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Non-abelian vortices in $\mathcal{N} = 2$ gauge theories

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Abstract. BPS non-abelian vortices can be constructed in the Higgs phase of N=2 theories for SU(N) and SO(N) gauge groups. An interesting relation between these vortices and non-abelian monopoles can be established through topological and flux-matching considerations.

1. Introduction

Abrikosov-Nielsen-Olesen vortex solitons are well-known to exist in $U(1)$ gauge theories in the Higgs phase. They are an important ingredient in the usual picture of confinement of magnetic charges in abelian superconductors. But their role in the confining phase of gauge theories with an unbroken non-abelian gauge group is still not clear.

In the basic mechanism proposed by t'Hooft and Mandelstam, confinement of electric charges at strong coupling can be understood in terms of confinement of magnetic charges in the Higgs phase of a dual theory and the chromoelectric string between a quark-antiquark pair corresponds to a magnetic vortex on the dual side. This mechanism is at work in $\mathcal{N} = 2$ SYM, where confinement is driven by condensation of dual quarks in the Seiberg-Witten dual theory. However in this theories dynamical abelianization takes place and the flux of BPS vortices of the dual theory is essentially abelian.

Non-abelian vortices have been introduced some years ago in the Higgs phase of $\mathcal{N} = 2$ theories with gauge group $U(N)$ [1] and $SU(N+1)$ [2]. Confined monopoles in this phase exist as string junctions between vortices of different orientation and can be seen as confined kinks in the worldsheet theory, explaining the correspondence between BPS spectra in 2d and 4d theories [3, 4]. However, in this paper we are interested in the non-abelian heavy monopoles appearing in the $SU(N+1)$ theory broken to $U(N)$ at an intermediate scale: these monopoles are confined by non-abelian vortices.

In section 2 we will study non-abelian vortex solutions in $SO(N) \times U(1)$ theories. These are the simplest example of vortices in theories with gauge group different from $SU(N)$. Then in section 3 we will discuss the relation between non-abelian vortices and heavy monopoles and its implication for the moduli space of non-abelian vortices. We will also show non-trivial examples of this correspondence.

2. Non-abelian vortices in $SO(N)$

We consider the bosonic part of an $\mathcal{N} = 2$ theory with gauge group $SO(2N) \times U(1)$ and $N_f = 2N$ hypermultiplets $(q_A, \tilde{q}_A^\dagger)$ in the $(\underline{2N}, +1)$ representation of the gauge group. We include a Fayet-Iliopoulos term ξ to break the gauge symmetry. The Higgs vacuum of the theory has a $SO(2N)_{C+F}$ color-flavor global symmetry. Using the ansatz $\tilde{q}^\dagger = \phi = 0$, the tension can be written in the Bogomolny form

$$T = \int d^2x \left\{ \left| \frac{1}{2\,g_{2N}}\, F^b_{ij} \mp g_{2N}\varepsilon_{ij} q^\dagger_A t^b q_A \right|^2 + \left| \frac{1}{2\,g_1} F^0_{ij} \pm \frac{g_1}{\sqrt{2}}\varepsilon_{ij}(q^\dagger_A q_A - \xi) \right|^2 + |\mathcal{D}_i q_A \mp i\varepsilon_{ij}\mathcal{D}_j q_A|^2 \pm \frac{\xi}{\sqrt{2}}\varepsilon_{ij}F^0_{ij} \right\} \tag{1}$$

and the ansatz for the solution [5] is $A_i = h_a(r)\, t^a\, \varepsilon_{ij}\, \frac{r_j}{r^2}$ for the gauge fields and

$$q_{iA}(r,\vartheta) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{in_1^+\vartheta}\varphi_1^+(r) & e^{in_1^-\vartheta}\varphi_1^-(r) & 0 & 0 & \cdots \\ ie^{in_1^+\vartheta}\varphi_1^+(r) & -ie^{in_1^-\vartheta}\varphi_1^-(r) & 0 & 0 & \cdots \\ 0 & 0 & e^{in_2^+\vartheta}\varphi_2^+(r) & e^{in_2^-\vartheta}\varphi_2^-(r) & \cdots \\ 0 & 0 & ie^{in_2^+\vartheta}\varphi_2^+(r) & -ie^{in_2^-\vartheta}\varphi_2^-(r) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{2}$$

for the squark fields, with the finite-energy conditions

$$\varphi_a^\pm(\infty) = \sqrt{\frac{\xi}{2N}} \qquad n_a^\pm = \frac{1}{\sqrt{2}}\,(h_0(\infty) \mp h_a(\infty)) \tag{3}$$

Vortex solutions obtained with this ansatz are therefore classified by $2N+1$ integers N_0,$n_a^\pm$ which satisfy the following conditions:

$$n_a^+ + n_a^- = N_0 \quad , \quad \mathrm{sign}(n_a^+) = \mathrm{sign}(n_a^-) = \mathrm{sign}(N_0) \qquad a = 1\ldots 2N \tag{4}$$

N_0 is related to the winding around the $U(1)$ part of the gauge group. The $U(1)$ factor is needed to stabilize the BPS solutions, therefore N_0 enters also the tension $T = 2\pi\xi|N_0|$.

The minimal solutions $N_0 = 1$ are classified by

$$\begin{pmatrix} n_1^+ & \cdots & n_N^+ \\ n_1^- & \cdots & n_N^- \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, \ldots \tag{5}$$

We can always apply $SO(2N)_{C+F}$ tranformations to these solutions, obtaining new solutions. However solutions of the form (2) belong to the same orbit if they are connected by an $SO(2N)$ transformation. All the solutions with $N_0 = 1$ lie in two different orbits of $SO(2N)_{C+F}$, generated from solutions of the form (2) with $\sum_a n_a^+$ even or odd. Therefore the moduli space corresponds to two copies of the quotient space $\mathcal{M} = SO(2N)/U(N)$.

More generally, two solutions which differ only by the exchange $(n_i^+, n_j^+) \leftrightarrow (n_i^-, n_j^-)$ or $(n_i^+, n_i^-) \leftrightarrow (n_j^+, n_j^-)$ for some i,j, belong to the same orbit. Note that it is possible that the moduli space for vortices of higher winding cannot be obtained simply by $SO(2N)_{C+F}$ and its structure is not known.

There is an interesting interpretation of these results. If we consider an high-energy $\mathcal{N} = 2$ theory with gauge group $SO(2N+2)$, $N_f = 2N$ hypermultiplets of mass m in the adjoint representation and a soft $\mathcal{N} = 2$-breaking term $\mu\phi^2$, and then $\langle\phi\rangle \sim m$ breaks the gauge group to $SO(2N) \times U(1)$, the low-energy theory below m corresponds to the one considered above [5]. In fact it is easy to get the tension (1) if we use the ansatz $q = \tilde{q}^\dagger$ and define $\xi = \mu m$.

Note that the high-energy theory at scale m contains heavy almost-BPS monopoles coming from the breaking pattern $SO(2N+2) \to SO(2N) \times U(1)$, while the low-energy theory at scale $\sqrt{\mu m}$ contains the vortices studied in this section, which are almost stable (they are unstable under creation of monopole-antimonopole pairs, but this process is heavily suppressed if $\mu \ll m$). Monopoles and vortices also appear with other gauge groups and symmetry breaking patterns in a similar way. In all these systems there is an apparent relation between monopoles and vortices, which will be discussed in the next section.

3. Monopole-vortex correspondence

We consider $\mathcal{N}=2$ gauge theories with gauge group G and N_f matter hypermultiplets $(q,\tilde{q})$ of mass m, usually in the fundamental representation (the model discussed above is an exception). We also add a small mass term $\mu\phi^2$ for the chiral superfield in the vector hypermultiplet which breaks softly $\mathcal{N}=2$ to $\mathcal{N}=1$. When $\mu \ll m$ there are vacua in the Higgs phase with a hierarchical pattern of symmetry breaking

$$G \xrightarrow{\langle\phi\rangle} H \xrightarrow{\langle q\rangle} 1 \tag{6}$$

where $\langle q\rangle \sim \sqrt{\mu m} \ll m \sim \langle\phi\rangle$. The simplest case is $G=SU(N+1)$ and $H=U(N)$, but examples of such patterns include $G=SO(2N)$ and $H=U(N)$ or $SO(2N-2)\times U(1)$, $G=SO(2N+1)$ and $H=U(N)$ or $SO(2N-1)\times U(1)$, $G=USp(2N)$ and $H=U(N)$ or $USp(2N-2)\times U(1)$.

In the high-energy theory at scale m there are regular monopoles coming from the symmetry breaking $G\to H$, which are not BPS because of $\sqrt{\mu/m}$ corrections. In the low-energy theory below scale m there are regular vortices which come from the breaking $H\to 1$ and are stable in the limit $m\to\infty$ with $\sqrt{\mu m}$ fixed.

We are interested in the case when H is non-abelian. In these systems the regular monopoles are Goddard-Nuyts-Olive non-abelian monopoles [6] which transform under the dual group $\tilde{H}$, while the vortices of the low-energy theory are non-abelian vortices in a theory with gauge group H and Fayet-Iliopoulos parameter $\xi \sim \sqrt{\mu m}$. There is an interesting relation between monopoles and vortices which has been discussed in [2, 7, 8].

We explain this relation starting from the topological classification of solitons in these theories. The relevant homotopy group for regular GNO monopoles is $\pi_2(G/H)$, while singular Dirac monopoles are classified by $\pi_1(G)$. After the breaking $H\to 1$, the only regular monopoles which are topologically stable are those classified by $\pi_2(G)$, which is trivial.

The fate of monopoles classified by a nontrivial element of $\pi_2(G/H)$ is related to the vortices coming from the breaking $H\to 1$. In fact the monopole magnetic flux cannot disappear, but it shrinks into a flux tube of width $1/\sqrt{\mu m}$ which is precisely a vortex of the low-energy theory. So for each monopole in the high-energy theory there should exist a vortex of the low-energy theory which carries the same flux.

This correspondence can be seen from a topological point of view. Vortex solutions are classified by $\pi_1(H)$. The topological relation

$$\pi_2(G/H) = \pi_1(H)/\pi_1(G) \tag{7}$$

has a simple interpretation if G is simply connected: in this case the homotopy groups for monopoles and vortices are the same. When $\pi_1(G)$ is nontrivial, the relation (7) states that regular monopoles are sources for vortices which correspond to trivial elements of $\pi_1(G)$.

A simple way, when possible, to establish this correspondence is flux matching [7]: the magnetic flux integrated over a plane orthogonal to the axis of the vortex should match the magnetic flux integrated over a sphere surrounding the corresponding monopole. Obviously, the abelian magnetic flux coming from the monopole and the flux carried by the vortex must match precisely, but this is only a check, because the $U(1)$ flux cannot determine the non-abelian orientation of the soliton.

Matching non-abelian fluxes is an effective way to match a monopole with the corresponding vortex. The problem with non-abelian flux is that it does not obey a conservation law as the abelian flux because of the term $i[A_j,A_k]$ in the magnetic field. Non-abelian flux matching could be reliable only for monopole and vortex solutions which satisfy $[A_j,A_k]=0$. This is the case for solutions obtained using an ansatz like (2) and the corresponding monopoles. All vortices of minimal winding belong to this case.

Unfortunately $[A_j, A_k] = 0$ is generally not true for vortices of higher winding, as can be seen from the explicit expression for vortex solutions of double winding in $U(2)$ [9]. The same problem occurs for the corresponding monopoles, whose explicit expression was discovered by E.Weinberg [10]. In this case flux matching can only be established in an approximate, thus not very useful, way.

The general claim of this section is that for each monopole in the high-energy theory there is a corresponding vortex in the low-energy theory and that their topological classification and fluxes should match. This claim leads to an interesting corollary about the moduli spaces of monopoles and vortices: in a theory where monopoles correspond to vortices of winding k, the internal moduli space of coaxial vortices of winding k should contain a subspace which has the same structure of the internal space of degenerate monopoles. This is an interesting point because the moduli space of non-abelian monopoles is not well-defined due to the non-normalizability of zero-modes, but it can be matched with the moduli space of vortices, which only have normalizable zero-modes.

In the next sections we will discuss some explicit examples of symmetry breaking patterns to check the correspondence discussed above.

3.1. $SU(N+1) \to U(N)$

This theory contains matter multiplets in the fundamental representation of $SU(N+1)$. The vacuum is invariant under a $SU(N)_{C+F}$ global symmetry. Non-abelian vortex solutions can be constructed with an ansatz similar to (2) and are classified by a set of positive integers $(n_1, n_2 \dots n_N)$ where $\sum_i n_i$ corresponds to the winding of the vortex. All the solutions with minimal winding belong to the same orbit of $SU(N)_{C+F}$. Monopoles are simply embeddings of 't Hooft-Polyakov monopoles in various $SU(2)$ subgroups. Vortices and monopoles in this theory are both classified by $\pi_2(SU(N+1)/U(N)) = \pi_1(U(N)) = \mathbb{Z}$, so fundamental monopoles correspond to vortices of minimal winding classified by $(1, 0 \dots 0)$ etc. The moduli space of these vortices is simply $\mathbb{C}P^{N-1}$ with $SU(N)$ isometry and Fubini-Study metric and it corresponds to the configuration space of monopoles. Flux matching can be easily checked and the correspondence works perfectly.

3.2. $SO(2N) \to U(N)$

This theory contains matter multiplets in the fundamental representation of $SO(2N)$. The vacuum respects a $SU(N)_{C+F}$ global symmetry. Vortex solutions are identical to the previous case, while monopoles are embeddings of 't Hooft-Polyakov ones in $SU(2) \subset SO(4)$ subgroups. The fact that $\pi_1(SO(2N)) = \mathbb{Z}_2$ and $\pi_2(SO(2N)/U(N)) = \pi_1(U(N))/\mathbb{Z}_2 = \mathbb{Z}/\mathbb{Z}_2$ implies that fundamental monopoles correspond to vortices of winding 2, while flux matching calculations suggest that they correspond precisely to the vortices classified by $(2, 0 \dots 0)$ etc. and their $SU(N)_{C+F}$ orbit. Both these vortices and the corresponding monopoles have a configuration space which is $\mathbb{C}P^{N-1}$ with $SU(N)$ isometry. However the moduli space of $k = 2$ vortices is much bigger [11]. In the simplest case $N = 2$ the moduli space is the weighted projective space $W\mathbb{C}P^2_{(2,1,1)}$ with $SU(2)$ isometry, which contains a $\mathbb{C}P^1$ corresponding to the vortices discussed above. So in this case the correspondence works correctly, but it seems unable to explain the presence of a bigger moduli space of vortices.

3.3. $SO(2N+1) \to U(N)$

This is the same theory as the previous case but with gauge group $SO(2N+1)$. They differ mainly because some of the monopoles are embeddings of 't Hooft-Polyakov monopoles in $SO(3)$ and $SU(2)$ subgroups, while others form a continuous family of solutions interpolating between these two embeddings [10]. In this case flux matching is only partially useful because $[A_j, A_k] \neq 0$ for the interpolating solutions. Topological arguments suggest that monopoles correspond to $k =$

2 vortices, because $\pi_1(SO(2N+1)) = \mathbb{Z}_2$ and $\pi_2(SO(2N+1)/U(N)) = \pi_1(U(N))/\mathbb{Z}_2 = \mathbb{Z}/\mathbb{Z}_2$ as in the previous case. Fluxes of $(2,0\dots0)$ vortices agree with those of monopoles embedded in $SU(2)$ subgroups, while fluxes of $(1,1\dots0)$ vortices agree with those of monopoles embedded in $SO(3)$ subgroups.

This case is an interesting check of the correspondence as both moduli spaces of monopoles and vortices are known in the $N=2$ case. The moduli space of vortices is $W\mathbb{C}P^2_{(2,1,1)}$, which is a $\mathbb{C}P^2$ with a conical singularity. It contains the $\mathbb{C}P^1$ discussed in the previous case and the rest of the moduli space corresponds to $\mathbb{C}^2/\mathbb{Z}^2$. The metric of this moduli space is unknown. The moduli spaces of monopoles and its metric have been found in [12]: it has the topological structure of $\mathbb{C}^2/\mathbb{Z}^2$, with a separated $\mathbb{C}P^1$ which represents monopoles with long-range magnetic fields. Therefore the correspondence seems to work also for this case and the existence of vortices which do not belong to $\mathbb{C}P^1$ finds a natural explanation in the existence of a large class of monopoles in this theory.

3.4. $SO(2N+2) \to SO(2N) \times U(1)$

This case has been discussed at the end of section 2. This theory contains matter hypermultiplets in the adjoint representation, but the only components which become massless after the breaking $SO(2N+2) \to SO(2N)$ transform in the fundamental representation of $SO(2N)$, so we end up with a low-energy theory containing squarks in the $(\underline{2N}, +1)$ representation. Non-abelian monopoles are embeddings of t'Hooft-Polyakov ones in $SU(2) \subset SO(4)$ subgroups. The topological structure of the groups in this breaking pattern is $SO(2N+2)/\mathbb{Z}_2 \to (SO(2N) \times U(1))/\mathbb{Z}_2 \to 1$ and therefore the relation $\pi_2(SO(2N+2)/(SO(2N)\times U(1))) = \pi_1((SO(2N)\times U(1))/\mathbb{Z}_2)/\pi_1(SO(2N+2)/\mathbb{Z}_2)$ implies that monopoles should correspond to vortices of winding $N_0 = 2$. Flux matching suggests that monopoles correspond to vortices in the $SO(2N)_{C+F}$ orbit of

$$\begin{pmatrix} 2 & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & \dots & 1 \end{pmatrix} \tag{8}$$

This orbit corresponds to the complex quadric surface $SO(2N)/(SO(2N-2)\times U(1)) = Q^{2N-2}(C)$ [5]. However, the whole moduli space of $N_0 = 2$ vortices is much bigger and its structure is not known.

4. Conclusions

Explicit non-abelian vortex solutions in theories with $SO(N)$ gauge group have been constructed. Their topological structure has been clarified, while the exploration of their moduli space is just starting. Hopefully there will be new theoretical developments along the line of the $SU(N)$ case. Note that vortices in $SO(N)$ theories were also discussed in different frameworks in [13], [14].

The correspondence between monopoles and vortices seems well established through considerations of topology and flux matching. The knowledge of the moduli space of $N_0 = 2$ vortices could provide a very interesting check.

It is interesting to stress that in this framework the degeneracy of monopoles is not related to their equivalence up to a gauge transformation. Instead the degeneracy is related to the global color-flavor transformations, which act simply as color transformations on the monopoles: this means that these monopole-vortex configurations represent physically distinct states. Monopoles should also transform under the dual group $\tilde{H}$, so the action of the color-flavor group on monopoles could be related to the action of the dual group [8].

There are many promising developments in the theory of non-abelian vortices which are not reviewed here. The most interesting direction is the study of vortices in theories with less supersymmetry (see for example [15]).

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