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Einstein's equations from entanglement entropy

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<p>Tiivistelmä – Referat – Abstract</p> <p>Since the inception of the AdS/CFT correspondence in 1997 there has been great interest in the holographic description of quantum gravity in terms of conformal field theory. Studying how classical gravity emerges in this framework helps us to understand the quantum foundations of general relativity. A fundamental concept is entanglement entropy which has a classical interpretation in terms of areas of minimal surfaces in general relativity, due to Ryu and Takayanagi.</p> <p>This thesis is a review on how Einstein's equations can be derived up to second order from the Ryu-Takayanagi formula in the context of AdS/CFT correspondence. It also serves as an introduction to entanglement entropy in quantum field theories and holography, while providing necessary mathematical ingredients to understand the derivation. We also review a related derivation, based on the entanglement equilibrium hypothesis, and discuss its extensions to higher order theories of gravity.</p>			
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1 Introduction

During the past century, there has been great advancement in the understanding of the laws of nature at all length scales. General theory of relativity, developed by Einstein in 1916, revolutionized our understanding of gravity by no longer treating it as a force, but as the curvature of spacetime itself. At the same time, quantum mechanics was developed to describe particles, the fundamental building blocks of matter, at the smallest scales possible, which eventually led to the development of quantum field theory and the Standard Model of particle physics. Both of these theories have agreed with experiments to extremely high precision. General relativity has passed numerous tests concerning large scale objects such as stars and black holes or even the universe as a whole. On the other hand, the Standard Model appears to be unbreakable with our current particle experiments. Nevertheless, this is not the end of the story, because these two descriptions of Nature are fundamentally incompatible, which is against our aspiration of a unified theory of everything.

The search for a unified description of standard model and general relativity has a long history. Such a description, a quantum theory of gravity, is sometimes called the holy grail of physics. It would unify the three forces of the standard model, electromagnetic, strong and weak forces, with gravity. The unification is made difficult by the fact that at high energies, general relativity cannot be quantized by the traditional rules of quantum field theory, which is the language of the Standard Model. The conversion of a classical theory to a quantum one was not a problem for example in the case of electromagnetism, because the quantum theory produced is sensible and produces finite predictions when treated properly. However for general relativity, the treatment that worked for electromagnetism is no longer enough, because we obtain too many infinities than we are able to handle. Hence something completely new is required to quantize gravity.

The quantization is achieved by string theory, which is a vast generalization of the framework of quantum field theory. Traditional quantum field theories describe point particles that propagate in spacetime tracing out one dimensional lines, worldlines, that intersect each other representing interactions. In string theory, the point particles are replaced by two dimensional extended objects, strings, which at large scales act like particles so that string theory agrees with all known physics at low energy scales. Instead of worldlines, the strings trace out smooth two dimensional manifolds, worldsheets, when propagating in spacetime. The worldsheets solve multiple problems: ultraviolet divergences are absent in string theory, because the sharp particle interaction vertices are smoothed out, but most notably, quantization of the worldsheet leads to a theory of quantum gravity in spacetime, which does not happen by quantizing the one dimensional worldlines of quantum field theory.

Because string theory is a theory of quantum gravity, spacetime is no longer fundamental, but emerges from the dynamics of the strings in suitable limits. A model built within string theory might contain multiple limits that describe different classical spacetimes and, in general, there is no classical spacetime at all. Making sense of how gravity and spacetime emerge is a complicated task, however, there has been substantial progress made in terms of dualities.

A duality can be thought of as a dictionary, which translates mathematical language between two a priori different theories. The duality maps physical quantities from one theory to another usually in a very non-trivial manner. Regardless, the existence of a duality implies that the theories are equivalent, even if they look completely different. They both describe the same physical phenomenon, but the information

is encoded in different ways so that one of the theories might be more suitable to extract physically meaningful information than the other. For example in some limit, the relevant calculations might get increasingly complex in one of the theories, while in the other they get simpler. Therefore dualities provide a useful tool in the study of physical systems.

In 1997, Maldacena discovered a duality relating a string theory to a conformal field theory (CFT), which became known as the AdS/CFT correspondence [1]. Theories related by an AdS/CFT correspondence live in different dimensional spacetimes: string theory lives in a curved anti-de Sitter (AdS) space (the bulk), while the CFT lives on its flat boundary, whose dimension compared to the bulk AdS is smaller by one. But what makes the duality so useful, is that string theory includes gravity, while the CFT does not. This means that quantum gravity is somehow encoded in the structure of the CFT! The duality is made more tolerable in a special limit of the CFT, in which the string theory reduces to classical general relativity in an AdS background with its curvature obeying Einstein's equations. Therefore using the AdS/CFT dictionary that relates certain CFT quantities to classical geometric quantities in general relativity, we are able to study how classical gravity emerges from quantum gravity. The key concept in this endeavor has turned out to be quantum entanglement.

In 1973, before the revolutions of string theory, Bekenstein argued based on a thought experiment that the entropy of a black hole in general relativity is proportional to the area of its horizon [2]. The following year, Hawking showed that in the presence of quantum fields, black holes radiate with a temperature consistent with Bekenstein's definition of entropy [3]. Today known as the Bekenstein-Hawking entropy, its area dependence is counterintuitive, because entropy in classical thermodynamics is an extensive quantity, growing linearly with the volume of the system. According to the statistical interpretation of entropy due to Boltzmann, the microstates of the black hole should hence live on its horizon. General relativity does not actually tell what these fundamental bits of spacetime are, but it appears to be a thermodynamical description of them.

Ordinary entropy is also extensive in quantum field theories, because they are local and the same amount of information is contained at each spacetime point. But quantum field theories also contain a very non-local property called entanglement, where the fields are correlated across long distances. These correlations can be measured by entanglement entropy and in 1986 it was shown that entanglement entropy of quantum fields outside of a black hole, when properly regulated, is proportional to the horizon area [4]. Therefore entanglement entropy of quantum fields might play a role in the origin of black hole entropy.

All of these ideas culminated in 2006, when Ryu and Takayanagi proposed that in the context of AdS/CFT, entanglement entropies of CFT fields living in a closed subregion are calculated by the area of a minimal surface in general relativity [5, 6]. The result is known as the Ryu-Takayanagi (RT) formula and it is an example of an entry in the AdS/CFT dictionary. When applied to a thermal state of the CFT, the minimal surface coincides with a black hole horizon in AdS space, which shows that Bekenstein-Hawking entropy of an AdS black hole is the same as CFT thermal entropy. This provides an interpretation for the microstates of an AdS black hole in terms of the quantum degrees of freedom of the dual CFT.

Area of a minimal surface is a property of classical spacetime so maybe it is entanglement that builds spacetime. By using the Ryu-Takayanagi formula, van Raamsdonk analysed the dual geometry and showed that, when the amount of CFT

entanglement is decreased, the spacetime is stretched like rubber with distances increasing and cross sections decreasing in area [7, 8]. Eventually in the limit of no entanglement, the spacetime snaps in the middle. Hence it looks like entanglement is the glue that holds the fabric of spacetime together.

Can we take these ideas one step further to include dynamics of the spacetime as well? Variations in the entanglement correspond to curvature variations in AdS space, but do these variations induced by entanglement obey Einstein's equations as dictated by general relativity? The answer is yes, at least for linear and second order perturbations in the geometry [9, 10, 11]. Also the reason why gravity couples to all kinds of matter equally is a result of the universality of entanglement [12]! These results inspired a similar derivation of Einstein's equations in Minkowski and de Sitter spaces [13], for which there are no known descriptions in terms of a dual field theory. A comprehensive review of these results and the required preliminaries is the topic of this thesis.

2 Entanglement entropy in quantum field theories

Originally entropy was defined in terms of thermodynamical quantities to describe a macroscopical system. It was shown that the entropy of an isolated system cannot decrease, a fact, which became known as the second law of thermodynamics. At the end of the 19th century, Boltzmann introduced the famous statistical interpretation of entropy in terms of the number of microstates available for the system.¹ In a thermodynamical system, there is a large variety of microstates corresponding to the same macroscopic configuration and entropy measures the number of these microstates. A macroscopic observer cannot keep track of the microstates and entropy is a measure of this ignorance. The second law of thermodynamics is therefore only a statistical result, true for a large number of particles, and violations are suppressed exponentially.

The original Boltzmann definition of entropy is based on the microcanonical ensemble describing an isolated system, whose microstates are all assumed to be equally probable. It is more common that the system is not fully isolated, but interacts with its environment by exchanging heat. When such a system reaches equilibrium, it is described by the canonical ensemble, where larger system energies are exponentially less likely according to the Boltzmann distribution. The uniform distribution of the isolated system is no longer applicable due to the random interactions between the system and its environment. Entropy in the canonical ensemble is measured by the Gibbs entropy, which generalizes Boltzmann entropy to arbitrary probability distributions.²

An analogous situation appears in the description of two quantum systems. The combined system is described by a state vector living in a tensor product space built from the Hilbert spaces of the individual systems. There are always entanglement correlations between the two systems due to the tensor product structure of the Hilbert space. Taking one of the systems to be the environment and tracing it out, we obtain an effective description for the second system. In the effective description, the entanglement correlations are ignored analogously to the environment interactions in the canonical ensemble and to measure these correlations, one defines an analogous entropy called entanglement entropy. Entanglement entropy is von Neumann entropy,

¹Boltzmann entropy is given by $S = \log W$ ($k_B = 1$), where W is the number of microstates of the system.

²Gibbs entropy of a probability distribution p_i is defined as $S = -\sum_i p_i \log p_i$, which reduces to the Boltzmann entropy given a uniform distribution $p_i = 1/W$.

which is just Gibbs entropy in the language of quantum theory.³

2.1 Definition of entanglement entropy

To formally define entanglement entropy, we need to introduce the concept of a density matrix, which introduces classical probabilities into the description of a quantum system. Consider a quantum system and let the states $|\Phi_i\rangle$ span its Hilbert space \mathcal{H} . Suppose the system is in a mixed state with p_i being the probability of the state $|\Phi_i\rangle$. The mixed state can be described in terms of a *density matrix* ρ , which is defined such that the expectation values take the form

$$\langle \mathcal{O} \rangle = \text{Tr}(\mathcal{O}\rho) = \sum_i p_i \langle \Phi_i | \mathcal{O} | \Phi_i \rangle, \quad (2.1)$$

where \mathcal{O} is the observable being measured. The probabilities appear as the eigenvalues of the density matrix:

$$\rho = \sum_i p_i |\Phi_i\rangle \langle \Phi_i|. \quad (2.2)$$

When the system is known to be in a pure state $|\Phi\rangle$, the density matrix is simply the projection operator

$$\rho = |\Phi\rangle \langle \Phi| \quad (2.3)$$

and (2.1) reduces to the ordinary quantum expectation value $\langle \Phi | \mathcal{O} | \Phi \rangle$.

Consider now a quantum field theory (QFT) on a discrete lattice so that the continuum description follows simply by taking the lattice spacing to zero. Suppose we now divide the lattice into two subregions \mathcal{A} and its complement \mathcal{A}^c . As a local theory, the Hilbert space of the theory factorizes into the spaces of the subregions.⁴

$$\mathcal{H} = \mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{A}^c}. \quad (2.4)$$

If the corresponding spaces are spanned by states $|\phi_i\rangle$ and $|\phi_i^c\rangle$, then an arbitrary pure state in \mathcal{H} can be expanded as

$$|\Phi\rangle = \sum_{i,j} c_{ij} |\phi_i\rangle \otimes |\phi_j^c\rangle, \quad \sum_{i,j} |c_{ij}|^2 = 1. \quad (2.5)$$

In general $|\Phi\rangle$ does not factorize into a product state, which is the origin of quantum entanglement between the two regions \mathcal{A} and \mathcal{A}^c . Because of the entanglement, the state of \mathcal{A} is not purely determined by $|\phi_i\rangle$ and additional information of the state of the complement is required. We can apply the above formalism of mixed states to encode the entanglement correlations in a special density matrix called the *reduced density matrix* $\rho_{\mathcal{A}}$. It is defined by tracing out the degrees of freedom of \mathcal{A}^c in the full density matrix (2.3):

$$\rho_{\mathcal{A}} = \text{Tr}_{\mathcal{A}^c} \rho \equiv \sum_i \langle \phi_i^c | \rho | \phi_i^c \rangle. \quad (2.6)$$

Now quantum the expectation value of an operator $\mathcal{O}_{\mathcal{A}}$ supported in \mathcal{A} can be calculated as

$$\langle \Phi | \mathcal{O}_{\mathcal{A}} \otimes 1_{\mathcal{A}^c} | \Phi \rangle = \text{Tr}(\mathcal{O}_{\mathcal{A}} \rho_{\mathcal{A}}). \quad (2.7)$$

³Von Neumann entropy is defined via a density matrix ρ as $S = -\text{Tr}(\rho \log \rho)$ (see section 2.1).

⁴In gauge theories, the factorization does not exist in general due to degrees of freedom associated to pairs of lattice points.

The amount of entanglement correlations is measured with von Neumann entropy

$$S_{\mathcal{A}} = -\text{Tr}(\rho_{\mathcal{A}} \log \rho_{\mathcal{A}}) = -\sum_i \lambda_i \log \lambda_i, \quad (2.8)$$

where λ_i are the eigenvalues of $\rho_{\mathcal{A}}$, the entanglement spectrum. The entropy is a non-negative quantity $S_{\mathcal{A}} \geq 0$ and vanishes only in the case of no entanglement (when \mathcal{A} is in a pure state). Hence $S_{\mathcal{A}}$ is called entanglement entropy (EE) of \mathcal{A} .

By taking the lattice spacing to zero, we obtain a continuum QFT, where entanglement entropy is now a divergent quantity. In [4, 14] spherically symmetric subregions were analyzed and it was shown that the leading order divergence of entanglement entropy is proportional to the area of the boundary of the sphere. Later it was proven [15] that the area law extends to regions \mathcal{A} of arbitrary shape. Formally⁵

$$S_{\mathcal{A}} = \gamma \frac{\text{Area}[\partial\mathcal{A}]}{a^{d-2}} + \dots \quad (2.9)$$

where d is the dimension of the spacetime, γ is a theory dependent constant (except in 2-dimensions (2.29)) and a is the UV cutoff, which corresponds to the lattice spacing. This result is physically intuitive, because as a local theory, most of the entanglement between the two regions is concentrated across the sharp boundary. The leading term is corrected by other divergent terms, whose structure is dependent on whether the dimension d is even or odd [16]. The expansion can be determined by holographic methods.⁶ From this expression it is clear that entanglement entropy violates the traditional extensivity of thermal entropy and is a candidate for the Bekenstein-Hawking entropy of a black hole. However, taming the divergence is a non-trivial task and would require knowledge of quantum gravitational physics in the UV.

Inequalities and the entanglement first law

EE satisfies a number of inequalities when calculated for various combinations of different subregions $\{\mathcal{A}_i\}$, which factor the Hilbert space as $\mathcal{H} = \otimes_i \mathcal{H}_{\mathcal{A}_i}$. For two regions ($i = 1, 2$) we have the inequality

$$|S_{\mathcal{A}_1} - S_{\mathcal{A}_2}| \leq S_{\mathcal{A}_1 \cup \mathcal{A}_2} \leq S_{\mathcal{A}_1} + S_{\mathcal{A}_2}. \quad (2.10)$$

The right inequality is called *subadditivity* and it shows that EE is not an extensive quantity in general. Subadditivity is automatically true in QFTs according to the area law (2.9). The left inequality is called the *Araki-Lieb* -inequality [17]. Together these inequalities imply classical monotonicity i.e. EE does not decrease as the spatial size of the system is increased.

For three subregions ($i = 1, 2, 3$) EE satisfies *strong subadditivity*

$$S_{\mathcal{A}_1 \cup \mathcal{A}_2} + S_{\mathcal{A}_2 \cup \mathcal{A}_3} \geq S_{\mathcal{A}_2} + S_{\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3} \quad (2.11)$$

for which the proof can be found in [18]. Strong subadditivity plays an important role in various proofs regarding entanglement entropy. In a holographic context, it provides a motivation for the correctness of the Ryu-Takayanagi formula discussed in section 3.2.

⁵Area refers to the volume of the codimension two surface $\partial\mathcal{A}$. In this case, it is $(d-2)$ -dimensional.

⁶See section 3.2.

We can also compare the entropies of different states in \mathcal{H} . Given two normalized density matrices ρ and σ corresponding to two states, we can define *relative entropy*, which is a non-negative quantity and vanishes only when $\rho = \sigma$ [19, 20]:

$$S(\rho||\sigma) = \text{Tr}(\rho \log \rho) - \text{Tr}(\rho \log \sigma) \geq 0. \quad (2.12)$$

We define the *modular Hamiltonian* K implicitly via

$$\sigma = \frac{1}{Z} e^{-K}, \quad (2.13)$$

where $Z = \text{Tr}(e^{-K})$ is the partition function ensuring normalization. By adding and subtracting a term $\text{Tr}(\sigma \log \sigma)$ in (2.12) we can write it as

$$\begin{aligned} S(\rho||\sigma) &= \text{Tr}(\rho \log \rho) - \text{Tr}(\sigma \log \sigma) - \text{Tr}(\rho \log \sigma) + \text{Tr}(\sigma \log \sigma) \\ &= -[-\text{Tr}(\rho \log \rho) + \text{Tr}(\sigma \log \sigma)] \\ &\quad + [\text{Tr}(\rho K) + \text{Tr}(\rho \log Z) - \text{Tr}(\sigma K) - \text{Tr}(\sigma \log Z)] \\ &= -[S(\rho) - S(\sigma)] + [\text{Tr}(\rho K) - \text{Tr}(\sigma K)] \\ &= \Delta\langle K \rangle - \Delta S \geq 0, \end{aligned} \quad (2.14)$$

where the terms involving $\log Z$ cancel due to the normalization $\text{Tr} \rho = \text{Tr} \sigma = 1$.

Consider a one parameter family of density matrices ρ_λ such that $\rho_0 = \sigma$ (λ can take both positive and negative values). By calculating the relative entropy for small λ , the positivity condition (2.12) amounts to

$$\left. \frac{d}{d\lambda} S(\lambda) \right|_{\lambda=0} \geq 0, \quad (2.15)$$

where $S(\lambda) = -\text{Tr}(\rho_\lambda \log \rho_\lambda)$. Relative entropy vanishes only at $\lambda = 0$ so $S(\lambda)$ has a local minimum at that point. Therefore for small differences the inequality (2.14) reduces to the equality:

$$\delta S = \delta\langle K \rangle. \quad (2.16)$$

This result is known as the *entanglement first law* as it resembles the first law of thermodynamics:

$$\delta S = (1/T) \delta Q. \quad (2.17)$$

The role of heat is played by the modular Hamiltonian, while with our conventions the inverse temperature is equal to one. Regardless, there is no deep relationship between the two and (2.16) is simply taken as a mathematical identity.

The derivation of entanglement first law above only required two different density matrices ρ and σ that lie infinitesimally close to each other so that ρ can result from all kinds of perturbations to the initial density matrix σ , the simplest being a perturbation to the state itself. Other examples include a perturbation of the shape of the spatial region or even a perturbation of the underlying Lagrangian of the theory.

2.2 Calculation of entanglement entropies

Consider a QFT on a d -dimensional globally hyperbolic Lorentzian manifold.⁷ Let Σ be the Cauchy slice of time $t = x^0 = 0$ and let $\mathcal{A} \subset \Sigma$ be a spatial region on the slice ($\mathcal{A} \cup \mathcal{A}^c = \Sigma$).⁸ We want to explicitly calculate the vacuum state entanglement

⁷The global hyperbolicity ensures the existence of a Cauchy slice at each instant of time.

⁸We will denote the coordinates by $x^\mu = (t, \vec{x})$ with $\mu = 0, 1, \dots, d-1$.

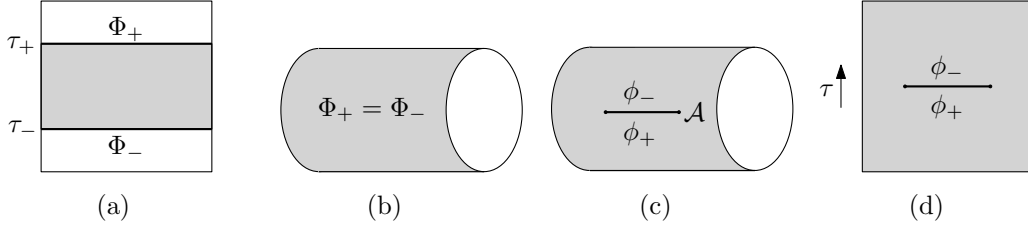


Figure 1: Visualization of the Euclidean functional integrals in terms of their integration manifolds [24]. The manifold (a) for (2.18) is simply the space bounded by the Cauchy slices at τ_{\pm} . In the partition function (2.19), the initial and final Cauchy slices are identified, which corresponds to gluing them together to form a cylinder (b) with circumference β . The manifold (c) for the thermal reduced density matrix (2.20) is obtained by removing \mathcal{A} along the line of identification. The vacuum density matrix is then obtained by integrating over the unwinded cylinder (d) with \mathcal{A} cut out at $\tau = 0$.

entropy of a spatial subregion $\mathcal{A} \in \Sigma$ in flat spacetime. The formula (2.8) cannot be utilized directly as it involves the logarithm, but luckily there exists a method known as the *replica trick* [21, 22], which circumvents the problem. The method we present here only applies to states that have time reflection symmetry and additional constructions are needed to cover time evolving states as well (see [23] for a review of these methods).

First we need an expression for the reduced density matrix $\rho_{\mathcal{A}}$ in the field vacuum state. Consider a transition amplitude between two states $|\Phi_{\pm}\rangle$ defined at times τ_{\pm} in the Euclidean description ($\beta = \tau_+ - \tau_-$):⁹

$$\langle \Phi_+ | e^{-\beta H} | \Phi_- \rangle = \frac{1}{Z} \int_{\Phi(\Sigma, \tau_-) = \Phi_-}^{\Phi(\Sigma, \tau_+) = \Phi_+} \mathcal{D}\Phi e^{-I_E[\Phi]}. \quad (2.18)$$

Here I_E is the Euclidean action and $\Phi(\Sigma, \tau) = \Phi(\vec{x}, \tau)$, $\vec{x} \in \Sigma$. The prefactor Z is the partition function and it is given as the trace of (2.18):

$$Z = \int_{\Phi(\Sigma, \tau_-) = \Phi(\Sigma, \tau_+)} \mathcal{D}\Phi e^{-I_E[\Phi]}. \quad (2.19)$$

We recognize $\rho_{\beta} = e^{-\beta H}$ as the density matrix of a thermal state with inverse temperature β . To obtain the reduced density matrix $\rho_{\mathcal{A}, \beta}$ of the thermal state, we trace over the complement as defined in (2.6):

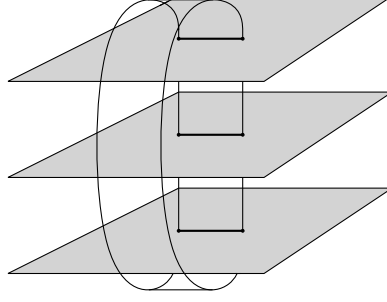
$$\langle \phi_+ | \rho_{\mathcal{A}, \beta} | \phi_- \rangle = \frac{1}{Z} \int_{\Phi(\mathcal{A}, \tau_-) = \phi_-, \Phi(\mathcal{A}^c, \tau_-) = \Phi(\mathcal{A}^c, \tau_+)}^{\Phi(\mathcal{A}, \tau_+) = \phi_+} \mathcal{D}\Phi e^{-I_E[\Phi]}. \quad (2.20)$$

This integral can be visualized as a cylinder of circumference β with a cut at $\{\mathcal{A}, \tau = 0\}$. The boundary conditions are given at both sides of the cut, see figure 1c.

By taking the limit $\beta \rightarrow \infty$ in (2.20), we get the reduced density matrix $\rho_{\mathcal{A}}$ of a vacuum state. This has the effect of unwinding the cylinder, see figure 1d. The cut remains at $\tau = 0$ with boundary conditions now given in the limits $\tau \rightarrow 0^{\pm}$:

$$\langle \phi_+ | \rho_{\mathcal{A}} | \phi_- \rangle = \frac{1}{Z} \int_{\Phi(\mathcal{A}, 0^+) = \phi_-}^{\Phi(\mathcal{A}, 0^-) = \phi_+} \mathcal{D}\Phi e^{-I_E[\Phi]}, \quad (2.21)$$

⁹Throughout this thesis we set $\hbar = c = k_B = 1$.

Figure 2: The replica surface \mathcal{B}_3 with a cyclic \mathbb{Z}_3 symmetry.

where Z is now the $\beta \rightarrow \infty$ limit of the original partition function. This is the desired formula for the reduced density matrix.

To proceed, we define *Rényi entropies*

$$S_{\mathcal{A}}^{(n)} = \frac{1}{1-n} \log \text{Tr} \rho_{\mathcal{A}}^n \quad (2.22)$$

for each natural number n such that entanglement entropy is the limit $n \rightarrow 1$:

$$S_{\mathcal{A}} = \lim_{n \rightarrow 1} S_{\mathcal{A}}^{(n)}. \quad (2.23)$$

Using this formula, the entanglement entropy can be obtained via an analytic continuation to non-integer values of n and some arguments can be given that the extension is unique [42].

Now comes the replica trick. The trace in (2.22) can be expanded as:

$$\text{Tr} \rho_{\mathcal{A}}^n = \int \mathcal{D}\phi_1 \langle \phi_1 | \rho_{\mathcal{A}}^n | \phi_1 \rangle \quad (2.24)$$

$$= \int \mathcal{D}\phi_1 \dots \int \mathcal{D}\phi_n \langle \phi_1 | \rho_{\mathcal{A}} | \phi_2 \rangle \langle \phi_2 | \rho_{\mathcal{A}} | \phi_3 \rangle \dots \langle \phi_n | \rho_{\mathcal{A}} | \phi_1 \rangle. \quad (2.25)$$

On the right hand side we have n copies of the matrix element (2.21). Substituting the path integral expression gives

$$\text{Tr} \rho_{\mathcal{A}}^n = \frac{1}{Z[\mathcal{B}_1]^n} \int_{\mathcal{B}_n} \mathcal{D}\Phi e^{-I_E[\Phi]} = \frac{Z[\mathcal{B}_n]}{Z[\mathcal{B}_1]^n}, \quad (2.26)$$

where $Z[\mathcal{B}_n]$ is the partition function on the *replica surface* \mathcal{B}_n and \mathcal{B}_1 is simply the original manifold in figure 1d without the cut. Therefore $Z[\mathcal{B}_1] = Z_0$ as used in equation (2.21). The replica surface consists of n copies of the manifolds 1d cyclically glued together at the cuts \mathcal{A} and is depicted in figure 2.

The power of the formula (2.26) is that it requires only the calculation of a partition function $Z[\mathcal{B}_n]$, which is not a simple task either, but can be done in some 2-dimensional conformal field theories. The replica trick becomes extremely useful in the context of AdS/CFT, where it is used to prove results regarding holographic entanglement entropy.

Replica trick in conformal field theories

A conformal field theory (CFT) is a QFT, which is symmetric under conformal transformations. A conformal transformation is a coordinate transformation $x \rightarrow x'$ that transforms the metric by an overall coordinate dependent factor:

$$g'_{\mu\nu}(x') = \Omega^2(x) g_{\mu\nu}(x). \quad (2.27)$$

Conformal transformations stretch distances, but preserve the angles between vectors.

The conformal group naturally contains the isometries of Minkowski space, the Poincaré group, as a subgroup. In addition, there are two additional generators that generate dilatations (scale transformations) and special conformal transformations. In dimension $d = p + q > 2$, the conformal algebra is isomorphic to $SO(p+1, q+1)$,¹⁰ while in dimension $d = 1 + 1$, the conformal algebra is the infinite dimensional Witt algebra [25].

In 2-dimensional CFTs one can explicitly calculate entanglement entropy of an interval $[-R, R]$ in the CFT thermal state of inverse temperature β using the replica trick. A thermal density matrix is the same as a vacuum density matrix on an Euclidean cylinder, which is periodic in the Euclidean time coordinate. In the case of CFT_2 , the cylinder can be conformally mapped to a plane, which allows the use of the replica trick. One can show [21, 22] that $\text{Tr} \rho_{\mathcal{A}}^n$ in (2.26) transforms as a two-point function of a primary operator with a known scaling dimension. Conformal symmetry completely fixes the form of the correlator, which leads to the entropy

$$S_{\mathcal{A},\beta} = \frac{c}{3} \log \left(\frac{\beta}{\pi a} \sinh \frac{\pi L}{\beta} \right). \quad (2.28)$$

where c is the central charge of the CFT, $L = 2R$ and a is the UV cutoff. At large temperatures ($L/\beta \gg 1$) the thermal nature of the state dominates and the entropy becomes extensive $S_{\mathcal{A},\beta} \sim L$. In the zero temperature limit $\beta \rightarrow \infty$, we obtain the vacuum entanglement entropy

$$S_{\mathcal{A}} = \frac{c}{3} \log \frac{L}{a}. \quad (2.29)$$

The form of the entropy is universal, same in all CFTs, and it only depends on the value of the central charge. Universality is a general property of entanglement entropies of ball-shaped regions, which will be discussed in section 3.3.

2.3 Modular flows in CFTs

In relativistic QFTs, entanglement entropy $S_{\mathcal{A}}$ is not uniquely associated with $\mathcal{A} \in \Sigma$, but instead we can always find another spacelike slice Σ' with a subregion \mathcal{A}' such that $S_{\mathcal{A}} = S_{\mathcal{A}'}$ [28]. Such regions have the same boundary $\partial\mathcal{A} = \partial\mathcal{A}'$ and share the same domain of dependence $D[\mathcal{A}'] = D[\mathcal{A}]$.¹¹ Hence entanglement entropy is a wedge operator: an operator, which is not uniquely associated with the Hilbert space of field states in \mathcal{A} , but with the Hilbert space of field states in $D[\mathcal{A}]$. The reduced density matrices $\rho_{\mathcal{A}}, \rho_{\mathcal{A}'}$ act on this wedge space and they are related by a unitary transformation $\rho_{\mathcal{A}'} = U \rho_{\mathcal{A}} U^\dagger$. The transformation U is a representation of a spacetime transformation that maps points of $D[\mathcal{A}]$ into each other. The generator of U is called the modular Hamiltonian $K_{\mathcal{A}}$:

$$U(\eta) = e^{iK_{\mathcal{A}}\eta}. \quad (2.30)$$

This transformation is a symmetry of the algebra of operators $\mathcal{O}_{\mathcal{A}}$ acting on the Hilbert space of $D[\mathcal{A}]$: the expectation values $\langle \mathcal{O}_{\mathcal{A}} \rangle = \text{Tr}(\rho_{\mathcal{A}} \mathcal{O}_{\mathcal{A}})$ are invariant under (2.30), which follows from the cyclicity of the trace. For example entanglement entropy $S_{\mathcal{A}} = -\langle \log \rho_{\mathcal{A}} \rangle$ is invariant. In addition, the correlation functions of two operators of the algebra satisfy the condition $\langle \mathcal{O}_1(i) \mathcal{O}_2 \rangle = \langle \mathcal{O}_2 \mathcal{O}_1 \rangle$, where

¹⁰The signature is taken to be (p, q) .

¹¹The domain of dependence $D[\mathcal{A}]$ of \mathcal{A} (or the causal diamond of \mathcal{A}) is the region of spacetime, whose points are causally connected to \mathcal{A} . In other words, a point p is in $D[\mathcal{A}]$ if there exists a timelike curve connecting p to a for all $a \in \mathcal{A}$.

$\mathcal{O}_1(i) = U(i)\mathcal{O}_1U(-i)$. This is known as the KMS (Kubo-Martin-Schwinger) periodicity relation and it implies that the expectation values have a formal thermal character, which is apparent in the definition of the modular Hamiltonian:

$$\rho_{\mathcal{A}} = \frac{1}{Z} e^{-K_{\mathcal{A}}}. \quad (2.31)$$

In general, the modular Hamiltonian is a highly non-local operator, but if it generates a symmetry of the theory, it takes a local form. Then it also generates a local spacetime flow, called modular flow, and the norm of the generating vector field vanishes on the boundary $\partial D[\mathcal{A}]$. To derive the local expression, we start by recalling the functional integral formula for the vacuum density matrix (2.21). It can be written as a flow in an Euclidean time coordinate s with the boundary values s^{\pm} at $\tau = 0^{\mp}$:

$$\langle \phi_+ | \rho_{\mathcal{A}} | \phi_- \rangle = \frac{1}{Z} \int_{\Phi(\mathcal{A}, s_-) = \phi_-}^{\Phi(\mathcal{A}, s_+) = \phi_+} \mathcal{D}\Phi e^{-I_E[\Phi]}. \quad (2.32)$$

If the flow in s is generated by an operator $\mathcal{K}_{\mathcal{A}}$ on the Hilbert space, we can write

$$\rho_{\mathcal{A}} = \frac{1}{Z} \mathcal{P} \exp \left(- \int_{s_-}^{s_+} ds \mathcal{K}_{\mathcal{A}}(s) \right), \quad (2.33)$$

where \mathcal{P} denotes path ordering in s . Given that the theory is symmetric under the flow in s , the operator $\mathcal{K}_{\mathcal{A}}$ is conserved and therefore independent of s so that (2.33) reduces to the expression (2.31) with

$$K_{\mathcal{A}} = (s_+ - s_-) \mathcal{K}_{\mathcal{A}}. \quad (2.34)$$

The flow along s is the Wick rotated modular flow ($\eta = is$), which in the Lorentzian signature is denoted by $x^{\mu}(\eta)$. Now the modular Hamiltonian can be written in a local form

$$K_{\mathcal{A}} = \int_{\mathcal{A}'} d\Sigma^{\mu} \zeta_{\mathcal{A}}^{\nu} T_{\mu\nu}, \quad \zeta_{\mathcal{A}}^{\mu} = (s_+ - s_-) \frac{dx^{\mu}}{d\eta}, \quad (2.35)$$

where $\zeta_{\mathcal{A}}^{\mu}$ is the vector field generating the modular flow, $T_{\mu\nu}$ is the stress-energy tensor of the theory and \mathcal{A}' is an arbitrary spacelike surface s.t. $D[\mathcal{A}'] = D[\mathcal{A}]$. On the $t = 0$ slice this can be written as

$$K_{\mathcal{A}} = \int_{\mathcal{A}} d^{d-1}x \zeta_{\mathcal{A}}^0 T_{00}(x). \quad (2.36)$$

For local flows the thermal character of $\rho_{\mathcal{A}}$ becomes physical, because for observers moving along the coordinate lines of the modular flow, the modular Hamiltonian (2.35) is simply the Hamiltonian. The corresponding inverse temperature of the observed thermal fluctuations is given by the coefficient $s_+ - s_-$.

Modular flow of a ball-shaped region

Consider the half-space $\mathcal{A}_R = \{x \in \mathbb{R}^{1,d-1} | x^1 > 0, t = 0\}$ of flat spacetime $\mathbb{R}^{1,d-1}$ in the CFT vacuum state. The Bisognano-Wichmann theorem [26] states that for this region the modular Hamiltonian is local and the modular flow is generated by the boost generator

$$\zeta_R = 2\pi(x^1\partial_t + t\partial_1). \quad (2.37)$$

Boosts correspond to rotations in Euclidean signature (see figure 3), which explains the factor of $2\pi = s_+ - s_-$ corresponding to a full rotation. In addition the modular

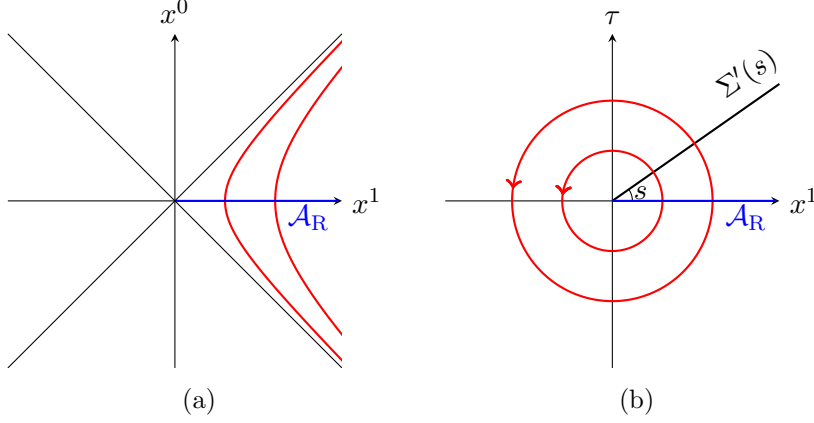


Figure 3: Half space modular flows (red) in (a) Lorentzian and (b) Euclidean signatures.

flow preserves the entangling surface located at $x^1 = 0$. The modular Hamiltonian is (with the normalization $Z = 1$)

$$\rho_R = e^{-K_R}, \quad K_R = 2\pi \int_{x^1 > 0} d^{d-1}x \, x^1 T_{00}(x). \quad (2.38)$$

The boost generator is one of the Killing vectors of $\mathbb{R}^{1,d-1}$, which means that an arbitrary Lorentz invariant QFT is actually symmetric under the flow. Now the thermal character of the density matrix is just the familiar Unruh effect [27]. To find the temperature, transform to the Rindler coordinates (η, z) :

$$\begin{aligned} t &= z \sinh(\eta/R) \\ x^1 &= z \cosh(\eta/R), \end{aligned} \quad (2.39)$$

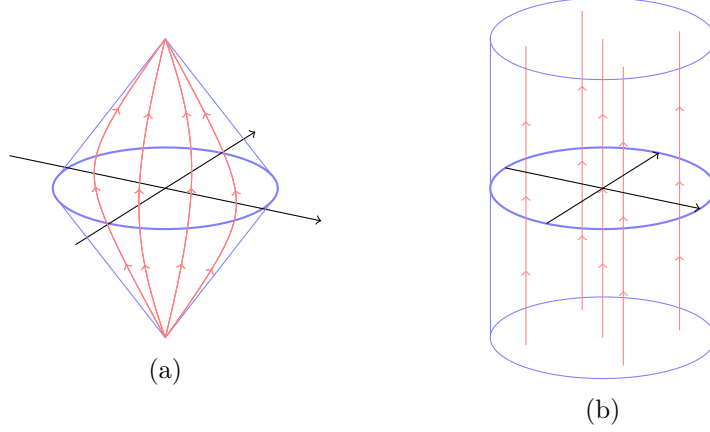
where R is just a constant with dimension of length. The Rindler coordinates parametrize the Rindler wedge $D[A_R]$ and the coordinate lines of constant z are hyperbolas. An observer with constant acceleration a in the direction x^1 travels along the hyperbola $(x^1)^2 - t^2 = 1/a^2$. Hence the Rindler coordinates define the rest frame of the accelerating observer and the proper time is given by η . In the Euclidean picture, the hyperbola of the observer corresponds to a circle of radius $1/a$ and of circumference $2\pi/a$. To ensure the correct periodicity of the Euclidean Rindler coordinates, we must set $R = 1/a$. Now in the rest frame of the observer, the boost generator takes the simple form $\zeta_R = (2\pi R)\partial_\eta = (2\pi/a)\partial_\eta$ (generator of translations in proper time) and as was prescribed above, the inverse of the coefficient is identified as the Unruh temperature $T = a/2\pi$.

In the case of a CFT, the above result for the half-space can be used to obtain the modular Hamiltonian of a ball-shaped region in the CFT vacuum state. This is done by applying a conformal transformation due to Casini, Huerta and Myers (CHM) [28], which can be decomposed as follows [23]. In spherical coordinates, the metric of Minkowski space is:

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_{d-2}^2. \quad (2.40)$$

A ball-shaped region $B(R, 0)$ of radius R (centered at the origin) is the region with $r = |\vec{x}| \leq R$. Consider the conformal transformation

$$t = \frac{R \sinh(\eta/R)}{\cosh u + \cosh(\eta/R)}, \quad r = \frac{R \sinh u}{\cosh u + \cosh(\eta/R)}, \quad (2.41)$$

Figure 4: The CHM map (a) \Rightarrow (b)

where η is to be identified with the Rindler time coordinate and $u \geq 0$ is a radial coordinate. This transformation maps the metric (2.40) to the metric of the Lorentzian hyperbolic cylinder $\mathbb{H}_{d-1} \times \mathbb{R}$

$$ds^2 = \Omega^2 [-d\eta^2 + R^2(du^2 + \sinh^2 u d\Omega_{d-2}^2)] \quad (2.42)$$

with the conformal factor

$$\Omega^2 = \frac{1}{[\cosh u + \cosh(\eta/R)]^2}. \quad (2.43)$$

The double cone $D[B] = \{r + t \leq R\} \cap \{r - t \leq R\}$ gets mapped to the entirety of the cylinder and the complement of the ball is sent to infinity.

By writing the metric of the hyperbolic cylinder (2.42) in Poincaré coordinates (see Appendix A), the metric becomes the metric of Rindler space (2.39) with $a = 1/R$:

$$ds^2 = \frac{R^2 \Omega^2}{z^2} \left[-\frac{z^2 d\eta^2}{R^2} + dz^2 + \sum_{i=2}^{d-2} (dx^i)^2 \right]. \quad (2.44)$$

The Poincaré coordinates cover half of the hyperbolic space, the Poincaré patch, which corresponds to the Rindler wedge $D[\mathcal{A}_R]$. Thus we have established a conformal map from the double cone $D[B]$ to $D[\mathcal{A}_R]$ and the corresponding operators are mapped to each other.

We can identify the generator of the Rindler modular flow to be the boost generator $\zeta_R = 2\pi R \partial_\eta$ as before. By inverting the conformal transformations, we can map this back to the generator of the ball modular flow, which in spherical coordinates is

$$\zeta_B = 2\pi \left[\left(\frac{R^2 - t^2 - r^2}{2R} \right) \partial_t - \frac{tr}{R} \partial_r \right]. \quad (2.45)$$

This vector field again preserves the entangling surface located at $r = R$. The boost generator is a Killing vector of Minkowski space, but the (2.45) is not a Killing vector, but only a conformal Killing vector. In other words, the boost generator is mapped to another generator of conformal symmetry, but in this case the new generator no longer generates a symmetry of the Minkowski spacetime. Using (2.36), the modular Hamiltonian of the ball becomes

$$K_B = 2\pi \int_{r \leq R} d^{d-1}x \frac{R^2 - r^2}{2R} T_{00}(x). \quad (2.46)$$

We can also use the CHM map to obtain an alternative interpretation for the vacuum entanglement entropy S_B of the ball. The generator ζ_B of the ball modular flow is mapped to the time evolution generator $2\pi R\partial_\eta$ on the hyperbolic cylinder. Therefore the reduced density matrix ρ_B of the ball is mapped to a density matrix of a thermal state on \mathbb{H}_{d-1} . These two density matrices are related by a unitary transformation that represents the conformal map in the Hilbert space. It is easy to see that entanglement entropy is invariant under unitary transformations of the density matrix, which means that the vacuum entanglement entropy of the ball is exactly equal to thermal entropy on \mathbb{H}_{d-1}

$$S_B = S_\beta(\mathbb{H}_{d-1}) \quad (2.47)$$

with inverse temperature proportional to the radius of the ball $\beta = 2\pi R$. This result has an interesting interpretation in terms of the AdS/CFT correspondence, which we discuss in section 3.2.

3 AdS/CFT and holographic entanglement entropy

AdS/CFT correspondence is a conjectured duality between a string theory and a conformal field theory. The term covers a large number of dualities between various string theories and CFTs, but they all share the same characteristics. The string theory lives on a $(d+1)$ -dimensional anti-de Sitter (AdS) spacetime, while the CFT lives on a lower dimensional d -dimensional flat spacetime. The string theory is a theory of quantum gravity and the CFT is an ordinary field theory without gravity. The duality states that the two theories are actually equivalent, even though their mathematical formalisms are different. In particular, the gravitational degrees of freedom of the string theory are implicitly encoded in the structure of the CFT.

To study how classical features of the string theory are encoded in the CFT, one must introduce coarse-graining on the CFT such that the string theory reduces to its low energy limit: a QFT coupled to classical gravity. These holographic CFTs, that are dual to classical gravity, are believed to contain large number of degrees of freedom (large- N) and to be strongly coupled. In the large- N limit of the CFT, the string dual is dominated by an effective classical string action that decouples to general relativity in the strong coupling limit of the CFT. The decoupling is due to the weak/strong nature of the AdS/CFT correspondence, which also makes the duality very useful in the study of different problems.

The setup of the two theories lies in the heart of the duality: the string theory lives in the bulk of the AdS space, while the CFT lives on its flat boundary. Diffeomorphisms in the bulk are elements of AdS isometry group $SO(2, d)$ that acts as conformal transformations on the d -dimensional boundary. This interplay between diffeomorphisms and conformal transformations is the reason why a diffeomorphism invariant classical gravity can be dual to a conformally symmetric theory. One can also use these symmetries to prove dualities between certain classical bulk geometries and CFT states. The simplest example is the CFT vacuum state, which is invariant under arbitrary bulk induced conformal transformations so that the bulk dual of the CFT vacuum state must be pure AdS spacetime. Small excitations then correspond to deformations of the AdS spacetime.

There are other ways to find spacetimes that correspond to various CFT states by using the holographic dictionary. For example a CFT thermal state can be shown to be dual to the AdS black hole with horizon surface gravity set by the state temperature. This result is of great interest, because it apparently solves the black hole information loss paradox: the CFT evolution is unitary, which means that the black

hole evolution should be as well. Regardless, it is hard to study how exactly the paradox is avoided and is currently an unsolved problem.

AdS/CFT is traditionally formulated in the Euclidean picture by using the partition functions of the two theories [29, 30]. The idea is that the CFT generating functional is directly related to the string theory partition function and the CFT correlation functions are sourced by the boundary values of bulk fields with the same spin. For example the CFT stress-energy tensor operator is sourced by the metric and CFT scalar operators by scalar fields in the bulk. This duality allows one to calculate CFT correlations functions by solving equations of motion in the bulk.

More complicated entries in the holographic dictionary have been discovered. The most important one for our purposes is the duality between entanglement entropy of a CFT spatial region and the area of a bulk minimal surface. This result is known as the Ryu-Takayanagi formula [5, 6] and is reviewed in section 3.2.

3.1 AdS/CFT correspondence

We will give a brief mathematical overview of the AdS/CFT correspondence [31, 24]. AdS_{d+1} space is the maximally symmetric solution of the Einstein's equations with a negative cosmological constant. It can be realized as a $(d+1)$ -dimensional hyperbola

$$-X_{-1}^2 - X_0^2 + X_1^2 + \dots + X_d^2 = -\ell^2 \quad (3.1)$$

embedded in a flat $\mathbb{R}^{2,d}$ spacetime with the metric

$$ds^2 = -dX_{-1}^2 - dX_0^2 + dX_1^2 + \dots + dX_d^2. \quad (3.2)$$

The hyperbola (3.1) is invariant under $SO(d, 2)$ transformations of the embedding space, which makes it the isometry group of AdS space.

The whole AdS space is covered by global coordinates (ρ, τ, Ω_i) :

$$X_{-1} = \ell \cosh \rho \cos \tau \quad (3.3)$$

$$X_0 = \ell \cosh \rho \sin \tau \quad (3.4)$$

$$X_i = \ell \sinh \rho \Omega_i \quad (3.5)$$

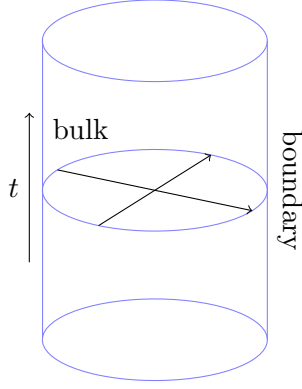
with the ranges $(\rho \geq 0, 0 \leq \tau < 2\pi)$. Here Ω_i parametrize the unit sphere S^{d-1} ($\sum_i \Omega_i^2 = 1$). The resulting metric is

$$ds^2 = \ell^2 (-\sinh^2 \rho d\tau^2 + \cosh^2 \rho d\rho^2 + \rho^2 d\Omega_{d-1}^2), \quad (3.6)$$

where $d\Omega_{d-1}$ is the metric on S^{d-1} . AdS space defined this way as an embedding is periodic in the time coordinate τ that parametrizes the circle over the timelike coordinates X_{-1} and X_0 . Therefore it contains closed timelike curves. We can circumvent this problem by considering the universal covering, which has the above metric (3.6), but with the range of τ unbounded. It is this universal cover that is actually referred to as AdS space. The cover has a timelike boundary at spatial infinity $\rho \rightarrow \infty$, which becomes obvious in the coordinates $\tan \theta = \sinh \rho$ ($0 \leq \theta < \pi/2$):

$$ds^2 = \frac{\ell^2}{\cos^2 \theta} (-d\tau^2 + d\theta^2 + \sin^2 \theta d\Omega_{d-1}^2). \quad (3.7)$$

The boundary is now located at $\theta = \pi/2$ and has the topology of $S^{d-1} \times \mathbb{R}$. From this metric it is clear that AdS space is conformally equivalent to a patch of $S^d \times \mathbb{R}$ (only a patch since $\theta \in [0, \pi/2[$ and not $[0, \pi]$) and can hence be depicted as a cylinder in figure 5.

Figure 5: Conformal diagram of AdS_{d+1} .

For large values of the coordinates in the hyperbola (3.1), the right hand side can be neglected. This defines the AdS boundary as the surface

$$-X_{-1}^2 - X_0^2 + X_1^2 + \dots + X_d^2 = 0 \quad (3.8)$$

subject to an additional scaling relation

$$(X_{-1}, X_0, X_i) \sim \lambda(X_{-1}, X_0, X_i), \quad \lambda > 0. \quad (3.9)$$

The scaling relation is the formal statement that the surface does not care how large the coordinates actually are. Embedding this surface into the ambient space (3.2), we see that it is equivalent to d -dimensional Minkowski space and that the scaling relation compactifies the space. Therefore the boundary is the conformal compactification of Minkowski space [29] and the action of $SO(2, d)$ on it is the same as the action of the conformal group [32].

The AdS/CFT correspondence is usually formulated in the Euclidean picture, where the bulk AdS space is Euclidean hyperbolic space. In bulk global coordinates the boundary has a topology of the sphere S^d . However, a bulk coordinate transformation can induce a boundary conformal transformation that changes the boundary topology up to a scaling of the metric, which can be removed due to conformal invariance of the CFT. Hence patches of the same bulk geometry can be encoded by a CFT on boundaries with different topology. For example S^d can be conformally mapped to \mathbb{R}^d so that the CFT vacuum states defined on these spaces are both dual to the same bulk geometry (pure AdS), but the S^d vacuum is dual to the bulk as a whole (covered by global coordinates), while the \mathbb{R}^d vacuum is only dual to a finite patch (the Poincaré patch).

At large- N limit of the CFT, the string theory is dominated by an on-shell classical action, which in general is a supergravity action. At strong-coupling, the action is the Einstein-Hilbert action

$$I_{\text{EH}} = \frac{1}{16\pi G_N} \int d^{d+1}x \sqrt{-g} \left[\frac{d(d-1)}{\ell^2} + R \right], \quad (3.10)$$

which is corrected by higher curvature terms at weaker coupling [23].

Solving Einstein's equations in AdS space is a bit different, because it is not globally hyperbolic due to the timelike boundary at spatial infinity. To have a well posed initial value problem, it is not enough to specify initial values on the bulk Cauchy slice, but we also need to set boundary conditions on the boundary. In fact, because Einstein's equations are second order in the metric, the solutions require two

boundary conditions: the boundary metric and its d^{th} -order derivative respect to the radial coordinate. Fefferman and Graham showed [33, 34, 31] that a general solution can be written as

$$ds^2 = G_{ab}dx^a dx^b = \frac{\ell^2}{z^2}(dz^2 + G_{\mu\nu}(z, x)dx^\mu dx^\nu), \quad (3.11)$$

where $z \geq 0$ is a radial coordinate with boundary at $z = 0$ and x^μ are the boundary coordinates.¹² They showed that the bulk metric $G_{\mu\nu}(z, x)$ can be expanded as a Taylor series in z starting from the boundary:

$$G_{\mu\nu}(z, x) = g_{\mu\nu}(x) + z^2 h_{\mu\nu}^{(2)}(x) + z^4 h_{\mu\nu}^{(4)}(x) + \dots + z^d h_{\mu\nu}^{(d)}(x) + \dots \quad (3.12)$$

The functions $h_{\mu\nu}^{(k)}(x)$ for $k \neq d$ are determined by Einstein's equations, but the boundary geometry $g_{\mu\nu}(x)$ and the term $h_{\mu\nu}^{(d)}(x)$ are fixed as boundary conditions [34]. It follows from the Einstein's equations that for flat boundary geometry $g_{\mu\nu}(x) = \eta_{\mu\nu}$, all the higher-order terms $h_{\mu\nu}^{(k)}(x)$ vanish, resulting in pure AdS geometry in Poincaré coordinates:

$$ds^2 = \frac{\ell^2}{z^2}(dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu). \quad (3.13)$$

These coordinates cover half of the AdS space, the Poincaré patch.

In the limit of strong CFT coupling, the string theory can be described as in Euclidean quantum gravity [35]: the string partition function $Z_{\text{STR}}[\mathcal{B}]$ is a functional integral over all bulk geometries that asymptote to the boundary geometry \mathcal{B} , which was set as a boundary condition. AdS/CFT correspondence is the statement that this partition function is equal to the CFT partition function $Z_{\text{CFT}}[\mathcal{B}]$ defined on the boundary geometry:

$$Z_{\text{STR}}[\mathcal{B}] \equiv Z_{\text{CFT}}[\mathcal{B}]. \quad (3.14)$$

This leads to a relationship between the generating functional of the CFT and the string partition function [29]. Given the bulk metric $G_{ab}(z, x)$ in Poincaré coordinates, it is written as

$$\left\langle \exp \left(- \int d^d x g_{\mu\nu}(x) T^{\mu\nu}(x) \right) \right\rangle_{\text{CFT}} \equiv Z_{\text{STR}}[\mathcal{B}], \quad (3.15)$$

where $T^{\mu\nu}(x)$ is the stress-energy tensor of the CFT and $G_{\mu\nu}(0, x) = g_{\mu\nu}(x)$, which is the metric of \mathcal{B} . The asymptotic bulk metric acts as a source that can be used to generate the CFT n -point functions of the stress-energy tensor from the string partition function. The same idea works for fields of arbitrary spin, in particular, a scalar field in the bulk sources correlation functions of CFT scalar operators on the boundary.

In the classical limit, the string partition function is dominated by the on-shell classical action

$$Z_{\text{STR}}[\mathcal{B}] \simeq e^{-I_{\text{EH}}[\mathcal{M}]}, \quad (3.16)$$

where \mathcal{M} is a solution of the Einstein's equations such that $\partial\mathcal{M} = \mathcal{B}$. In this saddle point approximation, we can explicitly calculate the CFT 1-point function $\langle T_{\mu\nu} \rangle$. When the bulk geometry \mathcal{M} has a vanishing Weyl tensor, the boundary geometry is conformally flat, which picks a specific vacuum state of the CFT. The result is [34]:

$$\langle T_{\mu\nu} \rangle = \frac{d \ell^{d-1}}{16\pi G_{\text{N}}} h_{\mu\nu}^{(d)}. \quad (3.17)$$

¹²The boundary coordinates are denoted by $x^\mu = (t, \vec{x})$, $\mu = 0, 1, \dots, d-1$, and the full set of AdS coordinates by $x^a = (x^\mu, z)$, $a = 0, 1, \dots, d$ with $x^d = z$.

This result can also be derived using the Ryu-Takayanagi formula [10]. The vanishing of $h_{\mu\nu}^{(d)}(x)$, as in (3.13), therefore implies that the CFT is in a vacuum state $\langle T_{\mu\nu} \rangle = 0$. This verifies the argument based on symmetry: pure AdS space is dual to the CFT vacuum state.

The classical limit of the string theory was obtained in the $N \rightarrow \infty$ limit of the CFT. Therefore $1/N$ basically acts as an effective Planck's constant $\hbar_{\text{eff}} \sim 1/N$ tuning the quantum corrections of the classical gravity theory. The classical contribution $\mathcal{O}(N)$ is of order $1/G_N$ in the bulk theory, which can be seen at the level of the action (3.10) and in the stress-energy tensor (3.17). The corrections in the gravity theory come in powers of G_N and the first order quantum correction, which is of order $\mathcal{O}(1)$ in G_N , comes in the form of quantum matter fields that couple to the classical spacetime geometry

$$Z_{\text{STR}}[\mathcal{M}] \simeq e^{-I_{\text{EH}}[\mathcal{M}]} Z_M[\mathcal{M}], \quad (3.18)$$

where Z_M is the partition function of the quantum matter fields on the classical geometry \mathcal{M} . Therefore the leading $1/N$ -correction to the CFT stress-energy 1-point function (3.17) comes from the bulk field stress-energy:

$$\langle T_{\mu\nu} \rangle = \frac{d \ell^{d-1}}{16\pi G_N} h_{\mu\nu}^{(d)} + C \langle T_{\mu\nu}^{\text{bulk}} \rangle. \quad (3.19)$$

Here C is a field independent constant that is not relevant to us.

3.2 Holographic Entanglement Entropy

Entanglement entropy is an example of a CFT quantity that has a classical bulk dual in the AdS/CFT correspondence. In 2006, Ryu and Takayanagi (RT) proposed [5, 6] that for static geometries, entanglement entropy of boundary spatial region is calculated as the area of a minimal surface in the bulk. They showed that for the CFT vacuum and thermal states, the area matches exactly with the formulas (2.29) and (2.28). This remarkable result was a year later generalized to general time dependent spacetimes by Hubeny, Rangamani and Takayanagi (HRT) [36], which generalized the minimal surface to an extremal surface with smallest area.

Consider a holographic CFT_d on the boundary \mathcal{B} of an asymptotically AdS_{d+1} spacetime \mathcal{M} . Let $\Sigma \subset \mathcal{B}$ be a boundary Cauchy slice and let $\mathcal{A} \subset \Sigma$ be a spatial region. Hubeny, Rangamani and Takayangi proposed that the entropy $S_{\mathcal{A}}$ can be calculated as the area of an extremal bulk codimension two surface anchored at the boundary.¹³ The proposal states that

$$S_{\mathcal{A}} = \min_{\mathcal{E}_{\mathcal{A}} \in \mathcal{R}_{\mathcal{A}}} \frac{\text{Area}[\mathcal{E}_{\mathcal{A}}]}{4G_N} \quad (3.20)$$

The set $\mathcal{R}_{\mathcal{A}}$ consists of bulk codimension two extremal surfaces $\mathcal{E}_{\mathcal{A}}$ that have the following properties:

- (i) The surface is connected to the boundary: $\partial \mathcal{E}_{\mathcal{A}} \subset \partial \mathcal{M} = \mathcal{B}$.
- (ii) The surface $\mathcal{E}_{\mathcal{A}}$ has the same boundary as \mathcal{A} : $\partial \mathcal{E}_{\mathcal{A}} = \partial \mathcal{A}$.
- (iii) The surface $\mathcal{E}_{\mathcal{A}}$ is homologous to \mathcal{A} : $\exists \Sigma_{\mathcal{A}} \subset \mathcal{M}$ s.t. $\partial \Sigma_{\mathcal{A}} = \mathcal{E}_{\mathcal{A}} \cup \mathcal{A}$.

¹³By an extremal surface we mean that the surface is a local extremum of the area functional (3.24). Area refers to the volume of the bulk codimension two surface which in the case of AdS_{d+1} is $(d-1)$ -dimensional.

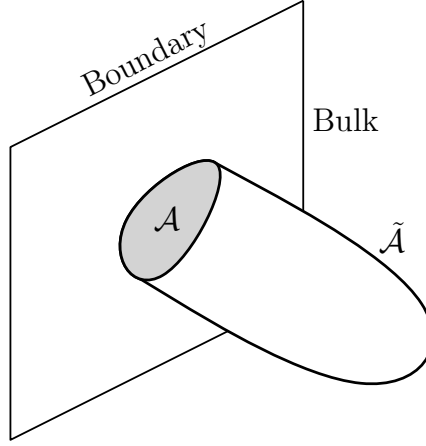


Figure 6: A spatial slice with a Ryu-Takayanagi surface $\tilde{\mathcal{A}}$ of a boundary spatial region \mathcal{A} in $\text{AdS}_4/\text{CFT}_3$.

If we denote the extremal surface $\mathcal{E}_{\mathcal{A}} \in \mathcal{R}_{\mathcal{A}}$ that has the smallest area by $\tilde{\mathcal{A}}$, we get the HRT formula:

$$S_{\mathcal{A}} = \frac{\text{Area}[\tilde{\mathcal{A}}]}{4G_{\text{N}}}. \quad (3.21)$$

The surface $\tilde{\mathcal{A}}$ will be referred to as the HRT surface.

In general, there is no unique extension of the boundary Cauchy slice Σ to the bulk. Instead, there is a whole family of bulk Cauchy slices, the FRW wedge [23], whose points are spacelike separated from Σ . Therefore the extremal surfaces $\mathcal{E}_{\mathcal{A}} \in \mathcal{R}_{\mathcal{A}}$ lie somewhere inside this FRW wedge. But suppose the bulk geometry is static (the CFT state is static), meaning that ∂_t is also a bulk Killing vector. Then one can naturally extend the boundary foliation, defined by ∂_t , to the bulk in a unique fashion. In such a situation, all of the extremal surfaces corresponding to a boundary region lie on a single bulk Cauchy slice [36]. The homology requirement is then automatically satisfied ($\Sigma_{\mathcal{A}}$ is a subset of the bulk Cauchy slice) and the problem reduces to finding the minimal surface anchored at $\partial\mathcal{A}$. This is the original Ryu-Takayanagi proposal [5, 6] and it is also applicable at a moment of time reflection symmetry. Then the minimal surface $\tilde{\mathcal{A}}$ is simply referred to as the RT surface.

The RT formula is a highly non-trivial result and should not be taken for granted. Both sides of the RT formula are divergent and it might seem impossible to regulate it consistently. Luckily, consistency is ensured by the UV/IR duality of the AdS/CFT correspondence. Consider a scale transformation $x^\mu \rightarrow \lambda x^\mu$ on the boundary. The CFT state is of course symmetric, meaning that the corresponding bulk geometry (3.11) must be as well. The invariance of (3.11) requires that the radial coordinate transforms in the same way $z \rightarrow \lambda z$. A transformation with small positive λ maps physics of the CFT to the UV, while on the bulk side everything is mapped to large scales (small z , large ρ), which is the UV/IR duality. The duality allows us to regulate the infinite area of the HRT surface and the UV divergence of CFT entanglement entropy at the same time, because there is a relation between the UV and IR cutoffs $a = \rho_0$. The regulated terms in the RT formula then always agree.

The above matching of the divergences is already suggestive that the proposal might be correct. More evidence comes from the fact that the areas of the HRT surfaces satisfy the strong subadditivity inequality. One can also explicitly calculate areas of extremal surfaces of various subregions in different CFT states and the results can be compared with pure field theoretic calculations. Both the RT and

the HRT proposal have passed multiple non-trivial checks. In fact, both the RT proposal [37, 38] and the HRT proposal [39] have been proven. See the book [23] for a comprehensive review of all of these topics.

The RT formula captures the classical $N \rightarrow \infty$ contribution to the entanglement entropy $S_{\mathcal{A}}$ by calculating a purely classical bulk quantity, the area. The leading $\mathcal{O}(1)$ -correction to the entanglement entropy can be calculated by considering quantum matter fields in the bulk. Faulkner, Lewkowycz and Maldacena (FLM) proposed [40] that the leading correction is due to the entanglement of the bulk fields over the RT surface $\tilde{\mathcal{A}}$ in the semi-classical approximation. Their proposal states that

$$S_{\mathcal{A}} = \frac{\text{Area}[\tilde{\mathcal{A}}]}{4G_N} + S_Q + \mathcal{O}(G_N) \quad (3.22)$$

$$S_Q = S_{\Sigma_{\mathcal{A}}}^{\text{bulk}} + \dots \quad (3.23)$$

where $S_{\Sigma_{\mathcal{A}}}^{\text{bulk}}$ is the *bulk* entanglement entropy of region $\Sigma_{\mathcal{A}}$ that is bounded by the RT surface and the boundary region: $\partial\Sigma_{\mathcal{A}} = \tilde{\mathcal{A}} \cup \mathcal{A}$. The role of the extra terms in S_Q (denoted by the ellipsis) is to cancel the UV divergences in the bulk entanglement entropy. Therefore the leading order correction to CFT entanglement entropy in $1/N$ is given by bulk entanglement entropy!

The proposal is not completely new [41, 42]. In the context of black hole entropy, it has been argued that the leading correction is due to the entanglement of quantum fields across the horizon. The FLM formula can be seen as a generalization of these arguments to arbitrary RT surfaces. In addition, it provides a microscopic interpretation in terms of the dual CFT, which is absent in the black hole considerations. The presence of AdS/CFT also ensures that the counter terms in S_Q exactly cancel the divergences of $S_{\Sigma_{\mathcal{A}}}^{\text{bulk}}$, because the bulk theory should be a UV finite theory of quantum gravity. The FLM formula can be proven analogously to the RT formula by extending the boundary replica construction to the bulk, but also keeping the bulk matter field contribution in the string partition function as in (3.18).

Entanglement entropy of a ball-shaped region

Calculating entanglement entropy of a boundary ball $B(R, \vec{x}_0)$ in the CFT vacuum state provides a nice example of the use of the HRT formula. The vacuum state is static so it is enough to apply the RT proposal and calculate the bulk minimal surface anchored to ∂B . In general, the bulk codimension-2 surface can be parametrized by σ^i so that the embedding Poincaré coordinates are $(z(\sigma), t = 0, x^i(\sigma))$. We choose the parametrization $\sigma^i = x^i$ giving the bulk surface as the function $z(\vec{x})$. The area functional is given by

$$\text{Area} = \int d^{d-1}x \sqrt{h} \quad (3.24)$$

where h is the determinant of the induced metric on the surface (sum over repeated is understood):¹⁴

$$h = \det \left[\frac{\ell^2}{z^2} \left(\delta_{ij} + \frac{\partial z}{\partial x^i} \frac{\partial z}{\partial x^j} \right) \right] = \left(\frac{\ell}{z} \right)^{2d-2} \left(1 + \frac{\partial z}{\partial x^i} \frac{\partial z}{\partial x^i} \right). \quad (3.25)$$

Given the boundary condition $z(\vec{x}) = 0$, when $|\vec{x} - \vec{x}_0| = R$, the Euler-Lagrange equations can be solved to give:

$$z^2 + |\vec{x} - \vec{x}_0|^2 = R^2. \quad (3.26)$$

¹⁴The determinant can be calculated as a product of eigenvalues. In this case, there is only a single eigenvalue with degeneracy $d - 1$. The eigenvalue of a dyadic matrix $M_{ij} = a_i a_j$ is easy to find: $M_{ij} a_j = (a_i a_j) a_j = (a_j a_j) a_i$.

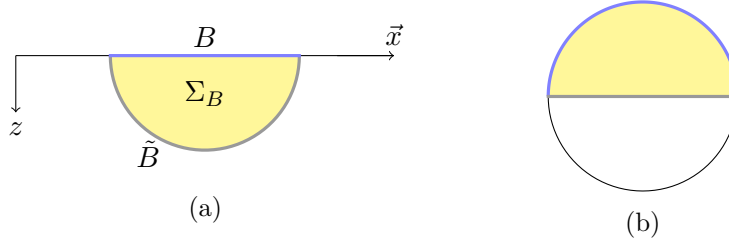


Figure 7: The spatial boundary \mathbb{R}^{d-1} can be compactified to S^{d-1} by a conformal transformation. The hemisphere \tilde{B} is mapped to a surface with least area, because it divides the sphere in half.

Hence in Poincaré coordinates, the RT surface is a bulk hemisphere $\tilde{B}(R, \vec{x}_0)$ (see figure 7a). Entanglement entropy of the ball is now obtained as the area of the hemisphere.

The result that the minimal surface is a hemisphere can also be obtained via a conformal transformation, which compactifies the boundary \mathbb{R}^{d-1} to S^{d-1} . The hemisphere is mapped to the surface that divides the sphere exactly in half and therefore having the least area [12] (see figure 7).

In 2-dimensions, given the relation between the central charge c of the CFT and the bulk parameters

$$c = \frac{3}{2} \frac{\ell}{G_N}, \quad (3.27)$$

the area of the RT surface exactly reproduces the formula (2.29) for the vacuum entanglement entropy of a closed interval $[-R, R]$, which was the $\beta \rightarrow \infty$ limit of the entanglement entropy of a thermal state (2.28). We can also reproduce the thermal state entropy by finding the corresponding dual geometry. The thermal state can be calculated by considering a vacuum CFT on a cylinder, which is periodic in the Euclidean time coordinate. Therefore the corresponding dual geometry must also be periodic in the Euclidean time coordinate. This is a property of black holes and indeed, the correct dual of a CFT thermal state in 2-dimensions is the 3-dimensional BTZ black hole [43, 44]. The temperature of the black hole is also determined by the periodicity of the Euclidean time coordinate. By calculating the ball RT surface in the BTZ geometry, one can explicitly show that the RT formula reproduces the result (2.28). The RT surface of the ball reaches out to the black hole horizon, which produces the dominating extensive behaviour of the entropy (2.28) at large temperature.

The relation between thermal states and black holes extends to vacuum entanglement entropy of higher dimensional balls. We saw in section 2.3 using the CHM map (2.41) that the CFT vacuum entanglement entropy of a ball is equal to thermal entropy on hyperbolic space \mathbb{H}_{d-1} . The bulk diffeomorphism inducing the CHM map on the boundary takes the initial bulk metric in global coordinates (3.6), which asymptotes to spherical coordinates on the boundary, to the metric

$$ds^2 = -\frac{\rho^2 - \ell^2}{R^2} d\eta^2 + \frac{\ell^2}{\rho^2 - \ell^2} d\rho^2 + \rho^2 (du^2 + \sinh^2 u d\Omega_{d-2}^2). \quad (3.28)$$

At large and constant ρ this is

$$ds^2 = \frac{\rho^2}{R^2} [-d\eta^2 + R^2 (du^2 + \sinh^2 u d\Omega_{d-2}^2)], \quad (3.29)$$

so that the metric (3.28) indeed asymptotes to the hyperbolic cylinder metric (2.42) on the boundary. This is the metric of AdS-Rindler space: pure AdS space written

in the coordinates of a uniformly accelerating observer and it is also a special case of the general hyperbolic black hole metric (see Appendix A). The black hole (Rindler) horizon is located at $\rho = \ell$ and has the topology of \mathbb{H}_{d-1} , which coincides with the RT surface \tilde{B} in Poincaré coordinates. In other words, the RT surface exactly matches with the black hole horizon \mathbb{H}_{d-1} , because the entropy is completely of thermal nature. In the AdS-Rindler coordinates, the ball vacuum entanglement entropy is thus given by (we emphasize the vacuum by adding a superscript)

$$S_B^{(0)} = \frac{\text{Area}[\mathbb{H}_{d-1}]}{4G_N}, \quad (3.30)$$

where area again refers to the volume of \mathbb{H}_{d-1} . We have established that the vacuum entanglement entropy of a ball is given as the Bekenstein-Hawking entropy of hyperbolic AdS black hole.

3.3 Higher curvature theories and ball universality

In the strong coupling limit of the large- N CFT, the string theory decouples to classical general relativity, but at weaker CFT coupling, the Einstein-Hilbert action is corrected by terms of higher order in curvature. These higher curvature theories of gravity in the AdS_{d+1} bulk have an action of the form

$$I_{\text{HC}} = \frac{1}{16\pi G_N} \int d^{d+1}x \sqrt{-g} \left[\frac{d(d-1)}{\ell^2} + R + f(\text{Riem}) \right], \quad (3.31)$$

where $f(\text{Riem})$ contain the higher curvature terms as contractions and covariant derivatives of the Riemann tensor.

On the CFT side at weaker coupling, one has corrections to the strongly coupled ball vacuum entanglement entropy. For any CFT in the vacuum state, it is calculated via the RT formula as the area of the horizon of a black hole (3.30). Now, because the corrections in the bulk come in the form of higher curvature terms, what is the classical geometric quantity that replaces area in the RT formula to produce the correct ball entanglement entropy in arbitrary CFT states? The answer is given by Wald entropy. In the paper [45], Wald proposed that the black hole entropy can be identified as a horizon integral of the Noether charge associated with the horizon generator. The integral is known as the Wald entropy functional and it can be extended to arbitrary theories of gravity with a diffeomorphism invariant Lagrangian (see section 4.1 for a review). Therefore the Wald functional provides a good candidate to generalize area functional to higher curvature theories, because in Einstein gravity, it reduces to the area functional. It turns out to be the correct generalization to calculate entanglement entropies of boundary ball-shaped regions, which is enough for our purposes.¹⁵

In Einstein gravity, the Wald functional is the area functional up to some parameter a_d^* , which is proportional to the single dimensionless parameter ℓ^{d-1}/G_N of the Einstein-Hilbert action. To match with (3.30), the parameter must be [28]

$$a_d^* = \frac{\pi^{d/2}}{\Gamma(d/2)} \frac{\ell^{d-1}}{8\pi G_N}. \quad (3.32)$$

¹⁵Given a CFT dual to a higher curvature theory of gravity in the bulk, the entanglement entropy of an arbitrary region (not just a ball) is calculated by a functional, which consists of the Wald functional plus additional terms proportional to the extrinsic curvature of the minimal surface [46]. The surface is obtained by minimizing this generalized functional as in the Ryu-Takayanagi prescription. For a boundary ball-shaped region, the hemisphere is still a minimal surface, because its extrinsic curvatures vanish and it minimizes the Wald functional. Thus for a ball-shaped region, the extension to the Wald functional is enough to correctly calculate the entanglement entropies.

This is known as the parameter that normalizes the universal part of the ball entanglement entropy, which is the vacuum entanglement entropy. As showed by (2.29) in 2-dimensions and (3.30) in general, the vacuum entropy is universal, which means that it has the same form in all CFTs. Of course not all CFTs are dual to Einstein gravity in the bulk, but ball entanglement entropies are not able to distinguish whether it is Einstein gravity or not and more complicated boundary regions would be required.

What about ball entanglement entropies calculated near the CFT vacuum state? More accurately, what kind of dual gravity theory is needed to reproduce CFT entanglement entropies via the generalized RT formula for perturbations of the vacuum state? It turns out that the CFT entanglement entropy is universal for second order perturbations of the CFT vacuum state as well, but the entropies cannot be calculated by Einstein gravity, but higher curvature terms (3.31) are needed. The universality up to second order can be seen as follows [11, 48].

Consider the action I_E of an arbitrary CFT and a perturbation to its vacuum state sourced by the linearized bulk metric:

$$I_E = I_E^{(0)} + \lambda \int d^d x T(x), \quad (3.33)$$

where $T = \delta g^{(1)\mu\nu} T_{\mu\nu}$ is the CFT stress-energy tensor contracted by the boundary value of the linearized bulk metric and $I_E^{(0)}$ is the action corresponding to the vacuum state. Based on the AdS/CFT correspondence, the bulk metric can be expressed in terms of the bulk-to-boundary propagator K and the boundary stress-energy one-point function [48]:

$$\delta g_{\mu\nu}^{(1)}(z, x) = \int_{D[B]} d^{d-1} x' K(z, x; x') \langle T_{\mu\nu}(x') \rangle. \quad (3.34)$$

Now expand the ball entanglement entropy up to second order in λ :

$$S_B = a_d^* S_B^{(0)} + \lambda \delta S_B^{(1)} + \lambda^2 \delta S_B^{(2)}, \quad (3.35)$$

where we have explicitly written the vacuum normalization a_d^* . The first order term is linear and the second order term is quadratic in the linearized bulk metric $\delta g^{(1)}$. Using the formula (3.34), we can thus write it schematically as [11]

$$S_B = a_d^* S_B^{(0)} + \lambda \int K_B^{(1)}(x) \langle T(x) \rangle + \frac{\lambda^2}{C_T} \int \int K_B^{(2)}(x_1, x_2) \langle T(x_1) \rangle \langle T(x_2) \rangle, \quad (3.36)$$

where C_T is the constant that normalizes the CFT two-point function $\langle T_{\mu\nu} T_{\rho\sigma} \rangle$. Therefore up to second order in the deformation parameter λ , the ball entanglement entropy depends only on two parameters: a_d^* and C_T . In other words, these parameters completely normalize the universal part of the ball entanglement entropy up to second order.

This result has implications for the dual gravity theory, which must correctly produce the ball entanglement entropies via the generalized RT formula with the Wald functional. In particular, Einstein gravity cannot reproduce the entropy for arbitrary values of a_d^* and C_T , because the Einstein-Hilbert action only contains a single dimensionless parameter, which is already fixed by the vacuum normalization (3.32). Therefore Einstein gravity is the correct dual of the CFT only if C_T is proportional to a_d^* as well. Explicitly the condition can be written as

$$a_d^* = \frac{\pi^d (d-1)}{\Gamma(d+2)} C_T \equiv \tilde{C}_T. \quad (3.37)$$

For CFTs with $\tilde{C}_T \neq a_d^*$, the dual gravity theory must contain an additional dimensionless parameter to account for \tilde{C}_T . A plethora of dimensionless parameters are available in higher curvature theories of gravity, where at each order one new dimensionful parameter is introduced. The simplest example to calculate ball entropies up to second order is the quadratic Gauss-Bonnet gravity with one additional parameter. At higher order perturbations of the vacuum state, one must keep the additional higher curvature terms in the gravity action to reproduce the ball entanglement entropies at second order via the generalized RT formula.

4 Einstein's equations from entanglement entropy

The RT formula relates boundary entanglement entropies to areas of minimal surfaces in classical geometry. The minimal surfaces probe different regions of the bulk spacetime depending on the boundary region in question. The shape of the classical geometry is thus encoded in the entanglement structure of the dual CFT and it should be possible to reconstruct the bulk geometry purely from the entanglement information. But we know from AdS/CFT correspondence that at strong coupling, the bulk geometry obeys Einstein's equations, or more generally at weaker coupling, the equations of motion of a higher curvature gravity theory. Therefore under perturbations of the CFT state, the entanglement structure should change in a special way to induce geometry perturbations that obey the correct equations of motion. This is a highly non-trivial statement and we should not expect it to hold for arbitrary CFTs. However, the entanglement entropy of a ball-shaped region is universal up to second order perturbations of the CFT vacuum state and it is controlled by the parameters a_d^* and \tilde{C}_T as shown in section 3.3. If the correct equations of motion are induced by the entanglement structure, deducing them from ball entropies should thus be possible up to second order at least, because the entropy is the same in all CFTs.

The strategy is to consider small perturbations of the CFT vacuum state and the corresponding change in the ball entanglement entropy, which at linear order obeys the entanglement first law. The first law can be translated to bulk language giving a constraint on the metric perturbation. Linear perturbations were first analysed in 2014 by explicitly calculating the change in the hemisphere area, from which linearized Einstein's equations around the AdS background were obtained [9]. In a follow-up paper [10], the result was extended to higher curvature theories of gravity by the use of the generalized RT formula with the Wald functional. The derivation is based on Iyer-Wald formalism [45, 49] and it allows one to consider all the theories of gravity simultaneously, while making the key aspects of the derivation clear. These derivations only give the linearized equations of motion without matter, because the RT formula only contains the gravitational $N \rightarrow \infty$ contribution. The correct semi-classical matter coupling at the linearized level was quickly obtained by using the FLM formula [12], where the $\mathcal{O}(1)$ -correction is included as bulk entanglement entropy.

At higher orders, one can no longer utilize the first law, but instead the whole non-linear expression for relative entropy is required. First order calculation is also simpler, because the first law does not depend on the explicit form of the perturbation and applies to arbitrary linear perturbations. At second order, one must explicitly construct a state perturbation that also corresponds to a classical bulk dual. Regardless of these additional difficulties, Einstein's equations can be shown to be obeyed by second order perturbations using the RT formula [11]. Their derivation only applies to perturbations with $\tilde{C}_T = a_d^*$, which in section 3.3 was argued to be the necessary

condition for Einstein gravity to be the correct bulk dual. The derivation was quickly extended to higher curvature theories of gravity by using the generalized RT formula leading to the correct equations of motion for arbitrary second order perturbations, including the $\tilde{C}_T \neq a_d^*$ case as well [50]. In the upcoming sections, we focus on the derivation of Einstein gravity, but provide the necessary tools for the generalization to higher curvature theories as well.

4.1 Iyer-Wald formalism and the black hole first law

In this section, we review how the black hole entropy can be defined as a Noether charge associated to the horizon generator [45, 49]. The Noether charge is given by Wald entropy and it can be defined for an arbitrary diffeomorphism invariant theory of gravity. On-shell variations of the Wald entropy can be written in terms of a variation of an asymptotic energy quantity: the black hole first law. It is analogous to the first law of thermodynamics and it will be of central importance in the derivation of linearized Einstein's equations in the next section.

Suppose we are working on a $(d+1)$ -dimensional background spacetime and let

$$\mathbf{L} = \mathcal{L}\epsilon \quad (4.1)$$

be the Lagrangian $(d+1)$ -form of a diffeomorphism invariant theory of gravity, such as a higher curvature theory, which includes matter fields. The form ϵ is the volume form¹⁶

$$\epsilon = \frac{1}{(d+1)!} \epsilon_{a_1 \dots a_{d+1}} dx^{a_1} \wedge \dots \wedge dx^{a_{d+1}}. \quad (4.3)$$

Under a general variation of the metric, the Lagrangian varies as (the contractions with the metric are left implicit)¹⁷

$$\delta \mathbf{L} = \mathbf{E}^g \delta g + d\Theta(g, \delta g), \quad (4.4)$$

where $\mathbf{E}^g = E_{ab}^g \epsilon^{ab} = 0$ are the gravitational equations of motion with matter and Θ is the boundary term, which is a function of the components of the metric. In particular, under a diffeomorphism generated by a vector field ξ^a , the variation is the Lie derivative $\delta_\xi \mathbf{L}$, which can be written as¹⁸

$$\delta_\xi \mathbf{L} = d(\xi \cdot \mathbf{L}). \quad (4.5)$$

Here $\xi \cdot \mathbf{L}$ denotes the contraction of ξ^a with the first index of \mathbf{L} .

A diffeomorphism is a local symmetry of the theory, so according to Noether's theorem there is a corresponding conserved current $J^a[\xi]$, whose Hodge dual is written as:

$$\mathbf{J}[\xi] = \Theta(g, \delta_\xi g) - \xi \cdot \mathbf{L}. \quad (4.6)$$

By the virtue of (4.4) and (4.5), it is indeed conserved on-shell:

$$d\mathbf{J}[\xi] = -\mathbf{E}^g \delta_\xi g = 0. \quad (4.7)$$

¹⁶We also use:

$$\epsilon_a = \frac{1}{d!} \epsilon_{aa_2 \dots a_d} dx^{a_2} \wedge \dots \wedge dx^{a_d}, \quad \epsilon_{ab} = \frac{1}{(d-1)!} \epsilon_{aba_3 \dots a_{d-1}} dx^{a_3} \wedge \dots \wedge dx^{a_{d-1}}. \quad (4.2)$$

¹⁷One could also vary respect to the matter fields and that adds an additional term proportional to the corresponding equations of motion.

¹⁸This follows from the identity $\delta_\xi \mathbf{L} = \xi \cdot d\mathbf{L} + d(\xi \cdot \mathbf{L})$ by using the fact that \mathbf{L} is a top form $d\mathbf{L} = 0$.

Therefore we can find a charge $(d-1)$ -form \mathbf{Q} such that

$$\mathbf{J}[\xi] = d\mathbf{Q}[\xi], \quad \text{on-shell.} \quad (4.8)$$

Off-shell we must include an additional term

$$\mathbf{J}[\xi] = d\mathbf{Q}[\xi] + \xi^a \mathbf{C}_a \quad (4.9)$$

that vanishes on-shell $\mathbf{C}_a = 2E_{ab}^g \epsilon^b$.

In the phase space of the gravity theory, the flow along ξ is generated by a Hamiltonian H_ξ and its variation along the corresponding phase space flow is governed by Hamilton's equations

$$\delta H_\xi = \Omega(\delta g, \delta_\xi g), \quad (4.10)$$

where δg is a perturbation of the metric and δH_ξ is the change of the Hamiltonian under this perturbation. The symplectic form can be written as an integral over a Cauchy surface Σ :

$$\Omega(\delta g, \delta_\xi g) = \int_\Sigma \omega(\delta g, \delta_\xi g). \quad (4.11)$$

Here ω is the symplectic current form:

$$\omega(\delta g, \delta_\xi g) = \delta \Theta(g, \delta_\xi g) - \delta_\xi \Theta(g, \delta g). \quad (4.12)$$

Clearly the current form ω vanishes if δg is generated by the Killing vector field $\delta g = \delta_\xi g$. In other words, the Hamiltonian generating translations along ξ is constant along its flow lines $\delta_\xi H_\xi = 0$. These are in accordance with the standard definitions of symplectic geometry.

Suppose the initial geometry, the geometry before the diffeomorphism, is a solution of the equations of motion $E_{ab}^g = 0$. Then we can calculate the variation by combining (4.4) and (4.6) as:¹⁹

$$\omega(\delta g, \delta_\xi g) = \delta \mathbf{J}[\xi] - d(\xi \cdot \Theta(g, \delta g)). \quad (4.13)$$

By writing the current in terms of the Noether charge (4.9) and using Stoke's theorem, we get

$$\int_\Sigma \omega(\delta g, \delta_\xi g) = \int_{\partial\Sigma} (\delta \mathbf{Q}[\xi] - \xi \cdot \Theta(g, \delta g)) + \int_\Sigma \xi^a \delta \mathbf{C}_a. \quad (4.14)$$

This is usually written in terms of a form

$$\chi = \delta \mathbf{Q}[\xi] - \xi \cdot \Theta(g, \delta g) \quad (4.15)$$

as

$$\int_\Sigma \omega(\delta g, \delta_\xi g) = \int_{\partial\Sigma} \chi + \int_\Sigma \xi^a \delta \mathbf{C}_a. \quad (4.16)$$

Consider now a black hole spacetime, where the horizon \tilde{A} of the black hole is a bifurcation Killing horizon generated by some vector field ξ_A .²⁰ The region outside of the black hole is denoted by Σ_A , whose boundary $\partial\Sigma_A$ is the union of the black hole horizon \tilde{A} and the asymptotic infinity \mathcal{A} . Wald proposed [45] that the entropy

¹⁹ $\delta \mathbf{J} = \delta \Theta - \xi \cdot \delta \mathbf{L} = \delta \Theta - \xi \cdot (\mathbf{E}^g \delta g + d\Theta) = \delta \Theta - (\delta_\xi \Theta - d(\xi \cdot \Theta)) = \omega + d(\xi \cdot \Theta)$.

²⁰ A Killing horizon is a null surface, on which the norm of a Killing vector vanishes. A bifurcation Killing horizon is the intersection of the future and past Killing horizons, where all the components of the Killing vector vanish.

of the black hole can be identified as the horizon integral of the conserved charge $Q[\xi_{\mathcal{A}}]$:

$$S_{\mathcal{A}}^{\text{Wald}} = \frac{2\pi}{\kappa} \int_{\bar{\mathcal{A}}} Q[\xi_{\mathcal{A}}], \quad (4.17)$$

where κ is the surface gravity of the horizon and it is usually normalized to 2π . Using the fact that the horizon is a bifurcation surface it was shown in [49] that the Wald entropy can also be written as ($\kappa = 2\pi$)

$$S_{\mathcal{A}}^{\text{Wald}} = -2\pi \int_{\bar{\mathcal{A}}} \sqrt{h} P^{abcd} n_{ab} n_{cd}, \quad P^{abcd} = \frac{\partial \mathcal{L}}{\partial R_{abcd}}, \quad (4.18)$$

where h is the induced metric on the horizon and n_{ab} is the binormal of the horizon normalized such that $n^{ab}n_{ab} = -2$. This is the Wald entropy functional mentioned in section 3.3. The equivalence of (4.17) and the functional (4.18) also holds for small perturbations of the geometry. In Einstein gravity (3.10)

$$P^{abcd} = \frac{1}{16\pi G_{\text{N}}} g^{ac} g^{bd} \quad (4.19)$$

so that

$$S_{\text{Einstein}}^{\text{Wald}} = -\frac{1}{8G_{\text{N}}} \int \sqrt{h} n^{ab} n_{ab} = \frac{1}{4G_{\text{N}}} \int \sqrt{h} \quad (4.20)$$

is the Bekenstein-Hawking entropy proportional to the area functional.

In the paper [49], Iyer and Wald also analysed the dynamics of the black hole and constructed the black hole first law, which relates the variation of the Wald entropy to the variation of the canonical energy at asymptotic infinity. Formally it states that ($\kappa = 2\pi$)

$$\delta S_{\mathcal{A}}^{\text{Wald}} = \delta E[\xi_{\mathcal{A}}] \quad (4.21)$$

for first order *on-shell* perturbations of the background spacetime. The quantity $E[\xi_{\mathcal{A}}]$ is the canonical energy of the black hole and it is defined as the conserved charge corresponding to diffeomorphisms along $\xi_{\mathcal{A}}$ at the asymptotic infinity \mathcal{A} (not to be confused with the symbol E_{ab}^g denoting the equations of motion). The variation of the canonical energy can be defined via the on-shell Hamiltonian (4.16) as:

$$\delta E[\xi_{\mathcal{A}}] = \int_{\mathcal{A}} \chi. \quad (4.22)$$

If there exists a form \mathbf{B} such that $\Theta = \delta \mathbf{B}$, then we can use (4.15) to write

$$E[\xi_{\mathcal{A}}] = \int_{\mathcal{A}} Q[\xi_{\mathcal{A}}] - \xi_{\mathcal{A}} \cdot \mathbf{B} \quad (4.23)$$

For a static asymptotically flat black hole in general relativity, $\xi_{\mathcal{A}}$ is the time evolution generator ∂_t at asymptotic infinity and $E[\xi_{\mathcal{A}}]$ can be shown to be equal to the ADM mass of the black hole [49].²¹

To prove the black hole first law (4.21), note that the variation of the Wald entropy (4.17) can also be written in terms of χ as

$$\delta S_{\mathcal{A}}^{\text{Wald}} = \int_{\bar{\mathcal{A}}} \chi, \quad (4.24)$$

²¹See [49] also for explicit formulas for \mathbf{L} , Θ , \mathbf{Q} , \mathbf{J} and $E[\partial_t]$ in general relativity.

because $\xi_{\mathcal{A}}$ vanishes on the bifurcation horizon \tilde{A} . For on-shell perturbations, Hamilton's equations (4.16) can thus be written as

$$\int_{\Sigma_{\mathcal{A}}} \omega(\delta g, \delta_{\xi_{\mathcal{A}}} g) = \int_{\partial \Sigma_{\mathcal{A}}} \chi = \int_{\mathcal{A}} \chi - \int_{\tilde{A}} \chi = \delta E[\xi_{\mathcal{A}}] - \delta S_{\mathcal{A}}^{\text{Wald}}. \quad (4.25)$$

Because $\xi_{\mathcal{A}}$ is a Killing vector and $\delta_{\xi_{\mathcal{A}}} g = 0$, then $\omega(\delta g, \delta_{\xi_{\mathcal{A}}} g) = 0$ as well [49], which gives us the black hole first law (4.21).

For off-shell perturbations we need to keep the additional term in (4.16), leaving us with

$$\delta S_{\mathcal{A}}^{\text{Wald}} - \delta E[\xi_{\mathcal{A}}] = - \int_{\Sigma_{\mathcal{A}}} \xi_{\mathcal{A}}^a \delta C_a = -2 \int_{\Sigma_{\mathcal{A}}} \xi_{\mathcal{A}}^a \delta E_{ab}^g \epsilon^b. \quad (4.26)$$

In Einstein gravity this takes the form

$$\frac{\delta \text{Area}[\tilde{A}]}{4G_{\text{N}}} - \delta E[\xi_{\mathcal{A}}] = -2 \int_{\Sigma_{\mathcal{A}}} \xi_{\mathcal{A}}^a \delta E_{ab}^g \epsilon^b, \quad (4.27)$$

which will be used in the next section.

4.2 Einstein's equations at first order

Consider a ball-shaped region $B(R, 0)$ on the flat boundary of AdS_{d+1} written in Poincaré coordinates. Let $|\Phi(\varepsilon)\rangle$ be a one parameter family of CFT states such that the vacuum state is located at $\varepsilon = 0$. We assume that in the vicinity of the vacuum, the states correspond to different classical bulk geometries $\mathcal{M}(\varepsilon)$ without matter fields, because we are working in the $N \rightarrow \infty$ limit of the CFT. At this order, we do not need to specify the explicit form of the states. The metric of $\mathcal{M}(\varepsilon)$ is expanded as

$$g(\varepsilon) = g^{(0)} + \varepsilon \delta g^{(1)} + \varepsilon^2 \delta g^{(2)} + \dots, \quad (4.28)$$

where $g^{(0)}$ is the metric of pure AdS, which is dual to the CFT vacuum state. The ball density matrix is expanded similarly:

$$\rho_B(\varepsilon) = \rho_B^{(0)} + \varepsilon \delta \rho_B^{(1)} + \varepsilon^2 \delta \rho_B^{(2)} + \dots \quad (4.29)$$

The relative entropy respect to the vacuum state is given by

$$S(\rho_B \| \rho_B^{(0)}) = \Delta(\langle K_B \rangle - S_B). \quad (4.30)$$

Differentiating with respect to ε :

$$\frac{d}{d\varepsilon} (\langle K_B \rangle - S_B) = \frac{d}{d\varepsilon} S(\rho_B \| \rho_B^{(0)}). \quad (4.31)$$

The left hand side can be translated to bulk language

$$\frac{d}{d\varepsilon} (E_B^{\text{grav}} - S_B^{\text{grav}}) = \frac{d}{d\varepsilon} S(\rho_B \| \rho_B^{(0)}), \quad (4.32)$$

where in Einstein gravity, the dual of entanglement entropy is given by the RT formula

$$S_B^{\text{grav}} = \frac{\text{Area}[\tilde{B}(\varepsilon)]}{4G_{\text{N}}}. \quad (4.33)$$

Here $\tilde{B}(\varepsilon)$ is the bulk extremal surface of the ball in the geometry $\mathcal{M}(\varepsilon)$.

The ball modular Hamiltonian can be translated to bulk language using the expression (2.46) involving the stress-energy tensor:

$$\langle K_B \rangle = 2\pi \int_B d^{d-1}x \frac{R^2 - r^2}{2R} \langle T_{tt}(x) \rangle. \quad (4.34)$$

The Poincaré invariance and scale invariance of the CFT imply the conservation law $\partial_\mu \langle T^{\mu\nu} \rangle = 0$ and tracelessness $\langle T^\alpha_\alpha \rangle = 0$. Using the correspondence (3.17) between the boundary stress-energy tensor and the z^d -coefficient of the Fefferman-Graham expansion (3.12), these conditions translate to

$$\partial_\mu h^{(d)\mu\nu} = 0, \quad h^{(d)\alpha}_\alpha = 0. \quad (4.35)$$

In particular, the second constraint implies $h^{(d)}_{tt} = h^{(d)}_{ii}$ with summation over the repeated index. Now expressing the stress-energy expectation value using the bulk metric (3.17), we get the gravitational analog in Einstein gravity:

$$E_B^{\text{grav}} = \frac{d \ell^{d-1}}{16G_N R} \int_B d^{d-1}x (R^2 - r^2) h^{(d)}_{ii}(x). \quad (4.36)$$

Equation (4.32) is now a constraint on the bulk geometry (4.28).

We can expand (4.32) around the vacuum state $\varepsilon = 0$. At first order, relative entropy vanishes giving the bulk dual of the CFT entanglement first law (2.16)

$$\delta^{(1)} S_B^{\text{grav}} - \delta^{(1)} E_B^{\text{grav}} = 0, \quad (4.37)$$

where $\delta^{(1)} = d/d\varepsilon|_{\varepsilon=0}$. For first order perturbations of the geometry, the change in the area of $\tilde{B}(\varepsilon)$ only comes from the change in the metric and not from the change in the coordinate location of $\tilde{B}(0)$, because $\tilde{B}(0)$ is an extremum by definition. Thus we get

$$\frac{\delta^{(1)} \text{Area}[\tilde{B}(0)]}{4G_N} - \delta^{(1)} E_B^{\text{grav}} = 0. \quad (4.38)$$

The hemisphere $\tilde{B}(0)$ is a bifurcation horizon of a bulk Killing vector ξ_B , which is the canonical extension of the boundary conformal Killing vector ζ_B (2.45). In Poincaré coordinates:

$$\xi_B = 2\pi \left[\left(\frac{R^2 - z^2 - t^2 - r^2}{2R} \right) \partial_t - \frac{t}{R} (z\partial_z + r\partial_r) \right], \quad (4.39)$$

where r is the boundary radial coordinate.

One can check that its norm vanishes on $\tilde{B}(0)$ and that it asymptotes to ζ_B (2.45) on the boundary. Together the horizon $\tilde{B}(0)$ and the boundary ball B bound the bulk spatial region Σ_B ($\partial\Sigma_B = \tilde{B} \cup B$), which makes up the AdS-Rindler patch presented in figure 8.

Equation (4.38) looks like the black hole first law (4.27) applied to the Killing horizon $\tilde{B}(0)$ and indeed it turns out that the energy quantities match:

$$\delta^{(1)} E_B^{\text{grav}} = \delta^{(1)} E[\xi_B]. \quad (4.40)$$

This can be verified by a direct calculation as follows.

The perturbation of the bulk AdS metric in Poincaré coordinates can be written as

$$ds^2 = \frac{\ell^2}{z^2} \left(dz^2 + \eta_{\mu\nu} dx^\mu dx^\nu + z^d H_{\mu\nu}(z, x) dx^\mu dx^\nu \right). \quad (4.41)$$

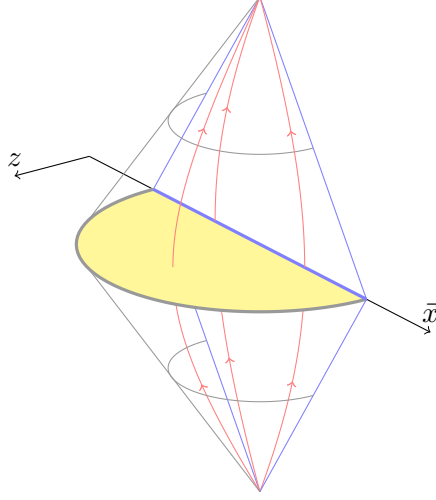


Figure 8: The AdS-Rindler patch (yellow) in Poincaré coordinates. The flow ξ_B is drawn in red and it asymptotes to the ball modular flow ζ_B on the boundary.

On the boundary, the perturbation $H_{\mu\nu}$ coincides with the Fefferman-Graham coefficient $h_{\mu\nu}^{(d)}$, which induces the boundary stress-energy tensor. Now in the AdS-Rindler slice Σ_B at $t = 0$, the form χ (4.15) can be written as [10]:

$$\chi|_{\Sigma_B} = \frac{z^d}{16\pi G_N} \left\{ \epsilon^t_z \left[\left(\frac{2\pi z}{R} + \frac{d}{z} \xi_B^t + \xi_B^t \partial_z \right) H_{ii} \right] + \epsilon^t_i \left[\left(\frac{2\pi x^i}{R} + \xi_B^t \partial_i \right) H_{jj} - \left(\frac{2\pi x^j}{R} + \xi_B^t \partial_j \right) H_{ij} \right] \right\} \quad (4.42)$$

with summation over i, j . On the corresponding boundary slice, we have explicitly

$$H_{ij}(0, x) = h_{ij}^{(d)}(x), \quad \xi_B^t = \frac{\pi}{R}(R^2 - r^2). \quad (4.43)$$

The volume forms are proportional to $(\ell/z)^{d-1}$ so that all the terms in (4.42) vanish at $z = 0$ except for the second term, which is independent of z . Therefore (4.42) reduces to

$$\chi|_B = \frac{d \ell^{d-1}}{16G_N R} (R^2 - r^2) h_{ii}^{(d)}(x) d^{d-1}x \quad (4.44)$$

that matches with (4.36).

We are now finally able to write the bulk first law (4.37) using the black hole first law (4.27) extended to off-shell perturbations:

$$\frac{\delta^{(1)} \text{Area}[\tilde{B}(0)]}{4G_N} - \delta^{(1)} E_B^{\text{grav}} = -2 \int_{\Sigma_B} d^d x \xi_B^t \delta E_{tt}^{(1)}, \quad (4.45)$$

where we have expanded Einstein's equations in ε :

$$\delta E_{ab}^g = \varepsilon \delta E_{ab}^{(1)} + \varepsilon^2 \delta E_{ab}^{(2)} + \dots \quad (4.46)$$

Now it is obvious that if Einstein's equations hold at first order, namely $\delta E_{tt}^{(1)} = 0$, the bulk entanglement first law holds as well. We can also reverse the implication, because the bulk first law (4.37) implies

$$-2 \int_{\Sigma_B} d^d x \xi_B^t \delta E_{tt}^{(1)} = 0 \quad (4.47)$$

for balls $B(R, \vec{x}_0)$ of arbitrary radius and position (generalizes the radial coordinate as $r \rightarrow |\vec{x} - \vec{x}_0|$). This is enough to make the integrand vanish [10], which gives the time-time component of linearized Einstein's equations at each bulk point:

$$\delta E_{tt}^{(1)} = 0. \quad (4.48)$$

If on the boundary we boost to an arbitrary Lorentz frame with a d -velocity u^μ , we get $u^\mu u^\nu \delta E_{\mu\nu}^{(1)} = 0$. This holds for arbitrary vectors u^μ so we get Einstein's equations along the boundary directions

$$\delta E_{\mu\nu}^{(1)} = 0. \quad (4.49)$$

The remaining components $\delta E_{z\mu}^{(1)} = 0$ and $\delta E_{zz}^{(1)} = 0$ arise as constraint equations from the identity

$$d\chi|_{\partial\mathcal{M}} = 0, \quad (4.50)$$

which follows from the conservation and tracelessness of the CFT stress-energy tensor [10]. All in all, we have established *full equivalence* between the entanglement first law and linearized Einstein's equations in the bulk.

Now that we have derived vacuum linearized Einstein's equations in the $N \rightarrow \infty$ limit of the CFT using the RT formula, it is natural to consider what kind of effects the $1/N$ -corrections have on the result. In section 3.1, we showed that the $\mathcal{O}(1)$ -correction of the CFT requires one to include semi-classical effects in the bulk. Thus one expects that the correction introduces a semi-classical matter coupling term, proportional to the bulk stress-energy $\langle T_{ab}^{\text{bulk}} \rangle$, in the linearized Einstein's equations. This is exactly what happens [12].

The bulk quantum effects cause corrections to both sides in the bulk first law (4.37). The leading correction to the variation $\delta^{(1)} S_B^{\text{grav}}$ is captured by the FLM proposal (3.22) as the variation of the bulk entanglement entropy of the surface Σ_B :

$$\delta^{(1)} S_B^{\text{grav}} = \frac{\delta^{(1)} \text{Area}[\tilde{B}(0)]}{4G_N} + \delta^{(1)} S_{\Sigma_B}^{\text{bulk}}. \quad (4.51)$$

The counter-terms that render the bulk entanglement entropy finite (3.23) are independent of the geometry and are thus cancelled in the above variation (they only depend on the regulator of the entanglement entropy).

According to AdS/CFT, the CFT state includes information of both the bulk geometry and the state of the matter fields. Therefore under the perturbation of the CFT state, the variation of $S_{\Sigma_B}^{\text{bulk}}$ is due to the following effects:

- (i) Variation in the bulk metric and the deformation of the initial extremal surface $\tilde{B}(0)$.
- (ii) Variation in the state of the bulk quantum fields.²²

The geometric effects (i) vanish: the variation of the bulk metric, while keeping $\tilde{B}(0)$ fixed, has no effect on the bulk entanglement, because it does not affect the state of the fields directly. In addition at first order, the deformation of $\tilde{B}(0)$ does not have an effect either, because Σ_B is an extremum of the bulk entropy functional $S_{\Sigma_B}^{\text{bulk}}$. The reason for this is that as a function of the bulk UV cutoff, the leading term of the bulk entanglement entropy is proportional to the area of $\tilde{B}(0)$, which is an

²²The quantum field variation does not contain graviton contribution as it would lead to double counting.

extremum. Another way to see this is by symmetry: the hemisphere can be mapped to a surface that divides the bulk in half (see figure 7).

The only source of variation in the bulk entanglement is due to the variation of the state (ii), which follows the entanglement first law. The modular flow of Σ_B is given by ξ_B (4.39) so the modular Hamiltonian has the form:

$$K_{\Sigma_B}^{\text{bulk}} = \int_{\Sigma_B} d^d x \xi_B^t T_{tt}^{\text{bulk}}. \quad (4.52)$$

Therefore according to entanglement first law:

$$\delta^{(1)} S_{\Sigma_B}^{\text{bulk}} = \int_{\Sigma_B} d^d x \xi_B^t \langle \delta T_{tt}^{(1)} \rangle, \quad (4.53)$$

where we have expanded

$$T_{ab}^{\text{bulk}} = \varepsilon \delta T_{ab}^{(1)} + \dots \quad (4.54)$$

Next we need the quantum correction to $\delta^{(1)} E_B^{\text{grav}}$ in the first law (4.37). According to (4.34), this amounts to finding corrections to the CFT 1-point function $\langle T_{\mu\nu} \rangle$. In section 3.1, we derived the result (3.19) that the leading correction is proportional to the bulk stress-energy tensor $\langle T_{ab}^{\text{bulk}} \rangle$. Based on locality, it is natural that it vanishes at infinity: $\langle T_{ab}^{\text{bulk}} \rangle|_{B=0} = 0$. Therefore the semi-classical effects add no corrections to $\delta^{(1)} E_B^{\text{grav}}$.

We have shown that the only correction to the boundary entanglement first law is given by bulk entanglement first law (4.53). Therefore solving S_B^{Wald} from (4.51) and recalling the off-shell black hole first law (4.27), we get

$$\left(\delta^{(1)} S_B^{\text{grav}} - \int_{\Sigma_B} d^d x \xi_B^t \langle \delta T_{tt}^{(1)} \rangle \right) - \delta^{(1)} E_B^{\text{grav}} = -2 \int_{\Sigma_B} d^d x \xi_B^t \delta E_{tt}^{(1)}. \quad (4.55)$$

Using the bulk first law (4.37), we get

$$\int_{\Sigma_B} d^d x \xi_B^t \left(\delta E_{tt}^{(1)} - \frac{1}{2} \langle \delta T_{tt}^{(1)} \rangle \right) = 0. \quad (4.56)$$

Again, because this holds for arbitrary balls, we have

$$\delta E_{tt}^{(1)} - \frac{1}{2} \langle \delta T_{tt}^{(1)} \rangle = 0, \quad (4.57)$$

which is exactly the time-time component of the linearized Einstein's equations with a semi-classical matter coupling (the factor of 1/2 is necessary). The rest of the components can be derived as before.

The result has deep implications. It shows that the universality of gravity, the fact that gravity couples to all kinds of matter equally, is a result of the universality of quantum entanglement. Since everything in nature is encoded in a Hilbert space, entanglement does not depend on the type of matter and is therefore universal. At least these facts are related in the context of AdS/CFT, but whether they are connected generally is not known. There have been recent attempts to generalize these ideas for example in [13], which we discuss in section 5.1, and in [51].

The definition of entanglement entropy for gauge fields is problematic, because the Hilbert space cannot be expressed as a tensor product of the spaces of the two regions. This might produce problems for the above derivation, when there are gauge fields present in the bulk, because then one is unable to define bulk entanglement

entropy.²³ However in [52] it was shown that the gauge effects cancel for relative entropy. Therefore the variation $\delta S_{\Sigma_B}^{\text{bulk}}$ is perfectly defined even for gauge fields and the derivation goes through. This is expected, because there are no problems in doing standard semi-classical gravity in the presence of gauge fields.

The power of using the Iyer-Wald formalism to derive the linearized Einstein's equations above, is that the derivation also extends to higher curvature theories of gravity. In higher curvature theories, the variation of the ball entanglement entropies are calculated by the generalized RT formula

$$\delta^{(1)} S_B^{\text{grav}} = \delta^{(1)} S_B^{\text{Wald}} + \delta^{(1)} S_{\Sigma_B}^{\text{bulk}} \quad (4.58)$$

with the Wald functional replacing the area functional of Einstein gravity. One can also show [10] that the equivalence $\delta^{(1)} E_B^{\text{grav}} = \delta^{(1)} E[\xi_B]$ extends to these theories as well. Therefore the formula (4.26) combined with the bulk first law (4.37) implies linearized higher curvature equations of motion with semi-classical matter coupling as in Einstein gravity above.

4.3 Einstein's equations at second order

We will now extend the above analysis to second order perturbations of the bulk metric as presented in [11]. The strategy is to expand relative entropy (4.32) up to second order in ε and then apply Iyer-Wald formalism with Einstein's equations E_{ab}^g expanded similarly. At first order the form of the state perturbations does not matter, because the entanglement first law is true for arbitrary small perturbations. However at second order, we must explicitly specify a one-parameter family of states that reduce to the vacuum state at $\varepsilon = 0$. The states considered are chosen to have the form:

$$\langle \Phi_+ | \Phi_\lambda(\varepsilon) \rangle = \int_{\Phi(\Sigma,0)=\Phi_+} \mathcal{D}\Phi \exp \left[- \int_{-\infty}^0 d\tau d^{d-1} \vec{x} (\mathcal{L}_{\text{CFT}} + \lambda_\alpha(x; \varepsilon) \mathcal{O}_\alpha(x)) \right]. \quad (4.59)$$

Recall the notation τ for the Euclidean time coordinate and Σ for the $\tau = 0$ Cauchy slice. The operators $\mathcal{O}_\alpha(x)$ are CFT primary operators (including stress-energy tensor) and all the ε dependence is in the deformation parameter $\lambda_\alpha(x; \varepsilon) = \varepsilon \lambda_\alpha(x) + O(\varepsilon^2)$ (summation over α is implied). At $\varepsilon = 0$ the lambdas vanish so that the path integral reduces to the vacuum state.²⁴ Therefore the above class of states are perturbations of the CFT vacuum state. The states are dual to quantum matter fields and coherent graviton fields in the bulk, leading to classical bulk geometry with semi-classical coupling to matter.

We want to apply Iyer-Wald formalism to connect CFT relative entropy to Einstein's equations in the bulk. There is an apparent problem however, because the coordinate location of the bulk minimal surface $\tilde{B}(0)$ is shifted at second order, meaning it no longer is a bifurcation horizon of the Killing vector ξ_B . This problem can be avoided by choosing a specific gauge for the perturbation $\delta g^{(1)}$ called the Hollands-Wald gauge. The gauge is defined such that the coordinate location of $\tilde{B}(0)$ is unchanged and that the vector field ξ_B in the original AdS coordinates stays as a

²³Gauge fields could arise when the CFT has a global symmetry, which gets lifted to a local symmetry in the bulk. It is generally argued that a theory of quantum gravity does not possess global symmetries.

²⁴A thermal state is a result of integration over a cylinder with radius given by the inverse temperature β (see section 2.2). Vacuum state is then a result of unwinding the cylinder ($\beta \rightarrow \infty$) leading to the half-space integral. The lambdas are also assumed to vanish sufficiently rapidly as $\tau \rightarrow 0$ to produce finite energy states [11].

Killing vector field along $\tilde{B}(0)$ under the perturbation [11]. The fact that such a gauge exists is non-trivial and details can be found in [53, 54].

Now that we have fixed the gauge, the extremal surface $\tilde{B}(0)$ is still a bifurcation horizon at second order. As we showed in the last section, we then have $\delta E_B^{\text{grav}} = \delta E[\xi_B]$, which allows us to write (4.32) as

$$\frac{d}{d\varepsilon} \left(E[\xi_B] - \frac{\text{Area}[\tilde{B}(0)]}{4G_N} \right) = \frac{d}{d\varepsilon} S(\rho_B \| \rho_B^{(0)}). \quad (4.60)$$

In the phase space formulation, the Hamilton's equations were given by (4.14) that can be written in our current notation in Einstein gravity as:

$$\int_{\Sigma_B} \omega \left(g, \frac{dg}{d\varepsilon}, \delta_{\xi_B} g \right) = \frac{d}{d\varepsilon} \left(E[\xi_B] - \frac{\text{Area}[\tilde{B}(0)]}{4G_N} \right) + \int_{\Sigma_B} \xi_B^a \delta \mathcal{C}_a. \quad (4.61)$$

Combining with (4.60) we get

$$\frac{d}{d\varepsilon} S(\rho_B \| \rho_B^{(0)}) = \int_{\Sigma_B} \omega \left(g, \frac{dg}{d\varepsilon}, \delta_{\xi_B} g \right) - \int_{\Sigma_B} 2\xi_B^a \delta E_{ab}^g \epsilon^b. \quad (4.62)$$

Expanding both sides to first order in ε , we find that $\delta g^{(1)}$ obeys linearized Einstein's equations with matter as we showed before (4.57). Expanding up to second order we get

$$\delta^{(2)} S(\rho_B \| \rho_B^{(0)}) = \int_{\Sigma_B} \omega \left(g^{(0)}, \delta g^{(1)}, \delta_{\xi_B} \delta g^{(1)} \right) - \int_{\Sigma_B} 2\xi_B^a \left(\delta E_{ab}^{(2)} - \frac{1}{2} \delta T_{ab}^{(2)} \right) \epsilon^b. \quad (4.63)$$

with the notation

$$\delta^{(2)} S(\rho_B \| \rho_B^{(0)}) \equiv \frac{1}{2} \frac{d^2}{d\varepsilon^2} S(\rho_B \| \rho_B^{(0)})|_{\varepsilon=0}. \quad (4.64)$$

Here the second order contribution to the bulk stress-energy tensor T_{ab}^{bulk} is a result of the matter fields sourcing the primary operators in the states (4.59).

The second order variation of relative entropy can be calculated directly in the CFT. The result of the calculation is [11]:

$$\delta^{(2)} S(\rho_B \| \rho_B^{(0)}) = \frac{\tilde{C}_T}{a_d^*} \int_{\Sigma_B} \omega \left(g^{(0)}, \delta g^{(1)}, \mathcal{L}_{\xi_B} \delta g^{(1)} \right), \quad (4.65)$$

where a_d^* is the normalization of the universal part of vacuum entanglement entropy (3.32) and \tilde{C}_T is the normalization of the stress-energy tensor two point function (3.37). Assuming that $\tilde{C}_T = a_d^*$ and combining with (4.63), we get

$$- \int_{\Sigma_B} 2\xi_B^a \left(\delta E_{ab}^{(2)} - \frac{1}{2} \delta T_{ab}^{(2)} \right) \epsilon^b = 0, \quad (4.66)$$

which applied to all balls as before implies that the metric obeys Einstein's equations at second order:

$$\delta E_{ab}^{(2)} - \frac{1}{2} \delta T_{ab}^{(2)} = 0. \quad (4.67)$$

Rest of the components follow in the same way as before.

We have established the following result: a state perturbation of a holographic CFT with $\tilde{C}_T = a_d^*$ is dual to a metric perturbation in the bulk obeying Einstein's equations at second order. This result is in agreement with the earlier discussion on the universality of the ball shaped region, where Einstein gravity was argued to produce correct ball entanglement entropies if $\tilde{C}_T = a_d^*$. Therefore to extend the above derivation to CFTs with $\tilde{C}_T \neq a_d^*$, one must consider higher curvature theories in the bulk. This was done in [50] to derive the correct equations of motion.

5 Einstein's equations in other spacetimes

We have now seen how classical gravity emerges from the entanglement structure of the CFT in the AdS/CFT correspondence. This begs the question whether gravity also emerges from entanglement in spacetimes other than AdS, namely, in other maximally symmetric spaces. Of course, there is no known dual field theory description for gravity in Minkowski or de Sitter space so to derive Einstein's equations, one needs to make assumptions on the quantum gravitational degrees of freedom in the UV. The ideas from AdS/CFT were put to use, when Jacobson introduced his maximum vacuum entanglement hypothesis (MVEH) that he applied to derive Einstein's equations, with semi-classical matter coupling, in any maximally symmetric space [13]. The hypothesis states that the lowest energy state of quantum gravity is in *entanglement equilibrium*, where the total entanglement entropy of a small geodesic ball is maximized. The equilibrium holds locally for balls centered at arbitrary spacetime points, which leads to Einstein's equations everywhere in the spacetime.

MVEH is motivated by an on-shell classical result in general relativity known as the first law of causal diamond mechanics. It states that the addition of matter energy to a geodesic ball decreases the area of the ball boundary at fixed volume. This first law is analogous to the black hole first law and can be derived from the phase space formulation using the Iyer-Wald formalism of section 4.1. Hence a generalization of entanglement equilibrium to higher curvature theories of gravity is possible [55].

5.1 Einstein's equations from entanglement equilibrium

We start by considering a geodesic ball B of radius R at a point p in d -dimensional flat Minkowski space.²⁵ Suppose we are working in the semi-classical limit of quantum gravity, where matter fields are quantum, but gravity is classical. In this limit, the quantum gravity Hilbert space is assumed to factorize into two components that are associated with UV and IR degrees of freedom of the theory:

$$\mathcal{H} = \mathcal{H}_{\text{UV}} \otimes \mathcal{H}_{\text{IR}}. \quad (5.1)$$

We can now define entanglement entropies S_{UV} and S_{IR} by tracing out the other component. The mutual information between the UV and IR degrees of freedom is taken to be negligible so that the total entanglement entropy is a simple sum

$$S_B = S_{\text{UV}} + S_{\text{IR}}. \quad (5.2)$$

The IR entropy is associated with entanglement of quantum matter fields across the ball boundary, while the UV entropy is due to unknown quantum gravity degrees of freedom and it is taken to satisfy the familiar area law

$$S_{\text{UV}} = \eta A, \quad (5.3)$$

where A is the area of ∂B and the constant η is the area density of vacuum entanglement entropy, which is assumed to be rendered finite by quantum gravitational effects. Jacobson's maximum vacuum entanglement hypothesis is the following statement:

When the geometry and the state of the matter fields are simultaneously varied from maximal symmetry, the total entanglement entropy S_B in a small geodesic ball is maximal at fixed volume.

²⁵A geodesic ball of radius R is defined as the set of points, whose geodesic distance from p is R .

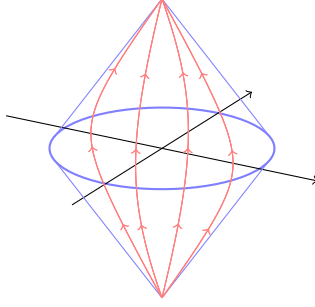


Figure 9: Causal diamond $D[B]$ of a geodesic ball.

The variation of geometry is classical, while the matter field variation is a variation in the quantum state. Maximal symmetry refers to the underlying spacetime being maximally symmetric. Formally MVEH states that under arbitrary off-shell variations of geometry and matter fields, δS_B vanishes:

$$\delta S_{UV} + \delta S_{IR} = 0, \quad (5.4)$$

where the area variation of ∂B is taken at fixed volume

$$\delta S_{UV} = \eta \delta A|_V. \quad (5.5)$$

These ingredients are enough to derive Einstein's equations as follows. The area variation of the geodesic ball at fixed volume is given by [13]

$$\delta A|_V = -\frac{\Omega_{d-2} R^d}{2(d^2 - 1)} \mathcal{R}, \quad (5.6)$$

where $\mathcal{R} = R_{ij}{}^{ij}$ is the spatial Ricci scalar of the perturbed spacetime and Ω_{d-2} is the volume of a $(d-2)$ -dimensional unit sphere. Note that this is a full non-perturbative change in the area. The Ricci scalar can be written in terms of the Einstein tensor as

$$\mathcal{R} = 2G_{00}. \quad (5.7)$$

This holds when the background spacetime is flat, but we want to consider an arbitrary maximally symmetric space (MSS), with a curvature scale Λ , as a background. At linear order in Λ , which is enough for small balls, the area variation respect to a MSS is obtained by replacing $G_{00} \rightarrow G_{00} - G_{00}^{\text{MSS}}$, where $G_{ab}^{\text{MSS}} = -\Lambda g_{ab}$. The result is

$$\delta S_{UV} = -\eta \frac{\Omega_{d-2} R^d}{d^2 - 1} (G_{00} + \Lambda g_{00}). \quad (5.8)$$

The IR entanglement entropy follows the familiar entanglement first law

$$\delta S_{IR} = \delta \langle K_B \rangle, \quad (5.9)$$

To proceed, we need assume that the matter fields are conformal so that the modular Hamiltonian K_B is given by the familiar result (2.46). Suppose that the radius R of the geodesic ball is smaller than the curvature scale of the spacetime, but larger than the excitation length of the quantum fields to have a well defined entropy. Then the stress-energy tensor of the CFT is approximately constant inside the ball:

$$\delta S_{IR} = \frac{\Omega_{d-2} R^d}{d^2 - 1} \delta \langle T_{00}(p) \rangle. \quad (5.10)$$

Substituting the expressions (5.8) and (5.10) to the MVEH (5.4) gives

$$G_{00} + \Lambda g_{00} = \frac{2\pi}{\eta} \langle T_{00} \rangle. \quad (5.11)$$

This is the time-time component of the Einstein's equations at the center of the ball if we identify $\eta = 1/4G_N$. Rest of the components are obtained by reproducing the calculation in all Lorentz frames. Because the derivation can be repeated for small balls at arbitrary points p , the Einstein's equations hold everywhere.

The above derivation is only applicable in the case of conformally symmetric fields, because of the use of the expression (5.10) for S_{IR} . Jacobson proposed [13] that it could be extended to QFTs with a conformal UV fixed point by the addition of a scalar:

$$\delta S_{\text{IR}} = \frac{\Omega_{d-2} R^d}{d^2 - 1} (\delta \langle T_{00} \rangle + \delta \langle X \rangle). \quad (5.12)$$

It was suggested that the scalar could be proportional to the trace of the stress-energy tensor, whose expectation value vanishes in CFTs. Holographic calculations in AdS/CFT have been done to check this proposal by considering perturbations to the conformal action by some scalar operator. The proposal works for some scaling dimensions Δ of the scalar operator, but the operators with $\Delta < d/2$ produce terms that dominate in the small radius limit violating the proposal [56, 57]. The problem could be avoided if, for example, MVEH only applies to linearized perturbations around the vacuum. This is also suggested by the fact that there exists non-perturbative families of CFT states that contain the vacuum state, but have a finite energy density with the same entanglement entropy.²⁶ For such states MVEH would be meaningless, because the change in the entanglement entropy is zero. In addition, MVEH applied to higher curvature theories of gravity only produces the equations of motion at the linearized level [55].

The constraint of taking the variation of S_{UV} at fixed volume is motivated by *the first law of causal diamond mechanics*, which is a classical on-shell statement about the spacetime dynamics within the geodesic ball [13]. Consider a simultaneous on-shell variation of the metric and the classical matter fields. According to Hamilton's equations (4.16) derived in section 4.1, the on-shell variation of the Hamiltonian generating the flow along ζ_B is given by Wald entropy

$$\delta H_{\zeta_B} = \int_{\partial B} \delta Q[\zeta_B] = \delta S_B^{\text{Wald}}. \quad (5.13)$$

In contrast to the black hole first law, the left hand side does not vanish, because ζ_B is not a pure Killing vector of Minkowski space, but only a conformal Killing vector. In Einstein gravity, the Wald entropy is proportional to the area of ∂B , while the Hamiltonian can be calculated from the explicit expression for ω [13]. The gravitational contribution to ω is proportional to the volume variation δV of the ball so that the gravitational part of (5.13) contains $\eta(\delta A - \delta V)$ up to a constant in δV . Turns out that this constant has just the right value so that the expression becomes area variation at fixed volume $\delta A|_V$. On the other hand, the matter contribution to ω is proportional to the matter stress-energy and therefore the first law of causal diamond mechanics states that the addition of matter energy to the diamond decreases the area of the boundary at fixed volume [13]:

$$\frac{\delta A|_V}{4G_N} + \delta E_{\zeta_B} = 0, \quad (5.14)$$

²⁶An example of such a family is given by coherent states [58].

where $E_{\zeta_B} = \int d^d x \zeta_B T_{tt}$ is the conformal boost energy resulting from matter. So we see that a similar expression to MVEH holds on-shell classically and this is the reason why Einstein's equations can be derived by reversing the argument.

Entanglement equilibrium can also be extended to higher curvature theories of gravity [55]. The strategy is to replace the area of the ball by the corresponding Wald functional of the theory, but now one also needs a “generalized volume”, which is taken to be fixed under the variation of the Wald functional. The generalized volume is obtained from the first law of causal diamond mechanics (5.13) applied to higher curvature theories and it is denoted by W . Entanglement equilibrium is assumed as before, but this time $\delta S_{UV} = \delta S_B^{\text{Wald}}|_W$.²⁷ For small balls, the correct equations of motion are obtained, but only at the linearized level, because the higher curvature terms are encoded at larger distance scales. Therefore one would hope that by increasing the radius of the ball, higher order corrections could be obtained. Turns out that this is not possible [55], because the Riemann normal coordinates, that parametrize the geodesics radiating from the center, contain terms second order in curvature at larger radius. These terms are exactly what the equations of motion would contain at the same order, breaking the perturbative expansion. It is suggested that this implies a break down of the effective field theory description.

5.2 Comparison with holography and discussion

There are some similarities between the holographic derivation of Einstein's equations reviewed in section 4 and the derivation of this section based on entanglement equilibrium. The vanishing of the total entanglement entropy of a geodesic ball is analogous to the bulk first law in AdS/CFT (the bulk dual of CFT entanglement first law), which was applied in the bulk region bounded by the RT surface (the AdS-Rindler patch). In both cases, semi-classical matter coupling follows from the entanglement of fields inside the region being considered, but the Einstein's equations itself are obtained differently. Entanglement equilibrium yields full non-linear Einstein's equations, while in holography, the equations are only obtained perturbatively. A geodesic ball is a well defined notion at all orders and that is why we are able to calculate the area variation (5.6) non-perturbatively, leading to the full non-linear equations. In holography, we do not have a non-perturbative constraint on the geometry, but only a perturbative expansion obtained from relative entropy (4.31).

The main difference when comparing Jacobson's derivation to holography is the absence of a dual theory. In Jacobson's derivation, the area law for the UV entropy is simply taken as an assumption, which supposedly captures the macroscopic behaviour of the UV degrees of freedom. In the holographic context, the interpretation of the area law for the hemisphere is perfectly clear and is given by the entanglement entropy of the dual CFT. There is also η that must be rendered finite by quantum gravitational effects. The mechanisms responsible should be similar to how the bulk entanglement entropy is rendered finite in the FLM formula (3.22), because its leading order divergence is also proportional to area. The entanglement equilibrium hypothesis is also taken as an assumption, which is of course well motivated by the first law of causal diamond mechanics, but it cannot be derived from a more fundamental principle. In holography the situation is again the opposite, because the bulk first law is dual to the entangled first law of the CFT.

²⁷There are ambiguities in the definitions that modify the Wald functional and generalized volume by terms S_{JKM} and W_{JKM} [59]. Nevertheless, the variation with the new functionals $\delta(S_B^{\text{Wald}} + S_{\text{JKM}})|_{W'}$ with $W' = W + W_{\text{JKM}}$ yields the correct linearized equations of motion.

In Jacobson’s derivation the shape of the geodesic ball and the conformal symmetry of the matter fields are essential, because together they produce the correct factor in the stress-energy tensor that cancels against the area variation in (5.11). It is the explicit form of the ball modular Hamiltonian of the conformal fields that produce the correct matter coupling in the Einstein’s equations. In the holographic derivation, the fields do not need to be conformal, because the canonical extension ξ_B of the boundary conformal Killing vector to the bulk is a real Killing vector of AdS space. Therefore entanglement first law can be applied regardless of the symmetries of the theory, when calculating the variation of bulk entanglement entropy in (4.53).

For completeness we also mention an older 1995 paper by Jacobson [60], where he derives Einstein’s equations as an equation of state. It is based on local Rindler horizons, for which one defines an entropy satisfying an area law. Matter energy is included as an integral of heat flux through the future horizon and it is assumed to follow a Clausius relation. The variation of the horizon entropy is taken to be equal to the variation of the matter entropy in the Clausius relation, leading to Einstein’s equations as a local equilibrium constraint. See appendix B for details. This “thermodynamical” derivation is entirely classical, but has some similarities with the derivations of this thesis. However, the derivations based on entanglement above are much better motivated and it is not really even clear what kind of entropy one is calculating in the thermodynamical case [61].

6 Summary

In this thesis, we reviewed how Einstein’s equations can be derived up to second order metric perturbations from the AdS/CFT correspondence. We started by introducing the concept of entanglement entropy and relative entropy in quantum field theories. Entanglement entropy of a spatial subregion can be calculated using the replica trick by means of a functional integral over a replicated surface. The method produces a universal entanglement entropy for two dimensional conformal field theories in thermal states. Thermal entanglement entropies also result from the vacuum state, when moving along non-inertial worldlines that generate a symmetry of the theory. In particular for a ball-shaped region in the CFT vacuum state, the entanglement entropy is equivalent to thermal entropy on a hyperbolic space.

We then moved on to the AdS/CFT correspondence, which is a duality between a string theory in the bulk of AdS space and a conformal field theory on its boundary. When the CFT is strongly coupled and has a large number of degrees of freedom, the string theory reduces to general relativity with quantum corrections being quantum fields that couple semi-classically. Entanglement entropy of the CFT is calculated via the Hubeny-Ryu-Takayanagi formula as the area of an extremal surface in the bulk, to which the leading correction is given by bulk entanglement entropy. For a boundary ball-shaped region in the CFT vacuum state, the formula reproduces the universal nature of the entanglement entropy, which is normalized by two parameters up to second order perturbations around the vacuum in any number of dimensions.

Variation of the ball entanglement entropy around the CFT vacuum state follows the entanglement first law. By using the Ryu-Takayanagi formula and other results from AdS/CFT, the first law can be translated to the bulk, where it acts as a constraint on the bulk geometry. This constraint is equivalent to linearized Einstein’s equations around AdS space. Semi-classical matter coupling is obtained by including the leading quantum correction, bulk entanglement entropy, in the RT formula. At second order the CFT first law is no longer enough, but instead the expression for relative entropy has to be used and expanded to second order, giving a second order

constraint on the bulk geometry. As a result, Einstein's equations at second order are obtained, but only for specific subset of perturbations. We discussed how these derivations can naturally be extended to higher curvature theories of gravity as well.

Last we reviewed how Einstein's equations can be derived from the maximum vacuum entanglement hypothesis. Total entanglement entropy of a geodesic ball is defined as a sum of UV and IR entropies of the quantum gravity Hilbert space. The UV entropy is assumed to follow an area law, while the IR entropy is normal entanglement entropy of the quantum matter fields. The hypothesis states that the total entanglement entropy of a small geodesic ball is maximized, which gives the full non-linear Einstein's equations inside the ball. Entanglement equilibrium is assumed to hold locally at every point, leading to Einstein's equations everywhere in the spacetime. The derivation is inspired by the analogs in AdS/CFT and we discussed their differences.

Appendices

Appendix A

Useful metrics

1 Poincaré metric

AdS_{d+1} and Euclidean AdS space \mathbb{H}_{d+1} (hyperbolic space) are defined as the embedding of

$$\pm X_0^2 + X_1^2 + \dots + X_d^2 - X_{d+1}^2 = -\ell^2 \quad (1.1)$$

in the $\mathbb{R}^{d,2}$ ambient spacetime

$$ds^2 = \pm dX_0^2 + dX_1^2 + \dots + dX_d^2 - dX_{d+1}^2. \quad (1.2)$$

AdS_{d+1} corresponds to the minus sign and \mathbb{H}_{d+1} to the plus sign in the coefficient of dX_0^2 above. Let

$$u = X_d + X_{d+1} \quad (1.3)$$

$$v = X_d - X_{d+1} \quad (1.4)$$

$$t = X_0/u \quad (1.5)$$

$$x^i = X_i/u, \quad i = 1, \dots, d-1. \quad (1.6)$$

With these coordinates, the hyperbola (1.1) becomes:

$$uv + u^2(\pm t^2 + \vec{x}^2) = -\ell^2. \quad (1.7)$$

We can solve for v and substitute it to the transformed ambient metric (1.2). The result is:

$$ds^2 = \frac{\ell^2}{u^2} du^2 + u^2(\pm dt^2 + d\vec{x}^2). \quad (1.8)$$

By defining $z = \ell/u$ with $z \geq 0$, we get the Poincaré metric

$$ds^2 = \frac{\ell^2}{z^2} (dz^2 \pm dt^2 + d\vec{x}^2) \quad (1.9)$$

with the minus sign being AdS_{d+1} and plus sign \mathbb{H}_{d+1} .

2 Hyperbolic blackhole metric

The metric of a general hyperbolic AdS_{d+1} black hole is [23]

$$ds^2 = -\frac{\ell^2}{R^2} f(\rho) d\eta^2 + \frac{1}{f(\rho)} d\rho^2 + \rho^2 (du^2 + \sinh^2 u d\Omega_{d-2}^2), \quad (2.1)$$

where

$$f(\rho) = \frac{\rho^2}{\ell^2} - 1 - \frac{\rho_+^{d-2}}{\rho^{d-2}} \left(\frac{\rho_+^2}{\ell^2} - 1 \right). \quad (2.2)$$

The temperature of the black hole is given by

$$T = \frac{\ell}{4\pi R} \left(\frac{d \rho_+}{\ell^2} - \frac{d-2}{\rho_+} \right) \quad (2.3)$$

Here ρ_+ corresponds to the the black hole mass and it fixes the radial location of the horizon $f(\rho_+) = 0$. AdS-Rindler space is the special case of (2.1) that is isometric to pure AdS. The Rindler horizon was shown to have an inverse temperature $\beta = 2\pi R$, which is obtained from (2.3) by setting $\rho_+ = \ell$. This corresponds to

$$f(\rho) = \frac{\rho^2}{\ell^2} - 1 \quad (2.4)$$

giving the AdS-Rindler metric (3.28).

Appendix B

Einstein's equations as an equation of state

The Einstein's equations imply the existence of a black hole entropy, which must be proportional to the area of the horizon, while the surface gravity determines the temperature. Without worrying about the microscopical degrees of freedom that the entropy describes, one could use the thermodynamical relationships and maybe derive Einstein's equations. This would allow us to view the Einstein's equations as an equation of state between the thermodynamical variables. Such a viewpoint was taken by Jacobson [60] to show that the Einstein's equations can indeed be derived as an equilibrium relation.

The idea is to consider the local Rindler horizon of a spacetime point p (the space is locally flat) and analyze the thermodynamical heat that flows through a spacelike 2-surface element \mathcal{P} , which is a thin strip extending from the horizon. In analogy with black hole entropy, a similar entropy can be defined for the Rindler horizon. There are subtleties however, because the Rindler horizon has an infinite area so that one instead looks at an infinitesimal piece of the horizon. The horizon has an Unruh temperature $T = a/2\pi$ and it is assumed that the change in entropy follows the familiar equation

$$\delta S = \eta \delta A, \tag{0.1}$$

where η is a finite constant and A is the area of the spacelike area element \mathcal{P} . In analogy with black hole entropy, the constant is taken to be $\eta = 1/4G_N$. Then we assume the first law like relation between the horizon entropy and a heat quantity:

$$\delta Q = T dS. \tag{0.2}$$

The heat Q is defined as the energy that flows through the domain of dependence of the Rindler horizon as measured by the Rindler observer. The horizon focuses the energy flow depending on how it is deformed so that the two equations (0.1) and (0.2) act as constraint on the curvature of the geometry at p . Turns out that this constraint is exactly the null components of Einstein's equations as we will show next. Applying the constraint at all points p then gives the equations everywhere in the spacetime.

The observer is assumed to hover near the horizon, meaning that the calculations are done in the infinite acceleration limit. In this limit, the boost generator ζ^μ (2.37), which generates the worldline of the observer, approximately also generates the local Rindler horizon of \mathcal{P} . Denoting the future horizon of \mathcal{P} by \mathcal{H} , the total heat measured by the observer can be defined in terms of the energy-momentum flux $\zeta^\nu T_{\mu\nu}$ through

the past horizon:

$$\delta Q = \int_{\mathcal{H}} d\Sigma^\mu \zeta^\nu T_{\mu\nu}. \quad (0.3)$$

Let $\lambda \in \mathbb{R}$ be an affine parameter that vanishes at \mathcal{P} and increases in the future direction. Then near \mathcal{P} we can approximate $\zeta^\mu = -a\lambda k^\mu$ and $d\Sigma^\mu = d\lambda dA k^\mu$ with k^μ being a tangent vector of \mathcal{H} . With these explicit formulas (0.3) becomes

$$\delta Q = -a \int_{\mathcal{H}} d\lambda dA \lambda T_{\mu\nu} k^\mu k^\nu. \quad (0.4)$$

The heat flow is assumed to travel along null geodesics that pass through the horizon \mathcal{H} . The change in the area of \mathcal{P} is proportional to the expansion θ of these geodesics:

$$\delta A = \int_{\mathcal{H}} d\lambda dA \theta. \quad (0.5)$$

Raychaudhuri equation is an equation, which relates the rate of change of the expansion to the spacetime curvature. Near p the null geodesics approximately travel along k^μ so the affine parameter of the geodesics is λ . Integrating the Raychaudhuri equation gives for small λ :

$$\theta = -\lambda R_{\mu\nu} k^\mu k^\nu. \quad (0.6)$$

The area variation becomes

$$\delta A = - \int_{\mathcal{H}} d\lambda dA \lambda R_{\mu\nu} k^\mu k^\nu. \quad (0.7)$$

The first law relation (0.2) is now equivalent to

$$R_{\mu\nu} k^\mu k^\nu = \frac{2\pi}{\eta} T_{\mu\nu} k^\mu k^\nu \quad (0.8)$$

for arbitrary null k^μ , which implies $R_{\mu\nu} + f g_{\mu\nu} = T_{\mu\nu}$ for some function f , because $g_{\mu\nu} k^\mu k^\nu$ vanishes. Using the Bianchi identity this is equivalent to Einstein's equations at p :

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu} \quad (0.9)$$

with Λ identified as the cosmological constant.

This derivation can also be extended to higher curvature theories of gravity [62]. The strategy is to replace the area of \mathcal{P} by the corresponding Wald functional of the theory. The key difference is that one is no longer able to use the Raychaudhuri equation in the derivation, but one instead uses the fact that for an exact Killing vector ζ^μ :

$$\nabla_\mu \nabla_\nu \zeta_\rho = R^\sigma_{\mu\nu\rho} \zeta_\sigma. \quad (0.10)$$

In our case, the ζ^μ as defined above is only an approximate Killing vector in the $a \rightarrow \infty$ limit and the identity (0.10) is taken to hold approximately. This is analogous to the approximation above that $\zeta^\mu = -a\lambda k^\mu$.

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