Symmetry of Brans-Dicke gravity as a novel solution-generating technique

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A little known symmetry group of Brans-Dicke gravity in the presence of conformally invariant matter (including electrovacuo) is used as a solution-generating technique, starting from a known solution as a seed. This novel technique is applied to generate, as examples, new spatially homogeneous and isotropic cosmologies, a 3-parameter family of spherical time-dependent spacetimes conformal to a Campanelli-Lousto geometry, and a family of cylindrically symmetric geometries.

Keywords: Brans-Dicke gravity, exact solutions, symmetry.

1. Introduction

Scalar-tensor gravity is the prototype theory alternative to general relativity (GR). The main motivation to modify gravity comes from cosmology: to explain the present acceleration of the universe, the GR-based ΛCDM model introduces a completely *ad hoc* dark energy.[1] Modifying gravity is an alternative.[2,3] The most popular approach is metric $f(R)$ gravity, which is an $\omega = 0$ Brans-Dicke (BD) theory (with a potential) in disguise. But this is not the only fundamental motivation: all attempts to quantize gravity introduce corrections to GR, consisting of quadratic terms in the curvature (giving rise to Starobinsky inflation[4]), scalar fields, or non-local terms (for example, the low-energy limit of bosonic string theory is an $\omega = -1$ BD theory[5,6]). What is more, Dirac's idea of varying fundamental "constants" of physics[7] is partially realized in scalar-tensor gravity, where the scalar degree of freedom $\phi \sim G^{-1}$ is dynamical. More recently, with quantum gravity in mind, it has been found that generalized BD solutions describe asymptotically Lifschitz black holes.[8]

When available, analytic solutions provide insight into various aspects of a theory, but they are relatively rare in scalar-tensor gravity. Therefore, it is valuable to find new solution-generating techniques. We explore a new one based on a 1-parameter symmetry group of BD theory. As an application, we found three new families of solutions of BD theory with potential $V \propto \phi^\beta$, and of $f(R) = R^n$ gravity.[9] They include Friedmann-Lemaître-Robertson-Walker (FLRW) cosmologies with power-law or exponential scale factor; spherical, time-dependent, asymptotically FLRW solutions; and axially symmetric (cosmic string-like) geometries.

2. Symmetry Group of Brans-Dicke Theory

The action of vacuum BD theory with a potential

$$S_{BD} = \int d^4x\sqrt{-g}\left[\phi R - \frac{\omega}{\phi}\, g^{ab}\nabla_a\phi\nabla_b\phi - V(\phi)\right] \tag{1}$$

is invariant in form under the transformation $(g_{ab}, \phi) \to \left(\tilde{g}_{ab}, \tilde{\phi}\right)$, where

$$\tilde{g}_{ab} = \Omega^2 g_{ab} = \phi^{2\alpha} g_{ab}\,, \tag{2}$$

$$\tilde{\phi} = \phi^{1-2\alpha}\,, \qquad \alpha \neq 1/2 \tag{3}$$

and where we follow the notation of Ref.[10]. Using the standard transformation properties under conformal transformations

$$\tilde{g}^{ab} = \Omega^{-2} g^{ab}\,, \qquad \sqrt{-\tilde{g}} = \Omega^4\sqrt{-g}\,, \tag{4}$$

$$\tilde{R} = \Omega^{-2}\left(R - \frac{6\Box\Omega}{\Omega}\right)\,, \tag{5}$$

one obtains

$$R = \phi^{2\alpha}\tilde{R} - \frac{6\alpha(1-\alpha)}{(1-2\alpha)^2}\,\phi^{6\alpha-2}\tilde{g}^{ab}\tilde{\nabla}_a\tilde{\phi}\tilde{\nabla}_b\tilde{\phi} + \frac{6\alpha}{1-2\alpha}\,\phi^{4\alpha-1}\tilde{\Box}\tilde{\phi}\,. \tag{6}$$

The last term contributes only a total divergence to $\sqrt{-g}\,\phi R$ in the action (1),

$$\frac{6\alpha}{1-2\alpha}\,\sqrt{-\tilde{g}}\,\tilde{\Box}\tilde{\phi} = \frac{6\alpha}{1-2\alpha}\,\partial_\mu\left(\sqrt{-\tilde{g}}\,\tilde{g}^{\mu\nu}\partial_\nu\tilde{\phi}\right) \tag{7}$$

and the BD action (1) then becomes

$$S_{BD} = \int d^4x\sqrt{-\tilde{g}}\left\{\tilde{\phi}\tilde{R} - \left[\frac{\omega}{(1-2\alpha)^2} + \frac{6\alpha(1-\alpha)}{(1-2\alpha)^2}\right] \cdot\frac{\tilde{g}^{ab}}{\tilde{\phi}}\tilde{\nabla}_a\tilde{\phi}\tilde{\nabla}_b\tilde{\phi} - \tilde{\phi}^{\frac{-4\alpha}{1-2\alpha}}V(\phi)\right\}\,. \tag{8}$$

By redefining the BD coupling and the potential according to

$$\tilde{\omega}(\omega,\alpha) = \frac{\omega + 6\alpha(1-\alpha)}{(1-2\alpha)^2}\,, \tag{9}$$

$$\tilde{V}(\tilde{\phi}) = \tilde{\phi}^{\frac{-4\alpha}{1-2\alpha}}V\left(\tilde{\phi}^{\frac{1}{1-2\alpha}}\right)\,, \tag{10}$$

we write

$$S_{BD} = \int d^4x \sqrt{-\tilde{g}} \left[\tilde{\phi}\tilde{R} - \frac{\tilde{\omega}}{\tilde{\phi}}\, \tilde{g}^{ab} \tilde{\nabla}_a \tilde{\phi} \tilde{\nabla}_b \tilde{\phi} - \tilde{V}(\tilde{\phi}) \right] , \tag{11}$$

i.e., the action (1) is invariant in form. The transformations (3) form a 1-parameter Abelian group.[11,12]

A special case is given by a power-law potential $V(\phi) = V_0 \phi^n$, which becomes $\tilde{V}(\tilde{\phi}) = V_0 \tilde{\phi}^{\tilde{n}}$ with $\tilde{n} = \frac{n-4\alpha}{1-2\alpha}$ (and is invariant if $n = 2$).

2.1. *Electrovacuum Brans-Dicke theory*

Electrovacuum BD theory is described by the action

$$S_{BD} = \int d^4x \sqrt{-g} \left[\phi R - \frac{\omega}{\phi}\, g^{ab} \nabla_a \phi \nabla_b \phi - V(\phi) - \frac{1}{4} F^{ab} F_{ab} \right] ; \tag{12}$$

since $\tilde{F}_{ab} = F_{ab}$ and $\sqrt{-g}\, F^{ab} F_{ab} = \sqrt{-\tilde{g}}\, \widetilde{F^{ab} F_{ab}}$, also $\sqrt{-g}\, \mathcal{L}_{(m)}$ remains invariant under the transformations (3).

2.2. *Conformally invariant matter*

For simplicity, let us use now the field equations

$$R_{ab} - \frac{1}{2}\, g_{ab} R = \frac{8\pi}{\phi}\, T_{ab} + \frac{\omega}{\phi^2} \left(\nabla_a \phi \nabla_b \phi - \frac{1}{2}\, g_{ab} g^{cd} \nabla_c \phi \nabla_d \phi \right) + \frac{1}{\phi} \left(\nabla_a \nabla_b \phi - g_{ab} \Box \phi \right) - \frac{V}{2\phi}\, g_{ab} \, , \tag{13}$$

$$\Box \phi = \frac{1}{2\omega + 3} \left[\frac{8\pi T}{\phi} + \phi\, \frac{dV}{d\phi} - 2V \right] . \tag{14}$$

The symmetry transformation (3) gives

$$\tilde{\Box} \tilde{\phi} = \frac{1}{2\tilde{\omega} + 3} \left[\frac{8\pi}{1 - 2\alpha}\, \tilde{\phi}^{\frac{-4\alpha}{1-2\alpha}} T + \tilde{\phi}\, \frac{d\tilde{V}}{d\tilde{\phi}} - 2\tilde{V} \right] \tag{15}$$

and the field equations are conformally invariant only if $T = 0$. Moreover,

$$\tilde{R}_{ab} - \frac{1}{2}\, \tilde{g}_{ab} \tilde{R} = \frac{8\pi}{\tilde{\phi}^{\frac{1}{1-2\alpha}}} T_{ab} + \frac{\tilde{\omega}}{\tilde{\phi}^2} \left(\tilde{\nabla}_a \tilde{\phi} \tilde{\nabla}_b \tilde{\phi} - \frac{1}{2}\, \tilde{g}_{ab} \tilde{g}^{cd} \tilde{\nabla}_c \tilde{\phi} \tilde{\nabla}_d \tilde{\phi} \right) + \frac{1}{\tilde{\phi}} \left(\tilde{\nabla}_a \tilde{\nabla}_b \tilde{\phi} - \tilde{g}_{ab} \tilde{\Box} \tilde{\phi} \right) - \frac{\tilde{V}}{2\tilde{\phi}}\, \tilde{g}_{ab} \tag{16}$$

where $\tilde{T}_{ab} = \Omega^{-2}\, T_{ab}$, so the first term in the right hand side becomes $8\pi \tilde{T}_{ab}/\tilde{\phi}$ and it is invariant. Hence, the field equations are invariant for arbitrary $V(\phi)$ but only for conformally invariant matter.

3. Example: Brans-Dicke Cosmology

In the case of FLRW cosmology, the Lagrangian reduces to a point-like one, but the symmetry (3) is not a Noether nor a Hojman symmetry. Moreover, the classical symmetry is broken by Wheeler-DeWitt quantization in minisuperspace: quantum effects cause an anomalous symmetry breaking similar to that occuring in condensed matter systems.[13]

This time our "seed" is[14,15]

$$ds^2 = -dt^2 + S^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2_{(2)}\right) \tag{17}$$

with power-law scale factor $S(t)$. We have vacuum BD theory with $V \equiv 0$ and

$$S(t) = S_0 t^p \,, \tag{18}$$

$$\phi(t) = \phi_0 t^q \,. \tag{19}$$

The symmetry transformation yields

$$d\tilde{s}^2 = -t^{2\alpha q} dt^2 + S_0^2 t^{2(p+\alpha q)}\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2_{(2)}\right) \, ; \tag{20}$$

we introduce a new time τ with $t = (\alpha q+1)^{\frac{1}{\alpha q+1}}\, \tau^{\frac{1}{\alpha q+1}}$, then the new solution is recast in the form

$$d\tilde{s}^2 = -d\tau^2 + \tilde{S}_0^2 \tau^{\frac{2(p+\alpha q)}{\alpha q+1}}\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2_{(2)}\right) , \tag{21}$$

$$\tilde{\phi}(\tau) = (\alpha q+1)^{\frac{q(1-2\alpha)}{\alpha q+1}}\, \phi_0^{1-2\alpha} \tau^{\frac{q(1-2\alpha)}{\alpha q+1}} , \tag{22}$$

or $\tilde{S}(\tau) = \tilde{S}_0 \tau^{\tilde{p}}$, $\tilde{\phi}(\tau) = \tilde{\phi}_0 \tau^{\tilde{q}}$ with

$$\tilde{p} = \frac{p+\alpha q}{\alpha q+1} \,, \qquad \tilde{q} = \frac{q(1-2\alpha)}{\alpha q+1} \,, \tag{23}$$

$$\tilde{S}_0 = (\alpha q+1)^{\frac{p+\alpha q}{\alpha q+1}}\, S_0 \,, \quad \tilde{\phi}_0 = (\alpha q+1)^{\frac{q(1-2\alpha)}{\alpha q+1}}\, \phi_0^{1-2\alpha} \,. \tag{24}$$

4. A New Family of Spherical Time-Dependent Solutions

Begin now from a special case of a family of spherical, time-dependent solutions of vacuum BD gravity conformal to the Fonarev[16] spacetime of GR[17]

$$\begin{aligned} ds^2 = {} & -A(r)^{\frac{1}{\sqrt{1+4d^2}}\left(2d-\frac{1}{\sqrt{|2\omega+3|}}\right)}\, \mathrm{e}^{4dat\left(2d-\frac{1}{\sqrt{|2\omega+3|}}\right)} dt^2 \\ & + \mathrm{e}^{2at\left(1-\frac{2d}{\sqrt{|2\omega+3|}}\right)} \Bigg[A(r)^{\frac{-1}{\sqrt{1+4d^2}}\left(2d+\frac{1}{\sqrt{|2\omega+3|}}\right)} dr^2 \\ & + A(r)^{1-\frac{1}{\sqrt{1+4d^2}}\left(2d+\frac{1}{\sqrt{|2\omega+3|}}\right)} r^2 d\Omega^2_{(2)} \Bigg] , \end{aligned} \tag{25}$$

$$\phi(t,r) = \phi_0\, \mathrm{e}^{\frac{4dat}{\sqrt{|2\omega+3|}}} A(r)^{\frac{1}{\sqrt{|2\omega+3|(1+4d^2)}}} , \tag{26}$$

where $A(r) = 1 - 2m/r$, $V(\phi) = V_0\phi^\beta$, $\beta = 2\left(1 - d\sqrt{|2\omega+3|}\right)$. Our "seed" is the special case with $a \neq 0$ and with time dependence eliminated by the parameter choice

$$d = \left(2\sqrt{|2\omega+3|}\right)^{-1} = \sqrt{|2\omega+3|}/2 \tag{27}$$

simultaneously, which gives $\omega = -1$ (this is the low-energy limit of the bosonic string, so presumably this solution has a stringy analogue). The scalar field remains time-dependent, $\beta = 1$, and $V(\phi) = V_0\phi$ (corresponding to a cosmological constant). Then

$$ds^2 = -dt^2 + A(r)^{-\sqrt{2}}dr^2 + A(r)^{1-\sqrt{2}}r^2 d\Omega_{(2)}^2\,, \tag{28}$$

$$\phi(t,r) = \phi_0\,\mathrm{e}^{2at}A(r)^{1/\sqrt{2}}\,, \tag{29}$$

which is a special case[17] of the Campanelli-Lousto geometry.[18] The symmetry transformation (3) applied to this seed now generates a new solution with

$$\tilde{V}(\tilde{\phi}) = V_0\,\tilde{\phi}^{\frac{1-4\alpha}{1-2\alpha}}\,, \qquad \tilde{\omega} = \frac{6\alpha(1-\alpha)-1}{(1-2\alpha)^2} \tag{30}$$

given by

$$d\tilde{s}^2 = -\mathrm{e}^{4\alpha a t}A(r)^{\alpha\sqrt{2}}dt^2 + \mathrm{e}^{4\alpha a t}\left[A(r)^{-\sqrt{2}(1-\alpha)}dr^2 + A(r)^{1-\sqrt{2}(1-\alpha)}r^2 d\Omega_{(2)}^2\right]\,, \tag{31}$$

$$\tilde{\phi}(t,r) = \tilde{\phi}_0\,\mathrm{e}^{2a(1-2\alpha)t}A(r)^{\frac{1-2\alpha}{\sqrt{2}}}\,, \qquad \tilde{\phi}_0 = \phi_0^{1-2\alpha}\,. \tag{32}$$

If $a \neq 0$, we can define the new time $\tau = \frac{\mathrm{e}^{2\alpha a t}}{2\alpha a}$ to obtain

$$d\tilde{s}^2 = -A(r)^{\alpha\sqrt{2}}d\tau^2 + (2\alpha a\tau)^2\left[A(r)^{-\sqrt{2}(1-\alpha)}dr^2 + A(r)^{1-\sqrt{2}(1-\alpha)}r^2 d\Omega_{(2)}^2\right]\,, \tag{33}$$

$$\tilde{\phi}(\tau,r) = \phi_*\,\tau^{\frac{1-2\alpha}{\alpha}}A(r)^{\frac{1-2\alpha}{\sqrt{2}}}\,, \qquad \tilde{\phi}_* = \left[(2\alpha a)^{1/\alpha}\,\phi_0\right]^{1-2\alpha}\,. \tag{34}$$

If $m \to 0$, or for $r \gg m$, this geometry reduces to a spatially flat FLRW universe with linear scale factor $S(\tau) = 2\alpha a\tau$ and the scalar field

$$\tilde{\phi}(\tau,r) = \phi_*\,\tau^{\frac{1-2\alpha}{\alpha}} \tag{35}$$

acts asymptotically as a perfect fluid with $P = -\rho/3$.

5. Conclusions

The 1-parameter Abelian symmetry group formed by the transformations (2) and (3) as α varies offers a new solution-generating technique for (electro)vacuum BD gravity with a potential $V(\phi)$. The symmetry transformation (3) introduces one new parameter α in addition to those already present in the seed solution. We have found a new 2-parameter family of spherical time-dependent, asymptotically FLRW solutions and also new FLRW solutions with power-law (or exponential[9]) scale factor. New cylindrical (cosmic string-like) solutions have also been found and they are reported in Ref.[9].

By using the BD representation of $f(R)$ gravity, it can be shown that these new solutions are also solutions of $f(R) = R^n$ theory.[9] The mathematical technique exposed does not, of course, guarantee that the new solutions are physically interesting, but it constitutes a new tool. Future work will explore new applications of the solution-generating technique presented here.

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