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Generic Three-Parameter Wormhole Solution in Einstein-Scalar Field Theory

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Abstract: An exact analytical, spherically symmetric, three-parametric wormhole solution has been found in the Einstein-scalar field theory, which covers the several well-known wormhole solutions. It is assumed that the scalar field is massless and depends on the radial coordinate only. The relation between the full contraction of the Ricci tensor and Ricci scalar has been found as $R_{\alpha\beta}R^{\alpha\beta} = R^2$. The derivation of the Einstein field equations have been explicitly shown, and the exact analytical solution has been found in terms of the three constants of integration. The several wormhole solutions have been extracted for the specific values of the parameters. In order to explore the physical meaning of the integration constants, the solution has been compared with the previously obtained results. The curvature scalar has been determined for all particular solutions. Finally, it is shown that the general solution describes naked singularity characterized by the mass, the scalar quantity and the throat.

Keywords: Einstein-scalar field theory; wormhole solution; scalar field; curvature invariant

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1. Introduction

The formation of spacetime around gravitationally compact objects such as black holes, wormholes, neutron stars, etc., is a quite common phenomenon in general relativity (GR) and alternative theories of gravity. The wormhole is an exact solution of the Einstein field equations describing a hypothetical bridge between two or more points in the Universe or between two different universes. From an astrophysical point of view, one has to mention that wormholes are much more exotic and mathematical objects in compaison to the concept of a black hole. However, wormholes also have interesting features and might represent similar exiting properties as black holes. From a practical point of view, wormholes have not been directly observed yet, however, from the theoretical point of view, they are good candidates for "time machines", which could provide the possibility to travel between the different universes and interstellar travel within different parts of our Universe.

The first idea about the wormhole was proposed by Flamm in 1916, just after the discovery of Schwarzschild's solution of the Einstein equations. Later on, in 1935 [1], Einstein and Rosen introduced a new formation of the spacetime between black holes, which is a regular wormhole solution, known as the Einstein–Rosen bridge in the present time. However, notice that Misner and Wheeler coined the term wormhole in 1957 [2]. The modern interest in a traversable wormhole was stimulated particularly by the pioneering works of Morris, Thorne and Yurtsever [3]. Notice that the Einstein–Rosen wormhole is not traversable, and traversable wormholes are described as having throats that connect two asymptotically flat regions of spacetime. The several properties of traversable wormholes have been investigated by Morris and Thorne [4], and Ellis constructed the famous eponymous wormhole solution in Einstein gravity coupled to a free phantom scalar [5].

There is a large number of wormhole solutions known in the literature. Particularly, one of the simple wormhole solutions has been found by Morris and Thorne (MT) [4]. Another well-known wormhole solution has been obtained by Janis–Newman–Winicour (JNW) [6], which is also known as the JNW naked singularity. The description of a so-called Ellis–Bronnikov (EB) wormhole solution can be found in [5]. The rotating wormhole solution has been discussed in [7]. Traversable wormhole solutions described by the exponential metric have been derived in [8,9].

On the other hand, there are known numerous exact analytical solutions of the Einstein field equations. In the pioneering work of [10], several static solutions of the Einstein field equations have been presented. The most interesting and known exact solutions of the Einstein field equations have been collected in textbooks [11,12]. In Ref [13], it is shown that the static spherical symmetric solution obtained by Wyman [14] is the same as Janis–Newman–Winicour's [6] spacetime. Generalizations of these solutions have been reviewed in [15–19]. The exact solution of the Einstein equations for the wormhole with the scalar field have been recently studied in [20,21]. The contribution of the scalar field in the spacetime of static [13,22] and rotating black holes [23] and rotating wormholes [24] have been also studied.

Here, we are interested in finding and exploring the spherically symmetric solutions of the Einstein-scalar field equations. First, we introduce the line element of space-time containing unknown metric functions. Then, using Einstein field equations, we obtain the second-order differential equations for the radial profile functions. The derived exact analytical solutions for the radial functions contain the constants of the integration which are explored. The paper is organized as follows. In Section 2, we provide the detailed derivations of Einstein-scalar fields equations. In Section 3, we obtain the general three-parametric wormhole solution by solving the Einstein-scalar field equations. In the next Section 4, we reproduce some of the well-known wormhole and black hole solutions. Section 5 is devoted to identifying the physical meaning of the constant of the integration. Finally, in Section 6, we summarize the main obtained results.

2. Einstein's Massless Scalar Field Equations

The action for Einstein's massless scalar field system reads as follows (Throughout the paper, we use the geometrized system of units in which $c = G = \hbar = 1$ and spacelike signature $(-,+,+,+)$):

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}(R - 2\partial_\alpha\Phi\partial^\alpha\Phi)\,, \tag{1}$$

where g is the determinant of the metric tensor of arbitrary spacetime, i.e., $g = ||g_{\alpha\beta}||$, R is the Ricci scalar, i.e., $R = R^\alpha_\alpha$, and Φ is the scalar field. By minimizing the action (1), equations of motion, namely, the Einstein field equations and the Klein–Gordon equation, can be obtained as:

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = T_{\alpha\beta}\,, \tag{2}$$

$$\Box\Phi = \frac{1}{\sqrt{-g}}\partial_\alpha\left(\sqrt{-g}\partial^\alpha\Phi\right) = 0\,, \tag{3}$$

where $G_{\alpha\beta}$ is the Einstein tensor, and $T_{\alpha\beta}$ is the energy-momentum tensor for the scalar field defined as:

$$T_{\alpha\beta} = 2\partial_\alpha\Phi\partial_\beta\Phi - g_{\alpha\beta}\partial_\mu\Phi\partial^\mu\Phi\,. \tag{4}$$

Hereafter, taking traces from both sides of Einsteins field Equation (2) and using the following summation condition $g^{\alpha}_{\alpha} = \delta^{\alpha}_{\alpha} = 4$, one can obtain the expression for the Ricci scalar as:

$$R = 2\partial_{\alpha}\Phi\partial^{\alpha}\Phi \,, \tag{5}$$

which is one of the curvature invariants. On the other hand, inserting the expression (5) into (2), one can obtain:

$$R_{\alpha\beta} = 2\partial_{\alpha}\Phi\partial_{\beta}\Phi \,, \tag{6}$$

which is also called the Einstein-scalar field equation. Notice that the above equation is equivalent to Einstein's field equations in the presence of a scalar field. We now wish to determine the Einstein "scalar", which is a product of the Einstein tensor (i.e., $G_{\alpha\beta}G^{\alpha\beta}$) and can be calculated as:

$$G_{\alpha\beta}G^{\alpha\beta} = \left(R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R\right)\left(R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R\right) = R_{\alpha\beta}R^{\alpha\beta} \,, \tag{7}$$

and, finally, using Equations (5) and (6), one can obtain the following expression:

$$G_{\alpha\beta}G^{\alpha\beta} = R_{\alpha\beta}R^{\alpha\beta} = 4\partial_{\alpha}\Phi\partial_{\beta}\Phi\partial^{\alpha}\Phi\partial^{\beta}\Phi = R^2 \,. \tag{8}$$

Keep in mind that Equation (8) is valid for massless scalar fields only. One has to emphasize that there exists one more curvature invariant, the so-called Kretschmann scalar, which is independent of the Lagrangian of the system and defined as $K = R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$, where $R_{\alpha\beta\mu\nu}$ is the Riemann tensor.

Now, we focus on obtaining the spherically symmetric solution for the Einstein-scalar field system. Assume that the scalar field depends on the radial coordinate only, $\Phi = \Phi(r)$, then the expression for the diagonal elements of the energy-momentum tensor in (4) can be written as:

$$T^t_t = -T^r_r = T^{\theta}_{\theta} = T^{\phi}_{\phi} = -\partial_r\Phi\partial^r\Phi \,, \tag{9}$$

which are equivalent to the following equations:

$$G^t_t = -G^r_r = G^{\theta}_{\theta} = G^{\phi}_{\phi} = -\partial_r\Phi\partial^r\Phi \,. \tag{10}$$

Simultaneously, the scalar field Φ satisfies the following equation:

$$\partial_r\left(\sqrt{-g}g^{rr}\partial_r\Phi\right) = 0 \,, \quad \text{or} \quad \sqrt{-g}g^{rr}\partial_r\Phi = \text{const} \,. \tag{11}$$

We are now in a position to find the solution of the Einstein-scalar field equations combined with the Klein–Gordon equation. In Boyer–Lindquist coordinates $x^{\alpha} = (t, r, \theta, \phi)$, the general form of the spherically symmetric, static spacetime metric is given by:

$$ds^2 = -e^{\nu(r)}dt^2 + e^{-\nu(r)}\left(dr^2 + e^{\lambda(r)}d\Omega^2\right) \,, \tag{12}$$

where $\nu(r)$ and $\lambda(r)$ are arbitrary radial functions, and $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. Taking into account Equation (12), the components of the Einstein tensor can be expressed as:

$$G^t_t = \frac{1}{4}e^{\nu}\left(4\lambda'' - 4\lambda'\nu' + 3\lambda'^2 - 4\nu'' + \nu'^2 - 4e^{-\lambda}\right) \,, \tag{13}$$

$$G^r_r = \frac{1}{4}e^{\nu}\left(\lambda'^2 - \nu'^2 - 4e^{-\lambda}\right) \,, \tag{14}$$

$$G^{\theta}_{\theta} = \frac{1}{4}e^{\nu}\left(\lambda'^2 + \nu'^2 + 2\lambda''\right) = G^{\phi}_{\phi} \,. \tag{15}$$

where prime and double primes denote the first- and second-order derivatives with respect to the radial coordinates. Now, using Equation (10), one can write the following equations for the radial functions:

$$G^r_r + G^\theta_\theta = 0 = \frac{1}{2}e^\nu\left(\lambda'' + \lambda'^2 - 2e^{-\lambda}\right), \tag{16}$$

$$G^r_r + G^t_t = 0 = e^\nu\left(\lambda'' + \lambda'^2 - 2e^{-\lambda} - \lambda'\nu' - \nu''\right), \tag{17}$$

$$G^\theta_\theta - G^t_t = 0 = \frac{1}{2}e^\nu\left(\lambda'' + \lambda'^2 - 2e^{-\lambda} - 2\lambda'\nu' - 2\nu''\right), \tag{18}$$

which allows one to write the following equations for unknown metric functions:

$$\frac{d^2}{dr^2}e^{\lambda(r)} - 2 = 0\,, \qquad \frac{d}{dr}\left(\nu'(r)e^{\lambda(r)}\right) = 0\,. \tag{19}$$

The solutions of Equation (19) are rather simple and can be found as follows:

$$e^{\lambda(r)} = r^2 + 2c_1 r + c_2\,, \tag{20}$$

$$\begin{aligned} \nu(r) &= 2c_3\int\frac{dr}{r^2 + 2c_1 r + c_2} \\ &= -\frac{2c_3}{\sqrt{c_1^2 - c_2}}\tanh^{-1}\left(\frac{r + c_1}{\sqrt{c_1^2 - c_2}}\right), \end{aligned} \tag{21}$$

where c_1, c_2 and c_3 are the constants of integration. The factor 2 in front of the constants c_1 and c_3 is introduced for further convenience. From Equation (20), one can easily see that the dimension of constants c_1 and c_3 is in length, while constant c_2 measures in square of length, i.e., $[c_1] = [c_3] =$ length, $[c_2] =$ length^2.

3. General Solution of Einstein-Scalar Field Equations

Finally, the general solution of the Einstein-scalar field equations can be expressed as:

$$\begin{aligned} ds^2 = &-\left(\frac{r + c_1 + \sqrt{c_1^2 - c_2}}{r + c_1 - \sqrt{c_1^2 - c_2}}\right)^{-\frac{c_3}{\sqrt{c_1^2 - c_2}}} dt^2 \\ &+ \left(\frac{r + c_1 + \sqrt{c_1^2 - c_2}}{r + c_1 - \sqrt{c_1^2 - c_2}}\right)^{\frac{c_3}{\sqrt{c_1^2 - c_2}}} dr^2 \\ &+ \left(r^2 + 2c_1 r + c_2\right)\left(\frac{r + c_1 + \sqrt{c_1^2 - c_2}}{r + c_1 - \sqrt{c_1^2 - c_2}}\right)^{\frac{c_3}{\sqrt{c_1^2 - c_2}}} d\Omega^2\,, \end{aligned} \tag{22}$$

and the associated scalar field is:

$$\Phi(r) = \frac{1}{2}\sqrt{1 - \frac{c_3^2}{c_1^2 - c_2}}\ln\left(\frac{r + c_1 + \sqrt{c_1^2 - c_2}}{r + c_1 - \sqrt{c_1^2 - c_2}}\right), \tag{23}$$

while the non-vanishing components of the energy-momentum tensor are:

$$T^t_t = -T^r_r = T^\theta_\theta = -\frac{c_1^2 - c_2 - c_3^2}{\left(r^2 + 2c_1 r + c_2\right)^2}\left(\frac{r + c_1 + \sqrt{c_1^2 - c_2}}{r + c_1 - \sqrt{c_1^2 - c_2}}\right)^{-\frac{c_3}{\sqrt{c_1^2 - c_2}}}.$$

The curvature invariants, namely, the Ricci scalar and Kretchmann scalar, are:

$$R = \frac{c_1^2 - c_2 - c_3^2}{(r^2 + 2c_1 r + c_2)^2} \left(\frac{r + c_1 + \sqrt{c_1^2 - c_2}}{r + c_1 - \sqrt{c_1^2 - c_2}} \right)^{-\frac{c_3}{\sqrt{c_1^2 - c_2}}}, \tag{24}$$

$$K = \frac{C_1 r^2 + C_2 r + C_3}{(r^2 + 2c_1 r + c_2)^4} \left(\frac{r + c_1 + \sqrt{c_1^2 - c_2}}{r + c_1 - \sqrt{c_1^2 - c_2}} \right)^{-\frac{2c_3}{\sqrt{c_1^2 - c_2}}}. \tag{25}$$

where:

$$C_1 = 48c_3^2, \tag{26}$$

$$C_2 = 32c_3[c_2 - (c_1 - 2c_3)(c_1 - c_3)], \tag{27}$$

$$C_3 = 4\left[7c_3^4 - 16c_1c_3^3 + 2\left(7c_1^2 - c_2\right)c_3^2 + 8c_1\left(c_2 - c_1^2\right)c_3 + 3\left(c_1^2 - c_2\right)^2\right]. \tag{28}$$

The curvature invariants at the center of gravitational source take the following values:

$$\lim_{r \to 0} R = \frac{2(c_1^2 - c_3^2 - c_2)}{c_2^2} \left(\frac{c_1 + \sqrt{c_1^2 - c_2}}{c_1 - \sqrt{c_1^2 - c_2}} \right)^{-\frac{c_3}{\sqrt{c_1^2 - c_2}}}, \tag{29}$$

$$\lim_{r \to 0} K = \frac{C_3}{c_2^4} \left(\frac{c_1 + \sqrt{c_1^2 - c_2}}{c_1 - \sqrt{c_1^2 - c_2}} \right)^{-\frac{2c_3}{\sqrt{c_1^2 - c_2}}}, \tag{30}$$

which show that the general wormhole solution (22) is regular. However, it might be singular for specific values of the constants of the integration. It is also easy to check that the obtained general wormhole solution (22) is asymptotically flat:

$$\lim_{r \to \infty} R = 0, \qquad \lim_{r \to \infty} K = 0. \tag{31}$$

So far, we have shown the general wormhole solution of the Einstein-scalar field equation with three constants of integration. Now, we focus on the particular cases of the general three-parameter solution (22).

4. Special Wormhole Solutions

4.1. Special Solution in the Case When $c_2 = c_1^2$

Let us start the analysis of solution (22) from the most simple case. Assume that $c_2 = c_1^2$. Then, the metric functions in (20) take the following form:

$$e^{\lambda(r)} = (r + c_1)^2, \quad \nu(r) = 2c_3 \int \frac{dr}{(r + c_1)^2} = -\frac{2c_3}{r + c_1}, \tag{32}$$

and two constants can be taken as the throat: $c_1 = a$ and mass $c_3 = M$ of the wormhole. Then, the spacetime metric can be expressed as:

$$ds^2 = -\exp\left(-\frac{2M}{r + a}\right)dt^2 + \exp\left(\frac{2M}{r + a}\right)\left[dr^2 + (r + a)^2 d\Omega^2\right], \tag{33}$$

and the associated scalar field is $\Phi(r) = iM/(r + a)$, which is responsible for the phantom field. The non-vanishing components of the energy-momentum tensor are:

$$G_t^t = -T_r^r = T_\theta^\theta = \frac{M^2}{(r + a)^4} \exp\left(-\frac{2M}{r + a}\right), \tag{34}$$

while curvature invariants take the form:

$$R = -\frac{2M^2}{(r+a)^4} \exp\left(-\frac{2M}{r+a}\right), \tag{35}$$

$$K = \frac{4M^2\left[12(r+a)^2 - 16M(r+a) + 7M^2\right]}{(r+a)^8} \exp\left(-\frac{4M}{r+a}\right), \tag{36}$$

which are regular at the origin (i.e., $r = 0$), and our analyses show that the expressions in (35) are regular for any values of the wormhole throat. Notice that, in the case when $a = 0$, the metric (33) reduces the wormhole solution obtained by Papopetrou [8], see also recent paper [9].

4.2. Special Wormhole Solutions When $c_1 = 0$

Now, we concentrate on the case when $c_1 = 0$. Then, the metric (22) reads:

$$ds^2 = -\left(\frac{r+\sqrt{-c_2}}{r-\sqrt{-c_2}}\right)^{-\frac{c_3}{\sqrt{-c_2}}} dt^2 + \left(\frac{r+\sqrt{-c_2}}{r-\sqrt{-c_2}}\right)^{\frac{c_3}{\sqrt{-c_2}}} dr^2 + \left(r^2 + c_2\right)\left(\frac{r+\sqrt{-c_2}}{r-\sqrt{-c_2}}\right)^{\frac{c_3}{\sqrt{-c_2}}} d\Omega^2 . \tag{37}$$

which in turn shows two separate solutions.

Assume that $c_2 = -M^2 < 0$ and $c_3 = M\gamma \in \mathrm{R}$, then one can obtain the well-known JNW solution [6,13,14]:

$$ds^2 = -\left(\frac{r+M}{r-M}\right)^{-\gamma} dt^2 + \left(\frac{r+M}{r-M}\right)^{\gamma} dr^2 + \left(r^2 - M^2\right)\left(\frac{r+M}{r-M}\right)^{\gamma} d\Omega^2 , \tag{38}$$

where the scalar field is:

$$\Phi(r) = \frac{1}{2}\sqrt{1 + \frac{1}{\gamma^2}} \ln\left(\frac{r+M}{r-M}\right), \tag{39}$$

the energy-momentum tensor is:

$$G^t_t = -T^r_r = T^\theta_\theta = \frac{M^2(1+\gamma^{-2})}{(r^2-M^2)^2}\left(\frac{r+M}{r-M}\right)^{-\gamma} \tag{40}$$

and the curvature invariants are:

$$R = \frac{2(1-\gamma^2)M^2}{(r^2-M^2)^2}\left(\frac{r-M}{r+M}\right)^{\gamma}, \tag{41}$$

$$K = \frac{4M^2}{(r^2-M^2)^4}\left(\frac{r-M}{r+M}\right)^{2\gamma}\left[12\gamma^2 r^2 - 8\gamma\left(2\gamma^2+1\right)Mr + \left(7\gamma^4 + 2\gamma^2 + 3\right)M^2\right]. \tag{42}$$

If the constant c_2 is positively identified, i.e., $c_2 = M^2 > 0$ and $D = c_3$, then one can obtain:

$$e^\lambda = r^2 + M^2 , \qquad \nu(r) = \frac{2c_3}{M}\tan^{-1}\left(\frac{r}{M}\right). \tag{43}$$

$$ds^2 = -\exp\left[\frac{2M}{\sqrt{D^2-M^2}}\tan^{-1}\frac{r}{\sqrt{D^2-M^2}}\right]dt^2 + \exp-\left[\frac{2M}{\sqrt{D^2-M^2}}\tan^{-1}\frac{r}{\sqrt{D^2-M^2}}\right]\left[dr^2 + (r^2 - D^2 - M^2)d\Omega^2\right], \tag{44}$$

which is the Ellis–Bronnikov wormhole solution.

4.3. Special Solutions in the Case When $c_2 = 0$

4.3.1. Schwarzschild Space-Time

The Schwarzschild space-time metric can be obtained by setting $c_3 = -c_1 = M$, which takes the following form:

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 , \tag{45}$$

with $\Phi(r) = 0$ and $G^t_t = -T^r_r = T^\theta_\theta = 0$. The curvature invariants are found as $R = 0$ and $K = 48M^2/r^6$.

4.3.2. Janis–Newman–Winicour Wormhole

In order to obtain the Janis–Newman–Winicour space-time metric, one can set $c_1 = -M/\gamma$ and $c_3 = M$, and then obtain the well-known JNW naked singularity:

$$ds^2 = -\left(1 - \frac{2M}{\gamma r}\right)^{\gamma} dt^2 + \left(1 - \frac{2M}{\gamma r}\right)^{-\gamma} dr^2 + r^2\left(1 - \frac{2M}{\gamma r}\right)^{1-\gamma} d\Omega^2 , \tag{46}$$

and the associated scalar field takes the following form:

$$\Phi(r) = \frac{\sqrt{1-\gamma^2}}{2} \ln\left(1 - \frac{2M}{\gamma r}\right) , \tag{47}$$

and the non-vanishing components of the energy-momentum tensor have the following form:

$$G^t_t = -T^r_r = T^\theta_\theta = -\frac{M^2(1-\gamma^{-2})}{r^4}\left(1 + \frac{2M}{\gamma r}\right)^{\gamma-2} . \tag{48}$$

The curvature invariants are found as:

$$R = \frac{2M^2(1-\gamma^{-2})}{r^4}\left(1 + \frac{2M}{\gamma r}\right)^{\gamma-2} , \tag{49}$$

$$K = \frac{48M^2}{r^6}\left(1 - \frac{2M}{\gamma r}\right)^{2\gamma-4}\left[1 - \frac{2(2\gamma+1)(\gamma+1)M}{3\gamma^2 r} + \frac{(7\gamma^2+2\gamma+3)(\gamma+1)^2M^2}{12\gamma^4 r^2}\right] . \tag{50}$$

4.3.3. Papapetrou Wormhole

Assume that $c_2 \to 0$ and $c_3 = M$. Then, the metric (37) reduces to the Papapetrou solution [8,9]:

$$ds^2 = -\exp\left(-\frac{2M}{r}\right)dt^2 + \exp\left(\frac{2M}{r}\right)\left[dr^2 + r^2 d\Omega^2\right] , \tag{51}$$

and when the scalar field reads $\Phi(r) = iM/r$, the energy-momentum tensor is:

$$G^t_t = -T^r_r = T^\theta_\theta = \frac{M^2}{r^4}\exp\left(-\frac{2M}{r}\right) , \tag{52}$$

while the curvature invariant takes the form:

$$R = -\frac{2M^2}{r^4}\exp\left(-\frac{2M}{r}\right) , \tag{53}$$

$$K = \frac{4M^2\left[12r^2 - 16Mr + 7M^2\right]}{r^8}\exp\left(-\frac{4M}{r}\right) . \tag{54}$$

4.4. Special Solutions in the Case When $c_3 = 0$

Finally, one can consider the case when $c_3 = 0$, then, the wormhole is described by two parameters c_1, c_2, and the space-time metric (22) reads:

$$ds^2 = -dt^2 + dr^2 + \left(r^2 + 2c_1 r + c_2\right) d\Omega^2 \,, \tag{55}$$

$$\Phi(r) = \frac{1}{2} \ln \left(\frac{r + c_1 + \sqrt{c_1^2 - c_2}}{r + c_1 - \sqrt{c_1^2 - c_2}} \right) . \tag{56}$$

Now, the question arises how two parameters are related to the physical quantities related to wormholes. To interpret these parameters, one can reconsider previously obtained and well-known solutions and relate them to measurable quantities.

Morris–Thorne Wormhole

One of the well-known wormhole solutions is described by Morris–Thorne's space-time metric with a single parameter, which is the so-called throat of the wormhole. Assume that $c_1 = 0$ and $c_2 = r_0^2$, where r_0 is the throat of the wormhole. Then, the general wormhole solution reduces the Morris–Thorne wormhole space-time metric in the following form [4]:

$$ds^2 = -dt^2 + dr^2 + \left(r^2 + r_0^2\right) d\Omega^2 \,, \tag{57}$$

and the associated scalar field takes the following form:

$$\Phi(r) = \frac{i}{2} \tan^{-1} \left(\frac{r}{r_0} \right) , \tag{58}$$

the components of the energy-momentum tensor have the form:

$$G_t^t = -T_r^r = T_\theta^\theta = \frac{r_0^2}{\left(r^2 + r_0^2\right)^2} \,. \tag{59}$$

The curvature invariants read:

$$R = -\frac{2r_0^2}{\left(r^2 + r_0^2\right)^2} \,, \qquad K = \frac{12 r_0^4}{\left(r^2 + r_0^2\right)^4} \,. \tag{60}$$

4.5. Solution in the Case When $c_1 = b$ and $c_2 = 0$

In the case when $c_1 = b$ and $c_2 = 0$, the solution is described by the scalar field only and reads as:

$$ds^2 = -dt^2 + dr^2 + r(r + 2b) d\Omega^2 \,, \tag{61}$$

with the associated scalar field:

$$\Phi(r) = \frac{1}{2} \ln \left(1 + \frac{2b}{r} \right) , \tag{62}$$

the components of the energy-momentum tensor have the form:

$$T_t^t = -T_r^r = T_\theta^\theta = -\frac{b^2}{r^2 (2b + r)^2} \,. \tag{63}$$

as well as the curvature invariants reading as:

$$R = \frac{2b^2}{r^2(r+2b)^2}\,, \qquad K = \frac{12b^4}{r^4(r+2b)^4}\,. \tag{64}$$

5. Relation of Integration Constants c_1, c_2 and c_3 to Physical Quantities

Our above-performed analyses shows that the physical meaning of constant c_1 is the mass of the wormhole $c_1 = M$, c_2 is responsible for the size of the throat of the wormhole $c_2 = r_0^2$ and $c_3 = \sigma$ is the scalar charge parameter.

Finally, the space-time metric for the tree parametric wormhole solution can be expressed as:

$$\begin{aligned} ds^2 = &-\left(\frac{r+M+\sqrt{M^2-r_0^2}}{r+M-\sqrt{M^2-r_0^2}}\right)^{-\frac{\sigma}{\sqrt{M^2-r_0^2}}} dt^2 \\ &+\left(\frac{r+M+\sqrt{M^2-r_0^2}}{r+M-\sqrt{M^2-r_0^2}}\right)^{\frac{\sigma}{\sqrt{M^2-r_0^2}}} dr^2 \\ &+\left(r^2+2Mr+r_0^2\right)\left(\frac{r+M+\sqrt{M^2-r_0^2}}{r+M-\sqrt{M^2-r_0^2}}\right)^{\frac{\sigma}{\sqrt{M^2-r_0^2}}} d\Omega^2\,, \end{aligned} \tag{65}$$

and the associated scalar field is:

$$\Phi(r) = \frac{1}{2}\sqrt{1-\frac{\sigma^2}{M^2-r_0^2}}\,\ln\left(\frac{r+M+\sqrt{M^2-r_0^2}}{r+M-\sqrt{M^2-r_0^2}}\right)\,. \tag{66}$$

6. Conclusions

It is well-known that in order to find the exact wormhole solutions to the field equations, one needs to assume the presence of exotic matter, for example, in the throat. In most cases, the exotic matter can be either scalar or phantom field. In general, it is difficult to find the exact analytical wormhole solutions to the field equations, however, the several particular wormhole solutions can be found for given forms of the action, which includes the exotic matter term. Some solutions including exotic matter might be also solutions for either the naked singularity or black holes. The main purpose of this paper is to demonstrate the derivation of three-parametric wormhole solutions in the framework of general relativity.

We have studied one of the fundamental problems related to the Einstein-scalar field equations in the framework of general relativity, assuming the scalar is massless. The Einstein-scalar field equations have been explicitly derived. The relation between the full contraction of the Ricci tensor and the Ricci scalar has been found as $R_{\alpha\beta}R^{\alpha\beta} = R^2$ for the massless scalar field. The general form of the spherically symmetric, static solution of the Einstein-scalar field equations has been found in terms of three constants of integration, which covers some of the well-known wormhole solutions, namely, the Janis–Newman–Winicour naked singularity, the Morris–Thorne wormhole and the Papapetrou exponential wormhole solutions in the specific value of those constants. In addition to that, several new wormhole solutions can also be generated.

In order to better understand the space-time formation, the curvature invariants, namely, the Ricci scalar, Ricci square, and Kretschmann scalar, have been determined for the particular solutions, as well as the components of the energy-momentum tensor. The exact analytical solution for the scalar field has been found.

As a future study, it is also possible to explore the Einstein–Maxwell-scalar field system. In our preceding paper [25], we have presented an exact analytical solution for the system of the Einstein–Maxwell-scalar field equations, which is a generalization based on a combination of well-known Reissner–Nordström and JNW space-time. It can also be shown that the special solution of the Einstein–Maxwell-scalar field equations might be obtained for charged wormholes described by exponential metrics.

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