. But, in case of AdS supergravity, due to the appearence of the so-called *Fayet-Iliopoulos (FI) term*, it follows that this group is broken yielding an effective $\mathrm{SO}(2) \cong \mathrm{U}(1)$ gauge symmetry of the theory [81,83,189]. Thus, it follows that pure $\mathcal{N} = 2$, $D = 4$ anti-de Sitter supergravity can be described as a super Cartan geometry modeled on the super Klein geometry $(\mathrm{OSp}(2|4), \mathrm{SO}(2) \times \mathrm{Spin}^+(1,3))$ corresponding to extended super anti-de Sitter space. In this case, since $\mathfrak{osp}(2|4) \cong \mathbb{R}^{1,3} \oplus \mathfrak{spin}^+(1,3) \oplus \Delta_{\mathbb{R}}^2 \oplus \mathfrak{u}(1)$, the super Cartan connection $\mathcal{A}$ takes the form

$$\mathcal{A} = e^I P_I + \frac{1}{2}\omega^{IJ} M_{IJ} + \hat{A}T + \Psi^\alpha_r Q^r_\alpha \tag{5.30}$$

In particular, besides the spin connection ω, the super Cartan connection contains an additional $\mathrm{U}(1)$ gauge field $\hat{A} \equiv \hat{A}T$ also referred to as the *graviphoton field* with $T := T^{12} = -T^{21}$. Moreover, the supermultiplet consists of two Majorana gravitinos which we denote by capital letters Ψ_r, $r = 1, 2$ to simplify notation. In this form, the R-symmetry index is raised and lowered w.r.t. the Kronecker symbol δ_{rs}. On the other hand, we denote the individual chiral components of the Majorana fermions by lower case letters ψ^r and ψ_r, respectively, where the position of the R-symmetry index now explicitly indicates the chirality:

$$\psi^r := \frac{\mathbb{1} + \gamma_*}{2}\Psi_r, \quad \text{and} \quad \psi_r := \frac{\mathbb{1} - \gamma_*}{2}\Psi_r \tag{5.31}$$

for $r=1,2$ denote the left-handed and right-handed components of the Majorana fermions, respectively. Similar to the $\mathcal{N}=1$ case, the horizontal 1-forms combine to the super soldering form $E:=e^I P_I+\Psi^\alpha_r Q^r_\alpha$ which provides a local identification of the underlying (curved) supermanifold $\mathcal{M}$ with the flat model given by the extended super anti-de Sitter space. In particular, it induces an isomorphism

$$E:\Gamma(T\mathcal{M})\xrightarrow{\sim}\Gamma(\mathrm{Ad}(\mathcal{P})),\ X\mapsto\langle X|E\rangle \tag{5.32}$$

between smooth vector fields on $\mathcal{M}$ and sections of the Adjoint-bundle $\mathcal{P}\times_{\mathrm{Ad}}\mathfrak{osp}(2|4)$, where $\mathcal{P}$ is the underlying $\mathrm{U}(1)\times\mathrm{Spin}^+(1,3)$ principal super fiber bundle over which the super Cartan geometry is defined. By the rheonomy principle, the fields are uniquely fixed by their pullback to the underlying ordinary smooth manifold M. Hence, choosing a local section $s:M\rightarrow P\subset\mathcal{P}$ of the underlying (bosonic) smooth subbundle P, the action of the theory takes the form

$$S^{\mathcal{N}=2}_{\mathrm{AdS}}(\mathcal{A})=\frac{L^2}{\kappa}\int_M s^*\mathscr{L} \tag{5.33}$$

where the Lagrangian $\mathscr{L}$ is a horizontal form living on the bundle which, when adapted to our choice of conventions and pulled back to M, takes the form[2] [81, 83, 189]

$$\begin{aligned}s^*\mathscr{L}=&\frac{1}{4L^2}\Sigma^{IJ}\wedge F(\omega)^{KL}\epsilon_{IJKL}-\frac{i}{2L^2}\bar{\Psi}^r\wedge\gamma\gamma_*\wedge\nabla\Psi_r\\&-\frac{1}{8L^3}\bar{\Psi}^r\wedge\gamma^{KL}\Psi_r\wedge\Sigma^{IJ}\epsilon_{IJKL}+\frac{1}{8L^4}\Sigma^{IJ}\wedge\Sigma^{KL}\epsilon_{IJKL}\\&+\frac{i}{4L^2}\left(\mathrm{d}\hat{A}+\frac{1}{4}\bar{\Psi}^r\wedge\Psi^s\epsilon_{rs}\right)\wedge\bar{\Psi}^p\wedge\gamma_*\Psi^q\epsilon_{pq}-\frac{1}{4L^2}\hat{F}\wedge\star\hat{F}\end{aligned} \tag{5.34}$$

where, again, $\Sigma^{IJ}=e^I\wedge e^J$ whereas $\nabla\Psi_r$ and $\hat{F}$ (resp. $\star\hat{F}$) are defined via Eq. (5.45) and (5.46) in Section 5.3.1 below. In contrast to the $\mathcal{N}=1$ case, supersymmetry transformations no longer have the simple interpretation in terms of gauge transformations on the associated $\mathrm{OSp}(2|4)$-bundle. Instead, according to the Castellani-D'Auria-Fré approach (see Section 3.4), one regards them as certain superdiffeomorphisms along the odd directions of the supermanifold. More precisely, in the presence of boundaries,

[2] Due to the appearance of the Hodge-star operator in the Maxwell-kinetic term in the Lagrangian (5.34), in this form, the Lagrangian can only be defined on the underlying spacetime manifold. This is related to the lack of top-degree forms on supermanifolds (for an alternative approach towards top-degree forms on supermanifolds using the concept of integral forms see e.g. [131, 132]). In order to extend (5.34) to the whole supermanifold, one works in the so-called first-order formalism in the $\mathrm{U}(1)$-sector by introducing additional fields (auxiliary fields). By solving the equations of motion of these additional fields, one regains the original action (5.33) (see e.g. [81, 83]).

SUSY transformations correspond to smooth vector fields $X \in \Gamma(T\mathcal{M})$ such that $i_X e^I = 0$ and

$$\delta_X \mathscr{L}|_P = L_X \mathscr{L}|_P = (i_X \mathrm{d}\mathscr{L} + \mathrm{d}i_X \mathscr{L})|_P = 0 \tag{5.35}$$

Recall that the super Cartan connection transforms via Eq. (3.60), i.e.,

$$\delta_X \mathcal{A} = i_X F(\mathcal{A}) + D^{(\mathcal{A})}\epsilon \tag{5.36}$$

with $\epsilon := \langle X|E\rangle$. Since X is horizontal, the curvature contribution in (5.36), in general, no longer vanishes in contrast to pure gauge transformations. Moreover, in order for X to describe a symmetry of the theory, this imposes constraints on the curvature, the so-called rheonomy conditions.

Instead of deriving the explicit form of the SUSY transformations of the theory in what follows (see for instance [72, 81, 83, 189] for more details), let us finally comment on the possible boundary terms to be added to (5.34) such that local supersymmetry is preserved. As argued in [81], the most general ansatz for the boundary term which is compatible with the symmetries of the bulk action (5.34) turns out to be of the form

$$\mathscr{L}_{\text{bdy}} = C_1 F(\omega)^{IJ} \wedge F(\omega)^{KL}\epsilon_{IJKL} + C_2 \mathrm{d}(\bar{\Psi}^r \wedge \gamma_* \nabla\Psi_r) + C_3 \mathrm{d}(\hat{A} \wedge \mathrm{d}\hat{A})$$

for any constant coefficients C_1, C_2 and C_3. Requiring invariance of the full action $\mathscr{L}_{\text{full}} := \mathscr{L}_{\text{bulk}} + \mathscr{L}_{\text{bdy}}$ under local supersymmetry, in case of the presence of a nontrivial boundary ∂M, this imposes the condition

$$(\iota_X \mathscr{L}_{\text{full}})|_{\partial M} = 0 \tag{5.37}$$

As shown in [81], it follows that condition (5.37) uniquely fixes the constants to the particular values $C_1 = \frac{1}{8}$, $C_2 = \frac{i}{2L}$ and $C_3 = 0$ yielding

$$\begin{aligned}\mathscr{L}_{\text{bdy}} =& \frac{1}{8} F(\omega)^{IJ} \wedge F(\omega)^{KL}\epsilon_{IJKL} + \frac{i}{2L}\nabla\bar{\Psi}^r \wedge \gamma_* \nabla\Psi_r \\ &- \frac{i}{8L} F(\omega)^{IJ} \wedge \bar{\Psi}^r \wedge \gamma_*\gamma_{IJ}\Psi_r - \frac{i}{4L^2}\mathrm{d}\hat{A} \wedge \bar{\Psi}^p \wedge \gamma_*\Psi^q\epsilon_{pq}\end{aligned} \tag{5.38}$$

Moreover, when added to (5.34), one then recognizes that the resulting action $\mathscr{L}_{\text{full}}$ has a very intriguing structure similar to the MacDowell-Mansouri action of $\mathcal{N} = 1$ AdS supergravity as discussed in the previous sections [81, 83].

5.3.1. Holst action for $\mathcal{N} = 2$ pure SUGRA

The content of this section has been reproduced from [3], with slight changes to account for the context of this thesis with the permission of Springer-Nature.

We want to derive a Holst variant of the action (5.34) corresponding to $\mathcal{N} = 2$, $D = 4$ AdS supergravity for arbitrary Barbero-Immirzi parameters β including the boundary terms (5.38). We therefore follow the ideas in Section 5.2.1 and introduce a β-deformed inner product. To this end, according to the decomposition (5.30) of the super Cartan connection, let us define an operator $\mathbf{P}_\beta : \Omega^2(M, \mathfrak{g}) \to \Omega^2(M, \mathfrak{g})$ on the space of differential 2-forms with values in the super Lie algebra $\mathfrak{g} := \mathfrak{osp}(2|4)$ (or rather the corresponding super Lie module $\Lambda \otimes \mathfrak{osp}(2|4)$) as follows

$$\mathbf{P}_\beta := \underline{\mathbf{0}} \oplus \mathcal{P}_\beta \oplus \mathbb{P}_\beta \oplus \mathcal{P}_\beta \oplus \mathcal{P}_\beta, \text{ where } \quad \mathbb{P}_\beta := \frac{1}{2\beta}\left(1 + \beta\star\right) \tag{5.39}$$

with $\star : \Omega^p(M) \to \Omega^{4-p}(M)$, for $0 \leq p \leq 4$ denoting the *Hodge star* operator on the bosonic spacetime manfifold M (trivially extended to $\mathfrak{g}$-valued, in fact even Grassmann-valued, differential forms) which, in case of Lorentzian signature and even spacetime dimensions, satisfies

$$\star^2 |_{\Omega^p(M)} = (-1)^{p+1}, \ \forall 0 \leq p \leq 4 \tag{5.40}$$

Similar to the general discussion in Section 5.6.1, the operator (5.39) can be used to introduce an inner product on $\Omega^2(M, \mathfrak{g})$ setting

$$\begin{aligned} \langle \cdot \wedge \cdot \rangle_\beta : \ \Omega^2(M, \mathfrak{g}) \times \Omega^2(M, \mathfrak{g}) &\to \Omega^4(M) \\ (\omega, \eta) &\mapsto \mathrm{str}(\omega \wedge \mathbf{P}_\beta \eta) \end{aligned} \tag{5.41}$$

Using this inner product, we define the Holst-MacDowell-Mansouri action of $\mathcal{N} = 2$, $D = 4$ AdS supergravity as follows

$$S^{\mathcal{N}=2}_{\text{H-MM}}(\mathcal{A}) = \frac{L^2}{\kappa} \int_M \langle F(\mathcal{A}) \wedge F(\mathcal{A}) \rangle_\beta \tag{5.42}$$

with $F(\mathcal{A})$ the associated Cartan curvature. Using the commutation relations (2.96)-(2.99) for $\mathcal{N} = 2$, it follows that the translational components of the curvature take the form

$$\begin{aligned} F(\mathcal{A})^I &= \mathrm{d}e^I + \omega^I{}_J \wedge e^J + \frac{1}{4}((-1)^{|Q_\alpha||Q_\beta|} \Psi^\alpha_r \wedge \Psi^\beta_s \otimes [Q^r_\alpha, Q^s_\beta])^I \\ &= \Theta^{(\omega)I} - \frac{1}{4}\bar{\Psi}^r \wedge \gamma^I \Psi_r \end{aligned} \tag{5.43}$$

since $(-1)^{|Q_\alpha||Q_\beta|} = -1$, with $\Theta^{(\omega)}$ the torsion 2-form associated to the spin connection ω. For the Lorentzian components, we find

$$\begin{aligned} F(\mathcal{A})^{IJ} &= \mathrm{d}\omega^{IJ} + \omega^I{}_K \wedge \omega^{KJ} + \frac{1}{2L^2} e^I \wedge e^J - \frac{1}{2}(\Psi^\alpha_r \wedge \Psi^\beta_s \otimes [Q^r_\alpha, Q^s_\beta])^{IJ} \\ &= F(\omega)^{IJ} + \frac{1}{L^2}\Sigma^{IJ} - \frac{1}{4L}\bar{\Psi}^r \wedge \gamma^{IJ}\Psi_r \end{aligned} \tag{5.44}$$

Moreover, for the odd part, we obtain, using $\hat{A} := \frac{1}{2}\hat{A}^{rs}T_{rs}$ for the U(1) gauge field,

$$\begin{aligned} F(\mathcal{A})^\alpha_r &= D^{(\omega)}\Psi^\alpha_r + \frac{1}{2L}\hat{A}\epsilon_{rs} \wedge \Psi^{\alpha s} - \frac{1}{2L} e^I \wedge (\gamma_I)^\alpha{}_\beta \Psi^\beta_r \\ &=: \nabla\Psi^\alpha_r - \frac{1}{2L} e^I \wedge (\gamma_I)^\alpha{}_\beta \Psi^\beta_r \end{aligned} \tag{5.45}$$

Finally, for the U(1) components, we get

$$\hat{F} := \frac{1}{2}F(\mathcal{A})^{rs}\epsilon_{rs} = \mathrm{d}\hat{A} + \frac{1}{2}\bar{\Psi}^r \wedge \Psi^s \epsilon_{rs} \tag{5.46}$$

We need to show that action (5.42) is indeed independent of the choice of the Barabero-Immirzi parameter. That is, we have to prove that the action at second order, i.e. provided ω satisfies its field equations, reduces to the action (5.34) together with the boundary term (5.38). This is equivalent to requiring that the supertorsion of $\mathcal{A}$ vanishes, i.e. $F(\mathcal{A})^I = 0$, and, when reinserting back into (5.42), all β-dependent terms become purely topological.

To this end, let us expand the action (5.42). Using the curvature expressions (5.44)-(5.46) as well as $\langle T, T\rangle_\beta = -\frac{1}{2L^2}$, which can be checked by direct computation using the explicit representation (2.92), we find

$$\begin{aligned} \langle F(\mathcal{A}) \wedge F(\mathcal{A})\rangle_\beta =& \frac{1}{4}F(\mathcal{A})^{IJ} \wedge F(\mathcal{A})^{KL}\langle M_{IJ}, M_{KL}\rangle_\beta \\ &- F(\mathcal{A})^\alpha_r \wedge F(\mathcal{A})^\delta_s \langle Q^r_\alpha, Q^s_\delta\rangle_\beta - \frac{1}{2L^2}\hat{F} \wedge \mathbb{P}_\beta \hat{F} \\ =& \langle F(\omega) \wedge F(\omega)\rangle_\beta + \frac{2}{L^2}\langle \Sigma \wedge F(\omega)\rangle_\beta \\ &- \frac{1}{2L}\langle F(\omega) \wedge \bar{\Psi}^r \wedge \gamma^{\bullet\bullet}\Psi_r\rangle_\beta - \frac{1}{2L^3}\langle \Sigma \wedge \bar{\Psi}^r \wedge \gamma^{\bullet\bullet}\Psi_r\rangle_\beta \\ &+ \frac{1}{L^4}\langle \Sigma \wedge \Sigma\rangle_\beta + \frac{1}{L}F(\mathcal{A})^\alpha_r \wedge F(\mathcal{A})^\delta_s \delta^{rs}(C\mathcal{P}_\beta)_{\alpha\delta} \\ &- \frac{1}{4L^2}\hat{F} \wedge \star\hat{F} - \frac{1}{4\beta L^2}\hat{F} \wedge \hat{F} \end{aligned} \tag{5.47}$$

where

$$\hat{F} \wedge \hat{F} = \mathrm{d}\hat{A} \wedge \mathrm{d}\hat{A} + \mathrm{d}\hat{A} \wedge \bar{\Psi}^r \wedge \Psi^s \epsilon_{rs} + \frac{1}{4}\bar{\Psi}^r \wedge \Psi^s \epsilon_{rs} \wedge \bar{\Psi}^p \wedge \Psi^q \epsilon_{pq} \quad (5.48)$$

Let us further expand the terms in (5.47) arising form the Lorentzian components of the curvature which gives

$$\begin{aligned} &\langle F(\omega) \wedge F(\omega)\rangle_\beta + \frac{2}{L^2}\langle \Sigma \wedge F(\omega)\rangle_\beta + \frac{i}{8L} F(\omega)^{IJ} \wedge \bar{\Psi}^r \wedge \gamma_* \gamma^{KL} \mathcal{P}_\beta \Psi_r \epsilon_{IJKL} \\ &+ \frac{i}{8L^3}\Sigma^{IJ} \wedge \bar{\Psi}^r \wedge \gamma_* \gamma^{KL} \mathcal{P}_\beta \Psi_r \epsilon_{IJKL} + \frac{1}{8L^4}\Sigma^{IJ} \wedge \Sigma^{KL} \epsilon_{IJKL} \\ &+ \frac{1}{32L^2}\bar{\Psi}^r \wedge \gamma^{IJ}\Psi_r \wedge \bar{\Psi}^s \wedge \gamma_{IJ}\mathcal{P}_\beta \Psi_s \end{aligned} \quad (5.49)$$

In contrast to the $\mathcal{N} = 1$ case, an additional Ψ^4-order term appears which, in general, no longer vanishes since the supermultiplet contains two independent Majorana fermions. In order to further evaluate this term, let us split the fermionic fields in their chiral components and use the following important identities stemming from the Fierz-rearrangement formula (4.8)

$$\psi_r \wedge \bar{\psi}_s = \frac{1}{2}\bar{\psi}_s \wedge \psi_r - \frac{1}{8}\gamma_{IJ}\bar{\psi}_s \wedge \gamma^{IJ}\psi_r \quad (5.50)$$

$$\psi_r \wedge \bar{\psi}^s = \frac{1}{2}\gamma_I \bar{\psi}^s \wedge \gamma^I \psi_r \quad (5.51)$$

In this way, we obtain (summation over repeated indices)

$$\begin{aligned} \bar{\Psi}^r \wedge \gamma^{IJ}\Psi_r \wedge \bar{\Psi}^s \wedge \gamma_{IJ}\Psi_s =& \bar{\psi}^r \wedge \gamma^{IJ}\psi^r \wedge \bar{\psi}^s \wedge \gamma_{IJ}\psi^s \\ &+ \bar{\psi}_r \wedge \gamma^{IJ}\psi_r \wedge \bar{\psi}_s \wedge \gamma_{IJ}\psi_s \\ =& 4\bar{\psi}^r \wedge \psi^s \epsilon_{rs} \wedge \bar{\psi}^p \wedge \psi^q \epsilon_{pq} \\ &+ 4\bar{\psi}_r \wedge \psi_s \epsilon^{rs} \wedge \bar{\psi}_p \wedge \psi_q \epsilon^{pq} \end{aligned} \quad (5.52)$$

where in the second equality we used that

$$\bar{\psi}^r \wedge \psi^s \wedge \bar{\psi}_s \wedge \psi_r = -\frac{1}{2}\bar{\psi}^r \wedge \psi^s \epsilon_{rs} \wedge \bar{\psi}_p \wedge \psi_q \epsilon^{pq} \quad (5.53)$$

On the other hand, we have

$$\begin{aligned} \bar{\Psi}^r \wedge \gamma^{IJ}\Psi_r \wedge \bar{\Psi}^s \wedge \gamma_{IJ}\gamma_*\Psi_s =& \bar{\psi}^r \wedge \gamma^{IJ}\psi^r \wedge \bar{\psi}^s \wedge \gamma_{IJ}\psi^s \\ &- \bar{\psi}_r \wedge \gamma^{IJ}\psi_r \wedge \bar{\psi}_s \wedge \gamma_{IJ}\psi_s \\ =& 4\bar{\Psi}^r \wedge \Psi^s \epsilon_{rs} \wedge \bar{\Psi}^p \wedge \gamma_* \Psi^q \epsilon_{pq} \end{aligned} \quad (5.54)$$

Thus, using (5.52) and (5.54), we can rewrite it as follows

$$\begin{aligned}\frac{1}{32L^2}\bar{\Psi}^r\wedge\gamma^{IJ}\Psi_r\wedge\bar{\Psi}^s\wedge\gamma_{IJ}\mathcal{P}_\beta\Psi_s=&\frac{i}{16L^2}\bar{\Psi}^r\wedge\Psi^s\epsilon_{rs}\wedge\bar{\Psi}^p\wedge\gamma_*\Psi^q\epsilon_{pq}\\&+\frac{1}{16\beta L^2}(\bar{\psi}^r\wedge\psi^s\epsilon_{rs}\wedge\bar{\psi}^p\wedge\psi^q\epsilon_{pq}\\&+\bar{\psi}_r\wedge\psi_s\epsilon^{rs}\wedge\bar{\psi}_p\wedge\psi_q\epsilon^{pq})\end{aligned}\tag{5.55}$$

Hence, if we combine this with the β-dependent $\hat{F}\wedge\hat{F}$-term in the expansion (5.47) given by the expression (5.48), this yields

$$\begin{aligned}&-\frac{1}{4\beta L^2}\hat{F}\wedge\hat{F}+\frac{1}{32L^2}\bar{\Psi}^r\wedge\gamma^{IJ}\Psi_r\wedge\bar{\Psi}^s\wedge\gamma_{IJ}\mathcal{P}_\beta\Psi_s\\=&-\frac{1}{4\beta L^2}\mathrm{d}(\hat{A}\wedge\mathrm{d}\hat{A})-\frac{1}{4\beta L^2}\mathrm{d}\hat{A}\wedge\bar{\Psi}^r\wedge\Psi^s\epsilon_{rs}\\&+\frac{i}{16L^2}\bar{\Psi}^r\wedge\Psi^s\epsilon_{rs}\wedge\bar{\Psi}^p\wedge\gamma_*\Psi^q\epsilon_{pq}-\frac{1}{8\beta L^2}\bar{\psi}^r\wedge\psi^s\epsilon_{rs}\wedge\bar{\psi}_p\wedge\psi_q\epsilon^{pq}\end{aligned}\tag{5.56}$$

The remaining terms in (5.47) can be treated as in the $\mathcal{N}=1$ case. For sake of completeness, let us repeat them here. Again, notice that

$$\begin{aligned}\frac{1}{L}F(\mathcal{A})^{\alpha}_r\wedge F(\mathcal{A})^{\delta}_s\delta^{rs}(C\mathcal{P}_\beta)_{\alpha\delta}=&\langle\nabla\Psi\wedge\nabla\Psi\rangle_\beta-\frac{1}{L^2}\bar{\Psi}^r\wedge\boldsymbol{\gamma}\wedge\mathcal{P}_\beta\nabla\Psi_r\\&-\frac{1}{4L^3}\bar{\Psi}^r\wedge\gamma_{IJ}\mathcal{P}_{-\beta}\Psi_r\wedge\Sigma^{IJ}\end{aligned}\tag{5.57}$$

Hence, using $\epsilon_{IJ}{}^{KL}\gamma_{KL}=2i\gamma_{IJ}\gamma_*$, the last term in (5.57) can be combined with the first term in the second line of (5.49) to give

$$\begin{aligned}&\frac{i}{8L^3}\Sigma^{IJ}\wedge\bar{\Psi}^r\wedge\gamma_*\gamma^{KL}\mathcal{P}_\beta\Psi_r\epsilon_{IJKL}-\frac{1}{4L^3}\bar{\Psi}^r\wedge\gamma_{IJ}\mathcal{P}_\beta\Psi_r\wedge\Sigma^{IJ}\\=&\frac{i}{8L^3}\bar{\Psi}^r\wedge\gamma_*\gamma^{KL}\left[\mathcal{P}_\beta+\mathcal{P}_{-\beta}\right]\Psi_r\wedge\Sigma^{IJ}\epsilon_{IJKL}=-\frac{1}{8L^3}\bar{\Psi}^r\wedge\gamma^{KL}\Psi_r\wedge\Sigma^{IJ}\epsilon_{IJKL}\end{aligned}\tag{5.58}$$

Thus, to summarize, it follows that the action (5.42) can be written in the equivalent form

$$\begin{aligned}S^{\mathcal{N}=2}_{\text{H-MM}}(\mathcal{A})=\frac{1}{2\kappa}\int_M\Bigg(&\frac{1}{2}\Sigma^{IJ}\wedge F(\omega)^{KL}\epsilon_{IJKL}-i\bar{\Psi}^r\wedge\boldsymbol{\gamma}\gamma_*\wedge\nabla\Psi_r\\&-\frac{1}{4L}\bar{\Psi}^r\wedge\gamma^{KL}\Psi_r\wedge\Sigma^{IJ}\epsilon_{IJKL}+\frac{1}{4L^2}\Sigma^{IJ}\wedge\Sigma^{KL}\epsilon_{IJKL}\end{aligned}$$

$$
\begin{aligned}
&+ \frac{i}{8}\bar{\Psi}^r \wedge \Psi^s \epsilon_{rs} \wedge \bar{\Psi}^p \wedge \gamma_* \Psi^q \epsilon_{pq} - \frac{1}{2}\hat{F} \wedge \star\hat{F} \\
&+ \frac{1}{4}F(\omega)^{IJ} \wedge F(\omega)^{KL}\epsilon_{IJKL} + iL\nabla\bar{\Psi}^r \wedge \gamma_* \nabla\Psi_r \\
&- \frac{iL}{4}F(\omega)^{IJ} \wedge \bar{\Psi}^r \wedge \gamma_*\gamma_{IJ}\Psi_r + \frac{1}{\beta}\mathscr{L}_\beta\Big)
\end{aligned}
\tag{5.59}
$$

where we have collected all terms depending on the Barbero-Immirzi parameter in the Lagrangian $\mathscr{L}_\beta$ given by

$$
\begin{aligned}
\mathscr{L}_\beta =& \frac{1}{2}F(\omega)^{IJ} \wedge F(\omega)_{IJ} + \Sigma^{IJ} \wedge F(\omega)_{IJ} - \frac{1}{4}\bar{\psi}^r \wedge \psi^s \epsilon_{rs} \wedge \bar{\psi}_p \wedge \psi_q \epsilon^{pq} \\
&- \bar{\Psi}^r \wedge \gamma \wedge \nabla\Psi_r + L\nabla\bar{\Psi}^r \wedge \nabla\Psi_r - \frac{L}{4}F(\omega)^{IJ} \wedge \bar{\Psi}^r \wedge \gamma_{IJ}\Psi_r \\
&- \frac{1}{2}\mathrm{d}\hat{A} \wedge \bar{\Psi}^r \wedge \Psi^s \epsilon_{rs} - \frac{1}{2}\mathrm{d}(\hat{A} \wedge \mathrm{d}\hat{A})
\end{aligned}
\tag{5.60}
$$

We have to show that this Lagrangian is indeed topological at second order, i.e. it takes the form of a boundary term provided the spin connection satisfies its field equations. For the second line in (5.60), this is an immediate consequence of the identity

$$
\begin{aligned}
L\nabla\bar{\Psi}^r \wedge \nabla\Psi_r =& \mathrm{d}(L\bar{\Psi}^r \wedge \nabla\Psi_r) + \bar{\Psi}^r \wedge \nabla\nabla\Psi_r \\
=& \mathrm{d}(L\bar{\Psi}^r \wedge \nabla\Psi_r) + \frac{L}{4}F(\omega)^{IJ} \wedge \bar{\Psi}^r \wedge \gamma_{IJ}\Psi_r + \frac{1}{2}\mathrm{d}\hat{A} \wedge \bar{\Psi}^r \wedge \Psi^s \epsilon_{rs}
\end{aligned}
\tag{5.61}
$$

For the first line, note that the EOM of ω are equivalent to the supertorsion constraint $F(\mathcal{A})^I = 0$, that is

$$
D^{(\omega)}e^I \equiv \Theta^{(\omega)I} = \frac{1}{4}\bar{\Psi}^r \wedge \gamma^I \Psi_r \tag{5.62}
$$

Thus, using (5.62), we can rewrite the last term in the first line of (5.60) as follows

$$
\begin{aligned}
\bar{\Psi}^r \wedge \gamma \wedge \nabla\Psi_r =& \frac{1}{2}\mathrm{d}(\bar{\Psi}^r \wedge \gamma \wedge \Psi_r) + \frac{1}{2}\bar{\Psi}^r \wedge D^{(\omega)}e^I\gamma_I \wedge \Psi_r \\
=& \frac{1}{2}\mathrm{d}(\bar{\Psi}^r \wedge \gamma \wedge \Psi_r) + \frac{1}{8}\bar{\Psi}^r \wedge \gamma_I\Psi_r \wedge \bar{\Psi}^s \wedge \gamma^I\Psi_s \\
=& \frac{1}{2}\mathrm{d}(\bar{\Psi}^r \wedge \gamma \wedge \Psi_r) + \frac{1}{2}\bar{\psi}_r \wedge \gamma_I \wedge \psi^r \wedge \bar{\psi}^s \wedge \gamma^I \wedge \psi_s \\
=& \frac{1}{2}\mathrm{d}(\bar{\Psi}^r \wedge \gamma \wedge \Psi_r) - \frac{1}{2}\bar{\psi}^r \wedge \psi^s \epsilon_{rs} \wedge \bar{\psi}_p \wedge \psi_q \epsilon^{pq}
\end{aligned}
\tag{5.63}
$$

On the other hand, according to (5.50)-(5.51) and (5.53), we have the important identity

$$\Theta^{(\omega)I}\wedge\Theta^{(\omega)}_{I}=\frac{1}{16}\bar{\Psi}^r\wedge\gamma_I\Psi_r\wedge\bar{\Psi}^s\wedge\gamma^I\Psi_s=-\frac{1}{4}\bar{\psi}^r\wedge\psi^s\epsilon_{rs}\wedge\bar{\psi}_p\wedge\psi_q\epsilon^{pq} \tag{5.64}$$

Hence, the last three terms in the first line of (5.60) can be re-expressed in the following way

$$\begin{aligned}&\Sigma^{IJ}\wedge F(\omega)_{IJ}-\frac{1}{4}\bar{\psi}^r\wedge\psi^s\epsilon_{rs}\wedge\bar{\psi}_p\wedge\psi_q\epsilon^{pq}-\bar{\Psi}^r\wedge\gamma\wedge\nabla\Psi_r\\=&\Sigma^{IJ}\wedge F(\omega)_{IJ}-\Theta^{(\omega)I}\wedge\Theta^{(\omega)}_{I}+2\mathrm{d}(e^I\wedge\Theta^{(\omega)}_{I})\end{aligned} \tag{5.65}$$

In fact, this can be simplified even further. To this end, we notice that the first two terms in equation (5.65) yield the so-called *Nieh-Yan toplogical invariant* $\mathrm{d}(e^I\wedge\Theta^{(\omega)}_I)$ [190]. This is easy to see using the properties of the covariant derivative which immediately gives[3]

$$\begin{aligned}\mathrm{d}(e^I\wedge\Theta^{(\omega)}_{I})&=D^{(\omega)}e^I\wedge\Theta^{(\omega)}_{I}-e^I\wedge D^{(\omega)}D^{(\omega)}e_I\\&=\Theta^{(\omega)I}\wedge\Theta^{(\omega)}_{I}-\Sigma^{IJ}\wedge F(\omega)_{IJ}\end{aligned} \tag{5.66}$$

Thus, to summarize, we observe that, provided that the spin connection satisfies its field equations, the Lagrangian (5.60) takes the final form

$$\mathscr{L}_\beta=\frac{1}{2}F(\omega)^{IJ}\wedge F(\omega)_{IJ}+\mathrm{d}\left(e^I\wedge\Theta^{(\omega)}_{I}+L\bar{\Psi}^r\wedge\nabla\Psi_r-\frac{1}{2}\hat{A}\wedge\mathrm{d}\hat{A}\right) \tag{5.67}$$

and therefore is indeed topological. Moreover, if we subtract this term from the full action (5.42), it follows that this action finally reduces to

$$\begin{aligned}S(\mathcal{A})=\frac{1}{2\kappa}\int_M\bigg(&\frac{1}{2}\Sigma^{IJ}\wedge F(\omega)^{KL}\epsilon_{IJKL}-i\bar{\Psi}^r\wedge\gamma\gamma_*\wedge\nabla\Psi_r\\&-\frac{1}{4L}\bar{\Psi}^r\wedge\gamma^{KL}\Psi_r\wedge\Sigma^{IJ}\epsilon_{IJKL}+\frac{1}{L^2}\Sigma^{IJ}\wedge\Sigma^{KL}\epsilon_{IJKL}\end{aligned}$$

---

[3] This can also be checked by direct computation. Indeed,

$$\begin{aligned}\mathrm{d}(e^I\wedge\Theta^{(\omega)}_{I})&=\mathrm{d}e^I\wedge\Theta^{(\omega)}_{I}-e^I\wedge\mathrm{d}\Theta^{(\omega)}_{I}=\mathrm{d}e^I\wedge\Theta^{(\omega)}_{I}+\omega^J{}_I\wedge e^I\wedge\mathrm{d}e_J-e^I\wedge e^J\wedge\mathrm{d}\omega_{IJ}\\&=\mathrm{d}e^I\wedge\Theta^{(\omega)}_{I}+\omega^J{}_I\wedge e^I\wedge\mathrm{d}e_J+\omega^K{}_J\wedge e^J\wedge\omega_K{}^I\wedge e_I-e^I\wedge e^J\wedge F(\omega)_{IJ}\\&=\Theta^{(\omega)I}\wedge\Theta^{(\omega)}_{I}-\Sigma^{IJ}\wedge F(\omega)_{IJ}\end{aligned}$$

$$+\frac{i}{2}\left(\mathrm{d}\hat{A} + \frac{1}{4}\bar{\Psi}^r \wedge \Psi^s \epsilon_{rs}\right) \wedge \bar{\Psi}^p \wedge \gamma_* \Psi^q \epsilon_{pq} - \frac{1}{2}\hat{F} \wedge \star\hat{F} + \mathscr{L}_{\text{bdy}}\Bigg) \tag{5.68}$$

with $\mathscr{L}_{\text{bdy}}$ given by (5.38) (times $2L^{-2}$). Hence, at second order, the Holst action leads back to the original action of $\mathcal{N} = 2$, $D = 4$ AdS supergravity as stated in [81, 83] as required. To summarize, the Holst-MacDowell-Mansouri action can be written in the form

$$\begin{aligned} S^{\mathcal{N}=2}_{\text{H-MM}}(\mathcal{A}) = \frac{1}{2\kappa}\int_M & \Sigma^{IJ} \wedge (P_\beta \circ F(\omega))^{KL}\epsilon_{IJKL} - \bar{\Psi}^r \wedge \gamma \frac{\mathbb{1} + i\beta\gamma_*}{\beta} \wedge \nabla\Psi_r \\ & -\frac{1}{4L}\bar{\Psi}^r \wedge \gamma^{KL}\Psi_r \wedge \Sigma^{IJ}\epsilon_{IJKL} + \frac{1}{L^2}\Sigma^{IJ} \wedge \Sigma^{KL}\epsilon_{IJKL} \\ & +\frac{i}{2}\left(\mathrm{d}\hat{A} + \frac{1}{4}\bar{\Psi}^r \wedge \Psi^s\epsilon_{rs}\right) \wedge \bar{\Psi}^p \wedge \gamma_*\Psi^q\epsilon_{pq} \\ & -\frac{1}{4\beta}\bar{\psi}^r \wedge \psi^s\epsilon_{rs} \wedge \bar{\psi}_p \wedge \psi_q\epsilon^{pq} - \frac{1}{2}\hat{F} \wedge \star\hat{F} + S_{\text{bdy}} \end{aligned} \tag{5.69}$$

with S_{bdy} a boundary action given by

$$S_{\text{bdy}}(\mathcal{A}) = \frac{L^2}{\kappa}\int_{\partial M} \left\langle \omega \wedge \mathrm{d}\omega + \frac{1}{3}\omega \wedge [\omega \wedge \omega]\right\rangle_\beta - \frac{1}{4\beta L^2}\hat{A} \wedge \mathrm{d}\hat{A} + \langle \Psi \wedge \nabla\Psi\rangle_\beta \tag{5.70}$$

In particular, according to the general discussion in Section 5.3, similar to the $\mathcal{N} = 1$ case, this boundary action is determined uniquely if one requires supersymmetry invariance of the full action at the boundary.

Again, as in the non-extended case, since the inner product in (5.70) is not invariant under the full supergroup $\mathrm{OSp}(2|4)$ but only under the action of its bosonic subgroup, the boundary action (5.70), in general, will not correspond to a super Chern-Simons action (see Section 5.6.1). As we will see in the following section, this changes however in case of the chiral theory where the boundary theory will generically be described in terms of a super Chern-Simons theory with gauge supergroup $\mathrm{OSp}(2|2)_{\mathbb{C}}$.

Nevertheless, one should emphasize that, at least in context of the standard underformed theory, one can construct models where this turns out to be true even in case of classical (real) variables. For instance, in [82], particular falloff conditions for the physical fields in the $\mathcal{N} = 2$ case where considered leading to a super Chern-Simons theory on the boundary corresponding to a $\mathrm{OSp}(2|2) \times \mathrm{SO}(1,2)$ gauge group. This model has also been studied in [191–193] which turned out to have interesting applications in condensed matter physics in the description of graphene near the Dirac points.

Remark 5.3.1. It is interesting to note that, via Definition (5.39) and (5.41), the Barbero-Immirzi parameter leads to an additional topological term in the $U(1)$ sector of the theory which is also known as the θ-term in Yang-Mills theory. Hence, in this sense, the Barbero-Immirzi parameter literally has the interpretation in terms of a θ-ambiguity. This supports the hypothesis of [186]. This may also have interesting consequences for the quantum theory of the U(1) sector (see Section 5.5 below).

5.4. Chiral supergravity and the super Ashtekar connection

5.4.1. The super Ashtekar connection

In the previous sections, we have derived the actions of $\mathcal{N}$-extended $D = 4$ anti-de Sitter supergravity for $\mathcal{N} = 1, 2$ including unique boundary terms considering it geometrically in terms of a super Cartan geometry. In particular, all the basic entities of the theory turn out to be completely encoded in the super Cartan connection

$$\mathcal{A} = e^I P_I + \frac{1}{2}\omega^{IJ} M_{IJ} + \frac{1}{2}\hat{A}_{rs}T^{rs} + \Psi^\alpha_r Q^r_\alpha \tag{5.71}$$

taking values in the super Lie algebra $\mathfrak{osp}(\mathcal{N}|4)$ corresponding to the underlying super Klein geometry.

In 1986 in [20], Ashtekar introduced his self-dual variables which give the canonical phase space of ordinary gravity the structure of a $SL(2, \mathbb{C})$ Yang-Mills theory. This construction is based on a particular structure of the internal symmetry algebra. In fact, the complexification of the Lie algebra of the orthochronous Lorentz group $SO^+(1, 3)$ has a decomposition of the form

$$\mathfrak{so}^+(1,3)_{\mathbb{C}} = \mathfrak{su}(2)_{\mathbb{C}} \oplus \mathfrak{su}(2)_{\mathbb{C}} \cong \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}) \tag{5.72}$$

and thus splits into two proper $\mathfrak{sl}(2, \mathbb{C})$ subalgebras (viewed as complex Lie algebras of complex $SL(2, \mathbb{C})$). This precisely corresponds to the decomposition of the spin connection ω into its self-dual A^+ and anti self-dual part A^-, respectively. In this sense, the self-dual variables can be regarded as chiral sub components of the $4D$ spin connection.

Hence, the natural question arises whether such a construction carries over to the super category. As we will see in what follows, this will be indeed the case, even for arbitrary

$\mathcal{N} \geq 1$. To this end, recall that the Ashtekar variables $A^{\pm}$ are defined as the (anti) self-dual part of the four-dimensional spin connection ω according to

$$A^{\pm} := \frac{1}{2}\left[\frac{1}{2}\left(\omega^{IJ} \mp \frac{i}{2}\epsilon^{IJ}{}_{KL}\omega^{KL}\right)\right]M_{IJ} \tag{5.73}$$

which takes values in the complexification $\mathfrak{spin}(1,3)_{\mathbb{C}}$ of the Lie algebra of the spin double cover $\mathrm{Spin}^{+}(1,3)$ of the orthochronous Lorentz group generated by M_{IJ}. After some simple algebra, it follows that

$$\begin{aligned} A^{\pm} &= \frac{1}{2}\left[\frac{1}{2}\left(\omega^{IJ} \mp \frac{i}{2}\epsilon^{IJ}{}_{KL}\omega^{KL}\right)\right]M_{IJ} \\ &= \frac{1}{2}\left(\frac{1}{4}\epsilon^{i}{}_{kl}\epsilon_{i}{}^{mn}\omega^{kl}M_{mn} \mp \frac{i}{2}\epsilon^{0i}{}_{kl}\omega^{kl}M_{0i} \mp \frac{i}{2}\epsilon_{0i}{}^{KL}\omega^{0i}M_{KL} + \omega^{0i}M_{0i}\right) \\ &= \left(-\frac{1}{2}\epsilon^{i}{}_{kl}\omega^{kl} \mp i\omega^{0i}\right)\frac{1}{2}\left(-\frac{1}{2}\epsilon_{i}{}^{kl}M_{kl} \pm iM_{0i}\right) =: A^{\pm i}T_i^{\pm} \end{aligned} \tag{5.74}$$

where $A^{\pm i} := \Gamma^{i} \mp iK^{i}$, $i = 1,\ldots,3$, with $\Gamma^{i} = -\frac{1}{2}\epsilon^{i}{}_{kl}\omega^{kl}$ the 3D spin connection and $K^{i} = \omega^{0i}$ the extrinsic curvature (cf. Section 4.3). Moreover, $T_i^{\pm}$ are given by

$$T_i^{\pm} = \frac{1}{2}(J_i \pm i\tilde{K}_i) \tag{5.75}$$

with $J_i = -\frac{1}{2}\epsilon_{i}{}^{jk}M_{jk}$ and $\tilde{K}_i = M_{0i}$, the generators of local rotations and boosts, respectively. These satisfy the commutation relations

$$[T_i^{\pm}, T_j^{\pm}] = \epsilon_{ij}{}^{k}T_k^{\pm} \tag{5.76}$$

and therefore generate the chiral $\mathfrak{sl}(2,\mathbb{C})$ subalgebras of $\mathfrak{spin}^{+}(1,3)_{\mathbb{C}}$. Since $\gamma_{0i} = \frac{i}{2}\epsilon_{ijk}\gamma^{jk}\gamma_{*}$, one has the important identity

$$\frac{1}{4}\left(-\epsilon_{i}{}^{jk}\gamma_{jk} \pm i\gamma_{0i}\right) = \frac{\gamma_{*} \pm \mathbb{1}}{2}\frac{i}{2}\gamma_{0i} \tag{5.77}$$

Hence, using (5.77), it immediately follows that the exterior covariant derivative induced by A^{+} (resp. A^{-}) acts on purely unprimed (resp. primed) spinor indices according to

$$D^{(A^{+})}\psi^{A} = \mathrm{d}\psi^{A} + A^{+A}{}_{B}\wedge\psi^{B}, \quad \text{and} \quad D^{(A^{-})}\psi_{A'} = \mathrm{d}\psi_{A'} + A^{-}{}_{A'}{}^{B'}\wedge\psi_{B'} \tag{5.78}$$

respectively, where $A^{+A}{}_{B} = A^{+i}(\tau_i)^{A}{}_{B}$ and $A^{-}{}_{A'}{}^{B'} = A^{-i}(\tau_i)_{A'}{}^{B'}$ (note that the second identity in (5.78) can be obtained taking the complex conjugate of the first one). Hence, focusing for the moment on the self-dual sector, let us consider the chiral sub components Q^r_A of the Majorana charges Q^r_α. From (2.96), it then follows, using again (5.77),

$$[T^+_k, Q^r_A] = Q^r_B(\tau_k)^B{}_A \tag{5.79}$$

that is, the Weyl fermions Q^r_A transform in the fundamental representation of $\mathfrak{sl}(2,\mathbb{C})$ as to be expected from (5.78). Next, let us consider the anticommutation relations between two Weyl fermions. In the Weyl representation, recall that the charge conjugation matrix C admits a block diagonal form given by $C = \mathrm{diag}(i\epsilon, i\epsilon)$. From this, we immediately deduce

$$\begin{aligned}(C\gamma^{IJ})_{AB}M_{IJ} &= \frac{i}{2}\left(\epsilon(\sigma^I\bar{\sigma}^J - \sigma^J\bar{\sigma}^I)\right)_{AB}M_{IJ}\\ &= 2i(\epsilon\sigma^i)_{AB}M_{0i} - \epsilon^{ij}{}_k(\epsilon\sigma^k)_{AB}M_{ij}\\ &= 2(\epsilon\sigma^i)_{AB}\left(iM_{0i} - \frac{1}{2}\epsilon_i{}^{jk}M_{jk}\right) = 2(\epsilon\sigma^i)_{AB}T^+_i\end{aligned} \tag{5.80}$$

Hence, using (2.99), it follows

$$[Q^r_A, Q^s_B] = \delta^{rs}\frac{1}{L}(\epsilon\sigma^k)_{AB}T^+_k - i\epsilon_{AB}T^{rs} \tag{5.81}$$

The R-symmetry generators T^{rs} do not mix the chiral components of the Majorana charges Q^r_α. Thus, to summarize, we have found that (T^+_i, T^{rs}, Q^r_A) indeed form a proper chiral sub super Lie algebra of $\mathfrak{osp}(\mathcal{N}|4)_\mathbb{C}$ with the graded commutation relations

$$[T^+_i, T^+_j] = \epsilon_{ij}{}^k T^+_k \tag{5.82}$$

$$[T^+_i, Q^r_A] = Q^r_B(\tau_i)^B{}_A \tag{5.83}$$

$$[Q^r_A, Q^s_B] = \delta^{rs}\frac{1}{L}(\epsilon\sigma^i)_{AB}T^+_i - i\epsilon_{AB}T^{rs} \tag{5.84}$$

$$[T^{pq}, Q^r_A] = \frac{1}{2L}(\delta^{qr}Q^p_A - \delta^{pr}Q^q_A) \tag{5.85}$$

which precisely coincide with the graded commutation relations of the complex orthosymplectic Lie superalgebra $\mathfrak{osp}(\mathcal{N}|2)_\mathbb{C}$, the extended supersymmetric generalization of the isometry algebra of $D = 2$ anti-de Sitter space [115]. Performing the Inönü-Wigner contraction, i.e., taking the limit $L \to \infty$, this yields the $\mathcal{N}$-extended $D = 2$ super Poincaré algebra also often denoted by $\overline{\mathfrak{osp}}(\mathcal{N}|2)_\mathbb{C}$. Similarly, considering the anti self-dual sector, one obtains a proper sub super Lie algebra generated by the anti chiral components $(T^-_i, T^{rs}, Q^{A'}_r)$ which again forms $\mathfrak{osp}(\mathcal{N}|2)_\mathbb{C}$.

For the construction of the super analog of Ashtekar's self-dual variables, in what follows, let us restrict to the cases $\mathcal{N} = 1, 2$ in which case we know that the theory is described in terms of the super Cartan connection $\mathcal{A} \in \Omega^1(\mathcal{P}_{/\mathcal{S}}, \mathfrak{osp}(\mathcal{N}|4))$ (5.71). Based on the above observations, we then introduce the following graded self-dual variables also called the *super Ashtekar connection*

$$\mathcal{A}^+ := A^{+i}T_i^+ + \frac{1}{2}\hat{A}_{rs}T^{rs} + \psi_r^A Q_A^r \quad \text{and} \quad \mathcal{A}^- := A^{-i}T_i^- + \frac{1}{2}\hat{A}_{rs}T^{rs} + \psi_{A'}^r Q_r^{A'} \tag{5.86}$$

defining $\mathcal{S}$-relative 1-forms on the $\mathcal{S}$-relative principal super fiber bundle $\mathrm{SO}(\mathcal{N}) \times \mathrm{Spin}^+(1,3) \to \mathcal{P}_{/\mathcal{S}} \to \mathcal{M}_{/\mathcal{S}}$ with values in the chiral sub superalgebra $\mathfrak{osp}(\mathcal{N}|2)_{\mathbb{C}}$ which, in the limit of a vanishing cosmological constant, yields the $\mathcal{N}$-extended $D = 2$ super Poincaré algebra.

Remark 5.4.1. The $\mathcal{N}$-extended $D = 2$ super Poincaré algebra has an equivalent description in terms of a direct sum super Lie algebra $\mathrm{sl}(2,\mathbb{C}) \oplus \mathbb{C}^{0|2\mathcal{N}}$, where $\mathbb{C}^{0|2\mathcal{N}}$ is regarded as a purely odd super vector space. Given $\mathcal{N}$ copies of the fundamental representation $\rho: \mathfrak{sl}(2,\mathbb{C}) \to \mathrm{End}(\mathbb{C}^2)$ of $\mathfrak{sl}(2,\mathbb{C})$, the graded commutation relations are given by

$$[(x,v),(x',v')] := ([x,x'], \rho^{\oplus\mathcal{N}}(x)(v) - \rho^{\oplus\mathcal{N}}(x')(v')) \tag{5.87}$$

$\forall (x,v),(x',v') \in \mathrm{sl}(2,\mathbb{C}) \oplus \mathbb{C}^{0|2\mathcal{N}}$. In the mathematical literature, such kind of superalgebras are usually called *generalized Takiff Lie superalgebras* [194].

For the rest of this section, let us focus on the chiral case (and the case of nonvanishing cosmological constant), the considerations for $\mathcal{A}^-$ are in fact completely analogous. Let us consider the complexification $\mathcal{P}^{\mathbb{C}}$ of $\mathcal{P}$ defined as the associated super $\mathrm{SO}(\mathcal{N}) \times \mathrm{Spin}^+(1,3)_{\mathbb{C}}$-bundle

$$\mathcal{P}^{\mathbb{C}} := \mathcal{P}[\mathrm{SO}(\mathcal{N}) \times \mathrm{Spin}^+(1,3)_{\mathbb{C}}] \tag{5.88}$$

via the obvious mapping $\mathrm{Spin}^+(1,3) \hookrightarrow \mathrm{Spin}^+(1,3)_{\mathbb{C}}$. Due to (5.72), this bundle can be reduced to a super $\mathrm{SO}(\mathcal{N}) \times \mathrm{SL}(2,\mathbb{C})$-bundle $\mathcal{Q} \hookrightarrow \mathcal{P}^{\mathbb{C}}$. It follows from the chiral nature of $\mathcal{A}^+$ that it can be reduced to a well-defined 1-form

$$\mathcal{A}^+ \in \Omega^1(\mathcal{Q}_{/\mathcal{S}}, \mathfrak{osp}(\mathcal{N}|2)_{\mathbb{C}})_0 \tag{5.89}$$

which, by construction, satisfies the conditions (i) and (ii) of Def. 3.3.3. Hence, $\mathcal{A}^+$ defines a *generalized super Cartan connection* on the $\mathcal{S}$-relative $\mathrm{SO}(\mathcal{N}) \times \mathrm{SL}(2,\mathbb{C})$-bundle $\mathcal{Q}_{/\mathcal{S}}$. For $\mathcal{N} = 1$, this is precisely the connection as first introduced in [63]. There, this connection arose by studying the constraint algebra of the canonical theory. Here, we have derived it using the geometrical description of $\mathcal{N}$-extended $D = 4$ supergravity in terms of super Cartan geometry and studying the chiral structure of the underlying supersymmetry algebra corresponding to the super Klein geometry. In particular, we have found that it has the interpretation in terms of a generalized super Cartan connection on the $\mathcal{S}$-relative $\mathrm{SO}(\mathcal{N})\times\mathrm{SL}(2,\mathbb{C})$-bundle $\mathcal{Q}_{/\mathcal{S}}$.

Thus, using Prop. 3.3.12, we can lift $\mathcal{A}^+$ to a super connection 1-form on the associated $\mathrm{OSp}(\mathcal{N}|2)_{\mathbb{C}}$-bundle

$$\mathcal{Q}[\mathrm{OSp}(\mathcal{N}|2)_{\mathbb{C}}]_{/\mathcal{S}} \tag{5.90}$$

In this way, it follows that the canonical phase space of $\mathcal{N}$-extended, $D=4$ supergravity inherits the structure of a $\mathrm{OSp}(\mathcal{N}|2)_{\mathbb{C}}$ Yang-Mills theory which is complete analogy to the standard self-dual variables in ordinary first-order gravity.

Remark 5.4.2. Let us emphasize that, since the above construction relied crucially on the chiral description of the theory, this construction cannot be carried over to *real* Barbero-Immirzi parameters! In fact, real β requires the consideration of both chiral components of the Majorana fermions Q^r_α. But, the anticommutator between Q^r_A and $Q_r^{A'}$ is proportional to P_I which is related to the soldering form e encoded in the dual electric field (see Section 4.3 or Section 5.4.4 below). Hence, this does not lead to a proper sub super Lie algebra and the super Ashtekar connection cannot be defined.

5.4.2. $\mathcal{N}=1$ chiral SUGRA: Chiral Palatini action and super Chern-Simons theory on the boundary

The content of this section has been reproduced from [3], with slight changes to account for the context of this thesis with the permission of Springer-Nature.

Having derived the most general form of the Holst action of $D=4$ AdS supergravity for the cases $\mathcal{N}=1,2$ in the presence of boundaries which also incorporates local supersymmetry invariance, in what follows, we want to focus on the special case of an imaginary Barbero-Immirzi parmareter $\beta=\pm i$ and $\mathcal{N}=1$ (the case $\mathcal{N}=2$ will be discussed in Section 5.4.3 below). As we will see, the resulting theory has many interesting properties and in fact leads to numerous intriguing structures which seem to be well-compatible with the underlying supersymmetry.

Hence, in what follows, let us set $\beta=-i$ (the other case can be treated in complete analogy). In this case, the operator (5.10) takes the form $\mathcal{P}_{\beta=-i}=i\frac{\mathbb{1}+\gamma_*}{2}$ so that $\mathbf{P}_{\beta=-i}=:i\mathbf{P}^+$ where, according to the general discussion in the previous section, $\mathbf{P}^+$ defines a projection

$$\mathbf{P}^+:\ \mathfrak{osp}(1|4)_{\mathbb{C}}\to\mathfrak{osp}(1|2)_{\mathbb{C}} \tag{5.91}$$

onto a proper sub superalgebra of $\mathfrak{osp}(1|4)$ given by the (complex) orthosymplectic algebra $\mathfrak{osp}(1|2)_{\mathbb{C}}$ corresponding to the superalgebra of $\mathcal{N}=1$, $D=2$ super anti-de Sitter space (in fact, it turns out that (5.91) even defines a morphism of superalgebras). It then follows that the inner product (5.11) reduces to the standard inner product

$\langle\cdot,\cdot\rangle$ on $\mathfrak{osp}(1|2)_{\mathbb{C}}$ given by the supertrace which, in particular, is invariant under the Adjoint representation of $\mathrm{OSp}(1|2)_{\mathbb{C}}$. Applying the projection (5.91) on the super Cartan connection (5.1), this yields the super Ashtekar connection (5.86) for $\mathcal{N}=1$

$$\mathcal{A}^{+}\equiv\mathbf{P}^{+}\mathcal{A}=A^{+i}T_{i}^{+}+\psi^{A}Q_{A} \tag{5.92}$$

As discussed in the previous section, the super Ashtekar connection defines a generalized super Cartan connection, so that, via the correspondence Cartan $\leftrightarrow$ Ehresmann (Prop. 3.3.12), it gives rise to a proper super connection 1-form on the associated $\mathrm{OSp}(1|2)_{\mathbb{C}}$-bundle. By applying the projection (5.91) on the Cartan curvature (5.13), we then find for the Lorentzian sub components

$$(\mathbf{P}^{+}F(\mathcal{A}))^{i}=F(A^{+})^{i}+\frac{i}{L}\psi^{A}\wedge\psi^{B}\tau_{AB}^{i}+\frac{1}{2L^{2}}\Sigma^{i}=F(\mathcal{A}^{+})^{i}+\frac{1}{2L^{2}}\Sigma^{i} \tag{5.93}$$

with $F(\mathcal{A}^{+})$ the associated curvature of $\mathcal{A}^{+}$. Here, $\Sigma^{AB}=\Sigma^{i}\tau_{i}^{AB}$ denotes the self-dual part of $\Sigma^{AA'BB'}:=e^{AA'}\wedge e^{BB'}$ which, due to antisymmetry, can be decomposed according to

$$\Sigma^{AA'BB'}=\epsilon^{AB}\Sigma^{A'B'}+\epsilon^{A'B'}\Sigma^{AB} \tag{5.94}$$

such that $\Sigma^{AB}:=\frac{1}{2}\epsilon_{A'B'}\Sigma^{AA'BB'}$. Moreover, for the chiral odd components, we find

$$(\mathbf{P}^{+}F(\mathcal{A}))^{A}=D^{(A^{+})}\psi^{A}+\frac{1}{2L}\chi^{A}=F(\mathcal{A}^{+})^{A}+\frac{1}{2L}\chi^{A} \tag{5.95}$$

where we set $\chi:=-\gamma\wedge\psi$ such that $\chi_{A}=e_{AA'}\wedge\bar{\psi}^{A'}$. Thus, defining $\mathcal{E}:=\Sigma^{i}T_{i}^{+}+L\chi^{A}Q_{A}$ which will also be called the *super electric field*, this yields

$$\mathbf{P}^{+}F(\mathcal{A})=F(\mathcal{A}^{+})+\frac{1}{2L^{2}}\mathcal{E} \tag{5.96}$$

Inserting this expression into the Holst-MacDowell-Mansouri action (5.12) for $\beta=-i$, this gives

$$\begin{aligned}\langle F(\mathcal{A})\wedge F(\mathcal{A})\rangle_{\beta}=&i\,\langle(F(\mathcal{A}^{+})+\frac{1}{2L^{2}}\mathcal{E})\wedge(F(\mathcal{A}^{+})+\frac{1}{2L^{2}}\mathcal{E})\rangle\\=&\frac{i}{L^{2}}\langle\mathcal{E}\wedge F(\mathcal{A}^{+})\rangle+\frac{i}{4L^{4}}\langle\mathcal{E}\wedge\mathcal{E}\rangle+i\,\langle F(\mathcal{A}^{+})\wedge F(\mathcal{A}^{+})\rangle\end{aligned} \tag{5.97}$$

such that

$$S_{\text{H-MM}}^{\mathcal{N}=1}(\mathcal{A})=\frac{i}{\kappa}\int_{M}\left(\langle\mathcal{E}\wedge F(\mathcal{A}^{+})\rangle+\frac{1}{4L^{2}}\langle\mathcal{E}\wedge\mathcal{E}\rangle\right)+S_{\text{bdy}} \tag{5.98}$$

with a boundary term S_{bdy} taking the form

$$\begin{aligned}S_{\mathrm{bdy}}(\mathcal{A}^+) &= \frac{iL^2}{\kappa}\int_M \langle F(\mathcal{A}^+)\wedge F(\mathcal{A}^+)\rangle \\ &= \frac{k}{4\pi}\int_{\partial M}\langle \mathcal{A}^+\wedge \mathrm{d}\mathcal{A}^+ + \frac{1}{3}\mathcal{A}^+\wedge[\mathcal{A}^+\wedge\mathcal{A}^+]\rangle \end{aligned} \tag{5.99}$$

where we used Eq. (5.291) in Section (5.6.1) which now holds due to $\mathrm{OSp}(1|2)_{\mathbb{C}}$-invariance. Thus, as we see, in the chiral theory, the Holst-MacDowell-Mansouri action becomes manifestly $\mathrm{OSp}(1|2)_{\mathbb{C}}$ gauge-invariant and the boundary term takes the form of a super Chern-Simons action with gauge supergroup $\mathrm{OSp}(1|2)_{\mathbb{C}}$ and (complex) Chern-Simons level $k = i4\pi L^2/\kappa = -i12\pi/\kappa\Lambda_{\cos}$ with $\Lambda_{\cos}$ the cosmological constant.

Finally, for the last part of this section, we want to explicitly show that the full action (5.98) is indeed invariant under local supersymmetry transformations. To this end, let us further evaluate the bulk term in (5.98). Using (5.93) and (5.95), we find

$$\langle \mathcal{E}\wedge F(\mathcal{A}^+)\rangle = \Sigma^{AB}\wedge F(A^+)_{AB} + \frac{i}{2L}\Sigma^{AB}\wedge\psi_A\wedge\psi_B + i\chi_A\wedge D^{(A^+)}\psi^A \tag{5.100}$$

as well as

$$\langle \mathcal{E}\wedge\mathcal{E}\rangle = \Sigma^{AB}\wedge\Sigma_{AB} + iL\chi_A\wedge\chi^A \tag{5.101}$$

so that the bulk contribuation in (5.98) can be written in the form

$$\begin{aligned}S_{\mathrm{bulk}}(\mathcal{A}) =& \frac{i}{\kappa}\int_M \Sigma^{AB}\wedge F(A^+)_{AB} + i\chi_A\wedge D^{(A^+)}\psi^A \\ &+ \frac{i}{2L}\Sigma^{AB}\wedge\psi_A\wedge\psi_B + \frac{i}{4L}\chi_A\wedge\chi^A + \frac{1}{4L^2}\Sigma^{AB}\wedge\Sigma_{AB}\end{aligned} \tag{5.102}$$

This is precisely the form of the action of chiral $\mathcal{N}=1$, $D=4$ AdS supergravity as stated, e.g., in [63, 180] and coincides with the Holst action $S^{\mathcal{N}=1}_{\text{H-AdS}}$ (Eq. (4.121)) for the special case $\beta = -i$.

In the Weyl representation of the gamma matrices, the Majorana spinor ϵ generating supersymmetry transformations splits into a left- and right-handed Weyl spinor $\epsilon = (\epsilon^A, \epsilon_{A'})^T$. We will say that transformations associated with the former are left-handed supersymmetry transformations, whereas the latter will be called right-handed supersymmetry transformations. According to the general discussion in Section 5.2, it follows that under left supersymmetry transformations corresponding to some left-handed Weyl spinor $\epsilon = (\epsilon^A, 0)^T$, the super Ashtekar connection and super electric field transform via

$$\delta_\epsilon \mathcal{A}^+ = D^{(\mathcal{A}^+)}\epsilon \quad \text{and} \quad \delta_\epsilon \mathcal{E} = -[\epsilon, \mathcal{E}] \tag{5.103}$$

respectively, and therefore correspond to ordinary $\mathrm{OSp}(1|2)_{\mathbb{C}}$-gauge transformations under which the action (5.98) is manifestly invariant. Note that this is true even off-shell, i.e. without ω satisfying its field equation. Thus, in the chiral theory, it follows that a sub part of the full SUSY transformations becomes a true gauge symmetry of the theory! It remains to show that the action is invariant under right SUSY transformations corresponding to some (anticommutative) right-handed Weyl spinor $\epsilon = (0, \epsilon_{A'})^T$. In that case, from (5.4) we deduce

$$\delta_\epsilon e^{AA'} = i\psi^A \epsilon^{A'}, \quad \delta_\epsilon \psi^A = -\frac{1}{2L} e^{AA'}\epsilon_{A'}, \quad \delta_\epsilon \psi_{A'} = D^{(A^-)}\epsilon_{A'}, \quad \delta_\epsilon A^+ = 0 \tag{5.104}$$

Using (5.104), it follows that the variation of the super electric field takes the form

$$\delta_\epsilon \chi_A = i\psi_A \epsilon_{A'} \wedge \bar{\psi}^{A'} + e_{AA'} \wedge D^{(A^-)}\epsilon_{A'} = D^{(A^+)}\eta_A + (D^{(\omega)} e_{AA'} + i\psi_A \wedge \bar{\psi}_{A'}) \tag{5.105}$$

where, similar as in [180], we introduced $\eta^A := e^{AA'}\epsilon_{A'}$ which furthermore yields

$$\delta_\epsilon \Sigma^{AB} = i\psi^{(A} \wedge \eta^{B)} \tag{5.106}$$

In what follows, we want to assume that the self-dual Ashtekar connection A^+ satisfies its field equations. In this context, it is important to note that the field equations of both A^+ and ψ^A are altered due to the appearence of additional boundary terms in the full action. More precisely, if one varies (5.98) w.r.t. the super Ashtekar connection $\mathcal{A}^+$, one finds that

$$\begin{aligned}\delta S^{\mathcal{N}=1}_{\text{H-MM}}(\mathcal{A}) &= \frac{i}{\kappa}\int_M \langle D^{(\mathcal{A}^+)}\delta\mathcal{A}^+ \wedge \mathcal{E}\rangle + \frac{2iL^2}{\kappa}\int_M \langle D^{(\mathcal{A}^+)}\delta\mathcal{A}^+ \wedge F(\mathcal{A}^+)\rangle \\ &= \frac{i}{\kappa}\int_M \langle \delta\mathcal{A}^+ \wedge D^{(\mathcal{A}^+)}\mathcal{E}\rangle + \frac{i}{\kappa}\int_{\partial M} \langle \delta\mathcal{A}^+ \wedge [\mathcal{E} + 2L^2 F(\mathcal{A}^+)]\rangle = 0\end{aligned} \tag{5.107}$$

where we have integrated by parts and used the Bianchi identity $D^{(\mathcal{A}^+)}F(\mathcal{A}^+) = 0$. Hence, the EOM of $\mathcal{A}^+$ are unaltered provided that the boundary contribution in (5.107) vanishes, i.e.

$$\underset{\Longleftarrow}{F(\mathcal{A}^+)} = -\frac{1}{2L^2}\underset{\Leftarrow}{\mathcal{E}} \tag{5.108}$$

where the arrow denotes the pullback to the boundary. In the following, let us assume that the boundary condition (5.108) is satisfied. Then, modulo the field equations of A^+ which, as will be discussed in detail in Section 5.4.4 below, turn out to be equivalent to the EOM of ω, i.e. $D^{(\omega)}e_{AA'} + i\psi_A \wedge \bar{\psi}_{A'} = 0$, it follows that

$$\delta_\epsilon(\Sigma^{AB} \wedge F(A^+)_{AB}) = \delta_\epsilon \Sigma^{AB} \wedge F(A^+)_{AB} = i\psi^{(A} \wedge \eta^{B)} \wedge F(A^+)_{AB} \tag{5.109}$$

On the other hand, we have

$$\begin{aligned}\delta_\epsilon(\chi_A \wedge D^{(A^+)}\psi^A) =& D\eta_A \wedge D^{(A^+)}\psi^A - \frac{1}{2L}\chi_A \wedge D^{(A^+)}\eta^A \\ =& \mathrm{d}(\eta_A \wedge D^{(A^+)}\psi^A) + \eta_A \wedge D^{(A^+)}D^{(A^+)}\psi^A \\ &- \frac{1}{2L}\chi_A \wedge D^{(A^+)}\eta^A \\ =& \mathrm{d}(\eta_A \wedge D^{(A^+)}\psi^A) - \psi_{(A} \wedge \eta_{B)} \wedge F(A^+)^{AB} \\ &- \frac{1}{2L}\chi_A \wedge D^{(A^+)}\eta^A \end{aligned} \tag{5.110}$$

as well as

$$\delta_\epsilon(\Sigma^{AB} \wedge \psi_A \wedge \psi_B) = -\frac{1}{L}\Sigma^{AB} \wedge \eta_{(A} \wedge \psi_{B)} \tag{5.111}$$

Finally, using $\delta_\epsilon(\chi_A \wedge \chi^A) = 2\chi_A \wedge D^{(A^+)}\eta^A$ and $\delta_\epsilon(\Sigma^{AB} \wedge \Sigma_{AB}) = 2i\Sigma^{AB} \wedge \eta_{(A} \wedge \psi_{B)}$, we finally obtain for the variation of the full Lagrangian $\mathscr{L}_{\text{full}}$ in (5.98) under right-handed SUSY transformations

$$\delta_\epsilon \mathscr{L}_{\text{full}} = i\mathrm{d}(\eta_A \wedge D^{(A^+)}\psi^A) + \delta_\epsilon \mathscr{L}_{\text{bdy}} \tag{5.112}$$

where, by the Bianchi identity, the variation of the boundary term can be written in the form

$$\begin{aligned}\delta_\epsilon \mathscr{L}_{\text{bdy}} &= L^2 \delta_\epsilon \langle F(\mathcal{A}^+) \wedge F(\mathcal{A}^+)\rangle = 2L^2 \langle D^{(\mathcal{A}^+)}\delta_\epsilon \mathcal{A}^+ \wedge F(\mathcal{A}^+)\rangle \\ &= 2L^2 \mathrm{d}\langle \delta_\epsilon \mathcal{A}^+ \wedge F(\mathcal{A}^+)\rangle + 2L^2 \langle \delta_\epsilon \mathcal{A}^+ \wedge D^{(\mathcal{A}^+)}F(\mathcal{A}^+)\rangle \\ &= 2L^2 \mathrm{d}\langle \delta_\epsilon \psi \wedge D^{(A^+)}\psi\rangle = -L\mathrm{d}\langle \eta \wedge D^{(A^+)}\psi\rangle = -i\mathrm{d}(\eta_A \wedge D^{(A^+)}\psi^A). \end{aligned} \tag{5.113}$$

Thus, when we combine the variations, we see that the variation of the boundary term cancels exactly with the respective contribution of the bulk Lagrangian, finally yielding

$$\delta_\epsilon \mathscr{L}_{\text{full}} = 0 \tag{5.114}$$

This proves that, provided boundary condition (5.108) is satisfied, the full action (5.98) is indeed invariant under local SUSY transformations at the boundary. Moreover, from the previous computations, we infer that, in the presence of boundaries, the boundary contributions (5.99) taking the form of a $\mathrm{OSp}(1|2)_{\mathbb{C}}$ super Chern-Simons action to the full action (5.98) are in fact unique if one requires both manifestly $\mathrm{OSp}(1|2)_{\mathbb{C}}$-gauge invariance and invariance under local right-handed SUSY transformations at the boundary. As we will see in the next sections, these observations even carry over to supergravity theories with extended supersymmetry.

Remark 5.4.3. As an aside, note that one can introduce a new independent 2-form field $\mathcal{B}$, also simply called the *B-field*, satisfying the simplicity constraint $\mathcal{B} := \mathcal{E}$. In this way, it then follows that one can rewrite the bulk term in (5.98) in terms of a constrained *super BF-action* with a nonvanishing cosmological constant [86, 181, 182].

As a last step, let us briefly comment on the canonical analysis of the theory (see also Section 5.4.4 below) as well as the boundary conditions which couple the bulk and boundary degrees of freedom. To this end, we split the full action (5.98) into a bulk and boundary term such that $S_{\mathrm{bulk}} + S_{\mathrm{bdy}}$ with S_{bdy} given by (5.99). Similar as above, the variation of the bulk contribution with respect to $\mathcal{A}^+$ then yields

$$\delta S_{\mathrm{bulk}} = \frac{i}{\kappa}\int_M \langle D^{(\mathcal{A}^+)}\delta\mathcal{A}^+ \wedge \mathcal{E}\rangle =: \mathrm{d}\Theta + \frac{i}{\kappa}\int_M \langle \delta\mathcal{A}^+ \wedge D^{(\mathcal{A}^+)}\mathcal{E}\rangle \tag{5.115}$$

Here, $\Theta(\delta)$ denotes the pre-symplectic potential inducing the bulk pre-symplectic structure $\Omega_{\mathrm{bulk}} = \mathbb{d}\Theta$

$$\Omega_{\mathrm{bulk}}(\delta_1, \delta_2) = \frac{2i}{\kappa}\int_\Sigma \langle \delta_{[1}\mathcal{A}^+ \wedge \delta_{2]}\mathcal{E}\rangle \tag{5.116}$$

and, as a consequence, $(\mathcal{A}^+, \mathcal{E})$ (or rather their pullback to Σ) define canonically conjugate variables of the canonical phase space. Moreover, from (5.115), we can immediately read off the Gauss constraint which takes the form

$$\mathscr{G}[\alpha] = \frac{i}{\kappa}\int_\Sigma \langle D^{(\mathcal{A}^+)}\mathcal{E}, \alpha\rangle \tag{5.117}$$

where α denotes a $\mathfrak{osp}(1|2)_{\mathbb{C}}$-valued smearing function defined on Σ. As can be checked by direct computation, the Gauss constraint satisfies the corresponding constraint algebra $\{\mathscr{G}[\alpha], \mathscr{G}[\beta]\} = \mathscr{G}[[\alpha, \beta]]$ and therefore generates local $\mathrm{OSp}(1|2)_{\mathbb{C}}$-gauge transformations. In this context, note that the form (5.117) of the super Gauss constraint, in general, is only valid in the absence of boundaries. In case of a nontrivial boundary $\partial M \neq \emptyset$, in order to account for functional differentiability, one either needs to assume that the smearing function α vanishes on the boundary or one has to add an additional boundary term. In the latter case, it follows that the Gauss constraint is instead given by

$$\mathscr{G}[\alpha]=-\frac{i}{\kappa}\int_{\Sigma}\langle\mathcal{E}\wedge D^{(\mathcal{A}^{+})}\alpha\rangle+\frac{i}{\kappa}\int_{\Delta}\langle\mathcal{E},\alpha\rangle \tag{5.118}$$

with Δ defined as $\Delta:=\Sigma\cap\partial M$. As already explained above, the boundary contribution to (5.98) is given by the action corresponding to a $\mathrm{OSp}(1|2)_{\mathbb{C}}$ super Chern-Simons theory. As a result, the pre-symplectic structure of the full theory takes the form

$$\Omega_{\Sigma}(\delta_1,\delta_2)=\frac{2i}{\kappa}\int_{\Sigma}\langle\delta_{[1}\mathcal{A}^{+}\wedge\delta_{2]}\mathcal{E}\rangle-\frac{k}{2\pi}\int_{\Delta}\langle\delta_{[1}\mathcal{A}^{+}\wedge\delta_{2]}\mathcal{A}^{+}\rangle \tag{5.119}$$

Furthermore, the decomposition of the full action into bulk and boundary terms leads to a matching condition on the boundary between bulk and boundary degrees of freedom. This is equivalent to requiring consistency with the equation of motion of the full theory, i.e. $\delta S^{\mathcal{N}=1}_{\text{H-MM}}=\delta S_{\text{bulk}}+\delta S_{\text{bdy}}=0$ which leads back to boundary condition (5.108). As we see, this condition arises quite naturally from the requirement of supersymmetry invariance at the boundary and in fact, based on the previous observations, even turns out to be unique.

Furthermore, it ensures that the pre-symplectic structure Ω_{Σ} of the full theory is conserved. More precisely, let Σ_i for $i=1,2$ be two Cauchy hypersurfaces and $B\subset\partial M$ be a subset of the boundary enclosed by Σ_1 and Σ_2. Then, since *on-shell* the pre-symplectic current of the bulk pre-symplectic structure defines a closed 2-form on field space [195], by Stokes' theorem, it follows that

$$\begin{aligned}&\Omega_{\Sigma_2}(\delta_1,\delta_2)-\Omega_{\Sigma_1}(\delta_1,\delta_2)\\&=-\frac{2i}{\kappa}\int_{B}\langle\delta_{[1}\mathcal{A}^{+}\wedge\delta_{2]}\mathcal{E}\rangle-\frac{2iL^2}{\kappa}\int_{\Delta_2}\langle\delta_{[1}\mathcal{A}^{+}\wedge\delta_{2]}\mathcal{A}^{+}\rangle\\&\quad+\frac{2iL^2}{\kappa}\int_{\Delta_1}\langle\delta_{[1}\mathcal{A}^{+}\wedge\delta_{2]}\mathcal{A}^{+}\rangle\end{aligned} \tag{5.120}$$

with $\Delta_i:=\Sigma_i\cap\partial M$ for $i=1,2$. According to boundary condition (5.108), the variation of the super electric field $\mathcal{E}$ on B is given by $\delta\mathcal{E}|_B=-2L^2\delta F(\mathcal{A}^{+})|_B=-2L^2D^{(\mathcal{A}^{+})}\delta\mathcal{A}^{+}|_B$. Hence, this implies that first term on the right-hand side of Eq. (5.120) can be written as

$$\begin{aligned}-\frac{2i}{\kappa}\int_{B}\langle\delta_{[1}\mathcal{A}^{+}\wedge\delta_{2]}\mathcal{E}\rangle&=\frac{2iL^2}{\kappa}\int_{B}\mathrm{d}\langle\delta_{[1}\mathcal{A}^{+}\wedge\delta_{2]}\mathcal{A}^{+}\rangle\\=\frac{2iL^2}{\kappa}\int_{\Delta_2}\langle\delta_{[1}\mathcal{A}^{+}\wedge\delta_{2]}\mathcal{A}^{+}\rangle&-\frac{2iL^2}{\kappa}\int_{\Delta_1}\langle\delta_{[1}\mathcal{A}^{+}\wedge\delta_{2]}\mathcal{A}^{+}\rangle\end{aligned} \tag{5.121}$$

Thus, when inserted back into (5.120), it follows immediately that the individual terms on the right-hand side cancel exactly finally proving that, on shell, $\Omega_{\Sigma_2}(\delta_1,\delta_2) = \Omega_{\Sigma_1}(\delta_1,\delta_2)$, that is, the pre-symplectic structure of the full theory is indeed conserved.

Remark 5.4.4. Note that one can also rewrite the boundary condition (5.108) in the equivalent form

$$\underset{\Longleftarrow}{F(\mathcal{A}^+)} + \frac{1}{2L^2}\underset{\Leftarrow}{\mathcal{E}} = 0 \quad \Leftrightarrow \quad \mathbf{P}^+\underset{\Longleftarrow}{F(\mathcal{A})} = 0 \tag{5.122}$$

according to the identity (5.96). Furthermore, taking the complex conjugate of (5.123) yields $\mathbf{P}^- F(\mathcal{A}) = 0$. Hence, when combined together, this in turn gives

$$\mathbf{P}^+\underset{\Longleftarrow}{F(\mathcal{A})} + \mathbf{P}^-\underset{\Longleftarrow}{F(\mathcal{A})} = 0 \quad \Leftrightarrow \quad \underset{\Longleftarrow}{F(\mathcal{A})}^{IJ} = 0 \text{ and } \underset{\Longleftarrow}{F(\mathcal{A})}^{\alpha} = 0 \tag{5.123}$$

that is, the curvature associated to the (full) super Cartan connection $\mathcal{A}$ is constrained to vanish at the boundary. This is precisely the boundary condition as derived in [81] in context of the non-chiral theory.

Remark 5.4.5. The derivation of the Holst-MacDowell-Mansouri action via the β-deformed inner product as described in Section 5.2.1 also gives an elegant approach to the "double chiral" action as considered e.g. in [86]. One notices that the standard action of $\mathcal{N}=1$ AdS SUGRA (modulo boundary terms) arises from (5.12) in the limit $\beta\to\infty$. On the other hand, one has $\mathbf{P}_i + \mathbf{P}_{-i} = 2\mathbf{P}_\infty$ where $\mathbf{P}_{\pm i} = \mp i\mathbf{P}^{\mp}$ with $\mathbf{P}^{\mp}$ defining projections from $\mathfrak{osp}(1|4)$ onto two chiral copies of $\mathfrak{osp}(1|2)_{\mathbb{C}}$. The action $S^{\mathcal{N}=1}_{\text{AdS}}(\mathcal{A})$ of $\mathcal{N}=1$ AdS SUGRA (Eq. (5.3)), modulo boundary terms, then decomposes as

$$2S^{\mathcal{N}=1}_{\text{AdS}}(\mathcal{A}) = S^{\mathcal{N}=1,\beta=+i}_{\text{H-MM}}(\mathcal{A}) + S^{\mathcal{N}=1,\beta=-i}_{\text{H-MM}}(\mathcal{A}) \tag{5.124}$$

and thus splits into two chiral acions of the form (5.98). These actions can be expressed in terms of the super Ashtekar connections $\mathcal{A}^-$ and $\mathcal{A}^+$, respectively.

5.4.3. $\mathcal{N}=2$ chiral SUGRA with boundaries

The content of this section has been reproduced from [3], with slight changes to account for the context of this thesis with the permission of Springer-Nature.

Let us finally consider the chiral limit setting $\beta=-i$ for the Barbero-Immirzi parameter. Then, the operator (5.39) takes the form $\mathbf{P}_{-i} = i\mathbf{P}^+$ with

$$\mathbf{P}^+ : \Omega^2(M,\mathfrak{osp}(2|4)_{\mathbb{C}}) \to \Omega^2(M,\mathfrak{osp}(2|2)_{\mathbb{C}}) \tag{5.125}$$

the projection onto differential forms on the bosonic spacetime manifold with values in the orthosymplectic subalgebra $\mathfrak{osp}(2|2)_{\mathbb{C}}$. In order to see the underlying $\mathrm{OSp}(2|2)_{\mathbb{C}}$-gauge symmetry of the theory, let us introduce the super Ashtekar connection (5.86) for $\mathcal{N}=2$ by projection the super Cartan connection (5.30) onto the chiral subalgebra yielding

$$\mathcal{A}^{+}\equiv\mathbf{P}^{\mathfrak{osp}(2|2)}\mathcal{A}=A^{+i}T_{i}^{+}+\hat{A}T+\psi_{r}^{A}Q_{A}^{r}\tag{5.126}$$

which, according to the general discussion in Section 5.4.1, defines a generalized super Cartan connection and therefore yields a proper connection on the associated $\mathrm{OSp}(2|2)_{\mathbb{C}}$-bundle. Applying the projection on the super curvature, we obtain for the Lorentzian sub components[4]

$$(\mathbf{P}^{\mathfrak{osp}(2|2)}F(\mathcal{A}))^{i}=F(A^{+})^{i}+\frac{i}{L}\psi_{r}^{A}\wedge\psi_{r}^{B}\tau^{i}_{\ AB}+\frac{1}{2L^{2}}\Sigma^{i}=F(\mathcal{A}^{+})^{i}+\frac{1}{2L^{2}}\Sigma^{i}\tag{5.127}$$

with $F(\mathcal{A}^{+})$ the curvature of $\mathcal{A}^{+}$ and Σ^{i} defined as in the $\mathcal{N}=1$ case. Moreover, for the chiral odd components, we find

$$(\mathbf{P}^{\mathfrak{osp}(2|2)}F(\mathcal{A}))_{r}^{A}=D^{(A^{+})}\psi_{r}^{A}+\frac{1}{2L}\hat{A}\epsilon_{rs}\wedge\psi_{s}^{A}+\frac{1}{2L}\chi_{r}^{A}=F(\mathcal{A}^{+})_{r}^{A}+\frac{1}{2L}\chi_{r}^{A}\tag{5.128}$$

where $\chi_{r}^{A}=-e^{AA'}\wedge\bar{\psi}^{s}_{A'}\delta_{rs}$. Finally, for the $\mathrm{U}(1)$-component, we get

$$\mathbf{P}^{\mathfrak{osp}(2|2)}\hat{F}=\hat{F}=\hat{F}^{+}+\frac{i}{2}\bar{\psi}^{r}_{A'}\wedge\bar{\psi}^{A's}\epsilon_{rs}\tag{5.129}$$

with $\hat{F}^{+}:=\mathrm{d}\hat{A}+\frac{i}{2}\psi^{r}_{A}\wedge\psi^{As}\epsilon_{rs}$. To summarize, we can decompose the super Cartan curvature in the following way

$$\mathbf{P}^{\mathfrak{osp}(2|2)}F(\mathcal{A})=:F(\mathcal{A}^{+})+\frac{1}{2L^{2}}\tilde{\mathcal{E}}\tag{5.130}$$

where $\tilde{\mathcal{E}}$ is a graded field which (in constrast to $\mathcal{E}$ to be defined below), as we would like to emphasize, *does not* have a simple transformation behavior as in the $\mathcal{N}=1$ case under left-handed supersymmetry transformations. This is due to the fact that, for $\mathcal{N}=2$, supersymmetry transformations have to be regarded as superdiffeomorphisms rather than gauge transformations leading to nontrivial curvature contributions in the SUSY variations of the super Cartan connection according to the general formula (5.36).

---

[4] We stick to our notation and write ψ_{r}^{A} and $\bar{\psi}^{r}_{A'}$ for the chiral and anti chiral components of the Majorana fermion fields, respectively. The position of the R-symmetry index for the chiral components stays fixed. Moreover, we will sum over repeated indices.

Let us insert (5.130) into (5.42) for $\beta = -i$ which gives the chiral Holst-MacDowell-Mansouri action

$$S_{\text{H-MM}}^{\mathcal{N}=2}(\mathcal{A}) = \frac{L^2}{\kappa}\int_M \langle [F(\mathcal{A}^+) + \frac{1}{2L^2}\tilde{\mathcal{E}}] \wedge \mathbf{P}^+[F(\mathcal{A}^+) + \frac{1}{2L^2}\tilde{\mathcal{E}}]\rangle \tag{5.131}$$

For reasons that will become clear in a moment, let us next subtract the topological term $\langle F(\mathcal{A}^+) \wedge F(\mathcal{A}^+)\rangle$ of the full action (5.131). If we then define the projection $\mathbb{P}^- := \underline{\mathbf{0}} \oplus \frac{1}{2}(1 + i\star) \oplus \underline{\mathbf{0}}$ projecting onto the anti self-dual part of the $\mathrm{U}(1)$-sub component of $F(\mathcal{A}^+)$, it follows that the bulk contribution in (5.131) takes the form

$$\begin{aligned} S_{\text{bulk}}(\mathcal{A}) &= \frac{i}{\kappa}\int_M -L^2\langle F(\mathcal{A}^+) \wedge \mathbb{P}^- F(\mathcal{A}^+)\rangle + \langle F(\mathcal{A}^+) \wedge \mathbf{P}^+\tilde{\mathcal{E}}\rangle \\ &\qquad + \frac{1}{4L^2}\langle \tilde{\mathcal{E}} \wedge \mathbf{P}^+\tilde{\mathcal{E}}\rangle \\ &= \frac{i}{\kappa}\int_M \langle F(\mathcal{A}^+) \wedge [\mathbf{P}^+\tilde{\mathcal{E}} - L^2\mathbb{P}^- F(\mathcal{A}^+)]\rangle + \frac{1}{4L^2}\langle \tilde{\mathcal{E}} \wedge \mathbf{P}^+\tilde{\mathcal{E}}\rangle \end{aligned} \tag{5.132}$$

As it turns out, (5.132) can be rewritten in a very intriguing form. In fact, let us define $\mathcal{E} := \mathbf{P}^+\tilde{\mathcal{E}} - 2L^2\mathbb{P}^- F(\mathcal{A}^+)$ for the *super electric field*. It then follows that the bulk action (5.132) is equivalent to

$$S_{\text{bulk}}(\mathcal{A}) = \frac{i}{\kappa}\int_M \langle F(\mathcal{A}^+) \wedge \mathcal{E}\rangle + \frac{1}{4L^2}\langle \mathcal{E} \wedge \mathcal{E}\rangle \tag{5.133}$$

This follows immediately from the fact that both $\mathbf{P}^+$ and $\mathbb{P}^-$ define projections projecting onto mutually orthogonal subspaces such that $\mathbf{P}^+ \circ \mathbb{P}^- = 0 = \mathbb{P}^- \circ \mathbf{P}^+$ which yields $\langle \mathcal{E} \wedge \mathcal{E}\rangle = \langle \tilde{\mathcal{E}} \wedge \mathbf{P}^+\tilde{\mathcal{E}}\rangle + 4L^4\langle F(\mathcal{A}^+) \wedge \mathbb{P}^- F(\mathcal{A}^+)\rangle$. Hence, the bulk action takes the form of a Palatini-type action with nontrivial cosmological constant written in chiral variables and $\mathrm{OSp}(2|2)_{\mathbb{C}}$ structure group. It is interesting to note that the subtraction of the CS-topological term from the full action was crucial for this result leading to the projection $\mathbb{P}^-$ which is orthogonal to the chiral projection $\mathbf{P}^+$.

In order to see that the super electric field $\mathcal{E}$ indeed defines the canonical conjugate of the super Ashtekar connection, let us go back to (5.132) and vary the action with respect to $\mathcal{A}^+$. In this way, it follows

$$\delta S_{\text{bulk}}(\mathcal{A}) = \frac{i}{\kappa}\int_M \langle D^{(\mathcal{A}^+)}\delta\mathcal{A}^+ \wedge \mathcal{E}\rangle =: \mathrm{d}\Theta + \frac{i}{\kappa}\int_M \langle \delta\mathcal{A}^+ \wedge D^{(\mathcal{A}^+)}\mathcal{E}\rangle \tag{5.134}$$

with pre-symplectic potential $\Theta(\delta)$ inducing the bulk pre-symplectic structure

$$\Omega_{\text{bulk}}(\delta_1, \delta_2) = \frac{2i}{\kappa}\int_\Sigma \langle \delta_{[1}\mathcal{A}^+ \wedge \delta_{2]}\mathcal{E}\rangle \tag{5.135}$$

Thus, indeed, $(\mathcal{A}^+, \mathcal{E})$ define the fundamental variables of the canonical phase space. Moreover, as discussed in the previous section, from (5.134), we deduce that the super Gauss constraint $\mathcal{G}[\alpha]$ in the presence of boundaries takes the form

$$\mathcal{G}[\alpha] = -\frac{i}{\kappa}\int_{\Sigma}\langle\mathcal{E}\wedge D^{(\mathcal{A}^+)}\alpha\rangle + \frac{i}{\kappa}\int_{\Delta}\langle\mathcal{E},\alpha\rangle \tag{5.136}$$

for arbitrary smooth $\mathfrak{osp}(2|2)_{\mathbb{C}}$-valued smearing function α defined on Σ. As a consequence, it follows that the super Gauss constraint satisfies the constraint algebra $\{\mathcal{G}[\alpha],\mathcal{G}[\beta]\} = \mathcal{G}[[\alpha,\beta]]$. That is, the Gauss constraint generates local $\mathrm{OSp}(2|2)_{\mathbb{C}}$-gauge transformations. The boundary action of the theory takes the form

$$\begin{aligned} S_{\mathrm{bdy}}(\mathcal{A}^+) &= \frac{iL^2}{\kappa}\int_M\langle F(\mathcal{A}^+)\wedge F(\mathcal{A}^+)\rangle \\ &= \frac{iL^2}{\kappa}\int_{\partial M}\langle\mathcal{A}^+\wedge \mathrm{d}\mathcal{A}^+ + \frac{1}{3}\mathcal{A}^+\wedge[\mathcal{A}^+\wedge\mathcal{A}^+]\rangle \end{aligned} \tag{5.137}$$

and thus, in particular, corresponds to the action of a $\mathrm{OSp}(2|2)_{\mathbb{C}}$ super Chern-Simons theory with (complex) Chern-Simons level $k = i4\pi L^2/\kappa = -i12\pi/\kappa\Lambda_{\mathrm{cos}}$. The presymplectic structure of the full theory is given by

$$\Omega(\delta_1,\delta_2) = \frac{2i}{\kappa}\int_{\Sigma}\langle\delta_{[1}\mathcal{A}^+\wedge\delta_{2]}\mathcal{E}\rangle - \frac{k}{2\pi}\int_{\partial\Sigma}\langle\delta_{[1}\mathcal{A}^+\wedge\delta_{2]}\mathcal{A}^+\rangle \tag{5.138}$$

As in the $\mathcal{N}=1$ case, due to the splitting of the full action into a bulk and boundary term, one needs to derive a matching condition relating bulk and boundary degrees of freedom at the boundary. This is equivalent to requiring consistency with the equation of motion of the full theory, i.e. $\delta S^{\mathcal{N}=2}_{\text{H-MM}} = \delta S_{\mathrm{bulk}} + \delta S_{\mathrm{bdy}} = 0$. From this we can immediately read off the boundary condition

$$\underset{\Leftarrow}{\mathcal{E}} = \frac{i\kappa k}{2\pi}\underset{\Longleftarrow}{F(\mathcal{A}^+)} \tag{5.139}$$

where, again, the arrow denotes the pullback to the boundary. This condition relates the super electric field $\mathcal{E}$ to the curvature of the super connection $\mathcal{A}^+$ corresponding to the $\mathrm{OSp}(2|2)_{\mathbb{C}}$ super Chern-Simons theory living on the boundary.

Remark 5.4.6. Note that boundary condition (5.139) can equivalently be rewritten in the following form

$$\underset{\Longleftarrow}{F(\mathcal{A}^+)} + \frac{1}{2L^2}\underset{\Leftarrow}{\mathcal{E}} = 0 \quad\Leftrightarrow\quad \mathbf{P}^+\underset{\Longleftarrow}{F(\mathcal{A})} = 0 \tag{5.140}$$

where we used that the U(1)-component $\hat{\mathcal{E}}$ of the super electric field can be written as $\hat{\mathcal{E}} = \hat{\hat{\mathcal{E}}} - L^2(1 + i\star)\hat{F}$ such that

$$\hat{F}^+ + \frac{1}{2L^2}\hat{\mathcal{E}} = \hat{F} - \frac{1}{2}(1 + i\star)\hat{F} = \frac{1}{2}(1 - i\star)\hat{F} \tag{5.141}$$

Thus, if we take the complex conjugate of (5.140) yielding $\mathbf{P}^- F(\mathcal{A}) = 0$, we find that condition (5.139) is equivalent to

$$\mathbf{P}^+ \underset{\Longleftarrow}{F(\mathcal{A})} + \mathbf{P}^- \underset{\Longleftarrow}{F(\mathcal{A})} = 0 \tag{5.142}$$

that is, the pullback of the curvature components $F(\mathcal{A})^{IJ}$, $F(\mathcal{A})^\alpha_r$ and $\hat{F}$ corresponding to the OSp$(2|4)$ super Cartan connection $\mathcal{A}$ to the boundary are constrained to vanish at the boundary in accordance with the boundary condition as derived in [81] in context of the non-chiral theory.

5.4.4. Reality conditions

According to the results of Section 5.4.2 and 5.4.3, the bulk pre-symplectic structure of chiral supergravity in $D = 4$ with $\mathcal{N}$-extended supersymmetry with $\mathcal{N} = 1, 2$ is given by

$$\Omega_{\text{bulk}}(\delta_1, \delta_2) = \frac{2i}{\kappa}\int_\Sigma \langle \delta_{[1}\mathcal{A}^+ \wedge \delta_{2]}\mathcal{E}\rangle = \frac{i}{\kappa}\int_\Sigma \mathrm{d}^3x \left(\delta_1\mathcal{A}^{+\underline{A}}_a \delta_2\mathcal{E}^a_{\underline{A}} - \delta_2\mathcal{A}^{+\underline{A}}_a \delta_1\mathcal{E}^a_{\underline{A}}\right) \tag{5.143}$$

where we have made the expansion $\mathcal{A}^+ = \mathcal{A}^{+\underline{A}}T_{\underline{A}}$ w.r.t. the real homogeneous basis $(T_{\underline{A}})_{\underline{A}} \equiv (T_i^+, Q^r_A, T^{rs})$ of OSp$(\mathcal{N}|2)$ and introduced the super electric field $\mathcal{E}^a_{\underline{A}}$ on Σ defined as

$$\mathcal{E}^a_{\underline{A}} := \frac{1}{2}\epsilon^{abc}\mathscr{T}_{\underline{BA}}\mathcal{E}^{\underline{B}}_{bc}, \quad \text{with } \mathscr{T}_{\underline{AB}} := \langle T_{\underline{A}}, T_{\underline{B}}\rangle \tag{5.144}$$

Hence, it follows that the pair $(\mathcal{A}^{+\underline{A}}_a, \mathcal{E}^a_{\underline{A}})$ build up a graded symplectic phase space with graded Poisson relations

$$\{\mathcal{E}^a_{\underline{A}}(x), \mathcal{A}^{+\underline{B}}_b(y)\} = i\kappa\delta^a_b\delta^{\underline{B}}_{\underline{A}}\delta^{(3)}(x, y) \tag{5.145}$$

where we used that pre-symplectic structure Ω_{bulk} defines an even 2-form on the phase space. As it turns out, these fundamental variables are, however, not fully independent but need to satisfy certain *reality conditions*. This is due to the fact that the initial conditions of the dynamical fields have to chosen in such a way such that their resulting dynamics as governed by the chiral action (5.98) (resp. (5.133)) are consistent with the dynamics of the ordinary (real) supergravity theory.

In what follows, following [62], we want to derive the explicit form of these reality conditions for the case $\mathcal{N}=1$ (the case $\mathcal{N}=2$ is similar yielding an additional condition for the $\mathrm{U}(1)$ component of the super electric field $\mathcal{E}^a_{\underline{A}}$). To this end, note that the super electric field decomposes via $\mathcal{E}^a_{\underline{A}}=(E^a_i,-i\pi^a_A)$ with $E^a_i=\sqrt{q}e^a_i$ the usual (bosonic) gravitational electric field conjugate to self-dual Ashtekar connection A^{+i} and π^a_A the canonically conjugate momentum of ψ^A_a given by

$$\pi^a_A=\epsilon^{abc}\bar{\psi}^{A'}_b e_{cAA'} \tag{5.146}$$

These are the reality conditions for the fermionic degrees of the freedom which allow to re-express the components of the complex conjugate Weyl spinor $\bar{\psi}^{A'}_a$ in terms of the fundamental variables. With respect to the canonically conjugate pairs (A^{+i}_a,E^a_i) and (ψ^A_a,π^a_A) the graded Poisson brackets (5.145) take the form

$$\{E^a_i(x),A^{+j}_b(y)\}=i\kappa\delta^a_b\delta^j_i\delta^{(3)}(x,y)\text{ and }\{\pi^a_A(x),\psi^B_b(y)\}=-\kappa\delta^a_b\delta^B_A\delta^{(3)}(x,y) \tag{5.147}$$

In order to find the respective reality conditions for the bosonic degrees of freedom, let us derive the equations of motion of the self-dual Ashtekar connection. Thus, varying the chiral bulk action (5.102) with respect to A^+, we find

$$D^{(A^+)}\Sigma^{AB}=\mathrm{d}\Sigma^{AB}+A^{+A}{}_C\wedge\Sigma^{CB}+A^{+B}{}_C\wedge\Sigma^{AC}=-i\bar{\psi}_{A'}\wedge\psi^{(A}\wedge e^{B)A'} \tag{5.148}$$

If we take the complex conjugate of (5.148) this yields, provided e is real, the respective equations involving $D^{(A^-)}\Sigma^{A'B'}$. Together with (5.148), this can then be combined to give the respective equations of motion for $\Sigma^{AA'BB'}$ which is equivalent to

$$D^{(\omega)}e^{AA'}\equiv\Theta^{(\omega)AA'}=-i\psi^A\wedge\bar{\psi}^{A'} \tag{5.149}$$

with $\Theta^{(\omega)AA'}$ the torsion two-form associated to the spin connection ω (written in spinor indices). Moreover, using the identity

$$D^{(\omega)}(D^{(\omega)}e^{AA'})=F(A^+)^A{}_B\wedge e^{BA'}-F(A^-)_{B'}{}^{A'}\wedge e^{AB'} \tag{5.150}$$

it follows, provided again that e is real and (5.149) is satisfied, that the imaginary part of the chiral action (5.102) takes the form

$$\begin{aligned}\Im(\mathscr{L}_{\text{bulk}})=&-\frac{1}{4\kappa}\Big(ie_{AA'}\wedge D^{(\omega)}D^{(\omega)}e^{AA'}-2e_{AA'}\wedge D^{(A^+)}\psi^A\wedge\bar{\psi}^{A'}\\&+2e_{AA'}\wedge\psi^A\wedge D^{(A^-)}\bar{\psi}^{A'}\Big)\\=&\frac{1}{4\kappa}e_{AA'}\wedge D^{(\omega)}(\psi^A\wedge\bar{\psi}^{A'})=-\frac{1}{4\kappa}\mathrm{d}(e_{AA'}\wedge\psi^A\wedge\bar{\psi}^{A'})\end{aligned} \tag{5.151}$$

and thus becomes a pure boundary term. Hence, it follows that, in this way, one indeed reobtains the field equations of ordinary real $\mathcal{N}=1$ supergravity. As discussed in detail in [62], in the canonical description of the theory, it follows that the reality conditions are equivalent to the requirement that the electric field E^a_i (depending on the co-frame $e\hat{\imath}$) is real and the $3D$ spin connection part $\Gamma^i = -\frac{1}{2}\epsilon^i{}_{jk}\omega^{jk}$ of A^+ satisfies the torsion equation

$$D^{(\Gamma)}e^i \equiv \mathrm{d}e^i + \epsilon^i{}_{jk}\Gamma^j \wedge e^k = \Theta^{(\Gamma)i} = \frac{i}{2}\psi^A \wedge \bar{\psi}^{A'}\sigma^i_{AA'} \tag{5.152}$$

This equation has the unique solution

$$\Gamma^i \equiv \Gamma^i(e) + C^i(e,\psi,\bar{\psi}) \tag{5.153}$$

with $\Gamma^i(e)$ the torsion-free metric connection

$$\Gamma^i_a(e) := -\epsilon^{ijk}e^b_j\left(\partial_{[a}e_{b]k} + \frac{1}{2}e^c_k e^l_a \partial_{[c}e_{b]l}\right) \tag{5.154}$$

and C^i the contorsion tensor (Eq. (4.95)) which can be written in the form

$$C^i_a = \frac{i}{4\sqrt{q}}\epsilon^{bcd}e^i_d\left(2\psi^A_{[a}\bar{\psi}^{A'}_{b]}e_{cAA'} - \psi^A_b\bar{\psi}^{A'}_c e_{aAA'}\right) \tag{5.155}$$

Thus, to summarize, the reality conditions for the bosonic degrees of freedom are given by

$$A^{+i}_a + (A^{+i}_a)^* = 2\Gamma^i_a(e) + 2C^i_a(e,\psi,\bar{\psi}), \quad E^a_i = \Re(E^a_i) \tag{5.156}$$

Provided that the initial conditions of the dynamical fields satisfy (5.156) as well as (5.146), this then ensures that the dynamical evolution remains in the real sector of the complex phase space, i.e., the phase space of ordinary real $\mathcal{N}=1$ supergravity.

5.5. The state space of chiral LQSG

5.5.1. The graded holonomy-flux algebra

As observed in the previous sections, reintroducing the underlying parametrizing supermanifold $\mathcal{S}$, the phase space of chiral supergravity for $\mathcal{N}=1$ and $\mathcal{N}=2$ is described in terms of the conjugate pair $(\mathcal{A}^{+\underline{A}}_a, \mathcal{E}^a_{\underline{A}})$ consisting of the pullback of the super Ashtekar connection to $\Sigma_{/\mathcal{S}}$ with Σ a Cauchy slice of the body $M := \mathbf{B}(\mathcal{M})$ of the underlying base supermanifold $\mathcal{M}$ as well as the dual super electric field $\mathcal{E}^a_{\underline{A}}$.

According to the general discussion in Section 5.4.1, the super Ashtekar connection defines a super-connection 1-form on the associated $\mathrm{OSp}(\mathcal{N}|2)_{\mathbb{C}}$-bundle. Thus, the phase of the theory turns out to be a graded generalization of the pure bosonic theory. This suggests to canonically quantize it using standard tools from loop quantum gravity and generalizing them to the super category. To this end, we first need to derive a graded analog of classical *holonomy-flux algebra* encoding the dynamical degrees of freedom and find a representation which is the starting point for the quantization of theory à la LQG.

In what follows, we would like to keep the discussion as general as possible and assume that we are given any locally supersymmetric field theory, such as a Yang-Mills gauge theory with gauge group given by a supergroup $\mathcal{G}$, such that its corresponding canonical description results into a canonically conjugate pair $(\mathcal{A}_a, \mathcal{E}^a_{\underline{A}})$ with $\mathcal{A}^{\underline{A}}_a$ the pullback to Σ of a super connection 1-form defined on a $\mathcal{S}$-relative principal $\mathcal{G}$-bundle $\mathcal{P}_{/\mathcal{S}}$ and a canonically conjugate momentum $\mathcal{E}^a_{\underline{A}}$ which is related to a $\mathrm{Lie}(\mathcal{G})$-valued 2-form $\mathcal{E}$ also referred to as the *super electric field* defined on the bundle via (5.144), that is,

$$\mathcal{E}^a_{\underline{A}} := \frac{1}{2}\epsilon^{abc}\,\mathscr{T}_{\underline{BA}}\mathcal{E}^{\underline{B}}_{bc} \tag{5.157}$$

Moreover, they satisfy the graded Poisson relations

$$\{\mathcal{E}^a_{\underline{A}}(s,x), \mathcal{A}^{\underline{B}}_b(s,y)\} = g\delta^a_b\delta^{\underline{B}}_{\underline{A}}\delta^{(3)}(x,y) \tag{5.158}$$

$\forall x, y \in \Sigma$ and $s \in \mathcal{S}$ for some *coupling constant* g which in the context of chiral supergravity is given by $g = i\kappa$. In Eq. (5.157), we have chosen an Ad-invariant super metric $\langle\cdot,\cdot\rangle$ on $\mathrm{Lie}(\mathcal{G})$ which is supposed to be non-degenerate and defined $\mathscr{T}_{\underline{AB}} := \langle T_{\underline{A}}, T_{\underline{B}}\rangle$ w.r.t. a real homogeneous basis $(T_{\underline{A}})_{\underline{A}}$ of $\mathfrak{g}$.

Finally, let us restrict the underlying parametrizing supermanifold $\mathcal{S}$ to be a superpoint, i.e., its body $\mathbf{B}(\mathcal{S}) = \{*\}$ just consists of a single point such that, according to Prop. 2.2.10, the parametrizing supermanifold can be regarded as an object of the category **Gr**. Due to categorial equivalence, in the following, we will interchangeably interpret supermanifolds as objects of the category of H^∞ or algebro-geometric supermanifolds. Thus, for instance, if $\mathcal{M}$ and $\mathcal{S}$ are regarded as objects in $\mathbf{SMan}_{\mathrm{Alg}}$, the set $\mathcal{M}(\mathcal{S}) := \{f : \mathcal{S} \to \mathcal{M}\}$ of smooth maps between supermanifolds can be identified with the $\mathcal{S}$-point of $\mathcal{M}$ (see Def. 2.2.5) which itself, since $\mathcal{S} \in \mathbf{Gr}$, carries the structure of a H^∞ supermanifold.

For the construction of the classical algebra, on $\mathcal{M}_{/\mathcal{S}}$, let us introduce the *path groupoid* $\mathbf{P}(\mathcal{M}_{/\mathcal{S}})$ whose set of objects $\mathbf{Ob}(\mathbf{P}(\mathcal{M}_{/\mathcal{S}}))$ is defined as the $\mathcal{S}$-point $\mathcal{M}(\mathcal{S})$ and where, for any objects $g, f : \mathcal{S} \to \mathcal{M}$, the morphisms $\mathrm{Hom}_{\mathbf{P}(\mathcal{M}_{/\mathcal{S}})}(f,g)$ is defined as the set of all piecewise smooth paths $\gamma : f \to g$ (see Def. 2.7.7). On the other hand, to $\mathcal{M}_{/\mathcal{S}}$ one can associate the *Atiyah groupoid* $\mathbf{At}(\mathcal{P}_{/\mathcal{S}})$ with objects given by $\mathbf{Ob}(\mathbf{At}(\mathcal{P}_{/\mathcal{S}})) = \mathcal{M}(\mathcal{S})$ and where, for any pair of objects $g, f : \mathcal{S} \to \mathcal{M}$, the set of morphisms

$\mathrm{Hom}_{\mathbf{At}(\mathcal{P}_{/\mathcal{S}})}(f,g)$ consists of smooth maps $\alpha : \Gamma(f^*\mathcal{P}) \to \Gamma(g^*\mathcal{P})$ that are $\mathcal{G}$-equivariant in the sense that $\alpha(f\cdot\phi) = \alpha(f)\cdot\phi$ for any $\mathcal{S}$-point $\phi : \mathcal{S}\to\mathcal{G}$.

According to the general discussion in Section 2.7.2, in particular Prop. 2.7.12 and Prop. 2.7.15, it then follows that the parallel transport map $\mathscr{P}^{\mathcal{A}}_{\mathcal{S}}$ corresponding to the super connection 1-form $\mathcal{A}$ induces a covariant functor

$$\mathscr{P}^{\mathcal{A}}_{\mathcal{S}} : \mathbf{P}(\mathcal{M}_{/\mathcal{S}}) \to \mathbf{At}(\mathcal{P}_{/\mathcal{S}}),\ (f \xrightarrow{\gamma} g) \mapsto (\Gamma(f^*\mathcal{P}) \xrightarrow{\mathscr{P}^{\mathcal{A}}_{\mathcal{S},\gamma}} \Gamma(g^*\mathcal{P})) \tag{5.159}$$

from the path groupoid to the Atiyah groupoid. Actually, adapting the conventions chosen in [18], it seems to be convenient to consider the corresponding contravariant counterpart $(\mathscr{P}^{\mathcal{A}}_{\mathcal{S}})^{\mathrm{op}} : \mathbf{P}(\mathcal{M}_{/\mathcal{S}}) \to \mathbf{At}(\mathcal{P}_{/\mathcal{S}})$ of (5.159) which to any piecewise smooth paths $e : f \to g$ associates the parallel transport map along the inverse path $e^{-1} : g \to f$, that is,

$$(\mathscr{P}^{\mathcal{A}}_{\mathcal{S}})^{\mathrm{op}}(e) := \mathscr{P}^{\mathcal{A}}_{\mathcal{S},e^{-1}} = (\mathscr{P}^{\mathcal{A}}_{\mathcal{S},e})^{-1} : \Gamma(g^*\mathcal{P}) \to \Gamma(f^*\mathcal{P}) \tag{5.160}$$

such that $(\mathscr{P}^{\mathcal{A}}_{\mathcal{S}})^{\mathrm{op}}(e'\circ e) = (\mathscr{P}^{\mathcal{A}}_{\mathcal{S}})^{\mathrm{op}}(e)\circ(\mathscr{P}^{\mathcal{A}}_{\mathcal{S}})^{\mathrm{op}}(e')$. Next, for the canonical description of the theory, let us to restrict $\mathbf{P}(\mathcal{M}_{/\mathcal{S}})$ to the subgroupoid $\mathbf{P}(\Sigma)$ of non-parametrized piecewise ordinary smooth paths[5] $e : x \to y$ between points $x, y \in \Sigma$ on the Cauchy slice Σ where we identified x with the constant map $c_x : \mathcal{S} \to \{x\} \subset \Sigma$. This in fact sufficient to resolve the physical degrees of freedom of the theory since, by the rheonomy principle, the super connection 1-form is uniquely determined by its pullback to the body of the supermanifold. Actually, as we will see later, for the construction of the graded holonomy-flux algebra, it turns out to be more convenient to work in the semianalytic category and thus to assume the edges e of a graph to be piecewise semianalytic (see Remark 5.5.1 below). Moreover, in the following, we want to work on a local trivialization of the relative principal super fiber bundle which, for sake of simplicity, we assume to be defined on all of Σ. This is reasonable as, in the quantum theory, we will restrict to gauge-invariant quantities anyway.

Thus, according to Example 2.7.16, for any smooth path e on Σ the parallel transport map along e is uniquely determined by the group-valued map $g_e[\mathcal{A}] : \mathcal{S} \to \mathcal{G}$ as defined via (2.269). Hence, in turn this implies that the functor $(\mathscr{P}^{\mathcal{A}}_{\mathcal{S}})^{\mathrm{op}}$ is uniquely determined by the $\mathcal{S}$-point

$$h_e[\mathcal{A}] := g_{e^{-1}}[\mathcal{A}] \equiv g_e(\cdot,1)^{-1} : \mathcal{S} \to \mathcal{G} \tag{5.161}$$

[5] Recall that the split functor **S** induces an equivalence of categories between the category **Man** of ordinary smooth manifolds and the subcategory $\mathbf{SMan}_0 \subset \mathbf{SMan}$ of bosonic supermanifolds with trivial odd dimensions.

which we call the *super holonomy* along e induced by $\mathcal{A}$. Splitting $\mathcal{A} =: A + \psi$ according to the even and odd part of the super Lie algebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with A the underlying bosonic connection, it follows by Example 2.7.17 that (5.161) explicitly takes the form

$$h_e[\mathcal{A}] = g_{e^{-1}}[\mathcal{A}] = \mathcal{P}\exp\left(\int_e \mathrm{Ad}_{h_e[A]}\psi\right) \cdot h_e[A] \tag{5.162}$$

with $h_e[A]$ the holonomy of the corresponding bosonic connection A. Under a local gauge transformation $\phi : \mathcal{S} \times \mathcal{M} \to \mathcal{G}$, it transforms as

$$h_e[\mathcal{A}] \to \phi(\cdot, b(e)) \cdot h_e[A] \cdot \phi(\cdot, f(e))^{-1} \tag{5.163}$$

where $b(e) := e(0)$ and $f(e) = e(1)$ are defined as the beginning and endpoints of the edge e, respectively. Using this identification of the parallel transport map in terms of its holonomies, we may equivalently describe it as a contravariant functor

$$(\mathscr{P}^{\mathcal{A}}_{\mathcal{S}})^{\mathrm{op}} : \mathbf{P}(\Sigma) \to \mathbf{G}(\mathcal{S}),\ (x \xrightarrow{e} y) \mapsto (y \xrightarrow{h_e[\mathcal{A}]} x) \tag{5.164}$$

from the path groupoid on Σ to the groupoid $\mathbf{G}(\mathcal{S})$ where $\mathbf{Ob}(\mathbf{G}(\mathcal{S})) = \Sigma$ and, for any $x, y \in \Sigma$, the arrows $x \to y$ are labeled by $\mathcal{S}$-points $g : \mathcal{S} \to \mathcal{G}$.

As usal in LQG, for the construction of the classical algebra, in the following we consider the whole set[6] $\mathrm{Hom}_{\mathbf{Cat}}(\mathbf{P}(\Sigma)^{\mathrm{op}}, \mathbf{G}(\mathcal{S}))$, that is, the set of *all* contravariant functors $H : \mathbf{P}(\Sigma) \to \mathbf{G}(\mathcal{S})$ from the path groupoid to the groupoid $\mathbf{G}(\mathcal{S})$. That is, we do not restrict to those functors arising from the parallel transport map of a smooth super connection 1-form. For this reason, we will also refer to a such functor H as a *generalized super connection*. Next, we are looking for a different description of the set of generalized super connections on the whole path goupoid $\mathbf{P}(\Sigma)$ in terms of subsets defined on supgroupoids $l(\gamma)$ generated by finite *graphs* γ.

To this end, following [18, 196], we define a graph γ as a collection of finitely many piecewise smooth paths e_i, $i = 1, \ldots, n$, embedded in Σ also called *edges* such that $\gamma = \bigcup_i \mathrm{im}(e_i)$ and the e_i are independent in the sense that they at most intersect at their endpoints also called *vertices*. For such a graph γ, let $E(\gamma)$ and $V(\gamma)$ denote the set of its underlying edges and vertices, respectively. Then, each graph γ in Σ induces a subgroupoid $l(\gamma)$ of the path groupoid $\mathbf{P}(\Sigma)$ with objects given by the set of vertices $V(\gamma)$ and morphisms generated by finitely many compositions of edges and their inverses. The collection $\mathcal{L}$ of all such subgroupoids l forms a partially ordered set $\mathcal{L} \equiv (\mathcal{L}, \leq)$

6 Here, **Cat** denotes the *category of small categories* with small categories C as objects and covariant functors $F : C \to \mathcal{D}$ between small categories as morphisms where a category C is called *small* if the collection of objects $\mathbf{Ob}(C)$ defines a set (see Appendix B). This category can be even lifted to a 2-*category* regarding natural transformations $\eta : F \to G$ between functors as 2-*morphisms*.

where $l \leq l'$ for any $l, l' \in \mathcal{L}$ iff l is a subgroupoid of l'. Since, we are working in the semianalytic category, it follows that this partially ordered set, in addition, is directed, i.e., $\forall l, l' \in \mathcal{L}$, there exists $l'' \in \mathcal{L}$ such that $l, l' \leq l''$ [18, 196].

Remark 5.5.1. Given two subgroupoids l, l' generated by graphs γ and γ' in Σ, respectively, one may try to define the groupoid l'' containing $l, l \leq l''$ as the groupoid generated by the graph $\gamma'' := \gamma \cup \gamma'$. For this to be well-defined, one needs to ensure that the so-constructed graph can again be subdivided into a finite number of edges. This is equivalent to requiring that two distinct edges only have a finite number of intersection points. This is generically not the case for arbitrary piecewise smooth edges. Similar issues arise by trying to consistently implement the (graded) holonomy-flux algebra, as one needs to ensure that edges only have a finite number of intersections with 2-dimensional surfaces. As it turns out, all these issues can be remedied simultaneously working instead in the *analytic* or even *semianalytic category* (see [18, 196] for more details; for a definition of *analytic supermanifolds* see [106]). One thus assumes that spatial manifold Σ allows for a (semi)analytic structure that is, a maximal smooth atlas so that transition functions are (semi)analytic. Consequently, (semi)analytic edges and surfaces are defined as 1- and 2-dimensional (semi)analytic submanifolds of Σ, respectively. In the semianalytic case, recall that, roughly speaking, a smooth function (or, more generally of class C^m with $m > 0$) defined on an open subset of $\mathbb{R}^n$ is called semianalytic, if it locally coincides with an analytic function defined on a slightly larger neighborhood. Perhaps, this definition can be generalized to the supermanifold category by requiring smooth functions $f_{\underline{I}}$ appearing in the expansion (C.1) of a H^∞ smooth function f to be semianalytic.

In the following, we would like to show that contravariant functors $H : \mathbf{P}(\Sigma) \to \mathbf{G}(\mathcal{S})$ defined on the whole path groupoid $\mathbf{P}(\Sigma)$ can equivalently be described in terms of their restrictions $H|_l$ on subgroupoids $l \in \mathcal{L}$. This will also enable us to equip this set with a topology which, under certain assumptions on the gauge group $\mathcal{G}$, turns out to be *projectively Hausdorff*. For this, for any $l \in \mathcal{L}$, let us define

$$\mathcal{A}_{\mathcal{S},l} := \mathrm{Hom}_{\mathbf{Cat}}(l^{\mathrm{op}}, \mathbf{G}(\mathcal{S})) \tag{5.165}$$

It is clear that a contravariant functor H on a subgroupoid $l \equiv l(\gamma)$ generated by a graph γ is uniquely determined by its images $(H(e_i))_{i=1,\ldots,n}$ of the underlying edges e_i. Hence, this yields a bijection

$$\mathcal{A}_{\mathcal{S},l} \xrightarrow{\sim} \mathcal{G}(\mathcal{S})^{|E(\gamma)|},\ H \mapsto (H(e_1), \ldots, H(e_n)) \tag{5.166}$$

Since the $\mathcal{S}$-point $\mathcal{G}(\mathcal{S})$ defines a topological space via the DeWitt topology, we can use (5.166) to induce a topology on $\mathcal{A}_{\mathcal{S},l}$. In fact, in this way, it follows that $\mathcal{A}_{\mathcal{S},l}$ in particular is projectively Hausdorff. In fact, in case that $\mathcal{S}$ corresponds to the Grassmann algebra $\mathbb{R} \in \mathbf{Ob}(\mathbf{Gr})$, the set

$$\mathcal{A}_{\mathbb{R},l} := \mathrm{Hom}_{\mathbf{Cat}}(l^{\mathrm{op}}, \mathbf{G}(\mathbb{R})) \tag{5.167}$$

can be identified with set of ordinary bosonic generalized holonomies on l with values in the bosonic subgroup $G := \mathbf{B}(\mathcal{G})$ which is Hausdorff. Hence, for this reason, we may call $\mathcal{A}_{\mathbb{R},l}$ the *body* of $\mathcal{A}_{\mathcal{S},l}$. For any $\mathcal{S} \in \mathbf{Ob}(\mathbf{Gr})$, this induces a projection

$$\mathcal{A}_{\mathcal{S},l} \to \mathcal{A}_{\mathbb{R},l} \tag{5.168}$$

By definition, $\mathcal{A}_{\mathbb{R},l}$ defines a topological Hausdorff space. Moreover, by construction, it follows that the topology on $\mathcal{A}_{\mathcal{S},l}$ is the coarsest topology such that projection (5.168) is continuous. Hence, in this sense, $\mathcal{A}_{\mathcal{S},l}$ indeed defines a projective Hausdorff space.

Next, as in the pure bosonic theory, for any $l, l' \in \mathcal{L}$ with $l \leq l'$, one has a surjective mapping

$$p_{ll'} : \mathcal{A}_{\mathcal{S},l'} \to \mathcal{A}_{\mathcal{S},l} \tag{5.169}$$

by simply restricting functors defined on l' to the subgroupid l. In this way, one obtains a *projective family* $(\mathcal{A}_{\mathcal{S},l}, p_{ll'})_{l,l' \in \mathcal{L}}$ (see Def. B.11 (ii)) to which one can associate the corresponding *projective limit*

$$\overline{\mathcal{A}}_{\mathcal{S}} := \varprojlim \mathcal{A}_{\mathcal{S},l} := \{(H_l)_{l\in\mathcal{L}} \in \prod_{l\in\mathcal{L}} \mathcal{A}_{\mathcal{S},l} |\, p_{ll'}(H_{l'}) = H_l \,\forall l \leq l'\} \tag{5.170}$$

which itself naturally inherits a topology via the *Tychonoff topology*. As in the classical bosonic theory, one can then prove that via restriction of functors this in fact yields a bijection

$$\mathrm{Hom}_{\mathbf{Cat}}(\mathbf{P}(\Sigma)^{\mathrm{op}}, \mathbf{G}(\mathcal{S})) \xrightarrow{\sim} \overline{\mathcal{A}}_{\mathcal{S}},\ H \to (H|_l)_{l\in\mathcal{L}} \tag{5.171}$$

so that, in this sense, the set of generalized holonomies can also be equipped with a topology. For $\mathcal{S} = \mathbb{R}$, we obtain the topological space

$$\overline{\mathcal{A}}_{\mathbb{R}} := \varprojlim \mathcal{A}_{\mathbb{R},l} \tag{5.172}$$

which can be identified with the subset of generalized bosonic connections with values in G. Again, by construction, the topology on $\overline{\mathcal{A}}_{\mathcal{S}}$ turns out to be coarsest topology such that the projection

$$\overline{\mathcal{A}}_{\mathcal{S}} \to \overline{\mathcal{A}}_{\mathbb{R}} \tag{5.173}$$

is continuous. If furthermore G (and thus $\mathcal{G}$) is compact, it follows from (5.166) for $\mathcal{S} = \mathbb{R}$ that $\mathcal{A}_{\mathbb{R},l}$ is also compact and Hausdorff for any $l \in \mathcal{L}$. Therefore, by the properties of the Tychnonoff toplogy, this implies that the projective limit $\overline{\mathcal{A}}_{\mathbb{R}}$ is compact and Hausdorff. But, due to (5.173), this then finally shows that $\overline{\mathcal{A}}_{\mathcal{S}}$ for any Grassmann algebra $\mathcal{S}$ defines a compact topological space which is projectively Hausdorff.

Remark 5.5.2. It is interesting to note that, since the topological spaces $\overline{\mathcal{A}}_{\mathcal{S}}$ transform covariantly under change of the parametrizing supermanifold $\mathcal{S}$, they naturally induce a functor

$$\overline{\mathcal{A}}: \mathbf{Gr} \to \mathbf{Top},\ \mathcal{S} \mapsto \overline{\mathcal{A}}_{\mathcal{S}} \tag{5.174}$$

Moreover, in case $\mathcal{G}$ is compact, it follows that the body $\overline{\mathcal{A}}(\mathbb{R}) = \overline{\mathcal{A}}_{\mathbb{R}}$ in particular defines a Hausdorff space. Thus, $\overline{\mathcal{A}}$ carries a structure which is intriguingly similar to a Molotkov-Sachse supermanifold (see Remark 2.2.13).

Using the identification (5.166), for any $l \equiv l(\gamma) \in \mathcal{L}$, let us introduce a set of smooth functions on $\mathcal{A}_{\mathcal{S},l}$ denoted by $\mathrm{Cyl}^{\infty}(\mathcal{A}_{\mathcal{S},l})$ such that

$$\mathrm{Cyl}^{\infty}(\mathcal{A}_{\mathcal{S},l}) \cong H^{\infty}(\mathcal{G}(\mathcal{S})^{|E(\gamma)|}, \mathbb{C}) \cong H^{\infty}(\mathcal{G}(\mathcal{S}), \mathbb{C})^{\hat{\otimes}_{\pi}|E(\gamma)|} \tag{5.175}$$

where $H^{\infty}(\mathcal{G}(\mathcal{S}), \mathbb{C}) := H^{\infty}(\mathcal{G}(\mathcal{S})) \otimes \mathbb{C}$. Then, for any $l, l' \in \mathcal{L}$ with $l \leq l'$, the pullback of the projection (5.169) induces a map $p_{ll'}^{*}: \mathrm{Cyl}^{\infty}(\mathcal{A}_{\mathcal{S},l}) \to \mathrm{Cyl}^{\infty}(\mathcal{A}_{\mathcal{S},l'})$. Thus, this in turn induces an *inductive family* $(\mathrm{Cyl}^{\infty}(\mathcal{A}_{\mathcal{S},l}), p_{ll'}^{*})_{l,l' \in \mathcal{L}}$ (see Def. B.11 (i)) to which we can associate the corresponding *inductive limit*

$$\mathrm{Cyl}^{\infty}(\overline{\mathcal{A}}_{\mathcal{S}}) := \varinjlim \mathrm{Cyl}^{\infty}(\mathcal{A}_{\mathcal{S},l}) := \coprod_{l \in \mathcal{L}} \mathrm{Cyl}^{\infty}(\mathcal{A}_{\mathcal{S},l}) \big/_{\sim} \tag{5.176}$$

which we will call the space of *cylindrical functions* on $\overline{\mathcal{A}}_{\mathcal{S}}$. In (5.176), for two functions $f_l \in \mathrm{Cyl}^{\infty}(\mathcal{A}_{\mathcal{S},l})$ and $f_{l'} \in \mathrm{Cyl}^{\infty}(\mathcal{A}_{\mathcal{S},l'})$, the equivalence relation is defined via $f_l \sim f_{l'}$ iff there exists $l, l' \leq l''$ such that $p_{ll''}^{*} f_l = p_{l'l''}^{*} f_{l'}$.

So far, we have focused on the choice of a particular superpoint $\mathcal{S}$ as a parametrizing supermanifold. According to the general discussion in Section 2.2, in particular Eq. (2.32), it follows that if the odd dimension of $\mathcal{S}$ is suitably large enough, that is, larger than the odd dimension of the underlying gauge supergroup $\mathcal{S}$, then the function sheaf $H^{\infty}(\mathcal{G}(\mathcal{S}))$ on the corresponding $\mathcal{S}$-point $\mathcal{G}(\mathcal{S})$ is isomorphic to the function sheaf on $\mathcal{G}$. Hence, here and in what follows, we will always implicitly assume that the dimension of the parametrizing supermanifold $\mathcal{S}$ is bounded by the odd dimension of $\mathcal{G}$. From (5.175), it then follows that the space of cylindrical functions corresponding to two different parametrizing supermanifolds satisfying this bound will always be isomorphic.

Finally, let us turn to the dual dynamical variables given by the super electric field $\mathcal{E}$. Since it defines a 2-form, one can smear it over two dimensional surfaces embedded in

Σ. Hence, let $S \subset \Sigma$ be a two-dimensional orientable submanifold which, in addition, we assume to be semianalytic and $n: S \to \mathfrak{g}$ be a $\mathfrak{g}$-valued smearing function[7] defined on the surface S. Then, we can integrate the super electric field over S yielding the Grassmann-valued quantity

$$\mathcal{E}_n(S) := \int_S \langle n, \mathcal{E} \rangle \tag{5.177}$$

which w.r.t. a local coordinate neighborhood $\phi: \mathbb{R}^3 \supset U \to \phi(U) \subset \Sigma$ of Σ adapted to S such that, for sake of simplicity, $S \subset \phi(U)$, explicitly takes the form

$$\begin{aligned}\mathcal{E}_n(S) = \int_U \phi^* \langle n, \mathcal{E} \rangle &= \int_U \frac{1}{2} n^{\underline{A}} \mathscr{S}_{\underline{BA}} \mathcal{E}^{\underline{B}}_{ab}\, \mathrm{d}\phi^a \wedge \mathrm{d}\phi^b \\ &= \int_U \mathrm{d}^2 u\, \frac{1}{2} n^{\underline{A}} \mathcal{E}^c_{\underline{A}} \epsilon_{cab} \partial_{u^1} \phi^a \partial_{u^2} \phi^b \end{aligned} \tag{5.178}$$

Via the graded Poisson bracket, the smeared quantities $\mathcal{E}_n(S)$ induce derivations $X(S): \mathrm{Cyl}^\infty(\overline{\mathcal{A}}_{\mathcal{S}}) \to \mathrm{Cyl}^\infty(\overline{\mathcal{A}}_{\mathcal{S}})$ on the space of cylindrical functions which we will call *super electric fluxes*. To find an explicit form of these fluxes, let us compute their action on holonomies corresponding to a smooth connection along certain edges e. In this case, we have

$$X_n(S)(h_e[\mathcal{A}]) := \{\mathcal{E}_n(S), h_e[\mathcal{A}]\} \tag{5.179}$$

In the following, by splitting edges appropriately, let us assume that e is adapted to the surface S in the sense that e intersects the surface only at a single point and such that e starts at S, i.e., $e \cap S = b(e)$. Then, performing a specific regularization scheme similar as in classical theory (see for instance [18]), it follows from the equivalent form (2.261) of the parallel transport map given in Example 2.7.16, the graded Poisson relation (5.158) as well as the fact that the super connection 1-form $\mathcal{A}$ is even that

$$X_n(S)(h_e[\mathcal{A}]) = \frac{g}{4} \epsilon(e, S) n^{\underline{A}}(b(e)) T_{\underline{A}} h_e[\mathcal{A}] \tag{5.180}$$

where $\epsilon(e, S) = +1, -1, 0$ if $\mathrm{vol}_\Sigma(v_1, v_2, \dot{e})$ is positive, negative or vanishing, respectively, at $b(e) \in S$ for any positive oriented basis (v_1, v_2) of $T_{b(e)}S$ with vol_Σ the volume form

[7] One may wonder whether one could allow for a more general class of $\mathcal{S}$-parametrized $\mathrm{Lie}(\mathcal{G})$-valued smearing function $n: \mathcal{S} \times S \to \mathrm{Lie}(\mathcal{G})$ defined on S. In fact, also in this case one could define flux operators via the Poisson bracket. However, by the general formula (5.185) to be derived below, this implies that these operators no longer preserve $\mathrm{Cyl}^\infty(\overline{\mathcal{A}}_{\mathcal{S}})$ but needs to be replaced by $\mathrm{Cyl}^\infty(\overline{\mathcal{A}}_{\mathcal{S}}) \otimes H^\infty(\mathcal{S})$. In the quantum theory to be discussed in Sec. 5.5.3, this would then require the choice of an additional inner product on $H^\infty(\mathcal{S})$ so that expectation values become real quantities. However, in the following, we would like to avoid this additional subtlety.

on Σ. From this, we can immediately read off the action of the super electric fluxes on the coordinate functions $x^i{}_j : \mathcal{G} \rightarrow \Lambda$ as defined in Example 2.3.14 which is given by

$$X_n(S)(x^i{}_j(h_e[\mathcal{A}]^+)) = \frac{g}{4}\epsilon(e,S)n^{\underline{A}}(b(e))x^i{}_j\left(T_{\underline{A}}h_e[\mathcal{A}]\right) \tag{5.181}$$

where, using (2.63), the right-hand side can be written in the form

$$\begin{aligned} x^i{}_j\left(T_{\underline{A}}h_e[\mathcal{A}]\right) =& x^i{}_k(T_{\underline{A}})\mathfrak{C}^{|e_i|+|e_k|}(x^k{}_j(h_e[\mathcal{A}])) \\ =&(-1)^{(|e_i|+|e_k|)(|e_k|+|e_j|)}(T_{\underline{A}})^i{}_k x^k{}_j(h_e[\mathcal{A}]) \end{aligned} \tag{5.182}$$

where we used that $x^i{}_k(T_{\underline{A}}) = (T_{\underline{A}})^i{}_k$ as $T_{\underline{A}}$ has purely real resp. complex coordinates. As it turns out, this can re-expressed in a very intriguing form. To see this, let $R_{\underline{A}} = T_{\underline{A}}\otimes\mathbb{1}\circ\mu^*$ and $L_{\underline{A}} = \mathbb{1}\otimes T_{\underline{A}}\circ\mu^*$ be the right- and left-invariant vector field, respectively, generated by $T_{\underline{A}}$. Then, using identity (2.66), its action on the coordinate functions $x^i{}_j$ yields

$$\begin{aligned} R_{\underline{A}}x^i{}_j(h_e[\mathcal{A}]) =&(-1)^{(|e_i|+|e_k|)(|e_k|+|e_j|)}(T_{\underline{A}}\otimes\mathbb{1})(x^i{}_k\otimes x^k{}_j)(h_e[\mathcal{A}]) \\ =&(-1)^{(|e_i|+|e_k|)(|e_k|+|e_j|)}(T_{\underline{A}})^i{}_k x^k{}_j(h_e[\mathcal{A}]) \end{aligned} \tag{5.183}$$

Comparing with (5.181), we thus conclude

$$X_n(S)(x^i{}_j(h_e[\mathcal{A}])) = \frac{g}{4}\epsilon(e,S)n^{\underline{A}}(b(e))R_{\underline{A}}x^i{}_j(h_e[\mathcal{A}]) \tag{5.184}$$

that is, the action of the super electric fluxes on coordinate functions is given by the action of right-invariant vector fields. Since, the coordinate functions generate the whole function sheaf $H^\infty(\mathcal{G})$, this immediately implies that (5.184) equally holds for any $f \in H^\infty(\mathcal{G})$. Thus, more generally, if $f_l \in [f_l] \in \mathrm{Cyl}^\infty(\overline{\mathcal{A}}_{\mathcal{S}})$ is a representative of an equivalence class of smooth cylindrical functions associated to a subgroupoid $l \equiv l(\gamma)$ generated by a graph γ adapted to S, this yields

$$X_n(S)(f_l) = \frac{g}{4}\sum_{e\in E(\gamma),\, e\cap S\neq\emptyset}\epsilon(e,S)n^{\underline{A}}(b(e))R^e_{\underline{A}}f_l \tag{5.185}$$

where we used the identification $\mathcal{A}_{\mathcal{S},l} \cong \mathcal{G}(\mathcal{S})^{|E(\gamma)|}$ such that $R^e_{\underline{A}}$ denotes the right-invariant vector field generated by $T_{\underline{A}}$ acting on the copy of $\mathcal{G}(\mathcal{S})$ labeled by e [18]. Similar as in in the classical non-supersymmetric case, one can prove that the action of the super electric flux via (5.185) is indeed well-defined, i.e., independent of the choice of a representative $f_l \in [f_l]$ defined w.r.t. a graph adapted to S.

From identity (5.185), we deduce the remarkable property that, for a given graph γ in Σ generating the subgroupoid $l \equiv l(\gamma)$, super electric fluxes corresponding to surfaces S which intersect the underlying edges only at their endpoints leave the space $\mathrm{Cyl}^{\infty}(\mathcal{A}_{\mathcal{S},l})$ of cylindrical functions on $\mathcal{A}_{\mathcal{S},l}$ invariant. Hence, if $V^{\infty}(\mathcal{A}_{\mathcal{S},l})$ denotes the superalgebra generated by the graded commutator of all such super electric flux operators, on this graph, we can define the *graded holonomy-flux algebra* $\mathfrak{A}^{\mathrm{gHF}}_{\mathcal{S},l}$ via

$$\mathfrak{A}^{\mathrm{gHF}}_{\mathcal{S},l} := \mathrm{Cyl}^{\infty}(\mathcal{A}_{\mathcal{S},l}) \rtimes V^{\infty}(\mathcal{A}_{\mathcal{S},l}) \tag{5.186}$$

which, in particular, forms a (infinite-dimensional) super Lie algebra according to

$$[(f,X),(f',Y)] := (X(f') - (-1)^{|Y||f|}Y(f), [X,Y]) \tag{5.187}$$

for any $f, f' \in \mathrm{Cyl}^{\infty}(\mathcal{A}_{\mathcal{S},l})$ and fluxes $X, Y \in V^{\infty}(\mathcal{A}_{\mathcal{S},l})$. Here, the parity $|X|$ of a homogeneous super electric flux X is defined in the usual way regarding it as a homogeneous derivation on $\mathrm{Cyl}^{\infty}(\mathcal{A}_{\mathcal{S},l})$. Thus, for instance, in case $X \equiv \mathcal{X}_n(S)$ with $\mathcal{X}_n(S)$ defined via (5.185), one has $|X| = |n|$ with $|n| =: i \in \mathbb{Z}_2$ the parity of the homogeneous smearing function $n : S \to \mathfrak{g}_i$. More generally, considering all possible graphs, we define the *graded holonomy-flux algebra* $\mathfrak{A}^{\mathrm{gHF}}_{\mathcal{S}}$ via

$$\mathfrak{A}^{\mathrm{gHF}}_{\mathcal{S}} := \mathrm{Cyl}^{\infty}(\overline{\mathcal{A}}_{\mathcal{S}}) \rtimes V^{\infty}(\overline{\mathcal{A}}_{\mathcal{S}}) \tag{5.188}$$

with $V^{\infty}(\overline{\mathcal{A}}_{\mathcal{S}})$ the superalgebra generated by the graded commutator of super electric fluxes on the inductive limit $\mathrm{Cyl}^{\infty}(\overline{\mathcal{A}}_{\mathcal{S}})$. Again, it follows that (5.188) forms a super Lie algebra. In context of the non-supersymmetric theory, this algebra is usually considered for quantization.

So far, we have not imposed any *-relation on the superalgebras (5.188) resp. (5.187) so that they form *-algebras. This is, however, necessary in order to identify physical quantities in terms of self-adjoint elements. In context of chiral supergravity, this may be achieved by re-expressing the reality conditions, such as (5.156) in the case $\mathcal{N} = 1$, in terms of holonomy and flux variables. However, since the reality conditions are highly non-linear, even in the purely bosonic theory, this turns out to be a nontrivial task. Hence, in the following, we do not want to comment further on the specific form of the reality conditions and the *-relations imposed on the graded holonomy-flux algebra. We will come back to this question in Chapter 6, specifically Sec. 6.6.1, in context of a symmetry reduced model where we will be able to find an explicit form of the *-relation induced by the reality conditions.

5.5.2. Haar measures on super Lie groups and super Hilbert spaces

Before we turn next towards quantization of the theory à la LQG to be discussed in the subsequent section, let us review some important facts concerning invariant measures on super Lie groups and the notion of super Hilbert spaces. In the literature, there exist various different approaches in this direction in both the algebraic or concrete approach to supermanifold theory. For instance, a first systematic approach in constructing invariant measures has been developed in the pure algebraic setting a long time ago in [197, 198]. There, one uses the fact that the function sheaf on a super Lie group, by its very definition, naturally inherits the structure of a *super Hopf algebra*. In this way, invariant measures have been found, e.g., for the series $\mathrm{OSp}(1|2n)$ and $\mathrm{OSp}(2|2n)$. In particular, it has been shown that these measures are indeed unique and both left- and right-invariant. Nevertheless, this construction turns out to be very abstract making it hardly accessible for concrete computations.

In [112], a more concrete approach has been developed for the construction of invariant measures for various real super Lie groups using their equivalent description in terms of *super Harish Chandra pairs* (see Theorem 2.3.9). In this way, Haar measures have been derived for the super unitary groups $\mathrm{U}(m|n)$, the orthosymplectic supergroups $\mathrm{OSp}(m|n)$ as well as their compact real forms given by the *unitary orthosymplectic supergroups* $\mathrm{UOSp}(m|n) := \mathrm{OSp}(m|n) \cap \mathrm{U}(m|n)$ (see also Section 5.5.3 below). However, in the algebraic setting, the correspondence between super Lie groups and super Harish-Chandra pairs, unfortunately, remains rather implicit. In the H^∞ category (or more generally for $\mathcal{A}$-manifolds), in [97], a concrete algorithm was given constructing invariant Haar measures for arbitrary (real) super Lie groups. This is based on the existence of a concrete relation between a super Lie group $\mathcal{G}$ and the data $(\mathbf{B}(\mathcal{G}), \mathfrak{g})$ provided by the diffeomorphism (2.45). This helps significantly in finding concrete formulae for invariant measures which can directly be used for computations.

Before, we state the basic definition of invariant measures, let us note that many super Lie groups that we are interested in are in fact non-compact. Hence, we need to integrate over a particular subclass of smooth functions on a super Lie group $\mathcal{G}$ that have support only on compact subsets. In this context, a function f on $\mathcal{G}$ is called of compact support iff $f|_{\mathbf{B}(\mathcal{G})}$ vanishes outside a compact subset of $\mathbf{B}(\mathcal{G})$. In the literature, invariant measures are usually defined involving the pullback of the group multiplication. However, the pullback, in general, does not preserve compact subsets. Hence, in the following, let us consider instead the smooth maps Θ_L and Θ_R on $\mathcal{G} \times \mathcal{G}$ defined as

$$\begin{aligned}\Theta_L := (\mathrm{id} \times \mu) \circ (d \times \mathrm{id}) :\ \mathcal{G} \times \mathcal{G} &\rightarrow \mathcal{G} \times \mathcal{G} \\ (g, h) &\mapsto (g, \mu(g, h))\end{aligned} \tag{5.189}$$

and

$$\Theta_R := (\mu \times \mathrm{id}) \circ (\mathrm{id} \times d) : \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G} \times \mathcal{G} \qquad (5.190)$$
$$(g, h) \mapsto (\mu(g, h), h)$$

respectively. The reason for choosing these maps is based on the fact they are *proper maps*, i.e., preimages of compact sets in $\mathcal{G} \times \mathcal{G}$ are compact in $\mathcal{G} \times \mathcal{G}$ (note that this is true for the group multiplication only in case $\mathcal{G}$ is compact). If X^L and X^R denote left- and right-invariant vector fields on $\mathcal{G}$, respectively, it is then easy to see that

$$\begin{aligned} \mathbb{1} \otimes X^L \circ \Theta_R^* &= \Theta_R^* \circ (X^L \otimes \mathbb{1} + \mathbb{1} \otimes X^L) \\ X^R \otimes \mathbb{1} \circ \Theta_L^* &= \Theta_L^* \circ (X^R \otimes \mathbb{1} + \mathbb{1} \otimes X^R) \end{aligned} \qquad (5.191)$$

We are now ready to define invariant measures on super Lie groups.

Definition 5.5.3. Let $\mathcal{G}$ be a H^∞ super Lie group and $H_c^\infty(\mathcal{G}, \mathbb{C}) := H_c^\infty(\mathcal{G}) \otimes \mathbb{C}$ denote the $\mathbb{C}$-vector space of smooth functions on $\mathcal{G}$ with compact support and values in $\Lambda^{\mathbb{C}}$. A $\mathbb{C}$-linear map $\int_{\mathcal{G}} : H_c^\infty(\mathcal{G}, \mathbb{C}) \rightarrow \mathbb{C}$ is called

(i) a *left-invariant integral* or *left-invariant Haar measure* of $\mathcal{G}$ if

$$\mathbb{1} \otimes \int_{\mathcal{G}} \circ \Theta_L^* = \mathbb{1} \otimes \int_{\mathcal{G}} \qquad (5.192)$$

(ii) a *right-invariant integral* or *right-invariant Haar measure* of $\mathcal{G}$ if

$$\int_{\mathcal{G}} \otimes \mathbb{1} \circ \Theta_R^* = \int_{\mathcal{G}} \otimes \mathbb{1} \qquad (5.193)$$

Moreover, $\int_{\mathcal{G}}$ is called simply *invariant integral* or *invariant (Haar) measure* if it is both left- and right-invariant.

The following proposition gives an equivalent characterization of invariant Haar measures using the correspondence between super Lie groups and super Harish-Chandra pairs. It provides a generalization of Theorem 6 stated in [199] to the case of non-compact super Lie groups.

Proposition 5.5.4. *Let $\mathcal{G}$ be a H^∞ super Lie group and $\int_{\mathcal{G}} : H_c^\infty(\mathcal{G}, \mathbb{C}) \rightarrow \mathbb{C}$ a $\mathbb{C}$-linear map. Then $\int_{\mathcal{G}}$ is a left- resp. right-invariant integral if and only if*

$$\mathrm{ev}_g \otimes \int_{\mathcal{G}} \circ \mu^* = \int_{\mathcal{G}} \quad \textit{resp.} \quad \int_{\mathcal{G}} \otimes \mathrm{ev}_g \circ \mu^* = \int_{\mathcal{G}} \qquad (5.194)$$

for all body points $g \in \mathbf{B}(\mathcal{G})$ *as well as*

$$\int_{\mathcal{G}} \circ X^R = 0 \quad \textit{resp.} \quad \int_{\mathcal{G}} \circ X^L = 0 \tag{5.195}$$

for all smooth right- resp. left-invariant vector fields $X^{R/L} \in \Gamma(T\mathcal{G})$.

Proof. Let us prove this proposition using Lemma 2.6.8. To do so, in the following, let us focus on the left-invariant case, the proof for the right-invariant case being similar.

Hence, suppose that $\int_{\mathcal{G}}$ is a left-invariant integral. Then, applying the evaluation morphism ev_g for any body point $g \in \mathbf{B}(\mathcal{G})$ on both sides of (5.192) immediately yields (5.194). On the other hand, using (5.191), the action of a smooth right-invariant vector field $X^R \in \mathfrak{g}^R$ on (5.192) gives

$$\begin{aligned} X^R \otimes \int_{\mathcal{G}} \circ \Theta_L^* &= \mathbb{1} \otimes \int_{\mathcal{G}} \circ X^R \otimes \mathbb{1} \circ \Theta_L^* = \mathbb{1} \otimes \int_{\mathcal{G}} \circ \Theta_L^* \circ (X^R \otimes \mathbb{1} + \mathbb{1} \otimes X^R) \\ &= X^R \otimes \int_{\mathcal{G}} + \mathbb{1} \otimes \left(\int_{\mathcal{G}} \circ X^R \right) \end{aligned} \tag{5.196}$$

But, by assumption, we have

$$X^R \otimes \int_{\mathcal{G}} \circ \Theta_L^* = X^R \otimes \int_{\mathcal{G}} \tag{5.197}$$

Thus, if we compare the right-hand sides of (5.196) and (5.197), this immediately gives (5.195).

Conversely, assume that (5.194) and (5.195) are satisfied for a $\mathbb{C}$-linear map $\int_{\mathcal{G}} : H_c^\infty(\mathcal{G}, \mathbb{C}) \to \mathbb{C}$. To show that this yields (5.192), let us define for any elementary tensors $\chi \otimes f \in H^\infty(\mathcal{G})_c \hat{\otimes}_\pi H_c^\infty(\mathcal{G})$ two smooth functions $F_{\chi\otimes f}, G_{\chi\otimes f} \in H_c^\infty(\mathcal{G})$ via $F_{\chi\otimes f} := \mathbb{1} \otimes \int_{\mathcal{G}} \Theta_L^*(\chi \otimes f)$ and $G_{\chi\otimes f} := \chi \int_{\mathcal{G}} f$. By (5.194), it follows that for any body point $g \in \mathbf{B}(\mathcal{G})$

$$\begin{aligned} F_{\chi\otimes f}(g) = \mathrm{ev}_g F_{\chi\otimes f} = \mathrm{ev}_g \otimes \int_{\mathcal{G}} \Theta_L^*(\chi \otimes f) &= \chi(g) \int_{\mathcal{G}} \mu_g^*(f) \\ &= \chi(g) \int_{\mathcal{G}} f = G_{\chi\otimes f}(g) \end{aligned} \tag{5.198}$$

for any $\chi, f \in H_c^\infty(\mathcal{G})$, where $\mu_g := \mu(g, \cdot)$ which is smooth as g has real coordinates. Hence, as both $F_{\chi\otimes f}$ and $G_{\chi\otimes f}$ are smooth for any $\chi, f \in H_c^\infty(\mathcal{G})$, the claim follows by Lemma 2.6.8 if we can show that $XF_{\chi\otimes f}|_{\mathbf{B}(\mathcal{G})}=XG_{\chi\otimes f}|_{\mathbf{B}(\mathcal{G})}$ for all $X\in\mathcal{U}(\mathfrak{g}^R)$. For $1\in\mathcal{U}(\mathfrak{g}^R)$ this follows from (5.198) above. So, let $X^R\in\mathfrak{g}^R$ for which we compute

$$\begin{aligned}X^RF_{\chi\otimes f}=X^R\otimes\int_{\mathcal{G}}\Theta_L^*(\chi\otimes f)&=\mathbb{1}\otimes\int_{\mathcal{G}}\Theta_L^*(X^R\chi\otimes f+(-1)^{|X||\chi|}\chi\otimes X^Rf)\\&=F_{X^R\chi\otimes f}+(-1)^{|X||\chi|}F_{\chi\otimes X^Rf}\end{aligned}\tag{5.199}$$

Hence, for $g\in\mathbf{B}(\mathcal{G})$ a body point, this implies, together with (5.198) above,

$$\begin{aligned}X^RF_{\chi\otimes f}(g)&=F_{X^R\chi\otimes f}(g)+(-1)^{|X||\chi|}F_{\chi\otimes X^Rf}(g)\\&=G_{X^R\chi\otimes f}(g)+(-1)^{|X||\chi|}G_{\chi\otimes X^Rf}(g)=(X^R\chi)(g)\int_{\mathcal{G}}f\end{aligned}\tag{5.200}$$

where, in the last step, condition (ii) was used. Thus, $X^RF_{\chi\otimes f}|_{\mathbf{B}(\mathcal{G})}=X^RG_{\chi\otimes f}|_{\mathbf{B}(\mathcal{G})}$. Following the same steps as before, by induction, it is then easy to see that $XF_{\chi\otimes f}(g)=XG_{\chi\otimes f}(g)$ for any body point $g\in\mathbf{B}(\mathcal{G})$ and $X\in\mathcal{U}(\mathfrak{g}^R)$. Hence, by Lemma 2.6.8, we have $F_{\chi\otimes f}=G_{\chi\otimes f}\,\forall\chi\otimes f\in H_c^\infty(\mathcal{G}\times\mathcal{G})$ so that $\int_{\mathcal{G}}$ indeed defines a left-invariant integral. □

Remark 5.5.5. Identifying a super Lie group $\mathcal{G}$ with its corresponding super Harish-Chandra pair $(G,\mathfrak{g})$ with $G:=\mathbf{B}(\mathcal{G})$. one can define the *left* and *right regular representation* ρ_L and ρ_R of $\mathcal{G}$ in terms of pairs of morphisms $\rho_{L/R}\equiv(|\rho_{L/R}|,\rho_{L/R*})$ with $|\rho_{L/R}|:\,G\rightarrow\mathrm{Aut}(H_c^\infty(\mathcal{G},\mathbb{C}))$ defined as

$$(|\rho_L|(g)f)(h):=f(g^{-1}h)\text{ and }(|\rho_R|(g)f)(h):=f(hg)\tag{5.201}$$

$\forall g\in G$ as well as super Lie algebra morphisms $\rho_{L/R*}:\,\mathfrak{g}\rightarrow\underline{\mathrm{End}}_R(H_c^\infty(\mathcal{G},\mathbb{C}))$ given by

$$\rho_{L*}(X):=-X^R\text{ and }\rho_{R*}(X):=X^L\tag{5.202}$$

$\forall X\in\mathfrak{g}$. Hence, by Prop. 5.5.4, it follows that $\int_{\mathcal{G}}:\,H^\infty(\mathcal{G},\mathbb{C})\rightarrow\mathbb{C}$ defines a left- resp. right-invariant integral iff it is invariant under the left resp. right regular representation of $\mathcal{G}$.

Integrals on supermanifolds can be formulated in terms of *Berezinian densities* (see [200–202] for more details). A Berezinian density on a supermanfold $\mathcal{M}$ is defined as a smooth section $\Gamma_c(\mathrm{Ber}(\mathcal{M}))$ with compact support of the *Berezin line bundle*[8] $\mathrm{Ber}(\mathcal{M}):=\mathscr{F}(\mathcal{M})\times_{\mathrm{Ber}}\Lambda^{\mathbb{C}}$ which is a bundle associated to the frame bundle $\mathscr{F}(\mathcal{M})$ via

[8] In the case of an ordinary C^∞ manifold M of dimension n, these can be identified with sections of the exterior bundle $\bigwedge^nT^*M$, i.e., top-degree forms $\Omega^n(M)$ on M. This, however, is no longer true in the case of supermanifolds as there a graded notion of a top-degree form turns out not to exist.

the one-dimensional dual representation $\mathrm{GL}(m|n,\Lambda)\ni A\mapsto \mathrm{Ber}(A)^{-1}\in\mathrm{Aut}(\Lambda^{\mathbb{C}})$ where $\dim\mathcal{M}=(m,n)$. For a super Lie group $\mathcal{G}$, it then follows that a Berezinian density $\nu\in\Gamma_c(\mathrm{Ber}(\mathcal{G}))$ induces a left-invariant integral iff its trivial extension $\hat{\nu}$ on $\mathcal{G}\times\mathcal{G}$ satisfies [109]

$$\Theta_L^*\hat{\nu}=\hat{\nu} \tag{5.203}$$

We will also refer to a Berezinian density satisfiying (5.203) as a *left-invariant (Haar) measure* on $\mathcal{G}$. To construct such an invariant measure note that, for any super Lie group $\mathcal{G}$, the tangent bundle $T\mathcal{G}$ is always trivializable with a global frame $\mathfrak{p}\in\Gamma(\mathscr{F}(\mathcal{G}))$ induced by a homogeneous basis (e_i,f_j) of left-invariant vector fields $e_i,f_j\in\mathfrak{g}$ on $\mathcal{G}$, $i=1,\ldots,m$ and $j=1,\ldots,n$ with $\dim\mathcal{G}=(m,n)$. In particular, this yields a global section $\nu_{\mathfrak{g}}:=[\mathfrak{p},1]\in\Gamma(\mathrm{Ber}(\mathcal{G}))$ of the associated Berezin line bundle which, by construction, automatically defines a left-invariant Haar measure. With respect to local coordinates (x,ξ) on $\mathcal{G}$, we can write $\mathfrak{p}=(\partial_{x^i},\partial_{\xi_j})\cdot X$ where X denotes the matrix representation of the left-invariant vector fields w.r.t. the induced coordinate derivatives. Thus, in local coordinates, the density $\nu_{\mathfrak{g}}$ then takes the form

$$\nu_{\mathfrak{g}}=[(\partial_{x^i},\partial_{\xi_j})\cdot X,1]=[(\partial_{x^i},\partial_{\xi_j}),\mathrm{Ber}(X)^{-1}] \tag{5.204}$$

To find an explicit expression for X, one can then use the equivalent description of $\mathcal{G}$ in terms of the corresponding super Harish-Chandra pair $(G,\mathfrak{g})$ via identification (2.45). This requires an intense use of the Baker-Campbell-Hausdorff formula and thus involves various powers of the (right) adjoint representation $\mathrm{ad}_R:\mathrm{Lie}(\mathcal{G})\to\mathrm{End}_R(\mathrm{Lie}(\mathcal{G}))$, $X\mapsto[X,\cdot]$. As shown in [109], the matrix representation then takes the form

$$X(x,\xi)=\begin{pmatrix}C(x) & C(x)\cdot H(\xi)\\ A(\xi) & B(\xi)\end{pmatrix} \tag{5.205}$$

where $C(x)$ as well as $H(\xi)$, $A(\xi)$ and $B(\xi)$ are submatrices depending purely on even and odd coordinates, respectively, and which are defined via

$$\begin{aligned}&\mathrm{ad}_R(v)(e_i)=:f_jA(\xi)^j{}_i,\quad b_+(\mathrm{ad}_R(v))f_j=:f_kB(\xi)^k{}_j\\ \text{and}\quad & h(\mathrm{ad}_R(v))f_j=e_iH(\xi)^i{}_j\end{aligned} \tag{5.206}$$

with $v:=f_j\xi^j\in(\mathfrak{g}_1\otimes\Lambda)_0$ and real functions

$$b_+(t):=\frac{t\cosh(t)}{\sinh(t)}=1+\frac{1}{3}t^2-\frac{1}{45}t^4+\ldots,\quad h(t):=\frac{e^t-1}{e^t+1}=\frac{1}{2}t-\frac{1}{24}t^3+\ldots \tag{5.207}$$

Moreover, $C(x)$ is determined via the matrix representation of the even left-invariant vector fields e_i via

$$e_i|_{(g,v)} = \partial_{x_j} C^j{}_i(x) + \partial_{\xi_k} A^k{}_i(\xi) \tag{5.208}$$

In particular, when restricting to the body, C can be identified with the matrix representation of the left-invariant vector fields on G. The left-invariant integral on $\mathbf{S}(\mathfrak{g}_1, G)$ for smooth functions $f \in H_c^\infty(\mathbf{S}(\mathfrak{g}_1, G)) \cong C_c^\infty(G) \otimes \bigwedge \mathfrak{g}_1^*$ then takes the form [109]

$$\begin{aligned}\int_{\mathcal{G}} f \nu_{\mathfrak{g}} &= \int \mathrm{d}^n x \int_B \mathrm{d}^n \xi \, f(x,\xi) \mathrm{Ber}(X)^{-1}(x,\xi) \\ &= \int \mathrm{d}^n x \, C(x)^{-1} \int_B \mathrm{d}^n \xi \, \frac{\det B(\xi)}{\det(\mathbb{1} - H(\xi) B(\xi)^{-1} A(\xi))} f(x,\xi) \\ &=: \int_G \mathrm{d}\mu_H(g) \int_B \mathrm{d}^n \xi \, \Delta(\xi) f(g,\xi)\end{aligned} \tag{5.209}$$

where μ_H is the induced left-invariant Haar measure on G and $\int_B$ denotes the usual *Berezin integral* on $\bigwedge \mathfrak{g}_1^*$. Hence, the derivation of the invariant integral on $\mathcal{G}$ boils down to the choice of an invariant Haar measure on the body G as well as the derivation of the density $\Delta(\xi)$ in the Berezin integral which, according to (5.206) and (5.207), only involves the computation of the matrix representation of the adjoint representation on the super Lie algebra $\mathfrak{g}$.

Example 5.5.6 (Invariant Haar measure on $\mathrm{OSp}(1|2)_{\mathbb{C}}$)**.** Let us apply the algorithm outlined above to compute the invariant Haar measure on the complex orthosymplectic group $\mathrm{OSp}(1|2)_{\mathbb{C}}$. In case of the real orthosymplectic group, this has been done explicitly already in [109] and in the algebraic category in [112]. Using the explicit matrix representation of the generators (T_i^+, Q_A) as resulting from (2.92) and (2.93), by Theorem 2.45, it follows that we can identify $\mathrm{OSp}(1|2)_{\mathbb{C}}$ with the split super Lie group $\mathbf{S}(\mathfrak{osp}(1|2)_1, \mathrm{SL}(2,\mathbb{C}))$ according to

$$\Phi(g,v) = g \exp(\xi^A Q_A) = g \cdot \begin{pmatrix} 1 - i\xi^+\xi^- & -i\xi^- & -i\xi^+ \\ -\xi^+ & 1 + \frac{i}{2}\xi^+\xi^- & 0 \\ -\xi^- & 0 & 1 + \frac{i}{2}\xi^+\xi^- \end{pmatrix} \tag{5.210}$$

for $g \in \mathrm{SL}(2,\mathbb{C})$ and $\xi^A \in \Lambda_1^{\mathbb{C}}$ for $A \in \{\pm\}$. For the derivation of the Haar measure, let us introduce a new real homogeneous basis $(J_3, J_\pm, V_\pm)$ of $\mathfrak{osp}(1|2)_{\mathbb{C}}$ defining

$$J_\pm := -i(T_1^+ \pm iT_2^+), \quad J_3 := iT_3^+, \quad V_\pm := \pm\frac{\sqrt{L}}{2}(i-1)Q_\pm \tag{5.211}$$

From (5.82)-(5.85), it follows that the commutators among the even generators satisfy

$$[J_3, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = 2J_3 \tag{5.212}$$

which are the standard commutation relations of $\mathfrak{sl}(2,\mathbb{C})$. For the remaining commutators, we find

$$[J_3, V_\pm] = \pm\frac{1}{2}V_\pm, \quad [J_\mp, V_\pm] = V_\mp, \quad [J_\pm, V_\pm] = 0 \tag{5.213}$$

$$[V_\pm, V_\pm] = \pm\frac{1}{2}J_\pm, \quad [V_+, V_-] = -\frac{1}{2}J_3 \tag{5.214}$$

These are precisely the graded commutation relations of *real* $\mathrm{OSp}(1|2)$ as stated for instance in [109]. Using the commutation relations above, it then follows immediately that the matrix representation of $\mathrm{ad}_R(\theta^A V_A)$ is given by

$$\mathrm{ad}_R(\theta^A V_A) = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{2}\theta^- & -\frac{1}{2}\theta^+ \\ 0 & 0 & 0 & \frac{1}{2}\theta^+ & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2}\theta^- \\ \frac{1}{2}\theta^+ & \theta^- & 0 & 0 & 0 \\ -\frac{1}{2}\theta^- & 0 & \theta^+ & 0 & 0 \end{pmatrix} \tag{5.215}$$

Hence, it follows from (5.206) as well as (5.208), using $\mathrm{ad}(\theta^A V_A)^n = 0$ for $n \geq 3$,

$$\begin{pmatrix} \mathbb{1} & H \\ A & B \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & -\frac{1}{4}\theta^- & -\frac{1}{4}\theta^+ \\ 0 & 1 & 0 & \frac{1}{4}\theta^+ & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{4}\theta^- \\ \frac{1}{2}\theta^+ & \theta^- & 0 & 1-\frac{1}{4}\theta^+\theta^- & 0 \\ -\frac{1}{2}\theta^- & 0 & \theta^+ & 0 & 1-\frac{1}{4}\theta^+\theta^- \end{pmatrix} \tag{5.216}$$

Actually, for the derivation of (5.209), it has been implicitly assumed that the super Lie group defines a real supermanifold. Hence, we need to view $\mathrm{OSp}(1|2)_\mathbb{C}$ as a real super Lie group. A homogeneous basis of the *realification* of $\mathfrak{g} := \mathfrak{osp}(1|2)_\mathbb{C}$ (resp. $\mathrm{Lie}(\mathcal{G}) := \mathfrak{g}\otimes\Lambda^\mathbb{C}$) is then given by $(J_3, J_\pm, V_\pm) \cup (iJ_3, iJ_\pm, iV_\pm)$. Let $\mathcal{R}: \underline{\mathrm{End}}_R(\mathrm{Lie}(\mathcal{G})) \to \underline{\mathrm{End}}_R(\mathrm{Lie}(\mathcal{G})_\mathbb{R})$ be the morphism which identifies any $X \in \underline{\mathrm{End}}_R(\mathrm{Lie}(\mathcal{G}))$ with the corresponding real endomorphism $\mathcal{R}(A)$ on the realification $\mathrm{Lie}(\mathcal{G})_\mathbb{R}$. For the density $\Delta \equiv \Delta(\xi, \bar{\xi}, \eta, \bar{\eta})$ in the Berezin integral, we then compute

$$\Delta = \frac{\det(\mathcal{R}(B))}{\det(\mathcal{R}(\mathbb{1} - H\cdot B^{-1}\cdot A))} = \left(1+\frac{1}{4}\theta^+\theta^-\right)\left(1+\frac{1}{4}\bar{\theta}^+\bar{\theta}^-\right) \tag{5.217}$$

Hence, it follows that the invariant integral for any smooth function $f \in V := C_c^\infty(\mathrm{SL}(2,\mathbb{C}),\mathbb{C}) \otimes \bigwedge[\theta^A, \bar{\theta}^{A'}]$ is given by

$$\int_{\mathrm{OSp}(1|2)_{\mathbb{C}}} f\nu_{\mathfrak{g}} = \int_{\mathrm{SL}(2,\mathbb{C})} \mathrm{d}\mu_H(g,\bar{g}) \int_B \mathrm{d}\mu(\theta,\bar{\theta}) f(g,\bar{g},\theta,\bar{\theta}) \tag{5.218}$$

with $\mathrm{d}\mu_H$ an invariant Haar measure on $\mathrm{SL}(2,\mathbb{C})$ and

$$\mathrm{d}\mu(\theta,\bar{\theta}) := \mathrm{d}\theta^A \mathrm{d}\bar{\theta}^{A'} \left(1 + \frac{1}{4}\theta^+\theta^-\right)\left(1 + \frac{1}{4}\bar{\theta}^+\bar{\theta}^-\right) \tag{5.219}$$

Finally, let us introduce the notion of a super Hilbert space. As already mentioned in the introduction, there are indeed many different approaches in formulating such a notion. Since, in this work, we are mainly interested in applications to LQSG, we would like to consider super vector spaces V given by the space $V := H^\infty(\mathcal{G},\mathbb{C})$ of smooth functions on super Lie groups $\mathcal{G}$ equipped with the super scalar product $\mathscr{S}$ (see Def. 2.3.12 (ii)) induced by the invariant Haar measure $\int_{\mathcal{G}}$ on $\mathcal{G}$ such that

$$\mathscr{S}(f|g) := \int_{\mathcal{G}} \bar{f} g \tag{5.220}$$

Unfortunately, it turns out that the induced super scalar product as defined via (5.220) will be in general indefinite yielding an indefinite inner product space $(V,\mathscr{S})$. Hence, at least a priori, the super scalar product cannot be used in order to extend V to a (super) Hilbert space. However, as shown in [109], one can always find an, not necessarily unique, endomorphism $J: V \to V$ such that

$$\langle\cdot|\cdot\rangle_J := \mathscr{S}(\cdot|J\cdot) \tag{5.221}$$

defines a positive definite inner product on V. Moreover, $\mathscr{S}$ turns out to be continuous w.r.t. the topology induced by $\langle\cdot|\cdot\rangle_J$ on V.

Definition 5.5.7. A *pre-super Hilbert space* is a triple $(\mathfrak{H},\mathscr{S},J)$ consisting of a super vector space $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_1$ together with a super scalar product $\mathscr{S}$ on $\mathfrak{H}$ as well as an endomorphism $J: \mathfrak{H} \to \mathfrak{H}$ such that the sesquilinear form $\langle\cdot|\cdot\rangle_J$ as defined via (5.221) yields a positive definite inner product on $\mathfrak{H}$, i.e., $(\mathfrak{H},\langle\cdot|\cdot\rangle_J)$ defines a ordinary pre-Hilbert space and $\mathscr{S}$ is continuous w.r.t. the topology induced by $\langle\cdot|\cdot\rangle_J$.

A pre-super Hilbert space $(\mathfrak{H},\mathscr{S},J)$ is called a *super Hilbert space* if $(\mathfrak{H},\langle\cdot|\cdot\rangle_J)$ defines a Hilbert space in the category of ordinary vector spaces, that is, if $\mathfrak{H}$, regarded as an ungraded vector space, is complete w.r.t. the topology induced by $\langle\cdot|\cdot\rangle_J$.

An interesting subclass of (pre-)super Hilbert spaces is provided by triples $(\mathfrak{H},\mathscr{S},J)$ where the endormorphism $J:\mathfrak{H}\rightarrow\mathfrak{H}$ satisfies additional properties yielding the notion of a super Krein space (for a definition of Krein spaces in the ungraded case see e.g. [203]).

Definition 5.5.8 (a reformulation of [204,205])**.** A (pre-)super Hilbert space $(\mathfrak{H},\mathscr{S},J)$ is called a *super Krein space* if the endomorphism $J:\mathfrak{H}\rightarrow\mathfrak{H}$ defines a *fundamental symmetry*, i.e., if it satisfies $J^4=\mathbb{1}$ as well as $\mathscr{S}(Jv|Jw)=\mathscr{S}(v|w)\;\forall v,w\in\mathfrak{H}$.

Following [205], given a super Krein space $(\mathfrak{H},\mathscr{S},J)$, the fundamental symmetry induces a decomposition of $\mathfrak{H}$ in the form

$$\mathfrak{H}=\mathfrak{H}_{[1]}\oplus\mathfrak{H}_{[i]}\oplus\mathfrak{H}_{[-1]}\oplus\mathfrak{H}_{[-i]}\tag{5.222}$$

with $\mathfrak{H}_{[i^k]}:=\ker(J-i^{-k}\mathbb{1})$ such that

$$\mathscr{S}(v|v)\in i^k\mathbb{R}_{\geq 0},\forall\text{homogeneous } v\in\mathfrak{H}_{[i^k]}\tag{5.223}$$

which we also refer to as a *super Krein decomposition*. Conversely, it is clear that any decomposition of the form (5.222) naturally induces a fundamental symmetry $J:\mathfrak{H}\rightarrow\mathfrak{H}$ via

$$J:=(P_{[1]}-iP_{[i]})-(P_{[-1]}-iP_{[-i]})\tag{5.224}$$

where $P_{[i^k]}:\mathfrak{H}\rightarrow\mathfrak{H}_{[i^k]}$ denotes the projection onto the subspace $\mathfrak{H}_{[i^k]}$. In the special case $\mathfrak{H}_{[-1]}=\{0\}=\mathfrak{H}_{[-i]}$, this leads back to original definition of super Hilbert spaces typically used in the literature [61,74,206]. The corresponding fundamental symmetry $J\equiv J_0$ then acquires the standard form [205]

$$J_0 v:=(-1)^{|v|}v,\;\forall\text{homogeneous } v\in\mathfrak{H}\tag{5.225}$$

Definition 5.5.9. A *(pre-)super Hilbert space* $(\mathfrak{H},\mathscr{S},J)$ is called *standard* if it is a super Krein space and if the fundamental symmetry J is of the standard form $J\equiv J_0$ (5.225).

Remark 5.5.10. By definition, it follows that any super vector space V equipped with a positive definite inner product $\langle\cdot|\cdot\rangle:V\times V\rightarrow\mathbb{C}$ such that $(V,\langle\cdot|\cdot\rangle)$ is an ordinary pre-Hilbert space naturally induces a corresponding standard pre-super Hilbert space $(V,\mathscr{S},J_0)$ with super scalar product $\mathscr{S}$ defined as $\mathscr{S}:=\langle\cdot|J_0^{-1}\cdot\rangle$. Hence, there exists a one-to-one correspondence between standard (pre-)super Hilbert spaces and ordinary (pre-)Hilbert spaces where the underlying vector space carries an additional $\mathbb{Z}_2$-grading.

Example 5.5.11. In the following, let us analyze the structure of inner product space $(V,\mathscr{S})$ with $V:=H^\infty(\mathrm{U}(1|1),\mathbb{C})$ the super vector space of smooth functions on super Lie group U(1|1) (see Example 2.3.16) equipped with a super scalar product $\mathscr{S}$ induced by the invariant Haar measure on U(1|1) defined via (5.220).

To this end, we first need to find an explicit form of the invariant Haar measure of U(1|1). In the algebraic setting, this has been discussed already in [112]. Here, we want to rederive it applying the concrete algorithm outlined above and originally developed in [109] using the super Harish-Chandra isomorphism (2.45). Hence, let $Y := \xi\Theta_1 + \eta\Theta_2$ with $\xi, \eta \in \Lambda_1$. Using (2.77), it follows that the adjoint representation ad_Y acquires the following matrix representation

$$\mathrm{ad}_Y = \begin{pmatrix} 0 & 0 & -2\xi & -2\eta \\ 0 & 0 & -2\xi & -2\eta \\ -\eta & \eta & 0 & 0 \\ \xi & -\xi & 0 & 0 \end{pmatrix} \tag{5.226}$$

which yields

$$\mathrm{ad}_Y^2 = \begin{pmatrix} 4\xi\eta & -4\xi\eta & 0 & 0 \\ 4\xi\eta & -4\xi\eta & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.227}$$

as well as $\mathrm{ad}_Y^n = 0$ for $n \geq 3$. Thus, from this, we deduce

$$\begin{pmatrix} \mathbb{1}_2 & H \\ A & B \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\xi & -\eta \\ 0 & 1 & -\xi & -\eta \\ -\eta & \eta & 1 & 0 \\ \xi & \xi & 0 & 1 \end{pmatrix} \tag{5.228}$$

so that the density takes the form

$$\Delta(\xi, \eta) = \frac{\det B}{\det(\mathbb{1}_2 - HB^{-1}A)} \equiv 1 \tag{5.229}$$

Thus, to summarize, up to a constant rescaling, the invariant integral $\int_{\mathrm{U}(1|1)} : V \to \mathbb{C}$ on the super Lie group U(1|1) is given by

$$\int_{\mathrm{U}(1|1)} f := \int_{\mathrm{U}(1)\times\mathrm{U}(1)} \mathrm{d}\mu_H \int i\mathrm{d}\psi\mathrm{d}\bar{\psi}\, f \tag{5.230}$$

for any $f \in V$. Moreover, using (5.230), we can introduce a corresponding super scalar product $\mathscr{S} : V \times V \to \mathbb{C}$ on V via

$$\mathscr{S}(f|g) = \int_{\mathrm{U}(1|1)} \bar{f} g \tag{5.231}$$

As it turns out, the tuple $(V, \mathscr{S})$ can be equipped with the structure of a pre-super Hilbert space such that, after completion, the resulting super Hilbert space has the structure of a super Krein space. In fact, defining $v_\pm := 1 \mp i\psi\bar{\psi}$ as well as $w_+ := \bar{\psi}$ and $w_- := \psi$ it follows that $\mathscr{S}(v_\pm|v_\pm) = \pm 2$ and $\mathscr{S}(w_\pm|w_\pm) = \pm i$ with all remaining combinations being zero. Since, the constant unit function $\mathbb{1}$ on $\mathrm{U}(1|1)$ can be written in the form $\mathbb{1} = \frac{1}{2}(v_+ + v_-)$, this implies

$$\int_{\mathrm{U}(1|1)} \mathbb{1} = 0 \tag{5.232}$$

that is, the constant unit function is not normalizable. Let us then define the super vector spaces

$$V_\pm := C^\infty(\mathrm{U}(1)^2, \mathbb{C})v_\pm \cup C^\infty(\mathrm{U}(1)^2, \mathbb{C})w_\pm \tag{5.233}$$

This yields the super Krein decomposition

$$V = V_+ \oplus V_- \tag{5.234}$$

which is precisely of the form (5.222). Thus, following the standard procedure, we can complete $(V, \mathscr{S})$ to a super Hilbert space. To do so, let $P^i_\pm : V \to (V_\pm)_i$ for $i \in \mathbb{Z}_2$ denote the projections onto the homogeneous subspaces $(V_\pm)_i$ and $J : V \to V$ the fundamental symmetry defined by

$$J := (P^0_+ - iP^1_+) - (P^0_- - iP^1_-) \tag{5.235}$$

Then, this induces a inner product $\langle\cdot,\cdot\rangle_J : V \times V \to \mathbb{C}$ on V setting

$$\langle f|g\rangle_J := \mathscr{S}(f|Jg), \quad \forall f, g \in V \tag{5.236}$$

which, by construction, is positive definite. In fact, for a general smooth function $f \in V = C^\infty(\mathrm{U}(1)^2, \mathbb{C}) \otimes \Lambda_2$ of the form $f = f_\emptyset + f_1\psi + f_2\bar{\psi} + f_{12}\psi\bar{\psi}$ one can decompose $f = \frac{1}{2}(f_\emptyset + if_{12})v_+ + \frac{1}{2}(f_\emptyset - if_{12})v_- + f_1 w_+ + f_2 w_-$. According to Definition (5.235), it then immediately follows that

$$\langle f|g\rangle_J = \sum_{\underline{I}} \langle\langle f_{\underline{I}}|g_{\underline{I}}\rangle\rangle \tag{5.237}$$

where $\langle\langle\cdot|\cdot\rangle\rangle$ denotes the ordinary positive definite inner product on $C^{\infty}(\mathrm{U}(1)^2,\mathbb{C})$ induced by the unique normalized invariant Haar measure μ_H on $\mathrm{U}(1)\times\mathrm{U}(1)$. Thus, this immediately implies that the completion of V w.r.t. the induced topology takes the form

$$\mathfrak{H}:=\overline{V}^{\|\cdot\|}=L^2(\mathrm{U}(1)^2,\mathrm{d}\mu_H)\otimes\Lambda_2^{\mathbb{C}} \tag{5.238}$$

Moreover, by construction, it follows that $\mathscr{S}$ is continuous so that it can be extended uniquely to $\mathfrak{H}$ yielding the super Hilbert space $(\mathfrak{H},\langle\cdot|\cdot\rangle_J,\mathscr{S})$ which, in particular, carries the structure of a super Krein space.

5.5.3. Loop quantization

5.5.3.1. General Scheme

With these preparations, let us finally turn towards the quantization of the theory adapting techniques from standard LQG. Again, in order to keep our considerations as general as possible, before focusing on the loop quantization of chiral supergravity, let us first discuss the general quantization scheme and suppose we are given a graded holonomy-flux algebra $\mathfrak{A}^{\mathrm{gHF}}_{\mathcal{S}}$ as constructed in Section 5.5.1 based on a general gauge supergroup $\mathcal{G}$. At this point, let us, however, emphasize that the final picture will remain rather incomplete. This is mainly due to the additional difficulties arising in the supersymmetric setting such as the indefiniteness of inner products induced by invariant integrals as well as their non-normalizability which is crucial in order to implement cylindrical consistency. Finally, in context of chiral supergravity to be discussed below, one needs to deal with the non-compactness of the gauge group and solve reality conditions which, even in case of the bosonic self-dual theory, still remains an open problem. Hence, in the following, we will only sketch the main idea behind the construction pointing out various difficulties arising along the way and discuss their possible resolutions.

For the quantization of the theory, we are looking for a faithful (grading preserving) morphism of superalgebras

$$\pi_{\mathcal{S}}:\,\mathfrak{A}^{\mathrm{gHF}}_{\mathcal{S}}\to\mathrm{Op}(\mathcal{D}_{\mathcal{S}},\mathfrak{H}_{\mathcal{S}}) \tag{5.239}$$

mapping from $\mathfrak{A}^{\mathrm{gHF}}_{\mathcal{S}}$ to the space $\mathrm{Op}(\mathcal{D}_{\mathcal{S}},\mathfrak{H}_{\mathcal{S}})$ of (un)bounded operators $T:\,\mathcal{D}_{\mathcal{S}}\subseteq\mathrm{dom}(T)\to\mathfrak{H}_{\mathcal{S}}$ on a super Hilbert space $\mathfrak{H}_{\mathcal{S}}\equiv(\mathfrak{H}_{\mathcal{S}},\mathscr{S},J)$ with domain $\mathrm{dom}(T)$ containing a dense graded subspace $\mathcal{D}_{\mathcal{S}}\subset\mathfrak{H}_{\mathcal{S}}$. Moreover, we require the representation to transform covariantly under change of parametrization $\mathcal{S}$. In fact, as we will see below, provided that $\mathcal{S}$ is large enough, the resulting quantum theory is completely independent of the choice of parametrization.

Following the standard procedure in the purely bosonic theory, for the quantization, we choose a *Ashtekar-Lewandowski-type representation* of $\mathfrak{A}^{\mathrm{gHF}}_{\mathcal{S}}$. To this end, for any subgroupoid $l \equiv l(\gamma) \in \mathcal{L}$ generated by a graph $\gamma \subset \Sigma$, let us consider the super vector space $V_{\mathcal{S},l} := \mathrm{Cyl}^{\infty}(\mathcal{A}_{\mathcal{S},l})$. On $V_{\mathcal{S},l}$ we then define the representation of $\mathfrak{A}^{\mathrm{gHF}}_{\mathcal{S},l}$ via

$$\pi_{\mathcal{S}}(f_l) := \widehat{f_l}, \quad \pi_{\mathcal{S}}(X_n(S)) := i\hbar X_n(S) \tag{5.240}$$

where $\widehat{f_l}$ acts as a multiplication operator by f_l. Using the identification (5.175), we can define a super scalar product $\mathscr{S}_l$ on $V_{\mathcal{S},l}$ choosing an invariant Haar measure $\int_{\mathcal{G}(\mathcal{S})}$ on the H^{∞} super Lie group $\mathcal{G}(\mathcal{S})$ which, via factorization, induces an invariant Haar measure on $\mathcal{G}(\mathcal{S})^{|E(\gamma)|}$ yielding

$$\mathscr{S}_l(g_l|f_l) := \int_{\mathcal{G}(\mathcal{S})^{|E(\gamma)|}} \bar{f}_l g_l \tag{5.241}$$

for any $f_l, g_l \in V_{\mathcal{S},l}$. As a next step, one then needs to check that the super scalar product as defined via (5.241) is indeed cylindrically consistent, i.e., independent of the choice of a graph γ. More precisely, given a graph γ' with $l(\gamma) \leq l(\gamma') =: l'$, one needs to check that

$$\mathscr{S}_{l'}(p^*_{ll'}f_l|p^*_{ll'}g_l) = \mathscr{S}_l(f_l|g_l) \tag{5.242}$$

$\forall f_l, g_l \in V_{\mathcal{S},l}$. To do so, it suffices to consider the following three distinct cases: (1) γ' arises from γ by adding a new edge not contained in γ, (2) an edge in γ can be written as a composition of two distinct edges in γ', (3) γ' arises from γ by inversion of the orientation of an edge.

As in the non-supersymmetric setting, consistency under these three cases turns out to be equivalent to requiring that the family of super scalar products $(\mathscr{S}_l)_l$ as defined via (5.241) are both left- and right-invariant and that the constant unit function $\mathbb{1} : \mathcal{G}(\mathcal{S}) \to \mathbb{C},\ g \to 1$ is contained in $H^{\infty}(\mathcal{G}(\mathcal{S}), \mathbb{C})$ and is normalized, i.e.,

$$\int_{\mathcal{G}(\mathcal{S})} \mathbb{1} = 1 \tag{5.243}$$

In fact, it follows that the last condition imposes severe restrictions on the super Lie group. On the one hand, as in the bosonic case, existence of the unit function implies that the super Lie group $\mathcal{G}(\mathcal{S})$ is compact which, by the DeWitt topology, is equivalent to requiring that the body $\mathcal{G}(\mathbb{R})$ is compact. While, in the bosonic theory, this condition is sufficient in order to ensure normalizability of the unit function, this, however, turns out to be no longer the case in the supersymmetric setting. We have indeed encountered an explicit example in Example 5.5.11 (see Eq. (5.232)). As shown in [198], requiring $\int_{\mathcal{G}(\mathcal{S})} \mathbb{1}$ to be different from zero implies that the finite-dimensional representations of $\mathcal{G}(\mathcal{S})$ are completely reducible which, for super Lie groups, is satisfied only in rare special cases. Nevertheless, there exists interesting candidates where all these conditions are satisfied such as, for instance, the *unitary orthosymplectic group* $\mathrm{UOSp}(1|2)$ defined as the intersection

$$\mathrm{UOSp}(1|2) = \mathrm{OSp}(1|2) \cap \mathrm{U}(1|2) \tag{5.244}$$

In particular, it defines a compact subgroup of $\mathrm{OSp}(1|2)$ and thus may be even of interest in context of chiral supergravity.

Hence, in the following, let us assume that the underlying super Lie group $\mathcal{G}(\mathcal{S})$ indeed satisfies all these conditions. It then follows that the family of super scalar products $(\mathscr{S}_l)_l$ can be lifted consistently to a super scalar product $\mathscr{S}$ on the inductive limit

$$V_{\mathcal{S}} := \varinjlim V_{\mathcal{S},l} \tag{5.245}$$

Thus, under these assumptions, we end up with an indefinite inner product space $(V_{\mathcal{S}}, \mathscr{S})$. For the quantum theory, we then finally need to extend $V_{\mathcal{S}}$ to a Hilbert space. To this end, for any $l \in \mathcal{L}$, one can apply the general results of [109] and construct an endomorphism $J_l : V_{\mathcal{S},l} \to V_{\mathcal{S},l}$ such that the induced inner product

$$\langle\cdot|\cdot\rangle_{J_l} := \mathscr{S}(\cdot|J_l\cdot) \tag{5.246}$$

is positive definite which can then be used in order to complete to $V_{\mathcal{S},l}$ to a Hilbert space

$$\mathfrak{H}_{\mathcal{S},l} := \overline{V}_{\mathcal{S}}^{\|\cdot\|_{J_l}} \tag{5.247}$$

and, in particular, $\mathscr{S}_l$ turns out to be continuous w.r.t. the induced topology. Thus, in this way, we indeed obtain a family of super Hilbert spaces $(\mathfrak{H}_{\mathcal{S},l}, \mathscr{S}_l, J_l)$. As a final and crucial step in the quantization scheme, we need to ask the question whether the family of inner products (5.246) can be lifted consistently to well-defined inner product $\langle\cdot|\cdot\rangle_J := \mathscr{S}(\cdot|J\cdot)$ induced by an endomorphism $J : \mathfrak{H}_{\mathcal{S}} \to \mathfrak{H}_{\mathcal{S}}$ on the inductive limit

$$\mathfrak{H}_{\mathcal{S}} := \varinjlim \mathfrak{H}_{\mathcal{S},l} \tag{5.248}$$

This, in general, turns out to be a difficult question to answer as the choice of endomorphisms J_l, a priori, is by far not unique. A sufficient criterion for this to be possible would be the existence of a choice of a family $(J_l)_l$ which commute with the pullbacks of the graph projections, i.e.,

$$p^*_{ll'} J_l = J_{l'} p^*_{ll'} \tag{5.249}$$

since then the cylindrical consistency of the induced inner product is indeed trivially satisfied. However, this condition seems to be rather restrictive. For an example of such a family of endomorphisms satisfying this condition see Section 5.5.4 below. Alternatively, one may try to find a suitable basis of the indefinite inner product spaces $(V_{\mathcal{S},l}, \mathscr{S}_l)$, such as a super spin network type basis (see Section 5.5.3.3), which can then be used in order to construct a canonical endomorphism J_l for any $l \in \mathcal{L}$. One then finally needs to check, whether the induced positive inner products are indeed cylindrically consistent. For the construction of such a basis, one needs a Peter-Weyl-type theorem for super Lie groups stating that matrix coefficients of irreducible finite-dimensional representations of the underlying super Lie group provide, at least in a specific sense, an orthonormal basis of the indefinite inner product spaces $(V_{\mathcal{S},l}, \mathscr{S}_l)$. Results in this direction have been studied for instance in [207] in context of $\mathrm{UOSp}(1|2)$. Interestingly, the basis constructed there seems to have almost the same properties as in the $\mathrm{SU}(2)$ case and therefore seem to be suitable in order to find such an endomorphism J.

Remark 5.5.12. Ultimately, one may follow a completely different strategy applying the methods of [208]. There, introducing a particular kind of topology on H^∞ super Lie groups different from the DeWitt topology, the authors of [208] are able to identify the underlying supermanifold with a corresponding ordinary real manifold. This approach then allows one to construct invariant integrals which, in particular, turn out to be positive definite. However, this construction depends crucially on the choice of the underlying Grassmann algebra. From the functor of points perspective, this means that one needs to make a particular choice of the parametrizing supermanifold $\mathcal{S}$. A suggestive candidate would be to choose the infinite-dimensional Grassmann algebra Λ_∞ as it arises in terms of an inductive limit of the family of finite-dimensional Grassmann algebras.

5.5.3.2. Application: Chiral supergravity

Having sketched the general strategy to canonically quantize field theories with gauge symmetry given by a supergroup in the framework LQG, we finally want to apply it in the context of chiral supergravity. However, there, one runs into several problems as the underlying gauge supergroups given by the (complex) orthosymplectic supergroups $\mathrm{OSp}(\mathcal{N}|2)_\mathbb{C}$ are non-compact. Moreover, one also needs to deal with the consistent implementation of the reality conditions as one is still dealing with a complex theory. An interesting and elegant possibility to solve the reality conditions would be to be adapt the ideas of [209] and to introduce some kind of a Wick rotation on the phase space so that the complex theory arises from an Euclidean counterpart corresponding to a real Barbero-Immirzi parameter $\beta \in \{\pm 1\}$ via a Wick transformation. However, the resulting gauge group given by the real orthosymplectic supergroup $\mathrm{OSp}(\mathcal{N}|2)$ is still non-compact. Adapting ideas in context of the purely bosonic theory (see for instance [210–214] and references therein for recent advances in this direction), this may be solved by going over instead to their corresponding compact form given by unitary orthosymplectic group $\mathrm{UOSp}(\mathcal{N}|2) = \mathrm{OSp}(\mathcal{N}|2) \cap \mathrm{U}(\mathcal{N}|2)$. As already mentioned in the previous section, for the special case $\mathcal{N}=1$, besides compactness, this group has very useful properties such as the existence of an invariant Haar measure with respect to which, in particular, the unit function is normalizable which is important in context of loop quantization in order to implement cylindrical consistency. Nevertheless, this last property turns out to be no longer satisfied in case of extended supersymmetry corresponding to higher $\mathcal{N}>1$.

Anyway, since, we want to explicitly include the extended case $\mathcal{N}=2$, in what follows, we will not discuss the question of how to impose cylindrical consistency and instead work on a single graph γ in Σ. As argued in [213], we may therefore assume that the graph under consideration is at least suitably fine enough to resolve the topology of Σ. Let $\mathfrak{A}^{\mathrm{cLQSG}}_{\mathcal{S},\gamma}:=\mathfrak{A}^{\mathrm{gHF}}_{\mathcal{S},l(\gamma)}$ denote the graded holonomy-flux algebra w.r.t. the graph γ and underlying gauge group given by $\mathrm{OSp}(\mathcal{N}|2)_{\mathbb{C}}$. The quantization of the theory then corresponds to a representation

$$\pi_{\mathcal{S},\gamma}:\,\mathfrak{A}^{\mathrm{cLQSG}}_{\mathcal{S},\gamma}\rightarrow\mathrm{Op}(\mathcal{D}_{\mathcal{S},\gamma},\mathfrak{H}^{\mathrm{cLQSG}}_{\mathcal{S},\gamma})\tag{5.250}$$

of $\mathfrak{A}^{\mathrm{cLQSG}}_{\mathcal{S},\gamma}$ on a super Hilbert space $\mathfrak{H}^{\mathrm{cLQSG}}_{\mathcal{S},\gamma}$. To construct this representation, as pre-Hilbert space, we consider the super vector space $V_{\mathcal{S},\gamma}:=\mathrm{Cyl}^{\infty}(\mathcal{A}_{\mathcal{S},l(\gamma)})$ which, according to (5.175), can be identified with

$$H^{\infty}(\mathcal{G}(\mathcal{S})^{|E(\gamma)|},\mathbb{C})\cong H^{\infty}(\mathcal{G}(\mathcal{S}),\mathbb{C})^{\otimes_{\pi}|E(\gamma)|}\tag{5.251}$$

or a suitable subspace thereof, if one restricts, for instance, to holomorphic functions as naturally arising from super holonomies induced by the super Ashtekar connection (see discussion below). The representation of the algebra on this vector space is then defined via (5.240). For the super scalar product $\mathscr{S}$ on $V_{\mathcal{S},\gamma}$ we make the ansatz

$$\mathscr{S}(f|g):=\int_{\mathcal{G}(\mathbb{R})}\mathrm{d}\mu_{H}(g,\bar{g})\int_{B}\mathrm{d}\mu(\theta,\bar{\theta})\rho(g,\bar{g},\theta,\bar{\theta})\,\bar{f}\,g\tag{5.252}$$

with $\mathrm{d}\mu_H$ the invariant Haar measure on the body $\mathcal{G}(\mathbb{R})=\mathrm{SL}(2,\mathbb{C})$ and the measure $\mathrm{d}\mu(\theta,\bar{\theta})$ in the Berezin integral which, in case $\mathcal{N}=1$, is given by expression (5.219). Here, $\rho\equiv\rho(g,\bar{g},\theta,\bar{\theta})$ denotes an additional density which has been chosen in order to deal with the non-compactness of the group. In this context, note that, generically, the matrix coefficients of the super holonomies (5.162), as part of the underlying algebra and thus of the resulting state space in the quantum theory, are functions of the form

$$f=\sum_{\underline{I}}f_{\underline{I}}\psi^{\underline{I}}=f_{\emptyset}+f_{A}\psi^{A}+\frac{1}{2}f_{+-}\psi_{A}\psi^{A}\tag{5.253}$$

with $f_{\underline{I}}$ Grassmann extensions of holomorphic functions on $\mathrm{SL}(2,\mathbb{C})$. But, by Liouville's theorem, if required to be nontrivial, general functions of this kind cannot be of compact support. This is of course problematic in context of integration theory and thus for the proper definition of the inner product. Hence, either one excludes holomorphic functions already in the definition of the classical algebra or the measure on $\mathrm{SL}(2,\mathbb{C})$ is changed appropriately by introducing a density ρ which is of compact support. We will study the last possibility in the following chapter in the context of symmetry reduced models. There, the measure turns out to be in fact distributional. In particular, we will see that this will also enable us to exactly implement the reality conditions (5.156). In context of the full theory with ordinary self-dual variables, this idea also been studied in [215] considering a specific subclass of the full reality conditions where it was found that the resulting density imposes a gauge-fixing onto the compact subgroup $\mathrm{SU}(2)$ of $\mathrm{SL}(2,\mathbb{C})$. Maybe, these results can be extended to the supersymmetric setting possibly involving the unitary orthosymplectic group $\mathrm{UOSp}(1|2)$ which, as explained above, has many interesting properties quite analogous to the purely bosonic theory.

Ultimately, for the construction of the super Hilbert space, we have to choose an endomorphism $J: V_{\mathcal{S},\gamma}\to V_{\mathcal{S},\gamma}$ such that the induced inner product $\langle\cdot|\cdot\rangle_J:=\mathscr{S}(\cdot|J\cdot)$ is positive definite. The choice of such an endomorphism is, of course, not unique but strongly restricted by the correct implementation of the reality conditions (see Section (5.5.4) or (6.6.2) in context of symmetry reduced models). Using this inner product, we can then complete $V_{\mathcal{S},\gamma}$ to a Hilbert space $\mathfrak{H}^{\mathrm{cLQSG}}_{\mathcal{S},\gamma}$ so that finally end up with the super Hilbert space $(\mathfrak{H}^{\mathrm{cLQSG}}_{\mathcal{S},\gamma},\mathscr{S},J)$.

5.5.3.3. Super spin networks and the super area operator

Having constructed the Hilbert space representation of the classical algebra underlying canonical chiral supergravity, we next have to select the proper subspace of *physical states* consisting of states in $\mathfrak{H}^{\mathrm{cLQSG}}_{\mathcal{S},\gamma}$ that are annihilated by the operators corresponding to the constraints of the canonical classical theory given by the *super Gauss*, the *right SUSY* and the *diffeomorphism constraint*, respectively, as well as the *Hamiltonian constraint*. In the following, let us only focus on the super Gauss constraint. The other constraints will be discussed in the context of the reduced theory in Section 6.4 below. In fact, the particular advantage of the loop representation as studied in this section is the rather straightforward implementation of the super Gauss constraint (5.118) (resp. (5.136) for $\mathcal{N}=2$) in the quantum theory implying invariance of physical states under local gauge transformations.

To this end, recall that the super Gauss constraint in the bulk theory, modulo boundary terms, takes the form

$$\mathscr{G}[\alpha]=-\frac{i}{\kappa}\int_{\Sigma}\langle D^{(\mathcal{A}^{+})}\alpha\wedge\mathcal{E}\rangle=-\frac{i}{\kappa}\int_{\Sigma}\mathrm{d}^{3}x\,(D_{a}^{(\mathcal{A}^{+})}\alpha^{\underline{A}})\mathcal{E}^{a}_{\underline{A}}=:-\frac{i}{\kappa}\mathcal{E}(D^{(\mathcal{A}^{+})}\alpha) \tag{5.254}$$

and thus resembles the definition of a super electric flux but smeared over a three-dimensional region instead of two-dimensional surfaces. Thus, for the corresponding operator in the quantum theory, we may set

$$\widehat{\mathscr{G}}[\alpha]:=\frac{\hbar}{\kappa}\{\mathcal{E}(D^{(\mathcal{A}^{+})}\alpha),\cdot\} \tag{5.255}$$

Following the same steps as in the purely bosonic theory, it is then immediate to see that the super Gauss constraint operator takes the form

$$\widehat{\mathscr{G}}[\alpha]=\frac{i\hbar}{2}\sum_{v\in V(\gamma)}\alpha^{\underline{A}}(v)\left[\sum_{e\in E(\gamma),b(e)=v}R^{e}_{\underline{A}}-\sum_{e\in E(\gamma),f(e)=v}L^{e}_{\underline{A}}\right] \tag{5.256}$$

In particular, due to its structure, the super Gauss constraint has a well-defined action on the super Hilbert space as it takes the standard form of a super electric flux operator and maps cylindrical functions to cylindrical functions. For a generic state $f\in\mathfrak{H}^{\mathrm{cLQSG}}_{\mathcal{S},\gamma}$ to be *physical*, this then yields the condition

$$\widehat{\mathscr{G}}[\alpha]f=0 \tag{5.257}$$

that is, according to (5.256) and Remark (5.5.5), physical states have to be invariant under both the left- and right-regular representation of $\mathcal{G}(\mathcal{S})$.

In standard loop quantum gravity, one considers a typical class of states satisfying the constraint equation (5.257) given by the so-called *spin network states*. These states are constructed via contraction of matrix coefficients of irreducible representations of the underlying gauge group. In fact, in case that the bosonic group is compact, it follows that these type of states form an orthonormal basis of the entire Hilbert space. This follows from the well-known *Peter-Weyl theorem* which is valid for compact bosonic groups. However, in case of general super Lie groups such a general statement, unfortunately, is not known. Nevertheless, the finite-dimensional irreducible representations of the orthosymplectic series $\mathrm{OSp}(\mathcal{N}|2)$ for $\mathcal{N}=1,2$ are well-known and have been intensively studied in the literature (for a summary see Appendix D). In particular, for the case $\mathcal{N}=1$, it follows that these type of representations form a subcategory which is closed under tensor product. In fact, the same applies to the extended case $\mathcal{N}=2$ if one restricts to a particular subclass of the so-called *typical representations* (see [216, 217] for

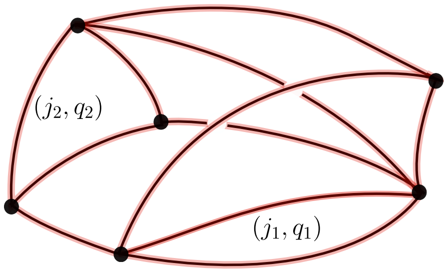


Figure 5.: A pictorial representation of a super spin network state for the case $\mathcal{N} = 2$. In contrast to the standard quantization scheme of fermions in LQG, the fermions are smeared over the one-dimensional edges of the graph which are labeled by admissible representations $\pi^{(j,q)} \in \mathcal{P}_{\text{adm}}$ of the supergroup $\mathrm{OSp}(2|2)$ with isospin j and charge quantum number q.

more details). We may call these kind of representations *admissible* in what follows. Thus, restricting to admissible representations, one can construct invariant states which lead to the notion of *super spin network states*. For $\mathcal{N} = 1$, these have been studied for instance in the References [84, 85]. For the rest of this section, let us briefly sketch the main idea behind their construction explicitly including the extended case $\mathcal{N} = 2$.

To this end, let $\mathcal{P}_{\text{adm}}$ denote the set of equivalence classes of admissible finite-dimensional irreducible representations of $\mathrm{OSp}(\mathcal{N}|2)$ with $\mathcal{N} = 1, 2$. For any subset $\vec{\pi} := \{\pi_e\}_{e\in E(\gamma)} \subset \mathcal{P}_{\text{adm}}$, we then define the cylindrical function $T_{\gamma,\vec{\pi},\vec{m},\vec{n}} \in \mathrm{Cyl}^{\infty}(\mathcal{A}_{\mathcal{S},\gamma})$ via

$$T_{\gamma,\vec{\pi},\vec{m},\vec{n}} := \prod_{e\in E(\gamma)} (\pi_e)^{m_e}{}_{n_e} \tag{5.258}$$

also called a *gauge-variant super spin network state* where, for any edge $e \in E(\gamma)$, $(\pi_e)^{m_e}{}_{n_e}$ denote certain matrix coefficients of the representation $\pi_e \in \mathcal{P}_{\text{adm}}$. By definition, it then follows from the general transformation law (5.163) of a super holonomy under local gauge transformations, that, at each vertex $v \in V(\gamma)$, the state (5.258) transforms under the following tensor product representation of $\mathcal{G}(\mathcal{S})$

$$\pi'_v := \left(\bigotimes_{e\in I(v)} \pi_e\right) \otimes \left(\bigotimes_{e\in F(v)} \pi_e^*\right) \tag{5.259}$$

where $\pi_e^* \in \mathcal{P}_{\text{adm}}$ denotes the right dual representation corresponding to π_e. Here, $I(v)$ and $F(v)$ are defined as subsets of $E(\gamma)$ consisting of all edges $e \in E(\gamma)$ which are beginning or ending at the vertex $v \in V(\gamma)$, respectively. Hence, in order to construct gauge-invariant states, at each vertex $v \in V(\gamma)$, we have to assume that the trivial representation π_0 appears in the decomposition of the product representation (5.259), i.e., $\pi_0 \in \pi'_v \, \forall v \in V(\gamma)$. For any $v \in V(\gamma)$, we can then choose an intertwiner I_v which contracted with the state (5.259) project onto the trivial representation at any vertex. As a consequence, the resulting state transforms trivially under local gauge transformations and thus indeed forms a gauge-invariant state which we call a *(gauge-invariant) super spin network state* (see Figure 5).

On the super Hilbert space $\mathfrak{H}^{\mathrm{cLQSG}}_{\mathcal{S},\gamma}$, one can introduce a gauge-invariant quantity in analogy to the area operator in ordinary LQG. More precisely, since the super electric field $\mathcal{E}$ defines a Lie($\mathcal{G}$)-valued 2-form, for any oriented (semianalytic) surface S embedded in Σ, one can define the *graded* or *super area* $\mathrm{gAr}(S)$ via

$$\mathrm{gAr}(S) := \int_S \|\mathcal{E}\| \tag{5.260}$$

where, generalizing the considerations in [45, 218, 219] in the context of the purely bosonic theory to the supersymmetric setting, the norm $\|\mathcal{E}\|$ is a 2-form on S defined as follows: Let $\iota_S : S \hookrightarrow \Sigma$ denote the embedding of the surface S in Σ. Since, $\iota_S^*\mathcal{E}$ defines a 2-form on S, it follows that there exists a unique Lie($\mathcal{G}$)-valued function $\mathcal{E}_S : \mathcal{S}\times S \to \mathrm{Lie}(\mathcal{G})$ such that $\iota_S^*\mathcal{E} = \mathcal{E}_S\,\mathrm{vol}_S$. The norm $\|\mathcal{E}\|$ is then given by

$$\|\mathcal{E}\| := \sqrt{\langle \mathcal{E}_S, \mathcal{E}_S\rangle} \tag{5.261}$$

For the special case $\mathcal{N} = 1$, it follows that the expression (5.260) coincides with the super area as considered in [85]. Note that, in case that the parametrizing supermanifold is trivial $\mathcal{S} = \{*\}$, the super area reduces, up to numerical factors, to the standard area of S in Riemannian geometry.

By definition, the quantity (5.260) solely depends on the super electric field which defines a phase space variable. Thus, we can implement it in the quantum theory. To do so, we first need to perform an appropriate regularization. Following [45], let us therefore assume that the surface S intersects the graph γ only in its vertices and is contained within a single coordinate neighborhood (U, ϕ_U) of Σ adapted to S. Furthermore, let $\mathcal{U}_\epsilon = \{U_i\}_i$ be a partition of U of fineness $\epsilon > 0$ such that S is covered by the $S_{U_i} := \phi_U(U_i)$. Then, for $\epsilon > 0$, we define

$$\mathrm{gAr}_\epsilon(S) := \sum_{V\in\mathcal{U}_\epsilon} \|\mathcal{E}(S_V)\| \equiv \sum_{V\in\mathcal{U}_\epsilon} \sqrt{\mathscr{T}^{\underline{AB}}\mathcal{X}_{\underline{B}}(S_V)\mathcal{X}_{\underline{A}}(S_V)} \tag{5.262}$$

where $\mathcal{X}_{\underline{A}}(S_V)$ denotes the super electric flux operator smeared over S_V with smearing function $n : S \to \mathfrak{g}$ satisfying $n^{\underline{B}} \equiv 1$ for $\underline{B} = \underline{A}$ and $n^{\underline{B}} = 0$ otherwise. In the limit

$\epsilon \to 0$, this then implies $\mathrm{gAr}(S) = \lim_{\epsilon\to 0}\mathrm{gAr}_{\epsilon}(S)$. Using this regularization, we can define the *super area operator* as follows

$$\widehat{\mathrm{gAr}}(S) = \lim_{\epsilon\to 0}\widehat{\mathrm{gAr}}_{\epsilon}(S), \qquad \widehat{\mathrm{gAr}}_{\epsilon}(S) = \sum_{V\in\mathcal{U}_{\epsilon}}\sqrt{\mathscr{T}^{\underline{AB}}\widehat{X}_{\underline{B}}(S_V)\widehat{X}_{\underline{A}}(S_V)} \tag{5.263}$$

Next, let us derive an explicit formula for its action on super spin network states. To this end, following again [45] in the context of purely bosonic theory, we compute

$$\begin{aligned}
\mathscr{T}^{\underline{AB}}\widehat{X}_{\underline{B}}(S_V)\widehat{X}_{\underline{A}}(S_V) &= \left(\frac{\hbar\kappa}{4}\right)^2\mathscr{T}^{\underline{AB}}\left(\sum_{e\cap S_V\neq\emptyset}\epsilon(e,S_V)R^e_{\underline{B}}\right)\left(\sum_{e\cap S_V\neq\emptyset}\epsilon(e,S_V)R^e_{\underline{A}}\right)\\
&= \left(\frac{\hbar\kappa}{4}\right)^2\mathscr{T}^{\underline{AB}}\left(R^{\mathrm{in}}_{\underline{B}} - R^{\mathrm{out}}_{\underline{B}}\right)\left(R^{\mathrm{in}}_{\underline{A}} - R^{\mathrm{out}}_{\underline{A}}\right)\\
&= \left(\frac{\hbar\kappa}{4}\right)^2\mathscr{T}^{\underline{AB}}\Big(2R^{\mathrm{in}}_{\underline{B}}R^{\mathrm{in}}_{\underline{A}} + 2R^{\mathrm{out}}_{\underline{B}}R^{\mathrm{out}}_{\underline{A}}\\
&\qquad - \left(R^{\mathrm{in}}_{\underline{B}} + R^{\mathrm{out}}_{\underline{B}}\right)\left(R^{\mathrm{in}}_{\underline{A}} + R^{\mathrm{out}}_{\underline{A}}\right)\Big)\\
&=: -\left(\frac{\hbar\kappa}{4}\right)^2(2\Delta_I + 2\Delta_F - \Delta_{I\cup F})
\end{aligned} \tag{5.264}$$

with $R^{\mathrm{in}}_{\underline{A}} = \sum_{e\text{ ingoing}} R^e_{\underline{A}}$ and $R^{\mathrm{out}}_{\underline{A}} = \sum_{e\text{ outgoing}} R^e_{\underline{A}}$. Moreover, $\Delta := -\mathscr{T}^{\underline{AB}}R_{\underline{B}}R_{\underline{A}}$ denotes the *super Laplace-Beltrami operator* of the super Lie group $\mathcal{G}$.

With these preliminary considerations, let us compute the action of the super area operator on a (gauge-invariant) super spin network $T_{\gamma,\vec{\pi},\vec{m},\vec{n}}$ for the special case $\mathcal{N}=1$. Suppose that the surface S intersects the graph γ in a single divalent vertex $v\in V(\gamma)$ so that, at this vertex, one has $\Delta_I = \Delta_F$ as well as $\Delta_{I\cup F}=0$. Using the results of Section D.1, up to numerical factors, we then find

$$\widehat{\mathrm{gAr}}(S)T_{\gamma,\vec{\pi},\vec{m},\vec{n}} \propto il_p^2\sqrt{j\left(j+\frac{1}{2}\right)}T_{\gamma,\vec{\pi},\vec{m},\vec{n}} \tag{5.265}$$

with j the spin quantum number labeling the edge $e\in E(\gamma)$ that intersects the vertex v. This coincides with the results of [85].

5.5.4. Comparison: Quantization of fermions in standard LQG

In the following, we would like to point out various similarities between the quantization of the combined boson-fermion system in chiral LQSG exploiting the enlarged gauge symmetry of the theory and the standard quantization scheme in the framework of loop quantum gravity using real Asthekar-Barbero variables [67, 80]. To this end, following [80,156] and explicitly taking into account the underlying parametrization supermanifold $\mathcal{S}$, let us assume we are given a canonical system of *anticommuting half-densitized* fermion fields θ^A (cf. Section 4.3.1), $A \in \{\pm\}$, defined on Σ and canonically conjugate momentum π_A satisfying the anti-Poisson relations

$$\{\pi_A(s,x), \theta^B(s,y)\} = -\delta^B_A \delta^{(3)}(x,y) \tag{5.266}$$

$\forall x, y \in \Sigma$ and $s \in \mathcal{S}$ together with certain reality conditions relating the momentum π to the corresponding complex conjugate Weyl spinor $\bar{\theta}$. In the standard literature, matter fields in loop quantum gravity are quantized by discretizing them over a finite number of points. As we will see below, the classical algebra has in fact an interesting interpretation in context of $\mathcal{S}$-parametrized field theories studying smooth functions on supermanifolds arising from the functor of points prescription.

To explain this in a bit more detail, in what follows, for a graph γ in Σ and a finite set of points $\{x_i\}_i \subset \Sigma$, let us define a *generalized graph* $\Gamma := \gamma \cup \{x_i\}_i$. Again, it follows that the collection of all such generalized graphs forms a partially ordered directed set by defining

$$\Gamma \leq \Gamma' :\Leftrightarrow l(\gamma) \leq l(\gamma') \text{ and } \{x_i\}_i \subseteq \{x'_j\}_j \tag{5.267}$$

for $\Gamma = \gamma \cup \{x_i\}_i$ and $\Gamma' = \gamma' \cup \{x'_j\}_j$. To a generalized graph $\Gamma = \gamma \cup \{x_i\}_i$, we then associate the set $\mathcal{A}^P_{\mathcal{S},\Gamma}$ of *pointed generalized super connections* via

$$\mathcal{A}^P_{\mathcal{S},\Gamma} := \mathrm{Hom}_{\mathbf{Cat}}(l(\gamma)^{\mathrm{op}}, \mathbf{G}(\mathcal{S})) \times \prod_{i=1}^{k} \mathbb{C}^{0|2}_{x_i}(\mathcal{S}) \tag{5.268}$$

where, for any $x \in \Sigma$, $\mathbb{C}^{0|2}_x$ denotes the superpoint $\mathbb{C}^{0|2}_x := (\{x\}, \Lambda^{\mathbb{C}}_2)$. In Definition (5.268), the first factor corresponds to the bosonic degrees of freedom given by the set $\mathcal{A}_{\mathcal{S},\gamma} := \mathrm{Hom}_{\mathbf{Cat}}(l(\gamma)^{\mathrm{op}}, \mathbf{G}(\mathcal{S}))$ of generalized bosonic connections on the graph γ with the underlying gauge gauge group given by the purely bosonic super Lie group $\mathbf{S}(\mathrm{SU}(2))$ which, as usual, we will identify with $G := \mathrm{SU}(2)$ to simplify notation. Moreover, the second factor encodes the fermionic degrees of freedom. In this context, note that, fixing a spatial point $x \in \Sigma$, it follows that the fermionic fields θ^A induce $\mathcal{S}$-points $\theta^A(x) \equiv \theta^A(\cdot, x) \in \mathbb{C}^{0|2}_x(\mathcal{S})$. Using the isomorphism (5.166) in case of a purely bosonic super Lie group, this implies

$$\mathcal{A}^P_{\mathcal{S},\Gamma} \cong \left(G^{|E(\gamma)|} \times \prod_{i=1}^{k} \mathbb{C}^{0|2}_{x_i}\right)(\mathcal{S}) =: \mathcal{M}(\mathcal{S}) \tag{5.269}$$

that is, $\mathcal{A}^P_{\mathcal{S},\Gamma}$ can be identified with the $\mathcal{S}$-point of a split supermanifold $\mathcal{M}$ (see Example 2.2.4). Hence, using this identification, we define the space $\mathrm{Cyl}^\infty(\mathcal{A}^P_{\mathcal{S},\Gamma})$ of *generalized cylindrical functions* on $\mathcal{A}^P_{\mathcal{S},\Gamma}$ in terms complex smooth functions on the H^∞ supermanifold $\mathcal{M}(\mathcal{S})$, i.e.,

$$\mathrm{Cyl}^\infty(\mathcal{A}^P_{\mathcal{S},\Gamma}) := H^\infty(\mathcal{M}(\mathcal{S}),\mathbb{C}) \cong C^\infty(G^{|E(\gamma)|},\mathbb{C}) \otimes \bigotimes_{i=1}^{k} \Lambda_N^{\mathbb{C}}(x_i) \tag{5.270}$$

where, for the last identity, it has been implicitly assumed that the parametrization supermanifold $\mathcal{S}$ is suitably large enough. It follows immediately from the definition that, given two generalized graphs Γ and Γ' with $\Gamma \leq \Gamma'$, the pullbacks $p^*_{ll'}$ of the morphisms $p_{ll'}: \mathcal{A}_{\mathcal{S},l'} \rightarrow \mathcal{A}_{\mathcal{S},l}$ as defined via (5.169) can be extended to morphisms $p^*_{\Gamma\Gamma'}$ on generalized cylindrical functions, i.e.,

$$p^*_{\Gamma\Gamma'}: \mathrm{Cyl}^\infty(\mathcal{A}^P_{\mathcal{S},\Gamma}) \rightarrow \mathrm{Cyl}^\infty(\mathcal{A}^P_{\mathcal{S},\Gamma'}) \tag{5.271}$$

satisfying the compatibility condition $p^*_{\Gamma'\Gamma''} \circ p^*_{\Gamma\Gamma'} = p^*_{\Gamma\Gamma''}$ for all $\Gamma \leq \Gamma' \leq \Gamma''$. Thus, in this way, it follows that $(\mathrm{Cyl}^\infty(\mathcal{A}^P_{\mathcal{S},\Gamma}), p^*_{\Gamma\Gamma'})$ defines an inductive family to which we can associate its corresponding inductive limit

$$\mathrm{Cyl}^\infty(\overline{\mathcal{A}}^P_{\mathcal{S}}) := \varinjlim \mathrm{Cyl}^\infty(\mathcal{A}^P_{\mathcal{S},\Gamma}) \tag{5.272}$$

For a generalized graph $\Gamma = \gamma \cup \{x_i\}_i$, we define $V^\infty(\mathcal{A}^P_{\mathcal{S},\Gamma})$ as the space $V^\infty(\mathcal{A}^P_{\mathcal{S},\Gamma}) \equiv V^\infty(\mathcal{A}_{\mathcal{S},l(\gamma)})$ of (bosonic) electric fluxes $X_n(S)$ acting on cylindrical functions on $\mathcal{A}_{\mathcal{S},l(\gamma)}$ via the Poisson bracket with the smeared (bosonic) electric flux $E_n(S)$ associated to the gravitational electric field E^i_a. Hence, again, it follows that electric fluxes leave the generalized graph unchanged so that they can be lifted consistently to electric fluxes $V^\infty(\overline{\mathcal{A}}^P_{\mathcal{S}})$ on the inductive limit. For the classical algebra, we then set

$$\mathfrak{B}_{\mathcal{S}} := \mathrm{Cyl}^\infty(\overline{\mathcal{A}}^P_{\mathcal{S}}) \rtimes V^\infty(\overline{\mathcal{A}}^P_{\mathcal{S}}) \tag{5.273}$$

It follows that $\mathfrak{B}_{\mathcal{S}}$ has the structure of a semi-direct Lie superalgebra with graded Lie bracket $[\cdot,\cdot]$ defined via

$$[(f,X),(g,Y)] := (X(g) - Y(f), [X,Y]) \tag{5.274}$$

where we used that the electric fluxes, by definition, are bosonic. By implementing the reality conditions, it follows that $\mathfrak{B}_{\mathcal{S}}$ even forms a *-algebra. As in the context of chiral supergravity, in order to quantize the theory, we are looking for a representation of $\mathfrak{B}_{\mathcal{S}}$ on a super Hilbert space $\mathfrak{H}_{\mathcal{S}}$, that is, a grading preserving morphism of Lie superalgebras

$$\pi_{\mathcal{S}}:\,\mathfrak{B}_{\mathcal{S}}\rightarrow\mathrm{Op}(\mathcal{D}_{\mathcal{S}},\mathfrak{H}_{\mathcal{S}}^{\mathrm{LQSG}}) \tag{5.275}$$

from $\mathfrak{B}_{\mathcal{S}}$ to the space of unbounded operators on the super Hilbert space $\mathfrak{H}_{\mathcal{S}}$ mutually defined on a common dense (graded) domain $\mathcal{D}_{\mathcal{S}}$.

To find a possible candidate for the super Hilbert space, for any generalized graph $\Gamma\cup\{x_i\}_i$, let us consider the super vector space $V_{\mathcal{S},\Gamma}:=\mathrm{Cyl}^{\infty}(\mathcal{A}^P_{\mathcal{S},\Gamma})$. Using the isormorphism (5.270), for elementary tensors of the form $f_{l(\gamma)}\otimes\theta^A(x_i)\in\mathrm{Cyl}^{\infty}(\mathcal{A}^P_{\mathcal{S},\Gamma})$, we then set

$$\pi_{\mathcal{S}}(f_{l(\gamma)}\otimes\theta^A(x_i)):=\widehat{f}_{l(\gamma)}\otimes\widehat{\theta}^A(x_i) \tag{5.276}$$

where $\widehat{f}_{l(\gamma)}$ as well as $\widehat{\theta}^A(x_i)$ act in terms of multiplication operators in the obvious way. For the electric fluxes, we set

$$\pi_{\mathcal{S}}(X_n(S)):=i\hbar\{E_n(S),\cdot\} \tag{5.277}$$

It is then immediate to see that the operators as defined via (5.276) and (5.277) indeed satisfy the correct graded commutation relations. Next, we need to extend $V_{\mathcal{S},\Gamma}$ to a super Hilbert space. To do so, note that the super scalar product has to be chosen in such a way so that it is invariant under local gauge transformations. If $\lambda:\,\mathcal{S}\times\Sigma\rightarrow G$ denotes such a local gauge transformation, this induces an action on the supermanifold $\mathcal{M}(\mathcal{S})$ via

$$\lambda\triangleright(\{g_e\},\{\theta(x_i)\}_i):=(\{\lambda(b(e))g_e\lambda(f(e))^{-1}\},\{\lambda(x_i)\cdot\theta(x_i)\}) \tag{5.278}$$

Hence, as a possible candidate for the super scalar product $\mathscr{S}'_{\Gamma}$, we may set

$$\mathscr{S}'_{\Gamma}(f_{\Gamma}|g_{\Gamma}):=\int_{G^{|E(\gamma)|}}\mathrm{d}^{|E(\gamma)|}\mu_H\int_B\mathrm{d}\Theta\,\bar{f}_{\Gamma}g_{\Gamma} \tag{5.279}$$

with μ_H the invariant Haar measure of G and where $\mathrm{d}\Theta$ is defined as the product measure $\mathrm{d}\Theta:=\mathrm{d}\Theta(x_1)\cdots\mathrm{d}\Theta(x_k)$ with $\mathrm{d}\Theta(x_i):=\mathrm{d}\theta^A(x_i)\mathrm{d}\bar{\theta}^{A'}(x_i)$. It then follows that this super scalar product indeed invariant under the action (5.278). However, $\mathscr{S}'_{\Gamma}$ turns out to be not cylindrically consistent, i.e., it cannot be lifted consistently on the inductive limit as the unit is not normalized. To fix this, recall that the representation given by (5.276) and (5.277) also has to preserve the *-relations imposed by the reality conditions. Interestingly, this turns out to be in fact equivalent to choosing a Krein completion of the indefinite inner product space $(V_{\mathcal{S},\Gamma},\mathscr{S}'_{\Gamma})$. For instance, for a generalized graph of the form $\Gamma=\gamma\cup\{x\}$ with $x\in\Sigma$ a single point, it follows that one can always find a fundamental symmetry $J_\Gamma \equiv J : V_{\mathcal{S},\Gamma} \to V_{\mathcal{S},\Gamma}$ such that the induced inner product

$$\langle\cdot|\cdot\rangle_\Gamma := \mathscr{S}'_\Gamma(\cdot|J\cdot) \tag{5.280}$$

takes the form

$$\langle f_\gamma, g_\gamma\rangle_\Gamma = \sum_{\underline{I}} \langle\langle f_{l(\gamma),\underline{I}}|g_{l(\gamma),\underline{I}}\rangle\rangle \tag{5.281}$$

where we have expanded $f_\Gamma = \sum_{\underline{I}} f_{l(\gamma),\underline{I}}\theta^{\underline{I}}(x)$ and similarly for g_Γ and $\langle\langle\cdot|\cdot\rangle\rangle$ denotes the standard inner product on $L^2(G)$ induced by the unique invariant Haar measure on G. As discussed already in a similar context in Section 4.5.1, this inner product is indeed invariant under local gauge transformations. Moreover, it is positive definite and correctly implements the reality conditions. For a general graph $\Gamma = \gamma\cup\{x_1,\ldots,x_k\}$, we can then set $J_\Gamma := J^{\otimes k}$. By construction, it follows that the induced inner product of the form (5.280) is again positive definite, gauge-invariant and solves the reality conditions. Moreover, by definition, the fundamental symmetry satisfies

$$p^*_{\Gamma\Gamma'}J_\Gamma = J_{\Gamma'}p^*_{\Gamma\Gamma'} \tag{5.282}$$

for any generalized graphs $\Gamma \leq \Gamma'$. Thus, since the unit is also normalized w.r.t. $\langle\cdot|\cdot\rangle_\Gamma$, this implies that the inner product can be lifted consistently to a positive definite inner product $\langle\cdot|\cdot\rangle$ on the inductive limit $V_{\mathcal{S}} := \lim_{\to} V_{\mathcal{S},\Gamma}$. Using this inner product, we can then complete $V_{\mathcal{S},}$ to a Hilbert space

$$\mathfrak{H}_{\mathcal{S}} := \overline{V}_{\mathcal{S}}^{\|\cdot\|} \tag{5.283}$$

so that, in this way, we end up with a standard super Hilbert space $(\mathfrak{H}_{\mathcal{S}},\langle\cdot|\cdot\rangle)$ (see Def. 5.5.9 and Remark 5.5.10) onto which the representation as defined via (5.276) and (5.277) can be lifted consistently and uniquely to a representation of the classical algebra $\mathfrak{B}_{\mathcal{S}}$. By definition, it follows that the super Hilbert space (5.283) has the tensor product structure

$$\mathfrak{H}_{\mathcal{S}} \cong \mathfrak{H}_{\mathcal{S},\mathrm{grav}} \otimes \mathfrak{H}_{\mathcal{S},f} \tag{5.284}$$

with $\mathfrak{H}_{\mathcal{S},\mathrm{grav}}$ and $\mathfrak{H}_{\mathcal{S},f}$ the (super) Hilbert spaces associated to the bosonic and fermionic degrees of freedom, respectively, where $\mathfrak{H}_{\mathcal{S},\mathrm{grav}}$ is defined via (5.248) according to the construction in Section 5.5.3 with super Lie group given by $\mathbf{S}(\mathrm{SU}(2))$.

Thus, to summarize, in the context of the standard quantization scheme of the gravity-fermion system in the framework LQG based on real Ashtekar-Barbero variables, one can construct a representation of the classical algebra that respects cylindrical consistency and, in particular, correctly implements the reality conditions.

5.6. The boundary theory

5.6.1. The super Chern-Simons action

Having discussed the quantization of the bulk theory of chiral supergravity in Section 5.5.3, in what follows, let us finally very briefly comment on the boundary theory. To this end, in this section, let us first recall the basic definition and structure of the super Chern-Simons action which, as we observed in Section 5.4, naturally appears, in fact even uniquely, as a boundary term in chiral supergravity in the presence of boundaries. An outlook on possible applications in the context of chiral LQSG such as the quantum description of supersymmetric black holes and black hole entropy will be given in Section 5.6.2 below. For more details on Chern-Simons theory with supergroup as a gauge group, let us refer to [166] as well as [220] studying the super Chern-Simons action in the geometric approach using integral forms. More details on integral forms and related concepts can be found, e.g., in [131, 132]. For notational simplification, in what follows, we will not explicitly mention the underlying parametrizing supermanifold.

Before we state the super Chern-Simons action, we need to introduce invariant inner products. Let $\mathcal{G}$ be a Lie supergroup. By the Super Harish-Chandra Theorem 2.3.9, the super Lie group has the equivalent characterization in terms of the super Harish-Chandra pair $(G,\mathfrak{g})$ with $G:=\mathbf{B}(\mathcal{G})$ the body and $\mathfrak{g}$ the super Lie algebra of $\mathcal{G}$ with $\mathfrak{g}_0=\mathrm{Lie}(G)$. According to Def. 2.3.12, a super metric on $\mathfrak{g}$ is a bilinear map $\langle\cdot,\cdot\rangle:\;\mathfrak{g}\times\mathfrak{g}\rightarrow\mathbb{C}$ that is non-degenerate and graded-symmetric, i.e. $\langle X,Y\rangle=(-1)^{|X||Y|}\langle Y,X\rangle$ for any homogeneous $X,Y\in\mathfrak{g}$. Moreover, it is called Ad-*invariant*, if

$$\langle\mathrm{Ad}_g X,\mathrm{Ad}_g Y\rangle=\langle X,Y\rangle\quad\forall g\in G\tag{5.285}$$

and

$$\langle[Z,X],Y\rangle+(-1)^{|X||Z|}\langle X,[Z,Y]\rangle=0\tag{5.286}$$

for all homogeneous $X,Y,Z\in\mathfrak{g}$. This can be extended to a bilinear form $\langle\cdot\wedge\cdot\rangle$: $\Omega^p(\mathcal{N},\mathfrak{g})\times\Omega^q(\mathcal{N},\mathfrak{g})\rightarrow\Omega^{p+q}(\mathcal{N})$ on differential forms on a supermanifold $\mathcal{N}$ with values in the super Lie module $\mathrm{Lie}(\mathcal{G})=\Lambda\otimes\mathfrak{g}$ (Def. 2.5.4). To this end, first note that the sheaf $\Omega^\bullet(\mathcal{N},\mathfrak{g})$ carries the structure of a $\mathbb{Z}\times\mathbb{Z}_2$-bigraded module, where, for any $\omega\in\Omega^p(\mathcal{N},\mathfrak{g})_i$, the parity $\epsilon(\omega)$ is defined as

$$\epsilon(\omega):=(p,i)\in\mathbb{Z}\times\mathbb{Z}_2\tag{5.287}$$

where we will also write $|\omega|:=i$ for the underlying $\mathbb{Z}_2$-grading. For homogeneous $\mathrm{Lie}(\mathcal{G})$-valued differential forms $\omega\in\Omega^p(\mathcal{N},\mathfrak{g})$ and $\eta\in\Omega^q(\mathcal{N},\mathfrak{g})$, we then set

$$\langle\omega\wedge\eta\rangle:=(-1)^{|\underline{A}|(|\eta|+|\underline{B}|)}\omega^{\underline{A}}\wedge\eta^{\underline{B}}\langle T_{\underline{A}},T_{\underline{B}}\rangle\tag{5.288}$$

where we have chosen a real homogeneous basis $(T_{\underline{A}})_{\underline{A}}$ of $\mathfrak{g}$ and simply wrote $|\underline{A}| := |T_{\underline{A}}|$ for the parity. A direct calculation yields

$$\begin{aligned}\langle\omega\wedge\eta\rangle &:= (-1)^{|\underline{A}|(|\eta|+|\underline{B}|)}\omega^{\underline{A}}\wedge\eta^{\underline{B}}\langle T_{\underline{A}},T_{\underline{B}}\rangle\\ &= (-1)^{pq}(-1)^{|\underline{A}||\eta|}(-1)^{(|\omega|+|\underline{A}|)(|\eta|+|\underline{B}|)}\eta^{\underline{B}}\wedge\omega^{\underline{A}}\langle T_{\underline{B}},T_{\underline{A}}\rangle\\ &= (-1)^{pq}(-1)^{|\omega||\eta|}\langle\eta\wedge\omega\rangle\end{aligned} \tag{5.289}$$

Finally, let us derive an important identity which plays a central role in many calculations. Using the Ad-invariance (5.286), one obtains

$$\begin{aligned}&\langle\omega\wedge[\eta\wedge\xi]\rangle\\ &=(-1)^{|\underline{A}|(|\eta|+|\xi|+|\underline{B}|+|\underline{C}|)}(-1)^{|\underline{B}|(|\xi|+|\underline{C}|)}\omega^{\underline{A}}\wedge\eta^{\underline{B}}\wedge\xi^{\underline{C}}\langle T_{\underline{A}},[T_{\underline{B}},T_{\underline{C}}]\rangle\\ &=(-1)^{|\underline{A}|(|\eta|+|\xi|+|\underline{B}|+|\underline{C}|)}(-1)^{|\underline{B}|(|\xi|+|\underline{C}|)}\omega^{\underline{A}}\wedge\eta^{\underline{B}}\wedge\xi^{\underline{C}}\langle[T_{\underline{A}},T_{\underline{B}}],T_{\underline{C}}\rangle\\ &=(-1)^{|\underline{A}|(|\eta|+|\underline{B}|)}\langle\omega^{\underline{A}}\wedge\eta^{\underline{B}}\otimes[T_{\underline{A}},T_{\underline{B}}]\wedge\xi\rangle\\ &=\langle[\omega\wedge\eta]\wedge\xi\rangle\end{aligned} \tag{5.290}$$

As we have seen in Section 5.4, the super Chern-Simons action naturally appears as a boundary term in the chiral limit of the Holst-MacDowell-Mansouri action of supergravity. In fact, as observed in [86,182], the super Chern-Simons action also arises as a boundary term by describing supergravity as a so-called (generalized) constrained topological field theory. In the chiral limit, we have shown, in particular, that this action is even uniquely fixed if one imposes supersymmetry invariance at the boundary.

To state this action, in what follows, let $\mathcal{A}$ be a super connection 1-form and $F(\mathcal{A}) = \mathrm{d}\mathcal{A} + \frac{1}{2}[\mathcal{A}\wedge\mathcal{A}]$ its corresponding curvature. Then, one has

$$\langle F(\mathcal{A})\wedge F(\mathcal{A})\rangle = \mathrm{d}\langle\mathcal{A}\wedge F(\mathcal{A}) - \frac{1}{6}\mathcal{A}\wedge[\mathcal{A}\wedge\mathcal{A}]\rangle \tag{5.291}$$

so the term in the exterior derivative is a natural generalization of the Chern-Simons 3-form to the present context. To prove (5.291), note that

$$\begin{aligned}&\mathrm{d}\langle\mathcal{A}\wedge F(\mathcal{A}) - \frac{1}{6}\mathcal{A}\wedge[\mathcal{A}\wedge\mathcal{A}]\rangle\\ &=\langle\mathrm{d}\mathcal{A}\wedge\mathrm{d}\mathcal{A} + \frac{1}{2}\mathrm{d}\mathcal{A}\wedge[\mathcal{A}\wedge\mathcal{A}] - \mathcal{A}\wedge[\mathrm{d}\mathcal{A}\wedge\mathcal{A}]\rangle - \frac{1}{6}\mathrm{d}\langle\mathcal{A}\wedge[\mathcal{A}\wedge\mathcal{A}]\rangle\\ &=\langle\mathrm{d}\mathcal{A}\wedge\mathrm{d}\mathcal{A} + \frac{1}{3}\mathrm{d}\mathcal{A}\wedge[\mathcal{A}\wedge\mathcal{A}] - \frac{2}{3}\mathcal{A}\wedge[\mathrm{d}\mathcal{A}\wedge\mathcal{A}]\rangle\end{aligned} \tag{5.292}$$

which directly leads to (5.291) using $\langle\mathcal{A}\wedge[\mathrm{d}\mathcal{A}\wedge\mathcal{A}]\rangle = -\langle\mathcal{A}\wedge[\mathcal{A}\wedge\mathrm{d}\mathcal{A}]\rangle = -\langle[\mathcal{A}\wedge\mathcal{A}]\wedge\mathrm{d}\mathcal{A}\rangle$ which is an immediate consequence of identity (5.290). In the following, suppose that the body of the supermanifold $\mathcal{N}$ which, following the standard conventions in the LQG literature, will be denoted by H is three-dimensional. Then, according to (5.291), the *super Chern-Simons action* is defined as follows

$$S_{\mathrm{CS}}(\mathcal{A}) := \frac{k}{4\pi}\int_H \langle \mathcal{A}\wedge \mathrm{d}\mathcal{A} + \frac{1}{3}\mathcal{A}\wedge[\mathcal{A}\wedge\mathcal{A}]\rangle \tag{5.293}$$

where k is referred to as the *level* of the Chern-Simons theory. Let us decompose $\mathcal{A} = \mathrm{pr}_{\mathfrak{g}_0}\circ\mathcal{A} + \mathrm{pr}_{\mathfrak{g}_1}\circ\mathcal{A} =: A + \psi$ w.r.t. the even and odd part of the super Lie algebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Inserting this into (5.293), this gives

$$\langle \mathcal{A}\wedge F(\mathcal{A})\rangle = \langle A\wedge F(A) + \frac{1}{2}A\wedge[\psi\wedge\psi]\rangle + \langle\psi\wedge(\mathrm{d}\psi + [A\wedge\psi])\rangle \tag{5.294}$$

On the other hand, using $\langle\psi\wedge[A\wedge\psi]\rangle = \langle\psi\wedge[\psi\wedge A]\rangle = \langle[\psi\wedge\psi]\wedge A\rangle$ according to (5.290), we find

$$\begin{aligned}\langle\mathcal{A}\wedge[\mathcal{A}\wedge\mathcal{A}]\rangle &= \langle A\wedge[A\wedge A] + A\wedge[\psi\wedge\psi]\rangle + 2\,\langle\psi\wedge[A\wedge\psi]\rangle\\ &= \langle A\wedge[A\wedge A] + A\wedge[\psi\wedge\psi]\rangle + 2\,\langle A\wedge[\psi\wedge\psi]\rangle\\ &= \langle A\wedge[A\wedge A] + 3A\wedge[\psi\wedge\psi]\rangle\end{aligned} \tag{5.295}$$

Thus, we can rewrite (5.293) in the following way

$$S_{\mathrm{CS}}(\mathcal{A}) = S_{\mathrm{CS}}(A) + \frac{k}{4\pi}\int_H \langle\psi\wedge D^{(A)}\psi\rangle \tag{5.296}$$

with $S_{\mathrm{CS}}(A)$ the Chern-Simons action of the bosonic connection A and $D^{(A)}$ the associated exterior covariant derivative.

5.6.2. Towards black hole entropy in LQSG – an outlook

In the following, let us derive the canonical decomposition of the super Chern-Simons action (5.293) defined on a three-dimensional smooth manifold H. To to do so, we can proceed similarly as in Section 4.3. Hence, following [221], for $t\in\mathbb{R}$, let Δ_t denote the 2-dimensional time slices in the foliation of H along the integral flow of the global time (null) vector field ∂_t. Furthermore, let $P^{\|}$ be the projection which projects any smooth vector field $X\in\mathfrak{X}(H)$ onto the subspace of smooth vector fields lying in the kernel of $\mathrm{d}t$, i.e., $\mathrm{d}t(P^{\|}(X)) = 0$ with $P^{\|}(X) := X - \mathrm{d}t(X)\partial_t$. This in turn induces a projection $P_{\|}$ on the space of covariant tensor fields T according to

$$P_{\|}T := T\circ P^{\|} \tag{5.297}$$

where $P^{\|}$ on the r.h.s. acts on each slot. For the derivation of the canonical decomposition, let us write

$$\underset{\leftarrow}{\mathcal{A}} := P_{\|}\mathcal{A} \tag{5.298}$$

in order to distinguish the projection $P_{\|}\mathcal{A}$ from its covariant counterpart. Hence, setting $\mathcal{A}_0 := \mathcal{A}(\partial_t)$, this yields

$$\mathcal{A} = \underset{\leftarrow}{\mathcal{A}} + \mathcal{A}_0 \mathrm{d}t \tag{5.299}$$

On the other hand, it is immediate to see that

$$\mathrm{d}\mathcal{A} = \mathrm{d}\underset{\leftarrow}{\mathcal{A}} - i_{\partial_t}\mathrm{d}\mathcal{A} \wedge \mathrm{d}t \tag{5.300}$$

A coordinate-free form of the time derivative of the connection along the timelike vector field ∂_t is given by

$$\underset{\leftarrow}{\dot{\mathcal{A}}} := P_{\|}L_{\partial_t}\mathcal{A} = P_{\|}(\mathrm{d}\mathcal{A}_0 + i_{\partial_t}\mathrm{d}\mathcal{A}) \tag{5.301}$$

Hence, combining (5.301) with (5.300) and using (5.299), it is immediate to see that the Chern-Simons 3-form can be written in the following way

$$\langle \mathcal{A} \wedge \mathrm{d}\mathcal{A} + \frac{1}{3}\mathcal{A} \wedge [\mathcal{A} \wedge \mathcal{A}]\rangle = \mathrm{d}t \wedge \left(\langle -\underset{\leftarrow}{\mathcal{A}} \wedge \underset{\leftarrow}{\dot{\mathcal{A}}}\rangle + 2\langle \mathcal{A}_0 F(\underset{\leftarrow}{\mathcal{A}})\rangle - \mathrm{d}\langle \mathcal{A}_0 \underset{\leftarrow}{\mathcal{A}}\rangle \right) \tag{5.302}$$

Thus, if we drop the arrow below the pulled back connection in order to simplify notation, we find that the 2+1-split of the super Chern-Simons action takes the form

$$S_{\mathrm{CS}}(\mathcal{A}) = \frac{k}{4\pi}\int_{\mathbb{R}}\mathrm{d}t\int_{\Delta_t}\langle -\mathcal{A} \wedge \dot{\mathcal{A}} + 2\mathcal{A}_0 F(\mathcal{A}) - \mathrm{d}(\mathcal{A}_0\mathcal{A})\rangle \tag{5.303}$$

As a consequence, the pre-symplectic structure of the canonical theory is given by

$$\Omega_{\mathrm{CS}}(\delta_1, \delta_2) = -\frac{k}{2\pi}\int_{\Delta}\langle \delta_{[1}\mathcal{A} \wedge \delta_{2]}\mathcal{A}\rangle \tag{5.304}$$

for variations $\delta\mathcal{A} \in T\mathscr{A}_{\Delta}$ where $\mathscr{A}_{\Delta}$ denotes the space of smooth super connection 1-forms on the induced $\mathcal{G}$-principal bundle $\mathcal{E} := \mathcal{P}|_{\Delta}$ over Δ (or rather the corresponding bosonic split supermanifold $\mathbf{S}(\Delta)$). Since the difference of two super connections defines an even horizontal 1-form of type $(\mathcal{G}, \mathrm{Ad})$, it follows that $T_{\mathcal{A}}\mathscr{A}_{\Delta}$ at any $\mathcal{A} \in \mathscr{A}_{\Delta}$ can be identified with $T_{\mathcal{A}}\mathscr{A}_{\Delta} \cong \Omega^1(\Delta, \mathrm{Ad}(\mathcal{E}))_0$. For the graded Poisson bracket, one obtains

$$\{\mathcal{A}^{\underline{A}}_{\underline{a}}(x), \mathcal{A}^{\underline{B}}_{\underline{b}}(y)\} = -\frac{2\pi}{k}\mathscr{S}^{\underline{AB}}\epsilon_{ab}\delta^{(2)}(x,y) \tag{5.305}$$

where $\mathscr{S}^{\underline{AB}}$ denotes the matrix components of the inverse super metric satisfying $\mathscr{S}_{\underline{CA}}\mathscr{S}^{\underline{CB}} = \delta^{\underline{B}}_{\underline{A}}$. Moreover, from the split action (5.303), we can read off the constraint

$$\mathcal{F}[\alpha] := \frac{k}{2\pi}\int_{\Delta}\langle\alpha F(\mathcal{A})\rangle \tag{5.306}$$

which imposes the condition $F(\mathcal{A}) = 0$, that is, the curvature of the super connection on Δ is constrained to vanish. For this reason, $\mathcal{F}[\alpha]$ is also referred to as the *flatness constraint*. Actually, since the curvature contains a term involving an exterior derivative, the flatness constraint (5.306), in general, turns out to be not functionally differentiable. In case that Δ has a nontrivial boundary $\partial\Delta$ which, in the context of two dimensions, we will refer to as the *corner* of Δ, one needs to require that the smearing function in (5.306) satisfies the condition $\alpha|_{\partial\Delta} \equiv 0$.

In the framework of LQG, singularities on the boundary typically arise from the intersection of the boundary with spin network states. Assuming that the spin network edges piercing the boundary have some infinitesimal but *nonzero* width, this induces infinitesimal holes at the punctures on the boundary, such that, at each puncture, $\partial\Delta$ becomes nontrivial and topologically equivalent to a 1-dimensional circle. As a consequence, this gives rise to new physical degrees of freedom on the boundary which are localised on the corner $\partial\Delta$. In the context of LQG, this was first observed in [222] and discussed more expansively, e.g., in [122–124, 195, 223]. As argued in [222], based on a general proposal formulated in [224, 225], these new degrees of freedom may also account for black hole entropy and thus may play a crucial role in the quantum description of the black holes. In fact, it turns out that these contain the physical degrees of freedom associated to the Hilbert spaces of conformal blocks which are usually considered in the context of black hole entropy computations in LQG.

While we have not yet been able to complete the definition of the Hilbert space for chiral LQSG, extrapolating from what we have it seems that all these observations carry over quite naturally to the context of the quantum description of chiral supergravity with $\mathcal{N}$-extended supersymmetry. In that case, we have described in Section 5.5 how the quantum excitations of the bulk degrees of freedom are represented by super spin network states associated to the gauge supergroup $\mathrm{OSp}(\mathcal{N}|2)_{\mathbb{C}}$. On the other hand, in Section 5.4, we have determined that the boundary theory is described in terms of a $\mathrm{OSp}(\mathcal{N}|2)_{\mathbb{C}}$ super Chern-Simons theory. Hence, it follows that, due to the quantization of super electric fluxes in the bulk, super spin network states induce singularities on the boundary. To see this, note that the Gauss constraint $\mathscr{G}_{\text{full}}[\alpha]$ of the full theory including both bulk and boundary degrees of freedom is given by the sum of the Gauss constraint (5.118) (resp. (5.136) for $\mathcal{N}=2$) in the bulk as well as the flatness constraint (5.306) on the boundary, that is,

$$\mathscr{G}_{\text{full}}[\alpha]=-\frac{i}{\kappa}\int_{\Sigma}\langle D^{(\mathcal{A}^+)}\alpha\wedge\mathcal{E}\rangle+\frac{i}{\kappa}\int_{\Delta}\langle\alpha[\mathcal{E}-\frac{i\kappa k}{2\pi}F(\mathcal{A}^+)]\rangle \tag{5.307}$$

for any Lie($\mathcal{G}$)-valued smearing function α. For a given finite graph γ embedded in Σ, we define the Hilbert space $\mathfrak{H}_{\text{full},\gamma}$ w.r.t. γ of the full theory as the tensor product

$$\mathfrak{H}_{\gamma}^{\text{full}}=\mathfrak{H}_{\gamma}^{\text{cLQSG}}\otimes\mathfrak{H}_{\gamma}^{\text{CS}} \tag{5.308}$$

with $\mathfrak{H}_{\gamma}^{\text{cLQSG}}$ the Hilbert space of the quantized bulk degrees of freedom as constructed in Section 5.5.3 and $\mathfrak{H}_{\gamma}^{\text{CS}}$ the Hilbert space corresponding to the quantized super Chern-Simons theory on the boundary.

As a next step, in order to implement the full Gauss constraint (5.307) in the quantum theory, we have to regularize it over the graph γ. To this end, at each *puncture* $p\in\mathcal{P}_{\gamma}:=\gamma\cap\Delta$, let us choose a disk $D_{\epsilon}(p)$ on Δ around p with radius $\epsilon>0$ and set

$$\mathcal{E}[\alpha](p):=\lim_{\epsilon\to 0}\int_{D_{\epsilon}(p)}\langle\alpha,\mathcal{E}\rangle,\quad F[\alpha](p):=\lim_{\epsilon\to 0}\int_{D_{\epsilon}(p)}\langle\alpha,F(\mathcal{A}^+)\rangle \tag{5.309}$$

By definition, these quantities (or suitable functions thereof) can be promoted to well-defined operators in the quantum theory. Thus, it follows that the Gauss constraint operator of the full theory takes the form

$$\widehat{\mathscr{G}}_{\text{full}}[\alpha]=\widehat{\mathscr{G}}[\alpha]-\hbar\kappa^{-1}\sum_{p\in\mathcal{P}_{\gamma}}\left(\widehat{\mathcal{E}}[\alpha]-\frac{i\kappa k}{2\pi}\widehat{F}[\alpha]\right)(p) \tag{5.310}$$

with $\widehat{\mathscr{G}}[\alpha]$ the Gauss constraint operator acting on the bulk Hilbert space given by (5.256). Assuming that the smearing function α vanishes on the boundary, the full constraint operator (5.307) reduces to the bulk Gauss constraint $\widehat{\mathscr{G}}[\alpha]$ implying gauge-invariance of the quantum state in the bulk. As a consequence, from (5.310), one obtains the additional constraint equation

$$\mathbb{1}\otimes\widehat{F}_{\underline{A}}(p)=-\frac{2\pi i}{\kappa k}\widehat{\mathcal{E}}_{\underline{A}}(p)\otimes\mathbb{1} \tag{5.311}$$

at each puncture $p\in\mathcal{P}_{\gamma}$. Note that, by definition, $\widehat{\mathcal{E}}_{\underline{A}}(p)$ can be related to the quantized super electric flux via $\widehat{\mathcal{E}}_{\underline{A}}(p)=\lim_{\epsilon\to 0}\widehat{\mathcal{X}}_{\underline{A}}(D_{\epsilon})$ and thus, according to (5.185), acts in terms of right- resp. left-invariant vector fields. Hence, from (5.311), we deduce that the Hilbert space of the quantized boundary degrees of freedom corresponds

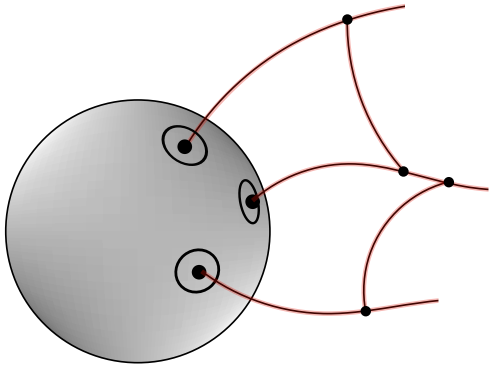


Figure 6.: A pictorial representation of a supersymmetric black hole in chiral LQSG. The super spin network states induce non-trivial Chern-Simons degrees of freedom (black circles) at the intersection points (punctures) with the boundary which can account for black hole entropy. In the case that the super spin network edges have some infinitesimal but nonzero width, these punctures are blown up to disks leading to new physical degrees of freedom living on the corner and which are associated to superconformal field theories.

to the Hilbert space of a quantized super Chern-Simons theory on Δ with punctures $\mathcal{P}_\gamma$ (see Figure 6). This leads to the well-known (super)conformal blocks. In the pure bosonic theory, these play an important role in the context of the computation of the black hole entropy.

As already outlined above, in [222], an alternative route in describing the entropy of black hole has been studied. More precisely, assuming that the edges piercing the boundary are of infinitesimal but nonzero width, this induces infinitesimal holes localized at the punctures on the boundary which then gives rise to new physical degrees of freedom that are localised at the corner $\partial\Delta$.

In the following, let us describe these new degrees of freedom in the context of chiral supergravity. To this end, generalizing the discussion in [195] in context of the bosonic theory to the super category, let us consider the following quantities defined on the canonical phase space of the super Chern-Simons theory

$$
\begin{aligned}
O[\alpha] &:= -\frac{k}{2\pi}\int_\Delta \langle \alpha F(\mathcal{A}^+)\rangle + \frac{k}{2\pi}\int_{\partial\Delta}\langle \alpha\mathcal{A}^+\rangle \\
&= \frac{k}{2\pi}\int_\Delta \langle \mathrm{d}\alpha\wedge\mathcal{A}^+ - \frac{1}{2}\alpha[\mathcal{A}^+\wedge\mathcal{A}^+]\rangle
\end{aligned} \tag{5.312}
$$

where α denotes an arbitrary Lie$(\mathcal{G})$-valued smearing function on Δ. In case that α vanishes on the corner, this quantity reduces to the flatness constraint (5.306), i.e.,

$\mathcal{O}[\alpha] \equiv \mathcal{F}[\alpha]$ if $\alpha|_{\partial\Delta} = 0$. Computing the graded Poisson bracket between $\mathcal{O}[\alpha]$ and the super connection, one finds

$$\{\mathcal{O}[\alpha], \mathcal{A}_a^{+\underline{A}}\} = D_a^{(\mathcal{A}^+)} \alpha^{\underline{A}} \tag{5.313}$$

This is in fact immediate to see using (5.305). For instance, direct calculation yields

$$\begin{aligned}&\{\frac{k}{2\pi}\int_\Delta \langle \mathrm{d}\alpha \wedge \mathcal{A}^+\rangle, \mathcal{A}_a^{+\underline{A}}(x)\}\\ =&\frac{k}{2\pi}\int_\Delta \mathrm{d}^2 y\, \epsilon^{bc} \mathscr{S}_{\underline{CB}} \partial_b \alpha^{\underline{B}}(y) \{\mathcal{A}_c^{+\underline{C}}(y), \mathcal{A}_a^{+\underline{A}}(x)\} = \partial_a \alpha^{\underline{A}}(x)\end{aligned} \tag{5.314}$$

On the other hand, one has

$$-\frac{k}{4\pi}\{\int_\Delta \langle \alpha[\mathcal{A}^+ \wedge \mathcal{A}^+]\rangle, \mathcal{A}_a^{+\underline{A}}(x)\} = [\mathcal{A}_a^+, \alpha]^{\underline{A}}(x) \tag{5.315}$$

which, together with (5.314), directly gives (5.313). With these preparations, let us next compute the Poisson algebra among the $\mathcal{O}[\alpha]$. Using identity (5.313), it follows for arbitrary smearing functions α and β that

$$\begin{aligned}\{\mathcal{O}[\alpha], \mathcal{O}[\beta]\} &= \frac{k}{2\pi}\int_\Delta (-1)^{|\alpha||\beta|} \langle \mathrm{d}\beta \wedge D^{(\mathcal{A}^+)}\alpha - \beta[\mathcal{A}^+ \wedge D^{(\mathcal{A}^+)}\alpha]\rangle\\ &= -\frac{k}{2\pi}\int_\Delta \langle D^{(\mathcal{A}^+)}\alpha \wedge D^{(\mathcal{A}^+)}\beta\rangle\end{aligned} \tag{5.316}$$

Since $D^{(\mathcal{A}^+)} D^{(\mathcal{A}^+)}\beta = [F(\mathcal{A}^+), \beta]$, one has

$$\begin{aligned}\langle D^{(\mathcal{A}^+)}\alpha \wedge D^{(\mathcal{A}^+)}\beta\rangle &= \mathrm{d}\langle \alpha D^{(\mathcal{A}^+)}\beta\rangle - \langle \alpha[F(\mathcal{A}^+), \beta]\rangle\\ &= \langle \mathrm{d}\alpha \wedge \mathrm{d}\beta\rangle - \mathrm{d}\langle [\alpha, \beta]\mathcal{A}^+\rangle + \langle [\alpha, \beta] F(\mathcal{A}^+)\rangle\end{aligned} \tag{5.317}$$

Thus, inserting (5.317) into (5.316) and assuming that α is vanishes on the corner $\partial\Delta$, it follows

$$\{\mathcal{F}[\alpha], \mathcal{O}[\beta]\} = \mathcal{F}[[\alpha, \beta]] \simeq 0 \tag{5.318}$$

where we used that $[\alpha, \beta]|_{\partial\Delta} = 0$. Thus, it follows that $\mathcal{O}[\alpha]$ weakly Poisson commutes with the flatness constraint. That is, $\mathcal{O}[\alpha]$ defines a weak *Dirac observable*. Moreover, for smearing functions α and and β with $\alpha|_{\partial\Delta} = \beta|_{\partial\Delta}$, one has

$$\mathcal{O}[\alpha] - \mathcal{O}[\beta] = \mathcal{O}[\alpha - \beta] \equiv -\mathcal{F}[\alpha - \beta] \simeq 0 \tag{5.319}$$

Hence, it follows that the observables $\mathcal{O}[\alpha]$ are localized on the corner. Furthermore, by (5.316) and (5.317), they satisfy the following graded Poisson relations

$$\{\mathcal{O}[\alpha],\mathcal{O}[\beta]\}=\mathcal{O}[[\alpha,\beta]]+\frac{k}{2\pi}\int_{\partial\Delta}\langle\mathrm{d}\alpha,\beta\rangle \tag{5.320}$$

Since, the last term on the right-hand side of Equation (5.320) is completely field-independent, it, in particular, Poisson commutes with all the corner observables $\mathcal{O}[\alpha]$. Thus, it follows that the Poisson algebra among the $\mathcal{O}[\alpha]$ is indeed closed up to a central term.

In this context, recall that, given an Abelian (bosonic) Lie algebra $\mathfrak{a}$, a *central extension* of a super Lie algebra $\mathfrak{g}$ (not necessarily finite-dimensional) by $\mathfrak{a}$ is defined as a short exact sequence [226]

$$0\longrightarrow\mathfrak{a}\longrightarrow\mathfrak{h}\stackrel{\pi}{\longrightarrow}\mathfrak{g}\longrightarrow 0 \tag{5.321}$$

with $\mathfrak{h}$ a super Lie algebra such that $[\mathfrak{a},\mathfrak{h}]=0$ and $\pi:\ \mathfrak{h}\longrightarrow\mathfrak{g}$ an even surjective super Lie algebra morphism yielding the identification $\mathfrak{h}/\mathfrak{a}\cong\mathfrak{g}$.

In our concrete situation, at each puncture, $\partial\Delta$ is topologically equivalent to a 1-dimensional circle. Thus, in this case, it follows that a basis of smearing functions α is given by functions $\alpha_N^{\underline{A}}$ of the form

$$\alpha_N^{\underline{A}}|_{\partial\Delta}:=e^{iN\theta}T^{\underline{A}},\quad \alpha_N^{\underline{A}}|_{\Delta\backslash\partial\Delta}\equiv 0 \tag{5.322}$$

where $\theta\in[0,2\pi]$ denotes the angle coordinate parametrizing the circle, $N\in\mathbb{Z}$ and $(T^{\underline{A}})_{\underline{A}}$ is a homogeneous basis of $\mathfrak{osp}(\mathcal{N}|2)_{\mathbb{C}}$. From (5.320), it then follows that the corresponding corner observables $q_N^{\underline{A}}:=\mathcal{O}[\alpha_N^{\underline{A}}]$ satisfy the Poisson relations

$$\{q_M^{\underline{A}},q_N^{\underline{B}}\}=f^{\underline{A}\underline{B}}{}_{\underline{C}}\,q_{M+N}^{\underline{C}}+N\delta_{M+N,0}(T^{\underline{A}},T^{\underline{B}}) \tag{5.323}$$

where $(T^{\underline{A}},T^{\underline{B}}):=ik\,\langle T^{\underline{A}},T^{\underline{B}}\rangle$ and $f^{\underline{A}\underline{B}}{}_{\underline{C}}$ denote the structure coefficients defined via

$$[T^{\underline{A}},T^{\underline{B}}]=f^{\underline{A}\underline{B}}{}_{\underline{C}}\,T^{\underline{C}} \tag{5.324}$$

Interestingly, (5.324) are precisely the graded commutation relations of a *Kac-Moody superalgebra* corresponding to the affinization of $\mathfrak{osp}(\mathcal{N}|2)_{\mathbb{C}}$ [226]. It follows via the so-called *Sugawara construction*, that the generators of the Kac-Moody superalgebra can be used in order to generate representations of the *super Virasoro algebra* [227]. Thus, to conclude, the singularities induced by the intersection of super spin networks with the boundary give rise to new physical degrees of freedom living on the corner which are associated to *superconformal field theories* and which, in analogy to [222] in context of the bosonic theory, may also account for black hole entropy and hence may play a role in the quantum description of supersymmetric black holes in the framework of LQG.

5.7. Some results on super Peter Weyl theory

In the standard quantization program of LQG, spin network states play a very prominent role as they provide a quite useful basis of the resulting Hilbert space of the theory. This is based on the famous Peter-Weyl theorem stating that, in context of compact Lie groups G, the matrix coefficients of the, necessarily finite-dimensional, unitary irreducible representations of G automatically define an orthonormal basis of the Hilbert space $L^2(G)$ induced by the unique normalized invariant Haar measure on G.

Unfortunately, such a strong statement seems, in general, no to be available in the context of super Lie groups. This may be related to the fact that finite-dimensional representations of compact super Lie groups turn out to be not necessarily reducible in contrast to the classical setting. There exist, however, some partial results in this direction stating that the matrix coefficients of the finite-dimensional representations of a super Lie group $\mathcal{G}$ form a dense subspace of the super vector space $H^\infty(\mathcal{G},\mathbb{C})$ w.r.t. a suitable topology (see for instance [228] for a general discussion as well as [229, 230] considering the special cases $\mathcal{G}=\mathbb{S}^{1|1}$ and $\mathcal{G}=\mathrm{SU}(1|1)$). But, in the context of LQG, we are rather interested in an "integral version" of the Peter-Weyl theorem, i.e., we would like to know whether the matrix coefficients also provide, at least in a specific sense, an orthonormal basis of the induced super Hilbert space. As mentioned already in Section 5.5.3, first results in this direction have been discussed in [207] where a Peter-Weyl-type basis for the unitary orthosymplectic group $\mathrm{UOSp}(1|2)$ has been constructed. This is based crucially on the fact that the underyling bosonic group is compact and, in particular, that the unit function is normalizable.

Hence, the question arises whether similar results also hold true if one considers more general compact super Lie groups for which the unit function is not normalizable and thus, as a consequence, finite-dimensional representations are not necessarily fully reducible. In the following, we want to address this question by considering a "simpler" example given by the super unitary group $\mathrm{U}(1|1)$ (see Example 5.5.11). To this end, adapting techniques developed in [230] in the context of the sub super Lie group $\mathrm{SU}(1|1):=\{g\in\mathrm{U}(1|1)|\,\mathrm{Ber}(g)=1\}$, we first analyze the finite-dimensional representations and, in particular, determine and classify irreducible representations of $\mathrm{U}(1|1)$. Hence, suppose that

$$\pi:\ \mathrm{U}(1|1)\to\mathrm{GL}(\mathcal{V}) \tag{5.325}$$

is an irreducible representation of $\mathrm{U}(1|1)$ on some finite-dimensional super Λ-vector space $\mathcal{V}=V\otimes\Lambda$ with V complex. Then, let us restrict to the bosonic super Lie subgroup $\mathrm{U}(1|1)_0 \cong \mathbf{S}(\mathrm{U}(1))^2$. As it is well known, the irreducible representations of $\mathrm{U}(1)$ are given by $\{(\pi_m, V_m)\}_{m\in\mathbb{Z}}$ where $V_m \cong \mathbb{C}$ and $\pi_m : \mathrm{U}(1) \to \mathrm{GL}(1,\mathbb{C}),\ z \mapsto z^m\ \forall m \in \mathbb{Z}$. Accordingly, the irreducible representations of $\mathrm{U}(1)\times\mathrm{U}(1)$ are given by $\{(\pi_{m,n}, V_{m,n})\}_{m,n\in\mathbb{Z}}$ with $V_{m,n} := V_m \otimes V_n \cong \mathbb{C}$ and

$$\pi_{m,n} : \mathrm{U}(1) \to \mathrm{GL}(1,\mathbb{C}),\ z \mapsto z^{m+n} \tag{5.326}$$

$\forall m,n \in \mathbb{Z}$. Hence, by applying the split functor, it follows immediately that the corresponding complete family of inequivalent irreducible representations of $\mathrm{U}(1|1)_0$ is given by $\{(\mathbf{S}(\pi_{m,n}), \mathcal{V}_{(m,n),0})\}_{m,n\in\mathbb{Z}} \cup \{(\mathbf{S}(\pi_{m,n}), \mathcal{V}_{0,(m,n)})\}_{m,n\in\mathbb{Z}}$ where $\mathcal{V}_{(m,n),0} := V_{m,n}\otimes\Lambda$ and $\mathcal{V}_{0,(m,n)} := \Pi V_{m,n}\otimes\Lambda\ \forall m,n\in\mathbb{Z}$ with $\Pi : \mathbf{SVec}\to\mathbf{SVec}$ the parity functor, i.e., $\Pi V_{m,n}$ is regarded as a purely odd super vector space. Thus, this leads to a decomposition of $\mathcal{V}$ of the form

$$\mathcal{V} \cong \bigoplus_{(m,n)\in\mathbb{Z}^2} k_{m,n}\mathcal{V}_{(m,n),0} \oplus \bigoplus_{(p,q)\in\mathbb{Z}^2} l_{p,q}\mathcal{V}_{0,(p,q)} \tag{5.327}$$

with multiplicities $k_{m,n}, l_{p,q} \in \mathbb{N}_0$, $(m,n),(p,q)\in\mathbb{Z}^2$. In order to find the irreducible submodule, note that, as $\mathrm{U}(1|1)$ is connected (in the DeWitt-topology) and π is supposed to be irreducible, it follows that the pushforward representation $\pi_* : \mathrm{Lie}(\mathrm{U}(1|1)) \to \underline{\mathrm{End}}_R(\mathcal{V})$ is also irreducible. Hence, in what follows, let us look for irreducible representations of the corresponding super Lie module. To this end, after complexification, let us consider the super Lie algebra elements $\Theta_\pm := \Theta_1 \pm i\Theta_2$ satisfying the following graded commutation relations

$$\begin{aligned}
&[X_1,\Theta_\pm] = \mp i\Theta_\pm, \quad [X_2,\Theta_\pm] = \pm i\Theta_\pm, \quad [\Theta_\pm,\Theta_\pm] = 0\\
&[\Theta_+,\Theta_-] = -4X := -4(X_1+X_2)
\end{aligned} \tag{5.328}$$

Let $(m,n)\in\mathbb{Z}^2$ with $k_{m,n}\neq 0$ and choose a nonzero vector $v\in k_{m,n}\mathcal{V}_{(m,n),0}$. Let $v_- := \pi_*^{\mathbb{C}}(\Theta_-)v$ and $v_+ := \pi_*^{\mathbb{C}}(\Theta_+)v$ where $\pi_*^{\mathbb{C}}$ denotes the complification of the pushforward representation π_*. Using the commutation relations (5.328), it is immediate to see that $v_-\in l_{m+1,n-1}\mathcal{V}_{0,(m+1,n-1)}$, $v_+\in l_{m-1,n+1}\mathcal{V}_{0,(m-1,n+1)}$ as well as $\pi_*^{\mathbb{C}}(\Theta_-)v_- = 0$ and $\pi_*^{\mathbb{C}}(\Theta_+)v_+ = 0$. Set $v' := \pi_*^{\mathbb{C}}(\Theta_+)v_- \in k_{m,n}\mathcal{V}_{(m,n),0}$. Then, again by (5.328), it follows that

$$\pi_*^{\mathbb{C}}(\Theta_-)v' = \pi_*^{\mathbb{C}}(\Theta_-)\pi_*^{\mathbb{C}}(\Theta_+)v_- = -4i(m+n)v_- \tag{5.329}$$

Finally, we compute

$$\begin{aligned}
\pi_*^{\mathbb{C}}(\Theta_-)v_+ = \pi_*^{\mathbb{C}}(\Theta_-)\pi_*^{\mathbb{C}}(\Theta_+)v &= -4\pi_*^{\mathbb{C}}(X)v - \pi_*^{\mathbb{C}}(\Theta_+)\pi_*^{\mathbb{C}}(\Theta_-)v\\
&= -4i(m+n)v - v'
\end{aligned} \tag{5.330}$$

Hence, we found that there exists an invariant submodule of the form

$$2\mathcal{V}_{(m,n),0} \oplus \mathcal{V}_{0,(m+1,n-1)} \oplus \mathcal{V}_{0,(m-1,n+1)} \tag{5.331}$$

generated by the vectors (v, v', v_-, v_+). In fact, this submodule can be reduced even further. To see this, similar as in [230] in the context of the sub super Lie group $\mathrm{SU}(1|1)$, consider the submodule $\mathcal{V}_{(m,n)} \cong \mathcal{V}_{(m,n),0} \oplus \mathcal{V}_{0,(m-1,n+1)}$ generated by the vectors (w_1, w_2) given by $w_1 := \pi_*^{\mathbb{C}}(\Theta_-)v_+$ and $w_2 := 2i\sqrt{i(m+n)}v_+$. By definition, it is then immediate to see that this submodule is also invariant under $\mathrm{U}(1|1)$ and that, w.r.t. that basis, $\pi_*(\Theta_1)$ and $\pi_*(\Theta_2)$ are given by

$$\pi_*(\Theta_1) = \begin{pmatrix} 0 & i\sqrt{i(m+n)} \\ i\sqrt{i(m+n)} & 0 \end{pmatrix}, \quad \pi_*(\Theta_2) = \begin{pmatrix} 0 & -\sqrt{i(m+n)} \\ \sqrt{i(m+n)} & 0 \end{pmatrix} \tag{5.332}$$

and hence π is in fact irreducible on $\mathcal{V}_{(m,n)}$. Let us denote this representation by $(\pi_{(m,n)}, \mathcal{V}_{(m,n)})$.

Finally, let us consider the submodule $\mathcal{V}'_{(m,n)} \cong \mathcal{V}_{(m,n),0} \oplus \mathcal{V}_{0,(m+1,n-1)}$ generated by the vectors (w_1', w_2') given by $w_1' := v' = \pi_*^{\mathbb{C}}(\Theta_+)v_-$ and $w_2' := 2i\sqrt{i(m+n)}v_-$. Again, it follows by direct computation that $\mathcal{V}'_{(m,n)}$ defines an irreducible submodule of π and that $\pi_*(\Theta_1)$ and $\pi_*(\Theta_2)$ have the following matrix representation

$$\pi_*(\Theta_1) = \begin{pmatrix} 0 & i\sqrt{i(m+n)} \\ i\sqrt{i(m+n)} & 0 \end{pmatrix}, \quad \pi_*(\Theta_2) = \begin{pmatrix} 0 & \sqrt{i(m+n)} \\ -\sqrt{i(m+n)} & 0 \end{pmatrix} \tag{5.333}$$

We denote this representation by $(\pi', \mathcal{V}'_{(m,n)})$. In fact, it turns out that $(\pi'_{(m,n)}, \mathcal{V}'_{(m,n)})$ can be related to the irreducible representation $(\pi_{(m+1,n-1)}, \mathcal{V}_{(m+1,n-1)})$ via the odd intertwining map

$$\begin{aligned} F: \mathcal{V}'_{(m,n)} \to \mathcal{V}_{(m+1,n-1)},\ & w_1' \mapsto i\sqrt{i(m+n)}w_2 \\ & w_2' \mapsto i\sqrt{i(m+n)}w_1 \end{aligned} \tag{5.334}$$

such that $\pi_{(m+1,n-1)} \circ F = F \circ \pi'_{(m,n)}$. Usually, in the mathematical literature, the intertwining map between two *equivalent* or *isomorphic* representations is defined as an *even* bijective morphism between super Λ-vector spaces. Thus, in this sense, the representations $(\pi'_{(m,n)}, \mathcal{V}'_{(m,n)})$ and $(\pi_{(m+1,n-1)}, \mathcal{V}_{(m+1,n-1)})$ would be inequivalent. However, since these representations do not generate new matrix coefficients, we will follow rather [109] and regard these representations also as equivalent (cf. Remark 4.4 in [109]). Thus, we arrive at the following result which provides an extension of the results obtained in [230] in the context of $\mathrm{SU}(1|1)$:

Theorem 5.7.1. *The irreducible representations of* $\mathrm{U}(1|1)$ *are, up to isomorphism, given by* $\{(\pi_{(m,n)},\mathcal{V}_{(m,n)})\}_{m,n\in\mathbb{Z}}$ *with* $\mathcal{V}_{(m,n)}=\mathcal{V}_{(m,n),0}\oplus\mathcal{V}_{0,(m-1,n+1)}\cong(\Lambda^{\mathbb{C}})^{1,1}$ *and* $\pi_{(m,n)}(X)$*, for* $g\in\mathrm{U}(1|1)$ *in the form* (2.80)*, given by*

$$\pi_{(m,n)}(g)=\begin{pmatrix}x^m y^n(1+\frac{i}{2}(m+n)\psi\bar{\psi}) & x^m y^n i\sqrt{i(m+n)}\psi\\ -x^{m-1}y^{n+1}i\sqrt{i(m+n)}\bar{\psi} & x^{m-1}y^{n+1}(1-\frac{i}{2}(m+n)\psi\bar{\psi})\end{pmatrix}\tag{5.335}$$

Moreover, the corresponding pushforward representation $\pi_{(m,n)*}:\mathrm{Lie}(\mathrm{U}(1|1))\to\underline{\mathrm{End}}_R(\mathcal{V}_{(m,n)})$ *of the super Lie module* $\mathrm{Lie}(\mathrm{U}(1|1))=\Lambda\otimes\mathfrak{u}(1|1)$ *takes the form*

$$\pi_{(m,n)*}(X_1)=\begin{pmatrix}im & 0\\ 0 & i(m-1)\end{pmatrix},\quad \pi_{(m,n)*}(X_2)=\begin{pmatrix}in & 0\\ 0 & i(n+1)\end{pmatrix}\tag{5.336}$$

for the even generators $X_1,X_2\in\mathfrak{u}(1|1)_0$ *as well as*

$$\pi_{(m,n)*}(\Theta_1)=\begin{pmatrix}0 & i\sqrt{i(m+n)}\\ i\sqrt{i(m+n)} & 0\end{pmatrix}\tag{5.337}$$

$$\pi_{(m,n)*}(\Theta_2)=\begin{pmatrix}0 & -\sqrt{i(m+n)}\\ \sqrt{i(m+n)} & 0\end{pmatrix}\tag{5.338}$$

for the odd generators $\Theta_1,\Theta_2\in\mathfrak{u}(1|1)_1$.

Proof. The following proof is inspired by the proof of Theorem 4.4 in [230] where an explicit form of the irreducible representations of the super Lie group $\mathrm{SU}(1|1)$ is derived (see also Remark 5.7.3 below). Recall from Equation (2.80) in Example 2.3.16, that, under the super Harish-Chandra isomorphism Φ, a generic group element $g\in\mathrm{U}(1|1)$ can be written in the form

$$g=\begin{pmatrix}xA & x\psi\\ iy\bar{\psi} & yA^{-1}\end{pmatrix}\tag{5.339}$$

with $x,y\in\mathbf{S}(\mathrm{U}(1))$, $\psi\in\Lambda_1$ and $A=1+\frac{i}{2}\psi\bar{\psi}$. Let us then try to split g in the following way

$$g=\begin{pmatrix}t & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & s\end{pmatrix}\begin{pmatrix}1 & \alpha\\ i\alpha & 1\end{pmatrix}\begin{pmatrix}1 & i\beta\\ \beta & 1\end{pmatrix}\tag{5.340}$$

for certain coefficients $t,s\in\Lambda_0$ and $\alpha,\beta\in\Lambda_1$. This yields the equation

$$\begin{pmatrix}xA & x\psi\\ iy\bar{\psi} & yA^{-1}\end{pmatrix}=\begin{pmatrix}t(1+\alpha\beta) & t(\alpha+i\beta)\\ is(\alpha-i\beta) & s(1-\alpha\beta)\end{pmatrix}\tag{5.341}$$

from which we can immediately read off $t = x$ and $s = y$ as well as $\alpha = \Re(\psi)$ and $\beta = \Im(\psi)$. By (5.336), we then deduce

$$\pi_{(m,n)}\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x^m & 0 \\ 0 & x^{m-1} \end{pmatrix}, \quad \pi_{(m,n)}\begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} y^n & 0 \\ 0 & y^{n+1} \end{pmatrix} \tag{5.342}$$

Moreover, by (5.337) and (5.338), it follows that

$$\begin{aligned} \pi_{(m,n)}\begin{pmatrix} 1 & \alpha \\ i\alpha & 1 \end{pmatrix} &= \pi_{(m,n)}\left(e^{\alpha\Theta_1}\right) = e^{\alpha\pi_{(m,n)*}(\Theta_1)} \\ &= \begin{pmatrix} 1 & i\sqrt{i(m+n)}\alpha \\ -i\sqrt{i(m+n)}\alpha & 1 \end{pmatrix} \end{aligned} \tag{5.343}$$

as well as

$$\begin{aligned} \pi_{(m,n)}\begin{pmatrix} 1 & i\beta \\ \beta & 1 \end{pmatrix} &= \pi_{(m,n)}\left(e^{\alpha\Theta_2}\right) = e^{\alpha\pi_{(m,n)*}(\Theta_2)} \\ &= \begin{pmatrix} 1 & -\sqrt{i(m+n)}\beta \\ -\sqrt{i(m+n)}\beta & 1 \end{pmatrix} \end{aligned} \tag{5.344}$$

Using (5.340), this then immediately yields (5.335). □

Corollary 5.7.2. *Let* $\{(\pi_{(m,n)}, \mathcal{V}_{(m,n)})\}_{m,n\in\mathbb{Z}}$ *be the irreducible representations of* $\mathrm{U}(1|1)$ *as stated in Theorem 5.7.1. Then, the coefficient functions* $(\pi_{(m,n)})^i{}_j$, $i, j = 1, 2$, *for any* $m, n \in \mathbb{Z}$ *are orthogonal w.r.t. the super scalar product* $\mathscr{S}$ *as defined via* (5.231). *More precisely, they satisfy the following identities*

$$\begin{aligned} \mathscr{S}((\pi_{(m,n)})^1{}_1, (\pi_{(m,n)})^1{}_1) &= -(m+n)\delta_{m,p}\delta_{n,q} \\ \mathscr{S}((\pi_{(m,n)})^2{}_2, (\pi_{(m,n)})^2{}_2) &= (m+n)\delta_{m,p}\delta_{n,q} \\ \mathscr{S}((\pi_{(m,n)})^1{}_2, (\pi_{(m,n)})^1{}_2) &= -i|m+n|\delta_{m,p}\delta_{n,q} \\ \mathscr{S}((\pi_{(m,n)})^2{}_1, (\pi_{(m,n)})^2{}_1) &= i|m+n|\delta_{m,p}\delta_{n,q} \end{aligned} \tag{5.345}$$

with all remaining combinations being zero.

Proof. This follows immediately by direct computation using the explicit form (5.335) of the matrix coefficients and the invariant Haar measure (5.230). For instance, we find

$$\begin{aligned}\mathscr{S}((\pi_{(m,n)})^1{}_1,(\pi_{(m,n)})^1{}_1)=&\int_0^{2\pi}\frac{\mathrm{d}\phi}{2\pi}\,e^{i(p-m)\phi}\int_0^{2\pi}\frac{\mathrm{d}\phi'}{2\pi}\,e^{i(q-n)\phi'}\times\\&\times\int_B i\mathrm{d}\psi\mathrm{d}\bar{\psi}\,\left(1+\frac{i}{2}(m+n)\psi\bar{\psi}\right)\left(1+\frac{i}{2}(p+q)\psi\bar{\psi}\right)\\=&\delta_{m,p}\delta_{n,q}\int_B i\mathrm{d}\psi\mathrm{d}\bar{\psi}\,\left(1+i(m+n)\psi\bar{\psi}\right)\\=&-(m+n)\delta_{m,p}\delta_{n,q}\end{aligned}\tag{5.346}$$

On the other hand, we have

$$\begin{aligned}\mathscr{S}((\pi_{(m,n)})^1{}_2,(\pi_{(m,n)})^1{}_2)=&\int_0^{2\pi}\frac{\mathrm{d}\phi}{2\pi}\,e^{i(p-m)\phi}\int_0^{2\pi}\frac{\mathrm{d}\phi'}{2\pi}\,e^{i(q-n)\phi'}\times\\&\times\int_B i\mathrm{d}\psi\mathrm{d}\bar{\psi}\,\sqrt{(m+n)(p+q)}\bar{\psi}\psi\\=&-|m+n|\delta_{m,p}\delta_{n,q}\int_B i\mathrm{d}\psi\mathrm{d}\bar{\psi}\,\psi\bar{\psi}\\=&-i|m+n|\delta_{m,p}\delta_{n,q}\end{aligned}\tag{5.347}$$

The rest follows similarly. □

In Example 5.5.11, we have constructed a super Hilbert space $(\mathfrak{H},\langle\cdot|\cdot\rangle_J,\mathscr{S})$ associated to the indefinite inner product space $(V,\mathscr{S})$ with $V:=H^\infty(\mathrm{U}(1|1),\mathbb{C})$ and $\mathscr{S}$ the super scalar product induced by the invariant Haar measure on $\mathrm{U}(1|1)$ defined via (5.231) performing a Krein completion. According to Corollary 5.7.2, it follows that the matrix coefficients of the finite-dimensional irreducible representations of $\mathrm{U}(1|1)$ indeed have the expected properties as they are normalizable up to signs and factors of $\pm i$ and are mutually orthogonal, i.e., they induce an orthonormal system in the super Krein space. However, they do not form a basis of $\mathfrak{H}$. To see this, note that the labels $(m,n)\in\mathbb{Z}^2$ of the irreducible representations $\pi_{(m,n)}$ need to satisfy the condition $m+n\neq 0$. As a consequence, the constant unit function $\mathbb{1}$ on $\mathrm{U}(1|1)$ as well as elementary functions of the form $x^m y^{-m}$ or $x^m y^{-m}\psi$ etc. with $m\in\mathbb{Z}$ are not contained in the corresponding subspace generated by these matrix coefficients. The remaining coefficients can be obtained, for instance, considering tensor product representations of the form

$$\pi_{(m,0)}\otimes\pi_{(0,-m)}\tag{5.348}$$

with $m\in\mathbb{Z}\setminus\{0\}$. These correspond to representations of $\mathrm{U}(1|1)$ with respect to which the bosonic generator $X=X_1+X_2$ of the diagonal subgroup $\mathrm{U}(1)\subset\mathrm{U}(1)\times\mathrm{U}(1)$

is trivially represented. By definition, they are orthogonal to the matrix coefficients of the irreps $\pi_{(m,n)}$. In particular, together with the trivial representation, it follows that this in fact provides a (overcomplete) set of elementary functions whose corresponding complex linear span is dense in $\mathfrak{H}$. Note, however, that representations corresponding to the zero eigenvalue of X are generically not semisimple. This has been discussed, e.g., in the context of SU$(1|1)$ in [230]. This is a phenomenon occuring in the supersymmetry setting and is directly related to the normalizability of the unit function leading to a classification of finite-dimensional representations into so-called *typical* and *atypical* representations.

Remark 5.7.3. Restricting the irreducible representations $\pi_{(m,n)}$ of the super unitary group $\mathrm{U}(1|1)$ with $(m,n) \in \mathbb{Z}^2$ and $m+n \neq 0$ as listed in Theorem 5.7.1 to the sub super Lie group $\mathrm{SU}(1|1)$ which corresponds to the choice $x = y$ in (2.80), it follows that these exactly reproduces the irreps $\pi_k := \pi_{(m,n)}|_{\mathrm{SU}(1|1)}$ with $k := m+n \in \mathbb{Z}\setminus\{0\}$ as derived in [230]. Again, it follows that the corresponding matrix coefficients are normalizable and mutually orthogonal thus inducing an orthonormal system on a super Krein space. Since the representation label k needs to be nonvanishing, it follows that the constant unit function $\mathbb{1}$ on $\mathrm{SU}(1|1)$ as well as functions of the form ψ, $\bar{\psi}$ and $\psi\bar{\psi}$ are not contained in the complex linear span of the matrix coefficients of π_k. As above, it follows that the remaining coefficients arise from finite-dimensional representations of $\mathrm{SU}(1|1)$ w.r.t. which the bosonic generator X is trivially represented.

Let π_0 denote such a representation and $\widehat{\Theta}_i := \pi_{0*}(\Theta_i)$ for $i = 1, 2$ and $\widehat{X} := \pi_{0*}(X)$. Then, since $\widehat{X} \equiv 0$, it follows from (2.77) that

$$\widehat{\Theta}_i^2 = 0 \text{ for } i = 1,2 \text{ and } \widehat{\Theta}_1\widehat{\Theta}_2 = -\widehat{\Theta}_2\widehat{\Theta}_1 \tag{5.349}$$

That is, π_0 can be identified with a representation of the Grassmann algebra Λ_2 generated by two Grassmann variables θ_i, $i = 1, 2$. A natural candidate of such a representation would be the standard representation on Λ_2 itself via left multiplication setting $\widehat{\Theta}_i = \theta_i$ for $i = 1, 2$. With respect to the homogeneous basis $(1, \theta_1\theta_2, \theta_1, \theta_2)$ of Λ_2, it follows that $\widehat{\Theta}_i$ acquire the following matrix representations

$$\widehat{\Theta}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \widehat{\Theta}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{5.350}$$

Hence, for $g \in \mathrm{SU}(1|1)$ of the form (2.80) with $x = y$, this gives

$$\pi_0(g) = (\mathbb{1} + \xi\widehat{\Theta}_1)(\mathbb{1} + \eta\widehat{\Theta}_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\xi\eta & 1 & -\eta & \xi \\ -\xi & 0 & 1 & 0 \\ -\eta & 0 & 0 & 1 \end{pmatrix} \tag{5.351}$$

which precisely yields the remaining matrix coefficients.

5.8. Discussion

In this chapter, we have addressed several questions concerning the classical description of (extended) supergravity theories in $D = 4$ in terms of a new type of action we called Holst-MacDowell-Mansouri action, in the presence of boundaries. For a special choice of the Barbero-Immirzi parameter, one obtains a description in terms of chiral Ashtekar-type variables in which case the theory has many interesting properties and which seem to be of particular relevance for applications in LQG. In particular, we considered the question of how to properly include boundary terms in the theory. This is crucial as the standard treatment of inner boundaries in (super)gravity theories expressed in terms of (real) Ashtekar-Barbero variables is based on the isolated horizon (IH) formalism which, so far, does not take into account local supersymmetry invariance at the boundary.[9]

Hence, we have followed a different route using new developments in the geometric approach to supergravity. More precisely, following [81, 83], we have discussed the most general ansatz of possible boundary terms to be added to the bulk action of $\mathcal{N} = 1$ and $\mathcal{N} = 2$ pure Holst-AdS supergravity in $D = 4$. [81, 83] show that the boundary terms are fixed uniquely if one requires invariance of the full action under supersymmetry transformations at the boundary. Moreover, it follows that the resulting action in both cases acquires a very intriguing form extending the well-known MacDowell-Mansouri action [138] even to supergravity theories with extended supersymmetry [81, 83].

Based on these results, we have derived a Holst variant of the MacDowell-Masouri action including topological terms for arbitrary Barbero-Immirzi paramters β for the cases $\mathcal{N} = 1, 2$. To this end, inspired by ideas of [183–185] in context of ordinary first-order Einstein gravity, we introduced a β-deformed inner product defined via a β-dependent operator $\mathbf{P}_\beta$ acting on super Lie algebra-valued forms. We have then shown that the resulting action is indeed independent of the Barbero-Immirzi parameter at second

9 In fact, the standard boundary conditions arising in this formalism even seem to break local supersymmetry.

order, i.e. provided that the spin connection satisfies its field equations, in the sense that all β-dependent terms become purely topological.

Moreover, for this result to be true, in case $\mathcal{N} = 2$, we have seen that this required the inclusion of an additional β-dependent topological term to the Maxwell-kinetic term in the Lagrangian corresponding to the graviphoton field which is commonly known as a θ-term in Yang-Mills theory. Hence, this supports the hypothesis as discussed e.g. in [186], that the Barbero-Immirzi parameter has to be regarded as kind of a θ-ambiguity. We have then studied the boundary terms arising from the Holst action. However, these boundary terms in general turn out to not correspond to a (super) Chern-Simons theory. This is, of course, in contrast to the results in context of ordinary gravity studying non-supersymmetric isolated horizon (IH) boundary conditions with Ashtekar-Barbero variables. There, one finds that, generically, the boundary theory is described via Chern-Simons theories. Nevertheless, one should emphasize that one can construct models where this turns out to be true even in the supersymmetric case. For instance, in [82,191], for classical variables, particular falloff conditions for the physical fields in the $\mathcal{N} = 2$ case where considered leading to a super Chern-Simons theory on the boundary corresponding to a $\mathrm{OSp}(2|2) \times \mathrm{SO}(1,2)$ gauge group. Hence, it is highly suggestive that similar models can also be constructed using real Ashtekar variables. This remains as a task for future investigation.

We have then turned towards the chiral limit of the theory corresponding to an imaginary $\beta = \pm i$ and have seen that the resulting theory has many interesting properties. On the one hand, it follows that the chiral action in both cases, i.e. $\mathcal{N} = 1$ and $\mathcal{N} = 2$, can be written in a way such that it is manifestly invariant under an enlarged gauge symmetry corresponding to the (complex) orthosymplectic group $\mathrm{OSp}(\mathcal{N}|2)_{\mathbb{C}}$ leading to the notion of a super Ashtekar connection. This generalizes and extends previous results obtained e.g. in [63, 84, 86, 179, 181]. In particular, it follows that the boundary action takes the form a super Chern-Simons action with $\mathrm{OSp}(\mathcal{N}|2)_{\mathbb{C}}$ as a gauge group. This confirms the prescient works [35, 86, 182] that saw a close connection between (super)gravity in the bulk and Chern-Simons theory on the boundary. For $\mathcal{N} = 1$, we have also shown that, at the boundary, the full action is indeed invariant under both left- and right-handed supersymmetry transformations. In fact, we have even proven this for arbitrary β in the end of Section 5.2.1. We could show explicitly that this requirement fixes the CS-term as the unique boundary term. In this context, we derived boundary conditions that couple bulk and boundary degrees of freedom. These turned out to be in strong similarity to the standard boundary conditions as typically considered in LQG as they imply coupling between the super electric field and the curvature of the super Ashtekar connection. In fact, similar boundary conditions have been encountered in [86, 182] for $\mathcal{N} = 1$ describing supergravity as a (generalized) constrained topological field theory. In this present work, however, they have been derived starting from the full (unconstrained) supergravity Lagrangian adapted to LQG for both cases $\mathcal{N} = 1$ and $\mathcal{N} = 2$ obtained as the chiral limit of the Holst-MacDowell-Mansouri action and including a discussion about the uniqueness of the boundary theory. Moreover, we were able to show that the boundary conditions are equivalent to the requirement that the chiral projections of the super Cartan curvature vanishes at the boundary which is consistent with the results obtained in [81] in the non-chiral theory. In the future, it would be interesting to investigate how the definition of IH have to be extended to the supersymmetric context to rederive the boundary conditions as studied in this chapter. Isolated horizons in the context of supergravity have been studied in [231]. There, however, one focuses on the purely bosonic sector and thus does not take fermionic degrees of freedom into account.

Using the structure of the chiral theory, in Section 5.5, we derived a graded analog of the holonomy-flux algebra. We have done this in a mathematically rigorous manner working in the category of enriched supermanifolds and using the parallel transport as constructed in Chapter 2. It follows that the functorial dependence on the underlying parametrization supermanifold $\mathcal{S}$ leads to an intriguing structure of the set $\overline{\mathcal{A}}_{\mathcal{S}}$ of generalized super connections which is in strong similarity to Molotkov-Sachse supermanifolds. Moreover, in case that the underlying gauge supergroup is compact, it follows that $\overline{\mathcal{A}}_{\mathcal{S}}$ is projectively Hausdorff.

Based on these observations, we sketched the quantization of the theory adapting standard tools of ordinary LQG with real variables. Moreover, for both cases $\mathcal{N} = 1, 2$, we constructed spin network states as particular kind of states in the corresponding super Hilbert space which for $\mathcal{N} = 1$ have been considered in [84–86]. However, the final picture remained incomplete due to certain difficulties related to the indefiniteness of the Haar measure on supergroups, the non-compactness of the gauge group $\mathrm{OSp}(\mathcal{N}|2)_{\mathbb{C}}$ as well as the open question of how to solve reality conditions. For this reason, in what follows, we will consider a symmetry-reduced model in Chapter 6. In fact, there, we will see that all these issues can be solved consistently. This gives hope that something similar could be achieved for the full theory. Finally, so far, it is not clear whether the spin network states provide a basis of the super Hilbert space as in case of standard LQG. This is due to the lack of a general (integral version of a) Peter-Weyl theorem for (compact) super Lie groups which has been studied only for rare special cases (see for instance [207, 228–230]). We therefore considered the special example $\mathrm{U}(1|1)$ generalizing the results of [230] obained for the $\mathrm{SU}(1|1)$ case to get an idea how such a theorem may look like in context of super Lie groups. In fact, such a generalization turns out to be non-straightforward due to the existence of so-called *atypical representations*. This has to be studied further in the future.

Ultimately, in Section 5.5.4, we compared this quantization scheme with the standard quantization scheme of LQG coupled to fermions [67,80,87] and observed many similarities. Among other things, it follows that the functorial dependence on the parametrization supermanifold requires a Berezin-type measure for the fermionic degrees of freedom in the inner product which is the usual measure considered in LQG. Moreover, the implementation of the reality condition selects a particular endormorphism J which is part of the definition of a super Hilbert space.

There are of course numerous open questions one should address in the future. For instance, this geometric approach to supergravity appears quite powerful in the appropriate description of boundaries as well as the correct implementation of locally supersymmetric boundary conditions. Moreover, as shown in [81], in this way one arrives at a very intriguing structure of the supergravity Lagrangian even in case of extended supersymmetry. It would be very interesting to see how these results could be extended to higher $\mathcal{N} > 2$, or even matter coupled supergravity theories. Moreover, as we have demonstrated in this chapter, this approach seems to be well-adapted to similar questions in LQG and may shed further light on the particularity of the (graded) self-dual variables as well as their possible generalizations to extended SUGRA theories.

On the other hand, these results provide a first step toward the quantum description of boundaries in supergravity in the framework of LQG and possible applications in the context of supersymmetric black holes. This requires a deeper understanding of Chern-Simons theories with supergroup as a gauge group. Super Chern-Simons theories are also of quite recent interest in context of string theory [166]. As explained in the introduction, there, one observes for certain brane configurations that the boundary theory is described in terms of a super Chern-Simons theory with gauge group including e.g. the supergroups $\mathrm{OSp}(m|n)$ and $\mathrm{U}(m|n)$. We suspect that a deeper analysis of super Chern-Simons theories in the framework of LQG may also shed further light on the relation between the quantum description of boundary theories in string theory and LQG.

6. Supersymmetric minisuperspace models in self-dual loop quantum cosmology

6.1. Introduction

As we have seen in Chapter 5, for both cases $\mathcal{N} = 1, 2$, the action of chiral $D = 4$ AdS supergravity with $\mathcal{N}$-extended supersymmetry takes a very intriguing form of a chiral Palatini-type action which is manifestly invariant under an enlarged $\mathrm{OSp}(\mathcal{N}|2)_{\mathbb{C}}$-gauge symmetry. As far as the bulk theory is concerned, this structure also carries over to the limit $L \to \infty$ corresponding to a vanishing cosmological constant in which case the enlarged gauge symmetry corresponds to the $\mathcal{N}$-extended super Poincaré group in $D = 2$. Based on these observations, in Section 5.5, a graded variant of the well-known holonomy-flux algebra was derived. Moreover, we sketched the quantization of the theory adapting standard tools of ordinary LQG with real variables. The final picture, however, remained incomplete due to various subtleties associated with cylindrical consistency as well as, in particular, the consistent solution of complicated reality conditions.

For this reason, in the following, we want to consider a symmetry reduced model of chiral $\mathcal{N} = 1$, $D = 4$ supergravity and investigate whether all these difficulties encountered in the full theory can be solved consistently. Furthermore, this will also provide a first approach to study implications of supersymmetry in the framework of loop quantum cosmology. Moreover, studying self-dual variables is of course an interesting topic in itself and our model will extend previous considerations by Wilson-Ewing [91] by including fermions and supersymmetry.

For this model, we apply a particular ansatz for the fermion fields as proposed by D'Eath et al. [88–90]. As we demonstrate in Section 6.5, by explicitly making use of the enlarged gauge symmetry observed in the chiral theory, this ansatz can be justified considering homogeneous isotropic super connection forms. The precise mathematical framework for studying these kind of connections systematically will be developed in Section 6.3. With this ansatz, we symmetry reduce the chiral action and derive the reduced left and right supersymmetry constraints as well as the Hamiltonian constraint. Moreover, it will be shown that the essential part of the constraint algebra in the classical theory closes. In particular, the (graded) Poisson bracket between the left and right supersymmetry constraint reproduces the Hamiltonian constraint modulo the right SUSY constraint. In Section 6.6, we will then go over to the quantum theory and construct the kinematical Hilbert space of loop quantum supercosmology. To this end, we will motivate the state space studying the super holonomies induced by the super Ashtekar connection. In this way, in Section 6.6.1, we derive a symmetry reduced variant of the graded holonomy-flux algebra (5.188) (resp. Eq, (5.186) in case of a fixed graph) as constructed in context of the full theory in Section 5.5.1. The quantization of this theory is then performed, by considering representations of this algebra on a super Hilbert space in Section 6.6.2. Since these representations are required to be grading preserving, this automatically yields the correct statistics for the bosonic and fermionic degrees of freedom. Finally, we study the implementation of the reality conditions.

In Section 6.6.4 we implement the dynamical constraints in the quantum theory given by the SUSY constraints and Hamiltonian constraint. For a certain subclass of these models, we will show that the (graded) commutator of the supersymmetry constraints exactly reproduces the classical Poisson relations. In particular, the trace of the commutator between the left and right supersymmetry constraint reproduces the Hamilton constraint. The requirement of this closure fixes some of the quantization ambiguities. In Section 6.6.5, we study the semi-classical limit of the theory in which quantum corrections arising from quantum geometry are supposed to be negligible. We derive the form of the left and right SUSY constraint in this limit and study their respective solutions. These solutions are then compared to other solutions obtained by different means in the literature. We close with a discussion and outlook in Section 6.7.

Again, let us note that in the following we will drop many mathematical details in order to simplify the notation and to make the following discussion easier accessible for the reader. In particular, we will not explicitly mention the underlying parametrization supermanifold except in Section 6.3 and 6.5.1 in context of the symmetry reduction of chiral supergravity where the parametrization turns out be essential.

A list of important symbols as well as an overview of our choice of conventions concerning indices, physical constants etc. can be found in the List of symbols, notations and conventions.

6.2. Preliminaries: Homogeneous isotropic cosmology

Before going over to the general discussion of symmetry reduction of field theories with local supersymmetry and its applications to chiral supergravity as well as supercosmology in the framework of LQG, in this section, we would like to briefly review some important aspects of the celebrated Friedmann-Lemaître-Robertson-Walker (FLRW) models in cosmology which will play a central role in the main part of this chapter. To this end, we will mainly follow [232].

Increasing observational evidence supports the hypothesis that the universe averaged over large scales can regarded as almost perfectly *homogeneous* and *isotropic* meaning that the universe at each point in space and each direction looks the same. Thereby, the universe is modeled in terms of a smooth globally hyperbolic Lorentzian manifold (M, g) of the form

$$M = I \times \Sigma \tag{6.1}$$

with I some open interval in $\mathbb{R}$ and where the *fiducial* spacelike Cauchy hypersurface Σ is supposed to be connected. Moreover, the universe contains a perfect fluid generated by the galaxies that move along integral curves

$$\gamma_p : I \to M,\ t \to (t, p) \tag{6.2}$$

at any $p \in \Sigma$ of the global timelike vector field $U := \partial_t$. Since the global time may be identified with the proper time of the comoving galaxies, the corresponding velocity vector field is supposed to satisfy $g(U, U) = -1$. Moreover, one assumes that the relative motion of the galaxies can be neglected such that Σ_t at any time $t \in I$ can be regarded as a common restspace of the galaxies leading to an ansatz for the metric of the form

$$g = -\mathrm{d}t^2 + q \tag{6.3}$$

with $q \equiv q(t)$ the Riemannian metric on the Cauchy slice Σ_t. Since M is supposed to be isotropic, this means that for any $x = (t, p) \in M$ and $v, v' \in T_p\Sigma_t$, there exists a local isometry ϕ on M of the form $\phi = \mathrm{id} \times \phi_\Sigma$ also referred to an *isotropy isometry* with ϕ_Σ a local isometry on Σ such that $\phi(x) = x$ and

$$D_x\phi(v) = v' \tag{6.4}$$

Thus, it follows that the group $\mathrm{ISO}_x(M)$ of local isometries at $x \in M$ contains a subgroup which is isomorphic to the full rotation group $\mathrm{SO}(3)$ in $\mathbb{R}^3$. Let $K(t)$ denote the sectional curvature (Def. E.1) of the Cauchy slice Σ_t. Since, $K(t) \circ \phi_* = K(t)$ for any local isometry ϕ on Σ_t, it follows immediately from the isotropy condition (6.4) that the sectional curvature $K(t)_p$ is constant at any $p \in \Sigma_t$. But, as M, and hence Σ_t, is supposed to be connected, it follows from Schur's Lemma (see, e.g., [233], Theorem 6.7) that $K(t)$ is actually constant on all of Σ_t, that is, Σ_t for any $t \in I$ is of constant curvature.

Next, one wants to relate Cauchy slices Σ_t corresponding to different times $t \in I$. For this purpose, for $s, t \in I$, one considers the natural diffeomorphism $\mu_{st} : \Sigma_s \to \Sigma_t$ defined as $\mu_{st}(s, p) = (t, p)\ \forall p \in \Sigma$. Using the fact that μ_{st} commutes with isotropy isometries, one concludes that μ_{st}, in fact, defines a *homothety* such that

$$\mu_{st}^* q_t = h(s, t)^2 q_s \tag{6.5}$$

for some smooth function $h \equiv h(s, t)$. Thus, this implies that the sectional curvatures associated to different time slices are related via $h(s, t)^2 K(t) = K(s)$ so that, in particular, the sectional curvatures do not change sign as h never becomes zero. As argued in [232], exploiting this property together with (6.5) as well as appropriately rescaling the metric on Σ, one can construct a function $a(t)$ called *scale factor* such that the embedding $\iota_t : \Sigma \to \Sigma_t$ of the fiducial Cauchy surface Σ into the time slice Σ_t for any $t \in I$ defines a homothety with scale factor $a(t)^2$. Thus, in particular, in this way, it follows that the metric q_t on Σ_t takes the form $q_t = a(t)^2 \mathring{q}$ with $\mathring{q}$ the fiducial metric on Σ so that, for the spacetime metric g, one obtains

$$g = -\mathrm{d}t^2 + a(t)^2 \mathring{q} \tag{6.6}$$

That is, the spacetime manifold M takes the form of a *warped product manifold* $M = I \times_a \Sigma$.

As we have seen, the isotropy assumption turns out to be a quite strong condition imposing strong restrictions on the geometric structure of the spacetime manifold. In fact, as argued in [234], the requirement that the group of local isometries contains the rotation group as a subgroup already implies that the spatial slice Σ (and thus Σ_t for any $t \in I$ by (6.6)) is homogeneous. Homogeneity implies that the group $\mathrm{ISO}(\Sigma)$ of (global) isometries of $(\Sigma, \mathring{q})$ acts transitively on Σ via the canonical left action

$$\mathrm{ISO}(\Sigma) \times \Sigma \to \Sigma,\ (\phi, p) \mapsto \phi(p) \tag{6.7}$$

Hence, if $H := \mathrm{ISO}(\Sigma)$ and $H_p := \{\phi \in H \,|\, \phi(p) = p\} \cong \mathrm{SO}(3)$ denotes the isotropy subgroup at some point $p \in \Sigma$, one can make the identification $\Sigma \cong H/H_p$. Thus, it follows that Σ has the structure of a Klein geometry (H, H_p) which canonically induces the corresponding (homogeneous) Cartan geometry $(H \to H_p, \theta^{(H)}_{\mathrm{MC}})$ with $\theta^{(H)}_{\mathrm{MC}} \in \Omega^1(H, \mathfrak{h})$ the Maurer-Cartan form on the Lie group H with Lie algebra $\mathfrak{h}$ satisfying the Maurer-Cartan structure equation

$$\mathrm{d}\theta^{(H)}_{\mathrm{MC}} + \frac{1}{2}[\theta^{(H)}_{\mathrm{MC}} \wedge \theta^{(H)}_{\mathrm{MC}}] = 0 \tag{6.8}$$

Let $\theta := \mathrm{pr}_{\mathfrak{h}/\mathfrak{h}_p} \circ \theta^{(H)}_{\mathrm{MC}}$ with $\mathfrak{h}_p := \mathrm{Lie}(H_p)$ be the corresponding soldering form. Via (3.9), it follows that θ induces a one-to-one correspondence between metrics on Σ and Ad-invariant metrics on $\mathfrak{h}/\mathfrak{h}_p$. Moreover, according to the general discussion in Section 3.5, we know that horizontal automorphisms on H given by right-invariant vector fields $X^R \in \mathfrak{aut}(H)$ with $X \in \mathfrak{h}$ satisfy $L_{X^R}\theta = 0$ so that the corresponding pushforward $\pi_* X^R \in \mathfrak{X}(\Sigma)$ defines a Killing vector field of the induced Riemannian geometry. Let $(X_I)_I$ be a basis of the Lie algebra $\mathfrak{h}$ with structure coefficients ${C_{IJ}}^K$.

Since, $[X^R, Y^R] = -[X, Y]^R$ it follows that the corresponding Killing vector fields $\xi_I := X_I^R$ satisfy the commutation relations

$$[\xi_I, \xi_J] = -C_{IJ}{}^K \xi_k \tag{6.9}$$

Remark 6.2.1. Note that, by antisymmetry, we can define $C^{KL} := \frac{1}{2} C_{IJ}{}^K \epsilon^{IJL}$ which itself can be split into a symmetric and antisymmetric part

$$C^{KL} = C^{(KL)} + C^{[KL]} =: n^{KL} + \epsilon^{KLM} a_M \tag{6.10}$$

Moreover, as n^{KL} is symmetric, one can perform a change of basis such that it acquires the diagonal form $n^{IK} = n^{(K)} \delta^{KL}$ (no summation over K). Hence, it follows that the structure of the homogeneous manifold Σ is encoded in the coefficients (n^I, a_J). This leads to the well-known *Bianchi classification* of homogeneous spacetimes (see, e.g., [235] for more details).

Since the fiducial Cauchy surface Σ defines a homogeneous Riemannian manifold, it follows, in particular, that Σ is complete (see Remark 9.37 in [232]). Finally, by possibly going over from Σ to its universal Riemannian covering manifold, one may assume that Σ is also simply connected. Hence, this implies that Σ defines a simply connected and complete Riemannian manifold of constant curvature $k = -1, 0$ or $+1$, that is, it defines a simply connected space form (see Definition E.4). But, by Theorem E.5 and Corollary E.7, this in turn implies that Σ is isometric isomorphic to a standard hyperquadric given by the hyperbolic space $\mathbb{H}^3$, Euclidean space $\mathbb{R}^3$ or three-sphere $\mathbb{S}^3$, respectively, depending on the curvature k.

Definition 6.2.2. A *Friedmann-Lemaître-Robertson-Walker (FLRW) model* is a Lorentzian spacetime manifold M which has the structure of a warped product manifold of the form

$$M \cong I \times_a \Sigma \tag{6.11}$$

with scale factor $a(t)$ and fiducial spacelike Cauchy surface Σ of constant sectional curvature $k = -1, 0, +1$ isometric isomorphic to a standard hyperquadric $\mathbb{H}^3$, $\mathbb{R}^3$ or $\mathbb{S}^3$.

6.3. Symmetry reduction in supersymmetric field theories

Since, chiral supergravity contains an enlarged gauge symmetry corresponding to a gauge supergroup, it seems suggestive to exploit this symmetry in order to construct symmetry reduced models by generalizing the notion of invariant connection 1-forms to the super category. The following discussion will provide a solid basis for the construction of (spatially) symmetry reduced models in the context of supersymmetric field theories. A respective discussion in the context of ordinary (bosonic) connection 1-forms defined on smooth principal fiber bundles can be found, e.g., in [236] (see also [235] for a nice introduction to this subject in the non-supersymmetric setting). We will use the following results in Section 6.5 to study minisuperspace models in the framework of loop quantum cosmology with local supersymmetry.

To this end, let us consider a general H^∞ supermanifold $\mathcal{M}$ as well as a super Lie group $\mathcal{H}$ which, in the most situations of interest, will correspond to the super Lie group of isometries of a super Riemannian manifold $(\mathcal{M}, g)$ (in fact, in most cases $\mathcal{M}$ will be a purely bosonic supermanifold corresponding to an ordinary smooth manifold). Suppose, $\mathcal{H}$ acts from the left on $\mathcal{M}$, i.e., there exists a smooth map

$$f : \mathcal{H} \times \mathcal{M} \to \mathcal{M} \tag{6.12}$$

such that

$$f \circ (\mathrm{id}_{\mathcal{H}} \times f) = f \circ (\mu_{\mathcal{H}} \times \mathrm{id}) \quad \text{and} \quad f_e(x) = x \quad \forall x \in \mathcal{M} \tag{6.13}$$

Furthermore, we assume that $\mathcal{H}$ acts transitively on $\mathcal{M}$. Hence, if $x \in \mathbf{B}(\mathcal{M})$ is a body point and $\mathcal{H}_x$ is the stabilizer subgroup of $\mathcal{H}$, one can identify $\mathcal{M} \cong \mathcal{H}/\mathcal{H}_x$ which we want to do in what follows. The left action of $\mathcal{H}$ is then given by its standard action on the coset space $\mathcal{H}/\mathcal{H}_x$ which still will be denoted by f.

Let $\mathcal{G} \to \mathcal{P} \xrightarrow{\pi} \mathcal{H}/\mathcal{H}_x$ be a principal super fiber bundle over $\mathcal{H}/\mathcal{H}_x$ with structure group $\mathcal{G}$ and $\mathcal{G}$-right action $\Phi : \mathcal{P} \times \mathcal{G} \to \mathcal{P}$. We want to the ask the question about the existence of a $\mathcal{H}$-left action $\hat{f} : \mathcal{H} \times \mathcal{P} \to \mathcal{P}$ on $\mathcal{P}$ such that $\hat{f}$ is a $\mathcal{G}$-equivariant bundle automorphism on $\mathcal{P}$ projecting to the left multplication of $\mathcal{H}$ on $\mathcal{H}/\mathcal{H}_x$, i.e.,

$$\hat{f} \circ (\mathrm{id}_{\mathcal{H}} \times \Phi) = \Phi \circ (\hat{f} \times \mathrm{id}_{\mathcal{G}}) \text{ and } \pi \circ \hat{f} = f \circ (\mathrm{id}_{\mathcal{H}} \times \pi) \tag{6.14}$$

Therefore, applying the forgetful functor $\mathbf{SMan}_{H^\infty} \to \mathbf{Set}$, we consider the set of abstract group homomorphisms $\lambda : \mathcal{H}_x \to \mathcal{G}$. On this set, we introduce the equivalence relation

$$\lambda \sim \lambda' :\Leftrightarrow \exists g \in \mathcal{G} : \lambda' = \mathrm{Ad}_g \circ \lambda \tag{6.15}$$

which yields the set of conjugacy classes $\mathrm{Conj}(\mathcal{H}_x \to \mathcal{G})$ of abstract group homomorphisms. An equivalence class $[\lambda] \in \mathrm{Conj}(\mathcal{H}_x \to \mathcal{G})$ will be called *smoothly admissible*, if it contains a H^∞-smooth super Lie group homomorphism as a representative. The set of such smoothly admissible conjugacy classes yields a proper subset $\mathrm{Conj}(\mathcal{H}_x \to \mathcal{G})_\infty \subset \mathrm{Conj}(\mathcal{H}_x \to \mathcal{G})$.

Proposition 6.3.1. *There exists a bijective correspondence between equivalence classes of principal $\mathcal{G}$-bundles over $\mathcal{H}/\mathcal{H}_x$ admitting an $\mathcal{H}$-left action which is $\mathcal{G}$-equivariant and* *projects to the standard left multiplication of $\mathcal{H}$ on the coset space $\mathcal{H}/\mathcal{H}_x$ and smoothly admissible conjugacy classes* $[\lambda] \in \mathrm{Conj}(\mathcal{H}_x \to \mathcal{G})_\infty$ *of group homomorphisms* $\lambda : \mathcal{H}_x \to \mathcal{G}$.

Proof. Suppose $\lambda : \mathcal{H}_x \to \mathcal{G}$ is a smooth representative of a smoothly admissible conjugacy class of super Lie group homomorphisms. Consider then the associated principal supper fiber bundle $\mathcal{H} \times_\lambda \mathcal{G}$ with structure group $\mathcal{G}$. On $\mathcal{H} \times \mathcal{G}$, we define the smooth left action

$$\mathcal{H} \times (\mathcal{H} \times \mathcal{G}) \to \mathcal{H} \times \mathcal{G}, \quad (\phi, (\psi, g)) \mapsto (\phi \circ \psi, g) \tag{6.16}$$

Since, $(\phi \circ (\psi \circ \phi'), \lambda(\phi')^{-1}(g)) = ((\phi \circ \psi) \circ \phi', \lambda(\phi')^{-1}(g))\ \forall \phi, \phi', \psi \in \mathcal{H}$ and $g \in \mathcal{G}$, it follows that (6.16) is constant on $\mathcal{G}$-orbits so that (6.16) induces a well-defined smooth $\mathcal{H}$-left action on $\mathcal{H} \times_\lambda \mathcal{G}$ which is $\mathcal{G}$-equivariant and projects to the multiplication of $\mathcal{H}$ on $\mathcal{H}/\mathcal{H}_x$.

Conversely, let $\hat{f} : \mathcal{H} \times \mathcal{P} \to \mathcal{P}$ be a $\mathcal{H}$-left action on $\mathcal{P}$. Let $p \in \mathbf{B}(\mathcal{P})$ be an element of the body. Since, the $\mathcal{G}$-right action on $\mathcal{P}$ is transitive on each fiber and $\hat{f}$ is fiber-preserving, for any $\phi \in \mathcal{H}_x$, there exists a unique $\lambda(\phi) \in \mathcal{G}$ such that

$$\hat{f}_\phi(p) = \Phi_{\lambda(\phi)}(p) \tag{6.17}$$

Moreover, since $p \in \mathbf{B}(\mathcal{P})$, the map $\mathcal{H} \to \mathcal{P}$, $\phi \mapsto f_\phi(p)$ is of class H^∞ proving that the map $\lambda : \mathcal{H}_x \to \mathcal{G}$, $\phi \mapsto \lambda(\phi)$ is smooth. By $\mathcal{G}$-equivariance (6.14), it follows for $\phi, \psi \in \mathcal{H}_x$

$$\begin{aligned} \hat{f}_{\phi\circ\psi}(p) &= f_\phi(f_\psi(p)) = f_\phi(\Phi_{\lambda(\psi)}(p)) = \Phi_{\lambda(\psi)}(f_\phi(p)) = \Phi_{\lambda(\phi)\circ\lambda(\psi)}(p) \\ &= \Phi_{\lambda(\phi\circ\psi)}(p) \end{aligned} \tag{6.18}$$

implying $\lambda(\phi \circ \psi) = \lambda(\phi) \circ \lambda(\psi)$, i.e.. λ is indeed a super Lie group homomorphism. If $p' \in \mathcal{P}$ is any other point, then, again by transitivity, there exists $g \in \mathcal{G}$ with $\Phi_g(p) = p'$. Hence,

$$\hat{f}_\phi(p') = \Phi_g(\hat{f}_\phi(p)) = \Phi_{\mathrm{Ad}_{g^{-1}}\lambda(\phi)}(p') \tag{6.19}$$

with $\mathrm{Ad}_{g^{-1}} \circ \lambda$ in the same equivalence class as λ. Finally, let $\mathcal{H} \times_\lambda \mathcal{G}$ be the associated principal $\mathcal{G}$-bundle with smooth $\mathcal{H}$-left action as constructed in the first part of this proof. For $p \in \mathbf{B}(\mathcal{P})$ a body point, consider the map

$$\mathcal{H} \times_\lambda \mathcal{G} \to \mathcal{P}, \quad [\phi, g] \mapsto \Phi_g(f_\phi(p)) \tag{6.20}$$

By (6.14), it follows immediately that (6.20) is well-defined and in fact yields an isomorphism of principal super fiber bundles. □

Proposition (6.3.1) provides a complete classification of principal super fiber bundles admitting such a smooth left action by equivalence classes of smooth super Lie group morphisms $\lambda:\mathcal{H}_x\rightarrow\mathcal{G}$. We next want to study connections on $\mathcal{P}$ that are invariant under this left action. Since, $\mathcal{M}$ is typically an ordinary smooth manifold and we would like to include fermionic degrees of freedom in our discussion, we go over to the category of relative supermanifolds. Hence, let us add a parametrizing supermanifold $\mathcal{S}$. We lift all objects and morphisms to the relative category in the obvious way. If $\mathcal{P}:=\mathcal{H}\times_{\lambda}\mathcal{G}$ is a principal super fiber bundle as in Prop. (6.3.1), it follows that $\mathcal{P}_{/\mathcal{S}}=\mathcal{H}_{/\mathcal{S}}\times_{\lambda}\mathcal{G}$. A $\mathcal{S}$-relative super connection 1-form $\mathcal{A}\in\Omega^1(\mathcal{P}_{/\mathcal{S}},\mathfrak{g})_0$ will be called $\mathcal{H}$*-invariant*, if

$$(\hat{f}_{\mathcal{S}})^*_{\phi}\mathcal{A}=\mathcal{A}\quad\forall\phi\in\mathcal{H}\tag{6.21}$$

with $\hat{f}_{\mathcal{S}}:\mathcal{H}\times\mathcal{P}_{/\mathcal{S}}\rightarrow\mathcal{P}_{/\mathcal{S}}$ the lift of the left multiplication $f_{\mathcal{S}}:\mathcal{H}\times\mathcal{H}_{/\mathcal{S}}\rightarrow\mathcal{H}_{/\mathcal{S}}:(\phi,(s,\psi))\mapsto(s,\phi\circ\psi)$ to a smooth $\mathcal{H}$-left action on $\mathcal{P}_{/\mathcal{S}}$ defined via (cf. proof of Prop. 6.3.1)

$$\hat{f}_{\mathcal{S}}\circ(\mathrm{id}_{\mathcal{H}}\times\hat{\pi})=\hat{\pi}\circ(f_{\mathcal{S}}\times\mathrm{id}_{\mathcal{G}})\tag{6.22}$$

with $\hat{\pi}:\mathcal{H}_{/\mathcal{S}}\times\mathcal{G}\rightarrow\mathcal{H}_{/\mathcal{S}}\times_{\lambda}\mathcal{G}=\mathcal{P}_{/\mathcal{S}}$ the canonical projection.

Proposition 6.3.2. *Let $\mathcal{P}:=\mathcal{H}\times_{\lambda}\mathcal{G}$ be the associated principal super fiber bundle induced by a smooth super Lie group homomorphism $\lambda:\mathcal{H}_x\rightarrow\mathcal{G}$. The $\mathcal{H}$-invariant super connection 1-forms on $\mathcal{P}_{/\mathcal{S}}$ are in one-to-one correspondence to smooth maps $\Xi\in H^{\infty}(\mathcal{S},\mathrm{Hom}_L(\mathrm{Lie}(\mathcal{H}),\mathrm{Lie}(\mathcal{G})))$ from the parametrizing supermanifold $\mathcal{S}$ to even left linear super Lie algebra homomorphisms $\mathrm{Hom}_L(\mathrm{Lie}(\mathcal{H}),\mathrm{Lie}(\mathcal{G}))$ satisfying*

$$\Xi(s)\big|_{\mathrm{Lie}(\mathcal{H}_x)}=\lambda_*\tag{6.23}$$

and

$$\mathrm{Ad}_{\phi^{-1}}\diamond\Xi(s)=\Xi(s)\diamond\mathrm{Ad}_{\lambda(\phi)^{-1}}\quad\text{on }\mathrm{Lie}(\mathcal{H})\tag{6.24}$$

$\forall s\in\mathcal{S}$ and $\phi\in\mathcal{H}_x$.

Proof. In the following, let $\hat{\iota}:\mathcal{H}_{/\mathcal{S}}\rightarrow\mathcal{P}_{/\mathcal{S}}$ be the smooth map defined via $\hat{\iota}(s,\phi):=[(s,\phi),e]$ and $\Phi_{\mathcal{S}}$ denote the $\mathcal{G}$-right action on $\mathcal{P}_{/\mathcal{S}}$. Suppose $\mathcal{A}\in\Omega^1(\mathcal{P}_{/\mathcal{S}},\mathfrak{g})_0$ is a $\mathcal{H}$-invariant super connection 1-form. Consider then $\mathcal{A}_{\mathcal{H}}:=\hat{\iota}^*\mathcal{A}\in\Omega^1(\mathcal{H}_{/\mathcal{S}},\mathfrak{g})_0$. Since $\hat{\iota}\circ f_{\mathcal{S}}=\hat{f}_{\mathcal{S}}\circ(\mathrm{id}_{\mathcal{H}}\times\hat{\iota})$ by (6.22), it follows from the $\mathcal{H}$-invariance of $\mathcal{A}$ that

$$(f_{\mathcal{S}})^*_{\phi}\mathcal{A}_{\mathcal{H}}=\hat{\iota}^*((\hat{f}_{\mathcal{S}})^*_{\phi}\mathcal{A})=\hat{\iota}^*\mathcal{A}=\mathcal{A}_{\mathcal{H}}\tag{6.25}$$

i.e., $\mathcal{A}_{\mathcal{H}}$ is left-invariant w.r.t. to the standard left-multiplication on $\mathcal{H}$. As a consequence, $\mathcal{A}_{\mathcal{H}}$ is uniquely determined by its restriction $\mathcal{A}_{\mathcal{H}}|_{T_e\mathcal{H}}:\mathcal{S}\times T_e\mathcal{H}\rightarrow\mathrm{Lie}(\mathcal{G})$. As this map is left linear in the second argument it follows similarly as in the proof of Lemma 2.4.7 that it defines an even smooth map $\mathcal{A}_{\mathcal{H}}|_{T_e\mathcal{H}}:\mathcal{S}\rightarrow\mathrm{Hom}_L(\mathrm{Lie}(\mathcal{H}),\mathrm{Lie}(\mathcal{G}))_0$. Moreover, since the Maurer-Cartan form $\theta_{\mathrm{MC}}^{(\mathcal{H})}|_{T_e\mathcal{H}}:T_e\mathcal{H}\rightarrow\mathrm{Lie}(\mathcal{H})$ on $T_e\mathcal{H}$ is the identity, it follows that

$$\mathcal{A}_{\mathcal{H}}(s)=\theta_{\mathrm{MC}}^{(\mathcal{H})}\diamond\Xi(s)\quad\forall s\in\mathcal{S}\tag{6.26}$$

on $T_e\mathcal{H}$ for some smooth map $\Xi\in H^\infty(\mathcal{S},\mathrm{Hom}_L(\mathrm{Lie}(\mathcal{H}),\mathrm{Lie}(\mathcal{G})))$. It follows by left-invariance that (6.26) indeed holds on all of $\mathcal{H}$.

Remains to proof that Ξ satisfies the properties (6.23) and (6.24) of the proposition. To this end, for $X\in\mathrm{Lie}(\mathcal{H}_x)$, we compute

$$\begin{aligned}\hat{\iota}_*(\mathbb{1}\otimes X)_p&=D_{(p,e_{\mathcal{G}})}\hat{\pi}(0_s,X_\phi,0_{e_{\mathcal{G}}})=D_{(p,e_{\mathcal{G}})}\hat{\pi}(0_p,\lambda_*(X))\\&=\widetilde{\lambda_*(X)}_{[p,e_{\mathcal{G}}]}\end{aligned}\tag{6.27}$$

$\forall p=(s,\phi)\in\mathcal{H}_{/\mathcal{S}}$, where in the second equality we used that the kernel of $\hat{\pi}_*$ is given by

$$\ker D_{(p,g)}\hat{\pi}=\{(\mathbb{1}\otimes Y_p,-R_{g*}\lambda_*(X))|Y\in\mathrm{Lie}(\mathcal{H}_x)\}\tag{6.28}$$

Using (6.27), this yields

$$\begin{aligned}\lambda_*(X)&=\langle\widetilde{\lambda_*(X)}|\mathcal{A}\rangle=\langle(\mathbb{1}\otimes X)_p|\mathcal{A}_{\mathcal{H}}\rangle\\&=\langle\langle X_\phi|\theta_{\mathrm{MC}}^{(\mathcal{H})}\rangle|\Xi(s)\rangle=\langle X|\Xi(s)\rangle\end{aligned}\tag{6.29}$$

$\forall X\in\mathrm{Lie}(\mathcal{H}_x)$. Finally, since $\hat{\iota}\circ(\mathrm{id}_{\mathcal{S}}\times R_\phi)=(\Phi_{\mathcal{S}})_{\lambda(\phi)}\circ\hat{\iota}$ with R_ϕ the right translation on $\mathcal{H}$ w.r.t. $\phi\in\mathcal{H}_x$, it follows that

$$\begin{aligned}\langle X|\mathrm{Ad}_{\phi^{-1}}\diamond\Xi(s)\rangle&=\langle\mathrm{Ad}_{\phi^{-1}}\langle X|\theta_{\mathrm{MC}}^{(\mathcal{H})}\rangle|\Xi(s)\rangle=\langle\langle R_{\phi*}X|\theta_{\mathrm{MC}}^{(\mathcal{H})}\rangle|\Xi(s)\rangle\\&=\langle R_{\phi*}X|\theta_{\mathrm{MC}}^{(\mathcal{H})}\diamond\Xi(s)\rangle=\langle R_{\phi*}|\mathcal{A}_{\mathcal{H}}(s)\rangle=\mathrm{Ad}_{\lambda(\phi)^{-1}}\langle X|\mathcal{A}_{\mathcal{H}}(s)\rangle\\&=\langle X|\Xi(s)\diamond\mathrm{Ad}_{\lambda(\phi)^{-1}}\rangle\end{aligned}\tag{6.30}$$

$\forall X\in\mathrm{Lie}(\mathcal{H})$ as required. Conversely, suppose one has given a smooth map $\Xi\in H^\infty(\mathcal{S},\mathrm{Hom}_L(\mathrm{Lie}(\mathcal{H}),\mathrm{Lie}(\mathcal{G})))$ satisfying (6.23) and (6.24) above. We have to show that there indeed exists a unique super connection 1-form $\mathcal{A}\in\Omega^1(\mathcal{P}_{/\mathcal{S}},\mathfrak{g})_0$ such that $\hat{\iota}^*\mathcal{A}(s)=\theta_{\mathrm{MC}}^{(\mathcal{H})}\diamond\Xi(s)$ for any $s\in\mathcal{S}$. This, in fact, follows along the lines of the proof of Prop. 3.3.12. As there, one can show that, if $\mathcal{A}$ exists, it necessarily has to be of the form

$$\langle D_{(p,g)}\hat{\pi}(X_p, Y_g)|\mathcal{A}_{[p,g]}\rangle = \mathrm{Ad}_{g^{-1}}\,\langle X_p|\theta_{\mathrm{MC}}^{(\mathcal{H})}\diamond \Xi(s)\rangle + \langle Y_g|\theta_{\mathrm{MC}}^{(\mathcal{G})}\rangle \tag{6.31}$$

Moreover, as $\hat{\pi}$ is a submersion, it is uniquely determined by (6.31). One then concludes that this indeed provides a well-defined super connection 1-form on $\mathcal{P}_{/\mathcal{S}}$. □

Remark 6.3.3. Note that if $\lambda : \mathcal{H}_0 \to \mathcal{G}$ is a group morphism corresponding to a left action of a bosonic super Lie group $\mathcal{H}_0$, i.e., a split super Lie group corresponding to an ordinary smooth symmetry group, then λ only takes values in the bosonic super Lie subgroup $\mathcal{G}_0 := \mathbf{S}(\mathbf{B}(\mathcal{G}))$ of $\mathcal{G}$. Thus, it follows that condition (6.12) only encodes super connections that are invariant under purely bosonic gauge transformations. This can be cured by considering a more general class of smooth $\mathcal{H}$-left actions $\hat{f} : \mathcal{H}\times\mathcal{P}_{/\mathcal{S}} \to \mathcal{P}_{/\mathcal{S}}$ on the $\mathcal{S}$-relative principal super fiber bundle $\mathcal{P}_{/\mathcal{S}}$ that are not merely trivial extensions of $\mathcal{H}$-left actions on $\mathcal{P}$ as considered above and which project to the left multiplication on $\mathcal{H}_{/\mathcal{S}}$, i.e., $\mathcal{H}\times\mathcal{H}_{/\mathcal{S}} \to \mathcal{H}_{/\mathcal{S}} : (\phi, (s, \psi)) \mapsto (s, \phi\circ\psi)$. It then follows that a classification of these type of actions is given by smooth maps of the form $\lambda' : \mathcal{S}\times\mathcal{H}_x \to \mathcal{G}$ satisfying

$$\lambda'(s, \phi\circ\psi) = \lambda'(s, \phi)\circ\lambda'(s, \psi) \tag{6.32}$$

$\forall (s, \phi), (s, \psi) \in \mathcal{S}\times\mathcal{H}_x$. Condition (6.21) for a $\mathcal{H}$-invariant super connection 1-form then again leads to (6.23) as well as (6.24), straightforwardly generalized to $\mathcal{S}$-parametrized group morphisms $\lambda' : \mathcal{S}\times\mathcal{H}_x \to \mathcal{G}$. In particular, since λ' now explicitly depends on the parametrization, it follows, in case that the symmetry group $\mathcal{H}$ is purely bosonic, that λ' can take values in the odd part of the gauge supergroup $\mathcal{G}$. Hence, in this way, one can model super connection 1-forms which are invariant under the spatial symmetry group up to super gauge transformations. In fact, as we will see in Section 6.5.1, this will play an important role in deriving symmetry reduced connections that contain nontrivial fermionic degrees of freedom.

6.4. Canonical decomposition of chiral $\mathcal{N} = 1$ supergravity

From here on, the content of this chapter has been reproduced from [4], with (slight) changes to account for the context of this thesis with the permission of Springer-Nature.

The canonical phase space of the chiral theory including a discussion about the reality conditions has been partially addressed already in Section 5.4.4. Here, in view of the symmetry reduction of the theory to be discussed in the subsequent sections, in what follows, let us summarize some of the most important facts and continue with the canonical analysis of the theory, in particular, with the derivation of the constraints.

Using the general results obtained in Section 4.3 and 4.4 for the special case $\beta = -i$, it follows that the chiral action (5.102) takes the form

$$S^{\mathcal{N}=1,\beta=-i}_{\text{H-AdS}}(e, \mathcal{A}^+) = \int_{\mathbb{R}} \mathrm{d}t \int_{\Sigma} \mathrm{d}^3x \left(\frac{i}{\kappa} E^a_i L_{\partial_t} A^{+i}_a - \pi^a_A L_{\partial_t} \psi^A_a \right. \\ \left. - A^{+i}_t G_i + N^a H_a + NH + \bar{\psi} S \right) \tag{6.33}$$

where, for later convenience, we have absorbed the factor $1/\sqrt{\kappa}$ in the Rarita-Schwinger field. Here, π^a_A is the canonically conjugate momentum of ψ^A_a which is related to the corresponding complex conjugate $\bar{\psi}^{A'}_a$ via the *reality condition*

$$\pi^a_A = \epsilon^{abc} \bar{\psi}^{A'}_b e_{cAA'} \tag{6.34}$$

Furthermore, $E^a_i = \sqrt{q} e^a_i$ is the electric field conjugate to A^{+i}_a. The canonical pairs (A^{+i}_a, E^a_i) and (ψ^A_a, π^a_A) build up a graded symplectic phase space with the nonvanishing Poisson brackets

$$\{E^i_a(x), A^{+j}_b(y)\} = i\kappa \delta^j_i \delta^{(3)}(x, y) \quad \text{and} \quad \{\pi^a_A(x), \psi^B_b(y)\} = -\delta^a_b \delta^B_A \delta^{(3)}(x, y) \tag{6.35}$$

Recall from Section 4.4 that, for arbitrary β, the SUSY constraint is given by

$$\begin{aligned} S =& \epsilon^{abc} \gamma_a \frac{\mathbb{1} + i\beta\gamma_*}{2\beta} D^{(\beta A)}_b \psi_c + \frac{\mathbb{1} + i\beta\gamma_*}{2\beta} D^{(\beta A)}_a \left(\epsilon^{abc} \gamma_b \psi_c \right) \\ &- \frac{1 + \beta^2}{2\beta} \epsilon^{abc} \gamma_0 \psi_c K_{ba} - \frac{1}{L} E^a_i \gamma^{0i} \psi_a \end{aligned} \tag{6.36}$$

In context of the chiral theory, let us bring this constraint in another form rewriting the second term in (6.36). To this end, using anticommutativity of the fermionic fields, it follows that

$$\begin{aligned} &\bar{\psi}_t \frac{\mathbb{1} + i\beta\gamma_*}{2\beta} D^{(\beta A)}_a \left(\epsilon^{abc} \gamma_b \psi_c \right) \\ =& \epsilon^{abc} \left[\bar{\psi}_t \frac{\mathbb{1} + i\beta\gamma_*}{2\beta} \partial_a \left(\gamma_b \psi_c \right) + \frac{i}{2} {}^{\beta}\! A^i_a \bar{\psi}_t \frac{\mathbb{1} + i\beta\gamma_*}{2\beta} \gamma_* \gamma_{0i} \gamma_b \psi_c \right] \\ =& \epsilon^{abc} \left[-\partial_a \left(\bar{\psi}_c \gamma_b \right) \frac{\mathbb{1} + i\beta\gamma_*}{2\beta} \psi_t + \frac{i}{2} {}^{\beta}\! A^i_a \bar{\psi}_c \gamma_b \gamma_* \gamma_{0i} \frac{\mathbb{1} + i\beta\gamma_*}{2\beta} \psi_t \right] \\ =& D^{(\beta A)}_a \left(\epsilon^{abc} \bar{\psi}_b \gamma_c \right) \frac{\mathbb{1} + i\beta\gamma_*}{2\beta} \psi_t \\ =& - \left(D^{(\beta A)}_a \pi^a \right) \psi_t \end{aligned} \tag{6.37}$$

with $D_a^{(\beta A)}\pi^b = \partial_a\pi^b - \frac{i}{2}\pi^b{}^\beta A^i_a\gamma_*\gamma_{0i}$ the exterior covariant derivative in the dual representation. This yields

$$\begin{aligned}\bar{\psi}_t S =&\bar{\psi}_t\epsilon^{abc}\gamma_a\frac{\mathbb{1}+i\beta\gamma_*}{2\beta}D_b^{(\beta A)}\psi_c - \bar{\psi}_t\frac{1}{L}E^a_i\gamma^{0i}\psi_a - \left(D_a^{(\beta A)}\pi^a\right)\psi_t\\ &-\frac{1+\beta^2}{2\beta}\epsilon^{abc}\gamma_0\psi_c K_{ba}\end{aligned} \tag{6.38}$$

Hence, for $\beta=-i$, it follows, together with the identity $\bar{\psi}_t\gamma_{0i}\psi_a = -2\psi_a^A(\epsilon\tau_i)_{AB}\psi_t^B - 2\overline{\psi}_{tA'}(\epsilon\tau_i)^{A'B'}\overline{\psi}_{aB'}$, that (6.38) can be split in the form

$$\bar{\psi}S = -S^L_A\psi^A_t - \bar{\psi}_{tA'}S^{RA'} \tag{6.39}$$

with S^L_A and $S^{RA'}$ the so-called *left* and *right supersymmetry (SUSY) constraints*, respectively, given by

$$S^L_A = D_a^{(A^+)}\pi^a_A + \frac{2}{L}E^{ai}\psi^B_a(\epsilon\tau_i)_{BA} \tag{6.40}$$

and

$$S^{RA'} = -\epsilon^{A'B'}\epsilon^{abc}e_{aAB'}D_b^{(A^+)}\psi^A_c + \frac{2}{L}E^{ai}(\epsilon\tau_i)^{A'B'}\bar{\psi}_{aB'} \tag{6.41}$$

respectively. According to (6.41), the right SUSY constraint depends on the complex conjugate Weyl spinor $\bar{\psi}_{A'}$. In order to re-express it in terms of the fundamental variables, one can apply the reality condition (6.34). In fact, using (4.30)-(4.33), it follows that

$$\begin{aligned}\pi^a_A e^{AA'}_a &= \epsilon^{abc}e^i_a\bar{\psi}^{B'}_b e^j_c\sigma_i^{AA'}\sigma_{jAB'}\\ &= -\sqrt{q}\epsilon^{ijk}e^b_k\bar{\psi}^{B'}_b\sigma_i^{AA'}\sigma_{jAB'}\\ &= -2i\sqrt{q}\bar{\psi}^{B'}_b n_{AB'}e^{bAA'}\end{aligned} \tag{6.42}$$

such that

$$\begin{aligned}-2E^{ai}(\epsilon\tau_i)^{A'B'}\bar{\psi}_{aB'} &= 2E^{ai}(\epsilon\tau_i\epsilon)^{A'}{}_{B'}\bar{\psi}^{B'}_a\\ &= -i\sqrt{q}\bar{\psi}^{B'}_a n_{AB'}e^{aAA'} = \frac{1}{2}\pi^a_A e^{AA'}_a\end{aligned} \tag{6.43}$$

Together with the identity $\epsilon_{abc}\epsilon^{ijk}E^b_jE^c_k = 2\sqrt{q}e^i_a$, we can thus rewrite the right supersymmetry constraint in the equivalent form

$$S^{RA'} = -\epsilon^{ijk}\frac{E^b_jE^c_k}{2\sqrt{q}}\sigma_i^{AA'}\left(2\epsilon_{AB}D_{[b}^{(A^+)}\psi^B_{c]} + \frac{1}{2L}\pi^a_A\epsilon_{abc}\right) \tag{6.44}$$

The remaining constraints can be obtained rather quickly using their general form derived in Section 4.3. For the Gauss and vector constraint, it follows that

$$G_i = \frac{i}{\kappa} D_a^{(A^+)} E_i^a - \pi_A^a (\tau_i)^A{}_B \psi_a^B \tag{6.45}$$

and

$$H_a := \frac{i}{\kappa} E_i^b F(A^+)^i_{ab} - \epsilon^{bcd} e_{aAA'} \bar{\psi}_b^{A'} D_c^{(A^+)} \psi_d^A \tag{6.46}$$

respectively. The Hamiltonian constraint reads[1]

$$\begin{aligned} H = &- \frac{E_i^a E_j^b}{2\kappa\sqrt{q}} \epsilon^{ij}{}_k F(A^+)^k_{ab} - \epsilon^{abc} \bar{\psi}_a^{A'} n_{AA'} D_b^{(A^+)} \psi_c^A \\ &+ \frac{E_i^a E_j^b}{2L\sqrt{q}} \epsilon^{ijk} (\psi_{aA} n_{BA'} \sigma_k^{AA'} \psi_b^B - \bar{\psi}_a^{A'} n_{AA'} \sigma_k^{AB'} \bar{\psi}_{bB'}) + \frac{3}{\kappa L^2} \sqrt{q} \end{aligned} \tag{6.47}$$

where $n^{AA'}$ is the spinor corresponding to the unit normal vector field n^μ orthogonal to the time slices Σ_t in the 3+1-decomposition.

Remark 6.4.1. Recall that the canonically conjugate momenta E_a^i and π_a^A can be combined to give the super electric field

$$\mathcal{E}^a = (E_i^a, -i\sqrt{\kappa}\pi_A^a) \tag{6.48}$$

As a consequence, it follows that the Gauss G_i and left supersymmetry constraint S_A^R arise in terms of the even and odd part of the super Gauss constraint (see Eq. (5.117) for the case $\mathcal{N}=1$)

$$\mathcal{G} = \frac{i}{\kappa} D_a^{(\mathcal{A}^+)} \mathcal{E}^a \tag{6.49}$$

which generates local $\mathrm{OSp}(1|2)_{\mathbb{C}}$-gauge transformations.

Finally, since they will play an important role in what follows, let us recall the reality conditions imposed on the canonical variables in order to recover ordinary real supegravity. According to (5.151), it follows that, provided e is real and A^+ satisfies the field equation (5.148) (resp. (5.149)), action (6.33) is purely real up to a boundary term. Hence, in the

[1] Of course, the vector and Hamiltonian constraint also have to be expressed in terms of the fundamental variables. This can be done in analogy to the right SUSY constraint. We will do so for the symmetry reduced expressions in the following section.

canonical theory, it follows that the reality conditions are equivalent to the requirement that the $3D$ spin connection part Γ^i of A^+ satisfies the torsion equation

$$D^{(\Gamma)}e^i \equiv \mathrm{d}e^i + \epsilon^i{}_{jk}\Gamma^j \wedge e^k = \Theta^{(\Gamma)i} = \frac{i\kappa}{2}\psi^A \wedge \bar{\psi}^{A'}\sigma^i_{AA'} \tag{6.50}$$

which has the unique solution

$$\Gamma^i \equiv \Gamma^i(e) + C^i(e,\psi,\bar{\psi}) \tag{6.51}$$

with $\Gamma^i(e)$ the torsion-free metric connection

$$\Gamma^i_a(e) = -\epsilon^{ijk}e^b_j\left(\partial_{[a}e_{b]k} + \frac{1}{2}e^c_k e^l_a \partial_{[c}e_{b]l}\right) \tag{6.52}$$

and C^i the contorsion tensor given by

$$C^i_a = \frac{i\kappa}{4\sqrt{q}}\epsilon^{bcd}e^i_d\left(2\psi^A_{[a}\bar{\psi}^{A'}_{b]}e_{cAA'} - \psi^A_b\bar{\psi}^{A'}_c e_{aAA'}\right) \tag{6.53}$$

Thus, to summarize, the reality conditions for the bosonic degrees of freedom take the form

$$A^{+i}_a + (A^{+i}_a)^* = 2\Gamma^i_a(e) + 2C^i_a(e,\psi,\bar{\psi}), \quad E^a_i = \Re(E^a_i) \tag{6.54}$$

These ensure that, provided the initial conditions satisfy (6.54), the dynamical evolution remains in the real sector of the complex phase space, i.e., the phase space of ordinary real $\mathcal{N} = 1$ supergravity.

6.5. The symmetry reduced model

6.5.1. Homogeneous (isotropic) super connection forms

Typically fermions in cosmological models are not compatible with isotropy. However, in case of $\mathcal{N} = 1$ supersymmetry, it turns out that there does exist an ansatz for the gravitino field which is consistent with the requirement of spatial isotropy. This is due to the intrinsic geometric nature of the Rarita-Schwinger field as well as the underlying supersymmetry of the theory and, in the context of chiral LQSG, can naturally be understood in terms of homogeneous (isotropic) super connection forms which we would like to explain it what follows.

We consider a spatial slice Σ in the spacetime manifold M and assume that Σ is homogeneous, i.e., the group $H := \mathrm{ISO}(\Sigma)$ of isometries of Σ acts transitively on it. Hence, if H_x denotes the stabilizer subgroup at some point $x \in \Sigma$, one can identify Σ with the coset space H/H_x. Here, in order to compare our results with other results in the literature, we are mainly interested in the standard homogeneous isotropic FLRW models, in particular, in the spatially flat case ($k = 0$) and the case with positive spatial curvature ($k = +1$). In both cases the isometry group takes the form of a semi-direct product $H \cong T \rtimes \mathrm{SO}(3)$ with T the subgroup of translations acting freely and transitively on Σ and the isotropy subgroup $\mathrm{SO}(3)$.

Let $\mathcal{H} := \mathbf{S}(H)$ be the corresponding bosonic super Lie group. As discussed in detail in Section 6.3, we are looking for a lift of the standard left action of $\mathcal{H}$ on $\mathcal{H}/\mathcal{H}_x$ to a left action on the $\mathcal{S}$-relative principal $\mathrm{OSp}(1|2)_{\mathbb{C}}$-bundle $\mathcal{P}_{/\mathcal{S}}$. This turns out to be equivalent to classifying conjugacy classes of super Lie group morphisms $\lambda : \mathcal{H} \to \mathrm{OSp}(1|2)_{\mathbb{C}}$, Given such a super Lie group morphism and the corresponding left action $\hat{f}_{\mathcal{S}} : \mathcal{H} \times \mathcal{P}_{/\mathcal{S}} \to \mathcal{P}_{/\mathcal{S}}$, one can study super connection 1-forms which are invariant w.r.t. the symmetry group, i.e., $\mathcal{H}$-invariant super connection 1-forms $\mathcal{A}^+ \in \Omega^1(\mathcal{P}_{/\mathcal{S}}, \mathfrak{osp}(1|2)_{\mathbb{C}})_0$ satisfying (6.21). According to Prop. 6.3.2, it follows that any invariant super connection $\mathcal{A}^+$ is uniquely determined by a ($\mathcal{S}$-parametrized) super Lie algebra morphism $\Xi : \mathfrak{h} \to \mathfrak{osp}(1|2)_{\mathbb{C}}$ (and trivially extended to the corresponding super Lie module $\mathrm{Lie}(\mathcal{H}) = \Lambda \otimes \mathfrak{h}$) satisfying $\Xi|_{\mathrm{Lie}(H_x)} = \lambda_*$ and

$$\mathrm{Ad}_{\phi^{-1}} \diamond \Xi = \Xi \diamond \mathrm{Ad}_{\lambda(\phi)^{-1}} \tag{6.55}$$

on $\mathrm{Lie}(\mathcal{H})$ for any $\phi \in \mathcal{H}_x$, such that

$$\hat{\imath}^* \mathcal{A}^+ = \theta_{\mathrm{MC}}^{(\mathcal{H})} \diamond \Xi \tag{6.56}$$

where $\hat{\imath} : \mathcal{H}_{/\mathcal{S}} \to \mathcal{P}_{/\mathcal{S}}$ is an embedding. Here, $\theta_{\mathrm{MC}}^{(\mathcal{H})} \in \Omega^1(\mathcal{H}, \mathfrak{h})$ is the Maurer-Cartan form on $\mathcal{H}$ which satisfies the Maurer-Cartan structure equation

$$\mathrm{d}\theta_{\mathrm{MC}}^{(\mathcal{H})} + \frac{1}{2}[\theta_{\mathrm{MC}}^{(\mathcal{H})} \wedge \theta_{\mathrm{MC}}^{(\mathcal{H})}] = 0 \tag{6.57}$$

In case of FLRW, we have $H \cong T \rtimes \mathrm{SO}(3)$. Let us first assume that $\mathcal{A}^+$ is homogeneous, that is, $\mathcal{A}^+$ is invariant under the translational subgroup T of the full symmetry group. Since T acts freely and transitively on $\Sigma \cong T$, we have $T_x \cong \{e\}$ and the only possible super Lie group morphism $\Xi : \mathbf{S}(\{e\}) \to \mathrm{OSp}(1|2)_{\mathbb{C}}$ consists of the identity morphism. Hence, in particular, condition (6.55) is empty and it follows that a homogeneous $\mathcal{A}^+$ is uniquely determined by a ($\mathcal{S}$-parametrized) super Lie algebra morphism $\Xi : \mathrm{Lie}(T) \to \mathfrak{osp}(1|2)_{\mathbb{C}}$ such that the pullback of $\mathcal{A}^+$ to Σ is given by

$$\mathcal{A}^+ = \theta_{\mathrm{MC}}^{(\mathbf{S}(T))} \diamond \Xi =: \phi_i^k T_k^+ \mathring{e}^i + \phi_i^A Q_A \mathring{e}^i \tag{6.58}$$

where, with respect to a basis $T_i \in \mathrm{Lie}(T)$ of $\mathrm{Lie}(T)$ (not to be confused with chiral generators T_i^+ of the bosonic subalgebra $\mathfrak{sl}(2, \mathbb{C})$ of $\mathfrak{osp}(1|2)_{\mathbb{C}}$), we set $\phi_i := \langle T_i | \Xi \rangle$. As

$\mathcal{A}^+$ is even, it follows that ϕ^i are bosonic and ϕ^A define odd fermion fields. Furthermore, $(\mathring{e}^i)_i$ is the induced basis of fiducial left-invariant one-forms[2] on Σ. This is the most general form of a homogeneous super connection 1-form.

If one requires that $\mathcal{A}^+$, in addition, is isotropic, it then follows that Ξ needs to satisfy (6.55) on the Lie algebra $\mathrm{Lie}(H_x) \cong \mathfrak{so}(3)$ of the isotropy subgroup which infinitesimally reads

$$\langle \mathrm{ad}_{\tau_i}(T_k)|\Xi\rangle = \mathrm{ad}_{\lambda_*(T_i^+)}\langle T_k|\Xi\rangle \tag{6.59}$$

$\forall \tau_i \in \mathfrak{so}(3)$ for some super Lie algebra morphism $\lambda_* : \mathfrak{so}(3) \to \mathfrak{osp}(1|2)_{\mathbb{C}}$. If one works in the standard category, it follows, as λ_* is even, that it only takes values in the even Lie subalgebra $\mathfrak{sl}(2,\mathbb{C})$. This corresponds to connections which are invariant under the spatial isotropy group up to ordinary gauge transformations. Let us focus first on these type of connections. We will then also consider a more general class below. Since $\mathfrak{so}(3)_{\mathbb{C}} = \mathfrak{sl}(2,\mathbb{C})$, it follows that the only nontrivial Lie algebra morphism, via this identification, is given by the identity morphism, i.e., $\lambda_* : \mathfrak{so}(3) \to \mathfrak{sl}(2,\mathbb{C})$, $\tau_i \mapsto T_i^+$. Using that the adjoint representation on the translational subgroup is given by $\mathrm{ad}_{\tau_i}(T_k) = \epsilon_{ik}{}^l T_l$, (6.59) leads to the condition

$$\epsilon_{ik}{}^l \phi_l = \mathrm{ad}_{T_i^+}(\phi_k^j T_j^+ + \phi_k^A Q_A) = \phi_k^j \epsilon_{ij}{}^m T_m^+ + \phi_k^A (\tau_i)^B{}_A Q_B \tag{6.60}$$

Restricting first on the even subalgebra, it follows immediately that the unique solution of the bosonic part of the connection has to be of the form

$$\phi_i^k = c\delta_i^k \tag{6.61}$$

for some complex, Grassmann-even number c. This is precisely the form of the isotropic self-dual Ashtekar connection as used in [91] (see also [235] for the discussion in case of real variables). Considering next the odd part of the super Lie algebra, one finds that the only possible solution to (6.60) for the fermionic degrees of freedom requires $\phi_i^A = 0$. Hence, in this restricted subclass of invariant super connections which are invariant only up to ordinary gauge transformations, it follows that one cannot make a purely isotropic ansatz in both bosonic and fermionic degrees of freedom.

For this reason, in what follows, let us consider a wider class of invariant super connections that are invariant up to gauge *and* (partial) supersymmetry transformations. To explain this in a bit more detail, let us focus on the special case of a vanishing cosmological constant, i.e., $L \to \infty$ such that, in this limit, $\mathfrak{osp}(1|2)_{\mathbb{C}}$ reduces to the super Poincaré algebra in $D = 2$ which we denote by $\overline{\mathfrak{osp}(1|2)}_{\mathbb{C}}$. According to Remark 6.3.3, super connection forms which are invariant up to $\overline{\mathrm{OSp}(1|2)}_{\mathbb{C}}$-gauge transformations

[2] We are adopting the notations in [88, 90] and denote the left-invariant 1-forms by $\mathring{e}^i$ instead of $\mathring{\omega}^i$ as usually done in the LQC literature.

are then classified by (even) $\mathcal{S}$-parametrized morphisms of super Lie modules of the form

$$\lambda_* : \mathcal{S} \times (\Lambda \otimes \mathfrak{so}(3)) \to \Lambda \otimes \overline{\mathfrak{osp}}(1|2)_{\mathbb{C}} \tag{6.62}$$

which, due to the additional parametrization supermanifold $\mathcal{S}$, can now take values in all of $\mathrm{Lie}(\overline{\mathrm{OSp}}(1|2)_{\mathbb{C}}) = \Lambda \otimes \overline{\mathfrak{osp}}(1|2)_{\mathbb{C}}$ not just in the bosonic sub module. It then follows that a more general class of such morphisms are of the form

$$\lambda_*^\theta(\tau_i) \equiv \lambda_*^\theta(\cdot, \tau_i) := T_i^+ + \sigma_i^{AA'}\bar{\theta}_{A'}Q_A \tag{6.63}$$

for some Grassmann-odd $\bar{\theta}_{A'} : \mathcal{S} \to \Lambda_1$. It is then immediate to see that this indeed defines a morphism of super Lie modules, since

$$\begin{aligned}
[\lambda_*^\theta(\tau_i), \lambda_*^\theta(\tau_j)] &= [T_i^+, T_j^+] + \bar{\theta}_{A'}[T_i^+, Q_A]\sigma_j^{AA'} + \bar{\theta}_{A'}[Q_A, T_j^+]\sigma_i^{AA'} \\
&= \epsilon_{ij}{}^k T_k^+ - i\bar{\theta}_{A'}Q_B(\sigma_{[i}\sigma_{j]})^{BA'} \\
&= \epsilon_{ij}{}^k (T_k^+ + \sigma_k^{AA'}\bar{\theta}_{A'}Q_A) = \epsilon_{ij}{}^k \lambda_*^\theta(\tau_k)
\end{aligned} \tag{6.64}$$

The form of the reduced connection again follows from the identity (6.60). Together with (6.63), it follows that the bosonic components of $\mathcal{A}^+$ need to satisfy

$$\epsilon_{ik}{}^l \phi_l^m = [\lambda_*^\theta(\tau_i), \phi_k^l T_l^+ + \phi_k^A Q_A]^m = \phi_k^l \epsilon_{il}{}^m \tag{6.65}$$

and thus again are of the form (6.61) for some complex, Grassmann-even number c. For the fermionic components, this yields

$$\begin{aligned}
\epsilon_{ik}{}^l \phi_l^A &= [\lambda_*^\theta(\tau_i), \phi_k^l T_l^+ + \phi_k^B Q_B]^A = \phi_k^B [T_i^+, Q_B]^A + c\sigma_i^{BB'}\bar{\theta}_{B'}[Q_B, T_k^+]^A \\
&= \phi_k^B (\tau_i)^A{}_B + \frac{c}{2}\epsilon_{ik}{}^l \sigma_l^{AA'}\bar{\theta}_{A'}
\end{aligned} \tag{6.66}$$

As may be easily checked by direct computation, this is solved by the following ansatz together with its complex conjugate

$$\phi_i^A = \sigma_i^{AA'}\bar{\psi}_{A'} \tag{6.67}$$

$$\bar{\phi}_i^{A'} = \sigma_i^{AA'}\psi_A \tag{6.68}$$

with $\bar{\psi}_{A'} := c\bar{\theta}_{A'}$ and ψ_A the corresponding complex conjugate. Interestingly, this is precisely the ansatz for the Rarita-Schwinger field as proposed by D'Eath at al. in [88–90]. Thus, we see that allowing for these general type of symmetry reduced connections, this leads to an ansatz that contains nontrivial fermionic contributions. Note, however, that these are not completely independent. This is due to the fact that this consideration only describes connections which are invariant up to left-handed supersymmetry transformations. Hence, for a full treatment, one also has to take right-handed supersymmetry transformations into account. Nevertheless, this demonstrates that in order to study symmetry reduced models with local supersymmetry, one should consider symmetry reduced connections which are invariant not only up to gauge but, at least in a specific sense, also supersymmetry transformations.

In the following sections, for the construction of symmetry reduced models, we will use the ansaetze (6.61) and (6.67) for the even and odd components of $\mathcal{A}^+$, respectively. In particular, we will allow the fermionic degrees of freedom to be independent from the bosonic ones. This is, in fact, consistent with the reality conditions. In this context, note that, making an isotropic ansatz for the bosonic degrees of freedom, this implies that the reality condition (6.54) which couples bosonic and fermionic degrees of freedom also has to be isotropic. In particular, the contorsion tensor necessarily has to be of the form

$$C_a^i \equiv C\mathring{e}_a^i \tag{6.69}$$

for some real, Grassmann-even number C. As we will see explicitly in Section 6.5.2, this turns out to be indeed the case using the ansatz (6.68).

Remark 6.5.1. In fact, one can argue that (6.67) and (6.68) are the most general ansatz for the fermionic fields consistent with the reality conditions. To this end, one notices that the Rarita-Schwinger field $\psi_i := (\phi_i^A, \bar{\phi}_{iA'})^T$ constructed out of the homogeneous fermionic components of $\mathcal{A}^+$ can be always split into a trace part $\phi := \gamma^i\psi_i$ as well as a trace-free part $\rho_i := \psi_i - \frac{1}{3}\gamma_i\phi$ w.r.t. the gamma matrices [67]. As the trace-free part carries internal indices, it then follows that the contorion tensor will generically not be of the form (6.69). The trace part, on the other hand, precisely leads back to (6.67) and (6.68). Thus, in this sense, (6.67) and (6.68) can be regarded as a necessary condition for odd components of $\mathcal{A}^+$ to provide consistency with the reality condition (6.54) in case of purely isotropic bosonic degrees of freedom.

6.5.2. Symmetry reduction of the chiral action

Having derived the general ansatz for the bosonic and fermionic degrees of freedom for homogeneous isotropic cosmology, we want to perform a symmetry reduction of the action (6.33) and determine the constraints in the symmetry reduced model. To this end, as already mentioned in the previous section, we are mainly interested in the FLRW models with positive ($k = 1$) and vanishing spatial curvature ($k = 0$), respectively. In the spatially flat case, the translational subgroup T of the isometry group is Abelian and a basis (co-frame) of fiducial left-invariant one-forms is obtained as the coefficients of the Maurer-Cartan form $\theta^{(T)}_{\mathrm{MC}} = \mathring{e}^i T_i$.

In the case $k = +1$, Σ is isomorphic to a three-sphere $\mathbb{S}^3$ which can be identified with the Lie group SU(2). Hence, the Maurer-Cartan form of SU(2) yields a canonical basis of fiducial left-invariant one-forms $\theta^{(\mathrm{SU}(2))}_{\mathrm{MC}} = \mathring{e}^i \tau_i$. For both situations, according to (6.57), the structure equation fulfilled by these sets of 1-forms can be written as

$$\mathrm{d}\mathring{e}^i + \frac{k}{2}\epsilon^i{}_{jk}\mathring{e}^j \wedge \mathring{e}^k = 0 \tag{6.70}$$

where $k = 1$ or $k = 0$ in case of a positive or vanishing spatial curvature. The corresponding fiducial frame fields $\mathring{e}_i$ dual to the 1-forms $\mathring{e}^i$ satisfy $\mathring{e}^i_a \mathring{e}^a_j = \delta^i_j$ and form a basis of left-invariant vector fields on Σ. The fiducial metric $\mathring{q}_{ab}$ is related to the co-frame via

$$\mathring{q}_{ab} = \delta_{ij}\mathring{e}^i_a \mathring{e}^j_b \tag{6.71}$$

where, for $k = 1$, as explained in [237] this metric corresponds to a three-sphere of radius $r_0 = 2$ so that the total volume of Σ is given by $V_0 = 2\pi^2 r_0^3 = 16\pi^2$. The co-frame e^i of a different spatial slice of the spacetime manifold is related to the fiducial one via rescaling $e^i = a\mathring{e}^i$ and similarly for the triad $e_i = a^{-1}\mathring{e}_i$ where a can be both positive or negative according to the handedness of the triad. Here and in the following, we will fix the sign of the internal three-form ϵ_{ijk} with the convention $\epsilon_{123} = 1$. The volume form of the spatial slices is then related to the internal 3-form via

$$\sqrt{q}\epsilon_{abc} = \epsilon_{ijk} e^i_a e^j_b e^k_c \tag{6.72}$$

According to (6.72), due to the conventions made for the internal 3-form, in case of a positive orientation of the triad, ϵ_{abc} is normalized to one, i.e., $\epsilon_{123} = +1$. Consequently, changing the orientation of the triad then changes the sign of the volume form such that $\epsilon_{abc} \longrightarrow -\epsilon_{abc}$ under $e^i_a \longrightarrow -e^i_a$. Finally, from the definition, it follows

$$\epsilon^{abc} e^i_a e^j_b e^k_c = \epsilon^2 \epsilon^{ijk} \tag{6.73}$$

where ϵ indicates the orientation of the triad. For convenience, following [91], we will keep track of the various sign factors appearing in the computations and therefore do not set $\epsilon^2 = 1$ at this stage, the reason being that this will simplify the implementation of the dynamical constraints in the quantum theory.

For the symmetry reduction of the theory, let us fix a fiducial cell $\mathcal{V}$ in Σ of finite volume V_0 as measured by (6.71) which will be the whole Σ in case $k = 1$ or a finite proper sub region in case $k = 0$ where, in the latter case, physics will be insensitive to this choice due to homogeneity. Furthermore, we introduce a length scale setting $\ell_0 := V_0^{\frac{1}{3}}$. With these definitions, it follows, using ansatz (6.61) for the reduced connection, that the fundamental variables corresponding to the bosonic degrees of freedom take the form

$$A^{+i}_a = \frac{c}{\ell_0}\mathring{e}^i_a \quad \text{and} \quad E^a_i = \frac{p}{\ell_0^2}\sqrt{\mathring{q}}\mathring{e}^a_i \tag{6.74}$$

for some Grasmann-even number p which, according to (6.54), needs to satisfy the reality condition $p = p^*$. For the fermionic degrees of freedom, we choose the ansatz (6.67) and (6.68). The symplectic potential in (6.33) then takes the form

$$\int_{\mathbb{R}} \mathrm{d}t \int_{\mathcal{V}} \mathrm{d}^3x \left(\frac{i}{\kappa}E^a_i\dot{A}^{+i}_a - \pi^a_A\dot{\psi}^A_a\right) = \int_{\mathbb{R}} \mathrm{d}t \left(\frac{3i}{\kappa}p\dot{c} - \bar{\pi}^{A'}\dot{\bar{\psi}}_{A'}\right) \tag{6.75}$$

with the canonically conjugate momentum

$$\bar{\pi}^{A'} := -6iV_0|a|\psi_A n^{AA'} \tag{6.76}$$

Hence, the canonically conjugate variables in the symmetry reduced theory are given by (c, p) and $(\bar{\psi}_{A'}, \bar{\pi}^{A'})$ satisfying the nonvanishing graded Poisson brackets

$$\{p, c\} = \frac{i\kappa}{3} \quad \text{and} \quad \{\bar{\pi}^{A'}, \bar{\psi}_{B'}\} = -\delta^{A'}_{B'} \tag{6.77}$$

where the complex conjugate ψ_A of $\bar{\psi}_{A'}$ is related to its canonically conjugate momentum through the reality condition (6.76). In order to derive the respective reality condition in the bosonic sector, let us first consider the metric connection part of (6.51) which, using (6.70), yields

$$\Gamma^i_a(e) = \frac{k}{4}\epsilon^{ijk}\mathring{e}^b_j\left(2\epsilon_{kmn}\mathring{e}^m_{[a}\mathring{e}^n_{b]} + \mathring{e}^c_k\mathring{e}^l_a\epsilon_{lmn}\mathring{e}^m_{[c}\mathring{e}^n_{b]}\right) = \frac{k}{2}\mathring{e}^i_a \tag{6.78}$$

Finally, for the torsion contribution we compute, using the identities (4.32) and (4.33) stated in Section 4.2,

$$\begin{aligned} C^i_a &= \frac{i\kappa}{4\sqrt{q}}\epsilon^{bcd}e^i_d\left(2\mathring{e}^{AB'}_{[a}\mathring{e}^{BA'}_{b]}e_{cAA'} - \mathring{e}^{AB'}_b\mathring{e}^{BA'}_c e_{aAA'}\right)\bar{\psi}_{B'}\psi_B \\ &= \frac{i\kappa\epsilon^2}{4a^2}\epsilon^{ijk}e^l_a\left(\sigma^{AB'}_l\sigma^{BA'}_j\sigma_{kAA'} - \sigma^{AB'}_j\sigma^{BA'}_l\sigma_{kAA'} - \sigma^{AB'}_j\sigma^{BA'}_k\sigma_{lAA'}\right)\bar{\psi}_{B'}\psi_B \\ &= \frac{i\kappa}{2a^2}e^l_a\left(-i\sigma^{AB'}_l n_{AA'}\sigma^{iBA'} + \epsilon^{ij}{}_l\sigma^{BB'}_j\right)\bar{\psi}_{B'}\psi_B \\ &= \frac{\kappa}{2a^2}e^i_a n^{BB'}\bar{\psi}_{B'}\psi_B \end{aligned} \tag{6.79}$$

which, together with (6.76), yields

$$C_a^i = -\frac{i\kappa}{12\ell_0 p}\mathring{e}_a^i \bar{\pi}^{A'}\bar{\psi}_{A'} \tag{6.80}$$

Hence, the reality condition (6.54) for self-dual Ashtekar connection in the reduced theory takes the form

$$c + c^* = k\ell_0 - \frac{i\kappa}{6p}\bar{\pi}^{A'}\bar{\psi}_{A'} \tag{6.81}$$

which, in particular, is isotropic in the torsion contribution consistent with the isotropic ansatz for the gravitational degrees of freedom.

Next, we have to compute the constraints in the reduced model. To this end, let us first consider the Gauss constraint (6.45). It is immediate to see that the first part depending on the covariant derivative yields a total derivative as the term proportional to the connection simply drops out due to the isotropic ansatz. Hence, only the fermionic contribution remains for which, using (6.34), we compute

$$\begin{aligned} G_i &= -\pi_A^a(\tau_i)^A{}_B\psi_a^B = -\epsilon^{abc}\mathring{e}_b^{CA'}\psi_C e_{cAA'}(\tau_i)^A{}_B\mathring{e}_a^{BC'}\bar{\psi}_{C'} \\ &= -|a|\sqrt{\mathring{q}}\epsilon^{jkl}\psi_C\bar{\psi}_{C'}\sigma_j^{CA'}\sigma_{kAA'}(\tau_i)^A{}_B\sigma_l^{BC'} \\ &= 2i|a|\sqrt{\mathring{q}}\psi_C\bar{\psi}_{C'}n_{AA'}\sigma^{lCA'}(\tau_i)^A{}_B\sigma_l^{BC'} \\ &= -2i|a|\sqrt{\mathring{q}}\psi_C\bar{\psi}_{C'}n^{CD'}(\tau_i)_{D'}{}^{C'} \end{aligned} \tag{6.82}$$

Hence, inserting (6.82) into the action (6.33), we find

$$\int_{\mathbb{R}}\mathrm{d}t\int_{\mathcal{V}}\mathrm{d}^3x\, A_t^i G_i = \int_{\mathbb{R}}\mathrm{d}t\,\frac{1}{3}A_t^i\left(-6i|a|V_0\psi_C n^{CD'}(\tau_i)_{D'}{}^{C'}\bar{\psi}_{C'}\right) \tag{6.83}$$

so that, due to (6.76), the reduced Gauss constraint can be written in the form

$$G_i = \bar{\pi}^{A'}(\tau_i)_{A'}{}^{B'}\bar{\psi}_{B'} \tag{6.84}$$

It is immediate from the homogeneous ansatz that the theory is invariant under diffeomorphism transformations. Hence, concerning the vector constraint, one may expect that (6.46) vanishes identically. However, as can be checked explicitly after some lengthy calculation, while the purely bosonic term vanishes identically due the isotropic ansatz for the gravitational degrees of freedom, the fermionic contribution turns out to be proportional to the Gauss constraint. This is in fact analogous to the full theory, were the Gauss constraint needs to be substracted from the vector constraint in order to obtain the infinitesimal generator of pure diffeomorphism transformations. Next, let us turn to the left supersymmetry constraint (6.40) which reads

$$S^L_A = \partial_a\pi^a_A - \pi^a_B(\tau_i)^B{}_A A^{+i}_a + 2L^{-1}E^{ai}\psi^B_a(\epsilon\tau_i)_{BA} \tag{6.85}$$

If we drop the total derivative in (6.85), it follows

$$\begin{aligned}
S^L_A &= -\frac{c}{\ell_0}\epsilon^{abc}\mathring{e}^i_a\bar{\psi}^{A'}_b e_{cBB'}(\tau_i)^B{}_A + 2\frac{p}{\ell_0^2 L}\sqrt{\mathring{q}}\mathring{e}^{ai}\mathring{e}^{BB'}_a\bar{\psi}_{B'}(\epsilon\tau_i)_{BA}\\
&= -\frac{c}{\ell_0 a^2}\epsilon^{abc}e^i_a e^j_b e^k_c\psi_C\sigma_j^{CB'}\sigma_{kBB'}(\tau_i)^B{}_A - \frac{ip}{\ell_0^2 L}\sqrt{\mathring{q}}\sigma^{iBB'}(\epsilon\tau_i)_{BA}\bar{\psi}_{B'}\\
&= \frac{i\epsilon^2 c}{\ell_0 a^2}|a|^3\sqrt{\mathring{q}}\epsilon^{ijk}\epsilon_{jk}{}^l n_{BB'}\sigma_l^{CB'}(\tau_i)^B{}_A\psi_C + \frac{3ip}{\ell_0^2 L}\sqrt{\mathring{q}}\bar{\psi}_{B'}\epsilon^{B'A'}n_{AA'}\\
&= \frac{3|a|\epsilon^2}{\ell_0}\sqrt{\mathring{q}}c\psi_A + \frac{3ip}{\ell_0^2 L}\sqrt{\mathring{q}}\bar{\psi}_{B'}\epsilon^{B'A'}n_{AA'}
\end{aligned} \tag{6.86}$$

which, together with identity (4.31), gives

$$S^L_A = 3\sqrt{\mathring{q}}n_{AA'}\left(\frac{|a|\epsilon^2 c}{\ell_0}\psi_B n^{BA'} + \frac{ip}{\ell_0^2 L}\bar{\psi}_{B'}\epsilon^{B'A'}\right) \tag{6.87}$$

Hence, if we insert the reduced expression (6.86) into (6.33), we find

$$\begin{aligned}
\int_{\mathbb{R}}\mathrm{d}t\int_{\mathcal{V}}\mathrm{d}^3x\, S^L_A\psi^A_t &= \int_{\mathbb{R}}\mathrm{d}t\left(\frac{|a|\epsilon^2 c}{\ell_0}V_0\psi_B n^{BA'} + \frac{i\ell_0}{L}p\bar{\psi}_{B'}\epsilon^{B'A'}\right)3n_{AA'}\psi^A_t\\
&= \int_{\mathbb{R}}\mathrm{d}t\left(-\frac{\epsilon^2 c}{\ell_0}6i|a|V_0\psi_B n^{BA'} + \frac{6\ell_0}{L}p\bar{\psi}_{B'}\epsilon^{B'A'}\right)\frac{i}{2}n_{AA'}\psi^A_t\\
&=: \int_{\mathbb{R}}\mathrm{d}t\left(\frac{\epsilon c}{\ell_0}\bar{\pi}^{A'} + \frac{6\ell_0}{L}|p|\bar{\psi}_{B'}\epsilon^{B'A'}\right)\epsilon\bar{\psi}_{tA'}
\end{aligned} \tag{6.88}$$

where in the last step the reality condition (6.76) was used. Therefore, in the reduced theory, left supersymmetry constraint takes the form

$$S^{LA'} = \frac{\epsilon c}{\ell_0}\bar{\pi}^{A'} + \frac{6\ell_0}{L}|p|\bar{\psi}_{B'}\epsilon^{B'A'} \tag{6.89}$$

For the right supersymmetry constraint (6.41), we first focus on the term depending on the covariant derivative which reads

$$
\begin{aligned}
&-\epsilon^{A'B'}\epsilon^{abc}e_{aAB'}D_b^{(A^+)}\psi_c^A = -\epsilon^{A'B'}\epsilon^{abc}e_{aAB'}(\partial_b\mathring{e}_c^j\sigma_j^{AC'}\bar{\psi}_{C'} + A_b^{+j}(\tau_j)^A{}_C\psi_c^C) \\
&= -\epsilon^{A'B'}\epsilon^{abc}e_a^i\partial_b\mathring{e}_c^j\sigma_{iAB'}\sigma_j^{AC'}\bar{\psi}_{C'} - \epsilon^{A'B'}\epsilon^{abc}\frac{c}{\ell_0}e_a^i\mathring{e}_b^j\mathring{e}_c^k\sigma_{iAB'}(\tau_j)^A{}_B\sigma_k^{BC'}\bar{\psi}_{C'}
\end{aligned}
\tag{6.90}
$$

Using again (6.70), this yields

$$
\begin{aligned}
&\frac{k}{2}\epsilon^j{}_{kl}\epsilon^{A'B'}\epsilon^{abc}e_a^i\mathring{e}_b^k\mathring{e}_c^l\sigma_{iAB'}\sigma_j^{AC'}\bar{\psi}_{C'} - \frac{c}{\ell_0}\epsilon^{A'B'}\epsilon^2|a|\epsilon^{ijk}\sigma_{iAB'}(\tau_j)^A{}_B\sigma_k^{BC'}\bar{\psi}_{C'} \\
&= -3k|a|\epsilon^2\sqrt{\mathring{q}}\epsilon^{A'B'}\bar{\psi}_{B'} + \frac{3|a|\epsilon^2 c}{\ell_0}\sqrt{\mathring{q}}\epsilon^{A'B'}\bar{\psi}_{B'} \\
&= \frac{3|a|\epsilon^2}{\ell_0}\sqrt{\mathring{q}}\epsilon^{A'B'}(c-k\ell_0)\bar{\psi}_{B'}
\end{aligned}
\tag{6.91}
$$

On the other hand, the term proportional to the cosmological constant gives

$$
\begin{aligned}
2L^{-1}E^{ai}(\epsilon\tau_i)^{A'B'}\psi_{aB'} &= \frac{2p}{\ell_0^2 L}\sqrt{\mathring{q}}\mathring{e}^{ai}(\epsilon\tau_i)^{A'B'}\mathring{e}_a^{CC'}\epsilon_{C'B'}\psi_C \\
&= \frac{2p}{\ell_0^2 L}\sqrt{\mathring{q}}(\epsilon\tau_i\epsilon)^{A'}{}_{C'}\sigma^{iCC'}\psi_C \\
&= \frac{2p}{\ell_0^2 L}\sqrt{\mathring{q}}(\tau_i)_{C'}{}^{A'}\sigma^{iCC'}\psi_C = \frac{3ip}{\ell_0^2 L}\sqrt{\mathring{q}}\psi_C n^{CA'}
\end{aligned}
\tag{6.92}
$$

where we used that $\epsilon\tau_i\epsilon = \tau_i^T$. To summarize, we found

$$
S^{RA'} = \epsilon^{A'B'}\sqrt{\mathring{q}}\left(\frac{3|a|\epsilon^2}{\ell_0}(c-k\ell_0)\bar{\psi}_{B'} + \frac{3ip}{\ell_0^2 L}\psi_A n^{AC'}\epsilon_{C'B'}\right) \tag{6.93}
$$

Inserting (6.93) into the action (6.33), this yields

$$
\begin{aligned}
\int_{\mathbb{R}}\mathrm{d}t\int_{\mathcal{V}}\mathrm{d}^3x\,\bar{\psi}_{tA'}S^{RA'} &= -\int_{\mathbb{R}}\mathrm{d}t\,\epsilon\bar{\psi}_t^{B'}\left(3a\ell_0^2(c-k\ell_0)\bar{\psi}_{B'} + \frac{3i|p|}{\ell_0^2 L}V_0\psi_A n^{AC'}\epsilon_{C'B'}\right) \\
&= -\int_{\mathbb{R}}\mathrm{d}t\,\epsilon\bar{\psi}_t^{B'}\left(3\epsilon\ell_0\sqrt{|p|}(c-k\ell_0)\bar{\psi}_{B'} - \frac{1}{2L}\sqrt{|p|}\bar{\pi}^{C'}\epsilon_{C'B'}\right)
\end{aligned}
\tag{6.94}
$$

so that, in the reduced theory, the right supersymmetry constraint can be written as

$$S^R_{A'} = 3\epsilon\ell_0\sqrt{|p|}(c - k\ell_0)\bar{\psi}_{A'} - \frac{1}{2L}\sqrt{|p|}\bar{\pi}^{B'}\epsilon_{B'A'} \tag{6.95}$$

Finally, we need to derive the reduced analog of the Hamiltonian constraint. As far as the purely bosonic term is concerned

$$H_b = -\frac{E^a_i E^b_j}{2\kappa\sqrt{q}}\epsilon^{ij}{}_k F(A^+)^k_{ab} \tag{6.96}$$

in case of the self-dual approach, a reduced expression is stated in [91]. For sake of completeness, let us derive it here in full detail. This will also clarify the positions of the sign factors. Again applying the structure equation (6.70), we find for the curvature of the self-dual Ashtekar connection

$$F(A^+)^k_{ab} = \frac{2c}{\ell_0}\partial_{[a}\mathring{e}^k_{b]} + \frac{c^2}{\ell_0^2}\epsilon^k{}_{lm}\mathring{e}^l_a\mathring{e}^m_b = -\frac{kc}{\ell_0}\epsilon^k{}_{lm}\mathring{e}^l_a\mathring{e}^m_b + \frac{c^2}{\ell_0^2}\epsilon^k{}_{lm}\mathring{e}^l_a\mathring{e}^m_b$$

such that (6.96) becomes

$$H_b = -\frac{p^2}{2\kappa|a|^3}\sqrt{\mathring{q}}\left(-\frac{6kc}{\ell_0} + \frac{6c^2}{\ell_0^2}\right) = -\frac{3\epsilon^2}{\kappa V_0}\sqrt{|p|}(c^2 - k\ell_0 c)\sqrt{\mathring{q}} \tag{6.97}$$

which is exactly the form of the bosonic part of the Hamiltonian constraint as stated in [91]. Next, let us turn to the fermionic contribution

$$H_f = -\epsilon^{abc}\bar{\psi}^{A'}_a n_{AA'}D^{(A^+)}_b\psi^A_c \tag{6.98}$$

which has in fact a similar form as the right supersymmetry constraint. Following the steps as before, we compute

$$\begin{aligned}
H_f =& -\epsilon^{abc}\bar{\psi}^{A'}_a n_{AA'}D^{(A^+)}_b\psi^A_c \\
=& -\epsilon^{abc}\bar{\psi}^B_a\mathring{e}^{BA'}_a n_{AA'}(\partial_b\mathring{e}^j_c\sigma^{AB'}_j\bar{\psi}_{B'} + A^{+j}_b(\tau_j)^A{}_C\mathring{e}^k_c\sigma^{CC'}_k\bar{\psi}_{C'}) \\
=&\frac{k}{2}\epsilon^j{}_{kl}\epsilon^{abc}\mathring{e}^i_a\mathring{e}^k_b\mathring{e}^l_c(\sigma_i\sigma_j)^{BB'}\psi_B\bar{\psi}_{B'} - \frac{c}{2i\ell_0}\epsilon^{abc}\mathring{e}^i_a\mathring{e}^k_b\mathring{e}^l_c(\sigma_i\sigma_j\sigma_k)^{BB'}\psi_B\bar{\psi}_{B'} \\
=&3k\epsilon\sqrt{\mathring{q}}n^{AA'}\psi_A\bar{\psi}_{A'} - \frac{3c}{\ell_0}\epsilon\sqrt{\mathring{q}}n^{AA'}\psi_A\bar{\psi}_{A'} \\
=& -\frac{3\epsilon}{\ell_0}\sqrt{\mathring{q}}(c - k\ell_0)n^{AA'}\psi_A\bar{\psi}_{A'}
\end{aligned} \tag{6.99}$$

Remains to compute the reduced expression of the cosmological contribution in (6.47). Some simple algebra reveals

$$
\begin{aligned}
H_\Lambda =& \sqrt{\mathring{q}}\frac{p^2}{2\ell_0^4 L|a|^3}\epsilon^{ijk}\mathring{e}_i^a\mathring{e}_j^b(\psi_{aA}n_{BA'}\sigma_k^{AA'}\psi_b^B - \bar{\psi}_a^{A'}n_{AA'}\sigma_k^{AB'}\bar{\psi}_{bB'}) + \sqrt{\mathring{q}}\frac{3|a|^3}{\kappa L^2} \\
=& \sqrt{\mathring{q}}\frac{\epsilon^2|a|}{2L}\mathring{\epsilon}^{ijk}(\sigma_i^{CC'}\epsilon_{CA}n_{BA'}\sigma_k^{AA'}\sigma_j^{BD'}\bar{\psi}_{C'}\bar{\psi}_{D'} \\
& - \sigma_i^{CA'}n_{AA'}\sigma_k^{AB'}\sigma_j^{DD'}\epsilon_{D'B'}\psi_C\psi_D) + \sqrt{\mathring{q}}\frac{3|p|^{\frac{3}{2}}}{V_0\kappa L^2} \\
=& \sqrt{\mathring{q}}\frac{3i\epsilon^2\sqrt{|p|}}{\ell_0 L}(\epsilon^{C'D'}\bar{\psi}_{C'}\bar{\psi}_{D'} + \epsilon^{CD}\psi_C\psi_D) + \sqrt{\mathring{q}}\frac{3|p|^{\frac{3}{2}}}{V_0\kappa L^2}
\end{aligned}
\tag{6.100}
$$

Thus, inserting (6.97), (6.99) and (6.100) into (6.33), we find

$$
\begin{aligned}
\int_{\mathbb{R}}\mathrm{d}t\int_{\mathcal{V}}\mathrm{d}^3x\, NH = \int_{\mathbb{R}}\mathrm{d}t\, N\Bigg(&-\frac{3\epsilon^2}{\kappa}\sqrt{|p|}(c^2 - k\ell_0 c) - \frac{\epsilon}{2}(c - k\ell_0)\frac{1}{\sqrt{|p|}}i\bar{\pi}^{A'}\bar{\psi}_{A'} \\
&+\frac{3i\epsilon^2\ell_0^2}{L}\sqrt{|p|}(\epsilon^{C'D'}\bar{\psi}_{C'}\bar{\psi}_{D'} + \epsilon^{CD}\psi_C\psi_D) + \frac{3|p|^{\frac{3}{2}}}{\kappa L^2}\Bigg)
\end{aligned}
\tag{6.101}
$$

Using the reality condition (6.76), one can express the term in the last line of (6.101) proportional to the complex conjugate ψ_A of the fundamental variable $\bar{\psi}_{A'}$ in terms of the corresponding canonically conjugate momentum. In fact, direct computation yields

$$
\begin{aligned}
\epsilon_{A'B'}\bar{\pi}^{A'}\bar{\pi}^{B'} &= -36|a|^2V_0^2 n^{AA'}\epsilon_{A'B'}n^{BB'}\psi_A\psi_B \\
&= -36|p|\ell_0^4\epsilon^{AB}\psi_A\psi_B
\end{aligned}
\tag{6.102}
$$

Hence, it follows that the Hamiltonian constraint in the reduced theory takes the form

$$
\begin{aligned}
H =& -\frac{3\epsilon^2}{\kappa}\sqrt{|p|}(c^2 - k\ell_0 c) - \frac{\epsilon}{2}(c - k\ell_0)\frac{1}{\sqrt{|p|}}i\bar{\pi}^{A'}\bar{\psi}_{A'} \\
&+ \frac{3i\epsilon^2\ell_0^2}{L}\sqrt{|p|}\epsilon^{C'D'}\bar{\psi}_{C'}\bar{\psi}_{D'} - \frac{\epsilon^2}{12\ell_0^2 L}\frac{1}{\sqrt{|p|}}\epsilon_{A'B'}\bar{\pi}^{A'}\bar{\pi}^{B'} + \frac{3}{\kappa L^2}|p|^{\frac{3}{2}}
\end{aligned}
\tag{6.103}
$$

6.5.3. Half densitized fermion fields

As proposed in [80] in case of real Ashtekar variables, in order to simplify the reality conditions (6.34) for the fermion fields, it is worthwhile to change their density weight by going over to half-densities (see Section 4.3.1). In the reduced theory, we can do something similar. According to (6.76), the complex conjugate of $\bar{\psi}_{A'}$ also depends on the scale factor and thus on the momentum conjugate to the reduced connection. Hence, it is suggestive to introduce the new variables[3]

$$\phi_{A'} := \sqrt{6|a|V_0}\bar{\psi}_{A'} = \sqrt{6}\ell_0|p|^{\frac{1}{4}}\bar{\psi}_{A'} \tag{6.104}$$

$$\pi_\phi^{A'} := \frac{1}{\sqrt{6|a|V_0}}\bar{\pi}^{A'} = \frac{1}{\sqrt{6}\ell_0|p|^{\frac{1}{4}}}\bar{\pi}^{A'} \tag{6.105}$$

Actually, in contrast to the full theory, we are not changing the effective density weight of the fermion fields since, due to the ansatz (6.67), the density weight has already been absorbed in the fiducial co-triad. Nevertheless, these new defined variables have the same dependence on the scale factor a as the half-densitized fields in the full theory (i.e. they are of order $\sim |a|^{\frac{1}{2}}$) which is the reason why we continue to call them half-densities in the reduced theory. Since the definition of the new fields explicitly involves the scale factor (and therefore the reduced electric field p), this, a priori, does not provide a canonical transformation. In fact, as will become clear in what follows, this also amounts to a redefinition of the bosonic degrees of freedom.

To this end, let us go back to the symplectic potential (6.75). Inserting the definitions (6.104) and (6.105), we find after some careful analysis

$$\begin{aligned}
\int_{\mathbb{R}}\mathrm{d}t\,\left(\frac{3i}{\kappa}p\dot{c}-\bar{\pi}^{A'}\dot{\bar{\psi}}_{A'}\right) &= \int_{\mathbb{R}}\mathrm{d}t\,\left(\frac{3i}{\kappa}p\dot{c}-\sqrt{6|a|V_0}\pi_\phi^{A'}L_{\partial_t}\left[\frac{1}{\sqrt{6|a|V_0}}\phi_{A'}\right]\right)\\
&= \int_{\mathbb{R}}\mathrm{d}t\,\left(\frac{3i}{\kappa}p\dot{c}-\pi_\phi^{A'}\dot{\phi}_{A'}+\frac{1}{2}|a|^{-1}\epsilon\dot{a}\pi_\phi^{A'}\phi_{A'}\right)\\
&= \int_{\mathbb{R}}\mathrm{d}t\,\left(\frac{3i}{\kappa}p\dot{c}-\pi_\phi^{A'}\dot{\phi}_{A'}+\frac{1}{4}\frac{2\epsilon a\dot{a}}{\epsilon a^2}\pi_\phi^{A'}\phi_{A'}\right)\\
&= \int_{\mathbb{R}}\mathrm{d}t\,\left(\frac{3i}{\kappa}p\dot{c}-\pi_\phi^{A'}\dot{\phi}_{A'}+\frac{1}{4}\frac{\dot{p}}{p}\pi_\phi^{A'}\phi_{A'}\right)\\
&= \int_{\mathbb{R}}\mathrm{d}t\,\left(\frac{3i}{\kappa}pL_{\partial_t}\left[c+\frac{i\kappa}{12p}\pi_\phi^{A'}\phi_{A'}\right]-\pi_\phi^{A'}\dot{\phi}_{A'}\right)
\end{aligned} \tag{6.106}$$

where in the third line we reinserted the definition $p = \epsilon a^2\ell_0^2$ and, from the fourth to the last line, we integrated by parts dropping a boundary term. Hence, according to (6.106), this suggests to define the transformation of the reduced connection via

$$\tilde{c} := c + \frac{i\kappa}{12p}\pi_\phi^{A'}\phi_{A'} \tag{6.107}$$

[3] For notational simplification, we will refrain from indicating the new fundamental variables with an additional bar in what follows. The complex conjugate of $\phi_{A'}$ will then be written as $\bar{\phi}_A$

Since signs matter in what follows, it is instructive to check explicitly that this indeed provides a canonical transformation. Observing that $\{c, |p|\} = \{c, \sqrt{p^2}\} = \epsilon\{c, p\}$, direct calculation yields

$$\begin{aligned}\{\tilde{c}, |p|^{\frac{1}{4}}\bar{\psi}_{A'}\} &= \{c + \frac{i\kappa}{12p}\bar{\pi}^{B'}\bar{\psi}_{B'}, |p|^{\frac{1}{4}}\bar{\psi}_{A'}\} \\ &= \{c, |p|^{\frac{1}{4}}\}\bar{\psi}_{A'} - \frac{i\kappa\epsilon}{12|p|^{\frac{3}{4}}}\bar{\psi}_{B'}\{\bar{\pi}^{B'}, \bar{\psi}_{A'}\} \\ &= \frac{1}{4|p|^{\frac{3}{4}}}\epsilon\{c, p\}\bar{\psi}_{A'} + \frac{i\kappa\epsilon}{12|p|^{\frac{3}{4}}}\bar{\psi}_{A'} = 0 \end{aligned} \tag{6.108}$$

proving that the Poisson bracket between $\tilde{c}$ and $\phi_{A'}$ indeed vanishes. On the other hand, we have

$$\begin{aligned}\{\tilde{c}, |p|^{-\frac{1}{4}}\bar{\pi}^{A'}\} &= \{c, |p|^{-\frac{1}{4}}\}\bar{\pi}^{A'} + \frac{i\kappa\epsilon}{12|p|^{\frac{5}{4}}}\bar{\pi}^{B'}\{\bar{\psi}_{B'}, \bar{\pi}^{A'}\} \\ &= \frac{i\kappa\epsilon}{12|p|^{\frac{5}{4}}}\bar{\pi}^{A'} - \frac{i\kappa\epsilon}{12|p|^{\frac{5}{4}}}\bar{\pi}^{A'} = 0 \end{aligned} \tag{6.109}$$

and therefore the Poisson bracket between $\tilde{c}$ and $\pi_{\phi}^{A'}$ is zero, as well. Hence, this proves that the transformation of the phase space variables as declared via (6.104), (6.105) as well as (6.107) indeed provides a canonical transformation.

To summarize, w.r.t. the half-densitized fermion fields, the new canonically conjugate variables are given by $(\tilde{c}, p)$ and $(\phi_{A'}, \pi_{\phi}^{A'})$, respectively, which satisfy the nonvanishing Poisson brackets

$$\{p, \tilde{c}\} = \frac{i\kappa}{3} \quad \text{and} \quad \{\pi_{\phi}^{A'}, \phi_{B'}\} = -\delta_{B'}^{A'} \tag{6.110}$$

The complex conjugate $\bar{\phi}_A$ is related to the canonically conjugate momentum $\pi_{\phi}^{A'}$ via the simplified reality condition

$$\pi_{\phi}^{A'} = -i\bar{\phi}_A n^{AA'} \quad \Leftrightarrow \quad \bar{\phi}_A = i\pi_{\phi}^{A'} n_{AA'} \tag{6.111}$$

Using (6.107), it follows that the respective reality conditions for the bosonic degrees of freedom take the form

$$\tilde{c} + (\tilde{c})^* = k\ell_0 \quad \text{and} \quad p^* = p \tag{6.112}$$

In particular, it follows that, in both parity even odd sector, the torsion contribution in the reality condition of the reduced connection simply drops out! This will drastically simplify their implementation in the quantum theory as will be studied in detail below.

Let us rewrite the dynamical constraints in the new fundamental variables. For the Hamiltonian constraint, one immediately finds

$$\begin{aligned}H=&-\frac{3\epsilon^2}{\kappa}\sqrt{|p|}(c^2-k\ell_0 c)-\frac{\epsilon}{2}(c-k\ell_0)\frac{1}{\sqrt{|p|}}i\pi_{\phi}^{A'}\phi_{A'}\\&+\frac{i\epsilon^2}{2L}\epsilon^{A'B'}\phi_{A'}\phi_{B'}-\frac{i\epsilon^2}{2L}\epsilon_{A'B'}\pi_{\phi}^{A'}\pi_{\phi}^{B'}+\frac{3}{\kappa L^2}|p|^{\frac{3}{2}}\end{aligned}\tag{6.113}$$

where we have implicitly performed the substitution $c:=\tilde{c}-\frac{i\kappa}{12|p|}\pi_{\phi}^{A'}\phi_{A'}$. The left and right supersymmetry constraints are given by

$$S^{LA'}=\epsilon c|p|^{\frac{1}{4}}\pi_{\phi}^{A'}+L^{-1}|p|^{\frac{3}{4}}\phi_{B'}\epsilon^{B'A'}\tag{6.114}$$

and

$$S^{R}_{A'}=3\epsilon|p|^{\frac{1}{4}}(c-k\ell_0)\phi_{A'}-3L^{-1}|p|^{\frac{3}{4}}\pi_{\phi}^{B'}\epsilon_{B'A'}\tag{6.115}$$

respectively.

Remark 6.5.2. Some comments on parity are in order. Due to our sign conventions made for the internal three-form ϵ_{ijk}, the metric connection (6.78) is even under the parity transformations $e^i_a\longrightarrow -e^i_a$ while the extrinsic curvature K^i_a is odd. Hence, c does not have a straightforward behavior under parity transformations. For LQC without fermions another set of conventions, in which the internal three-form ϵ_{ijk} changes sign under internal parity, is very useful [238]. Under those conventions c does transform in a well-defined way. In the present situation, it turns out that those conventions lead to unwanted sign factors in the Dirac algebra, effectively breaking supersymmetry. We suspect that the root of the problem is that the Ashtekar connection has to lift to a connection on both, the frame bundle and the spin structure, in a consistent way.

For completeness, let us also consider the parity transformations of the fermionic fields. Since the Rarita-Schwinger field $\phi:=(\bar{\phi}^A,\phi_{A'})^T$, in particular, is a Majorana fermion, one would impose a parity transformation of the form[4] $\phi\longrightarrow\gamma^0\phi$ which, in terms of the fundamental variables, reads

$$\Pi(\phi)_{A'}:=i\pi_{\phi}^{B'}\epsilon_{B'A'}\quad\text{and}\quad\Pi(\pi_{\phi})^{A'}:=-i\phi_{B'}\epsilon^{B'A'}\tag{6.116}$$

This yields

$$\Pi(\pi_{\phi}\phi)=\phi_{A'}\pi_{\phi}^{A'}=-\pi_{\phi}\phi\tag{6.117}$$

[4] Note that, in the mostly plus convention, $(\gamma^0)^2=-\mathbb{1}$. Hence, the parity transformation is not involutive but acquires an additional phase of the form $e^{i\frac{\pi}{2}}$. In fact, one cannot simply redefine the parity transformation replacing γ^0 by $i\gamma^0$ since this turns out to be not compatible with the Majorana condition (4.20), i.e., the transformed field will be no longer a Majorana fermion.

Hence, it follows that, under these parity transformations, the contorsion tensor is now parity-even. However, in any case, the reduced connection does not seem to have a straightforward behavior under parity transformations.

6.5.4. Constraint algebra

We want to study the Poisson relations between the left and the right supersymmtery constraints. To this end, note that the canonical phase space $\mathscr{P}$ turns out to have the structure of a *Poisson supermanifold* (see, e.g., [239] for an introduction to Poisson supermanifolds in the algebro-geometric framework). In fact, for any homogenenous supersmooth functions $f, g, h \in \mathcal{O}(\mathscr{P})$ on the phase space, one has

$$\{f, gh\} = \{f, g\}h + (-1)^{|f||g|}g\{f, h\} \tag{6.118}$$

with $|f| \in \mathbb{Z}_2$ the parity of f, i.e., $|f| = 0$ resp. $|f| = 1$ if f is Grassmann-even resp. Grassmann-odd. That is, according to (6.118), the Poisson bracket defines a super right derivation on $\mathcal{O}(\mathscr{P})$. Moreover, the Poisson bracket is also graded skew-symmetric, i.e., $\{f, g\} = -(-1)^{|f||g|}\{g, h\}$. Using (6.118), this yields

$$\{fg, h\} = f\{g, h\} + (-1)^{|g||h|}\{f, h\}g \tag{6.119}$$

and therefore it also defines a super left derivation. In fact, it even follows that the Poisson bracket satisfies a graded analog of the Jacobi identity. Hence, $(\mathcal{O}(\mathscr{P}), \{\cdot, \cdot\})$ has the structure of a super Lie module.

With these preparations, let us compute the Poisson bracket between the left and right supersymmetry constraints. To this end, one often needs to compute Poisson brackets of the form $\{c, \pi_\phi^{A'}\}$ and $\{c, \phi_{A'}\}$ which yields

$$\{c, \pi_\phi^{A'}\} = \{\tilde{c} - \frac{i\kappa}{12p}\pi_\phi^{B'}\phi_{B'}, \pi_\phi^{A'}\} = -\frac{i\kappa}{12p}\pi_\phi^{B'}\{\phi_{B'}, \pi_\phi^{A'}\} = \frac{i\kappa}{12p}\pi_\phi^{A'} \tag{6.120}$$

as well as

$$\{c, \phi_{A'}\} = \{\tilde{c} - \frac{i\kappa}{12p}\pi_\phi^{B'}\phi_{B'}, \phi_{A'}\} = \frac{i\kappa}{12p}\phi_{B'}\{\pi_\phi^{B'}, \phi_{A'}\} = -\frac{i\kappa}{12p}\phi_{A'} \tag{6.121}$$

Finally, using

$$\{\tilde{c} - \frac{i\kappa}{12p}\pi_\phi^{B'}\phi_{B'}, p\} = \{c, p\} = -\frac{i\kappa}{3} \tag{6.122}$$

as well as the fact that $(\mathcal{O}(\mathscr{P}),\{\cdot,\cdot\})$ defines a super Lie module, it follows for the trace part of the matrix $\{S^{LA'},S^{R}_{B'}\}$, in case of a vanishing cosmological constant,

$$
\begin{aligned}
&\{\epsilon c|p|^{\frac{1}{4}}\pi_{\phi}^{A'},3\epsilon|p|^{\frac{1}{4}}(c-k\ell_0)\phi_{A'}\}\\
=&-6\epsilon^2\sqrt{|p|}(c^2-k\ell_0 c)+3\epsilon^2\{c,|p|^{\frac{1}{4}}\}(c-k\ell_0)|p|^{\frac{1}{4}}\pi_{\phi}\phi\\
&+3\epsilon^2|p|^{\frac{1}{4}}c\{|p|^{\frac{1}{4}},c\}\pi_{\phi}\phi+3\epsilon^2\sqrt{|p|}\pi_{\phi}^{A'}\{c,\phi_{A'}\}(c-k\ell_0)\\
&+3\epsilon^2 c\sqrt{|p|}\{\pi_{\phi}^{A'},c\}\phi_{A'}\\
=&-6\epsilon^2\sqrt{|p|}(c^2-k\ell_0 c)-\frac{i\kappa\epsilon}{4}\frac{(c-k\ell_0)}{\sqrt{|p|}}\pi_{\phi}\phi\\
&+\frac{i\kappa\epsilon}{4}\frac{c}{\sqrt{|p|}}\pi_{\phi}\phi-\frac{i\kappa\epsilon}{4}\frac{(c-k\ell_0)}{\sqrt{|p|}}\pi_{\phi}\phi-\frac{i\kappa\epsilon}{4}\frac{c}{\sqrt{|p|}}\pi_{\phi}\phi\\
=&2\kappa(H_b+H_f)+\frac{i\kappa\epsilon}{2}\frac{(c-k\ell_0)}{\sqrt{|p|}}\pi_{\phi}\phi\\
=&2\kappa H+\frac{i\kappa}{6|p|^{\frac{3}{4}}}\pi_{\phi}^{A'}\left(3\epsilon|p|^{\frac{1}{4}}(c-k\ell_0)\phi_{A'}\right)
\end{aligned}
\tag{6.123}
$$

which precisely consists of the sum of the Hamiltonian constraint as well as the right SUSY constraint in case of a vanishing cosmological constant. For $0<L<\infty$, this yields

$$
\begin{aligned}
&\{\epsilon c|p|^{\frac{1}{4}}\pi_{\phi}^{A'}+L^{-1}|p|^{\frac{3}{4}}\phi_{B'}\epsilon^{B'A'},3\epsilon|p|^{\frac{1}{4}}(c-k\ell_0)\phi_{A'}-3L^{-1}|p|^{\frac{3}{4}}\pi_{\phi}^{B'}\epsilon_{B'A'}\}\\
=&2\kappa(H_b+H_f)+\frac{i\kappa}{6|p|^{\frac{3}{4}}}\pi_{\phi}^{A'}\left(3\epsilon|p|^{\frac{1}{4}}(c-k\ell_0)\phi_{A'}\right)\\
&+6L^{-2}|p|^{\frac{3}{2}}+3L^{-1}\epsilon|p|^{\frac{1}{4}}\{c,|p|^{\frac{3}{4}}\}\pi_{\phi}^{A'}\pi_{\phi}^{B'}\epsilon_{A'B'}-3L^{-1}\epsilon|p|\pi_{\phi}^{A'}\{c,\pi_{\phi}^{B'}\}\epsilon_{B'A'}\\
&+3L^{-1}\epsilon|p|^{\frac{1}{4}}\{c,|p|^{\frac{3}{4}}\}\phi_{A'}\phi_{B'}\epsilon^{B'A'}+3L^{-1}\epsilon|p|\phi_{A'}\{c,\phi_{B'}\}\epsilon^{B'A'}\\
=&2\kappa(H_b+H_f)+\frac{i\kappa}{6|p|^{\frac{3}{4}}}\pi_{\phi}^{A'}\left(3\epsilon|p|^{\frac{1}{4}}(c-k\ell_0)\phi_{A'}\right)\\
&+6L^{-2}|p|^{\frac{3}{2}}-\frac{3i\kappa}{4L}\epsilon^2\pi_{\phi}^{A'}\pi_{\phi}^{B'}\epsilon_{A'B'}+\frac{i\kappa}{4L}\epsilon^2\pi_{\phi}^{A'}\pi_{\phi}^{B'}\epsilon_{B'A'}\\
&+\frac{3i\kappa}{4L}\epsilon^2\phi_{A'}\phi_{B'}\epsilon^{A'B'}+\frac{i\kappa}{4L}\epsilon^2\phi_{A'}\phi_{B'}\epsilon^{A'B'}\\
=&2\kappa H-\frac{i\kappa}{2L}\epsilon^2\pi_{\phi}^{A'}\pi_{\phi}^{B'}\epsilon_{A'B'}+\frac{i\kappa}{6|p|^{\frac{3}{4}}}\pi_{\phi}^{A'}\left(3\epsilon|p|^{\frac{1}{4}}(c-k\ell_0)\phi_{A'}\right)
\end{aligned}
$$

$$=2\kappa H + \frac{i\kappa}{6|p|^{\frac{3}{4}}}\pi_{\phi}^{A'}\left(3\epsilon|p|^{\frac{1}{4}}(c-k\ell_0)\phi_{A'} - 3L^{-1}|p|^{\frac{3}{4}}\pi^{B'}\epsilon_{B'A'}\right) \tag{6.124}$$

so that we found

$$\{S^{LA'}, S^R_{A'}\} = 2\kappa H + \frac{i\kappa}{6|p|^{\frac{3}{4}}}\pi_{\phi}^{A'}S^R_{A'} \tag{6.125}$$

Let us emphasize that, in the above calculation, the sign factors ϵ depending on the orientation of the triad matched up exactly to give the algebra (6.125) proving that the theory is indeed consistent with local supersymmetry in both sectors. Next, we want to compute the off-diagonal entries of the matrix $\{S^{LA'}, S^R_{B'}\}$. Following the same steps as before, we immediately find for $A' \neq B'$

$$\{\epsilon c|p|^{\frac{1}{4}}\pi_{\phi}^{A'}, 3\epsilon|p|^{\frac{1}{4}}(c-k\ell_0)\phi_{B'}\} = -\frac{i\kappa\epsilon}{2}\frac{(c-k\ell_0)}{\sqrt{|p|}}\pi_{\phi}^{A'}\phi_{B'} \tag{6.126}$$

By anticommutativity of fermionic fields, one has $\pi_{\phi}^{A'}\pi_{\phi}^{C'}\epsilon_{C'B'} = 0$ for $A' \neq B'$ and similarly for ϕ. Hence, this yields

$$\begin{aligned}&\{\epsilon c|p|^{\frac{1}{4}}\pi_{\phi}^{A'} + L^{-1}|p|^{\frac{3}{4}}\phi_{C'}\epsilon^{C'A'}, 3\epsilon|p|^{\frac{1}{4}}(c-k\ell_0)\phi_{B'} - 3L^{-1}|p|^{\frac{3}{4}}\pi_{\phi}^{D'}\epsilon_{D'B'}\}\\ &= -\frac{i\kappa\epsilon}{2}\frac{(c-k\ell_0)}{\sqrt{|p|}}\pi_{\phi}^{A'}\phi_{B'} = -\frac{i\kappa}{6|p|^{\frac{3}{4}}}\pi_{\phi}^{A'}\left(3\epsilon|p|^{\frac{1}{4}}(c-k\ell_0)\phi_{B'} - 3L^{-1}|p|^{\frac{3}{4}}\pi^{C'}\epsilon_{C'B'}\right)\end{aligned} \tag{6.127}$$

such that

$$\{S^{LA'}, S^R_{B'}\} = -\frac{i\kappa}{6|p|^{\frac{3}{4}}}\pi_{\phi}^{A'}S^R_{B'} \tag{6.128}$$

Thus, to summarize, we found that

$$\{S^{LA'}, S^R_{B'}\} = \left(\kappa H + \frac{i\kappa}{6|p|^{\frac{3}{4}}}\pi_{\phi}^{A'}S^R_{A'}\right)\delta^{A'}_{B'} - \frac{i\kappa}{6|p|^{\frac{3}{4}}}\pi_{\phi}^{A'}S^R_{B'} \tag{6.129}$$

Equation (6.129) provides a very strong relation between the Hamiltonian and supersymmetry constraint which will play a central role in Section 6.6.4 in the construction of the physical sector of the kinematical Hilbert space and the study of the resulting dynamics of the theory.

6.6. Quantum theory

6.6.1. Construction of the classical algebra

We want to motivate the kinematical Hilbert space in the reduced theory. To this end, we follow the standard procedure in LQC and compute holonomies along straight edges of the fiducial cell $\mathcal{V}$ which are parallel to integral flows Φ^i_τ generated by the basis of left-invariant vector fields $(\mathring{e}_i)_i$ dual to the fiducial co-frame. Choosing the new variables as constructed in the previous section, we combine them to the super connection $\tilde{\mathcal{A}}^+$ defined as

$$\tilde{\mathcal{A}}^+ = \tilde{A}^+ + \tilde{\psi} := \frac{\tilde{c}}{\ell_0}\mathring{e}^i T_i^+ + \ell_0^{-\frac{3}{2}}\phi_{A'}\mathring{e}^{AA'}Q_A \tag{6.130}$$

where, in the second term, the factor $\ell_0^{-\frac{3}{2}}$ has been included for dimensional reasons. By definition, it follows

$$\{\tilde{\mathcal{A}}^{+\underline{A}}_a, \tilde{\mathcal{A}}^{+\underline{B}}_b\} = 0 \tag{6.131}$$

for $\underline{A}, \underline{B} \in \{i, A\}$. Since, in the manifest approach to canoncial loop quantum supergravity, we have this super connection at our disposal, we would like to motivate the kinematical Hilbert space of the reduced theory studying the corresponding super holonomies. To this end, for sake of convenience, we will not adapt the conventions of Section 5.5.1 and go back to the original definition of the parallel transport map regarded as a *covariant* functor on the path groupoid $\mathbf{P}(\Sigma)$, i.e., let us replace $h_e[\tilde{\mathcal{A}}^+] \to h_e[\tilde{\mathcal{A}}^+]^{-1}$. Hence, according to Example 2.7.16, it follows that these holonomies $h_e[\tilde{\mathcal{A}}^+]$ along edges $e \subset \Sigma$ embedded in Σ (in case of super matrix Lie groups such as, in the present situation, $\mathrm{OSp}(1|2)_{\mathbb{C}}$) satisfy the differential equation

$$\partial_\tau h_e[\tilde{\mathcal{A}}^+] = -\alpha\tilde{\mathcal{A}}^{+e} h_e[\tilde{\mathcal{A}}^+] \tag{6.132}$$

with $\tilde{\mathcal{A}}^{+e}(\tau) := e^*\tilde{\mathcal{A}}^+(\tau) = \dot{e}^a(\tau)\tilde{\mathcal{A}}^+_a(e(\tau))$ the pullback of $\tilde{\mathcal{A}}^+$ w.r.t. e and $\alpha \in \mathbb{C}$ some complex number. For a standard holonomy corresponding to a proper parallel transport map induced by a super connection 1-form one has $\alpha = 1$. But, following [91], in view of the solution of the reality conditions in the quantum theory (and thus regain the solutions of ordinary real $\mathcal{N} = 1$ supergravity) we do not fix this constant to a specific value at this stage. Adopting the terminology of [91] to the supersymmetric setting, we will call them *generalized super holonomies*. As shown in Example 2.7.17, in a specific gauge, one can decompose the generalized super holonomy in the form $h_e[\tilde{\mathcal{A}}^+] = h_e[\tilde{A}^+] \cdot h_e[\tilde{\psi}]$ with $h_e[\tilde{A}^+]$ the generalized (bosonic) holonomy generated by the bosonic part $\tilde{A}^+$ of the super connection 1-form, such that Eq. (6.132) turns out to be equivalent to

$$\partial_\tau h_e[\tilde{A}^+] = -\alpha\tilde{A}^{+e} h_e[\tilde{A}^+] \tag{6.133}$$

$$\partial_\tau h_e[\tilde{\psi}] = -\alpha(\mathrm{Ad}_{h[\tilde{A}^+]^{-1}}\tilde{\psi}^e) h_e[\tilde{\psi}] \tag{6.134}$$

where, for $g \in \mathrm{SL}(2,\mathbb{C})$, the adjoint representation on the odd part of the super Lie algebra $\mathfrak{osp}(1|2)_{\mathbb{C}}$ is given by the fundamental representation of $\mathrm{SL}(2,\mathbb{C})$ such that

$$\mathrm{Ad}_g\tilde{\psi} = \tilde{\psi}^A Q_B g^B{}_A = (g^B{}_A\tilde{\psi}^A)Q_B \tag{6.135}$$

Let us consider edges $\gamma_i:\ [0,b]\to\Sigma,\ \tau\mapsto\gamma_i(\tau)$ that are parallel along integral flows Φ^i_τ generated by the basis of left-invariant vector fields $(\mathring{e}_i)_i$. We choose τ as the proper time (length) as measured w.r.t. fiducial metric $\mathring{q}$ on Σ. Hence, we can identify $b=\ell_0$. With respect to γ_i, it immediately follows from (6.133) that $h_i[\tilde{A}^+]\equiv h_{\gamma_i}[\tilde{A}^+]$ is given by

$$h_i[\tilde{A}^+](\tau) = \exp\left(-\frac{\alpha\tilde{c}\tau}{\ell_0}T_i^+\right) = \cosh\left(\frac{\alpha\mathring{\mu}\tilde{c}}{2i}\right)\mathbb{1} + \sinh\left(\frac{\alpha\mathring{\mu}\tilde{c}}{2i}\right)2iT_i^+ \tag{6.136}$$

where we used that in the fundamental representation of $\mathrm{OSp}(1|2)_{\mathbb{C}}$ one has $(T_i^+)^2 = -\frac{1}{4}\mathbb{1}$ and set $\mathring{\mu} := \frac{\tau}{\ell_0}$. Inserting (6.136) into (6.134) it follows that the solution of this equation is given by a path ordered exponential yielding

$$\begin{aligned}
h_i[\tilde{\psi}](\tau) &= \mathcal{P}\exp\left(-\int_0^\tau \mathrm{d}\tau'\,\alpha\left(e^{\frac{\alpha\tilde{c}\tau'}{\ell_0}T_i^+}\right)^B{}_A \phi_{A'}\,\sigma_i^{AA'}Q_B\right)\\
&= \mathbb{1} - \int_0^\tau \mathrm{d}\tau'\,\alpha\ell_0^{-\frac{3}{2}}\left(e^{\frac{\alpha\tilde{c}\tau'}{\ell_0}T_i^+}\right)^B{}_A \phi_{A'}\,\sigma_i^{AA'}Q_B\\
&\quad + \int_0^\tau \mathrm{d}\tau'\int_0^{\tau'}\mathrm{d}\tau''\alpha^2\ell_0^{-3}\left(e^{\frac{\alpha\tilde{c}\tau'}{\ell_0}T_i^+}\right)^D{}_C\left(e^{\frac{\alpha\tilde{c}\tau''}{\ell_0}T_i^+}\right)^B{}_A \phi_{C'}\phi_{A'}\,\sigma_i^{CC'}\sigma_i^{AA'}Q_DQ_B
\end{aligned} \tag{6.137}$$

where the sum terminates at second order due to the homogeneity and nilpotency of the fermionic variables. To see how a typical matrix element in (6.137) looks like, let us compute the first integral which gives

$$\int_0^\tau \mathrm{d}\tau'\,\alpha e^{\frac{\alpha\tilde{c}\tau'}{\ell_0}T_i^+} = \frac{4\ell_0}{\tilde{c}}\left(e^{\alpha\tilde{c}\mathring{\mu}T_i^+} - \mathbb{1}\right)T_i^+ \tag{6.138}$$

and thus contains terms of the form (6.136) as well as smooth functions on $\mathbb{C}$ vanishing at infinity. If we first consider the purely bosonic contributions to the generalized super holonomy $h[\tilde{\mathcal{A}}^+]$, it follows that the matrix elements can be equivalently be encoded in terms of holomorphic functions (or rather their Grassmann extensions) f on $\mathbb{C}$ of the form $f(z)=e^{Cz}$ with $C\in\mathbb{C}$. In fact, functions of this kind play a special role in group theory in mathematics. To this end, consider $(\mathbb{C},+)$ as an additive group. A *generalized character* ρ on $(\mathbb{C},+)$ is defined as a group morphism

$$\rho:\ (\mathbb{C},+)\rightarrow \mathrm{GL}(1,\mathbb{C})\cong\mathbb{C}^{\times} \tag{6.139}$$

This implies that $\rho(z+z')=\rho(z)\rho(z')$ for any $z,z'\in\mathbb{C}$ and therefore

$$\rho(z+z')-\rho(z)=\rho(z)\rho(z')-\rho(z)=(\rho(z')-1)\rho(z) \tag{6.140}$$

Hence, if assume that ρ is differentiable and set $C:=\rho'(0)\in\mathbb{C}$, it immediately follows from (6.140)

$$\frac{\mathrm{d}}{\mathrm{d}z}\rho(z)=C\rho(z) \tag{6.141}$$

which, due to $\rho(1)=e$, has the unique solution

$$\rho(z)=e^{Cz},\quad C\in\mathbb{C} \tag{6.142}$$

That is, the matrix elements of the bosonic part of the super holonomies can be described in terms of generalized characters on $\mathbb{C}$ as an additive group.

Remark 6.6.1. This is, in fact, in complete analogy to LQC with real variables. There, it turns out that the matrix elements of holonomies can be encoded in terms ordinary characters of $\mathbb{R}$ regarded as an additive group, i.e. group morphisms

$$\rho:\ (\mathbb{R},+)\rightarrow \mathrm{U}(1)\subset\mathrm{GL}(1,\mathbb{C}) \tag{6.143}$$

The real line $(\mathbb{R},+)$ is the universal covering of $\mathrm{U}(1)$ which is compact. Taking the complexification on both sides of (6.143), this immediately leads to the notion of a generalized character on the complex plane as introduced above since $\mathrm{U}(1)_{\mathbb{C}}=\mathbb{C}^{\times}$ which is non-compact similarly as the complexification $\mathrm{SL}(2,\mathbb{C})$ of $\mathrm{SU}(2)$ is non-compact.

The whole set of generalized characters on $\mathbb{C}$ is probably too large. In fact, according to (6.136), since the fiducial length $\mathring{\mu}\ell_0$ is a real number, it may be already sufficient to restrict to generalized characters (6.142) labeled by real numbers $C\in\mathbb{R}$. This corresponds to the requirement of a purely imaginary $\alpha\in i\mathbb{R}$. As we will see in the next section, this choice is consistent with the reality conditions. Hence, following [91], we will set $\alpha=i$. Based on the above observations, in order to construct the algebra corresponding to the purely bosonic degrees of freedom, we define a subalgebra $H_{\mathrm{AP}}(\mathbb{C})\subset H(\mathbb{C})$ of the algebra of holomorphic functions on $\mathbb{C}$, called *almost periodic holomorphic functions*, generated by complex linear combinations of generalized characters of $\mathbb{C}$ labeled by real numbers. That is, a general element $T \in H_{\mathrm{AP}}(\mathbb{C})$ will be of the form

$$T(z) = \sum_{i=1}^{N} a_i e^{\mu_i z} \tag{6.144}$$

with $a_i \in \mathbb{C}$ and $\mu_i \in \mathbb{R}$. This can be completed to a Fréchet algebra. To this end, consider the compact exhaustion $\mathbb{C} = \bigcup_n K_n$ of the complex plane by the compact sets $K_n := \{z \in \mathbb{C} | |z| \leq n\}$, $n \in \mathbb{N}$. The space $H(\mathbb{C})$ of holomorphic functions on $\mathbb{C}$ can then be given the structure of a locally convex space endowing it with the topology of uniform convergence on the K_n. In fact, in this way, it turns out that $H(\mathbb{C})$ even has the structure of a *uniform Fréchet algebra* (see [240] and references therein) via pointwise multiplication of holomorphic functions. As a consequence, the closure $\overline{H}_{\mathrm{AP}}(\mathbb{C}) := \overline{H_{\mathrm{AP}}(\mathbb{C})}$ in $H(\mathbb{C})$ inherits the structure of a uniform Fréchet algebra. However, note that it does not define a *-algebra. Regarding the standard construction of the state space of LQG (or LQC) via cylindrical functions, we would like to interpret $H_{\mathrm{AP}}(\mathbb{C})$ in terms of (continuous) functions on a group. To this end, as in case of Banach algebras, one can define the *spectrum* $\mathrm{Spec}\,\overline{H}_{\mathrm{AP}}(\mathbb{C})$ given by the set of all nonzero continuous algebra homomorphisms $\phi : \overline{H}_{\mathrm{AP}}(\mathbb{C}) \to \mathbb{C}$. Any $f \in \overline{H}_{\mathrm{AP}}(\mathbb{C})$ canonically induces a linear map on the spectrum via

$$\hat{f}(\phi) := \phi(f), \quad \forall \phi \in \mathrm{Spec}\,\overline{H}_{\mathrm{AP}}(\mathbb{C}) \tag{6.145}$$

called the *Gelfand transform* of f. We equip $\mathrm{Spec}\,\overline{H}_{\mathrm{AP}}(\mathbb{C})$ with the *Gelfand topology* given by the coarsest topology such that the Gelfand transforms (6.145) are continuous. Since $\overline{H}_{\mathrm{AP}}(\mathbb{C})$ is a uniform Fréchet algebra, it follows that $\mathrm{Spec}\,\overline{H}_{\mathrm{AP}}(\mathbb{C})$ is a *hemicompact space* [240]. As a consequence, the space $C(\mathrm{Spec}\,\overline{H}_{\mathrm{AP}}(\mathbb{C}))$ of continuous functions on the spectrum endowed with the compact open topology also has the structure of a uniform Fréchet algebra. Consider the map

$$\begin{aligned} \Gamma : \overline{H}_{\mathrm{AP}}(\mathbb{C}) &\to C(\mathrm{Spec}\,\overline{H}_{\mathrm{AP}}(\mathbb{C})) \\ f &\mapsto \hat{f} \end{aligned} \tag{6.146}$$

called the *Gelfand transformation*. It is immediate that Γ defines a homomorphism of algebras. In particular, as $\overline{H}_{\mathrm{AP}}(\mathbb{C})$ defines a uniform Fréchet algebra, it follows that Γ defines an injective toplogical algebra homomorphism identifying $\overline{H}_{\mathrm{AP}}(\mathbb{C})$ with a closed subalgebra $\Gamma(\overline{H}_{\mathrm{AP}}(\mathbb{C}))$ of $C(\mathrm{Spec}\,\overline{H}_{\mathrm{AP}}(\mathbb{C}))$.

The spectrum naturally carries the structure of an abstract group via pointwise multiplication

$$\begin{aligned}\operatorname{Spec}\overline{H}_{\mathrm{AP}}(\mathbb{C})) \times \operatorname{Spec}\overline{H}_{\mathrm{AP}}(\mathbb{C}) &\to \operatorname{Spec}\overline{H}_{\mathrm{AP}}(\mathbb{C}) \\ (\phi,\psi) &\mapsto \phi\cdot\psi\end{aligned} \tag{6.147}$$

and unit $1:\ \overline{H}_{\mathrm{AP}}(\mathbb{C}) \to \mathbb{C},\ f \mapsto 1$. Hence, in this way, we can identify $\overline{H}_{\mathrm{AP}}(\mathbb{C})$ in terms of functions on a group. This is similar to ordinary LQC with real variables. However, it is not clear whether this also forms a topological group. This would follow immediately if, in the definition of $H_{\mathrm{AP}}(\mathbb{C})$, we would have restricted to the subset of ordinary characters $\chi:\ (\mathbb{C},+) \to \mathrm{U}(1)$. In this case, it follows that the spectrum of the closure has the structure of a (compact, as $\mathrm{U}(1)$ compact) topological group, the so-called *Bohr-compactification of the complex plane*. We leave it as a task for future investigations to show whether this is also true for $\operatorname{Spec}\overline{H}_{\mathrm{AP}}(\mathbb{C})$.

With these observations, let us go back to the generalized holonomies $h[\tilde{\mathcal{A}}^+]$ of the super connection. Mimicking the standard procedure of LQC, due to (6.138) as well as (6.137), we may identify the matrix elements of the generalized holonomies as functions in

$$H_{\mathrm{AP}}(\mathbb{C}) \cup C_0(\mathbb{C}) \otimes \bigwedge[\phi_{A'}] \tag{6.148}$$

with $C_0(\mathbb{C})$ the space of continuous functions on $\mathbb{C}$ that vanish at infinity. Interestingly, this is very similar to LQC with real variables. In fact, in [241] it has been found that these type of functions, already in the pure bosonic sector, i.e. $\phi_{A'} = 0$, arise if one considers (bosonic) holonomies along more general edges which are not simply straight edges along integral flows of the left-invariant vector fields but which may also contain small kinks. Here, we observe that these type of functions appear in the fermionic components of the super holonomies computed along straight edges.

In what follows, we will consider a simplified model in which, based on the considerations of [242], we will drop functions which are contained in $C_0(\mathbb{C})$. Of course, this means a drastic simplification of the present model which will also break the manifest supersymmetry of the theory. However, this is what is normally done in the literature about canonical minisuperspace models in the framework of supergravity and, as we will see, this model already has a lot of interesting physical implications. One may come back to the more general model for future investigations. Hence, for the rest of this chapter, we will take as the classical algebra of the theory the following super vector space

$$\mathfrak{A} := H_{\mathrm{AP}}(\mathbb{C}) \otimes \bigwedge[\phi_{A'}] \tag{6.149}$$

which, in particular, has the structure of a super commutative Lie superalgebra. To this superalgebra, we need to add the algebra generated by the canonically conjugate momenta both encoded in the super electric field $\mathcal{E} = (p, \pi_\phi^{A'})$ (see Remark 6.4.1) which, following the standard procedure in LQG (and LQC), should be implemented as a (super) electric flux operator, i.e., in terms of derivations on $\mathfrak{A}$. More precisely, for any $f \in \mathfrak{A}$, we define

$$p(f) \equiv \{p, f\}, \quad \pi_\phi^{A'}(f) \equiv \{\pi_\phi^{A'}, f\} \tag{6.150}$$

As the Poisson bracket is a super derivation, it follows that these define elements of the superalgebra $\mathrm{Der}(\mathfrak{A})$ of super derivations on $\mathfrak{A}$. Hence, as the total algebra[5] $\mathfrak{A}^{\mathrm{cLQSC}}$ of the classical symmetry reduced theory, we take

$$\mathfrak{A}^{\mathrm{cLQSC}} := \mathfrak{A} \oplus \mathrm{Der}(\mathfrak{A}) \tag{6.151}$$

This again has the structure of a Lie superalgebra. In fact, $\mathfrak{A}^{\mathrm{cLQSC}}$ defines an ideal w.r.t the action of the super electric fluxes as defined via (6.150) which, due to the fact the Poisson defines a super right derivations, in particular, defines a representation of Lie superalgebras. Hence, we impose a graded Lie bracket on $\mathfrak{A}^{\mathrm{cLQSC}}$ setting

$$[(f, X), (g, Y)] := (X(g) - (-1)^{|f||Y|} Y(f), [X, Y]) \tag{6.152}$$

for any $(f, X), (g, Y) \in \mathfrak{A}^{\mathrm{cLQSC}}$. So far, $\mathfrak{A}^{\mathrm{cLQSC}}$ does not define a *-algebra as the obvious choice of an involution via complex conjugation would lead to anti holomorphic functions and derivations which, for physical reasons, have not been included into the definition of $\mathfrak{A}^{\mathrm{cLQSC}}$. However, one needs to define an involution in order to classify physical quantities in terms of self-adjoint elements. For this reason, let us go back to the reality conditions (6.111) and (6.112). Re-expressing them in terms of the fundamental variables, we may thus impose a *-relation on $\mathfrak{A}^{\mathrm{cLQSC}}$ setting

$$(e^{\mu\bar{c}})^\star := e^{-\mu\bar{c}} e^{\mu k \ell_0}, \qquad p^\star := p \tag{6.153}$$

$$\phi_A^\star := i\pi_\phi^{A'} n_{AA'}, \quad \pi_\phi^{\star A} := i\phi_{A'} n^{AA'} \tag{6.154}$$

It is immediate to see that this in fact provides an involution on $\mathfrak{A}^{\mathrm{cLQSC}}$. For instance, one has

$$(e^{\mu\bar{c}})^{\star\star} = (e^{-\mu\bar{c}})^\star e^{\mu k \ell_0} = e^{\mu\bar{c}} e^{-\mu k \ell_0} e^{\mu k \ell_0} = e^{\mu\bar{c}} \tag{6.155}$$

On the other hand, it follows

$$\phi_{A'}^{\star\star} = -i\pi_\phi^{\star A} n_{AA'} = \phi_{B'} n^{AB'} n_{AA'} = \phi_{A'} \tag{6.156}$$

[5] Here, the label "cLQSC" in $\mathfrak{A}^{\mathrm{cLQSC}}$ refers to *chiral loop quantum supercosmology*

as required. Hence, we have constructed a consistent classical Lie *-superalgebra $\mathfrak{A}^{\text{cLQSC}}$ of symmetry reduced chiral $\mathcal{N}=1$ supergravity which may be regarded as the symmetry reduced counterpart of the graded holonomy-flux algebra as constructed in context of the full theory in Section 5.5.1.

6.6.2. The kinematical Hilbert space

In order to construct the state space of the symmetry reduced model of chiral $\mathcal{N}=1$ LQSG, we need to find a representation of the graded holonomy-flux algebra $\mathfrak{A}^{\text{cLQSC}}$ on a super Hilbert space. More precisely, we are looking for a faithful Lie *-superalgebra morphism

$$\pi_0:\ \mathfrak{A}^{\text{cLQSC}}\to\operatorname{Op}(\mathcal{D},\mathfrak{H}^{\text{cLQSC}}) \tag{6.157}$$

from $\mathfrak{A}^{\text{cLQSC}}$ to a subset of possibly unbounded operators on a common dense domain $\mathcal{D}$ of a super Hilbert space $\mathfrak{H}^{\text{cLQSC}}$ such that $\pi_0(X^{\star})=\pi_0(X)^{\dagger}$ and $\pi_0(f^{\star})=\pi_0(f)^{\dagger}$ $\forall(f,X)\in\mathfrak{A}^{\text{cLQSC}}$.

Remark 6.6.2. Requiring π_0 to be faithful, in particular, means that π_0 preserves the grading. This immediately implies that bosons and fermions *automatically satisfy the correct statistics*. This is in fact a direct consequence of the graded structure of $\mathfrak{A}^{\text{cLQSC}}$ and is rooted in the underlying supersymmetry of the theory since, classically, supersymmetry requires commuting bosonic and anticommuting fermionic fields.

As an obvious candidate for a pre-super Hilbert space V, we take

$$V:=H_{\text{AP}}(\mathbb{C})\otimes\bigwedge[\phi_{A'}] \tag{6.158}$$

Following the standard procedure in LQG and LQC, we define a representation of $\mathfrak{A}^{\text{cLQSC}}$ on this super vector space via

$$\widehat{f}:=\pi_0(f)=f,\quad \widehat{p}:=\pi_0(p):=i\hbar\{p,\cdot\}=-\frac{\hbar\kappa}{3}\frac{\mathrm{d}}{\mathrm{d}\bar{c}} \tag{6.159}$$

$$\widehat{\pi}_{\phi}^{A'}:=\pi_0(\pi_{\phi}^{A'})=i\hbar\{\pi_{\phi}^{A'},\cdot\}=\frac{\hbar}{i}\frac{\partial}{\partial\phi_{A'}} \tag{6.160}$$

such that the operators corresponding to $f\in\mathfrak{A}^{\text{cLQSC}}$ and the super electric fluxes p and $\pi_{\phi}^{A'}$ act as multiplication operators and derivations, respectively. Requiring commutativity with the involution then implies

$$\widehat{e^{\mu\bar{c}}}^{\dagger}=\widehat{e^{-\mu\bar{c}}e^{\mu k\ell_0}},\qquad \widehat{p}^{\dagger}=\widehat{p} \tag{6.161}$$

$$\widehat{\phi}^{\dagger}_{A}=i\widehat{\pi}_{\phi}^{A'}n_{AA'},\quad \widehat{\pi}_{\phi}^{\dagger A}=i\widehat{\phi}_{A'}n^{AA'} \tag{6.162}$$

$\forall \mu \in \mathbb{R}$. This requires a suitable choice of an inner product on V. Hence, we see that the proposal of [91, 243] to implement the classical reality conditions in terms of a suitable inner product follows very naturally if we interpret them as defining equations for a consistent involution on the classical algebra.

To find such an inner product, let us go back to the full theory. As discussed in detail in Section 5.5, since the theory has underlying $\mathrm{OSp}(1|2)_{\mathbb{C}}$ gauge symmetry, a natural choice of an inner product is given via an invariant measure on $\mathrm{OSp}(1|2)_{\mathbb{C}}$ as a super Lie group. The induced inner product then generically consists of ordinary invariant integral on the bosonic Lie subgroup, i.e., $\mathrm{SL}(2,\mathbb{C})$, as well as some variant of a Berezin integral w.r.t. the odd degrees of freedom. Hence, a natural choice for an inner product $\mathscr{S}$ on V is given by the super scalar product

$$\mathscr{S}(f|g) := \int \mathrm{d}\nu(\tilde{c},\bar{\tilde{c}}) \int_B \mathrm{d}\phi_A \mathrm{d}\bar{\phi}_{A'}\, \bar{f}(\bar{\tilde{c}},\bar{\phi}_A) g(\tilde{c},\phi_{A'}) \tag{6.163}$$

where $\int_B$ denotes the standard translation-invariant Berezin integral on $\mathbb{C}^{0|2}$ regarded as purely odd super vector space. This defines a super scalar product on V which, a priori, is indefinite turning $(V,\mathscr{S})$ into an indefinite inner product space. However, it turns out that this can be completed to a Hilbert space. More precisely, according to the general discussion in Section 5.5.2, one can always find an endomorphism $J: V \to V$ such that $\mathscr{S}(\cdot|J\cdot)$ defines a positive definite scalar product on V. This is a standard fact about invariant measures on super Lie groups proven in [109]. The choice of such an endomorphism is a priori completely arbitrary but may be fixed by the requirement of a consistent implementation of the reality conditions (6.162). A typical choice of J would be

$$J := \exp\left(\frac{1}{\hbar}\bar{\phi}_A \phi_{A'} n^{AA'}\right) \tag{6.164}$$

If $\langle\cdot|\cdot\rangle := \mathscr{S}(\cdot|J\cdot)$, it follows that via the identification $V \cong (H_{\mathrm{AP}}(\mathbb{C}))^{\otimes 4}$, one has

$$\langle f|g\rangle = \frac{1}{\hbar^2}\langle\langle f^{\emptyset}|g^{\emptyset}\rangle\rangle + \frac{1}{\hbar}\langle\langle f^{+}|g^{+}\rangle\rangle + \frac{1}{\hbar}\langle\langle f^{-}|g^{-}\rangle\rangle + \langle\langle f^{+-}|g^{+-}\rangle\rangle \tag{6.165}$$

where, for $f \in V$, we made the decomposition $f = f^{\emptyset} + f^{A'}\phi_{A'} + \frac{1}{2}f^{+-}\phi_{A'}\phi^{A'}$ and $\langle\langle\cdot|\cdot\rangle\rangle$ denotes the inner product on $H_{\mathrm{AP}}(\mathbb{C})$, that is,

$$\langle\langle f^{I}|g^{I}\rangle\rangle := \int \mathrm{d}\nu(c,\bar{c})\, \bar{f}^{I}(\bar{c}) g^{I}(c) \tag{6.166}$$

for some ordered multi-index $\underline{I}$ of length $0 \leq |\underline{I}| \leq 2$. As shown in [88, 146], with respect to this scalar product, one indeed has

$$\widehat{\phi}_A^{\dagger} = \hbar n_{AA'} \frac{\partial}{\partial \theta_{A'}} = i \widehat{\pi}_{\phi}^{A'} n_{AA'} \tag{6.167}$$

and thus the reality condition (6.162) is satisfied. In fact, it turns out that J is even uniquely determined by this requirement.

Finally, we need to implement the reality conditions (6.161) for the bosonic degrees of freedom. Since the torsion contribution to the reduced connection simply drops off, this can be done in complete analogy to the matter-free case. Hence, following [91], we make the following ansatz

$$\langle\langle f|g\rangle\rangle = \lim_{D\to\infty} \frac{1}{2D} \int_{-D}^{D} \mathrm{d}\tilde{c}_R \int_{-D}^{D} \mathrm{d}\tilde{c}_I \, \delta\big(\tilde{c}_R - \frac{k\ell_0}{2}\big) \bar{f}(\bar{\tilde{c}}) g(\tilde{c}) \tag{6.168}$$

with $f, g \in H_{\mathrm{AP}}(\mathbb{C})$ where we made the decomposition $\tilde{c} = \tilde{c}_R + i\tilde{c}_I$ with $\tilde{c}_R, \tilde{c}_I \in \mathbb{R}$ the real and imaginary part of $\tilde{c}$, respectively. Moreover, we have added the spatial curvature $k = 0, +1$ of the FLRW spacetime in order to treat both cases simultaneously. Note that, in contrast to [91], we do not have to consider the states with positive and negative frequency separately. This is due our sign convention made for the internal 3-form ϵ^{ijk}. Of course, the prize to pay is that then $\tilde{c}$ does not have a straightforward behavior under parity transformations in the purely bosonic case. We however made this choice due to the inclusion of parity-violating fermionic matter degrees of freedom into the theory.

Let us verify that this indeed correctly implements the reality conditions (6.161). To this end, note that for elementary states of the form $f = e^{\nu\tilde{c}}$ and $g = e^{\nu'\tilde{c}}$ one has

$$\begin{aligned}\langle\langle f|g\rangle\rangle &= \lim_{D\to\infty} \frac{1}{2D} \int_{-D}^{D} \mathrm{d}\tilde{c}_R \int_{-D}^{D} \mathrm{d}\tilde{c}_I \, \delta\big(\tilde{c}_R - \frac{k\ell_0}{2}\big) e^{(\nu+\nu')\tilde{c}_R} e^{i(\nu'-\nu)\tilde{c}_I} \\ &= e^{(\nu+\nu')\frac{k\ell_0}{2}} \lim_{D\to\infty} \frac{1}{2D} \int_{-D}^{D} \mathrm{d}\tilde{c}_I \, e^{i(\nu'-\nu)\tilde{c}_I} = e^{\nu k\ell_0} \delta_{\nu,\nu'}\end{aligned} \tag{6.169}$$

so that

$$\langle\langle f|e^{\mu k\ell_0} e^{-\mu\tilde{c}} g\rangle\rangle = e^{\mu k\ell_0} \langle\langle e^{\nu\tilde{c}}|e^{(\nu'-\mu)\tilde{c}}\rangle\rangle = e^{\nu' k\ell_0} \delta_{\mu+\nu,\nu'} = \langle\langle e^{\mu\tilde{c}} f|g\rangle\rangle \tag{6.170}$$

$\forall \mu \in \mathbb{R}$. By linearity, it follows that this holds on all of V so that this indeed provides an implementation of (6.161). As argued in [91], this choice is in fact unique. Thus, we have constructed a unique positive definite inner product on V which we can use to complete V to a Hilbert space $\mathfrak{H} := \overline{V}^{\|\cdot\|}$. This Hilbert space has the tensor product structure

$$\mathfrak{H} = \mathfrak{H}_{\text{grav}} \otimes \mathfrak{H}_f = \mathfrak{H}_{\text{grav}} \otimes \bigwedge[\phi_{A'}] \tag{6.171}$$

and therefore naturally carries a $\mathbb{Z}_2$-grading where $\mathfrak{H}_{\text{grav}}$ is the Hilbert space of the purely gravitational degrees of freedom obtained via completion of $H_{\text{AP}}(\mathbb{C})$ w.r.t. the inner product $\langle\langle\cdot|\cdot\rangle\rangle$. Hence, in this way, we end up with a standard super Hilbert space $(\mathfrak{H}, \langle\cdot|\cdot\rangle)$ (see Def. 5.5.9). According to (6.169), an orthonormal basis of $\mathfrak{H}_{\text{grav}}$ is given by the states $\psi_\mu \in \mathfrak{H}_{\text{grav}}$ of the form

$$\psi_\mu(\tilde{c}) := e^{-\frac{\mu k}{2\ell_0}} e^{\mu\tilde{c}} \tag{6.172}$$

Since

$$\widehat{p}\psi_\mu = -\frac{\hbar\kappa\mu}{3}\psi_\mu \tag{6.173}$$

it follows that these are eigenstates of the bosonic electric flux operator $\widehat{p}$ with eigenvalue $-\frac{\hbar\kappa\mu}{3} \in \mathbb{R}$. Since $\widehat{p}$ factorizes as $\widehat{p} \equiv \widehat{p} \otimes \mathbb{1}$ w.r.t. the tensor product structure (6.171) of the super Hilbert space $\mathfrak{H}$, this implies that $\widehat{p}$ is a densely defined unbounded and symmetric operator on $\mathfrak{H}$ with spectrum contained in the reals, that is, $\widehat{p}$ is self-adjoint. Hence, we have successfully implemented all the reality conditions (6.161) and (6.162) on the Hilbert space $\mathfrak{H}$.

6.6.3. Solution of the residual Gauss constraint

We finally have to implement the remaining kinematical constraint given by the residual Gauss constraint (6.84) given by

$$G_i = \pi_\phi^{A'}(\tau_i)_{A'}{}^{B'}\phi_{A'} \tag{6.174}$$

imposing invariance of the fermionic degrees of freedom under local SL$(2,\mathbb{C})$ gauge transformations. Note that, due to the hybrid ansatz, it solely depends on the fermionic degrees of freedom as the bosonic variables have been chosen to be isotropic. For the quantization of this constraint, we order π_ϕ to the right so that this yields

$$\widehat{G}_i = (\tau_i)_{A'}{}^{B'}\widehat{\phi}_{B'}\widehat{\pi}_\phi^{A'} \tag{6.175}$$

We then require that kinematical states are annihiliated by $\widehat{G}_i$. Writing $f = f^{\emptyset} + f^{A'}\phi_{A'} + \frac{1}{2}f^{+-}\phi_{A'}\phi^{A'}$ for a general state $f \in \mathfrak{H}$, it follows immediately from the definition that purely bosonic states $f \equiv f^{\emptyset} \in \mathfrak{H}$ are solutions of the constraint equation. For a fully occupied state $f \equiv f^{+-}\phi\phi \in \mathfrak{H}$, it follows

$$\begin{aligned}\widehat{G}_i f^{+-}\phi\phi &= -i\hbar f^{+-}(\tau_i)_{A'}{}^{B'}\phi_{B'}\phi_{D'}\epsilon^{A'D'} = i\hbar f^{+-}\tau_i^{B'D'}\phi_{B'}\phi_{D'} \\ &= i\hbar f^{+-}\tau_i^{(B'D')}\phi_{B'}\phi_{D'} = 0\end{aligned} \tag{6.176}$$

as the fermions are anticommutative while $\epsilon\tau$ is symmetric. On the other hand, for a single fermionic state $f \equiv f^{A'}\phi_{A'}$, one immediately finds for $i = 3$ that $\widehat{G}_3 f^{A'}\phi_{A'} = 0$ if and only if $f^{A'} = 0\ \forall A'$. Hence, solutions to the Gauss constraint are given by kinematical states $f \in \mathfrak{H}$ of the form

$$f = f^{\emptyset} + \frac{1}{2}f^{+-}\phi_{A'}\phi^{A'} \tag{6.177}$$

which is in fact in complete analogy with the results obtained via the approach of D'Eath et al. in [88, 90]. States of the form (6.177) form the kinematical Hilbert space which we denote by $\mathfrak{H}^{\mathrm{cLQSC}}$. This completes the construction of the kinematical Hilbert space $\mathfrak{H}^{\mathrm{cLQSC}}$ of chiral loop quantum cosmology with local $\mathcal{N} = 1$ supersymmetry.

6.6.4. The SUSY constraints and the quantum algebra

Having constructed the kinematical Hilbert space of the theory and successfully implemented all the reality conditions, we finally need to determine the physical states. To this end, note that, according to the constraint algebra (6.129), the SUSY constraints generate the Hamiltonian constraint. This is a particular property of the canonical description of fields theories with local supersymmetry. This means that, if the SUSY constraints have been succesfully implemented in the quantum theory such that (6.129) holds in the quantum theory, then these are superior to the Hamiltonian constraint in the sense that, once they are solved, this automatically leads to the solution of the latter. More precisely, if $\Psi \in \mathfrak{H}^{\mathrm{cLQSC}}$ with $\widehat{S}^{LA'}\Psi = 0 = \widehat{S}^R_{B'}\Psi\ \forall A', B'$, then

$$0 = [\widehat{S}^{LA'}, \widehat{S}^R_{B'}]\Psi = \widehat{H}\Psi \tag{6.178}$$

This is an important feature in canonical quantum supergravity and the Poisson relation (6.129) poses strong reglementations on the quantization of the constraints and therefore may fix also some of the quantization ambiguities. In Chapter 4, the (quantum) SUSY constraint has been investigated in the full theory of LQG with real variables. However, the precise relation to the Hamiltonian constraint via studying the quantum algebra has not been considered yet, due to the complexity of the resulting quantum operators. This changes in the symmetry reduced setting where we will be able to study the quantum algebra in detail.

In order to implement the SUSY constraints on $\mathfrak{H}^{\mathrm{cLQSC}}$, we have to regularize the connection $\tilde{c}$ in the classical expressions, as it not represented as a well-defined operator in the quantum theory. Following [91], as typically done in LQC, one therefore fixes some minimal length ℓ_m which is of the order of the Planck length and which arises from quantum geometry in the full theory. The connection $\tilde{c}$ will then be approximated via generalized holonomies $e^{\mu\tilde{c}}$ computed along paths of that minimal length ℓ_m. According to (6.136), the proper length τ as measured by the generalized holonomy $e^{\mu\tilde{c}}$ w.r.t. the fiducial metric $\mathring{q}$ is given by $\tau = \mathring{\mu}\ell_0 = 2\mu\ell_0$. The physical length as measured w.r.t. the metric $a^2\mathring{q}$ is then given by $\tau|a| = 2\mu\sqrt{|p|}$ which we require to be the minimal length ℓ_m from quantum geometry. Hence, this yields

$$\mu = \frac{\lambda_m}{\sqrt{|p|}} \tag{6.179}$$

where we set $\lambda_m := \ell_m/2$. A regularization of the connection $\tilde{c}$ in terms of these generalized holonomies is then given by

$$\tilde{c} = \frac{\sqrt{|p|}}{2\lambda_m}\left(e^{\lambda_m\tilde{c}/\sqrt{|p|}} - e^{-\lambda_m\tilde{c}/\sqrt{|p|}}\right) \tag{6.180}$$

To study the action of the corresponding operator in the quantum theory, we introduce the new classical variables

$$\tilde{\beta} := \frac{\tilde{c}}{\sqrt{|p|}} \quad \text{and} \quad V := \epsilon|p|^{\frac{3}{2}} \tag{6.181}$$

which satisfy the classical Poisson bracket

$$\{\tilde{\beta}, V\} = -\frac{i\kappa}{2} \tag{6.182}$$

and thus defines a canonically conjugate pair. From this, it follows

$$\{e^{\lambda_m\tilde{\beta}}, V\} = \sum_{k=1}^{\infty}\frac{(\lambda_m)^k}{k!}\{\tilde{\beta}^k, V\} = -\frac{i\kappa\lambda_m}{2}e^{\lambda_m\tilde{\beta}} \tag{6.183}$$

On $\mathfrak{H}^{\mathrm{cLQSC}}$, let us introduce the states

$$|V\rangle := e^{\frac{k\mu(V)}{2\ell_0}}\psi_{\mu(V)}, \quad \text{with} \quad \mu(V) := -\frac{3}{\hbar\kappa}\operatorname{sign}(V)|V|^{\frac{2}{3}} \tag{6.184}$$

for $V \in \mathbb{R}$ where ψ_μ is the orthonormal basis of $\mathfrak{H}_{\mathrm{grav}}$ as defined via (6.172). The states (6.184) are eigenstates of the volume operator $\widehat{V} := \epsilon|\widehat{p}|^{\frac{3}{2}}$ with eigenvalue V. Note that they are normalized only in case of vanishing spatial curvature. We will however use them for both cases as the action of the holonomy operators on these states is much simpler [91]. According to (6.183), it follows[6]

$$V\left(e^{\lambda_m\bar{\beta}}\,|V\rangle\right)=[V,e^{\lambda_m\bar{\beta}}]\,|V\rangle+Ve^{\lambda_m\bar{\beta}}\,|V\rangle=\left(V-\frac{\hbar\kappa\lambda_m}{2}\right)e^{\lambda_m\bar{\beta}}\,|V\rangle \tag{6.185}$$

Setting $v:=\frac{4}{\hbar\kappa}V$, we thus obtain

$$e^{\lambda_m\bar{\beta}}\,|v\rangle=|v-2\lambda_m\rangle \tag{6.186}$$

For the implementation of the SUSY constraints (6.114) and (6.115) as well as the Hamiltonian constraint (6.113), we exploit quantization ambiguities to quantize expressions of the form $\epsilon\bar{c}$ in the most symmetric way. In fact, as we will see, symmetric ordering will lead to a correct implementation of the classical constraint algebra (6.129). Hence, using the regularization (6.180), for the quantum analog of $\epsilon\bar{c}$, we take

$$\epsilon\bar{c}=\frac{\sqrt{|p|}}{2\lambda_m}\left(\mathcal{N}_--\mathcal{N}_+\right) \tag{6.187}$$

where

$$\mathcal{N}_\pm:=\frac{1}{2}\left(e^{\mp\lambda_m\bar{c}/\sqrt{|p|}}\epsilon+\epsilon e^{\mp\lambda_m\bar{c}/\sqrt{|p|}}\right) \tag{6.188}$$

Due to (6.186), the action of these operators on volume eigenstates are given by

$$\mathcal{N}_\pm\,|v\rangle=\frac{1}{2}\big(\mathrm{sign}(v)+\mathrm{sign}(v\pm 2)\big)\,|v\pm 2\lambda_m\rangle \tag{6.189}$$

and thus $\mathcal{N}_-\,|v\rangle$ resp. $\mathcal{N}_+\,|v\rangle$ vanishes in case $v\in(0,2\lambda_m)$ resp. $v\in(-2\lambda_m,0)$. Next, we have to implement the classical quantity $i\phi_\phi^{A'}\phi_{A'}$ which appears, for instance, in Eq. (6.107) relating the reduced connections c and $\bar{c}$ after having performed a canonical transformation to half-densitized fermionic fields. Sticking again to symmetric ordering, we define

$$\widehat{\Theta}:=\frac{i}{2}(\widehat{\pi}_\phi^{A'}\widehat{\phi}_{A'}-\widehat{\phi}_{A'}\widehat{\pi}_\phi^{A'})=\hbar-i\widehat{\phi}_{A'}\widehat{\pi}_\phi^{A'} \tag{6.190}$$

By definition, it follows that $\widehat{\Theta}$ is self-adjoint, since, due to reality conditions (6.162),

$$\left(i\widehat{\phi}_{A'}\widehat{\pi}_\phi^{A'}\right)^\dagger=-i\widehat{\pi}_\phi^{\dagger A}\widehat{\phi}_A^\dagger=i\widehat{\phi}_{B'}\widehat{\pi}_\phi^{C'}n^{AB'}n_{AC'}=i\widehat{\phi}_{A'}\widehat{\pi}_\phi^{A'} \tag{6.191}$$

6 As common in the LQC literature, for notational simiplification, we will drop hats indicating (bosonic) operator expressions in what follows

Moreover, $\widehat{\Theta}$ annihilates states with one single fermionic excitation, that is, $\widehat{\Theta}\phi_{A'} = 0$. In fact, $\widehat{\Theta}$ is related to the *particle number operator* $\widehat{N}$ via $\widehat{N} = \mathbb{1} - \frac{1}{\hbar}\widehat{\Theta}$, where $\widehat{N}$ counts the number of fermionic excitations in a quantum state. Using (6.190) as well as (6.187), it follows that the quantum analog of (6.107) takes the form

$$\begin{aligned} \epsilon c &= \epsilon\tilde{c} - \frac{\kappa\epsilon}{12p}\frac{i}{2}(\widehat{\pi}_{\phi}^{A'}\widehat{\phi}_{A'} - \widehat{\phi}_{A'}\widehat{\pi}_{\phi}^{A'}) \\ &= \frac{g^{\frac{1}{3}}|v|^{\frac{1}{3}}}{2\lambda_m}\left((\mathcal{N}_{-} - \mathcal{N}_{+}) - \frac{\kappa\lambda_m}{6g|v|}\widehat{\Theta}\right) \end{aligned} \tag{6.192}$$

where we used that $|p| = (2\pi G\hbar)^{\frac{2}{3}}|v|^{\frac{2}{3}} =: g^{\frac{2}{3}}|v|^{\frac{2}{3}}$. To implement the dynamical constraints in the quantum theory, for sake of simplicity, we want to restrict to the special case $k = 0$ and $L \to \infty$, i.e., vanishing spatial curvature and cosmological constant. This will simplify the derivation of the quantum algebra and also indicate very clearly how the classical algebra can be maintained in the quantum theory. The case with nonvanishing spatial curvature will be discussed in the following section in the context of the semi-classical limit of the theory.

If we use symmetric ordering, it follows that the Hamiltonian constraint operator in the quantum theory can be defined in the following way

$$\begin{aligned} \widehat{H} = &- \frac{3g}{4\kappa\lambda_m^2}|v|^{\frac{1}{4}}\left((\mathcal{N}_{-} - \mathcal{N}_{+}) - \frac{\kappa\lambda_m}{6g|v|}\widehat{\Theta}\right)|v|^{\frac{1}{2}}\left((\mathcal{N}_{-} - \mathcal{N}_{+}) - \frac{\kappa\lambda_m}{6g|v|}\widehat{\Theta}\right)|v|^{\frac{1}{4}} \\ &- \frac{1}{8\lambda_m}|v|^{\frac{1}{4}}\left((\mathcal{N}_{-} - \mathcal{N}_{+}) - \frac{\kappa\lambda_m}{6g|v|}\widehat{\Theta}\right)|v|^{-\frac{1}{4}}\widehat{\Theta} \\ &- \frac{1}{8\lambda_m}|v|^{-\frac{1}{2}}|v|^{\frac{1}{4}}i\widehat{\pi}_{\phi}^{A'}\left((\mathcal{N}_{-} - \mathcal{N}_{+}) - \frac{\kappa\lambda_m}{6g|v|}\widehat{\Theta}\right)|v|^{\frac{1}{4}}\widehat{\phi}_{A'} \end{aligned} \tag{6.193}$$

The bosonic part of $\widehat{H}$ is almost the same as in [91]. However, unlike [91], we did not change the overall power of the rescaled volume operator by redefining the lapse function N. This is due to the fact that the Hamiltonian constraint should be related to the SUSY constraints via a quantum analog of (6.129). Hence, redefining the Hamiltonian constraint requires a redefinition of the SUSY constraints which may then change the resulting quantum algebra which we would like avoid.

Note that each of the two fermionic contributions in (6.193) indeed yield half of H_f in the classical limit $\hbar \to 0$ in which case the terms $|v|^{\frac{1}{4}}$ and $|v|^{-\frac{1}{4}}$ simply cancel each other. Moreover, in the last line of (6.193), the term $|v|^{-\frac{1}{4}}$ on the left-hand side of the bracket has been regularized replacing it by the equivalent expression $|v|^{-\frac{1}{2}}|v|^{\frac{1}{4}}$ for $v \neq 0$. In this way, it follows that the Hamiltonian constraint is well-defined by acting on all states $|v\rangle$ with $v \neq 0$. In fact, suppose $\widehat{H}$ acts on the state $|v\rangle = |2\rangle$, then this state will mapped to zero volume state $|0\rangle$ by the shift operator $\mathcal{N}_-$ which will subsequently be annihilated by $|v|^{\frac{1}{2}}$. A similar kind of reasoning applies to the sector with $v < 0$. Hence, the Hamiltonian has a well-defined action on the states $|v\rangle = |\pm 2\rangle$. As will become clear in what follows, this kind of regularization of the Hamiltonian constraint operator is not imposed artificially but turns out to be even mandatory if one requires consistency with the classical Poisson relation (6.129). Therefore, let us implement the SUSY constraints in the quantum theory. If we stick to symmetric ordering, this immediately gives

$$\widehat{S}^{LA'} = \frac{g^{\frac{1}{2}}}{2\lambda_m}|v|^{\frac{1}{4}}\left((\mathcal{N}_- - \mathcal{N}_+) - \frac{\kappa\lambda_m}{6g|v|}\widehat{\Theta}\right)|v|^{\frac{1}{4}}\widehat{\pi}_{\phi}^{A'} \tag{6.194}$$

for the left supersymmetry constraint as well as

$$\widehat{S}^{R}_{A'} = \frac{3g^{\frac{1}{2}}}{2\lambda_m}|v|^{\frac{1}{4}}\left((\mathcal{N}_- - \mathcal{N}_+) - \frac{\kappa\lambda_m}{6g|v|}\widehat{\Theta}\right)|v|^{\frac{1}{4}}\widehat{\phi}_{A'} \tag{6.195}$$

for the right supersymmetry constraint. By definition, these operators are both well-defined while acting on states $|v\rangle$ with $v \neq 0$. For the computation of the quantum algebra among the SUSY constraints, we have to calculate various commutators of the form $[\widehat{\Theta}, \widehat{\pi}_{\phi}^{A'}]$ and $[\widehat{\Theta}, \widehat{\phi}_{A'}]$ which are given by

$$[\widehat{\Theta}, \widehat{\pi}_{\phi}^{A'}] = \hbar\widehat{\pi}_{\phi}^{A'} \quad \text{and} \quad [\widehat{\Theta}, \widehat{\phi}_{A'}] = -\hbar\widehat{\phi}_{A'} \tag{6.196}$$

Thus, using (6.196), we find that the trace part of the operator-valued matrix $[\widehat{S}^{LA'}, \widehat{S}^{R}_{B'}]$ is given by

$$\begin{aligned}
&[\widehat{S}^{LA'}, \widehat{S}^{R}_{A'}] = \\
&= \frac{3g}{4\lambda_m^2}|v|^{\frac{1}{4}}\left((\mathcal{N}_- - \mathcal{N}_+) - \frac{\kappa\lambda_m}{6g|v|}\widehat{\Theta}\right)|v|^{\frac{1}{2}}\left((\mathcal{N}_- - \mathcal{N}_+) - \frac{\kappa\lambda_m}{6g|v|}\widehat{\Theta}\right)|v|^{\frac{1}{4}}[\widehat{\pi}_{\phi}^{A'}, \widehat{\phi}_{A'}] \\
&+ \frac{3g}{4\lambda_m^2}|v|^{\frac{1}{4}}\left((\mathcal{N}_- - \mathcal{N}_+) - \frac{\kappa\lambda_m}{6g|v|}\widehat{\Theta}\right)|v|^{\frac{1}{4}}\left[\widehat{\pi}_{\phi}^{A'}, |v|^{\frac{1}{4}}\left((\mathcal{N}_- - \mathcal{N}_+) - \frac{\kappa\lambda_m}{6g|v|}\widehat{\Theta}\right)|v|^{\frac{1}{4}}\right]\widehat{\phi}_{A'} \\
&- \frac{3g}{4\lambda_m^2}|v|^{\frac{1}{4}}\left((\mathcal{N}_- - \mathcal{N}_+) - \frac{\kappa\lambda_m}{6g|v|}\widehat{\Theta}\right)|v|^{\frac{1}{4}}\left[|v|^{\frac{1}{4}}\left((\mathcal{N}_- - \mathcal{N}_+) - \frac{\kappa\lambda_m}{6g|v|}\widehat{\Theta}\right)|v|^{\frac{1}{4}}, \widehat{\phi}_{A'}\right]\widehat{\pi}_{\phi}^{A'} \\
&= 2i\hbar\kappa\widehat{H}_b - \frac{\kappa}{8\lambda_m}|v|^{\frac{1}{4}}\left((\mathcal{N}_- - \mathcal{N}_+) - \frac{\kappa\lambda_m}{6g|v|}\widehat{\Theta}\right)|v|^{-\frac{1}{4}}[\widehat{\pi}_{\phi}^{A'}, \widehat{\Theta}]\widehat{\phi}_{A'} \\
&+ \frac{\kappa}{8\lambda_m}|v|^{\frac{1}{4}}\left((\mathcal{N}_- - \mathcal{N}_+) - \frac{\kappa\lambda_m}{6g|v|}\widehat{\Theta}\right)|v|^{-\frac{1}{4}}[\widehat{\Theta}, \widehat{\phi}_{A'}]\widehat{\pi}_{\phi}^{A'}
\end{aligned}$$

$$
\begin{aligned}
&=2i\hbar\kappa\left(\widehat{H}_b-\frac{1}{8\lambda_m}|v|^{\frac{1}{4}}\left((\mathcal{N}_--\mathcal{N}_+)-\frac{\kappa\lambda_m}{6g|v|}\widehat{\Theta}\right)|v|^{-\frac{1}{4}}\widehat{\Theta}\right)\\
&=2i\hbar\kappa\widehat{H}-\frac{\hbar\kappa}{4\lambda_m|v|^{\frac{1}{2}}}\widehat{\pi}_\phi^{A'}\left(|v|^{\frac{1}{4}}\left((\mathcal{N}_--\mathcal{N}_+)-\frac{\kappa\lambda_m}{6g|v|}\widehat{\Theta}\right)|v|^{\frac{1}{4}}\widehat{\phi}_{A'}\right)
\end{aligned}
\tag{6.197}
$$

that is

$$
[\widehat{S}^{LA'},\widehat{S}^R_{A'}]=2i\hbar\kappa\widehat{H}-\frac{\hbar\kappa}{6g^{\frac{1}{2}}|v|^{\frac{1}{2}}}\widehat{\pi}_\phi^{A'}\widehat{S}^R_{A'}
\tag{6.198}
$$

which is precisely the quantum analog of (6.125). To compute the off-diagonal entries of $[\widehat{S}^{LA'},\widehat{S}^R_{B'}]$ note that, by definition, the right SUSY constraint operator $\widehat{S}^R_{B'}$ creates a fermionic state $\phi_{B'}$. On the other hand, $\widehat{S}^{LA'}$ annihilates a fermionic state labelled with A'. Hence, if $[\widehat{S}^{LA'},\widehat{S}^R_{B'}]$ acts on a fermionic vacuum state $f\equiv f^{\emptyset}\in\mathfrak{H}^{\mathrm{cLQSC}}$, this immediately gives for $A'\neq B'$

$$
[\widehat{S}^{LA'},\widehat{S}^R_{B'}]f^{\emptyset}=\widehat{S}^{LA'}(\widehat{S}^R_{B'}f^{\emptyset})=0=\frac{\hbar\kappa}{6g^{\frac{1}{2}}|v|^{\frac{1}{2}}}\widehat{\pi}_\phi^{A'}(\widehat{S}^R_{B'}f^{\emptyset})
\tag{6.199}
$$

On the other hand, in case $f\equiv f^{+-}\phi\phi\in\mathfrak{H}^{\mathrm{cLQSC}}$ is a fully occupied state, it follows

$$
[\widehat{S}^{LA'},\widehat{S}^R_{B'}]f=-\widehat{S}^R_{B'}(\widehat{S}^{LA'}f^{+-}\phi\phi)=0=\frac{\hbar\kappa}{6g^{\frac{1}{2}}|v|^{\frac{1}{2}}}\widehat{\pi}_\phi^{A'}(\widehat{S}^R_{B'}f)
\tag{6.200}
$$

since $\widehat{S}^R_{B'}(\widehat{S}^{LA'}f^{+-}\phi\phi)\propto(\phi_{B'})^2=0$ for $A'\neq B'$ by anticommutativity of the fermions. Hence, to summarize, we found that the quantum algebra between the left and right supersymmetry constraint on $\mathfrak{H}^{\mathrm{cLQSC}}$ takes the form

$$
[\widehat{S}^{LA'},\widehat{S}^R_{B'}]=\left(i\hbar\kappa\widehat{H}-\frac{\hbar\kappa}{6g^{\frac{1}{2}}|v|^{\frac{1}{2}}}\widehat{\pi}_\phi^{C'}\widehat{S}^R_{C'}\right)\delta^{A'}_{B'}+\frac{\hbar\kappa}{6g^{\frac{1}{2}}|v|^{\frac{1}{2}}}\widehat{\pi}_\phi^{A'}\widehat{S}^R_{B'}
\tag{6.201}
$$

and thus exactly reproduces the classical Poisson relation (6.129). This also justifies the symmetric ordering chosen for the Hamiltonian constraint. This is in fact a standard quantization scheme used in LQC [244] and follows here requiring consistency with the SUSY constraints.

As a final step, one needs to find physical states in $\mathfrak{H}^{\mathrm{cLQSC}}$ annihilated by the constraint operators in order to study the dynamics of the theory. As already explained above, to this end, it suffices to solve the SUSY constraints as, via (6.201), this immediately leads to a solution of the Hamiltonian constraint. However, in order to introduce a relational clock, one cannot simply add a scalar field to the theory as usually done in LQC. This is due to the fact that ordinary scalar fields will not contribute to the SUSY constraints leading to inconsistent dynamics according to (6.201). Hence, instead, one needs to consider local supersymmetric matter coupled to supergravity, i.e., scalar fields with additional spin-$\frac{1}{2}$ matter fields.

Remark 6.6.3. Let us briefly comment on the dynamics of the classical theory and the suitability of fermionic fields as relational clocks (see also Remark 6.6.4 below). It is an immediate consequence of supersymmetry that classical fermionic fields have to be anticommuting. This may be regarded as the classical limit of the well-known spin-statistics theorem in quantum field theory. As a result, the fermionic fields are odd Grassmann-valued functions on phase space and thus, in particular, are nilpotent. Recall that, generally, any Grassmann number has an ordinary real- (resp. complex-) valued scalar coefficient at lowest order, i.e., its body. These are the numbers one has access to in classical experiments. However, as fermionic fields are Grassman-odd, it follows that their body has to be zero. For this reason, fermionic fields do not serve as good "clocks" in cosmology even at the classical level.

Taking the body of the chiral supergravity action (5.102), the fermionic contributions immediately drop off due to nilpotency leading back to the standard chiral Palatini action of first-order Einstein gravity with cosmological constant. Hence, classically, one may interpret this limit in terms of the evolution of fermions on the classical bosonic background. Nevertheless, one may consider instead bosonic quantities (or rather their expectation values) derived from the fermionic fields. For instance, one can consider the current $J_0 := i\pi_{\phi}^{A'}\phi_{A'}$ or the associated fermionic energy density ρ_f which, in case of a vanishing cosmological constant, is related to the current via[7] $\rho_f \sim J_0^2/a^6$. Further bosonic quantities, in case of a nonvanishing cosmological constant, are given by the, in general complex, currents $J_{\phi} := i\epsilon^{A'B'}\phi_{A'}\phi_{B'}$ and $\bar{J}_{\phi} := -i\epsilon_{A'B'}\pi_{\phi}^{A'}\pi_{\phi}^{B'}$, respectively. The classical Hamiltonian constraint may then be written as

$$H = -\frac{3\epsilon^2}{\kappa}\sqrt{|p|}(c^2 - k\ell_0 c) - \frac{\epsilon}{2\sqrt{|p|}}(c - k\ell_0)J_0 - \frac{\epsilon^2}{L}\Re(J_{\phi}) + \frac{3}{\kappa L^2}|p|^{\frac{3}{2}}$$

However, in the classical theory and without coupling it to additional locally supersymmetric matter fields, it turns out to be hard to find nontrivial solutions of this pure graviton-gravitino model with nontrivial fermion currents which lead to a non-static dynamical universe (this is mainly due to the constraints imposed by the SUSY and

[7] Probably, the easiest way to see this is to use the Hamiltonian constraint $H = 0$ from which one can read off the imaginary part $c_I = \Im(c)$ of the symmetry reduced Ashtekar connection. Since $\dot{a} = \{a, H\} = c_I$ this yields an expression for the *Hubble parameter* $\dot{a}/a$ which, via the *Friedmann equation* for a homogeneous isotropic universe, is related to the fermionic energy density via $(\dot{a}/a)^2 = \kappa\rho_f/3 - (k\ell_0)^2/4a^2$.

Hamiltonian constraints on the initial conditions). This coincides with the observations in [88,90], where it is argued that this pure graviton-gravitino model has to be considered quantum theoretically. Hence, for various reasons, it is desirable to study the system coupled to further locally supersymmetric matter fields. According to the discussions in [88], this may in fact lead to very interesting dynamics.

Remark 6.6.4. As already argued in previous remark, even in the classical theory, fermionic fields do not serve as good relational clocks. One may then consider instead derived bosonic quantities such as the fermionic current J_0 which in the quantum theory, up to ordering ambiguities, is represented by the fermionic particle number operator $\widehat{N} = \mathbb{1} - \frac{1}{\hbar}\widehat{\Theta}$. However, this operator has pure discrete spectrum consisting of the three eigenvalues $\{0, 1, 2\}$. Hence, in the quantum theory, this quantity also does not serve as a good relational clock. Alternatively, one may use p as a clock. The fermionic state then becomes a function of this "gravitational" time. We will do something like this in the discussion of the semi-classical limit in Section 6.6.5 below.

Based on the observations in Remark 6.6.3 and 6.6.4, we leave it as a task for future investigations to study the full dynamics of the theory including local supersymmetric matter fields as a relational clock. Nevertheless, one can already make some qualitative statements concerning the singularity resolution. In fact, according to (6.194) and (6.195), by acting with the quantum SUSY constraints, irrespective of the number of fermionic excitations, the volume eigenstates $|v\rangle = |\pm 2\rangle$ are mapped to the zero volume state $|0\rangle$ or $|\pm 4\rangle$ by the shift operators $\mathcal{N}_{\pm}$. The zero volume state is then subsequently annihilated by the volume operator $|v|^{\frac{1}{4}}$. Moreover, states $|v\rangle$ with $v \in (0, 2\lambda_m)$ are annihilated by $\mathcal{N}_-$ whereas states $|v\rangle$ with $v \in (-2\lambda_m, 0)$ are mapped to zero by $\mathcal{N}_+$. Hence, it follows that the vacuum state decouples from the dynamics and accordingly the cosmic singularity is resolved in this model.

6.6.5. The semi-classical limit

So far, for the derivation of the quantum algebra (6.201), we have restricted to the special case of a vanishing spatial curvature and vanishing cosmological constant. In order to compare our model with other supersymmetric minisuperspace models in the literature, we next want to consider the case of a positive spatial curvature, i.e., $k = 1$, and study the semi-classical limit of the theory. As observed already in the previous section, in order to ensure consistency with classical Poisson algebra, it is worthwhile to choose a symmetric ordering for the dynamical constraints. Hence, we define

$$\widehat{S}^{LA'} = \frac{g^{\frac{1}{2}}}{2\lambda_m}|v|^{\frac{1}{4}}\left((\mathcal{N}_- - \mathcal{N}_+) - \frac{\kappa\lambda_m}{6g|v|}\widehat{\Theta}\right)|v|^{\frac{1}{4}}\widehat{\pi}_{\phi}^{A'} \tag{6.202}$$

for the left supersymmetry constraint which is equivalent to (6.195) and respectively

$$\widehat{S}^R_{A'} = \frac{3g^{\frac{1}{2}}}{2\lambda_m}|v|^{\frac{1}{4}}\left((\mathcal{N}_- - \mathcal{N}_+) - \frac{\kappa\lambda_m}{6g|v|}\widehat{\Theta}\right)|v|^{\frac{1}{4}}\widehat{\phi}_{A'} - 3\epsilon g^{\frac{1}{6}}|v|^{\frac{1}{6}}\ell_0\widehat{\phi}_{A'} \tag{6.203}$$

for the right supersymmetry constraint. Since $\widehat{S}^R_{B'}$ and $\widehat{S}^{LA'}$ for $A' \neq B'$ create resp. annihilate fermions with opposite quantum numbers, as in the previous section, it is immediate to see that on the subspace of gauge invariant states the commutator $[\widehat{S}^{LA'}, \widehat{S}^R_{B'}]$ simply vanishes such that

$$[\widehat{S}^{LA'}, \widehat{S}^R_{B'}] = \frac{\hbar\kappa}{6g^{\frac{1}{2}}|v|^{\frac{1}{2}}}\widehat{\pi}^{A'}_{\phi}\widehat{S}^R_{A'} = 0 \tag{6.204}$$

on $\mathfrak{H}^{\text{cLQSC}}$ for $A' \neq B'$as required. The trace part can then be considered as a defining equation of the Hamiltonian constraint. We will not derive an explicit expression of Hamiltonian constraint in what follows. Instead, we want to turn to the semi-classical limit of the theory. To this end, note that the minimal length λ_m (resp. ℓ_m) should arise from quantum geometry of the full theory of self-dual LQG. More precisely, it should be related to the discrete spectrum of the area operator in terms of the square root of the minimal possible area eigenvalue. Hence, following [29, 91], this suggests that $\lambda_m^2 = 16\sqrt{3}\pi G\hbar$.

In this section, we are interested in the limit in which effects from quantum geometry are negligible, that is, in which case the quantum area spectrum becomes nearly continuous. Hence, we consider the limit $\lambda_m \to 0$.

Since the SUSY constraints are superior to the Hamiltonian constraint in canonical quantum supergravity, it suffices to find the semi-classical solutions to (6.202) and (6.203). Hence, let us consider a state $\Upsilon \in \mathfrak{H}^{\text{cLQSC}}$ of the form

$$\Upsilon = \sum_v \psi(v)\,|v\rangle + \left(\sum_v \psi'(v)\,|v\rangle\right) \otimes \phi_{A'}\phi^{A'} \tag{6.205}$$

with certain coefficients $\psi(v), \psi'(v) \in \mathbb{C}$ which we assume to correspond to at least once continuously differentiable functions $\psi : v \mapsto \psi(v)$ and $\psi' : v \mapsto \psi'(v)$ on some open subset of $\mathbb{R}_{>0}$ where we have restricted to the sector of positive volume. It follows that Υ is a physical state if and only if $\widehat{S}^{LA'}\Upsilon = 0 = \widehat{S}^R_{B'}\Upsilon\,\forall A', B'$. Concerning the right supersymmetry constraint, this yields, using the fact that $\widehat{\Theta}$ annihilates states with one single fermionic excitation,

$$\begin{aligned}\widehat{S}^R_{A'}\Upsilon =&\frac{3g^{\frac{1}{2}}}{2\lambda_m}\sum_v\Big(|v|^{\frac{1}{4}}|v-2\lambda_m|^{\frac{1}{4}}\psi(v)\,|v-2\lambda_m\rangle - |v|^{\frac{1}{4}}|v+2\lambda_m|^{\frac{1}{4}}\psi(v)\,|v+2\lambda_m\rangle\\ &-3g^{\frac{1}{6}}|v|^{\frac{1}{6}}\ell_0\psi(v)\,|v\rangle\Big)\otimes\phi_{A'}\\ =&\frac{3g^{\frac{1}{2}}}{2\lambda_m}\sum_v\Big[\Big(|v+2\lambda_m|^{\frac{1}{4}}|v|^{\frac{1}{4}}\psi(v+2\lambda_m) - |v-2\lambda_m|^{\frac{1}{4}}|v|^{\frac{1}{4}}\psi(v-2\lambda_m)\Big)\,|v\rangle\\ &-3g^{\frac{1}{6}}|v|^{\frac{1}{6}}\ell_0\psi(v)\Big]\,|v\rangle\otimes\phi_{A'}\end{aligned}\tag{6.206}$$

where we have performed an index shift in order to obtain the last line. Hence, setting $F(v) := |v|^{\frac{1}{4}}\psi(v)$, it follows that Υ is a solution to the right SUSY constraint operator if and only if F satisfies the difference equation

$$\frac{F(v+2\lambda_m)-F(v-2\lambda_m)}{4\lambda_m}=\frac{\ell_0}{2}\frac{1}{g^{\frac{1}{3}}|v|^{\frac{1}{3}}}|v|^{\frac{1}{4}}\psi(v)=\frac{\ell_0}{2}\frac{1}{g^{\frac{1}{3}}|v|^{\frac{1}{3}}}F(v)\tag{6.207}$$

Thus, in the limit where effects from quantum geometry are negligible, one can approximate the difference on the l.h.s. of Eq. (6.207) by a derivative yielding

$$F'(v)=\lim_{\lambda_m\to 0}\frac{F(v+2\lambda_m)-F(v-2\lambda_m)}{4\lambda_m}=\frac{\ell_0}{2}\frac{1}{g^{\frac{1}{3}}|v|^{\frac{1}{3}}}F(v)\tag{6.208}$$

which after separating variables and integrating on both sides gives

$$\ln F(v)=\frac{3\ell_0}{4g^{\frac{1}{3}}}|v|^{\frac{2}{3}}=\frac{3a^2V_0}{\kappa\hbar}+C\tag{6.209}$$

where C is some constant fixed by the initial conditions. Hence, it follows that in the semi-classical limit the unique solution to the right SUSY constraint, up to a constant multiple, is given by

$$F(v)=\exp\left(\frac{3a^2V_0}{\kappa\hbar}\right)\quad\Leftrightarrow\quad\psi(v)=|v|^{-\frac{1}{4}}\exp\left(\frac{3a^2V_0}{\kappa\hbar}\right)\tag{6.210}$$

Finally, the solution $\widehat{S}^{LA'}\Upsilon=0$ to the left SUSY constraint are obtained following the same steps as before which yields $\psi'(v)=|v|^{-\frac{1}{4}}$. Hence, we found a general solution of the quantum SUSY constraints in the semi-classical limit $\lambda_m \to 0$ in which effects from quantum geometry are negligible take the form

$$C|v|^{-\frac{1}{4}} \exp\left(\frac{3a^2V_0}{\kappa\hbar}\right) + D|v|^{-\frac{1}{4}}\phi_{A'}\phi^{A'} \tag{6.211}$$

with some constant coefficients $C, D \in \mathbb{C}$. For $D = 0$ the exponential term is exactly the semi-classical state as found in [88, 90] and, as will be explained in more detail below, turns out to be a *Hartle-Hawking* state of the theory. In the present case, the additional correction term $|v|^{-\frac{1}{4}}$ arises from the symmetric ordering chosen for the definition of the quantum SUSY constraints and thus is simply a quantization ambiguity. In fact, if the regularized connection in (6.203) would have been ordered to the right in front of the volume operator $|v|^{\frac{1}{4}}$, then this term would not appear as a correction to (6.211). Moreover, in [88, 90], in contrast to the present situation, the exponential term appears in the maximally fermion occupied state. This is simply due to the fact that there $\phi_{A'}$ has been implemented as a derivation whereas $\bar{\phi}_A$ and thus $\pi_{\phi}^{A'}$ is quantized as a multiplication operator. Since this choice and the choice made here just correspond to two different representations of the CAR *-algebra, they are simply related via a *Bogoliubov transformation*.

More precisely, in [88, 90], one considers a semi-classical approximation of the Hartle-Hawking wave function

$$\Psi := \int_C \mathcal{D}[e^I_\mu]\mathcal{D}[\psi^A_\mu]\mathcal{D}[\bar{\psi}^{A'}_\mu] \exp\left(-\frac{1}{\hbar}I\right) \tag{6.212}$$

with I the Euclidean action and C a class of four-dimensional metrics and fermionic matter fields satisfying certain prescribed boundary conditions on a given surface. It is then argued, under certain assumptions on the initial conditions and provided that, in the reduced setting, $\phi_{A'}$ is quantized as a multiplication operator, a semi-classical approximation of (6.212) is indeed proportional to (6.211) in case $D = 0$.

6.7. Discussion

In this chapter we have quantized a class of symmetry reduced models of $\mathcal{N} = 1$ supergravity using self-dual variables. We have tried to keep the supersymmetry manifest as far as possible, and used ideas and techniques from loop quantum gravity. In particular:

- We reduced the full theory to a homogeneous and quasi-isotropic one and showed that the essential part of the constraint algebra in the classical theory closes.
- We calculated the elementary super holonomies in the reduced theory and showed that they can be obtained from simple building blocks which form a superalgebra.

- We found a representation of the graded holonomy-flux algebra of the symmetry reduced theory on a super Hilbert space that satisfies the reality conditions. The residual Gauss constraint can be implemented and selects a sub super Hilbert space.
- The supersymmetry constraints and the Hamiltonian constraint can be implemented. Choosing the ordering appropriately reproduces the relevant part of the (super) Dirac algebra of the classical theory.
- We showed that the zero volume state decouples from the dynamics, hence the classical singularity is resolved in this sense.
- In the limit of sending the area gap to zero, a part of the solution space obtained in another approach is recovered.

We would like to highlight the following points.

As pointed out in Section 5.5, some of the apparent difficulties with quantizing the full chiral theory while maintaining manifest supersymmetry are the necessity of a non-compact structure group, the complicated reality conditions, the fact that Haar measures on quantum groups are typically not positive definite, and the complicated constraints. However, one can take some encouragement from the fact that in the model we considered, these problems turned out to be solvable. In particular, reality conditions select a suitable measure and lead to a Hilbert space that is very close to the one used in standard LQC coupled to fermions. For the bosonic sector this was already observed in [91], but here we see that it extends to the fermions as well. Similarly, the closure of the constraint algebra actually reduces the quantization ambiguities and thus is a helpful criterion in the process of writing down the quantum theory.

We started with a theory with manifest supersymmetry but ended with one where parts of it live on in the constraint algebra, but it is not manifest anymore. It is interesting to see where this change happened. Note that (6.137) still contains (the matrix elements of) a super holonomy, i.e., an element of the relevant supergroup and covariant under supersymmetry transformations. For simplicity, and to obtain a result that can be compared to standard LQC, we then considered an algebra of "building blocks" for the matrix elements. But these lack the fine tuning between odd and even degrees of freedom that is necessary for supersymmetry. In the future, it would be interesting to make a different choice here, and use the methods of Section 5.5 (resp. Section 5.5.3) to quantize the super holonomies directly on a suitable space of functions on the supergroup.

Another point that we would like to note is that both orientations of the triad are treated on an equal footing throughout the work. In fact, factors of $\mathrm{sign}(\det(e^i_a))$ enter in many places and are important for the consistency of the theory in both sectors. However, as we have discussed in Section 6.6.4, the dynamics as we have written it does not mix the two sectors. It appears that parity does not act in a well-defined way on c (resp. $\bar{c}$) and hence in the theory. We will leave it to future investigation if there is a way to make parity a well-defined operation in these models.

We have seen that the dynamics of the system resolves the classical singularity in the sense that the zero volume quantum state decouples. We leave it as a task for future investigations to study the full dynamics of the theory including local supersymmetric matter fields as a relational clock. We expect that this would lead to the appearance of bounce cosmologies as in the non-supersymmetric models of LQC [48, 245, 246].

Finally, while we have seen that some solutions to the constraints have similar behavior to those of D'Eath et al. [88, 90] in a certain limit, there are also profound differences, such as the nature of the states and quantum constraint equations. Moreover, we seem to see the Hartle-Hawking state, but not the wormhole state of D'Eath et al. This shows that the use of chiral variables and the quantization principles of LQG have interesting implications that should be understood better in the future.

7. Conclusions and outlook

In this final chapter, let us summarize the results obtained in this work and outline possible future research projects.

7.1. Summary of the results

One primary goal of this work has been to develop a mathematically rigorous approach to classical and quantum supergravity. To achieve this, we decided to follow the "group geometric" approach also commonly known as the Castellani-D'Auria-Fré approach to supergravity. This seems to be the most appropriate for the context of LQG. We tried to put this formalism on a mathematically rigorous foundation. To this end, in Chapter 2, we gave a detailed account of supermanifold and super fiber bundle theory. The focus has mainly been on the Rogers-DeWitt approach as this approach seems to be easier for direct physical applications, although a concrete link to the Berezin-Kostant-Leites and Molotkov-Sachse approach has also been established.

Within this formalism, we constructed the parallel transport map corresponding to super connection forms defined on principal super fiber bundles. In the context of the Berezin-Kostant-Leites approach, the parallel transport map associated to covariant derivatives on super vector bundles has been studied in [78, 79]. As it turns out, in the ordinary category of supermanifolds **SMan**, the parallel transport map, in general, does not provide an isomorphism between the different fibers of a super fiber bundle which is in complete contrast to the classical theory of smooth manifolds. A resolution is given by adding a parametrization supermanifold $\mathcal{S}$ and hence by the *enriched* or *relative category* $\mathbf{SMan}_{/\mathcal{S}}$. As a result, within this category, it follows that the parallel transport $\mathscr{P}^{\mathcal{A}}_{\mathcal{S},\gamma}$ induced by a super connection 1-form $\mathcal{A}$ along a path γ indeed defines an isomorphism between the fibers over the boundary of the path. In particular, it follows that $\mathscr{P}^{\mathcal{A}}_{\mathcal{S},\gamma}$ transforms covariantly under change of parametrization $\mathcal{S}' \to \mathcal{S}$. In fact, generically, it follows from the definition of the relative category, that physical quantities are well-behaved under change of parametrization. This can be regarded as the mathematical realization of the physical requirement that physics should not depend on a particular choice of $\mathcal{S}$. Interestingly, exploiting this property, one can provide a concrete link to the description of anticommutative fermionic fields in pAQFT [76, 77]. This is based on an idea first formulated by Schmitt in [73] and which, in the context of the Molotkov-Sachse approach, has been sketched explicitly in Section 3.6.

Finally, in the case of super matrix Lie groups $\mathcal{G}$, for a particular choice of a gauge, we derived an explicit form of the parallel transport map along paths γ embedded in the underlying bosonic sub supermanifold which turned out to be particularly useful for direct physical applications studied later in this work. It is represented by the $\mathcal{S}$-parametrized $\mathcal{G}$-valued map $g_\gamma[\mathcal{A}]: \mathcal{S} \times \mathcal{I} \to \mathcal{G}$ given by

$$g_\gamma[\mathcal{A}](s,t) = g_\gamma[\omega](s,t) \cdot \mathcal{P}\exp\left(-\int_0^t \mathrm{d}\tau\,(\mathrm{Ad}_{g_\gamma[\omega]^{-1}}\psi^\gamma)(s,\tau)\right) \tag{7.1}$$

where ω and ψ denote the bosonic and fermionic part of the super connection $\mathcal{A}$, respectively, such that $\mathcal{A} = \omega + \psi$ and $g_\gamma[\omega]$ corresponds to the parallel transport map associated to ω. When evaluated at the endpoints of the path γ, this yields a map $g_\gamma[\mathcal{A}] \equiv g_\gamma[\mathcal{A}](\cdot,1): \mathcal{S} \to \mathcal{G}$, that is, an $\mathcal{S}$-point $g_\gamma[\mathcal{A}] \in \mathcal{G}(\mathcal{S})$ of the generalized supergroup $\mathcal{G}(\mathcal{S})$ which, provided that $\mathcal{S}$ is suitably large enough, itself carries the structure of a Rogers-DeWitt supermanifold. In case that the paramaterization supermanifold $\mathcal{S}$ is absent, ψ becomes trivial so that (7.1) reduces to the parallel transport map of the ordinary bosonic connection ω. Hence, the parametrization is necessary in order to resolve the fermionic degrees of freedom of the theory.

Next, in Chapter 3, we turned to the application of these methods for the purpose of a mathematically rigorous approach towards geometric supergravity. To this end, we introduced the notion of a *super Cartan geometry* in analogy of the purely bosonic theory. Again, in order to consistently resolve the fermionic degrees of freedom of the theory, it follows that one needs to work within enriched categories. A super Cartan geometry is mainly described in terms of a 1-form $\mathcal{A}$ called *super Cartan connection* defined on a $\mathcal{S}$-relative principal super fiber bundle $\mathcal{P}_{/\mathcal{S}}$ which, e.g., for $\mathcal{N} = 1$ splits in the form

$$\mathcal{A} = e^I P_I + \frac{1}{2}\omega^{IJ} M_{IJ} + \psi^\alpha Q_\alpha \tag{7.2}$$

and thus encodes all the physical degrees freedom of the theory. We then embedded the Castellani-D'Auria-Fré approach into the present formalism and discussed the Cartan geometric approach to $\mathcal{N} = 1$, $D = 4$ Poincaré supergravity. In this approach, it follows that when certain conditions are imposed on the physical fields, then supersymmetry transformations can be described in terms of a particular subclass of superdiffeomorphisms on the base supermanifold of the bundle. In fact, using the Cartan geometric interpretation, it follows, using the strong relation between Cartan connections and Ehresmann connections, that SUSY transformations can also be interpreted as infinitesimal gauge transformations on associated bundles. This observation turned out to be important in the context of the chiral theory studied later in Chapter 5 as, there, it follows that half of the supersymmetry appears as a group of gauge transformations.

We then also extended the formalism to include a nontrivial cosmological constant and extended supersymmetry, yielding a geometric description of $\mathcal{N}$-extended pure anti-de Sitter-supergravity theories with $\mathcal{N} = 1, 2$. In this context, we also explicitly included the possibility of the existence of a nontrivial boundary. Moreover, in view of applications in the framework of LQG, we also considered a finite Barbero-Immirzi parameter β. To this end, we adapted the techniques developed in [81–83], and asked the question: What is the general boundary term consistent with the symmetries of the bulk action and such that the full action (including both bulk and boundary contributions) is invariant under SUSY transformations at the boundary. In the limit $\beta \to \infty$, this question has been answered in [81] with the result that the boundary theory is in fact uniquely fixed by the requirement of SUSY-invariance at the boundary. In particular, it follows that the full action acquires a very intriguing form given by a MacDowell-Mansouri-type action in both cases $\mathcal{N} = 1, 2$. In case of a *finite* Barbero-Immirzi parameter, we have shown that the Holst action of pure $\mathcal{N} = 1, 2$ AdS supergravity in the presence of boundaries can again be written in the Yang-Mills-like form

$$S^{\mathcal{N}}_{\text{H-MM}}(\mathcal{A}) = \frac{L^2}{\kappa}\int_M \langle F(\mathcal{A}) \wedge F(\mathcal{A})\rangle_\beta \tag{7.3}$$

with $F(\mathcal{A})$ the curvature of super Cartan connection $\mathcal{A}$ encoding the physical degrees of freedom of the theory. However, here, $\langle\cdot,\cdot\rangle_\beta$ denotes a $\mathrm{Spin}^+(1,3)$-invariant inner product on $\Omega^2(\mathcal{M}_{/\mathcal{S}},\mathfrak{g})$ induced by a β-dependent operator

$$\mathbf{P}_\beta : \Omega^2(\mathcal{M}_{/\mathcal{S}},\mathfrak{g}) \to \Omega^2(\mathcal{M}_{/\mathcal{S}},\mathfrak{g}) \tag{7.4}$$

where $\mathfrak{g} := \mathfrak{osp}(\mathcal{N}|4)$ denotes the super Lie algebra of the super anti-de Sitter group $\mathrm{OSp}(\mathcal{N}|4)$. In particular, for $\mathcal{N} = 2$, we have shown that the restriction of (7.4) to the $\mathfrak{u}(1)$ subalgebra of $\mathfrak{g}$ is given by $\mathbf{P}_\beta|_{\mathfrak{u}(1)} = (1+\beta\star)/2\beta$. As a consequence, the boundary theory acquires an additional $\mathrm{U}(1)$-contribution depending on β also known as the θ-term in Yang-Mills theory. Hence, in this framework, it follows that β literally has the interpretation of the θ parameter of the θ-ambiguity of QED.

The chiral limit of the theory corresponding to the choices $\beta = \pm i$ is special. In this case, the full action remains manifestly invariant under an enlarged gauge symmetry given by the (complex) orthosymplectic supergroup $\mathrm{OSp}(\mathcal{N}|2)_{\mathbb{C}}$ which is a chiral subgroup of the complexified super anti-de Sitter group. More precisely, for both $\mathcal{N} = 1$ and $\mathcal{N} = 2$, the chiral action can be re-written in the form

$$S^{\mathcal{N},\beta=-i}_{\text{H-MM}}(\mathcal{A}) = \frac{i}{\kappa}\int_M \left(\langle F(\mathcal{A}^+) \wedge \mathcal{E}\rangle + \frac{1}{4L^2}\langle \mathcal{E} \wedge \mathcal{E}\rangle\right) + S_{\text{bdy}}(\mathcal{A}^+) \tag{7.5}$$

with $\mathcal{A}^+$ the chiral subpart of $\mathcal{A}$ defining a *generalized* super Cartan connection and $\mathcal{E}$ is called the super electric field. $\mathcal{A}^+$ is a generalization of the Ashtekar connection [20] to the context of supergravity, and hence we called it the *super Ashtekar connection*. A further sign that the chiral case is quite special came when considering the action (7.3) in the presence of boundaries. In this case, the unique boundary action takes the form of a super Chern-Simons action $S_{\mathrm{bdy}}(\mathcal{A}^+) \equiv S_{\mathrm{CS}}(\mathcal{A}^+)$ with gauge supergroup $\mathrm{OSp}(\mathcal{N}|2)_{\mathbb{C}}$ and complex Chern-Simons level. This extends results obtained in [63, 84–86, 182] by including extended supersymmetry $\mathcal{N} = 2$, real Barbero-Immirzi parameters as well as a general discussion about boundary theory and the role of supersymmetry.

In the canonical theory, it follows that $(\mathcal{A}^+, \mathcal{E})$ defines a canonically conjugate pair building up a graded symplectic phase space. Using this fact together with the parallel transport map as derived in Chapter 2, we then constructed a graded analog of the well-known holonomy-flux algebra. In this context, we found that the configuration space of generalized super connections can be identified with the limit of a projective family $\mathcal{A}_{\mathcal{S},\gamma}$ which, besides finite graphs γ embedded in the spatial Cauchy slices of the bosonic sub supermanifold as in the classical non-supersymmetric theory, are labeled by the additional parametrization supermanifold $\mathcal{S}$. Moreover, by construction, the projective family is well-behaved under change of parametrization. More precisely, in case of a finite subgraph $\gamma' \subset \gamma$ and parametrization $\mathcal{S}'$ with $\mathcal{S}' \subset \mathcal{S}$, one obtains the following commutative diagram

$$\begin{array}{ccc} \mathcal{A}_{\mathcal{S},\gamma} & \longrightarrow & \mathcal{A}_{\mathcal{S},\gamma'} \\ \downarrow & & \downarrow \\ \mathcal{A}_{\mathcal{S}',\gamma} & \longrightarrow & \mathcal{A}_{\mathcal{S}',\gamma'} \end{array} \tag{7.6}$$

Based on these observations, we then sketched the quantization of the theory by choosing an Ashtekar-Lewandowski-type representation of the graded holonomy-flux algebra on a super Hilbert space. However, the final picture remained rather incomplete due to several difficulties related to the implementation of cylindrical consistency due to the non-compactness of the gauge group, as well as due to the indefiniteness of the inner products induced by the Haar measures on supergroups. At least for the special case $\mathcal{N} = 1$, a possible resolution seems to be given by considering the compact real form $\mathrm{UOSp}(1|2)$ of $\mathrm{OSp}(1|2)$ as there, according to [207], a Peter-Weyl-type basis seems to exist which has similar properties as for $\mathrm{SU}(2)$. Of course, ultimately, one needs to solve reality conditions as, a priori, one is dealing with a complex theory. But, even in case of the purely bosonic theory, this remains a rather open problem.

We also compared this manifestly supersymmetric approach with the standard quantization techniques of LQG coupled to fermions using real variables [67, 80, 87]. We therefore introduced the notion of pointed generalized super connections and derived a graded holonomy-flux-type algebra. The resulting picture then turned out to share many similarities with the manifest approach. In particular, in this framework, it follows that the all the previously mentioned difficulties can be solved consistently.

Finally, in Chapter 5, we applied these methods to study a class of symmetry reduced models of $\mathcal{N} = 1$, $D = 4$ supergravity. We exploited the enlarged gauge symmetry of the chiral theory and studied super connection 1-forms that are homogeneous and isotropic up to local super gauge transformations. We found that such an invariant connection can be written in the form

$$\mathcal{A}^+ = c\mathring{e}^i T_i^+ + \psi_{A'}\mathring{e}^{AA'} Q_A \tag{7.7}$$

for some Grassmann-even and -odd numbers c and $\psi_{A'}$, respectively. Interestingly, the fermionic part of (7.7) precisely coincides with the ansatz of the gravitino field as proposed by D'Eath et al. in [88–90]. In fact, we have argued that this is the most general ansatz which is consistent with the reality conditions assuming homogeneity of the bosonic degrees of freedom. Using the form (7.7) of the invariant super Ashtekar connection, we then performed a symmetry reduction of the chiral theory and derived symmetry reduced expressions of the constraints with explicit consideration of parity. In particular, we studied the constraint algebra and showed that the essential part given by the graded Poisson bracket between the left and right SUSY constraints indeed closes and reproduces the Hamiltonian constraint.

We then turned to the quantization of the theory. Following the standard procedure in loop quantum cosmology in studying the holonomies (7.1) induced by the super Ashtekar connection (7.7), we defined a symmetry reduced variant of the graded holonomy-flux algebra. Moreover, we were able to express the symmetry reduced form of the reality conditions in terms of adjointness relations that gave the algebra the structure of a Lie *-superalgebra. For the quantization, we then chose an Ashtekar-Lewandowski-type representation of the algebra on a super Hilbert space. Following the ideas of [91] in the context of the purely bosonc theory, we were able to solve the reality conditions and obtained a unique inner product.

As a next step, we implemented the dynamical constraints in the quantum theory and, at least for a specific subclass of the symmetry reduced models under consideration, studied explicitly the quantum constraint algebra. It turned out that imposing conistency with the classical Poisson relations required a symmetric ordering in the definition of the constraint operators in accordance with [244] in the context of the non-supersymmetric theory. This also fixed some of the quantization ambiguities. In this way, we found that the anticommutator between the left and right SUSY constraint operators $\widehat{S}^{LA'}$ and $\widehat{S}^R_{A'}$ takes the form

$$[\widehat{S}^{LA'}, \widehat{S}^R_{B'}] = \left(i\hbar\kappa\widehat{H} - \frac{\hbar\kappa}{6g^{\frac{1}{2}}|v|^{\frac{1}{2}}}\widehat{\pi}_\phi^{C'}\widehat{S}^R_{C'}\right)\delta^{A'}_{B'} + \frac{\hbar\kappa}{6g^{\frac{1}{2}}|v|^{\frac{1}{2}}}\widehat{\pi}_\phi^{A'}\widehat{S}^R_{B'} \tag{7.8}$$

which, in particular, exactly reproduces the classical Poisson relations.

In Section 6.6.5, we considered the semi-classical limit of the theory assuming that physical implications of the quantum area gap can be neglected. In this limit, we found that a subclass of physical states Ψ_{phys} annihilated by the dynamical constraints are of the form

$$\Psi_{\text{phys}} \sim \exp\left(\frac{3a^2 V_0}{\kappa\hbar}\right) \tag{7.9}$$

where the right-hand side exactly corresponds to a state as derived by D'Eath et al. [88–90] using standard variables and standard minisuperspace techniques. This state is a stationary action-type approximation of the symmetry reduced Hartle-Hawking state. However, another physical state with different initial conditions ("wormhole state") obtained there turns out to be not part of the physical Hilbert space in the present formalism. This shows that chiral variables have interesting properties that should be explored further in the future.

Finally, in the context of the full theory, we derived a compact expression of the classical SUSY constraint in Chapter 4 using real Ashtekar-Barbero variables. There, an implementation of this constraint in the quantum theory has been discussed proposing a specific regularization scheme. In particular, explicit expressions for its action on spin network states have been derived. These results provide a starting point for the computation of the commutator of the SUSY constraint in the full theory and to check whether a similar strong relation between the SUSY constraint and Hamilton operator such as (7.8) can also be obtained in the full theory. In particular, it would be interesting to see whether this also fixes some of the quantization ambiguities.

7.2. Future research

As outlined already in the previous chapters, there exist many interesting and important research directions in which the present work could be developed further in the near future. In the following, let us give a short summary of, in our opinion, some of the most important points:

Black hole entropy: In Chapter 5, adapting the techniques developed in [81–83], the most general form of the boundary action compatible with local supersymmetry for $\mathcal{N}$-extended AdS supergravity in $D = 4$ with $\mathcal{N} = 1, 2$ has been derived including a finite Barbero-Immirzi parameter. In the chiral limit, this boundary action turns out to take the form of a super Chern-Simons action. The quantization of the bulk theory adapting tools from standard LQG has also been sketched in Section 5.5.3. As a next step, it would be very interesting to use these results to study the quantum theory of supersymmetric (charged) black holes in the framework of LQG, in particular, for the special case $\mathcal{N} = 2$ (see also Section 5.6.2). Supersymmetric black holes play a very prominent role in superstring theory. There, a derivation of the black hole entropy consistent with the Bekenstein-Hawking area law for a specific class of supersymmetric (charged) extremal black holes has been achieved [6–13].

It would be interesting to see whether similar results can be obtained in the framework of LQG and, in particular, whether these results could be related to the superstring computations. Perhaps, the recent results of [166] may help to provide such a link. There, for specific string-brane configurations, it has been observed that the boundary theory also carries the structure of a super Chern-Simons theory. In any case, all these observations suggest that this requires a deeper understanding of super Chern-Simons theories and its relation to both LQG and superstring theory.

Hilbert space of chiral LQSG: Related to the previous point, it would be highly desirable to complete the construction of the super Hilbert space of chiral LQSG as outlined in Section 5.5.3. In this context, one needs to find a way to consistently implement cylindrical consistency and to deal with the additional difficulties arising in the supersymmetric context related to the indefiniteness of Haar measures on super Lie groups. Furthermore, one has to solve reality conditions which, even in the purely bosonic self-dual theory, is an open problem. As we have seen in Chapter 6 in the context of symmetry reduced models, there, the problem of indefiniteness and reality conditions can indeed be solved consistently implying that the measure has to be distributional. Perhaps, these results can be extended to the full theory. In fact, recent developments in the framework of the full bosnic self-dual theory [215] also suggest that the reality conditions (at least some subclass thereof) can be solved by choosing the measure appropriately. Moreover, the measure induces a gauge fixing to the compact subgroup $\mathrm{SU}(2)$ of $\mathrm{SL}(2,\mathbb{C})$. Maybe, these results can be generalized to the supersymmetric setting. In particular, it would be interesting to see whether this leads to a gauge fixing to the unitary orthosymplectic group $\mathrm{UOSp}(\mathcal{N}|2)$ which, at least for the special case $\mathcal{N}=1$, has very similar properties as its corresponding bosonic counterpart $\mathrm{SU}(2)$.

SUGRA with $\mathcal{N}>2$: It would be interesting to extend the present considerations to include supergravity theories in the presence of boundaries with higher supersymmetry $\mathcal{N}\geq 3$ in $D=4$ and even higher spacetime dimensions. In particular, it would be interesting to see whether also there the boundary theory turns out to be fixed uniquely if one imposes SUSY-invariance at the boundary and whether the resulting action of the full theory again acquires an intriguing geometrical form similarly as for the cases $\mathcal{N}=1,2$. Moreover, one needs to check whether the fascinating structures observed in the chiral theory also carry over to higher $\mathcal{N}$.

In this context, the special case of maximal $\mathcal{N} = 8$, $D = 4$ SUGRA would be of particular interest due to interesting results suggesting its perturbative finiteness [247–250]. One may note that $\mathcal{N} = 8$ SUGRA can be derived via Kaluza-Klein compactification from the unique maximal $\mathcal{N} = 1$, $D = 11$ SUGRA which, as discovered in [129, 130], can be described geometrically in terms of a higher super Cartan geometry. On the other hand, Ashtekar-Barbero-type variables for arbitrary higher spacetime dimensions have been derived in [67–69]. Hence, it would be interesting to see whether these variables, at least in a certain limit, can be described geometrically similarly as in the context of chiral SUGRA in $D = 4$ for $\mathcal{N} = 1, 2$.

Hamiltonian dynamics: In the recent papers [251, 252], a new approach has been proposed to study the Hamiltonian dynamics in canonical general relativity expressed in terms of self-dual variables by introducing the notion of a *generalized gauge covariant Lie derivative*. Expressed in this way, the Hamiltonian dynamics acquire an intriguing simple structure which may considerably simplify the corresponding dynamics in the quantum theory. Moreover, the authors suggest that this approach may provide a concrete link to the double copy pattern that relates structures in gravitational theory to that of Yang-Mills with double the number of fields.

In fact, double copy ideas have intensively been studied in the context of perturbative quantum supergravity, in particular, in the context of maximal $\mathcal{N} = 8$, $D = 4$ SUGRA [247–250], in order to simplify calculations of scattering amplitudes. Thus, it would be very interesting to know whether this alternative description of the Hamiltonian dynamics can also be extended to (chiral) supergravity. Preliminary calculations suggest that this may indeed be possible. This might help to implement double copy ideas to non-perturbative quantum supergravity.

Dynamics in LQSC: In Chapter 6, the classical and quantum theory of a class of symmetry reduced models of chiral $\mathcal{N} = 1$, $D = 4$ SUGRA has been studied. Moreover, it has been argued in Section 6.6.4, due to symmetric ordering of the dynamical constraints, that the big bang singularity is resolved in the quantum theory. However, it was not possible to develop an approximate spacetime picture confirming a bouncing geometry, since the Rarita-Schwinger field, as being a fermionic field, cannot be used as a relational clock. Since, in the context of locally supersymmetric field theory, the dynamics is governed by the SUSY constraint(s), this thus requires the inclusion of further locally supersymmetric matter fields to the theory which may then serve as relational clocks. It would be interesting to understand how locally supersymmetric matter enters to the constraints and how, also in this framework, the strong relationship between the dynamical constraints as observed in Section 6.6.4 can be maintained in the quantum theory.

Finally, it would be interesting to see how these results can be compared to the results of standard homogeneous isotropic models in (self-dual) LQC. In a sense, local supersymmetry simplifies the conisderations as the SUSY constraint(s) already correspond to a kind of a "square root" of the Hamiltonian constraint operator.

etc. etc.

Appendix

A. Super linear algebra

This section, following [97, 106], is meant to fix some terminology of important aspects in super linear algebra used in the main text. Therefore, we will exclusively focus on $\mathbb{Z}_2$ grading as these are commonly used in physics in the context of (supersymmetric) field theories modeling commuting bosonic and anticommuting fermionic fields. For more details on this fascinating subject, the interested reader may be referred to the great references of [97, 106].

Definition A.1. A $\mathbb{Z}_2$*-graded* or simply *super vector space* V is a vector space over a field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) of the form

$$V = V_0 \oplus V_1 \tag{A.1}$$

together with a map $|\cdot| : \bigcup_{i\in\mathbb{Z}_2} V_i \to \mathbb{Z}_2$ called *parity map* such that $|v| := i\ \forall v \in V_i$. Elements in V_i are called *homogeneous with parity* $i \in \mathbb{Z}_2$. If the dimension of V_0 and V_1 are given by $\dim V_0 = m$ and $\dim V_1 = n$, respectively, then the dimension of V is denoted by $\dim V = m|n$.

A morphism $\phi : V \to W$ between super vector spaces is a linear map between vector spaces preserving the parity, i.e., $\phi(V_i) \subseteq W_i$ for $i \in \mathbb{Z}_2$. The set of such super vector space morphisms is denoted by $\mathrm{Hom}(V, W)$. In case $V = W$, we also write $\mathrm{End}(V) := \mathrm{Hom}(V, V)$.

Remark A.2. Instead of just looking at parity preserving morphisms between super vector spaces V and W, one can also consider all possible linear maps between the underlying vector spaces. This yields the internal $\underline{\mathrm{Hom}}(V, W)$ which has the structure of a super vector space with the even and odd part $\underline{\mathrm{Hom}}(V, W)_0$ and $\underline{\mathrm{Hom}}(V, W)_1$ given by the parity preserving and parity reversing linear maps between V and W, respectively. Hence, $\underline{\mathrm{Hom}}(V, W)_0$ coincides with $\mathrm{Hom}(V, W)$ in definition A.1.

Example A.3. A trivial but also very important example of a super vector space is given by $\mathbb{R}^{m|n} := \mathbb{R}^m \oplus \mathbb{R}^n$ (or, more generally $\mathbb{K}^{m|n}$ for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$) called *superspace* of dimension $m|n$. In fact, any super vector space $V = V_0 \oplus V_1$ is isomorphic to a superspace. If $\dim V = m|n$, let $\{e_i\}_{i=1,\dots,m+n}$ be a basis of V such that $\{e_i\}_{i=1,\dots,m}$ is a basis of V_0 and $\{e_j\}_{j=m,\dots,m+n}$ is a basis of V_1. Such a basis is called a *homogeneous basis of* V. Then, V is isomorphic, as a super vector space, to the superspace $\mathbb{R}^{m|n}$.

Definition A.4. A *superalgebra* A is a super vector space $A = A_0 \oplus A_1$ together with a bilinear map $m : A \times A \to A$ such that

$$m(A_i, A_j) =: A_i \cdot A_j \subseteq A_{i+j} \quad \forall i, j \in \mathbb{Z}_2 \tag{A.2}$$

The superalgebra A is called *super commutative* if

$$a \cdot b = (-1)^{|a||b|} b \cdot a \tag{A.3}$$

for all homogeneous $a, b \in A$.

Definition A.5. Let A be a superalgebra. A *super left A-module* $\mathcal{V}$ is a super vector space which, in addition, has the structure of a left A-module such that

$$A_i \cdot \mathcal{V}_j \subseteq \mathcal{V}_{i+j} \quad \forall i, j \in \mathbb{Z}_2 \tag{A.4}$$

A morphism $\phi : \mathcal{V} \to \mathcal{W}$ between super left A-modules is a map between the underlying super vector spaces such that $\phi(a \cdot v) = a\phi(v)\ \forall a \in A, v \in \mathcal{V}$. Analogously, one defines a *super right A-modules* and morphisms between them.

Remark A.6. Given super left A-module $\mathcal{V}$ one can also turn it into a super right A-module setting

$$v \cdot a := (-1)^{|v||a|} a \cdot v \tag{A.5}$$

for homogeneous $a \in A$ and $v \in \mathcal{V}$. For this reason, in the following, we will simply say super A-module if we do not want to specify whether it should be regarded as a super left or right A-module. If $\mathcal{V}$ and $\mathcal{W}$ are super commutative super A-modules, their tensor product $\mathcal{V} \otimes \mathcal{W}$ is defined viewing $\mathcal{V}$ as a left and $\mathcal{W}$ as a right A-module.

Given a super left A-modules $\mathcal{V}$ and $\mathcal{W}$ we denote the set of left A-module morphisms $\phi : \mathcal{V} \to \mathcal{W}$ by $\mathrm{Hom}_L(\mathcal{V}, \mathcal{W})$ (and similarly $\mathrm{Hom}_R(\mathcal{V}, \mathcal{W})$ for right A-module morphisms). As in remark A.2, instead of just looking at parity preserving morphisms, one can also consider all possible linear maps $\phi : \mathcal{V} \to \mathcal{W}$ between the underlying vector spaces satisfying

$$\phi(a \cdot v) = a\phi(v), \quad \forall v \in \mathcal{V},\ a \in A \tag{A.6}$$

This again yields an internal $\underline{\mathrm{Hom}}_L(\mathcal{V}, \mathcal{W})$ which has the structure of a super right A-module with $\underline{\mathrm{Hom}}_L(\mathcal{V}, \mathcal{W})_0 = \mathrm{Hom}_L(\mathcal{V}, \mathcal{W})$. In case $\mathcal{V} = \mathcal{W}$, we also write $\underline{\mathrm{End}}_L(\mathcal{V}) := \underline{\mathrm{Hom}}_L(\mathcal{V}, \mathcal{V})$ with $\mathrm{End}_L(\mathcal{V}) = \underline{\mathrm{End}}_L(\mathcal{V})_0$ and likewise for right linear morphisms $\underline{\mathrm{End}}_R(\mathcal{V}) := \underline{\mathrm{Hom}}_R(\mathcal{V}, \mathcal{V})$. As usual, we denote the evaluation of a morphism $\phi \in \underline{\mathrm{Hom}}_L(\mathcal{V}, \mathcal{W})$ at $v \in \mathcal{V}$ by

$$\langle v|\phi\rangle \in \mathcal{W} \tag{A.7}$$

This has the advantage that one does not need to care about signs due to super commutativity after right multiplication with elements $a \in A$, i.e., $\langle v|\phi a\rangle = \langle v|\phi\rangle\, a \;\forall a \in A$, $v \in \mathcal{V}$. Finally, let us define via $\phi \diamond \psi \in \underline{\mathrm{Hom}}_L(\mathcal{V}, \mathcal{W}')$ the composition of two left linear morphisms $\phi : \underline{\mathrm{Hom}}_L(\mathcal{V}, \mathcal{W})$ and $\psi : \underline{\mathrm{Hom}}_L(\mathcal{W}, \mathcal{W}')$ given by

$$\langle v|\phi \diamond \psi\rangle := \langle \langle v|\phi\rangle\, |\psi\rangle\,, \quad \forall v \in \mathcal{V} \tag{A.8}$$

which, by definition, is well-behaved under right multiplication.

Definition A.7. For A a superalgebra, a *super A-Lie module* (or *super Lie algebra* or *Lie superalgebra* if $A = \mathbb{K}$) is a super A-module L with a bilinear map $m \equiv [\cdot,\cdot] : L \times L \to L$, also called the *(Lie) bracket*, that is graded skew-symmetric, i.e.,

$$[a, b] = -(-1)^{|a||b|}[b, a] \tag{A.9}$$

and satisfies the graded Jacobi identity

$$[a, [b, c]] + (-1)^{|a|(|b|+|c|)}[b, [c, a]] + (-1)^{|c|(|a|+|b|)}[c, [a, b]] = 0 \tag{A.10}$$

for all homogeneous $a, b, c \in L$.

Example A.8. (i) If V is a vector space (finite- or infinite-dimensional), then the exterior algebra $\bigwedge V := \bigoplus_{k=0}^{\infty} \bigwedge^k V$ also called *Grassmann algebra* naturally defines a superalgebra with even and odd part given by $(\bigwedge V)_0 := \bigoplus_{k=0}^{\infty} \bigwedge^{2k} V$ and $(\bigwedge V)_1 := \bigoplus_{k=0}^{\infty} \bigwedge^{2k+1} V$, respectively. The Grassmann algebra is super commutative, associative and unital with unit $1 \in \mathbb{R} = \bigwedge^0 V$.

(ii) For A a superalgebra, the tensor product $A^{m|n} := A \otimes \mathbb{K}^{m|n}$ defines a super A-module with grading $(A^{m|n})_0 = A_0 \otimes \mathbb{K}^m \oplus A_1 \otimes \mathbb{K}^n$ and $(A^{m|n})_1 = A_0 \otimes \mathbb{K}^n \oplus A_1 \otimes \mathbb{K}^m$.

Definition A.9. A super A-module $\mathcal{V}$ is called *free* if it contains a homogeneous basis $\{e_i\}_{i=1,\ldots,m+n}$ for some $m, n \in \mathbb{N}_0$ such that any element $v \in \mathcal{V}$ can be written in the form $v = a^i e_i$ with coefficients $a^i \in A\; \forall i = 1, \ldots, m+n$. Equivalently, $\mathcal{V}$ is a free super super A-module iff it is isomorphic to $A^{m|n} = A \otimes \mathbb{K}^{m|n}$. In this case, the dimension of $\mathcal{V}$ will be denoted by $\dim \mathcal{V} = m|n$.

Definition A.10. (i) Let $\mathcal{V}$ be a free super A-module. Two homogeneous bases $\{e_i\}_i$ and $\{f_j\}_j$ of $\mathcal{V}$ are called equivalent if they are related to each other by scalar coefficients, i.e., there exists real or complex numbers $a_i^{\ j} \in \mathbb{K}$ such that $e_i = a_i^{\ j} f_j \ \forall i.j$ (for a proof that this indeed defines an equivalence relation see [97]).

(ii) A free super A-module $\mathcal{V}$ together with a distinguished equivalence class $[(e_i)_i]$ of homogeneous bases of $\mathcal{V}$ is called a *super A-vector space*. A representative $(e_i) \in [(e_i)_i]$ will be called a *real* resp. *complex homogeneous basis* of $\mathcal{V}$. A morphism $\phi : \mathcal{V} \to \mathcal{W}$ between super A-vector spaces is a morphism of super A-modules such that ϕ maps the equivalence class of bases of $\mathcal{V}$ to the real resp. complex vector space spanned by the equivalence class of bases of $\mathcal{W}$.

Remark A.11. If $\mathcal{V}$ is a super A-vector space with equivalence class $[\{e_i\}_i]$ of homogeneous bases of $\mathcal{V}$, then $\mathcal{V} \cong A \otimes V$ with V the super vector space spanned by $\{e_i\}_i$ which, in particular, is independent on the choice of a representative of that equivalence class. Hence, the choice of such an equivalence class yields a well-defined super vector space V also called the *body of* $\mathcal{V}$. On the other hand, if $\mathcal{V}$ is a free super A-module, one can always choose a homogeneous basis $\{e_i\}_i$ of $\mathcal{V}$ such that $\mathcal{V}$ becomes a super A-vector space w.r.t. the equivalence class $[\{e_i\}_i]$. However, such a choice may not be canonical and various different bases exist which are not related by scalar coefficients.

B. Categories, sheaves and locally ringed spaces

This chapter is meant to summarize some important aspects of category theory and algebraic geometry as this abstract language turns out to play a crucial role in properly defining the notion of a *supermanifold* and related concepts. Moreover, we will use this opportunity in order to fix some terminology used in the main part of this work. To this end, we will mainly follow Reference [253]. We start with the definition of the notion of a *category*.

Definition B.1. A *category* C consists of a collection $\mathbf{Ob}(C)$ of *objects* and, for each pair of objects $X, Y \in \mathbf{Ob}(C)$, of a set $\mathrm{Hom}(X, Y) \equiv \mathrm{Hom}_C(X, Y)$ of *morphisms* (or *arrows*) $f : X \to Y$ together with a law of composition

$$\begin{aligned} \mathrm{Hom}(Y, Z) \times \mathrm{Hom}(X, Y) &\to \mathrm{Hom}(X, Z) \\ (g, f) &\mapsto g \circ f \end{aligned} \tag{B.1}$$

for any objects $X, Y, Z \in \mathbf{Ob}(C)$ such that the following conditions are satisfied:

(i) The composition of morphisms is associative.

(ii) For all $X \in \mathbf{Ob}(C)$ there is a (unique) morphism $\mathrm{id}_X \in \mathrm{Hom}(X, X)$ called *identity-morphism* such that for any morphisms $f \in \mathrm{Hom}(X, Y)$ and $g \in \mathrm{Hom}(Y, X)$, $f \circ \mathrm{id}_X = f$ and $\mathrm{id}_X \circ g = g$.

In case $X = Y$, a morphism $f : X \to Y$ is also called an *endomorphism*. If $f : X \to Y$ is invertible, i.e., there exists $g : Y \to X$ with $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$, then f is also called an *isomorphism*.

Definition B.2. A category C is called *small* if the collection of objects $\mathbf{Ob}(C)$ forms a set. A small category C is called a *groupoid* if any morphism $f : X \to Y$ between objects $X, Y \in \mathbf{Ob}(C)$ is invertible.

In the following, let us give some important examples of categories. In fact, we will encounter various further examples in the main text.

Example B.3. (i) the category **Set** of *sets* with sets X as objects and maps $f : X \to Y$ between sets as morphisms

(ii) the category **Grp** of *groups* with groups G as objects and group morphisms $\phi : G \to H$ as morphisms

(iii) the category **Ring** of *rings* with rings R as objects and ring morphisms $\phi : R \to R'$ as morphisms

(iv) the category $\mathbf{Mod}_R$ of *R-modules* with R a ring with R-modules M_R as objects and R-module morphisms $\phi : M_R \to N_R$ as morphisms

(v) the category **Top** of *toplogical spaces* with topological spaces X as objects and continuous maps $F : X \to Y$ between topological spaces as morphisms

(vi) the category **Man** of *real smooth manifolds* with smooth manifolds M as objects and smooth maps $f : M \to N$ between manifolds as morphisms.

(vii) Real smooth vector bundles (E, M) (or $E \to M$) together with smooth vector bundle morphisms $(\phi, f) : (E, M) \to (F, N)$, i.e., smooth maps $\phi : E \to F$ and $f : M \to N$ such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{\phi} & F \\ {\scriptstyle \pi_E}\downarrow & & \downarrow{\scriptstyle \pi_F} \\ M & \xrightarrow{f} & N \end{array}$$

is commutative and $\phi_x : E_x \to F_{f(x)}$, $x \in M$, is linear on each fiber, form the category $\mathbf{Vect}_{\mathbb{R}}$ of *real smooth vector bundles*.

(viii) the category **Cat** of all *small categories* with small categories as objects and functors (see Definition B.4 below) between small categories as morphisms.

(ix) To any category $\mathcal{C}$, one can assign the corresponding *opposite category* $\mathcal{C}^{\mathrm{op}}$ consisting of the same collection of objects $\mathbf{Ob}(\mathcal{C}^{\mathrm{op}}) = \mathbf{Ob}(\mathcal{C})$ and, for each objects $X, Y \in \mathbf{Ob}(\mathcal{C}^{\mathrm{op}})$, the set of morphisms $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y) := \mathrm{Hom}_{\mathcal{C}}(Y, X)$.

Definition B.4. Let $\mathcal{C}, \mathcal{D}$ be categories. A *covariant functor* $F : \mathcal{C} \to \mathcal{D}$ from the category $\mathcal{C}$ to the category $\mathcal{D}$ is a map that assigns to each object $X \in \mathbf{Ob}(\mathcal{C})$ an object $F(X) \in \mathbf{Ob}(\mathcal{D})$ and to each morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ in $\mathcal{C}$ a morphism $F(f) \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ such that

(i) $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for all $X \in \mathcal{C}$.

(ii) $F(f \circ g) = F(f) \circ F(g)$ for composable morphisms g and f in $\mathcal{C}$.

A *contravariant functor* $F : \mathcal{C} \to \mathcal{D}$ between the category $\mathcal{C}$ to the category $\mathcal{D}$ is defined as a covariant functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ on the opposite category $\mathcal{C}^{\mathrm{op}}$. Hence, roughly speaking, a contravariant functor reverses the arrows.

Definition B.5. A (covariant) functor $F : \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ is called

(i) *faithful* resp. *full* if, for any pair of objects $X, Y \in \mathbf{Ob}(\mathcal{C})$, the induced map

$$F : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y)),\ f \mapsto F(f) \tag{B.2}$$

between the sets $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ and $\mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ is injective resp. surjective.

(ii) *fully faithful* if it is both full and faithful, i.e., the induced map (B.2) is bijective.

Definition B.6. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}$ be two (covariant) functors between categories $\mathcal{C}$ and $\mathcal{D}$. A *functor morphism* or *natural transformation* $\eta : F \to G$ between F and G is a collection of morphisms $\eta_X : F(X) \to G(X)$, $X \in \mathbf{Ob}(\mathcal{C})$, such that for any morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, the following diagram commutes

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ {\scriptstyle \eta_X}\downarrow & & \downarrow{\scriptstyle \eta_Y} \\ G(X) & \xrightarrow{G(f)} & G(Y) \end{array}$$

If the morphisms $\eta_X : F(X) \to G(X)$ are isomorphisms for any object $X \in \mathcal{C}$, then $\eta : F \to G$ is called a *natural isomorphism*. In this case, the functors F and G are called *equivalent* or *isomorphic*. Finally, two categories C and $\mathcal{D}$ are called *equivalent* if there exists functors $F : C \to \mathcal{D}$ and $G : \mathcal{D} \to C$ such that the functors $G \circ F : C \to C$ and $F \circ G : \mathcal{D} \to \mathcal{D}$ are equivalent to the identity functors $\mathrm{id}_C : C \to C$ and $\mathrm{id}_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}$, respectively. In this case, the functor F (or G) is called an *equivalence of categories*.

Example B.7. (i) The *base functor* $\mathbf{b} : \mathbf{Vect}_{\mathbb{R}} \to \mathbf{Man}$ is the full functor from the category of real smooth vector bundles to the category of smooth manifolds which on objects is defined via $\mathbf{b}(E, M) := M$ and on morphisms $(\phi, f) : (E, M) \to (F, N)$ is given by $\mathbf{b}(\phi, f) := f$. Thus, the base functor associates to each vector bundle the corresponding base and to vector bundle morphism the underlying morphism between the bases.

(ii) The *inclusion functor* $\mathbf{i} : \mathbf{Man} \to \mathbf{Vect}_{\mathbb{R}}$ is the faithful functor from the category of real smooth manifolds to the category of real smooth vector bundles which associates to a smooth manifold M the trivial vector bundle $\mathbf{i}(M) := (\{0\}, M)$ and to a smooth function $f : M \to N$ the vector bundle morphism $\mathbf{i}(f) := (\mathbf{0}, f) : (\{0\}, M) \to (\{0\}, N)$. Since $\mathbf{b} \circ \mathbf{i} = \mathrm{id}_{\mathbf{Man}}$, the base functor defines a (left) inverse to the inclusion functor and yields an equivalence of categories restricting $\mathbf{Vect}_{\mathbb{R}}$ to the subcategory $\mathbf{Vect}_{\mathbb{R},0} \subset \mathbf{Vect}_{\mathbb{R}}$ of real smooth vector bundles of rank 0.

We next turn towards the definition of a (pre)sheaf which can be regarded as a generalization of the concept of function spaces on topological spaces which are consistent in a certain sense under restriction to open subsets. This is crucial since supermanifolds in the algebraic sense are defined using sheaves of supercommuting rings, which play the role of algebras of (coordinate) functions on supermanifolds. To this end, for a topological space X, let us define the category $\mathbf{Opn}(X)$ of *open subsets of* X with objects $\mathbf{Ob}(\mathbf{Opn}(X))$ given by the collection of open subsets $U \subseteq X$ and, for each pair of open subsets $U, V \subseteq X$, the set of morphisms $\mathrm{Hom}(U, V)$ defined as

$$\mathrm{Hom}(U, V) := \begin{cases} \{\iota_{U,V} : U \hookrightarrow V\} & \text{if } U \subseteq V \\ \emptyset & \text{if } U \nsubseteq V \end{cases}$$

where, for $U \subseteq V$, $\iota_{U,V} : U \hookrightarrow V$ denotes the standard inclusion of U in V.

Definition B.8. Let X be a topological space. A *presheaf* $\mathcal{F}$ of sets (resp. rings, groups, modules, super rings, ...) on a topological space X is defined as a contravariant functor

$$\mathcal{F} : \mathbf{Opn}(X) \to C \tag{B.3}$$

from the category $\mathbf{Opn}(X)$ of open subsets of X to the category C with C given by the category **Set** of sets (resp. **Ring**, **Grp**, $\mathbf{Mod}_R$, super rings **SRing**,...). Hence, the presheaf $\mathcal{F}$ assigns to any open subset $U \subseteq X$ an object $\mathcal{F}(U)$ in the category C and to any two open subsets $U, V \subseteq X$ with $U \subseteq V$ a morphism $\rho_{U,V} := \mathcal{F}(\iota_{U,V}) : \mathcal{F}(V) \to \mathcal{F}(U)$ called *restriction morphism* such that

(i) $\rho_{U,U} = \mathrm{id} : \mathcal{F}(U) \to \mathcal{F}(U)\ \forall U \subseteq X$ open

(ii) $\rho_{U,V} \circ \rho_{V,W} = \rho_{U,W} : \mathcal{F}(W) \to \mathcal{F}(U)$ for open subsets $U \subseteq V \subseteq W$ of X.

The presheaf $\mathcal{F}$ is called a *sheaf* if, in addition, for any open subset $U \subseteq X$ and open covering $\{U_\alpha\}_{\alpha\in\Upsilon}$ of U the following conditions are satisfied

(i) if $f, g \in \mathcal{F}(U)$ and $\rho_{U_\alpha,U} f = \rho_{U_\alpha,U}\, g\ \forall \alpha \in \Upsilon$, then $f = g$.

(ii) if $\{f_\alpha\}_{\alpha\in\Upsilon}$ is a family of sections $f_\alpha \in \mathcal{F}(U_\alpha)$, $\alpha \in \Upsilon$, with $\rho_{U_\alpha\cap U_\beta,U} f_\alpha = \rho_{U_\alpha\cap U_\beta,U}\, g_\alpha\ \forall \alpha, \beta \in \Upsilon$, then there exists a (unique) $f \in \mathcal{F}(U)$ such that $\rho_{U_\alpha,U} f = f_\alpha\ \forall \alpha \in \Upsilon$.

In other words, a preasheaf $\mathcal{F}$ is a sheaf, if the following short sequence

$$\mathcal{F}(U) \to \prod_{\alpha\in\Upsilon} \mathcal{F}(U_\alpha) \rightrightarrows \prod_{\alpha,\alpha'\in\Upsilon} \mathcal{F}(U_\alpha \cap U_{\alpha'}) \tag{B.4}$$

is exact for any open subset $U \subseteq X$ where the arrows are defined by the obvious restriction morphisms.

Definition B.9. A morphism ϕ between (pre)sheaves $\mathcal{F}$ and $\mathcal{G}$ on a topological space X is a natural transformation $\phi : \mathcal{F} \to \mathcal{G}$ between the respective covariant functors $\mathcal{F} : \mathbf{Opn}(X)^{\mathrm{op}} \to C$ and $\mathcal{G} : \mathbf{Opn}(X)^{\mathrm{op}} \to C$ with C given by the category **Set** (resp. **Ring**, **Grp**, $\mathbf{Mod}_R$, **SRing**,...). That is, a morphism of (pre)sheaves is a collection of morphisms $\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ (in the respective category), $U \subseteq X$ open, such that $\forall U \subseteq V \subseteq X$ open, the following diagram commutes

$$\begin{array}{ccc} \mathcal{F}(V) & \xrightarrow{\phi_V} & \mathcal{G}(V) \\ {\scriptstyle \rho_{U,V}}\downarrow & & \downarrow{\scriptstyle \rho_{U,V}} \\ \mathcal{F}(U) & \xrightarrow{\phi_U} & \mathcal{G}(U) \end{array}$$

Moreover, ϕ is called an *isomorhism* if it defines a natural isomorphism between the respective functors, i.e., $\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is an isomorphism $\forall U \subseteq X$ open.

Remark B.10. A typical example of a preasheaf is the presheaf C^{∞}_M of smooth real-valued functions on a smooth manifold M (or even just continuous functions on topological spaces) which assign an open subset $U \subseteq M$ to the set $C^{\infty}(U)$ of smooth functions on U and to two open subsets $U \subseteq V \subseteq M$ the restriction map $C^{\infty}(V) \ni f \mapsto f|_U \in C^{\infty}(U)$. Since smoothness is a local property, it follows that C^{∞}_M is also a sheaf. In the main text, in particular in Chapter 2, we will also encounter further examples of sheaves in the super category. There, the sheaf property (B.4) turns out to be important as, for instance, it allows to extend statements proven locally to the whole supermanifold. There exist various (trivial) examples showing that not every presheaf is in fact a sheaf. However, it turns out that every presheaf can be extended uniquely to a sheaf using a procedure called *sheafification* (see [253] for more details).

Next, let us introduce the notion of a *inductive* and *projective family*. These play a crucial role in the context of preasheaves and sheaves and also in the framework of loop quantum (super)gravity for the definition of the (graded) holonomy-flux algebra to be discussed in Section 5.5.1. Let $(I, \leq)$ be a *partially ordered index set* equipped with a binary relation $\leq$ called *preorder* satisfying *reflexivity* and *transitivity*, that is, $i \leq i$ $\forall i \in I$ and

$$i \leq j \text{ and } j \leq k \Rightarrow i \leq k, \text{ for } i, j, k \in I \tag{B.5}$$

Moreover, we require I to be nonempty and *directed* in the sense that

$$\forall i, j \in I \, \exists k \in I : \, i \leq k \text{ and } j \leq k \tag{B.6}$$

Definition B.11. Let $(X_i)_{i \in I}$ be a family of objects in the category **Set** (resp. **Ring**, **Grp**, $\mathbf{Mod}_R$, **SRing**,…) together with

(i) morphisms (in the respective category) $p_{ij} : X_i \to X_j$ for any $i, j \in I$ with $i \leq j$ satisfying $p_{ii} = \mathrm{id}$ as well as the *compatibility condition*

$$p_{jk} \circ p_{ij} = p_{ik}, \, \forall i, j, k \in I \text{ with } i \leq j \leq k \tag{B.7}$$

then $(X_i, p_{ij})_{i,j \in I}$ is called an *inductive system*.

(ii) morphisms (in the respective category) $p_{ij} : X_j \to X_i$ for any $i, j \in I$ with $i \leq j$ satisfying $p_{ii} = \mathrm{id}$ as well as the *compatibility condition*

$$p_{ij} \circ p_{jk} = p_{ik}, \, \forall i, j, k \in I \text{ with } i \leq j \leq k \tag{B.8}$$

then $(X_i, p_{ij})_{i,j \in I}$ is called a *projective system*.

Definition B.12. (i) Let $(X_i, p_{ij})_{i,j \in I}$ be an inductive system of objects in the category C. An object $X \in \mathbf{Ob}(C)$ together with morphisms $p_i : X_i \to X$

is called an *inductive limit* of $(X_i, p_{ij})_{i,j\in I}$ if it satisfies the following *universal property*: For any $Y \in \mathbf{Ob}(C)$ and morphisms $g_i : X_i \to Y, i \in I$, such that $g_i = g_j \circ p_{ij} \; \forall i \leq j$ there exists an unique morphism $g : X \to Y$ such that $g_i = g \circ p_i \; \forall i \in I$.

It follows from the universal property that an inductive limit, if it exists, will be unique and will be often denoted by

$$X =: \varinjlim X_i \tag{B.9}$$

(ii) Let $(X_i, p_{ij})_{i,j\in I}$ be a projective system objects in the category C. An object $X \in \mathbf{Ob}(C)$ together with morphisms $f_i : X \to X_i$ is called a *projective limit* of $(X_i, p_{ij})_{i,j\in I}$ if it satisfies the following *universal property*: For any $Y \in \mathbf{Ob}(C)$ and morphisms $g_i : Y \to X_i, i \in I$, such that $g_i = p_{ij} \circ g_j \; \forall i \leq j$ there exists an unique morphism $g : Y \to X$ such that $g_i = p_i \circ g \; \forall i \in I$.

It follows from the universal property that a projective limit, if it exists, will be unique and will be often denoted by

$$X =: \varprojlim X_i \tag{B.10}$$

Proposition B.13. *Let $(X_i, p_{ij})_{i,j\in I}$ be an inductive (resp. a projective) system, then the inductive (resp. projective) limit exists in the category* **Set**, **Ring**, **Grp**, $\mathbf{Mod}_R$ *and* **SRing**.

Sketch of proof. In case $(X_i, p_{ij})_{i,j\in I}$ defines an inductive system of objects in the category **Set**, one may define the inductive limit X as the quotient

$$X \equiv \varinjlim X_i := \coprod_{i\in I} X_i \Big/ \sim \tag{B.11}$$

where two elements $x_i \in X_i$ and $x_j \in X_j$ with $i, j \in I$ are defined as equivalent, in symbol $x_i \sim x_j$, iff there exists $k \in I$ with $i, j \leq k$ such that $p_{ik}(x_i) = p_{jk}(x_j)$. Using the compatibility condition of the morphisms $p_{ij} : X_i \to X_j$ it is easy to see that this indeed defines an equivalence relation. For any $i \in I$, the canonical embeddings $X_i \hookrightarrow \coprod_k X_k$ induce maps $p_i : X_i \to X$. It then follows X together with the morphisms p_i indeed satisfies the properties of an inductive limit in the category **Set**.

In case that the X_i carry additional structure, i.e., they define objects in the category **Ring**, **Grp**, $\mathbf{Mod}_R$ or **SRing** one can apply a standard procedure and use the morphisms p_{ij} and f_i to extend these additional structures to the inductive limit X as defined via

(B.11) so that X in fact defines an object in the respective category (see [253] for more details).

The projective case is simpler. More precisely, if $(X_i, p_{ij})_{i,j\in I}$ defines a projective system, one may define the corresponding projective limit X as the subset

$$X \equiv \varprojlim X_i := \{(x_i)_i \in \prod_{i\in I} X_i |\, p_{ij}(x_j) = x_i\, \forall i \leq j\} \tag{B.12}$$

of the cartesian product $\prod_i X_i$. Restricting the canonical projections $\prod_k X_k \to X_i$ to the subset X, this induces maps $p_i : X \to X_i$ which, in particular, automatically define morphisms in the respective category. By construction, it then follows immediately that X together with the morphisms p_i indeed satisfies the properties of a projective limit. □

Let X be a topological space and $x \in X$ an arbitrary but fixed point. Then, x induces a partially ordered set $(\mathcal{U}_x, \leq)$ of open subsets $U \subseteq X$ with $x \in U$ where $U \leq V :\Leftrightarrow U \subseteq V$ for U, V open in X. Given a presheaf $\mathcal{F}$ on a topological space X and $x \in X$, this yields an inductive system $(\mathcal{F}(U), p_{UV} := \rho_{V,U})_{U,V\in\mathcal{U}_x}$. With these preparations, we can define the following.

Definition B.14. Let $\mathcal{F}$ be a presheaf on a topological space X and $x \in X$ an arbitrary but fixed point. The *stalk* $\mathcal{F}_x$ of $\mathcal{F}$ at x is defined as the inductive limit

$$\mathcal{F}_x := \varinjlim \mathcal{F}(U) \tag{B.13}$$

of the inductive system $(\mathcal{F}(U), p_{UV})_{U,V\in\mathcal{U}_x}$.

Proposition B.15. *Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves on a topological space X. Then, for any $x \in X$, there exists a unique morphism $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ between the stalks at x such that the following diagram commutes*

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\phi_U} & \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}_x & \xrightarrow{\phi_x} & \mathcal{G}_x \end{array} \tag{B.14}$$

Proof. Composing ϕ_U with the morphism $p_U : \mathcal{G}(U) \to \mathcal{G}_x$ for any $U \subseteq X$ open, it follows that the resulting morphisms $\psi_U := p_U \circ \phi_U : \mathcal{F}(U) \to \mathcal{G}_x$ satisfy

$$\psi_U \circ p_{VU} = p_U \circ \phi_U \circ \rho_{U,V} = p_U \circ \rho_{U,V} \circ \phi_V = p_U \circ p_{VU} \circ \phi_V = p_V \circ \phi_V = \psi_V \tag{B.15}$$

for any $U \subseteq V$ open. Hence, by universal property of the inductive limit, there exists an unique morphism $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ such that

$$p_U \circ \phi_U = \phi_x \circ p_U \tag{B.16}$$

$\forall x \in U \subseteq X$ open, that is, such that the diagram (B.14) commutes. □

Before, we finally turn towards the introduction of the notion of a locally ringed space, let us prove an important proposition stating that a morphism between sheaves is uniquely characterizeed in terms of its restrictions on a certain base of open subsets of the underlying topological space. This allows to extend local data to a global object. An example of such a base of open subsets that we will typically be interested in are local coordinate neighborhoods on a (super)manifold. The result of the following proposition has been implicitly used in the arguments of the Reference [95]. In the following, we want to state it more formally and give an explicit proof.

Definition B.16. Let X be a topological space. A *base of the topology* of X is a system $\mathfrak{B}$ of open subsets of X such that

(i) $U, V \in \mathfrak{B} \Rightarrow U \cap V \in \mathfrak{B}$

(ii) every open subset of X is a union of subsets from $\mathfrak{B}$.

Lemma B.17. *Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of sets (resp. rings, modules, super rings,...) on a topological space X and $\mathfrak{B}$ be a base of the topology of X. Furthermore, let $K := \{K_U\}_{U\in\mathfrak{B}}$ be collection of morphisms $K_U : \mathcal{F}(U) \to \mathcal{G}(U)$ commuting with restrictions, i.e. $\rho_{U,V} K_V = K_U \rho_{U,V}$ for $U, V \in \mathfrak{B}$ with $U \subset V$. Then, K can uniquely be extended to a sheaf morphism $\bar{K} : \mathcal{F} \to \mathcal{G}$ such that $\bar{K}_U = K_U$ for all $U \in \mathfrak{B}$.*

Proof. It is clear, by the uniqueness property of sheaves, that such an extension of K, provided it exists, will be unique. Hence, we only have to prove its existence. To this end, let $W \subseteq X$ be an arbitrary open subset. Then, there exists a collection $\{U_\alpha\}_{\alpha\in\Upsilon}$ of open subsets in $\mathfrak{B}$ such that $W = \bigcup_{\alpha\in\Upsilon} U_\alpha$. For $f \in \mathcal{F}(W)$, consider the sections $\tilde{f}_\alpha := K_{U_\alpha} \rho_{U_\alpha,W} f \in \mathcal{G}(U_\alpha)$, $\alpha \in U_\alpha$. Since

$$\begin{aligned} \tilde{f}_\alpha|_{U_\alpha\cap U_\beta} &= \rho_{U_\alpha\cap U_\beta,U_\alpha} K_{U_\alpha} \rho_{U_\alpha,W} f = K_{U_\alpha\cap U_\beta} \rho_{U_\alpha\cap U_\beta,W} f \\ &= \rho_{U_\alpha\cap U_\beta,U_\beta} K_{U_\beta} \rho_{U_\beta,W} f = \tilde{f}_\beta|_{U_\alpha\cap U_\beta} \end{aligned} \tag{B.17}$$

$\forall \alpha, \beta \in \Upsilon$, by the sheaf property of $\mathcal{G}$, there exists a unique $\tilde{f} \in \mathcal{G}(W)$ with $\tilde{f}|_{U_\alpha} = \tilde{f}_\alpha$ $\forall \alpha \in \Upsilon$. We define the map $\bar{K}_W : \mathcal{F}(W) \to \mathcal{G}(W)$ by setting $\bar{K}_W(f) := \tilde{f}$

$\forall f \in \mathcal{F}(W)$. Let $U \in \mathfrak{B}$ with $U \subset W$, then $\{U \cap U_\alpha\}_{\alpha\in\Upsilon}$ is an open covering of U. Since

$$\begin{aligned}\rho_{U\cap U_\alpha,W}\bar{K}_W(f) &= \rho_{U\cap U_\alpha,U_\alpha}\rho_{U_\alpha,W}\bar{K}_W(f) = \rho_{U\cap U_\alpha,U_\alpha}\tilde{f}_\alpha \\ &= \rho_{U\cap U_\alpha,U_\alpha}K_{U_\alpha}(\rho_{U_\alpha,W}f) = K_{U\cap U_\alpha}(\rho_{U\cap U_\alpha,W}f) \\ &= \rho_{U\cap U_\alpha,U}K_U(\rho_{U,W}f) \end{aligned} \tag{B.18}$$

$\forall \alpha \in \Upsilon$, this shows, by the uniqueness property of sheaves, that $\rho_{U,W}\bar{K}_W(f) = K_U(\rho_{U,W}f)\ \forall f \in \mathcal{F}(W)$. In particular, this implies that the definition of $\bar{K}_W$ is independent on the choice of a covering $\{U_\alpha\}$ of W. Finally, the uniqueness property also yields that $\bar{K}_W$ defines a morphism in the respective category.

It follows that the collection $\bar{K} := \{\bar{K}_W\}_W$ indexed by open subsets $W \subseteq X$ defines a morphism of sheaves $\bar{K} : \mathcal{F} \to \mathcal{G}$. Indeed, let $V, W \subseteq X$ be open subsets with $V \subset W$ and $\{U_\alpha\}_{\alpha\in\Upsilon}$ be an open covering of V with $U_\alpha \in \mathfrak{B}\ \forall \alpha \in \Upsilon$. Then, for $f \in \mathcal{F}(W)$, we compute

$$\rho_{U_\alpha,V}\rho_{V,W}\bar{K}_W(f) = \rho_{U_\alpha,W}\bar{K}_W(f) = K_{U_\alpha}\rho_{U_\alpha,W}f = \rho_{U_\alpha,V}\bar{K}_V(\rho_{V,W}f) \tag{B.19}$$

for any $\alpha \in \Upsilon$, so that, again by the uniqueness property, we can conclude $\rho_{V,W}\bar{K}_W(f) = \bar{K}_V(\rho_{V,W}f)$ proving that $\bar{K}$ is a morphism of sheaves. Hence $\bar{K}$ defines the unique extension of K. □

Definition B.18. A *locally ringed space* is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X as well as a sheaf $\mathcal{O}_X$ of rings on X called *structure sheaf* such that, for any $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a local ring.[1] A morphism $f = (|f|, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of locally ringed spaces consists of a continuous map $|f| : X \to Y$ between topological spaces as well as a morphism $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves of rings on Y called *pullback* such that, $\forall x \in X$, the induced morphism $f^\sharp_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ (see Def. B.14 and Prop B.15) is local, i.e., $f^\sharp_x$ maps the maximal ideal of $\mathcal{O}_{Y,f(x)}$ to the maximal ideal of $\mathcal{O}_{X,x}$. Here, $f_*\mathcal{O}_Y$ is a sheaf over X called the *pushforward of* $\mathcal{O}_Y$ w.r.t. f given by $f_*\mathcal{O}_Y(U) := \mathcal{O}_Y(f^{-1}(U))$, $U \subseteq X$ open.

Example B.19. A smooth manifold M naturally induces a locally ringed space (M, C^∞_M) with C^∞_M the sheaf of smooth functions on M. For each $x \in M$, the maximal ideal of the stalk $C^\infty_{M,x}$ is given by $I_x := \{[f]_x \in C^\infty_{M,x} | f(x) = 0, f \in [f]_x\}$. Moreover, choosing a local coordinate neighborhood U of M, it follows that $(U, \mathcal{O}_M|_U)$ is isomorphic to the locally ringed space $(V, C^\infty_{\mathbb{R}^n}|_V)$ where $V \subseteq \mathbb{R}^n$ is an open subset of $\mathbb{R}^n$.

[1] an ideal $\mathfrak{m}$ of a ring R is called *maximal* if $R/\mathfrak{m}$ is a field. A ring is called *local* if it has a unique maximal ideal.

In fact, it follows that smooth manifolds are uniquely characterized by this property. More precisely, it follows that there exists an equivalence of categories between **Man** and the category of locally ringed spaces that are locally isomorphic to the flat model $(V, C^{\infty}_{\mathbb{R}^n}|_V)$ with $V \subseteq \mathbb{R}^n$ open (see, e.g., [117, 253] for more details).

C. Rogers-DeWitt supermanifolds

In this section, we want to briefly review the basic definition of H^{∞} supermanifolds in the Rogers-DeWitt approach in order to fix some terminology used in the main text. We mostly follow the standard References [94, 96, 97].

Definition C.1. Let Λ be a Grassmann algebra (finite- or infinite-dimensional) and $\Lambda^{m,n} := \Lambda_0^m \times \Lambda_1^n$ be the *superdomain* of dimension (m, n) for any $m, n \in \mathbb{N}_0$. On $\Lambda^{m,n}$ one has the *body map* given by the projection $\epsilon_{m,n} : \Lambda^{m,n} \to \mathbb{R}^m$ onto the real subspace $\mathbb{R}^m$. We equip $\Lambda^{m,n}$ with the *DeWitt-topology* defined as the coarsest topology such that the body map is continuous. For any open subset $U \subseteq \mathbb{R}^m$, a function $f : \epsilon_{m,n}^{-1}(U) \to \Lambda$ is called *(H^{∞}-)smooth*, if there exists ordinary real smooth functions $f_{\underline{I}} \in C^{\infty}(U)$ for any ordered multi-index $\underline{I}$ of length $0 \leq |\underline{I}| \leq n$, such that

$$f(x, \theta) = \sum_{\underline{I}} \mathbf{G}(f_{\underline{I}})(x)\theta^{\underline{I}} \tag{C.1}$$

$\forall (x, \theta) \in \epsilon_{m,n}^{-1}(U)$ where $\mathbf{G}(f_{\underline{I}})$ is the so-called *Grassmann analytic continuation* or simply *Grassmann extension* of $f_{\underline{I}}$ defined as

$$\mathbf{G}(f_{\underline{I}})(x) := \sum_{\underline{J}} \frac{1}{\underline{J}!} \partial_{\underline{J}} f_{\underline{I}}(\epsilon_{m,n}(x)) s(x)^{\underline{J}} \tag{C.2}$$

where the sum runs over all unordered multi-indices $\underline{J}$, $s(x) := x - \epsilon_{m,n}(x)$ is the *soul* of $x \in \Lambda_0^m$ and

$$\partial_{\underline{I}} := \frac{\partial^{|\underline{I}|}}{\partial x^{\underline{I}}} \equiv \frac{\partial^{|\underline{I}|}}{\partial x_1^{i_1} \cdots \partial x_k^{i_k}} \tag{C.3}$$

for some multi-index $\underline{I} = (i_1, \ldots, i_k)$.

Remark C.2. Following [97], a topological space M together with a C^{∞}-smooth structure will be called a *proto C^{∞} manifold*. That is, a proto C^{∞} manifold is just an ordinary smooth manifold without requiring the underlying topological space to be Hausdorff and second countable.

Definition C.3. Let $\mathcal{M}$ be a topological space. A local *superchart* on $\mathcal{M}$ is defined as a pair (U, ϕ_U) that consists of an open subset $U \subseteq \mathcal{M}$ as well as a homeomorphism

$\phi_U : U \to \phi_U(U) \subseteq \Lambda^{m,n}$ onto an open subset of the superdomain $\Lambda^{m,n}$. A (m,n)-dimensional H^∞-smooth atlas on $\mathcal{M}$ is a family $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in\Upsilon}$ of supercharts $\phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha) \subseteq \Lambda^{m,n}$, $\alpha \in \Upsilon$, of $\mathcal{M}$ such that $\bigcup_{\alpha\in\Upsilon} U_\alpha = \mathcal{M}$ and the supercharts are smoothly compatible. That is, for any $\alpha, \beta \in \Lambda$ with $U_\alpha \cap U_\beta \neq \emptyset$, the *transition functions*

$$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta) \tag{C.4}$$

are of class H^∞. An atlas on $\mathcal{M}$ is called *maximal* if any superchart (U, ϕ_U) of $\mathcal{M}$ that is smoothly compatible with any superchart of the given atlas is already contained in this atlas.

Definition C.4. A *proto H^∞ supermanifold* of dimension (m,n) is a topological space $\mathcal{M}$ furnished with a (m,n)-dimensional H^∞-smooth atlas.

Definition C.5. Let $\mathcal{M}$ and $\mathcal{N}$ be proto H^∞ supermanifolds. A continuous map $f : \mathcal{M} \to \mathcal{N}$ is called *smooth* if, for any local charts (U, ϕ_U) and (V, ψ_V) of $\mathcal{M}$ and $\mathcal{N}$, respectively, with $U \cap f^{-1}(V) \neq \emptyset$, the map

$$\psi_V \circ f \circ \phi_U^{-1} : \phi_U(U \cap f^{-1}(V)) \to \psi_V(V) \tag{C.5}$$

is of class H^∞.

Definition C.6. Let $\mathcal{M}$ be a proto H^∞ supermanifold of dimension $\dim \mathcal{M} = (m,n)$ with maximal atlas $\left\{(V_\alpha, \psi_\alpha)\right\}_{\alpha\in\Gamma}$. On $\mathcal{M}$, one can introduce an equivalence relation $\sim$ via (for a proof that this indeed defines an equivalence relation see [96])

$$p \sim q :\Leftrightarrow \exists \alpha \in \Gamma : \; p, q \in V_\alpha \text{ and } \epsilon_{m,n}(\psi_\alpha(p)) = \epsilon_{m,n}(\psi_\alpha(q)) \tag{C.6}$$

It immediately follows from the definition that, for any $p \in \mathcal{M}$, there exists a unique element $\mathbf{B}(p) \in \mathcal{M}$ such that $p \sim \mathbf{B}(p)$ and $f(p) \in \mathbb{R}\; \forall f \in H^\infty(\mathcal{M})$ which we call its *body representative*. This yields a proper subset

$$\mathbf{B}(\mathcal{M}) := \{p \in \mathcal{M} | f(p) \in \mathbb{R}, \forall f \in H^\infty(\mathcal{M})\} \tag{C.7}$$

called the *body* of $\mathcal{M}$, together with a surjective map $\mathbf{B} : \mathcal{M} \to \mathbf{B}(\mathcal{M})$, $p \mapsto \mathbf{B}(p)$ called the *body map* where elements $p \in \mathbf{B}(\mathcal{M})$ will also be referred to as *body points* of $\mathcal{M}$. For any smooth map $f : \mathcal{M} \to \mathcal{N}$ between H^∞ supermanifolds, one has $f(\mathbf{B}(\mathcal{M})) \subseteq \mathbf{B}(\mathcal{N})$. Hence, the body map can be extended to morphisms setting $\mathbf{B}(f) := f|_{\mathbf{B}(\mathcal{M})} : \mathbf{B}(\mathcal{M}) \to \mathbf{B}(\mathcal{N})$. This yields a functor $\mathbf{B} : \mathbf{SMan}_{proto,H^\infty} \to \mathbf{Set}$ which we call the *body functor*.

Lemma C.7. *Let $p \in \mathcal{M}$ be a point on a proto H^∞ supermanifold $\mathcal{M}$ and $\sim$ the equivalence relation as in definition C.6. Then, any other point $q \in \mathcal{M}$ in the equivalence class of p will be contained in an arbitrary small open neighborhood U of p in $\mathcal{M}$.*

Proof. Let $p \in U$ be an open neighborhood of p in $\mathcal{M}$ and (V_α, ψ_α) a coordinate neighborhood of p with $q \in V_\alpha$. Then, $p \in U \cap V_\alpha$ and $\psi_\alpha(U \cap V_\alpha)$ is open in $\Lambda^{m,n}$ so that, by the definition of the DeWitt-topology, there is a $\tilde{U} \subset \mathbb{R}^m$ open with $\psi_\alpha(U \cap V_\alpha) = \epsilon_{m,n}^{-1}(\tilde{U})$. As $p \sim q$ we have $\epsilon_{m,n}(\psi_\alpha(q)) = \epsilon_{m,n}(\psi_\alpha(p)) \in \tilde{U}$ and thus $\psi_\alpha(q) \in \psi_\alpha(U \cap V_\alpha)$. But, since ψ_α is injective this implies $q \in U \cap V_\alpha$ and therefore $q \in U$. □

Lemma C.8. *Let $\mathcal{M}$ be a proto H^∞ supermanifold and $U, V \subseteq \mathcal{M}$ open subsets in $\mathcal{M}$. Then $U \subseteq V \Leftrightarrow \mathbf{B}(U) \subseteq \mathbf{B}(V)$ and thus, in particular, $U = V \Leftrightarrow \mathbf{B}(U) = \mathbf{B}(V)$. Moreover, any open subset $U \subseteq \mathcal{M}$ is of the form $U = \mathbf{B}^{-1}(W)$ with W open in $\mathbf{B}(\mathcal{M})$ (in fact $W = \mathbf{B}(U)$) which can be thought of as a generalization of the DeWitt-topology.*

Proof. It is clear that $U \subseteq V$ implies $\mathbf{B}(U) \subseteq \mathbf{B}(V)$. Hence, suppose that $\mathbf{B}(U) \subseteq \mathbf{B}(V)$. Then $x \in U$ yields $\mathbf{B}(x) \in \mathbf{B}(U) \subseteq \mathbf{B}(V)$ such that there exists $y \in V$ with $x \sim y$. But, by Lemma C.7, this implies $x \in V$ and thus indeed $U \subseteq V$. Next, for any open subset $U \subseteq \mathcal{M}$ define $U' := \mathbf{B}^{-1}(\mathbf{B}(U))$ yielding $U \subseteq U'$. For any $x \in V$ one has $\mathbf{B}(x) \in \mathbf{B}(V) = \mathbf{B}(U)$. Thus, similarly as above, this implies $x \in U$ as U is open and therefore $U' \subseteq U$, that is, $U' = U$. □

Definition C.9. A H^∞ supermanifold $\mathcal{M}$ is a proto H^∞ supermanifold whose body $\mathbf{B}(\mathcal{M})$, equipped with the trace toplogy, defines a second countable Hausdorff topological space and thus defines an ordinary C^∞-smooth manifold. Supermanifolds together with smooth maps between them form a category $\mathbf{SMan}_{H^\infty}$ called the *category of H^∞ supermanifolds*.

Proposition C.10. *Let $f: \mathcal{M} \to \mathcal{N}$ be a smooth map between H^∞ supermanifolds $\mathcal{M}$ and $\mathcal{N}$. Then, f is surjective if and only if $\mathbf{B}(f): \mathbf{B}(\mathcal{M}) \to \mathbf{B}(\mathcal{N})$ is surjective.*

Proof. Let $M := \mathbf{B}(\mathcal{N})$ and $N := \mathbf{B}(\mathcal{N})$. If f is surjective, it follows $\mathbf{B}(f)(M) = \mathbf{B}(f(\mathcal{M})) = \mathbf{B}(\mathcal{N}) = N$ and thus $\mathbf{B}(f)$ is surjective. Conversely, if $\mathbf{B}(f)$ be surjective, it follows from $N = \mathbf{B}(f)(M) = \mathbf{B}(f(\mathcal{M}))$ that $\mathbf{B}(f(\mathcal{M}))$ is open and thus $f(\mathcal{M})$ is open in $\mathcal{N}$ implying $\mathbf{B}(f(\mathcal{M})) = \mathbf{B}(f)(M) = N = \mathbf{B}(\mathcal{N})$. Hence, by Lemma C.8, this yields $f(\mathcal{M}) = \mathcal{N}$, that is, f is surjective. □

Example C.11. To any vector bundle $V \to E \to M$ with $\dim V = n$ and $\dim M = m$, it follows that one can associate a H^∞ supermanifold $\mathbf{S}(E, M)$ of dimension (m, n)

called the *split supermanifold*[2] Moreover, any morphism $(\phi, f) : (E, M) \to (F, N)$ between two vector bundles induces a morphism $\mathbf{S}(\phi, f) : \mathbf{S}(E, M) \to \mathbf{S}(F, N)$ between the corresponding split supermanifolds. Hence, this yields a functor

$$\mathbf{S} : \mathbf{Vect}_{\mathbb{R}} \to \mathbf{SMan}_{H^\infty} \tag{C.8}$$

between the category of real vector bundles to the category of H^∞ supermanifolds which we call the *split functor*. It is a general result due to Batchelor [107] that any algebro-geometric supermanfold is isomorphic to a split supermanifold, i.e., (C.8) is surjective on objects. However, the split functor is *not* full, i.e., not every morphism $f : \mathbf{S}(E, M) \to \mathbf{S}(F, N)$ between split manifolds arises from a morphism between the respective vector bundles $(E, M), (F, N) \in \mathbf{Ob}(\mathbf{Vect}_{\mathbb{R}})$. Hence, the structure of morphisms between supermanifolds in general turns out to be much richer than for ordinary vector bundles.

A smooth manifold M can be identified with the trivial vector bundle $M \times \{0\} \to M$, i.e., one has the *inclusion functor* $\mathbf{i} : \mathbf{Man} \to \mathbf{Vect}_{\mathbb{R}}$ such that $\mathbf{i}(M) := (M \times \{0\}, M)$ and $\mathbf{i}(f) := (\mathbf{0}, f)$ for any $M \in \mathbf{Ob}(\mathbf{Man})$ and morphisms $f \in \mathrm{Hom}_{\mathbf{Man}}(M, N)$. Combined with the split functor, this yields another functor

$$\mathbf{S} \equiv \mathbf{S} \circ \mathbf{i} : \mathbf{Man} \to \mathbf{SMan}_{H^\infty} \tag{C.9}$$

mapping an ordinary smooth manifold M to a supermanifold $\mathbf{S}(M)$ with trivial odd dimensions, also called a *bosonic supermanifold*. Conversely, given a bosonic supermanifold $\mathcal{M}$, it follows that the corresponding split supermanifold $\mathcal{M}_0 := \mathbf{S}(\mathbf{B}(\mathcal{M}))$ is isomorphic to $\mathcal{M}$. Hence, in this way, one obtains an equivalence of categories between bosonic supermanifolds and ordinary C^∞-smooth manifolds.

D. Irreducible representations of $\mathrm{OSp}(\mathcal{N}|2)$

In the following, let us summarize some main results about finite-dimensional irreducible representations of the orthosymplectic Lie supergroups $\mathrm{OSp}(\mathcal{N}|2)$ for $\mathcal{N} = 1, 2$ which play a role in the chiral description of the pure AdS supergravity theories in $D = 4$ to be discussed in Chapter 5. To this end, we will discuss the representation theory on the level of corresponding Lie superlalgebras $\mathfrak{osp}(\mathcal{N}|2)$. The respective representations of the underlying supergroup can be obtained using the super Harish-Chandra isomorphism (2.45).

[2] the explicit construction turns out to be a bit technical via Grassmann extensions of transition functions (see e.g. [97]). Alternatively, the split supermanifold may be obtained via the functor of points prescription applied to the algebro-geometric supermanifold $(M, \Gamma(\bigwedge E^*))$ (see Example 2.2.4 for more details).

D.1. Representation theory of OSp(1|2)

The finite-dimensional irreducible representations of $\mathfrak{osp}(1|2)$ are discussed in various references. Here, we follow [207, 216, 254, 255]. Starting from the general definition (5.82)-(5.85) of $\mathfrak{osp}(\mathcal{N}|2)$ for $\mathcal{N} = 1$ generated by (T_i^+, Q_A) with $i \in \{1, 2, 3\}$ and $A \in \{\pm\}$, one can arrive at the Cartan-Weyl basis $(J_3, J_\pm, V_\pm)$ of the superalgebra setting

$$J_\pm := -i(T_1^+ \pm iT_2^+), \quad J_3 := iT_3^+, \quad V_\pm := \pm\frac{\sqrt{L}}{2}(i-1)Q_\pm \tag{D.1}$$

It then follows from (5.82)-(5.85) that the commutators among the even generators satisfy

$$[J_3, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = 2J_3 \tag{D.2}$$

which are the standard commutation relations of $\mathfrak{sl}(2,\mathbb{C})$. For the remaining commutators, it follows

$$[J_3, V_\pm] = \pm\frac{1}{2}V_\pm, \quad [J_\mp, V_\pm] = V_\mp, \quad [J_\pm, V_\pm] = 0 \tag{D.3}$$

$$[V_\pm, V_\pm] = \pm\frac{1}{2}J_\pm, \quad [V_+, V_-] = -\frac{1}{2}J_3 \tag{D.4}$$

The quadratic Casimir operator C_2 which commutes with all the generators of the super Lie algebra $\mathfrak{osp}(1|2)$ takes the form [216]

$$C_2 = \vec{J}^2 + V_+V_- - V_-V_+ \tag{D.5}$$

where $\vec{J} := (J_1, J_2, J_3)^T$ with $J_1 := \frac{1}{2}(J_+ + J_-)$ and $J_2 = \frac{1}{2i}(J_+ - J_-)$. A finite-dimensional representation of $\mathfrak{osp}(1|2)$ is a grading preserving superalgebra morphism

$$\pi : \mathfrak{osp}(1|2) \to \mathfrak{gl}(V) \tag{D.6}$$

with $\mathfrak{gl}(V) \equiv \underline{\mathrm{End}}(V)$ the super Lie algebra of endomorphisms on a finite-dimensional super vector space $V = V_0 \oplus V_1$. By restriction, each irreducible representation of $\mathfrak{osp}(1|2)$ induces a corresponding (reducible) representation of the bosonic subalgebra. Let (ρ^j, W^j) with $j \in \frac{1}{2}\mathbb{N}_0$ denote the finite-dimensional irreducible representations of $\mathfrak{sl}(2,\mathbb{C})$. Set

$$V^{\lambda,j} := \Pi^\lambda W^j \tag{D.7}$$

where Π : **SVec** $\to$ **SVec** denotes the *parity functor* on the category **SVec** of (real) super vector spaces which, on objects $\mathbb{R}^{m|n} \in \mathbf{Ob}(\mathbf{SVec})$, is defined as $\Pi(\mathbb{R}^{m|n}) := \mathbb{R}^{n|m}$. Thus, $V^{\lambda,j}$ for $\lambda = 0$ resp. $\lambda = 1$ is regarded as a purely even resp. odd super vector space. It follows that finite-dimensional irreducible representations of $\mathfrak{osp}(1|2)$ are of the form (π^j, V^j) with

$$V^j := V^{\lambda,j} \oplus V^{\lambda+1,j-1} \tag{D.8}$$

for $j \in \frac{1}{2}\mathbb{N}_0$. A homogeneous basis of the super vector space is provided by states of the form

$$|j, j, m, \lambda\rangle\,, \quad |j, j-1, m, \lambda+1\rangle \tag{D.9}$$

with m the magnetic quantum number which, in the former case, takes values $m \in \{-j, -j+1, \ldots, j\}$ and, in the latter, $m \in \{-j+1, -j+2, \ldots, j-1\}$. Hence, in particular, it follows that (π^j, V^j) has (ungraded) dimension $4j+1$. The irreducible representations are classified by the quadratic Casimir operator (D.5) which, when restricted to the representation spaces (D.8), is given by

$$C_2 = j\left(j + \frac{1}{2}\right)\mathbb{1} \tag{D.10}$$

The equivalence classes of irreducible representations of the form (π^j, V^j) with $j \in \frac{1}{2}\mathbb{N}_0$ form a subcategory which is closed under tensor product. In order to derive the Clesch-Gordan decomposition of the tensor product of two such representations, one needs to define a suitable inner product on the representation space w.r.t. which the operators representing the generators J_i are self-adjoint and two inequivalent irreps are orthogonal. For the Lie superalgebra $\mathfrak{osp}(1|2)$, it follows that this requires a generalization of the adjointness relation in a suitable sense leading to the notion of a so-called *grade star representation* [216, 254]. It follows that, given two irreducible representations (π^{j_1}, V^{j_1}) and (π^{j_2}, V^{j_2}), the tensor product representation decomposes as [216]

$$\pi^{j_1} \otimes \pi^{j_2} = \bigoplus_{j=|j_1-j_2|}^{j_1+j_2} \pi^j \tag{D.11}$$

which almost looks like as in the bosonic theory with the crucial difference that the direct sum runs over all positive *half-integers* j satisfying the inequality $|j_1 - j_2| \leq j \leq j_1 + j_2$.

D.2. Representation theory of OSp(2|2)

In this section, we follow the References [216, 217] to discuss the finite irreducible representations of $\mathfrak{osp}(2|2)$. Again, starting from the general definition (5.82)-(5.85) of $\mathfrak{osp}(\mathcal{N}|2)$ for the special case $\mathcal{N} = 2$ generated by (T_i^+, T^{12}, Q_A^r) with $i \in \{1, 2, 3\}$, $A \in \{\pm\}$ and $r = 1, 2$, one can arrive at the Cartan-Weyl basis $(J_3, J_\pm, Q, V_\pm^r)$ of the superalgebra setting $Q := iLT^{12}$ also called *charge* together with (5.211) for the bosonic generators and

$$V^1_\pm := \pm\frac{\sqrt{L}}{2}(i-1)(Q^1_\pm - iQ^2_\pm) \text{ and } V^2_\pm := \pm\frac{\sqrt{L}}{2}(i-1)(Q^1_\pm + iQ^2_\pm) \tag{D.12}$$

for the fermionic generators, respectively. The nontrivial commutation relations among the even generators are again given by (D.2) corresponding to the Lie algebra $\mathfrak{sl}(2,\mathbb{C})\oplus\mathbb{C}$ which is the complexification of the corresponding compact real form $\mathfrak{su}(2)\oplus\mathfrak{u}(1)$. The mixed commutators between even and odd generators are given by

$$[J_3, V^r_\pm] = \pm\frac{1}{2}V^r_\pm, \quad [J_\mp, V^r_\pm] = V^r_\mp, \quad [J_\pm, V^r_\pm] = 0 \tag{D.13}$$

$$[Q, V^1_\pm] = \frac{1}{2}V^1_\pm, \qquad [Q, V^2_\pm] = -\frac{1}{2}V^2_\pm \tag{D.14}$$

and, finally, for the odd generators, one obtains

$$[V^r_\pm, V^r_\pm] = [V^r_\pm, V^r_\mp] = 0, \quad [V^1_\pm, V^2_\pm] = \pm J_\pm, \quad [V^1_\pm, V^2_\mp] = -J_3 + Q \tag{D.15}$$

These are precisely the graded commutation relations as stated for instance in [216, 217]. The superalgebra admits two Casimir operators which commute with all the generators given by the quadratic and cubic Casimir operators C_2 and C_3, respectively. For instance, the quadratic Casimir operator takes the form [216]

$$C_2 = \vec{J}^2 - Q^2 + \frac{1}{2}(V^1_+V^2_- - V^1_-V^2_+ - V^2_+V^1_- - V^2_-V^1_+) \tag{D.16}$$

where, again, $\vec{J} := (J_1, J_2, J_3)^T$. Let $(\rho^{(j,q)}, W^{(j,q)})$ with $j \in \frac{1}{2}\mathbb{N}_0$ and $q \in \mathbb{C}$ denote the finite-dimensional irreducible representations of the bosonic subalgebra $\mathfrak{sl}(2,\mathbb{C})\oplus\mathbb{C}$. The finite-dimensional representations of $\mathfrak{osp}(2|2)$ are more complicated than for the non-extended case $\mathfrak{osp}(1|2)$. In fact, the representation fall into two different categories called *typical* and *atypical representations* [216]. The typical representations $(\pi^{(j,q)}, V^{(j,q)})$ labeled by isospin $j \in \frac{1}{2}\mathbb{N}_0$ and charge quantum number $q \in \mathbb{C}$ with $j \neq \pm q$ are irreducible and classified by the Casimir operators C_2 and C_3. They consist of four $\mathfrak{sl}(2,\mathbb{C})\oplus\mathbb{C}$ multiplets such that, up to parity (see discussion below),

$$V^{(j,q)} = W^{(j,q)} \oplus W^{(j-\frac{1}{2},q-\frac{1}{2})} \oplus W^{(j-\frac{1}{2},q+\frac{1}{2})} \oplus W^{(j-1,q)} \tag{D.17}$$

with homogeneous basis given by states of the form

$$|q, j, m\rangle, \quad |q-\frac{1}{2}, j-\frac{1}{2}, m\rangle, \quad |q+\frac{1}{2}, j-\frac{1}{2}, m\rangle, \quad |q, j-1, m\rangle \tag{D.18}$$

Consequently, the typical representation has (ungraded) dimension $8j$. When applied on the states (D.18), the Casimir operators respectively take the form $C_2 = j^2 - q^2$ and $C_3 = q(j^2 - q^2)$.

On the other hand, in case of the atypical representations $(\pi^{(j,q)}, V^{(j,q)})$ corresponding to the special cases $j = \pm q$, it follows that the Casimir operators simply vanish and therefore cannot be used for their classification. The atypical representations fall into two subcategories, the so-called *atypical irreducible* and the *atypical not fully reducible* ones. While the latter are more complicated to describe (see e.g. [216, 217] for more details) the former type of representations split into two $\mathfrak{sl}(2,\mathbb{C}) \oplus \mathbb{C}$ multiplets of the form

$$V^{(j,\pm j)} = W^{(j,\pm j)} \oplus W^{(j-\frac{1}{2},\pm j\pm\frac{1}{2})} \tag{D.19}$$

and therefore have ungraded dimension $4j + 1$.

Finally, let us discuss the Clebsch-Gordan decomposition of the tensor product of two irreducible representations of $\mathfrak{osp}(2|2)$. To this end, one needs to introduce an inner product on the representations spaces such that the operators corresponding to the bosonic generators iT_i^+ and Q are self-adjoint and w.r.t. which two inequivalent irreducible representations are orthogonal. In contrast to the $\mathcal{N} = 1$-case, it turns out that, provided that $q \in \mathbb{R}$ and $\pm q \geq j$, one can in fact introduce a positive definite inner product satisfying all the above mentioned requirements such that the irreducible representation become so-called *star representations* [216, 254]. In particular, due to this property, it follows that the Clebsch-Gordan decomposition does not depend on the choice of parity of the subspaces appearing in the definition (D.17).

Moreover, it follows that equivalence classes of typical representations $(\pi^{(j,q)}, V^{(j,q)})$ with $q \in \mathbb{R}$ and $\pm q \geq j$ form of subcategory which is closed under tensor product. More precisely, given two such irreducible typical representations $\pi^{(j_1,q_1)}$ and $\pi^{(j_2,q_2)}$, the corresponding tensor product representations decomposes as [216]

$$\begin{aligned}\pi^{(j_1,q_1)} \otimes \pi^{(j_2,q_2)} = &\bigoplus_{j=|j_1-j_2|}^{j_1+j_2} \pi^{(j,q_1+q_2)} \oplus \bigoplus_{j=|j_1-j_2|+\frac{1}{2}}^{j_1+j_2-\frac{1}{2}} \pi^{(j,q_1+q_2+\frac{1}{2})} \\ &\oplus \bigoplus_{j=|j_1-j_2|+\frac{1}{2}}^{j_1+j_2-\frac{1}{2}} \pi^{(j,q_1+q_2-\frac{1}{2})} \oplus \bigoplus_{j=|j_1-j_2|+1}^{j_1+j_2-1} \pi^{(j,q_1+q_2)}\end{aligned} \tag{D.20}$$

where the index j in the direct sums runs in integer steps.

E. Space forms

In this section, following closely [232], we review the basic definition of the so-called *space forms* which can be thought of as the simplest possible models of semi-Riemannian manifolds. As will be demonstrated frequently in the main text, these type of manifolds turn out to have very important applications in general relativity and cosmology.

Before we start with the main definition, we first need to introduce the sectional curvature K of a semi-Riemannian manifold.

Definition E.1. Let (M, g) be a semi-Riemannian manifold. A two-dimensional subspace $\Pi \subset T_pM$ of the tangent space at $p \in M$ is called *non-degenerate* if $g|_\Pi$ is non-degenerate. This is equivalent to saying that, for any bases (v, w) of Π, one has $Q(v, w) \neq 0$ where

$$Q(v, w) := \langle v, v \rangle \langle w, w \rangle - \langle v, w \rangle^2 \tag{E.1}$$

Let $\Pi \subset T_pM$ be a two-dimensional non-degenerate subspace of the tangent space at $p \in M$. For any bases (v, w) of Π, the *sectional curvature* $K(\Pi)$ of Π is defined as

$$K(\Pi) = \frac{\langle R(v, w)v, w \rangle}{Q(v, w)} \tag{E.2}$$

where $R \in \Gamma((T^*M)^3) \otimes \Gamma(TM)$ denotes the Riemann curvature tensor corresponding to the Levi-Civita connection $\nabla \equiv \nabla^{LC}$ of (M, g).

Proof. We have to prove that (E.2) is independent of the choice of a basis (v, w) of the non-degenerate tangent plane Π. To this end, let (v, w) and (x, y) be two bases of Π. Then, there exists an invertible matrix A of rank two such that $(v, w) = (x, y)A^T$. By the symmetry properties of Q as well as the Riemann curvature tensor, it is immediate to see that $\langle R(v, w)v, w \rangle = \det A^2 \langle R(x, y)x, y \rangle$ and $Q(v, w) = \det A^2 Q(x, y)$. Hence, this proves the independence of (E.2) on the choice of a basis. □

Definition E.2. A semi-Riemannian manifold (M, g) is called of *constant curvature*, if the sectional curvature is constant, i.e., there exists some real number $C \in \mathbb{R}$ such that $K(\Pi) = C$ for any non-degenerate tangent plane Π.

Proposition E.3. *For a semi-Riemannian manifold (M, g) of constant curvature $C \in \mathbb{R}$, the Riemann curvature tensor R takes the form*

$$R(X, Y)Z = C(\langle Z, X \rangle Y - \langle Z, Y \rangle X) \tag{E.3}$$

for any vector fields $X, Y, Z \in \Gamma(TM)$.

Proof. At $p \in M$, consider the map

$$F : (T_pM)^4 \to \mathbb{R}, \ (v, w, x, y) \mapsto C(\langle v, x\rangle \langle y, w\rangle - \langle v, y\rangle \langle x, w\rangle) \tag{E.4}$$

such that $F(v, w, v, w) = CQ(v, w)$. Hence, for a non-degenerate tangent plane generated by the basis (v, w), this yields

$$K(\Pi) \equiv C = \frac{F(v, w, v, w)}{Q(v, w)} \tag{E.5}$$

implying $\Delta(v, w, v, w) = 0$ where $\Delta : (T_pM)^4 \to \mathbb{R}$ is defined as $\Delta(v, w, x, y) := \langle R(v, w)x, y\rangle - F(v, w, x, y)$. Using an approximation argument, it is easy to see that this implies $\Delta(v, w, v, w) = 0$ for any, i.e., not necessarily linearly independent tangent vectors $v, w \in \Pi$. Then, via polarization, that is, considering the quantity $\Delta(v, w + x, v, w + x) = 0$ and exploiting the symmetry properties induced by the Riemann curvature tensor and Q, one concludes that $\Delta \equiv 0$ yielding (E.3). ◻

Definition E.4. A semi-Riemannian manifold is called *complete* if any maximal geodesic is defined on the entire real line. A complete connected semi-Riemannian manifold of constant curvature is called a *space form*.

Theorem E.5. *A simply connected space form is uniquely determined, up to isometry, by the triple (d, ν, C) consisting of its dimension, index and curvature, respectively.*

Sketch of Proof. One direction is immediate. So suppose that M and N are two semi-Riemannian manifolds with same dimension, index and curvature. This immediately implies that for arbitrary but fixed points $p \in M$ and $q \in N$, there exists a linear isometry $L : T_pM \to T_qN$ satisfying $\langle L(v), L(w)\rangle = \langle v, w\rangle \ \forall v, w \in T_pM$. Moreover, since the curvature is constant, it follows in particular that L preserves the curvature. By Theorem 8.17 in [232], there thus exists a unique semi-Riemannian covering map[3] $\phi : M \to N$ such that $D_p\phi = L$. Since both M and N are supposed to simply connected, this implies that ϕ is in fact an isometry. ◻

Definition E.6 (Hyperquadrics)**.** For $d \geq 2$ and $0 \leq \nu \leq d - 1$ let $(\mathbb{R}^{d+1}_\nu, \eta)$ be the $d + 1$-dimensional Minkowski spacetime with Minkowski metric η with index ν. Then

(i) the d-dimensional *(pseudo) sphere* $\mathbb{S}^d_\nu(r)$ of radius $r > 0$ in $\mathbb{R}^{d+1}_\nu$ is defined as

$$\mathbb{S}^d_\nu(r) := \{x \in \mathbb{R}^{d+1}_\nu \,|\, \eta(x, x) = r^2\} \tag{E.6}$$

[3] A smooth map $\phi : M \to N$ between smooth manifolds M and N is called a *covering map* if it is surjective and for any $p \in M$ there exists a connected open neighborhood $p \in U \subset N$ such that ϕ defines a diffeomorphism onto U on each connected component of $\phi^{-1}(U)$.

(ii) the d-dimensional *(pseudo) hyperbolic space* $\mathbb{H}^d_\nu(r)$ of radius $r > 0$ in $\mathbb{R}^{d+1}_{\nu+1}$ is defined as

$$\mathbb{H}^d_\nu(r) := \{x \in \mathbb{R}^{d+1}_{\nu+1} |\, \eta(x,x) = -r^2\} \tag{E.7}$$

Corollary E.7 (Hopf)**.** *Up to isometry, the complete simply connected d-dimensional Riemannian manifolds of constant curvature C are given by*

the sphere $\mathbb{S}^d(r)$	*if* $C = 1/r^2$
the Euclidean space $\mathbb{R}^d$	*if* $C = 0$
the hyperbolic space $\mathbb{H}^d(r)$	*if* $C = -1/r^2$

Corollary E.8. *Up to isometry, the complete simply connected d-dimensional Lorentzian spacetime manifolds of constant curvature C are given by*

the de Sitter space $\mathrm{dS}_d \equiv \mathbb{S}^d_1(r)$	*if* $C = 1/r^2$ *and* $d \geq 3$
Minkowski space $\mathbb{R}^{1,d-1}$	*if* $C = 0$
the universal anti-de Sitter space $\widetilde{\mathrm{AdS}}_d \equiv \widetilde{\mathbb{H}}^d_1(r)$	*if* $C = -1/r^2$

where $\widetilde{M}$ denotes the universal covering of a smooth semi-Riemannain manifold M.

F. Proof of Proposition 2.6.10

Proof. Since, for any $p \in \mathcal{P}$, the map $\Phi_{p*} : \mathrm{Lie}(\mathcal{G}) \rightarrow \mathcal{V}_p \subset T_p\mathcal{P}$ is an isomorphism of free super Λ-modules and $\mathcal{A} \in \Omega^1(\mathcal{P}, \mathfrak{g})_0$ is linear, condition (i) in Definition 2.5.19 of a connection 1-form and condition (i) of Proposition 2.6.10 are equivalent. In the following, we can thus restrict on condition (ii).

Hence, suppose $\mathcal{A} \in \Omega^1(\mathcal{P}, \mathfrak{g})_0$ is a connection 1-form on $\mathcal{P}$. Then, by restriction on body points, the first part of condition (ii) is immediate, i.e., $\Phi^*_g\mathcal{A} = \mathrm{Ad}_{g^{-1}} \circ \mathcal{A}$ for all $g \in \mathbf{B}(\mathcal{G})$. For the second part, let us extend $\mathcal{A}$ to a smooth $\mathrm{Lie}(\mathcal{G})$-valued 1-form $\tilde{\mathcal{A}} \in \Omega^1(\mathrm{Lie}(\mathcal{G})_0 \times \mathcal{P}, \mathfrak{g})$ on $\mathrm{Lie}(\mathcal{G})_0 \times \mathcal{P}$ by setting $\langle (Z, Z')|\tilde{\mathcal{A}}\rangle := \langle Z'|\mathcal{A}\rangle$ $\forall (Z, Z') \in T(\mathrm{Lie}(\mathcal{G})_0 \times \mathcal{P}) \cong \mathrm{Lie}(\mathcal{G}) \times T\mathcal{P}$. Moreover, we consider the extended $\mathcal{G}$-right action $\tilde{\Phi} : (\mathrm{Lie}(\mathcal{G})_0 \times \mathcal{P}) \times \mathcal{G} \rightarrow \mathrm{Lie}(\mathcal{G})_0 \times \mathcal{P}$ on $\mathrm{Lie}(\mathcal{G})_0 \times \mathcal{P}$ defined via $\tilde{\Phi}((Y, p), g) := (Y, \Phi(p, g))$. It is then immediate to see that that condition (ii) in Definition 2.5.19 of a connection 1-form is equivalent to

$$\tilde{\Phi}^*_g\tilde{\mathcal{A}} = \mathrm{Ad}_{g^{-1}} \circ \tilde{\mathcal{A}}, \quad \forall g \in \mathcal{G} \tag{F.1}$$

Consider the even smooth vector field $\widetilde{\mathcal{Z}} \in \Gamma(T(\mathrm{Lie}(\mathcal{G})_0 \times \mathcal{P}))$ given by

$$\widetilde{\mathcal{Z}}_{(X,p)} := (0_Y, D_{(p,e)}\Phi(0_p, Y_e)) = (0_Y, \widetilde{Y}_p)$$

$\forall (Y,p) \in \mathrm{Lie}(\mathcal{G})_0 \times \mathcal{P}$ parametrizing the (not necessarily H^∞-smooth) fundamental vector fields on $\mathcal{P}$. The flow $\phi^{\widetilde{\mathcal{Z}}} : \Lambda_0 \times (\mathrm{Lie}(\mathcal{G})_0 \times \mathcal{P}) \to \mathrm{Lie}(\mathcal{G})_0 \times \mathcal{P}$ of $\widetilde{\mathcal{Z}}$ is of the form $\phi_t^{\widetilde{\mathcal{Z}}}(Y,p) = (Y, \Phi(p, e^{tY})) = \tilde{\Phi}_{e^{tY}}(Y,p)$. Using ([97], Prop. V.7.27), (F.1) then yields

$$\begin{aligned}\langle (0_Y, Z_p) | L_{\widetilde{\mathcal{Z}}} \tilde{\mathcal{A}}_{(Y,p)} \rangle &= \frac{\partial}{\partial t}\bigg|_{t=0} \langle (0_Y, Z_p) | ((\phi_t^{\widetilde{\mathcal{Z}}})^* \tilde{\mathcal{A}})_{(Y,p)} \rangle \\ &= \frac{\partial}{\partial t}\bigg|_{t=0} \langle (0_Y, Z_p) | (\Phi^*_{e^{tY}} \mathcal{A})_{(Y,p)} \rangle \\ &= \frac{\partial}{\partial t}\bigg|_{t=0} \mathrm{Ad}_{e^{-tY}} \langle (0_Y, Z_p) | \tilde{\mathcal{A}}_{(Y,p)} \rangle \\ &= -\mathrm{ad}_Y \langle Z_p | \mathcal{A}_p \rangle \end{aligned} \tag{F.2}$$

$\forall (X,p) \in \mathrm{Lie}(\mathcal{G})_0 \times \mathcal{P}$ and smooth homogeneous $Z \in \Gamma(T\mathcal{P})$. On the other hand, one has

$$\begin{aligned}\langle (0_Y, Z_p) | L_{\widetilde{\mathcal{Z}}} \tilde{\mathcal{A}}_{(Y,p)} \rangle &= \widetilde{\mathcal{Z}}_{(Y,p)} \langle (0,Z) | \tilde{\mathcal{A}} \rangle - \langle [\widetilde{\mathcal{Z}}, (0,Z)]_{(Y,p)} | \tilde{\mathcal{A}}_{(Y,p)} \rangle \\ &= \widetilde{Y}_p \langle Z | \mathcal{A} \rangle - \langle [\widetilde{\mathcal{Z}}, (0,Z)]_{(Y,p)} | \tilde{\mathcal{A}}_{(Y,p)} \rangle \end{aligned} \tag{F.3}$$

for any $(Y,p) \in \mathrm{Lie}(\mathcal{G})_0 \times \mathcal{P}$. If $Y = X \in \mathfrak{g}_0 \subseteq \mathrm{Lie}(\mathcal{G})_0$, it follows $[\widetilde{\mathcal{Z}}, (0,Z)]_{(X,p)} = [\widetilde{X}, Z]_p$ yielding

$$\widetilde{X}_p \langle Z | \mathcal{A} \rangle - \langle [\widetilde{X}, Z]_p | \mathcal{A}_p \rangle = \langle Z_p | L_{\widetilde{X}} \mathcal{A}_p \rangle \tag{F.4}$$

and thus $\langle Z_p | L_{\widetilde{X}} \mathcal{A}_p \rangle = \langle Z_p | -\mathrm{ad}_X \circ \mathcal{A}_p \rangle \, \forall p \in \mathcal{P}$. Since this holds for any smooth homogeneous vector field Z, this implies

$$L_{\widetilde{X}} \mathcal{A} = -\mathrm{ad}_X \circ \mathcal{A}, \text{ for } X \in \mathfrak{g}_0 \tag{F.5}$$

On the other hand, if $Y = \tau X$ with $X \in \mathfrak{g}_1$ and $\tau \in \Lambda_1$, it follows $[\widetilde{\mathcal{Z}}, (0,Z)]_{(Y,p)} = \tau [\widetilde{X}, Z]_p$ such that

$$\tau \widetilde{X}_p \langle Z | \mathcal{A} \rangle - \tau \langle [\widetilde{X}, Z]_p | \mathcal{A}_p \rangle = (-1)^{|Z|} \tau \langle Z_p | L_{\widetilde{X}} \mathcal{A}_p \rangle \tag{F.6}$$

and therefore $\tau\langle Z_p|L_{\widetilde{X}}\mathcal{A}_p\rangle = -(-1)^{|Z|}\tau\mathrm{ad}_X\langle Z_p|\mathcal{A}_p\rangle = \tau\langle Z_p|-\mathrm{ad}_X\circ\mathcal{A}_p\rangle$ $\forall p\in\mathcal{P}$. Since this holds for any $\tau\in\Lambda_1$, we thus have

$$L_{\widetilde{X}}\mathcal{A} = -\mathrm{ad}_X\circ\mathcal{A},\ \text{for } X\in\mathfrak{g}_1 \tag{F.7}$$

Conversely, suppose $\mathcal{A}\in\Omega^1(\mathcal{P},\mathfrak{g})_0$ satisfies the conditions (i) and (ii) of the above Proposition. For any smooth homogeneous vector field $Z\in\Gamma(T\mathcal{P})$ consider the $\mathrm{Lie}(\mathcal{G})$-valued H^∞-smooth functions $F_Z, G_Z\in H^\infty(\mathcal{G}\times\mathcal{P})\otimes\mathrm{Lie}(\mathcal{G})$ defined as

$$F_Z(g,p) := \langle Z_p|\Phi_g^*\mathcal{A}_p\rangle = \langle D_{(p,g)}\Phi(Z_p,0_g)|\mathcal{A}_{p\cdot g}\rangle \tag{F.8}$$

as well as

$$G_Z(g,p) := \mathrm{Ad}_{g^{-1}}\langle Z_p|\mathcal{A}_p\rangle \tag{F.9}$$

$\forall(g,p)\in\mathcal{G}\times\mathcal{P}$. Since $H^\infty(\mathcal{G}\times\mathcal{P})\cong H^\infty(\mathcal{G})\hat{\otimes}_\pi H^\infty(\mathcal{P})$, it follows from Lemma 2.6.8 that $\mathcal{A}$ defines a connection 1-form on $\mathcal{P}$ if and only if we can show

$$(X\otimes\mathbb{1})F_Z(g,p) = (X\otimes\mathbb{1})G_Z(g,p) \tag{F.10}$$

$\forall p\in\mathcal{P}$ and body points $g\in\mathbf{B}(\mathcal{G})$ as well as $X\in\mathcal{U}(\mathfrak{g})$ and smooth homogeneous vector fields $Z\in\Gamma(T\mathcal{P})$. For $X=1\in\mathcal{U}(\mathfrak{g})$, this is an immediate consequence of the first part of condition (ii).

Using the extension $\tilde{\mathcal{A}}\in\Omega^1(\mathrm{Lie}(\mathcal{G})_0\times\mathcal{P},\mathrm{Lie}(\mathcal{G}))$ of $\mathcal{A}$ on $\mathrm{Lie}(\mathcal{G})_0\times\mathcal{P}$ as well as the $\mathcal{G}$-right action $\tilde{\Phi}:(\mathrm{Lie}(\mathcal{G})_0\times\mathcal{P})\times\mathcal{G}\to\mathrm{Lie}(\mathcal{G})_0\times\mathcal{P}$ as defined above, we may extend F_Z and G_Z to H^∞-smooth functions $\tilde{F}_Z$ and $\tilde{G}_Z$ on $\mathrm{Lie}(\mathcal{G})_0\times\mathcal{G}\times\mathcal{P}$ by setting

$$\begin{aligned}\tilde{F}_Z(Y,g,p) :&= \langle\tilde{\Phi}_{g*}(0_Y,Z_p)|\tilde{\mathcal{A}}_{\tilde{\Phi}(Y,p,g)}\rangle = \langle D_{(Y,p,g)}\tilde{\Phi}(0_Y,Z_p,0_g)|\tilde{\mathcal{A}}_{\tilde{\Phi}(Y,p,g)}\rangle\\ &= \langle D_{(p,g)}\Phi(Z_p,0_g)|\mathcal{A}_{p\cdot g}\rangle = F_Z(g,p)\end{aligned} \tag{F.11}$$

as well as

$$\tilde{G}_Z(Y,g,p) := \mathrm{Ad}_{g^{-1}}\langle(0_Y,Z_p)|\tilde{\mathcal{A}}_{(Y,p)}\rangle = \mathrm{Ad}_{g^{-1}}\langle Z_p|\mathcal{A}_p\rangle = G_Z(g,p) \tag{F.12}$$

$\forall(Y,g,p)\in\mathrm{Lie}(\mathcal{G})_0\times\mathcal{G}\times\mathcal{P}$. Let $\mathcal{Z}^L\in\Gamma(T(\mathrm{Lie}(\mathcal{G})_0\times\mathcal{G}))$ be the H^∞-smooth homogeneous vector field on $\mathrm{Lie}(\mathcal{G})_0\times\mathcal{G}$ defined as $\mathcal{Z}^L_{(Y,g)} := (0_Y, D_{(g,e)}\mu(0_g,Y_e))$ $\forall(Y,g)\in\mathrm{Lie}(\mathcal{G})_0\times\mathcal{G}$. Similarly as above, using the explicit form of the flow ϕ^L of $\mathcal{Z}^L$, this yields

$$\begin{aligned}((\phi^L_t)^*\tilde{F}_Z)(Y,g,p) &= \langle\tilde{\Phi}_{ge^{tY}*}(0_Y,Z_p)|\tilde{\mathcal{A}}_{\tilde{\Phi}(Y,p,g\cdot e^{tY})}\rangle\\ &= \langle\tilde{\Phi}_{e^{tY}*}\circ\tilde{\Phi}_{g*}(0_Y,Z_p)|\tilde{\mathcal{A}}_{\tilde{\Phi}(Y,p,g\cdot e^{tY})}\rangle\end{aligned}$$

$$= \langle (0_Y, \Phi_{g*} Z_p) | (\phi_t^{\tilde{Z}})^* \tilde{\mathcal{A}}_{\phi_t^{\tilde{Z}}(Y, p\cdot g)} \rangle \tag{F.13}$$

Taking the derivative, we thus conclude

$$\begin{aligned}(Y \otimes \mathbb{1}) F_Z(g,p) &= \frac{\partial}{\partial t}\bigg|_{t=0} \langle (0_Y, \Phi_{g*} Z_p) | (\phi_t^{\tilde{Z}})^* \tilde{\mathcal{A}}_{\phi_t^{\tilde{Z}}(Y, p\cdot g)} \rangle \\ &= \langle (0_Y, \Phi_{g*} Z_p) | L_{\tilde{Z}} \tilde{\mathcal{A}}_{(Y, p\cdot g)} \rangle \end{aligned} \tag{F.14}$$

Following exactly the same steps as above, one then concludes

$$(X \otimes \mathbb{1}) F_Z(g,p) = (-1)^{|Z||X|} \langle \Phi_{g*} Z_p | L_{\tilde{X}} \mathcal{A}_{p\cdot g} \rangle, \quad \forall \text{homogeneous } X \in \mathfrak{g} \tag{F.15}$$

For G_Z we proceed similarly and compute

$$\begin{aligned}((\phi_t^L)^* \tilde{G}_Z)(Y,g,p) &= \mathrm{Ad}_{(ge^{tY})^{-1}} \langle (0_Y, Z_p) | \tilde{\mathcal{A}}_{(Y,p)} \rangle \\ &= \mathrm{Ad}_{e^{-tY}} (\mathrm{Ad}_{g^{-1}} \langle Z_p | \mathcal{A}_p \rangle) = \mathrm{Ad}_{e^{-tY}} (G_Z(g,p)) \end{aligned} \tag{F.16}$$

which yields

$$\begin{aligned}(Y \otimes \mathbb{1}) F_Z(g,p) &= \frac{\partial}{\partial t}\bigg|_{t=0} \mathrm{Ad}_{e^{-tY}} (G_Z(g,p)) = -\mathrm{ad}_Y G_Z(g,p) \\ &= -\mathrm{ad}_Y (\mathrm{Ad}_{g^{-1}} \langle Z_p | \mathcal{A}_p \rangle) \end{aligned} \tag{F.17}$$

Hence, it follows for $\forall X \in \mathfrak{g}$

$$(X \otimes \mathbb{1}) G_Z(g,p) = -\mathrm{ad}_X (\mathrm{Ad}_{g^{-1}} \langle Z_p | \mathcal{A}_p \rangle) \tag{F.18}$$

$\forall g \in \mathcal{G},\ p \in \mathcal{P}$. If $g \in \mathbf{B}(\mathcal{G})$ is a body point this, together with condition (ii), yields

$$\begin{aligned}(X \otimes \mathbb{1}) G_Z(g,p) &= -\mathrm{ad}_X (\mathrm{Ad}_{g^{-1}} \langle Z_p | \mathcal{A}_p \rangle) = -\mathrm{ad}_X \langle \Phi_{g*} Z_p | \mathcal{A}_{p\cdot g} \rangle \\ &= (-1)^{|Z||X|} \langle \Phi_{g^*} Z_p | L_{\tilde{X}} \mathcal{A}_{p\cdot g} \rangle = (X \otimes \mathbb{1}) F_Z(g,p) \end{aligned} \tag{F.19}$$

proving (F.10) in case $X \in \mathfrak{g}$. Next, let $Y \circ X \in \mathcal{U}(\mathfrak{g})$ with homogeneous $Y, X \in \mathfrak{g}$. In a similar way as above, one finds

$$(Y \circ X \otimes \mathbb{1}) F_Z(g,p) = (-1)^{|Z|(|X|+|Y|)} \langle \Phi_{g*} Z_p | L_{\tilde{Y}} L_{\tilde{X}} \mathcal{A}_{p\cdot g} \rangle \tag{F.20}$$

as well as

$$\begin{aligned}(Y \circ X \otimes \mathbb{1}) G_Z(g,p) &= (Y \otimes \mathbb{1}) \mathrm{ad}_X \circ G_Z(g,p) \\ &= (-1)^{|X||Y|} \mathrm{ad}_X ((Y \otimes \mathbb{1}) G_Z(g,p)) \end{aligned}$$

$$
\begin{aligned}
&= (-1)^{|X||Y|}\mathrm{ad}_X \circ \mathrm{ad}_Y \circ G_Z(g,p) \\
&= (-1)^{|X||Y|}\mathrm{ad}_X \circ \mathrm{ad}_Y(\mathrm{Ad}_{g^{-1}}\langle Z_p|\mathcal{A}_p\rangle) \qquad \text{(F.21)}
\end{aligned}
$$

$\forall g \in \mathcal{G},\ p \in \mathcal{P}$. Taking the Lie derivative on both sides of the second part of condition (ii), one obtains

$$
L_{\widetilde{Y}} L_{\widetilde{X}}\mathcal{A} = -L_{\widetilde{Y}}(-\mathrm{ad}_X \circ \mathcal{A}) = -(-1)^{|X||Y|}\mathrm{ad}_X \circ L_{\widetilde{Y}}\mathcal{A} = (-1)^{|X||Y|}\mathrm{ad}_X \circ \mathrm{ad}_Y \circ \mathcal{A} \qquad \text{(F.22)}
$$

Hence, inserting (F.21) in (F.22) and restricting on body points $g \in \mathbf{B}(\mathcal{G})$, it follows

$$
\begin{aligned}
(Y \circ X \otimes \mathbb{1})G_Z(g,p) &= (-1)^{|X||Y|}\mathrm{ad}_X \circ \mathrm{ad}_Y(\mathrm{Ad}_{g^{-1}}\langle Z_p|\mathcal{A}_p\rangle) \\
&= (-1)^{|X||Y|}\mathrm{ad}_X \circ \mathrm{ad}_Y(\langle \Phi_{g*}Z_p|\mathcal{A}_{p\cdot g}\rangle) \\
&= (-1)^{|Z|(|X|+|Y|)}(\langle \Phi_{g*}Z_p|(-1)^{|X||Y|}\mathrm{ad}_X \circ \mathrm{ad}_Y \circ \mathcal{A}_{p\cdot g}\rangle) \\
&= (-1)^{|Z|(|X|+|Y|)}(\langle \Phi_{g*}Z_p|L_{\widetilde{Y}} L_{\widetilde{X}}\mathcal{A}_{p\cdot g}\rangle) \\
&= (Y \circ X \otimes \mathbb{1})F_Z(g,p) \qquad \text{(F.23)}
\end{aligned}
$$

proving (F.10) in case $Y \circ X \in \mathcal{U}(\mathfrak{g})$ with $Y, X \in \mathfrak{g}$. Thus, by induction, one concludes that (F.10) holds for any $X \in \mathcal{U}(\mathfrak{g})$ and hence $\mathcal{A}$ indeed defines a connection 1-form on $\mathcal{P}$. □

List of symbols, notations and conventions

l_p	Planck length, page 1
$\kappa = 8\pi G$	gravitational coupling constant, page 108
β	Barbero-Immirzi parameter, page 133
$L, \Lambda_{\cos} = -3/L^2$	anti-de Sitter radius and corresponding cosmological constant, page 34
ℓ_0, V_0	fiducial length/volume, page 297
Λ, Λ_N	Grassmann algebra, Grassmann algebra of dimension $N \in \mathbb{N}_0$, page 356
$\Lambda^{\mathbb{C}} := \Lambda \otimes \mathbb{C}$	complexified Grassmann algebra, page 29
$\Lambda^{m,n} = \Lambda_0^m \times \Lambda_1^n$	superdomain of dimension (m, n), page 356
$\mathbb{K}^{m\|n}$	superspace of dimension $m\|n$ with $\mathbb{K} = \mathbb{R}, \mathbb{C}$, page 343
$A^{m\|n}$	$=: A \otimes \mathbb{K}^{m\|n}$ for A a superalgebra and $\mathbb{K}$ a field ($\mathbb{K} = \mathbb{R}, \mathbb{C}$), page 345
$\mathcal{S}, \mathcal{S}'$	parametrization supermanifold, page 54
$\lambda : \mathcal{S} \to \mathcal{S}'$	change of parametrization, page 55
$I, J, \ldots = 0, \ldots, 3$	local Lorentz indices in $4D$, page 129
$i, j, \ldots = 1, 2, 3$	local $\mathfrak{su}(2)$ indices, page 137
$\mu, \nu, \ldots = 0, \ldots, 3$	local spacetime indices, page 134
$a, b, \ldots = 1, 2, 3$	local indices on a Cauchy slice Σ, page 137
$\alpha, \beta, \ldots$	$4D$ Dirac/Majorana spinor indices, page 131
$A, B, \ldots = +, -$	left-handed Weyl spinor indices, page 131

$A', B', \ldots = +, -$	right-handed Weyl spinor indices, page 131
$p, q, r, s, \ldots = 1, \ldots, \mathcal{N}$	R-symmetry index, page 36
$\underline{I}, \underline{J}, \ldots$	multi-indices (ordered or unordered), page 12
$\epsilon^{IJKL} = -\epsilon_{IJKL}$	completely antisymmetric symbol in $D = 4$ with $\epsilon^{0123} := 1$, page 128
$\epsilon^{ijk} := \epsilon^{0ijk}$	completely antisymmetric symbol in $D = 3$ with $\epsilon^{123} = 1$, page 138
$\epsilon^{AB}, \epsilon^{A'B'}$	completely antisymmetric symbol in the space of left- resp. right-handed Weyl spinors, page 131
η	Minkowski metric on $\mathbb{R}^{1,3}$, signature $(-+++)$, page 27
$\sigma_i, i = 1, 2, 3$	Pauli matrices, page 129
$\tau_i := \frac{1}{2i}\sigma_i, i = 1, 2, 3$	generators of $\mathfrak{su}(2)$, page 141
$J^i, i = 1, 2, 3$	angular momentum operator generated by $\tau_i \in \mathfrak{su}(2)$, $i = 1, 2, 3$, page 162
$P_I, I = 0, \ldots, 3$	infinitesimal spacetime translations, page 28
$M_{IJ}, I, J = 0, \ldots, 3$	generators of $\mathfrak{spin}^+(1,3)$ (infinitesimal Lorentz transformations), page 28
Q^r_α	fermionic generators of $\mathfrak{osp}(\mathcal{N}\vert 4)$, page 36
$T^{rs}, r, s = 1, \ldots, \mathcal{N}$	generators of R-symmetry subgroup of $\mathrm{OSp}(\mathcal{N}\vert 4)$, page 36
$T^\pm_i$	chiral components of M_{IJ}, page 207
$[\cdot, \cdot]$	graded commutator on a Lie superalgebra, page 24
$[\cdot, \cdot]_-, [\cdot, \cdot]_+$	(standard) commutator resp. anticommutator on a ring or associative algebra, page 27
$[\alpha \wedge \beta]$	wedge-product of $\mathrm{Lie}(\mathcal{G})$-valued forms, page 67
$\{\cdot, \cdot\}$	graded Poisson bracket, page 221

$\{\cdot,\cdot\}_{\mathrm{DB}}$	graded Dirac bracket, page 152
γ^I	gamma matrices, page 128
$\gamma_{I_1 I_2 \cdots I_k}$	$=: \gamma_{[I_1}\gamma_{I_2}\cdot\ldots\cdot\gamma_{I_k]}$, page 128
γ_*	highest rank Clifford algebra element, page 128
γ	path on a ($\mathcal{S}$-relative) supermanifold, page 75
γ	finite graph embedded in Σ, page 226
γ^{hor}	horizontal lift of a path γ, page 75
Γ	generalized graph, page 254
$E(\gamma), V(\gamma)$	set of edges e resp. vertices v of a graph γ, page 226
$l \equiv l(\gamma)$	subgroupoid generated by a finite graph γ, page 227
$p_{ll'}$	surjective mapping from $\mathcal{A}_{\mathcal{S},l'}$ to $\mathcal{A}_{\mathcal{S},l}$ for any subgroupoids $l \leq l'$, page 228
$a \equiv a(t)$	scale factor, page 282
A^+, A^-	self- resp. anti self-dual Ashtekar connection, page 207
${}^{\beta}A$	Ashtekar-Barbero connection, page 138
$\Gamma^i, i=1,2,3$	$3D$ spin connection, page 138
$K^i, i=1,2,3$	extrinsic curvature, page 138
$\mathscr{A}$	super connection 1-form on a $\mathcal{S}$-relative principal super fiber bundle, page 63
$\mathscr{A}$	(generalized) super Cartan connection, page 100
$\mathscr{A}^+$	super Ashtekar connection, page 209
θ_{MC}	Maurer-Cartan form, page 44
$\Theta^{(\omega)}$	torsion 2-form associated to a connection 1-form ω, page 96
$F(\omega)$	curvature of the connection 1-form ω, page 96

$F(\mathcal{A})$	curvature of a super connection 1-form $\mathcal{A}$, page 67
$F(\mathcal{A})$	the super Cartan curvature of a super Cartan connection $\mathcal{A}$, page 107
$h_e[\mathcal{A}]$	super holonomy along an edge e induced by a super connection 1-form $\mathcal{A}$, page 225
$H_e[{}^{\beta}A]$	holonomy induced by ${}^{\beta}A$ in the $\mathfrak{su}(2)$-sub representation of the real Majorana representation $\kappa_{\mathbb{R}*}$, page 158
$(e_I), (e^I), I = 0, \ldots, 3$	(local) frame and co-frame on a $4D$ Lorentzian manifold, page 97
$(\mathring{e}_i), (\mathring{e}^i), i = 1, 2, 3$	basis of fiducial vector fields resp. left-invariant one-forms on fiducial Cauchy slice Σ of a FLRW model, page 294
e	soldering form, page 94
e	$:= \det(e^I_\mu)$ with e^I a local co-frame, page 135
e	edge of a graph γ, page 226
$b(e), f(e)$	beginning and endpoints of an edge $e \in E(\gamma)$, page 226
Σ	spacelike Cauchy hypersurface in a globally hyperbolic space-time manifold, page 136
$\Sigma := e \wedge e$	$\mathfrak{spin}^+(1, 3)$-valued 2-form generated by the soldering form e, page 133
E^a_i	(gravitational) electric field canonically conjugate to the connection ${}^{\beta}A$, page 138
E	super soldering form or supervielbein, page 101
$\mathcal{E}$	super electric field canonically conjugate to $\mathcal{A}^+$, page 211
$\mathcal{X}_n(S)$	super electric flux smeared over a two-dimensional surface S and super Lie algebra-valued smearing function n on S, page 230
$\widehat{V}^{\mathrm{AL}}, \widehat{V}^{\mathrm{RS}}$	Asthekar-Lewandowski resp. Rovelli-Smolin volume operator, page 161

(c, p)	symmetry reduced self-dual Asthekar connection and dual electric field, page 298
$(\phi_{A'}, \pi_{\phi}^{A'})$	Rarita-Schwinger field and conjugate momentum in symmetry reduced theory, page 304
$\tilde{c}$	canonically transformed symmetry reduced self-dual Asthtekar connection, page 304
C	charge conjugation matrix, page 128
C^i_a	contorsion tensor, page 144
$\mathbf{P}_\beta$	β-dependent operator on the super Lie algebra $\mathfrak{osp}(1\|4)$ (resp. the space of 2-forms with values in $\mathrm{Lie}(\mathrm{OSp}(2\|4))$), page 191
$\mathcal{P}_\beta$, ${P_\beta}^{IJ}{}_{KL}$	β-dependent operator on $\mathfrak{spin}^+(1,3)$, page 191
q_{ab}	pullback metric on Cauchy slice Σ, page 138
$\mathring{q}_{ab}$	fiducial metric on fiducial Cauchy slice Σ of a FLRW model, page 282
$\mathbf{SMan}_{\mathrm{Alg}}$	category of algebro-geometric supermanifolds (generic objects denoted by $\mathcal{M}, \mathcal{N}, \ldots$ group objects denoted by $\mathcal{G}, \mathcal{H}, \ldots$), page 12
$\mathbf{SMan}_{H^\infty}$	category of H^∞ supermanifolds (generic objects denoted by $\mathcal{M}, \mathcal{N}, \ldots$ group objects denoted by $\mathcal{G}, \mathcal{H}, \ldots$), page 358
$\mathbf{SMan}_{/\mathcal{S}}$	category of $\mathcal{S}$-relative supermanifolds (objects denoted by $\mathcal{M}_{/\mathcal{S}}, \mathcal{N}_{/\mathcal{S}}, \ldots$), page 54
$\mathbf{Gr}$	category of (finite-dimensional) Grassmann algebras, page 18
$\mathbf{G}(\mathcal{S})$	groupoid with points on a Cauchy slice Σ as objects and smooth maps $g : \mathcal{S} \to \mathcal{G}$ as morphisms, page 226
$\mathbf{P}(\Sigma)$	path groupoid on a Cauchy slice Σ of a globally hyperbolic spacetime manifold $M = \mathbb{R} \times \Sigma$, page 225
$\mathbf{B}$	body functor, page 357
$\mathbf{S}$	split functor, page 14

$\mathbf{A}$	functor from $\mathbf{SMan}_{H^\infty}$ to $\mathbf{SMan}_{\mathrm{Alg}}$ mapping a H^∞ supermanifold $\mathcal{M}$ to the correponding algebro-geometric supermanifold $(\mathbf{B}(\mathcal{M}), \mathbf{B}_* H^\infty_{\mathcal{M}})$, page 22
$\mathbf{H}_N$	functor from $\mathbf{SMan}_{\mathrm{Alg}}$ to $\mathbf{Set}$ mapping an algebro-geometric supermanifold $\mathcal{M}$ to the Λ_N-point $\mathcal{M}(\Lambda_N)$, page 19
Π	parity functor, page 360
$\mathfrak{g}$ (resp. $\mathfrak{g}^R$)	super Lie algebra of left-invariant (resp. right-invariant) vector fields on a super Lie group $\mathcal{G}$, page 24
$\mathrm{Lie}(\mathcal{G}) = \Lambda \otimes \mathfrak{g}$	super Lie module of a H^∞ super Lie group $\mathcal{G}$ modeled over a Grassmann algebra Λ, page 26
$\mathrm{GL}(\mathcal{V})$, $\mathfrak{gl}(\mathcal{V})$	the general linear supergroup and the corresponding super Lie algebra on a super Λ-vector space $\mathcal{V}$, page 30
$\mathrm{GL}(m\|n, \Lambda)$	the general linear supergroup on the super Λ-vector space $\mathbb{R}^{m\|n} \otimes \Lambda$, page 32
$\mathrm{OSp}(\mathcal{V})$, $\mathfrak{osp}(\mathcal{V})$	orthosymplectic supergroup and its corresponding Lie superalgebra on a super Λ-vector space $\mathcal{V}$, page 34
$\mathrm{OSp}(m\|n)$	orthosymplectic supergroup in standard representation, page 34
$\mathrm{U}(m\|n)$, $\mathfrak{u}(m\|n)$	super unitary group and its corresponding super Lie algebra on the super Λ-vector space $\mathbb{C}^{m\|n} \otimes \Lambda$, page 32
$\mathrm{UOSp}(m\|n)$	unitary orthosymplectic group, page 233
$\mathrm{ISO}(\mathbb{R}^{1,3\|4})$, $\mathfrak{iso}(\mathbb{R}^{1,3\|4})$	super Poincaré group and corresponding super Lie algebra, page 28
$\mathcal{T}^{1,3\|4}$, $\mathfrak{t} \equiv \mathfrak{t}^{1,3\|4}$	super translation group and corresponding super Lie algebra, page 27
$\mathfrak{aut}(\mathcal{P}_{/\mathcal{S}})$	infinitesimal automorphisms on $\mathcal{P}_{/\mathcal{S}}$, page 108
$\mathscr{G}(\mathcal{P}_{/\mathcal{S}})$	set of global gauge transformations on a $\mathcal{S}$-relative principal super fiber bundle $\mathcal{P}_{/\mathcal{S}}$, page 82
$\mathfrak{gau}(\mathcal{P}_{/\mathcal{S}})$	vertical infinitesimal automorphisms or infinitesimal gauge transformations on $\mathcal{P}_{/\mathcal{S}}$, page 109

$\mathfrak{k}(\mathcal{M}, g)$	Lie superalgebra of Killing vector field on a super Riemannian manifold $(\mathcal{M}, g)$, page 116
$(\mathcal{M}, g)$	super Riemannian manifold consisting of a supermanifold $\mathcal{M}$ and a super metric g on $\mathcal{M}$, page 116
$\mathcal{F} \to \mathcal{E} \xrightarrow{\pi} \mathcal{M}$	super fiber bundle with total space $\mathcal{E}$, base $\mathcal{M}$, typical fiber $\mathcal{F}$ and canonical projection $\pi : \mathcal{E} \to \mathcal{M}$ (also simply denoted by $\mathcal{E}$ if base and typical fiber are clear from the context), page 37
$\mathcal{F} \to f^*\mathcal{E} \xrightarrow{\pi_f} \mathcal{N}$	pullback bundle of $\mathcal{F} \to \mathcal{E} \xrightarrow{\pi} \mathcal{M}$ with respect to the morphism $f : \mathcal{M} \to \mathcal{N}$, page 39
$\mathcal{V} \to \mathcal{E} \xrightarrow{\pi} \mathcal{M}$	super vector bundle with super Λ-vector space $\mathcal{V}$ as typical fiber, page 40
${}^*\mathcal{V} \to {}^*\mathcal{E} \to \mathcal{M}$	left dual super vector bundle, page 43
$\mathcal{V}^* \to \mathcal{E}^* \to \mathcal{M}$	right dual super vector bundle, page 43
$\mathcal{G} \to \mathcal{P} \xrightarrow{\pi} \mathcal{M}$	principal super fiber bundle with structure group $\mathcal{G}$ (or simply $\mathcal{G}$-bundle), page 45
$\mathscr{F}(\mathcal{E})$	frame bundle of the super vector bundle $\mathcal{E}$, page 46
$\mathscr{F}(\mathcal{M}) \equiv \mathscr{F}(T\mathcal{M})$	frame bundle of a supermanifold $\mathcal{M}$, page 47
$\mathcal{P} \times_\rho \mathcal{F}$	super fiber bundle associated to the principal super bundle $\mathcal{P}$ w.r.t. a $\mathcal{G}$-left action $\rho : \mathcal{G} \times \mathcal{F} \to \mathcal{F}$ on a supermanifold $\mathcal{F}$, page 47
$\mathcal{P}[\mathcal{G}] \equiv \mathcal{P} \times_\mathcal{H} \mathcal{G}$	$\mathcal{G}$-extension of the $\mathcal{H}$-bundle $\mathcal{P}$, page 53
$\mathcal{V} \to \mathcal{E}_{/\mathcal{S}} \xrightarrow{\pi_\mathcal{S}} \mathcal{M}_{/\mathcal{S}}$	$\mathcal{S}$-relative super vector bundle, page 57
$\mathcal{G} \to \mathcal{P}_{/\mathcal{S}} \xrightarrow{\pi_\mathcal{S}} \mathcal{M}_{/\mathcal{S}}$	$\mathcal{S}$-relative principal super fiber bundle, page 57
$T_p(\mathcal{M}_{/\mathcal{S}})$	tangent module of the $\mathcal{S}$-relative supermanifold $\mathcal{M}_{/\mathcal{S}}$ at $p \in \mathcal{M}_{/\mathcal{S}}$, page 56
$\mathscr{H}$	principal connection or Ehresmann connection on a $\mathcal{S}$-relative principal super fiber bundle $\mathcal{P}_{/\mathcal{S}}$, page 62

$\mathscr{V}, \mathscr{V}_p$	vertical tangent bundle and vertical tangent module at a point $p \in \mathcal{P}_{/\mathcal{S}}$, page 60
$\mathcal{A}_{\mathcal{S},l}$	set of generalized super connections w.r.t. the subgroupoid l, page 227
$\overline{\mathcal{A}}_{\mathcal{S}}$	set of generalized super connections, page 228
$\mathcal{A}^{P}_{\mathcal{S},\Gamma}$	set of pointed generalized super connections on a generalized graph Γ, page 254
$\mathrm{Cyl}^{\infty}(\mathcal{A}_{\mathcal{S},l})$	space of smooth cylindrical functions on $\mathcal{A}_{\mathcal{S},l}$, page 229
$\mathrm{Cyl}^{\infty}(\overline{\mathcal{A}}_{\mathcal{S}})$	space of cylindrical functions on $\overline{\mathcal{A}}_{\mathcal{S}}$, page 229
$\mathrm{Cyl}^{\infty}(\mathcal{A}^{P}_{\mathcal{S},\Gamma})$	space of generalized cylindrical functions on $\mathcal{A}^{P}_{\mathcal{S},\Gamma}$, page 255
$\mathrm{Cyl}^{\infty}(\overline{\mathcal{A}}^{P}_{\mathcal{S}})$	space of generalized cylindrical functions on $\overline{\mathcal{A}}^{P}_{\mathcal{S}}$, page 255
$V^{\infty}(\mathcal{A}_{\mathcal{S},l(\gamma)})$	space of super electric fluxes w.r.t. a graph γ, page 232
$V^{\infty}(\overline{\mathcal{A}}_{\mathcal{S}})$	space of super electric fluxes on the inductive limit $\mathrm{Cyl}^{\infty}(\overline{\mathcal{A}}_{\mathcal{S}})$, page 232
$\mathfrak{A}^{\mathrm{gHF}}_{\mathcal{S},l(\gamma)}$	graded holonomy-flux algebra w.r.t. a finite graph γ, page 232
$\mathfrak{A}^{\mathrm{gHF}}_{\mathcal{S}}$	graded holonomy-flux algebra, page 232
$\mathfrak{A}^{\mathrm{cLQSC}}$	graded holonomy-flux algebra of symmetry reduced theory, page 315
$\mathfrak{A}^{\mathrm{cLQSG}}_{\mathcal{S},\gamma}$	graded holonomy-flux algebra of chiral LQSG w.r.t. a finite graph γ, page 248
$(\pi : P \to M, A; \langle\cdot,\cdot\rangle)$	metric reductive Cartan geometry, page 94
$(\pi_{\mathcal{S}} : \mathcal{P}_{/\mathcal{S}} \to \mathcal{M}_{/\mathcal{S}}, \mathscr{A})$	super Cartan geometry, page 100
$\mathcal{P}_{\mathrm{adm}}$	set of equivalence classes of admissible finite-dimensional representations of $\mathrm{OSp}(\mathcal{N}\|2)$, $\mathcal{N} = 1, 2$, page 251
$\mathcal{I}$	super interval (connected subset of $\Lambda^{1,1}$ or $\Lambda^{1,0}$), page 74

$\mathcal{M}(\mathcal{T})$	$\mathcal{T}$-point of $\mathcal{M}$ for algebro-geometric supermanifolds $\mathcal{M},\mathcal{T}$, page 14
$\mathcal{O}(\mathcal{M}) := \mathcal{O}_M(M)$	superalgebra of global sections of the structure sheaf $\mathcal{O}_M$ of an algebro-geometric supermanifold $\mathcal{M}=(M,\mathcal{O}_M)$, page 15
$\mathrm{MaxSpec}_{\mathbb{R}}(\mathcal{O}(\mathcal{M}))$	maximal spectrum of $\mathcal{O}(\mathcal{M})$, page 16
$\mathrm{Spec}_{\mathbb{R}}(\mathcal{O}(\mathcal{M}))$	real spectrum of $\mathcal{O}(\mathcal{M})$, page 16
$\mathfrak{H} \equiv (\mathfrak{H},\mathscr{S},J)$	(pre-)super Hilbert space, page 240
$\mathrm{Op}(\mathcal{D},\mathfrak{H})$	space of (un)bounded operators $T:\ \mathcal{D}\subseteq\mathrm{dom}(T)\rightarrow\mathfrak{H}$ on a super Hilbert space $\mathfrak{H}$ with domain $\mathrm{dom}(T)$ containing a dense graded subspace $\mathcal{D}\subset\mathfrak{H}$, page 244
$\Gamma(\mathcal{E}_{/\mathcal{S}})$	smooth sections of the $\mathcal{S}$-relative super vector bundle $\mathcal{E}_{/\mathcal{S}}$, page 57
$\mathfrak{X}(\mathcal{M}_{/\mathcal{S}})$	super $H^\infty(\mathcal{S}\times\mathcal{M})$-module of smooth vector fields on a $\mathcal{S}$-relative supermanifold $\mathcal{M}_{/\mathcal{S}}$, page 56
$\underline{\mathrm{Hom}}_{L/R}(\mathcal{V},\mathcal{W})$	super $\mathcal{A}$-modules of left/right linear morphisms (both parity preserving and reversing) between super left/right $\mathcal{A}$-modules $\mathcal{V},\mathcal{W}$ with $\mathcal{A}$ a superalgebra, page 345
$\mathrm{Hom}_{L/R}(\mathcal{V},\mathcal{W})$	even part of $\underline{\mathrm{Hom}}_{L/R}(\mathcal{V},\mathcal{W})$, page 345
$\underline{\mathrm{End}}_{L/R}(\mathcal{V})$	$:= \underline{\mathrm{Hom}}_{L/R}(\mathcal{V},\mathcal{V})$, page 345
${}^*\mathcal{V}$	$:= \underline{\mathrm{Hom}}_{L}(\mathcal{V},\Lambda)$, left dual of a super Λ-module $\mathcal{V}$, page 42
$\mathcal{V}^*$	$:= \underline{\mathrm{Hom}}_{R}(\mathcal{V},\Lambda)$, right dual of a super Λ-module $\mathcal{V}$, page 42
$\Omega^k(\mathcal{M}_{/\mathcal{S}})$	super $H^\infty(\mathcal{S}\times\mathcal{M})$-module of k-forms on the $\mathcal{S}$-relative supermanifold $\mathcal{M}_{/\mathcal{S}}$, page 56
$\Omega^k(\mathcal{M}_{/\mathcal{S}},\mathcal{V})$	k-forms on $\mathcal{M}_{/\mathcal{S}}$ with values in a super Λ-vector space $\mathcal{V}$, page 56
$\Omega^k(\mathcal{M}_{/\mathcal{S}},\mathfrak{g})$	k-forms on $\mathcal{M}_{/\mathcal{S}}$ with values in the super Lie module $\mathrm{Lie}(\mathcal{G}) = \Lambda\otimes\mathfrak{g}$ of a super Lie group $\mathcal{G}$, page 56
$\Omega^k_{hor}(\mathcal{P}_{/\mathcal{S}},\mathcal{V})^{(\mathcal{G},\rho)}$	horizontal $\mathcal{V}$-valued k-forms of type $(\mathcal{G},\rho)$, page 65

$\Omega^k(\mathcal{M}_{/\mathcal{S}}, \mathcal{E}_{/\mathcal{S}})$	k-forms on $\mathcal{M}_{/\mathcal{S}}$ with values in an associated $\mathcal{S}$-relative super vector bundle $\mathcal{E}_{/\mathcal{S}}$, page 87
$H^\infty_{\mathcal{M}}$	sheaf of H^∞-smooth functions on a H^∞ supermanifold $\mathcal{M}$, page 21
$H_{\mathrm{AP}}(\mathbb{C})$	space of almost periodic holomorphic functions on the complex plane $\mathbb{C}$, page 312
$T_{\gamma,\vec{\pi},\vec{m},\vec{n}}$	(gauge-variant) spin network state, page 251
$\alpha_{\mathcal{S}}$	isomorphism from the set $\mathrm{Hom}_{\mathbf{SMan}_{/\mathcal{S}}}(\mathcal{M}_{/\mathcal{S}}, \mathcal{N}_{/\mathcal{S}})$ of $\mathcal{S}$-relative morphisms to $\mathrm{Hom}_{\mathbf{SMan}_{H^\infty}}(\mathcal{S}\times\mathcal{M}, \mathcal{N})$, page 55
$\epsilon_{m,n} : \Lambda^{m,n} \to \mathbb{R}^m$	body map, page 356
$\int_{\mathcal{G}}$	invariant integral on a super Lie group $\mathcal{G}$, page 234
$\int_B$	Berezin integral, page 154
L_X	Lie derivative along a smooth vector field $X \in \mathfrak{X}(\mathcal{M}_{/\mathcal{S}})$, page 56
ι_X	interior derivative w.r.t. a smooth vector field $X \in \mathfrak{X}(\mathcal{M}_{/\mathcal{S}})$, page 56
$\mathbf{G}(f)$	Grassmann analytic continuation or Grassmann extension of a C^∞-smooth function f, page 356
$\mathcal{D} = \partial_\theta + \theta\partial_t$	right-invariant vector field on the super translation group, page 74
$\mathfrak{C}$	involution on a super Λ-module, page 67
Ad, ad	Adjoint/adjoint representation of a super Lie group $\mathcal{G}$/super Lie module Lie($\mathcal{G}$), page 61
$D^{(\mathcal{A})}$	covariant derivative induced by a super connection 1-form $\mathcal{A}$, page 65
$\mathrm{d}_{\mathcal{A}}$	exterior covariant derivative induced by $\mathcal{A}$, page 87
$\nabla^{(\mathcal{A})}$	exterior covariant derivative induced by $\mathcal{A}$ restricted to sections of an associated $\mathcal{S}$-relative super vector bundle, page 87

$\nabla\Psi^{\alpha}_{r}$	covariant derivative of a Majorana spinor with values in the Lie algebra of the R-symmetry subgroup of $\mathrm{OSp}(\mathcal{N}\vert 4)$, page 200
ev_x	evaluation morphism at the point x, page 13
$\mathscr{P}^{\mathcal{A}}_{\mathcal{S},\gamma}$	parallel transport map along a path γ w.r.t. a super connection 1-form $\mathcal{A}$, page 80
$\mathscr{P}^{\mathcal{E},\mathcal{A}}_{\mathcal{S},\gamma}$	parallel transport map on an associated $\mathcal{S}$-relative super vector bundle $\mathcal{E}_{/\mathcal{S}}$ along a path γ with respect to a super connection 1-form $\mathcal{A}$, page 88
$W_\gamma[\mathcal{A}]$	super Wilson loop along a path γ induced by a super connection 1-form $\mathcal{A}$, page 86
$\mathscr{S}, \mathscr{T}$	super metric resp. Hermitian super metric or super scalar product, page 29
$\overline{\omega}$	k-form with values on associated $\mathcal{S}$-relative super vector bundle $(\mathcal{P}\times_\rho\mathcal{V})_{/\mathcal{S}}$ corresponding to $\omega\in\Omega^k_{hor}(\mathcal{P}_{/\mathcal{S}},\mathcal{V})^{(\mathcal{G},\rho)}$, page 87
$\phi\diamond\psi$	composition of two left linear morphisms ϕ, ψ, page 345
$\phi_{\mathcal{T}}$	natural transformation between $\mathcal{T}$-points $\phi_{\mathcal{T}}:\mathcal{M}(\mathcal{T})\to\mathcal{N}(\mathcal{T})$ induced by a morphism $\phi:\mathcal{M}\to\mathcal{N}$, page 15
ρ_L, ρ_R	left resp. right regular representation of a super Lie group $\mathcal{G}$, page 236
$\widetilde{X}$	fundamental vector field generated by $X\in\mathfrak{g}$, page 60
$\widetilde{X}_p$	fundamental tangent vector at a point p of a $\mathcal{S}$-relative principal super fiber bundle $\mathcal{P}_{/\mathcal{S}}$ generated by $X\in\mathrm{Lie}(\mathcal{G})$, page 60
L_g, R_g	left resp. right translation on a super Lie group $\mathcal{G}$ w.r.t. $g\in\mathcal{G}$, page 44
$L_{\underline{A}}, R_{\underline{A}}$	left- resp. right-invariant vector field generated by an element $T_{\underline{A}}$ of a real homogeneous basis $(T_{\underline{A}})_{\underline{A}}$ of a super Lie algebra, page 231
$(\kappa_{\mathbb{R}}, \Delta_{\mathbb{R}})$	Majorana representation of $\mathrm{Spin}^+(1,3)$, page 130

$\langle \cdot, \cdot \rangle_\beta$	β-deformed inner product induced by $\mathbf{P}_\beta$, page 192
$\langle \cdot \vert \cdot \rangle_J = \mathscr{S}(\cdot \vert J \cdot)$	positive definite inner product on a super Hilbert space induced by the endomorphism $J : \mathfrak{H} \to \mathfrak{H}$, page 240
χ_Δ	characteristic function of a Borel set Δ, i.e., $\chi_\Delta(x) = 1$ if $x \in \Delta$ and $\chi_\Delta(x) = 0$ if $x \notin \Delta$, page 158

Bibliography

[1] K. Eder, "Super fiber bundles, connection forms, and parallel transport," J. Math. Phys. **62** (2021), 063506 doi:10.1063/5.0044343 [arXiv:2101.00924 [math.DG]].

[2] K. Eder and H. Sahlmann, "$\mathcal{N} = 1$ Supergravity with loop quantum gravity methods and quantization of the SUSY constraint," Phys. Rev. D **103** (2021) no.4, 046010 doi:10.1103/PhysRevD.103.046010 [arXiv:2011.00108 [gr-qc]].

[3] K. Eder and H. Sahlmann, "Holst-MacDowell-Mansouri action for (extended) supergravity with boundaries and super Chern-Simons theory," JHEP **07** (2021), 071 doi:10.1007/JHEP07(2021)071 [arXiv:2104.02011 [gr-qc]].

[4] K. Eder and H. Sahlmann, "Supersymmetric minisuperspace models in self-dual loop quantum cosmology," JHEP **21** (2020), 064 doi:10.1007/JHEP03(2021)064 [arXiv:2010.15629 [gr-qc]].

[5] K. Eder, "Super Cartan geometry and the super Ashtekar connection," [arXiv:2010.09630 [gr-qc]].

[6] A. Strominger and C. Vafa, "Microscopic origin of the Bekenstein-Hawking entropy," Phys. Lett. B **379** (1996), 99-104 doi:10.1016/0370-2693(96)00345-0 [arXiv:hep-th/9601029 [hep-th]].

[7] C. G. Callan and J. M. Maldacena, "D-brane approach to black hole quantum mechanics," Nucl. Phys. B **472** (1996), 591-610 doi:10.1016/0550-3213(96)00225-8 [arXiv:hep-th/9602043 [hep-th]].

[8] J. M. Maldacena and L. Susskind, "D-branes and fat black holes," Nucl. Phys. B **475** (1996), 679-690 doi:10.1016/0550-3213(96)00323-9 [arXiv:hep-th/9604042 [hep-th]].

[9] K. Behrndt, G. Lopes Cardoso, B. de Wit, R. Kallosh, D. Lust and T. Mohaupt, "Classical and quantum N=2 supersymmetric black holes," Nucl. Phys. B **488** (1997), 236-260 doi:10.1016/S0550-3213(97)00028-X [arXiv:hep-th/9610105 [hep-th]].

[10] M. Cvetic and D. Youm, "Dyonic BPS saturated black holes of heterotic string on a six torus," Phys. Rev. D **53** (1996), 584-588 doi:10.1103/PhysRevD.53.R584 [arXiv:hep-th/9507090 [hep-th]].

[11] M. Cvetic and A. A. Tseytlin, "Solitonic strings and BPS saturated dyonic black holes," Phys. Rev. D **53** (1996), 5619-5633 [erratum: Phys. Rev. D **55** (1997), 3907] doi:10.1103/PhysRevD.53.5619 [arXiv:hep-th/9512031 [hep-th]].

[12] R. Kallosh and B. Kol, "E(7) symmetric area of the black hole horizon," Phys. Rev. D **53** (1996), R5344-R5348 doi:10.1103/PhysRevD.53.R5344 [arXiv:hep-th/9602014 [hep-th]].

[13] M. Cvetic and C. M. Hull, "Black holes and U duality," Nucl. Phys. B **480** (1996), 296-316 doi:10.1016/S0550-3213(96)00449-X [arXiv:hep-th/9606193 [hep-th]].

[14] J. M. Maldacena, "The Large N limit of superconformal field theories and supergravity," Adv. Theor. Math. Phys. **2** (1998), 231-252 doi:10.1023/A:1026654312961 [arXiv:hep-th/9711200 [hep-th]].

[15] E. Witten, "Anti-de Sitter space and holography," Adv. Theor. Math. Phys. **2** (1998), 253-291 doi:10.4310/ATMP.1998.v2.n2.a2 [arXiv:hep-th/9802150 [hep-th]].

[16] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, "Large N field theories, string theory and gravity," Phys. Rept. **323** (2000), 183-386 doi:10.1016/S0370-1573(99)00083-6 [arXiv:hep-th/9905111 [hep-th]].

[17] A. Ashtekar and J. Lewandowski, "Background independent quantum gravity: A Status report," Class. Quant. Grav. **21** (2004), R53 doi:10.1088/0264-9381/21/15/R01 [arXiv:gr-qc/0404018 [gr-qc]].

[18] T. Thiemann, "Modern Canonical Quantum General Relativity," Cambridge University Press, 2007.

[19] A. Sen, "GRAVITY AS A SPIN SYSTEM," Phys. Lett. B **119**, 89-91 (1982) doi:10.1016/0370-2693(82)90250-7

[20] A. Ashtekar, "New Variables for Classical and Quantum Gravity," Phys. Rev. Lett. **57** (1986), 2244-2247 doi:10.1103/PhysRevLett.57.2244

[21] J. F. Barbero G., Phys. Rev. D **51**, 5507-5510 (1995) doi:10.1103/PhysRevD.51.5507 [arXiv:gr-qc/9410014 [gr-qc]].

[22] G. Immirzi, "Real and complex connections for canonical gravity," Class. Quant. Grav. **14**, L177-L181 (1997) doi:10.1088/0264-9381/14/10/002 [arXiv:gr-qc/9612030 [gr-qc]].

[23] S. Holst, "Barbero's Hamiltonian derived from a generalized Hilbert-Palatini action," Phys. Rev. D **53** (1996), 5966-5969 doi:10.1103/PhysRevD.53.5966 [arXiv:gr-qc/9511026 [gr-qc]].

[24] C. Rovelli and L. Smolin, "Knot Theory and Quantum Gravity," Phys. Rev. Lett. **61** (1988), 1155 doi:10.1103/PhysRevLett.61.1155

[25] C. Rovelli and L. Smolin, "Loop Space Representation of Quantum General Relativity," Nucl. Phys. B **331** (1990), 80-152 doi:10.1016/0550-3213(90)90019-A

[26] A. Ashtekar, J. Lewandowski, D. Marolf, J. Mourao and T. Thiemann, "Quantization of diffeomorphism invariant theories of connections with local degrees of freedom," J. Math. Phys. **36** (1995), 6456-6493 doi:10.1063/1.531252 [arXiv:gr-qc/9504018 [gr-qc]].

[27] D. Marolf, J. Mourao and T. Thiemann, "The Status of diffeomorphism superselection in Euclidean (2+1) gravity," J. Math. Phys. **38** (1997), 4730-4740 doi:10.1063/1.532118 [arXiv:gr-qc/9701068 [gr-qc]].

[28] L. Smolin, "Recent developments in nonperturbative quantum gravity," [arXiv:hep-th/9202022 [hep-th]].

[29] C. Rovelli and L. Smolin, "Discreteness of area and volume in quantum gravity," Nucl. Phys. B **442**, 593-622 (1995) [erratum: Nucl. Phys. B **456**, 753-754 (1995)] doi:10.1016/0550-3213(95)00150-Q [arXiv:gr-qc/9411005 [gr-qc]].

[30] A. Ashtekar and J. Lewandowski, "Quantum theory of geometry. 1: Area operators," Class. Quant. Grav. **14** (1997), A55-A82 doi:10.1088/0264-9381/14/1A/006 [arXiv:gr-qc/9602046 [gr-qc]].

[31] J. Lewandowski, "Volume and quantizations," Class. Quant. Grav. **14** (1997), 71-76 doi:10.1088/0264-9381/14/1/010 [arXiv:gr-qc/9602035 [gr-qc]].

[32] A. Ashtekar and J. Lewandowski, "Quantum theory of geometry. 2. Volume operators," Adv. Theor. Math. Phys. **1** (1998), 388-429 doi:10.4310/ATMP.1997.v1.n2.a8 [arXiv:gr-qc/9711031 [gr-qc]].

[33] A. Perez, "The Spin Foam Approach to Quantum Gravity," Living Rev. Rel. **16** (2013), 3 doi:10.12942/lrr-2013-3 [arXiv:1205.2019 [gr-qc]].

[34] C. Rovelli "Quantum Gravity," Cambridge: Cambridge University Press (2004).

[35] L. Smolin, "Linking topological quantum field theory and nonperturbative quantum gravity," J. Math. Phys. **36**, 6417 (1995) [arXiv:gr-qc/9505028].

[36] A. Ashtekar, C. Beetle and S. Fairhurst, "Mechanics of isolated horizons," Class. Quant. Grav. **17** (2000), 253-298 doi:10.1088/0264-9381/17/2/301 [arXiv:gr-qc/9907068 [gr-qc]].

[37] A. Ashtekar, J. C. Baez and K. Krasnov, "Quantum geometry of isolated horizons and black hole entropy," Adv. Theor. Math. Phys. **4**, 1-94 (2000) doi:10.4310/ATMP.2000.v4.n1.a1 [arXiv:gr-qc/0005126 [gr-qc]].

[38] R. K. Kaul and P. Majumdar, "Logarithmic correction to the Bekenstein-Hawking entropy," Phys. Rev. Lett. **84**, 5255 (2000) [arXiv:gr-qc/0002040].

[39] J. Engle, K. Noui, A. Perez and D. Pranzetti, "Black hole entropy from an SU(2)-invariant formulation of Type I isolated horizons," Phys. Rev. D **82**, 044050 (2010) [arXiv:1006.0634 [gr-qc]].

[40] C. Rovelli, "Black hole entropy from loop quantum gravity," Phys. Rev. Lett. **77**, 3288 (1996) [arXiv:gr-qc/9603063].

[41] A. Ashtekar, J. Baez, A. Corichi and K. Krasnov, "Quantum geometry and black hole entropy," Phys. Rev. Lett. **80**, 904-907 (1998) doi:10.1103/PhysRevLett.80.904 [arXiv:gr-qc/9710007 [gr-qc]].

[42] M. Domagala and J. Lewandowski, "Black hole entropy from quantum geometry," Class. Quant. Grav. **21**, 5233 (2004) [arXiv:gr-qc/0407051].

[43] J. Engle, A. Perez and K. Noui, "Black hole entropy and SU(2) Chern-Simons theory," Phys. Rev. Lett. **105**, 031302 (2010) doi:10.1103/PhysRevLett.105.031302 [arXiv:0905.3168 [gr-qc]].

[44] I. Agullo, J. Fernando Barbero, E. F. Borja, J. Diaz-Polo and E. J. S. Villasenor, "Detailed black hole state counting in loop quantum gravity," Phys. Rev. D **82**, 084029 (2010) [arXiv:1101.3660 [gr-qc]].

[45] K. Eder and H. Sahlmann, "Quantum theory of charged isolated horizons," Phys. Rev. D **97** (2018) no.8, 086016 doi:10.1103/PhysRevD.97.086016 [arXiv:1801.00747 [gr-qc]].

[46] M. Bojowald, "Isotropic loop quantum cosmology," Class. Quant. Grav. **19** (2002), 2717-2742 doi:10.1088/0264-9381/19/10/313 [arXiv:gr-qc/0202077 [gr-qc]].

[47] A. Ashtekar, M. Bojowald and J. Lewandowski, "Mathematical structure of loop quantum cosmology," Adv. Theor. Math. Phys. **7** (2003) no.2, 233-268 doi:10.4310/ATMP.2003.v7.n2.a2 [arXiv:gr-qc/0304074 [gr-qc]].

[48] A. Ashtekar, T. Pawlowski and P. Singh, "Quantum nature of the big bang," Phys. Rev. Lett. **96**, 141301 (2006) doi:10.1103/PhysRevLett.96.141301 [arXiv:gr-qc/0602086 [gr-qc]].

[49] M. Bojowald, "Loop quantum cosmology," Living Rev. Rel. **11** (2008), 4

[50] A. Ashtekar and P. Singh, "Loop Quantum Cosmology: A Status Report," Class. Quant. Grav. **28** (2011), 213001 doi:10.1088/0264-9381/28/21/213001 [arXiv:1108.0893 [gr-qc]].

[51] I. Agullo and P. Singh, "Loop Quantum Cosmology," doi:10.1142/9789813220003_0007 [arXiv:1612.01236 [gr-qc]].

[52] A. Ashtekar and M. Bojowald, "Quantum geometry and the Schwarzschild singularity," Class. Quant. Grav. **23** (2006), 391-411 doi:10.1088/0264-9381/23/2/008 [arXiv:gr-qc/0509075 [gr-qc]].

[53] L. Modesto, "Loop quantum black hole," Class. Quant. Grav. **23** (2006), 5587-5602 doi:10.1088/0264-9381/23/18/006 [arXiv:gr-qc/0509078 [gr-qc]].

[54] R. Gambini, J. Olmedo and J. Pullin, "Quantum black holes in Loop Quantum Gravity," Class. Quant. Grav. **31** (2014), 095009 doi:10.1088/0264-9381/31/9/095009 [arXiv:1310.5996 [gr-qc]].

[55] E. Bianchi, M. Christodoulou, F. D'Ambrosio, H. M. Haggard and C. Rovelli, "White Holes as Remnants: A Surprising Scenario for the End of a Black Hole," Class. Quant. Grav. **35** (2018) no.22, 225003 doi:10.1088/1361-6382/aae550 [arXiv:1802.04264 [gr-qc]].

[56] A. Ashtekar, J. Olmedo and P. Singh, "Quantum Transfiguration of Kruskal Black Holes," Phys. Rev. Lett. **121** (2018) no.24, 241301 doi:10.1103/PhysRevLett.121.241301 [arXiv:1806.00648 [gr-qc]].

[57] A. Ashtekar, J. Olmedo and P. Singh, "Quantum extension of the Kruskal spacetime," Phys. Rev. D **98** (2018) no.12, 126003 doi:10.1103/PhysRevD.98.126003 [arXiv:1806.02406 [gr-qc]].

[58] N. Bodendorfer, F. M. Mele and J. Münch, "Effective Quantum Extended Spacetime of Polymer Schwarzschild Black Hole," Class. Quant. Grav. **36** (2019) no.19, 195015 doi:10.1088/1361-6382/ab3f16 [arXiv:1902.04542 [gr-qc]].

[59] S. R. Coleman and J. Mandula, "All Possible Symmetries of the S Matrix," Phys. Rev. **159** (1967), 1251-1256 doi:10.1103/PhysRev.159.1251

[60] R. Haag, J. T. Lopuszanski and M. Sohnius, "All Possible Generators of Supersymmetries of the s Matrix," Nucl. Phys. B **88** (1975), 257 doi:10.1016/0550-3213(75)90279-5

[61] C. Carmeli, G. Cassinelli, A. Toigo and V. S. Varadarajan, "Unitary representations of super Lie groups and applications to the classification and multiplet structure of super particles," Commun. Math. Phys. **263** (2006), 217-258 [erratum: Commun. Math. Phys. **307** (2011), 565-566] doi:10.1007/s00220-005-1452-0 [arXiv:hep-th/0501061 [hep-th]].

[62] T. Jacobson, "New Variables for Canonical Supergravity," Class. Quant. Grav. **5** (1988), 923 doi:10.1088/0264-9381/5/6/012

[63] G. Fulop, "About a superAshtekar-Renteln ansatz," Class. Quant. Grav. **11** (1994), 1-10 doi:10.1088/0264-9381/11/1/005 [arXiv:gr-qc/9305001 [gr-qc]].

[64] M. Sawaguchi, "Canonical formalism of N=1 supergravity with the real Ashtekar variables," Class. Quant. Grav. **18** (2001), 1179-1186 doi:10.1088/0264-9381/18/7/303

[65] M. Tsuda, "Generalized Lagrangian of N=1 supergravity and its canonical constraints with the real Ashtekar variable," Phys. Rev. D **61** (2000), 024025 doi:10.1103/PhysRevD.61.024025 [arXiv:gr-qc/9906057 [gr-qc]].

[66] N. Bodendorfer, T. Thiemann and A. Thurn, "Towards Loop Quantum Supergravity (LQSG)," Phys. Lett. B **711**, 205-211 (2012) doi:10.1016/j.physletb.2012.04.003 [arXiv:1106.1103 [gr-qc]].

[67] N. Bodendorfer, T. Thiemann and A. Thurn, "Towards Loop Quantum Supergravity (LQSG) I. Rarita-Schwinger Sector," Class. Quant. Grav. **30** (2013), 045006 doi:10.1088/0264-9381/30/4/045006 [arXiv:1105.3709 [gr-qc]].

[68] N. Bodendorfer, T. Thiemann and A. Thurn, "Towards Loop Quantum Supergravity (LQSG) II. p-Form Sector," Class. Quant. Grav. **30** (2013), 045007 doi:10.1088/0264-9381/30/4/045007 [arXiv:1105.3710 [gr-qc]].

[69] N. Bodendorfer, T. Thiemann and A. Thurn, "New Variables for Classical and Quantum Gravity in all Dimensions I. Hamiltonian Analysis," Class. Quant. Grav. **30** (2013), 045001 doi:10.1088/0264-9381/30/4/045001 [arXiv:1105.3703 [gr-qc]].

[70] Y. Ne'eman and T. Regge, "Gravity and Supergravity as Gauge Theories on a Group Manifold," Phys. Lett. B **74** (1978), 54-56 doi:10.1016/0370-2693(78)90058-8

[71] R. D'Auria and P. Fre, "Geometric Supergravity in d = 11 and Its Hidden Supergroup," Nucl. Phys. B **201** (1982), 101-140 [erratum: Nucl. Phys. B **206** (1982), 496] doi:10.1016/0550-3213(82)90281-4

[72] L. Castellani, R. D'Auria and P. Fre, "Supergravity and superstrings: A Geometric perspective. Vol. 1: Mathematical foundations," World Scientific (1991) 1-603, Singapore.

[73] T. Schmitt, "Supergeometry and quantum field theory, or: What is a classical configuration?," Rev. Math. Phys. **9** (1997), 993-1052 doi:10.1142/S0129055X97000348 [arXiv:hep-th/9607132 [hep-th]].

[74] P. Deligne, P. Etingof, D. Freed, L. Jeffrey, D. Kazhdan, J. Morgan, D. Morrison and E. Witten, "Quantum fields and strings: A course for mathematicians. Vol. 1"

[75] E. Keßler, "Supergeometry, Super Riemann Surfaces and the Superconformal Action Functional," Lect. Notes Math. **2230** (2019), pp. doi:10.1007/978-3-030-13758-8

[76] K. Rejzner, "Fermionic fields in the functional approach to classical field theory," Rev. Math. Phys. **23** (2011), 1009-1033 doi:10.1142/S0129055X11004503 [arXiv:1101.5126 [math-ph]].

[77] K. Rejzner, "Perturbative Algebraic Quantum Field Theory: An Introduction for Mathematicians," Math. Phys. Stud. (2016) doi:10.1007/978-3-319-25901-7

[78] F. Dumitrescu, "Superconnections and parallel transport," Pacific Journal of Mathematics 236.2 (2008): 307-332.

[79] J. Groeger, "Super Wilson Loops and Holonomy on Supermanifolds," [arXiv:1312.4745 [math-ph]].

[80] T. Thiemann, "Kinematical Hilbert spaces for Fermionic and Higgs quantum field theories," Class. Quant. Grav. **15** (1998), 1487-1512 doi:10.1088/0264-9381/15/6/006 [arXiv:gr-qc/9705021 [gr-qc]].

[81] L. Andrianopoli and R. D'Auria, "N=1 and N=2 pure supergravities on a manifold with boundary," JHEP **08** (2014), 012 doi:10.1007/JHEP08(2014)012 [arXiv:1405.2010 [hep-th]].

[82] L. Andrianopoli, B. L. Cerchiai, R. D'Auria and M. Trigiante, "Unconventional supersymmetry at the boundary of AdS_4 supergravity," JHEP **04** (2018), 007 doi:10.1007/JHEP04(2018)007 [arXiv:1801.08081 [hep-th]].

[83] L. Andrianopoli, B. L. Cerchiai, R. Matrecano, O. Miskovic, R. Noris, R. Olea, L. Ravera and M. Trigiante, "$\mathcal{N} = 2$ AdS_4 supergravity, holography and Ward identities," JHEP **02** (2021), 141 doi:10.1007/JHEP02(2021)141 [arXiv:2010.02119 [hep-th]].

[84] R. Gambini, O. Obregon and J. Pullin, "Towards a loop representation for quantum canonical supergravity," Nucl. Phys. B **460** (1996), 615-631 doi:10.1016/0550-3213(95)00582-X [arXiv:hep-th/9508036 [hep-th]].

[85] Y. Ling and L. Smolin, "Supersymmetric spin networks and quantum supergravity," Phys. Rev. D **61** (2000), 044008 doi:10.1103/PhysRevD.61.044008 [arXiv:hep-th/9904016 [hep-th]].

[86] Y. Ling and L. Smolin, "Holographic formulation of quantum supergravity," Phys. Rev. D **63**, 064010 (2001) doi:10.1103/PhysRevD.63.064010 [arXiv:hep-th/0009018 [hep-th]].

[87] H. A. Morales-Tecotl and C. Rovelli, "Fermions in quantum gravity," Phys. Rev. Lett. **72** (1994), 3642-3645 doi:10.1103/PhysRevLett.72.3642 [arXiv:gr-qc/9401011 [gr-qc]].

[88] P. D. D'Eath, "Supersymmetric quantum cosmology," doi:10.1017/CBO9780 511524424

[89] P. D. D'Eath and D. I. Hughes, "Supersymmetric Minisuperspace," Phys. Lett. B **214**, 498-502 (1988) doi:10.1016/0370-2693(88)90107-4

[90] P. D. D'Eath and D. I. Hughes, "Minisuperspace with local supersymmetry," Nucl. Phys. B **378**, 381-409 (1992) doi:10.1016/0550-3213(92)90013-2

[91] E. Wilson-Ewing, "Loop quantum cosmology with self-dual variables," Phys. Rev. D **92** (2015) no.12, 123536 doi:10.1103/PhysRevD.92.123536 [arXiv:1503.07855 [gr-qc]].

[92] F. Berezin and D. Leites "Supermanifolds," Soviet Maths Doknaja **16**, pp. 1218-1222.

[93] B. Kostant, "Graded Manifolds, Graded Lie Theory, and Prequantization," Lect. Notes Math. **570** (1977), 177-306 doi:10.1007/BFb0087788

[94] B. DeWitt, "Supermanifolds," Cambridge University Press, Cambridge 1984.

[95] A. Rogers, "A global theory of supermanifolds," Journal of Mathematical Physics 21.6 (1980): 1352-1365.

[96] A. Rogers, "Supermanifolds: Theory and Applications," World Scientific (2007).

[97] G. M. Tuynman, "Supermanifolds and Supergroups-Basic Theory," Kluwer Academic Publishers (2004), doi:10.1007/1-4020-2297-2

[98] V. Molotkov, "infinite-dimensional and colored supermanifolds," Journal of Nonlinear Mathematical Physics, 17:sup1, 375-446 (2010), doi: 10.1142/S140 292511000088X

[99] C. Sachse, "A categorical formulation of superalgebra and supergeometry," arXiv:0802.4067v1 [math.AG].

[100] C. Sachse, "Global Analytic Approach to Super Teichmueller Spaces," arXiv:0902.3289v1 [math.AG].

[101] T. P. Hack, F. Hanisch and A. Schenkel, "Supergeometry in locally covariant quantum field theory," Commun. Math. Phys. **342** (2016) no.2, 615-673 doi:10.1007/s00220-015-2516-4 [arXiv:1501.01520 [math-ph]].

[102] T. Stavracou, "Theory of connections on graded principal bundles," Rev. Math. Phys. **10** (1998), 47-80 doi:10.1142/S0129055X98000033 [arXiv:dg-ga/9605006 [math.DG]].

[103] S. Garnier and T. Wurzbacher, "The geodesic flow on a Riemannian supermanifold," J. Geom. Phys. **62** (2012) 1489–1508.

[104] M. J. D. Hamilton, "Mathematical Gauge Theory: With Applications to the Standard Model of Particle Physics," Universitext. Springer International Publishing, 2017.

[105] H. Baum, "Eichfeldtheorie–Eine Einführung in die Differentialgeometrie auf Faserbündeln," Springer Spektrum, Berlin Heidelberg, 2014.

[106] C. Carmeli, L. Caston, R. Fioresi, "Mathematical Foundations of Supersymmetry," EMS Series of Lectures in Mathematics 15, European Mathematical Society, 2011.

[107] M. Batchelor, "The structure of supermanifolds," Transactions of the American Mathematical Society 253 (1979): 329-338.

[108] M. Batchelor, "Two approaches to supermanifolds," Transactions of the American Mathematical Society 258.1 (1980): 257-270.

[109] G. M. Tuynman, "Super unitary representations revisited," arXiv:1711.00233v2 [math.DG]

[110] J. Schütt, "Infinite-Dimensional Supermanifolds via Multilinear Bundles," arXiv preprint arXiv:1810.05549 (2018).

[111] S. J. Cheng and W. Wang, "Dualities and Representations of Lie Superalgebras," Graduate Studies in Mathematics 144, Amer. Math. Soc., Providence, RI, 2012.

[112] K. Coulembier and R. B. Zhang. "Invariant integration on orthosymplectic and unitary supergroups," Journal of Physics A: Mathematical and Theoretical 45.9 (2012): 095204.

[113] H. Nicolai, "REPRESENTATIONS OF SUPERSYMMETRY IN ANTI-DE SITTER SPACE," CERN-TH-3882.

[114] D. Z. Freedman and H. Nicolai, "Multiplet Shortening in Osp(N,4)," Nucl. Phys. B **237** (1984), 342-366 doi:10.1016/0550-3213(84)90164-0

[115] A. Wipf, "Introduction to Supersymmetry," Vorlesungsskript, Universität Jena, 2016.

[116] J. Jost, E. Keßler and J. Tolksdorf, "Super Riemann surfaces, metrics and gravitinos," Adv. Theor. Math. Phys. **21** (2017), 1161-1187 doi:10.4310/ATMP.2017.v21.n5.a2 [arXiv:1412.5146 [math-ph]].

[117] C. Bartocci, U. Bruzzo, D. Hernandez-Ruiperez, "The Geometry of Supermanifolds," Kluwer Academic Publishers, 1991.

[118] L. Mason and D. Skinner, "The Complete Planar S-matrix of N=4 SYM as a Wilson Loop in Twistor Space," JHEP **12** (2010), 018 doi:10.1007/JHEP12(2010)018 [arXiv:1009.2225 [hep-th]].

[119] U. Schreiber, "Differential cohomology in a cohesive infinity-topos," [arXiv: 1310.7930 [math-ph]].

[120] J. C. Baez and J. Huerta, "An Invitation to Higher Gauge Theory," Gen. Rel. Grav. **43** (2011), 2335-2392 doi:10.1007/s10714-010-1070-9 [arXiv:1003.4485 [hep-th]].

[121] E. De Paoli and S. Speziale, "A gauge-invariant symplectic potential for tetrad general relativity," JHEP **07** (2018), 040 doi:10.1007/JHEP07(2018)040 [arXiv:1804.09685 [gr-qc]].

[122] L. Freidel, M. Geiller and D. Pranzetti, "Edge modes of gravity. Part I. Corner potentials and charges," JHEP **11** (2020), 026 doi:10.1007/JHEP11(2020)026 [arXiv:2006.12527 [hep-th]].

[123] L. Freidel, M. Geiller and D. Pranzetti, "Edge modes of gravity. Part II. Corner metric and Lorentz charges," JHEP **11** (2020), 027 doi:10.1007/JHEP11(2020)027 [arXiv:2007.03563 [hep-th]].

[124] L. Freidel, M. Geiller and D. Pranzetti, "Edge modes of gravity. Part III. Corner simplicity constraints," JHEP **01** (2021), 100 doi:10.1007/JHEP01(2021)100 [arXiv:2007.12635 [hep-th]].

[125] P. B. Aneesh, S. Chakraborty, S. J. Hoque and A. Virmani, "First law of black hole mechanics with fermions," Class. Quant. Grav. **37** (2020) no.20, 205014 doi:10.1088/1361-6382/aba5ab [arXiv:2004.10215 [hep-th]].

[126] V. Iyer and R. M. Wald, "Some properties of Noether charge and a proposal for dynamical black hole entropy," Phys. Rev. D **50** (1994), 846-864 doi:10.1103/PhysRevD.50.846 [arXiv:gr-qc/9403028 [gr-qc]].

[127] K. Prabhu, "The First Law of Black Hole Mechanics for Fields with Internal Gauge Freedom," Class. Quant. Grav. **34** (2017) no.3, 035011 doi:10.1088/1361-6382/aa536b [arXiv:1511.00388 [gr-qc]].

[128] D. Z. Freedman, P. van Nieuwenhuizen and S. Ferrara, "Progress Toward a Theory of Supergravity," Phys. Rev. D **13** (1976), 3214-3218 doi:10.1103/PhysRevD.13.3214

[129] D. Fiorenza, H. Sati and U. Schreiber, "Super Lie n-algebra extensions, higher WZW models, and super p-branes with tensor multiplet fields," Int. J. Geom. Meth. Mod. Phys. **12** (2014), 1550018 doi:10.1142/S0219887815500188 [arXiv:1308.5264 [hep-th]].

[130] H. Sati and U. Schreiber, "Lie n-algebras of BPS charges," JHEP **03** (2017), 087 doi:10.1007/JHEP03(2017)087 [arXiv:1507.08692 [math-ph]].

[131] L. Castellani, R. Catenacci and P. A. Grassi, "Supergravity Actions with Integral Forms," Nucl. Phys. B **889** (2014), 419-442 doi:10.1016/j.nuclphysb.2014.10.023 [arXiv:1409.0192 [hep-th]].

[132] R. Catenacci, P. A. Grassi and S. Noja, "Superstring Field Theory, Superforms and Supergeometry," J. Geom. Phys. **148** (2020), 103559 doi:10.1016/j.geomphys.2019.103559 [arXiv:1807.09563 [hep-th]].

[133] D. K. Wise, "MacDowell-Mansouri gravity and Cartan geometry," Class. Quant. Grav. **27** (2010), 155010 doi:10.1088/0264-9381/27/15/155010 [arXiv:gr-qc/0611154 [gr-qc]].

[134] R. W. Sharpe, "Differential Geometry - Cartan's Generalisation of Klein's Erlangen Program," Springer-Verlag New York, 1997.

[135] A. Čap and J. Slovák, "Parabolic Geometries I: Background and general theory," Mathematical Surveys and Monographs, vol. 154, American Mathematical Society, 2009.

[136] M. Hammerl, "Homogeneous Cartan Geometries ," Diplomarbeit, Universität Wien, 2006.

[137] G. Catren, "Geometric foundations of Cartan gauge gravity," Int. J. Geom. Meth. Mod. Phys. **12** (2015) no.04, 1530002 doi:10.1142/S0219887815300020 [arXiv:1407.7814 [gr-qc]].

[138] S. W. MacDowell and F. Mansouri, "Unified Geometric Theory of Gravity and Supergravity," Phys. Rev. Lett. **38** (1977), 739 [erratum: Phys. Rev. Lett. **38** (1977), 1376] doi:10.1103/PhysRevLett.38.739

[139] O. Goertsches, "Riemannian supergeometry," Mathematische Zeitschrift 260.3 (2008): 557-593.

[140] D. V. Alekseevsky and P. W. Michor, "Differential geometry of Cartan connections," Publ. Math. Debrecen **47** (1995), 349-375.

[141] L. Castellani, "OSp(1|4) supergravity and its noncommutative extension," Phys. Rev. D **88** (2013) no.2, 025022 doi:10.1103/PhysRevD.88.025022 [arXiv:1301.1642 [hep-th]].

[142] L. Castellani, "Supergravity in the Group-Geometric Framework: A Primer," Fortsch. Phys. **66** (2018) no.4, 1800014 doi:10.1002/prop.201800014 [arXiv:1802.03407 [hep-th]].

[143] P. van Nieuwenhuizen, "Supergravity as a Yang–Mills Theory," doi:10.1142/9789812567147_0018 [arXiv:hep-th/0408137 [hep-th]].

[144] D. V. Alekseevsky, V. Cortes, C. Devchand and U. Semmelmann, "Killing spinors are Killing vector fields in Riemannian supergeometry," J. Geom. Phys. **26** (1998), 37-50 doi:10.1016/S0393-0440(97)00036-3 [arXiv:dg-ga/9704002 [math.DG]].

[145] V. Cortés, C. I. Lazaroiu and C. S. Shahbazi, "$\mathcal{N} = 1$ Geometric Supergravity and Chiral Triples on Riemann Surfaces," Commun. Math. Phys. **375** (2019) no.1, 429-478 doi:10.1007/s00220-019-03476-7 [arXiv:1810.12353 [hep-th]].

[146] P. D. D'Eath, "The Canonical Quantization of Supergravity," Phys. Rev. D **29**, 2199 (1984) [erratum: Phys. Rev. D **32**, 1593 (1985)] doi:10.1103/PhysRevD.29.2199

[147] E. R. Livine and R. Oeckl, "Three-dimensional Quantum Supergravity and Supersymmetric Spin Foam Models," Adv. Theor. Math. Phys. **7** (2003) no.6, 951-1001 doi:10.4310/ATMP.2003.v7.n6.a2 [arXiv:hep-th/0307251 [hep-th]].

[148] R. K. Kaul, "Holst Actions for Supergravity Theories," Phys. Rev. D **77** (2008), 045030 doi:10.1103/PhysRevD.77.045030 [arXiv:0711.4674 [gr-qc]].

[149] A. Macias, O. Obregón, and M. P. Ryan, "Quantum cosmology: the supersymmetric square root," Classical and Quantum Gravity, **4** (1987), 1477 doi:10.1088/0264-9381/4/6/007

[150] P. D. D'Eath, S. W. Hawking and O. Obregón, "Supersymmetric Bianchi models and the square root of the Wheeler-DeWitt equation," Phys. Lett. B **300** (1993), 44–48 doi:10.1016/0370-2693(93)90746-5

[151] O. Obregón and C. Ramírez, "Dirac-like formulation of quantum supersymmetric cosmology," Phys. Rev. D **57** (1998), 1015–1026 doi:10.1103/PhysRevD.57.1015

[152] D. Z. Freedman and A. Van Proeyen, "Supergravity," Cambridge Univ. Press, 2012.

[153] D. Giulini, "Dynamical and Hamiltonian formulation of General Relativity," [arXiv:1505.01403 [gr-qc]].

[154] Y. Tavakoli, "Lecture III: Ashtekar variables for general relativity." Lecture notes (2015), https://www.cosmo-ufes.org/uploads/1/3/7/0/13701821/lect.notes-3.pdf

[155] M. Bojowald and R. Das, "Canonical gravity with fermions," Phys. Rev. D **78**, 064009 (2008) doi:10.1103/PhysRevD.78.064009 [arXiv:0710.5722 [gr-qc]].

[156] T. Thiemann, "QSD 5: Quantum gravity as the natural regulator of matter quantum field theories," Class. Quant. Grav. **15** (1998), 1281-1314 doi:10.1088/0264-9381/15/5/012 [arXiv:gr-qc/9705019 [gr-qc]].

[157] T. Thiemann, "Quantum spin dynamics (QSD)," Class. Quant. Grav. **15** (1998), 839-873 doi:10.1088/0264-9381/15/4/011 [arXiv:gr-qc/9606089 [gr-qc]].

[158] R. De Pietri and C. Rovelli, "Geometry eigenvalues and scalar product from recoupling theory in loop quantum gravity," Phys. Rev. D **54** (1996), 2664-2690 doi:10.1103/PhysRevD.54.2664 [arXiv:gr-qc/9602023 [gr-qc]].

[159] J. Brunnemann and T. Thiemann, "Simplification of the spectral analysis of the volume operator in loop quantum gravity," Class. Quant. Grav. **23** (2006), 1289-1346 doi:10.1088/0264-9381/23/4/014 [arXiv:gr-qc/0405060 [gr-qc]].

[160] A. R. Edmonds, "Angular Momentum in Quantum Mechanics," Princeton University Press, Princeton, New Jersey, 1974.

[161] M. Assanioussi, J. Lewandowski and I. Mäkinen, "New scalar constraint operator for loop quantum gravity," Phys. Rev. D **92** (2015) no.4, 044042 doi:10.1103/PhysRevD.92.044042 [arXiv:1506.00299 [gr-qc]].

[162] J. D. Bekenstein, "Black holes and entropy," Phys. Rev. D **7**, 2333-2346 (1973) doi:10.1103/PhysRevD.7.2333

[163] J. M. Bardeen, B. Carter and S. W. Hawking, "The Four laws of black hole mechanics," Commun. Math. Phys. **31**, 161-170 (1973) doi:10.1007/BF01645742

[164] S. W. Hawking, "Particle Creation by Black Holes," Commun. Math. Phys. **43**, 199-220 (1975) [erratum: Commun. Math. Phys. **46**, 206 (1976)] doi:10.1007/BF02345020

[165] L. Susskind, "The World as a hologram," J. Math. Phys. **36** (1995), 6377-6396 doi:10.1063/1.531249 [arXiv:hep-th/9409089 [hep-th]].

[166] V. Mikhaylov and E. Witten, "Branes And Supergroups," Commun. Math. Phys. **340** (2015) no.2, 699-832 doi:10.1007/s00220-015-2449-y [arXiv:1410.1175 [hep-th]].

[167] G. W. Gibbons and S. W. Hawking, "Action Integrals and Partition Functions in Quantum Gravity," Phys. Rev. D **15** (1977), 2752-2756 doi:10.1103/PhysRevD.15.2752

[168] I. G. Avramidi and G. Esposito, "Gauge theories on manifolds with boundary," Commun. Math. Phys. **200** (1999), 495-543 doi:10.1007/s002200050539 [arXiv:hep-th/9710048 [hep-th]].

[169] P. van Nieuwenhuizen and D. V. Vassilevich, "Consistent boundary conditions for supergravity," Class. Quant. Grav. **22** (2005), 5029-5051 doi:10.1088/0264-9381/22/23/008 [arXiv:hep-th/0507172 [hep-th]].

[170] D. V. Belyaev and P. van Nieuwenhuizen, "Tensor calculus for supergravity on a manifold with boundary," JHEP **02** (2008), 047 doi:10.1088/1126-6708/2008/02/047 [arXiv:0711.2272 [hep-th]].

[171] P. Concha, L. Ravera and E. Rodríguez, "On the supersymmetry invariance of flat supergravity with boundary," JHEP **01** (2019), 192 doi:10.1007/JHEP01(2019)192 [arXiv:1809.07871 [hep-th]].

[172] M. C. Ipinza, P. K. Concha, L. Ravera and E. K. Rodríguez, "On the Supersymmetric Extension of Gauss-Bonnet like Gravity," JHEP **09** (2016), 007 doi:10.1007/JHEP09(2016)007 [arXiv:1607.00373 [hep-th]].

[173] P. K. Concha and E. K. Rodríguez, "N = 1 Supergravity and Maxwell superalgebras," JHEP **09** (2014), 090 doi:10.1007/JHEP09(2014)090 [arXiv:1407.4635 [hep-th]].

[174] A. Ashtekar, C. Beetle, O. Dreyer, S. Fairhurst, B. Krishnan, J. Lewandowski and J. Wisniewski, "Isolated horizons and their applications," Phys. Rev. Lett. **85**, 3564-3567 (2000) doi:10.1103/PhysRevLett.85.3564 [arXiv:gr-qc/0006006 [gr-qc]].

[175] A. Ashtekar, C. Beetle and J. Lewandowski, "Geometry of generic isolated horizons," Class. Quant. Grav. **19**, 1195-1225 (2002) doi:10.1088/0264-9381/19/6/311 [arXiv:gr-qc/0111067 [gr-qc]].

[176] N. Bodendorfer, T. Thiemann and A. Thurn, "New Variables for Classical and Quantum Gravity in all Dimensions V. Isolated Horizon Boundary Degrees of Freedom," Class. Quant. Grav. **31**, 055002 (2014) doi:10.1088/0264-9381/31/5/055002 [arXiv:1304.2679 [gr-qc]].

[177] A. Ashtekar and B. Krishnan, "Isolated and dynamical horizons and their applications," Living Rev. Rel. **7**, 10 (2004) doi:10.12942/lrr-2004-10 [arXiv:gr-qc/0407042 [gr-qc]].

[178] E. R. Livine and J. P. Ryan, "N=2 supersymmetric spin foams in three dimensions," Class. Quant. Grav. **25** (2008), 175014 doi:10.1088/0264-9381/25/17/175014 [arXiv:0710.3540 [gr-qc]].

[179] M. Tsuda and T. Shirafuji, "The Canonical formulation of N=2 supergravity in terms of the Ashtekar variable," Phys. Rev. D **62** (2000), 064020 doi:10.1103/PhysRevD.62.064020 [arXiv:gr-qc/0003010 [gr-qc]].

[180] T. Sano, "The Ashtekar formalism and WKB wave functions of N=1, N=2 supergravities," [arXiv:hep-th/9211103 [hep-th]].

[181] K. Ezawa, "Ashtekar's formulation for N=1, N=2 supergravities as constrained BF theories," Prog. Theor. Phys. **95** (1996), 863-882 doi:10.1143/PTP.95.863 [arXiv:hep-th/9511047 [hep-th]].

[182] Y. Ling, R. S. Tung and H. Y. Guo, "Supergravity and Yang-Mills theories as generalized topological fields with constraints," Phys. Rev. D **70** (2004), 044045 doi:10.1103/PhysRevD.70.044045 [arXiv:hep-th/0310141 [hep-th]].

[183] A. Randono, "Generalizing the Kodama state. II. Properties and physical interpretation," [arXiv:gr-qc/0611074 [gr-qc]].

[184] D. K. Wise, "Symmetric space Cartan connections and gravity in three and four dimensions," SIGMA **5** (2009), 080 doi:10.3842/SIGMA.2009.080 [arXiv:0904.1738 [math.DG]].

[185] L. Freidel and A. Starodubtsev, "Quantum gravity in terms of topological observables," [arXiv:hep-th/0501191 [hep-th]].

[186] O. Obregon, M. Ortega-Cruz and M. Sabido, "Immirzi parameter and theta ambiguity in de Sitter MacDowell-Mansouri supergravity," Phys. Rev. D **85** (2012), 124061 doi:10.1103/PhysRevD.85.124061

[187] J. A. Nieto, J. Socorro and O. Obregon, "Gauge theory of supergravity based only on a selfdual spin connection," Phys. Rev. Lett. **76** (1996), 3482-3485 doi:10.1103/PhysRevLett.76.3482

[188] A. Perez, "Black Holes in Loop Quantum Gravity," Rept. Prog. Phys. **80** (2017) no.12, 126901 doi:10.1088/1361-6633/aa7e14 [arXiv:1703.09149 [gr-qc]].

[189] L. Andrianopoli, M. Bertolini, A. Ceresole, R. D'Auria, S. Ferrara, P. Fre and T. Magri, "N=2 supergravity and N=2 superYang-Mills theory on general scalar manifolds: Symplectic covariance, gaugings and the momentum map," J. Geom. Phys. **23** (1997), 111-189 doi:10.1016/S0393-0440(97)00002-8 [arXiv:hep-th/9605032 [hep-th]].

[190] H. T. Nieh and M. L. Yan, "An Identity in Riemann-cartan Geometry," J. Math. Phys. **23** (1982), 373 doi:10.1063/1.525379

[191] P. D. Alvarez, M. Valenzuela and J. Zanelli, "Supersymmetry of a different kind," JHEP **04** (2012), 058 doi:10.1007/JHEP04(2012)058 [arXiv:1109.3944 [hep-th]].

[192] A. Guevara, P. Pais and J. Zanelli, "Dynamical Contents of Unconventional Supersymmetry," JHEP **08** (2016), 085 doi:10.1007/JHEP08(2016)085 [arXiv:1606.05239 [hep-th]].

[193] P. D. Alvarez, P. Pais and J. Zanelli, "Unconventional supersymmetry and its breaking," Phys. Lett. B **735** (2014), 314-321 doi:10.1016/j.physletb.2014.06.031 [arXiv:1306.1247 [hep-th]].

[194] J. Greenstein and V. Mazorchuk, "Koszul duality for semidirect products and generalized Takiff algebras.," Algebras and Representation Theory 20.3 (2017): 675-694.

[195] M. Geiller, "Edge modes and corner ambiguities in 3d Chern–Simons theory and gravity," Nucl. Phys. B **924** (2017), 312-365 doi:10.1016/j.nuclphysb.2017.09.010 [arXiv:1703.04748 [gr-qc]].

[196] J. Lewandowski, A. Okolow, H. Sahlmann and T. Thiemann, "Uniqueness of diffeomorphism invariant states on holonomy-flux algebras," Commun. Math. Phys. **267** (2006), 703-733 doi:10.1007/s00220-006-0100-7 [arXiv:gr-qc/0504147 [gr-qc]].

[197] M. Scheunert, R. B. Zhang, "Invariant integration on classical and quantum Lie supergroups." J. Math. Phys. 42, 3871–3897 (2001).

[198] M. Scheunert, R. B. Zhang, "Integration on Lie supergroups: a Hopf algebra approach." J. Algebra 292, 324–342 (2005).

[199] K. Coulembier, "The orthosymplectic superalgebra in harmonic analysis." J. Lie Theory 23, 55–83 (2013) [arXiv:1202.0668 [math-ph]].

[200] D. A. Leites, "Introduction to the theory of supermanifolds," Russian Math. Surveys 35(1980), 1-64.

[201] M. J. Rothstein, "Integration on noncompact supermanifolds," Trans. A.M.S. 299(1987), 387-396.

[202] V. S. Varadarajan, "Supersymmetry for mathematicians: An introduction," Courant Lecture Notes 11, AMS, Providence, RI, 2004.

[203] J. Bognar, "Indefinite inner product spaces." Springer-Verlag, 1974.

[204] P. Bieliavsky, A. de Goursac, G. M. Tuynman, "Deformation quantization for Heisenberg supergroup," J. Func. Analysis 263 (2012), 549–603 [arXiv:1011.2370 [math.QA]].

[205] A. de Goursac, J. P. Michel, Superunitary representations of Heisenberg supergroups, arXiv:1601.07387 [math.RT].

[206] A. M. El Gradechi and L. M. Nieto, "Supercoherent states, superKahler geometry and geometric quantization," Commun. Math. Phys. **175** (1996), 521-564 doi:10.1007/BF02099508 [arXiv:hep-th/9403109 [hep-th]].

[207] F. A. Berezin, V. N. Tolstoy, "The group with Grassmann structure UOSP(1|2)," Commun. Math. Phys. 78 (1981) 409.

[208] D. Williams and J. F. Cornwell, "The Haar Integral for Lie Supergroups," J. Math. Phys. **25** (1984), 2922 doi:10.1063/1.526040

[209] T. Thiemann, "Reality conditions inducing transforms for quantum gauge field theory and quantum gravity," Class. Quant. Grav. **13** (1996), 1383-1404 doi:10.1088/0264-9381/13/6/012 [arXiv:gr-qc/9511057 [gr-qc]].

[210] E. Frodden, M. Geiller, K. Noui and A. Perez, "Black Hole Entropy from complex Ashtekar variables," EPL **107** (2014) no.1, 10005 doi:10.1209/0295-5075/107/10005 [arXiv:1212.4060 [gr-qc]].

[211] M. Han, "Black Hole Entropy in Loop Quantum Gravity, Analytic Continuation, and Dual Holography," [arXiv:1402.2084 [gr-qc]].

[212] J. Ben Achour, A. Mouchet and K. Noui, "Analytic Continuation of Black Hole Entropy in Loop Quantum Gravity," JHEP **06** (2015), 145 doi:10.1007/JHEP06(2015)145 [arXiv:1406.6021 [gr-qc]].

[213] J. Ben Achour, "Towards self dual Loop Quantum Gravity," [arXiv:1511.07332 [gr-qc]].

[214] N. Bodendorfer and Y. Neiman, "Imaginary action, spinfoam asymptotics and the 'transplanckian' regime of loop quantum gravity," Class. Quant. Grav. **30** (2013), 195018 doi:10.1088/0264-9381/30/19/195018 [arXiv:1303.4752 [gr-qc]].

[215] R. Seeger and H. Sahlmann, "Revisiting loop quantum gravity with selfdual variables," in preparation.

[216] M. Scheunert, W. Nahm and V. Rittenberg, "Irreducible Representations of the OSP(2,1) and SPL(2,1) Graded Lie Algebras," J. Math. Phys. **18** (1977), 155 doi:10.1063/1.523149

[217] M. Marcu, "The Representations of Spl(2,1): An Example of Representations of Basic Superalgebras," J. Math. Phys. **21** (1980), 1277 doi:10.1063/1.524576

[218] A. Corichi, U. Nucamendi and D. Sudarsky, "Einstein-Yang-Mills isolated horizons: Phase space, mechanics, hair and conjectures," Phys. Rev. D **62** (2000), 044046 doi:10.1103/PhysRevD.62.044046 [arXiv:gr-qc/0002078 [gr-qc]].

[219] A. Ashtekar, S. Fairhurst and B. Krishnan, "Isolated horizons: Hamiltonian evolution and the first law," Phys. Rev. D **62** (2000), 104025 doi:10.1103/Phys RevD.62.104025 [arXiv:gr-qc/0005083 [gr-qc]].

[220] C. A. Cremonini and P. A. Grassi, "Pictures from Super Chern-Simons Theory," JHEP **03** (2020), 043 doi:10.1007/JHEP03(2020)043 [arXiv:1907.07152 [hep-th]].

[221] K. Eder, "Quantum theory of charged black hole horizons," Master thesis, Ludwig-Maximilians-Universtität München, 2017.

[222] A. Ghosh and D. Pranzetti, "CFT/Gravity Correspondence on the Isolated Horizon," Nucl. Phys. B **889** (2014), 1-24 doi:10.1016/j.nuclphysb.2014.10.002 [arXiv:1405.7056 [gr-qc]].

[223] L. Freidel, A. Perez and D. Pranzetti, "Loop gravity string," Phys. Rev. D **95** (2017) no.10, 106002 doi:10.1103/PhysRevD.95.106002 [arXiv:1611.03668 [gr-qc]].

[224] S. Carlip, "Symmetries, Horizons, and Black Hole Entropy," Gen. Rel. Grav. **39** (2007), 1519-1523 doi:10.1142/S0218271808012401 [arXiv:0705.3024 [gr-qc]].

[225] S. Carlip, "Effective Conformal Descriptions of Black Hole Entropy," Entropy **13** (2011), 1355-1379 doi:10.3390/e13071355 [arXiv:1107.2678 [gr-qc]].

[226] K. Iohara, Y. Koga, "Central Extensions of Lie superalgebras," Comment. Math. Helv. **76** (2001), 110-154 doi:10.4171/CMH/246

[227] P. Goddard, A. Kent and D. I. Olive, "Unitary Representations of the Virasoro and Supervirasoro Algebras," Commun. Math. Phys. **103** (1986), 105-119 doi:10.1007/BF01464283

[228] T. Quella and V. Schomerus, "Free fermion resolution of supergroup WZNW models," JHEP **09** (2007), 085 doi:10.1088/1126-6708/2007/09/085 [arXiv:0706.0744 [hep-th]].

[229] C. Carmeli, R. Fioresi and S. D. Kwok, "SUSY structures, representations and Peter-Weyl theorem for $S^{1|1}$," J. of Geom. Phys. Vol. 95, 144-158 (2015) doi:10.1016/j.geomphys.2015.05.005 [arXiv:1407.2706 [math.RT]]

[230] C. Carmeli, R. Fioresi and S. D. Kwok, "The Peter-Weyl Theorem for SU(1|1)," arXiv:1509.07656 [math.RT].

[231] I. Booth and T. Liko, "Supersymmetric isolated horizons in ADS spacetime," Phys. Lett. B **670** (2008), 61-66 doi:10.1016/j.physletb.2008.10.020 [arXiv:0808.0905 [gr-qc]].

[232] B. O'Neill, "Semi-Riemannian geometry with applications to relativity," San Diego: Acad. Press, 1983.

[233] W. Kühnel, "Differentialgeometrie," Wiesbaden: Springer, 2013.

[234] N. Straumann, "General Relativity," doi:10.1007/978-94-007-5410-2

[235] M. Bojowald, "Canonical Gravity and Applications: Cosmology, Black Holes, and Quantum Gravity," Cambridge University Press, Cambridge, 2010.

[236] J. P. Harnad, L. Vinet and S. Shnider, "Group Actions on Principal Bundles and Invariance Conditions for Gauge Fields," J. Math. Phys. **21** (1980), 2719 doi:10.1063/1.524389

[237] A. Ashtekar, T. Pawlowski, P. Singh and K. Vandersloot, "Loop quantum cosmology of k=1 FRW models," Phys. Rev. D **75**, 024035 (2007) doi:10.1103/PhysRevD.75.024035 [arXiv:gr-qc/0612104 [gr-qc]].

[238] A. Ashtekar and E. Wilson-Ewing, "Loop quantum cosmology of Bianchi type II models," Phys. Rev. D **80** (2009), 123532 doi:10.1103/PhysRevD.80.123532 [arXiv:0910.1278 [gr-qc]].

[239] F. Cantrijn and L. A. Ibort, "Introduction to Poisson supermanifolds," Differential Geometry and its Applications, 1(2) (1991), 133-152 doi:10.1016/0926-2245(91)90027-7

[240] H. Goldmann, "Uniform Fréchet Algebras," North-Holland Math. Stud., 162. North-Holland, Amsterdam, (1990).

[241] C. Fleischhack, "Kinematical Foundations of Loop Quantum Cosmology," doi:10.1007/978-3-319-26902-3_11 [arXiv:1505.04400 [math-ph]].

[242] J. Engle, M. Hanusch and T. Thiemann, "Uniqueness of the Representation in Homogeneous Isotropic LQC," Commun. Math. Phys. **354**, no.1, 231-246 (2017) [erratum: Commun. Math. Phys. **362**, no.2, 759-760 (2018)] doi:10.1007/s00220-017-2881-2 [arXiv:1609.03548 [gr-qc]].

[243] A. Ashtekar, "Lectures on nonperturbative canonical gravity," Adv. Ser. Astrophys. Cosmol. **6** (1991), 1-334 doi:10.1142/1321

[244] M. Martin-Benito, G. A. M. Marugan and J. Olmedo, "Further Improvements in the Understanding of Isotropic Loop Quantum Cosmology," Phys. Rev. D **80** (2009), 104015 doi:10.1103/PhysRevD.80.104015 [arXiv:0909.2829 [gr-qc]].

[245] M. Bojowald, "Absence of singularity in loop quantum cosmology," Phys. Rev. Lett. **86**, 5227-5230 (2001) doi:10.1103/PhysRevLett.86.5227 [arXiv:gr-qc/0102069 [gr-qc]].

[246] A. Ashtekar, T. Pawlowski and P. Singh, "Quantum Nature of the Big Bang: Improved dynamics," Phys. Rev. D **74**, 084003 (2006) doi:10.1103/Phys RevD.74.084003 [arXiv:gr-qc/0607039 [gr-qc]].

[247] Z. Bern, J. J. M. Carrasco and H. Johansson, "New Relations for Gauge-Theory Amplitudes," Phys. Rev. D **78** (2008), 085011 doi:10.1103/PhysRevD.78.085011 [arXiv:0805.3993 [hep-ph]].

[248] Z. Bern, L. J. Dixon and R. Roiban, "Is N = 8 supergravity ultraviolet finite?," Phys. Lett. B **644** (2007), 265-271 doi:10.1016/j.physletb.2006.11.030 [arXiv:hep-th/0611086 [hep-th]].

[249] Z. Bern, J. J. Carrasco, L. J. Dixon, H. Johansson, D. A. Kosower and R. Roiban, "Three-Loop Superfiniteness of N=8 Supergravity," Phys. Rev. Lett. **98** (2007), 161303 doi:10.1103/PhysRevLett.98.161303 [arXiv:hep-th/0702112 [hep-th]].

[250] Z. Bern, L. J. Dixon, D. C. Dunbar, M. Perelstein and J. S. Rozowsky, "On the relationship between Yang-Mills theory and gravity and its implication for ultraviolet divergences," Nucl. Phys. B **530** (1998), 401-456 doi:10.1016/S0550-3213(98)00420-9 [arXiv:hep-th/9802162 [hep-th]].

[251] A. Ashtekar and M. Varadarajan, "Gravitational Dynamics—A Novel Shift in the Hamiltonian Paradigm," Universe **7**, no.1, 13 (2021) doi:10.3390/universe7010013 [arXiv:2012.12094 [gr-qc]].

[252] M. Varadarajan, "Euclidean LQG Dynamics: An Electric Shift in Perspective," [arXiv:2101.03115 [gr-qc]].

[253] S. Bosch, "Algebraic Geometry and Commutative Algebra," Universitext, Springer, London, 2013.

[254] M. Scheunert, W. Nahm and V. Rittenberg, "Graded Lie Algebras: Generalization of Hermitian Representations," J. Math. Phys. **18** (1977), 146 doi:10.1063/1.523148

[255] P. Minnaert and M. Mozrzymas, "Racah-Wigner calculus for the superrotation algebra. 1.," J. Math. Phys. **33** (1992), 1582-1593 doi:10.1063/1.529683

Superstring theory and loop quantum gravity (LQG) are promising approaches towards the formulation of a quantum theory of gravity. Superstring theory aims at unification of all fundamental forces of nature, predicting supersymmetry and even higher spacetime dimensions. LQG, on the other hand, takes a more conservative viewpoint by proposing new quantization techniques that take seriously the central principles of general relativity.

The goal of this work is to relate ideas from LQG and superstring theory by combining LQG with the concept of supersymmetry. To achieve this, the mathematical apparatus for a mathematically rigorous description of the underlying geometric structures of supergravity theories, i.e., supersymmetric extensions of Einstein's theory of gravity, will be developed. Among other things, this approach leads to a reformulation of the theory in which (part of) supersymmetry manifests itself in terms of a gauge symmetry.

Using the interpretation of supergravity in terms of a super Cartan geometry, the Holst variant of the MacDowell-Mansouri action for (extended) AdS supergravity in D=4 for arbitrary values of the Barbero-Immirzi parameter - a free parameter of the theory - will be derived. Moreover, it will be demonstrated that these actions provide unique boundary terms that ensure local supersymmetry invariance at boundaries.

The chiral case is special: The action is invariant under an enlarged gauge symmetry, the boundary theory is a topological super Chern-Simons theory, and a chiral connection emerges that is the natural generalization of the Ashtekar connection to the supersymmetric context. Making use of the enlarged gauge symmetry, a quantization of the theory generalizing standard tools of LQG will be proposed.

These results provide a starting point for applications in the context of supersymmetric black holes and quantum cosmology. There, the enhanced gauge symmetry proves to be a promising tool which in the future may shed a lot of insights on how to relate results from these different approaches.

ISBN 978-3-96147-529-2
9 783961 475292

FAU UNIVERSITY PRESS 2022