Aspects of Gauge/Gravity Dualities

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Abstract

This dissertation is composed of several studies of gauge/gravity dualities aimed at providing evidence for the dualities we investigate, and exploring possible applications.

First we discuss the universal scaling function $f(g)$, which appears in the dimensions of high-spin operators of the $\mathcal{N}=4$ Super Yang-Mills theory. We study numerically an integral equation that implements a resummation of the complete planar perturbative expansion, and find a smooth function which for large coupling constant g matches with high accuracy the asymptotic form predicted by string theory. Furthermore, we give an exact analytic solution of the strong coupling limit of the integral equation.

Next, we study the $\mathcal{N}=1$ supersymmetric warped deformed conifold, which has two bosonic massless modes, a scalar and a pseudoscalar, that are dual to the modulus and phase of the baryonic condensates in the cascading $\mathrm{SU}(k(M+1)) \times \mathrm{SU}(kM)$ gauge theory. We generalize both perturbations to include non-zero 4-dimensional momentum, and find the mass spectra of $S^{PC}=0^{+-}$ and $S^{PC}=0^{--}$ glueballs. We argue that these massive modes belong to 4-dimensional axial vector or vector supermultiplets.

Extending our discussion to the complete baryonic branch of the gauge theory, we show that the action of a Euclidean D5-brane wrapping all six deformed conifold directions measures the baryon expectation values. We demonstrate that this is consistent with its coupling to the scalar and pseudoscalar massless modes, and reproduces the scaling dimension of baryon operators. We also derive an expression for the variation of the baryon expectation values along the supergravity dual of the baryonic branch.

Finally we turn to 3-dimensional gauge theories with $\mathcal{N}=3$ supersymmetry, and calculate the non-abelian R-charges of BPS monopole operators. In the UV limit they are described by classical backgrounds, and this allows us to find their exact $\mathrm{SU}(2)_R$ charges in a one-loop computation, by quantizing an $\mathrm{SU}(2)/\mathrm{U}(1)$ collective coordinate. We show that monopole operators with vanishing scaling dimensions exist in the ABJM theory, which is essential for matching its spectrum with supergravity on $\mathrm{AdS}_4 \times \mathrm{S}^7/\mathbb{Z}_\mathrm{k}$.

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To my parents,

who put up with my stubbornness,

and never tried to dissuade me from the path I chose,

even when there was no telling where it would lead.

Contents

Chapter 1

A Brief Review of Select Topics in Gauge/Gravity Dualities

While string theory is often thought of primarily as a candidate theory of everything, aiming to unify all fundamental interactions including gravity in a single quantum theory, it has transpired during the past twelve years that it is also a framework in which quantum field theory and general relativity appear as complementary, dual descriptions of the same physics. This allows us to use gravity to study strongly coupled field theories, or conversely, employ field theory ideas to define what we mean by quantum gravity.

Let us begin by reviewing some well-known facts about gauge/gravity dualities. We shall first discuss the prototype of such dualities, the celebrated $\mathcal{N} = 4$ AdS_5/CFT_4 correspondence [1, 2, 3], before we move on to examples with less supersymmetry and without conformal symmetry. Finally, we shall turn to some more recent developments concerning AdS_4/CFT_3 dualities and the worldvolume theory of M2-branes.

This introductory chapter is based on material from work in collaboration with I. R. Klebanov, T. Klose and M. Smedbäck [4, 5]. For further background on the topics touched upon in Section 1.1 we refer the reader to the reviews [6, 7, 8, 9], while summaries of much of the material discussed in Section 1.2 can be found in [10, 11, 12].

1.1 A Sketch of the AdS/CFT Correspondence

Even though the elementary excitations of perturbative string theory are (as the name suggests) 1+1 dimensional objects, whose world-sheet dynamics is governed by the Nambu-Goto area action

$$\mathcal{S}_{\mathrm{NG}} = -\frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{-\det\, \partial_a X^\mu \partial_b X_\mu}\ , \tag{1.1}$$

it was realized a long time ago that supersymmetric theories of closed strings also contain higher dimensional objects [13]. While closed superstrings move freely in the 9+1 dimensional spacetime, open strings are bound to end on p+1 dimensional hyperplanes known as Dirichlet p-branes, usually denoted as Dp-branes. This terminology refers to the Dirichlet boundary conditions which open string have to satisfy at their endpoints in the 9$-p$ directions orthogonal to the D-brane, while Neumann boundary conditions apply for the p+1 coordinate directions parallel to it.

This seemingly innocent observation has profound consequences, because the world-volumes of D-branes support gauge theories. If we consider a single brane, open strings beginning and ending on it have massless excitations identical to those of a maximally supersymmetric U(1) gauge theory in p+1 dimensions. There are 9$-p$ scalar Goldstone bosons, since the brane breaks translation invariance in the transverse directions, and a massless vector which arises from the string excitations in the p+1 directions parallel to the brane. Together with the appropriate fermions they combine to form a vector supermultiplet.

Extending these considerations to multiple parallel branes, one finds that their pair-wise separations are determined by the expectation values of the transverse scalar fields associated with strings stretching between two different branes. In particular, if all scalar expectation values vanish, the branes are stacked on top of each other. If this is the case, then N parallel branes support N^2 massless modes (the number of distinct choices of beginning and endpoint for an open string), which transform in the adjoint representation of U(N). The upshot is that there is a p+1 dimensional U(N) gauge theory with 16 supercharges living on such a collection of branes.

The gravitational background of a stack of D3-branes. For large N a stack of parallel Dp-branes is a heavy object and will backreact onto the spacetime around it. The so-called black p-branes, gravitational backgrounds corresponding to such situations, were originally discovered as supersymmetric soliton solutions of the low energy supergravity theory, which carry electric charge with respect to the p+1 form Ramond-Ramond (RR) potential C_{p+1} [14]. In type IIA string theory (and the associated supergravity theory) p is even, while for type IIB p is odd.

Since we will mostly consider 3+1 dimensional U(N) gauge theory we will be particularly interested in stacks of D3-branes. This case is special insofar as the dilaton field of the type IIB theory is constant in such a background, and the RR field strength $\tilde{F}_5$ the brane couples to is self-dual (the supergravity equations relevant to this case are reviewed in Appendix A.1). The metric of this classical solution is given by [14]

$$ds^2 = h^{-1/2}(r)\left[-f(r)(dx^0)^2 + \sum_{i=1}^{3}(dx^i)^2\right] + h^{1/2}(r)\left[f^{-1}(r)dr^2 + r^2 d\Omega_5^2\right] , \qquad (1.2)$$

where $d\Omega_5^2$ is the metric of a unit 5-dimensional sphere S^5 and

$$h(r) = 1 + \frac{L^4}{r^4} , \qquad f(r) = 1 - \frac{r_0^4}{r^4} . \qquad (1.3)$$

We would like to consider two successive limits of this black 3-brane background. First we take the extremal limit $r_0 \to 0$ of the solution (1.2), in which the area of the event horizon at constant radial coordinate $r = r_0$ vanishes. We have $f(r) \to 1$ and thus

$$ds^2 = h^{-1/2}(r)\eta_{\mu\nu}dx^\mu dx^\nu + h^{1/2}(r)\left(dr^2 + r^2 d\Omega_5^2\right) . \qquad (1.4)$$

In this limit the mass (per unit volume) of the 3-brane saturates a BPS-type bound and it can be shown that the extremal solution preserves 16 of the 32 supersymmetries of the type IIB theory, as it must if it is to describe a stack of parallel D3-branes.

When r is large compared to L it is easy to see that (1.4) simply approaches 9+1 dimensional flat space. In the opposite, near horizon limit $r \to 0$ we find that, defining a new variable $z = L^2/r$, the metric (1.4) simplifies to

$$ds^2 = \frac{L^2}{z^2}\left(dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu\right) + L^2 d\Omega_5^2 \ . \tag{1.5}$$

This metric describes the direct product of a five-sphere S^5, and the Poincaré wedge of the 5-dimensional Anti-de Sitter space, AdS_5, both with radius of curvature L.

By comparing the tension of the extremal 3-brane solution, parameterized by the length scale L, to that of a stack of N D3-branes, characterized by the string length scale $\sqrt{\alpha'}$, one finds that

$$L^4 = g_{\mathrm{YM}}^2 N \alpha'^2 \ , \tag{1.6}$$

where g_{YM} is the coupling constant of the supersymmetric Yang-Mills theory that the brane supports. The five-form flux piercing the S^5 that is required to satisfy the supergravity equations of motions is then simply

$$\int_{\mathrm{S}^5} *\tilde{F}_5 = N \ , \tag{1.7}$$

i.e. the number of flux units equals the number of colors in the gauge theory picture.

Large N and the 't Hooft coupling. The combination $\lambda \equiv g_{\mathrm{YM}}^2 N$ that appears on the right hand side of (1.6) is known as the 't Hooft coupling, and is familiar to field theorists from 't Hooft's generalization of SU(3) gauge theory (i.e. QCD) to SU(N) gauge group [15]. Increasing the number of colors, while keeping λ constant, implies that each Feynman diagram acquires a topological factor of N raised to the power of the Euler characteristic of the graph. Hence in the large N limit Feynman graphs that can be drawn on a sphere (called planar) dominate, since they carry a factor of N^2, while those that can be drawn on a torus are subleading (of order N^0) and those requiring higher genus surfaces even more suppressed. Thus making N large significantly simplifies field theory; if in addition λ is small one can perform calculations perturbatively.

It has been argued that the sum over Feynman graphs with a given Euler characteristic can be thought of as a sum over string world-sheets. In such an interpretation the string coupling constant would be proportional to N^{-1}, such that large N would lead to a weakly coupled string theory. While this argument had never been made precise before the arrival of the AdS/CFT correspondence, one can indeed check that in the full string theory the string loop corrections to the classical solution considered above proceed in powers of N^{-2}. Thus taking N to infinity allows us to use a classical description of string theory on $\mathrm{AdS}_5 \times \mathrm{S}^5$, and for the remainder of this subsection we will assume the large N limit with λ held fixed.

Now from (1.6) we see that the curvature radius L of the near-horizon geometry is equal to $\lambda^{1/4}$ in units of the string scale. Since the supergravity approximation to type IIB string theory is only applicable for weakly curved spaces we require $L \gg \sqrt{\alpha'}$, which implies $\lambda \gg 1$. Hence the gravitational calculation is reliable precisely in the limit in which field theory computations are not perturbative and therefore difficult.

The $\mathbf{AdS_5/CFT_4}$ correspondence with $\mathcal{N} = 4$ supersymmetry. At this point we have arrived at two superficially completely different descriptions of 3-branes. On the one hand, we have a field theory picture of a stack of N D3-branes in terms of $\mathcal{N} = 4$ supersymmetric SU(N) gauge theory that is most useful for small λ. In the low-energy limit the field theory decouples from the bulk closed string theory, and the massless modes of the open strings living on the branes adequately describe the situation. On the other hand, the supergravity picture of a black 3-brane is applicable for large

λ, and here the $AdS_5 \times S^5$ region (at $r \ll L$) decouples from the asymptotically flat large r spacetime in the low-energy limit, as confirmed by investigations of so-called greybody factors describing the cross-section for the absorption of radiation by the brane [16, 17, 18]. This second description is purely gravitational, i.e. in terms of the massless excitations of closed strings.

These two complementary points of view are at the core of the AdS/CFT correspondence. Maldacena [1] made the seminal conjecture that type IIB string theory on $AdS_5 \times S^5$, of radius L as in (1.6), is dual to the $\mathcal{N} = 4$ SU(N) SYM theory.[1] This can be understood as a duality between open and closed string descriptions of the same phenomena. Crucially, it is also a weak-strong duality, since the two pictures apply for different regimes of λ. This renders the AdS/CFT correspondence very hard to prove rigorously, but at the same time exceedingly useful as a computational tool.

The original conjecture of the AdS/CFT correspondence was partially motivated by symmetry considerations [1]. The isometry supergroup of the $AdS_5 \times S^5$ background is PSU$(2,2|4)$, which perfectly matches the $\mathcal{N} = 4$ superconformal symmetry. The maximal bosonic subgroup of this supergroup is SO$(2,4) \times$ SO(6). The first factor is nothing but the conformal group in 3+1 dimensions, which coincides with the isometry group of AdS_5, since Anti-de Sitter space is the maximally symmetric space with (constant) negative curvature. The second factor of SO(6) $\sim$ SU(4) appears as the R-symmetry of the $\mathcal{N} = 4$ SYM theory, while in the geometry it is realized simply as the isometry group of the S^5.

The development of a detailed dictionary that allows one to translate from conformal field theory to string theory language was initiated in [2, 3]. Each gauge invariant operator in the conformal field theory corresponds to a particular field in the $AdS_5 \times S^5$ background (or in some cases, as e.g. for the baryonic operators we will discuss below, to an extended object such a D-brane), in such a way that its scaling dimension Δ is

[1]In its strongest form the conjecture states that this holds even for finite N.

related to the mass of the dual field. E.g. scalar operators satisfy $\Delta(\Delta-4)=m^2L^2$.

For the fields in AdS_5 that come from the type IIB supergravity modes, including the Kaluza-Klein excitations on the 5-sphere, the masses are of order $1/L$. Hence, it is consistent to assume that their operator dimensions are independent of L, and therefore independent of λ. This is due to the fact that such operators commute with some of the supercharges and are thus protected by supersymmetry. Perhaps the simplest such operators are the chiral primaries which are traceless symmetric polynomials in the six real scalar fields. These operators are dual to spherical harmonics on S^5 which mix the graviton and RR 4-form fluctuations. Their masses are $m_k^2 = k(k-4)/L^2$, where $k = 2, 3, \ldots$. These masses reproduce the operator dimensions $\Delta = k$ which are the same as in the free theory. The situation is completely different for operators dual to the massive string modes: $m_n^2 = 4n/\alpha'$. In this case the AdS/CFT correspondence predicts that the operator dimension grows at strong coupling as $2n^{1/2}\lambda^{1/4}$.

In order to compute correlation functions of operators in the conformal field theory using its dual fields φ one identifies a certain gauge theory quantity W with a string theory quantity Z_{string} [2, 3]

$$W[\varphi_0(\vec{x})] = Z_{\text{string}}[\varphi_0(\vec{x})] \ . \tag{1.8}$$

Here W is the generator of the connected Euclidean Green's function of a gauge theory operator $\mathcal{O}$,

$$W[\varphi_0(\vec{x})] = \langle \exp \int d^4x \, \varphi_0 \, \mathcal{O} \rangle \ . \tag{1.9}$$

On the other hand, Z_{string} is the string theory path integral calculated as a functional of φ_0, the boundary condition on the field φ related to $\mathcal{O}$ by the AdS/CFT duality. In the large N limit the string theory becomes classical, which implies

$$Z_{\text{string}} \sim e^{-\mathcal{S}[\varphi_0(\vec{x})]} \ , \tag{1.10}$$

where $\mathcal{S}[\varphi_0(\vec{x})]$ is the extremum of the classical string action which is again a functional of φ_0. If we are further interested in correlation functions at very large 't Hooft coupling, the problem of extremizing the classical string action reduces to solving the equations of motion in type IIB supergravity.

Long operators and the cusp anomalous dimension. Operators with large quantum numbers provide a simple setting in which the AdS/CFT correspondence can be tested very efficiently. E.g. the single-trace operators examined in [19] carry a large R-charge (dual to a string angular momentum J on the compact space S^5) and the dual states are almost point-like closed strings moving rapidly on the five-sphere [20]. If we consider such a long operator with a second large quantum number the dual object may be a macroscopic semi-classical string [21], which allows for quantitative checks of the correspondence (and in addition strongly suggests that string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ Super Yang-Mills theory are integrable [22, 23]).

We can also consider operators with large spin S, such as the twist-2 operator [24, 25]

$$\mathrm{Tr}\ F_{+\mu}\mathcal{D}_+^{S-2}F_+{}^{\mu}\ . \tag{1.11}$$

The scaling dimension Δ of such twist-2 operators at large values of the spin S is characterized by the universal scaling function (or cusp anomalous dimension) $f(g)$:

$$\Delta - S = f(g)\ln S + \mathcal{O}(S^0)\ , \tag{1.12}$$

with $g = \sqrt{g_{YM}^2 N}/4\pi$. The logarithmic dependence of the dimension on large Lorentz spin is a generic feature that has been independently observed for both $\mathcal{N} = 4$ SYM and its string theory dual, see e.g. [26, 27, 28]. Importantly, such a scaling behavior stems [29, 30] from the large spin limit of the Bethe equations [31, 32, 33, 34] which underlie the integrable structures of gauge and string theories.

Due to the universal role played by the function $f(g)$, one would like to compute it in the $\mathcal{N} = 4$ SYM theory. In this case we can consider operators in the SL(2) sector, of the form

$$\text{Tr } \mathcal{D}^S Z^J + \dots \ , \tag{1.13}$$

where Z is one of the complex scalar fields, the R-charge J is the twist, and the dots serve as a reminder that the operator is a linear combination of such terms with the covariant derivatives acting on the scalars in all possible ways. The object dual to such a high-spin twist-2 operator is a folded string [20] spinning around the center of AdS_5; its generalization to large J was found in [28]. The result (1.12) is generally applicable when J is held fixed while S is sent to infinity [29].

Though the function $f(g)$ for $\mathcal{N} = 4$ SYM is not the same as for QCD, its perturbative expansion is in fact related by the conjectured transcendentality principle [35], which states that each expansion coefficient has terms of definite degree of transcendentality (namely the exponent of g minus two), and that the QCD result contains the same terms (in addition to others which have lower degree of transcendentality).

An interesting problem is to smoothly match the explicit predictions of string theory for large g to those of gauge theory at small g. During the past few years methods of integrability in AdS/CFT [36, 37, 38] have led to major progress in addressing this question. In an impressive series of papers [30, 39, 40] a linear integral equation has been derived, which allows one to compute the universal scaling function $f(g)$ to any desired order in perturbation theory. We will refer to it as the BES equation.

It was obtained from the asymptotic Bethe ansatz for the SL(2) sector by considering the limit $S \to \infty$ with J finite, and extracting the piece proportional to $\ln S$, which is manifestly independent of J. Taking the spin to infinity, the discrete Bethe equations can be rewritten as an integral equation for the density of Bethe roots in rapidity space.

In Chapter 2 we shall discuss how to extract the strong coupling behavior of $f(g)$ from the BES equation, and show that it agrees with the predictions from string theory.

1.2 Generalizations to Lower Supersymmetry and the Non-Conformal Case

1.2.1 Reducing Supersymmetry and the Conifold

To obtain an AdS/CFT duality with less than the maximal $\mathcal{N} = 4$ supersymmetry, we consider a stack of D3-branes located at the singularity of a 6-dimensional Ricci-flat cone [41, 42, 43, 44]. The metric is then given by

$$ds^2 = h^{-1/2}(r)\eta_{\mu\nu}dx^\mu dx^\nu + h^{1/2}(r)\left(dr^2 + r^2 ds_Y^2\right) \ . \qquad (1.14)$$

where ds_Y^2 is the metric of the 5-dimensional compact space Y_5, which is an Einstein manifold (i.e. $R_{ab} = 4g_{ab}$) and forms the base of the cone. The geometry dual to the conformal field theory supported by the D3-branes at the tip of the cone emerges in the near horizon limit. It is simply $\mathrm{AdS}_5 \times Y_5$, which still has the $\mathrm{SO}(2,4)$ conformal symmetry manifest in the geometry, even though the R-symmetry group which must be contained in the isometries of Y_5 will in general be smaller than SU(4).

In order to find gauge theories with $\mathcal{N} = 1$ superconformal symmetry the Ricci-flat cone must be a Calabi-Yau 3-fold [44, 45] whose base Y_5 is called a Sasaki-Einstein space. Among the simplest examples of these is $Y_5 = T^{1,1}$. The corresponding Calabi-Yau cone is called the conifold.

The conifold is a singular non-compact Calabi-Yau three-fold [46]. Its importance arises from the fact that the generic singularity in a Calabi-Yau three-fold locally looks like the conifold, described by the quadratic equation in $\mathbb{C}^4$:

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0 \ . \qquad (1.15)$$

This homogeneous equation defines a real cone over the 5-dimensional manifold $T^{1,1}$.

The topology of $T^{1,1}$ can be shown to be $\mathrm{S}^2 \times \mathrm{S}^3$ and its metric [47] is

$$\begin{aligned} ds^2_{T^{1,1}} = & \frac{1}{9}\left(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2\right)^2 \\ & + \frac{1}{6}(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2) + \frac{1}{6}(d\theta_2^2 + \sin^2\theta_2 d\phi_2^2) \ . \end{aligned} \tag{1.16}$$

$T^{1,1}$ is a homogeneous space, and can be described as the coset $\mathrm{SU}(2) \times \mathrm{SU}(2)/\mathrm{U}(1)$. The metric on the cone is then $ds_6^2 = dr^2 + r^2 ds^2_{T^{1,1}}$.

Symmetries and field theory interpretation. We can introduce different complex coordinates on the conifold, a_i and b_j, as follows:

$$\begin{aligned} Z = & \begin{pmatrix} z_3 + iz_4 & z_1 - iz_2 \\ z_1 + iz_2 & -z_3 + iz_4 \end{pmatrix} = \begin{pmatrix} a_1 b_1 & a_1 b_2 \\ a_2 b_1 & a_2 b_2 \end{pmatrix} \\ = & r^{\frac{3}{2}} \begin{pmatrix} -c_1 s_2 \, e^{\frac{i}{2}(\psi + \phi_1 - \phi_2)} & c_1 c_2 \, e^{\frac{i}{2}(\psi + \phi_1 + \phi_2)} \\ -s_1 s_2 \, e^{\frac{i}{2}(\psi - \phi_1 - \phi_2)} & s_1 c_2 \, e^{\frac{i}{2}(\psi - \phi_1 + \phi_2)} \end{pmatrix} , \end{aligned} \tag{1.17}$$

where $c_i = \cos\frac{\theta_i}{2},\ s_i = \sin\frac{\theta_i}{2}$ (see [46] for more details on the z and angular coordinates). The equation defining the conifold is now $\det Z = 0$.

The a_i, b_j coordinates above will be of particular interest to us because the symmetries of the conifold are most apparent in this basis. The conifold equation has $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ symmetry since under these symmetry transformations,

$$\det L Z R^\dagger = \det e^{i\alpha} Z = 0. \tag{1.18}$$

This is also a symmetry of the metric presented above where each SU(2) acts on θ_i, ϕ_i, ψ (thought of as Euler angles on S^3) while the U(1) acts by shifting ψ. This symmetry can be identified with $\mathrm{U}(1)_R$, the R-symmetry of the dual gauge theory, in the conformal case. The action of the $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)_R$ symmetry on a_i, b_j defined in (1.17) is given by

$$\text{SU}(2)\times\text{SU}(2)\ \text{symmetry}: \quad \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \to L \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} , \tag{1.19}$$

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \to R \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} , \tag{1.20}$$

$$\text{R-symmetry}: \quad (a_i, b_j) \to e^{i\frac{\alpha}{2}} (a_i, b_j) \ , \tag{1.21}$$

i.e. a and b transform as $(1/2, 0)$ and $(0, 1/2)$ under $\text{SU}(2)\times\text{SU}(2)$ and with R-charge $1/2$ each. We can thus describe the singular conifold as the manifold parametrized by a and b, but from (1.17), we see that there is some redundancy in the a, b coordinates. Namely, the transformation

$$a_i \to \lambda\, a_i \ , \quad b_j \to \frac{1}{\lambda}\, b_j \ , \quad (\lambda \in \mathbb{C}) \ , \tag{1.22}$$

results in the same z coordinates in (1.17). Thus we impose the additional constraint $|a_1|^2 + |a_2|^2 - |b_1|^2 - |b_2|^2 = 0$ to fix the magnitude in the above transformation. To account for the remaining phase, we describe the singular conifold as the quotient of the a, b space with the above constraint by the relation $a \sim e^{i\alpha} a, b \sim e^{-i\alpha} b$.

The importance of the coordinates a_i, b_j is that in the gauge theory on D3-branes at the tip of the conifold they are promoted to chiral superfields. The low-energy gauge theory on N D3-branes is an $\mathcal{N} = 1$ supersymmetric $\text{SU}(N)\times\text{SU}(N)$ gauge theory with bifundamental chiral superfields A_i, B_j $(i, j = 1, 2)$ in the $(\mathbf{N}, \overline{\mathbf{N}})$ and $(\overline{\mathbf{N}}, \mathbf{N})$ representations of the gauge groups, respectively [44, 45]. The superpotential for this gauge theory is

$$W \sim \text{Tr}\det A_i B_j = \text{Tr}\ (A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1) \ . \tag{1.23}$$

The continuous global symmetries of this theory are $\text{SU}(2)\times\text{SU}(2)\times\text{U}(1)_B\times\text{U}(1)_R$, where the SU(2) factors act on A_i and B_j respectively, $\text{U}(1)_B$ is a baryonic symmetry under which the A_i and B_j have opposite charges, and $\mathrm{U}(1)_R$ is the R-symmetry with charges of the same sign $R_A = R_B = \frac{1}{2}$. This assignment ensures that W is marginal, and one can also show that the gauge couplings do not run. Hence this theory is superconformal for all values of gauge couplings and superpotential coupling [44, 45].

Resolution of the conifold. A simple way to understand the resolution of the conifold is to deform the modulus constraint above into

$$|b_1|^2 + |b_2|^2 - |a_1|^2 - |a_2|^2 = u^2 \ , \tag{1.24}$$

where u is a real parameter which controls the resolution. The resolution corresponds to a blow up of the S^2 at the bottom of the conifold. In the dual gauge theory turning on u corresponds to a particular choice of vacuum [48]. After promoting the a, b fields to the bifundamental chiral superfields of the dual gauge theory, we can define the operator $\mathcal{U}$ as

$$\mathcal{U} = \frac{1}{N}\mathrm{Tr}(B_1^\dagger B_1 + B_2^\dagger B_2 - A_1^\dagger A_1 - A_2^\dagger A_2) \ . \tag{1.25}$$

Thus, the singular conifolds correspond to gauge theory vacua where $\langle\mathcal{U}\rangle = 0$, while the warped resolved conifolds correspond to vacua where $\langle\mathcal{U}\rangle \neq 0$. In the latter case, some VEVs for the bifundamental fields A_i, B_j must be present. Since these fields are charged under the $U(1)_B$ symmetry, the warped resolved conifolds correspond to vacua where this symmetry is broken [48]. In this thesis we shall not discuss the resolved conifold further (except in the context of the baryonic branch), but we shall be interested instead in the deformation of the conifold singularity, which is the subject of the following subsection.

1.2.2 Deformation of the Conifold

We have seen above that the singularity of the cone over $T^{1,1}$ can be replaced by an S^2 through resolving the conifold (1.15) as in (1.24). An alternative supersymmetric blow-up, which replaces the singularity by an S^3, is the deformed conifold [46]

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 = \varepsilon^2 \ . \tag{1.26}$$

To achieve the deformation, one needs to turn on M units of RR 3-form flux. This modifies the dual gauge theory to $\mathcal{N} = 1$ supersymmetric $SU(N) \times SU(N+M)$ theory with chiral superfields A_1, A_2 in the $(\mathbf{N}, \overline{\mathbf{N}+\mathbf{M}})$ representation, and B_1, B_2 in the $(\overline{\mathbf{N}}, \mathbf{N}+\mathbf{M})$ representation.

The necessary 3-form field strength linking the three-cycle of the conifold can be turned on by wrapping M D5-branes on the S^2 at the tip of the cone. They are often referred to as fractional branes. Since their number is dual to the difference in the ranks of the two gauge groups, the deformation, unlike the resolution, cannot be achieved in the context of the $SU(N) \times SU(N)$ gauge theory.

The KS solution. The 10-dimensional metric takes the following form [49]:

$$ds_{10}^2 = h^{-1/2}(\tau)\, \eta_{\mu\nu} dx^\mu dx^\nu + h^{1/2}(\tau)\, ds_6^2 \ , \tag{1.27}$$

where $h(\tau)$ is a warp factor to be discussed below, and ds_6^2 is the Calabi-Yau metric of the deformed conifold:

$$\begin{aligned} ds_6^2 = \frac{\varepsilon^{4/3}}{2} K(\tau) \Big[& \sinh^2\left(\frac{\tau}{2}\right) \left[(g^1)^2 + (g^2)^2\right] \\ & + \cosh^2\left(\frac{\tau}{2}\right) \left[(g^3)^2 + (g^4)^2\right] + \frac{1}{3K^3(\tau)} \left[d\tau^2 + (g^5)^2\right] \Big] \ , \end{aligned} \tag{1.28}$$

with $K(\tau) \equiv (\sinh\tau \cosh\tau - \tau)^{1/3} / \sinh\tau$.

For $\tau \gg 1$ we may introduce another radial coordinate r defined by

$$r^2 = \frac{3}{2^{5/3}} \varepsilon^{4/3} e^{2\tau/3} \ , \tag{1.29}$$

and in terms of this coordinate we find $ds_6^2 \to dr^2 + r^2 ds^2_{T^{1,1}}$.

The basis one-forms g^i in terms of which this metric is diagonal are defined by

$$g^1 \equiv \frac{e_2 - \epsilon_2}{\sqrt{2}} \ , \quad g^2 \equiv \frac{e_1 - \epsilon_1}{\sqrt{2}} \ , \tag{1.30}$$

$$g^3 \equiv \frac{e_2 + \epsilon_2}{\sqrt{2}} \ , \quad g^4 \equiv \frac{e_1 + \epsilon_1}{\sqrt{2}} \ , \tag{1.31}$$

$$g^5 \equiv \epsilon_3 + \cos\theta_1 d\phi_1 \ , \tag{1.32}$$

where the e_i are one-forms on S^2

$$e_1 \equiv d\theta_1 \ , \qquad e_2 \equiv -\sin\theta_1 d\phi_1 \ , \tag{1.33}$$

and the ϵ_i a set of one-forms on S^3

$$\epsilon_1 \equiv \sin\psi \sin\theta_2 d\phi_2 + \cos\psi d\theta_2 \ , \tag{1.34}$$

$$\epsilon_2 \equiv \cos\psi \sin\theta_2 d\phi_2 - \sin\psi d\theta_2 \ , \tag{1.35}$$

$$\epsilon_3 \equiv d\psi + \cos\theta_2 d\phi_2 \ . \tag{1.36}$$

The NSNS two-form is given by

$$B_2 = \frac{g_s M\alpha'}{2} \frac{\tau\coth\tau - 1}{\sinh\tau} \left[\sinh^2\left(\frac{\tau}{2}\right) g^1 \wedge g^2 + \cosh^2\left(\frac{\tau}{2}\right) g^3 \wedge g^4\right] \ , \tag{1.37}$$

and the RR fluxes are most compactly written as

$$F_3 = \frac{M\alpha'}{2}\left\{g^3 \wedge g^4 \wedge g^5 + d\left[\frac{\sinh\tau - \tau}{2\sinh\tau}(g^1 \wedge g^3 + g^2 \wedge g^4)\right]\right\}, \tag{1.38}$$

$$\tilde{F}_5 = dC_4 + B_2 \wedge F_3 = (1 + *)\,(B_2 \wedge F_3) \ . \tag{1.39}$$

Note that the complex three-form field of this BPS supergravity solution is imaginary self-dual:

$$*_6 G_3 = iG_3 \ , \qquad\qquad G_3 = F_3 - \frac{i}{g_s} H_3 \ , \tag{1.40}$$

where $*_6$ denotes the Hodge dual with respect to the unwarped metric ds_6^2. This guarantees that the dilaton is constant, and we set $\phi = 0$.

The above expressions for the NSNS- and RR-forms follow by making a simple ansatz consistent with the symmetries of the problem, and solving a system of differential equations, which owing to the supersymmetry of the problem are only first order [49]. The warp factor is then found to be completely determined up to an additive constant, which is fixed by demanding that it go to zero at large τ:

$$h(\tau) = (g_s M\alpha')^2 2^{2/3} \varepsilon^{-8/3} I(\tau) \ , \tag{1.41}$$

$$I(\tau) \equiv 2^{1/3} \int_\tau^\infty dx \frac{x \coth x - 1}{\sinh^2 x} (\sinh x \cosh x - x)^{1/3} \ . \tag{1.42}$$

For small τ the warp factor approaches a finite constant since $I(0) \approx 0.71805$. This implies confinement because the chromo-electric flux tube, described by a fundamental string at $\tau = 0$, has tension

$$T_s = \frac{1}{2\pi\alpha' \sqrt{h(0)}} \ . \tag{1.43}$$

The KS solution [49] is SU(2) $\times$ SU(2) symmetric and the expressions above can be written in an explicitly SO(4) invariant way. It also possesses a $\mathbb{Z}_2$ symmetry $\mathcal{I}$, which exchanges (θ_1, ϕ_1) with (θ_2, ϕ_2) accompanied by the action of $-I$ of SL(2, $\mathbb{Z}$), changing the signs of the three-form fields.

Examining the metric ds_6^2 for $\tau = 0$ we see that it degenerates into

$$d\Omega_3^2 = \frac{1}{2} \varepsilon^{4/3} (2/3)^{1/3} \left[(g^3)^2 + (g^4)^2 + \frac{1}{2} (g^5)^2 \right] \ , \tag{1.44}$$

which is the metric of a round S^3, while the S^2 spanned by the other two angular coordinates, and fibered over the S^3, shrinks to zero size. In the ten-dimensional metric

(1.27) this appears multiplied by a factor of $h^{1/2}(\tau)$, and thus the radius squared of the three-sphere at the tip of the conifold is of order $g_s M\alpha'$. Hence for $g_s M$ large, the curvature of the S^3, and in fact everywhere in this manifold, is small and the supergravity approximation reliable.

The field theory interpretation of the KS solution exhibits some unusual features. The deformation breaks the conformal symmetry of the singular conifold and thus the gauge couplings now run. At a certain point along the RG flow, the coupling of the higher rank $SU(N+M)$ group diverges and one is forced to perform a Seiberg duality transformation [50].

Interchanging the two gauge groups, the theory becomes $SU(\tilde{N}) \times SU(\tilde{N}+M)$ with the same bifundamental field content and superpotential, but with a reduced number of colors $\tilde{N} = N - M$. Since this otherwise looks exactly like field theory we started with, the procedure can be iterated many times, gradually reducing the rank of the gauge groups along the RG flow. For a careful field theoretic discussion of this quasi-periodic RG flow, see [12]. In particular, if $N = (k+1)M$ for some integer k, the so-called duality cascade stops after k steps, resulting in an $SU(M) \times SU(2M)$ gauge theory. This IR field theory exhibits a number of interesting effects reflected in the dual supergravity background, which include confinement and chiral symmetry breaking.

The KT solution. Let us briefly note here some formulas describing the KT solution [51], which corresponds to the large τ limit of the more general KS solution. For simplicity we take $g_s = \alpha' = 1$, $M = 2$ and $N = 0$. In terms of the radial coordinate $r \sim \varepsilon^{2/3} e^{\tau/3}$ the KT background is given by

$$ds^2 = \frac{1}{\sqrt{h(r)}}\,\eta_{\mu\nu}dx^\mu dx^\nu + \sqrt{h(r)}(dr^2 + r^2 ds^2_{T^{11}})\,, \tag{1.45}$$

$$H_3 = \frac{3}{r}\,dr \wedge \omega_2\,, \qquad B_2 \;=\; 3\,\log\frac{r}{r_*}\,\omega_2\,, \qquad F_3 = \omega_3\,, \tag{1.46}$$

$$\tilde{F}_5 = (1+*)\,B_2 \wedge F_3 \;=\; 3\log\frac{r}{r_*}\left[\omega_2 \wedge \omega_3 - \frac{54}{h^2 r^5}\,d^4x \wedge dr\right]. \tag{1.47}$$

The warp factor differs from the AdS case by an additional logarithmic term

$$h(r) = \frac{81}{8r^4}\left(1 + 4\log\frac{r}{r_*}\right) \ , \tag{1.48}$$

and the metric of the base of the conifold can be expressed concisely as

$$ds^2_{T^{11}} = \frac{1}{9}(g^5)^2 + \frac{1}{6}\sum_{i=1}^{4}(g^i)^2 \, . \tag{1.49}$$

The volume form is given by

$$\text{vol} = \frac{\sqrt{h}r^5}{54}\, d^4x \wedge \omega_2 \wedge \omega_3 \wedge dr \, . \tag{1.50}$$

In the expressions above we have introduced two harmonic forms,

$$\omega_2 = \frac{1}{2}(g^1 \wedge g^2 + g^3 \wedge g^4) = \frac{1}{2}(\sin\theta_1 d\theta_1 \wedge d\phi_1 - \sin\theta_2 d\theta_2 \wedge d\phi_2) \ , \tag{1.51}$$

and $\omega_3 = \omega_2 \wedge g^5$.

1.2.3 Massless Modes of the Warped Throat

As we shall see below, the U(1) baryonic symmetry of the warped deformed conifold is in fact spontaneously broken, since baryonic operators acquire expectation values. The corresponding Goldstone boson is a massless pseudoscalar supergravity fluctuation which has non-trivial monodromy around D-strings at the bottom of the warped deformed conifold [52, 53]. Like fundamental strings they fall to the bottom of the conifold (corresponding to the IR of the field theory), where they have non-vanishing tension. But while F-strings are dual to confining strings, D-strings are interpreted as global solitonic strings in the dual cascading $\text{SU}(M(k+1)) \times \text{SU}(Mk)$ gauge theory.

Thus the warped deformed conifold naturally incorporates a supergravity description of the supersymmetric Goldstone mechanism. Below we review the supergravity dual of a pseudoscalar Goldstone boson, as well as its superpartner, a massless scalar glueball [52, 53]. In the gauge theory they correspond to fluctuations in the phase and magnitude of the baryonic condensates, respectively.

The Goldstone mode. A D1-brane couples to the three-form field strength F_3, and therefore we expect a four-dimensional pseudoscalar $p(x)$, defined so that $*_4 dp = \delta F_3$, to experience monodromy around the D-string.

The following ansatz for a linear perturbation of the KS solution

$$\begin{aligned} \delta F_3 &= *_4 dp + f_2(\tau)\, dp \wedge dg^5 + f_2'(\tau)\, dp \wedge d\tau \wedge g^5 \ , \\ \delta \tilde{F}_5 &= (1+*)\delta F_3 \wedge B_2 = (*_4 dp - \frac{\varepsilon^{4/3}}{6K^2(\tau)} h(\tau)\, dp \wedge d\tau \wedge g^5) \wedge B_2 \ , \end{aligned} \tag{1.52}$$

where $f_2' = df_2/d\tau$, falls within the general class of supergravity backgrounds discussed by Papadopoulos and Tseytlin [54]. The metric, dilaton and B_2 field remain unchanged. This can be shown to satisfy the linearized supergravity equations [52, 53], provided that $d *_4 dp = 0$, i.e. $p(x)$ is massless, and $f_2(\tau)$ satisfies

$$-\frac{d}{d\tau}[K^4 \sinh^2 \tau f_2'] + \frac{8}{9K^2} f_2 = \frac{(g_s M \alpha')^2}{3\varepsilon^{4/3}} (\tau \coth \tau - 1) \left(\coth \tau - \frac{\tau}{\sinh^2 \tau} \right) \ . \tag{1.53}$$

The normalizable solution of this equation is given by [52, 53]

$$f_2(\tau) = -\frac{2c}{K^2 \sinh^2 \tau} \int_0^\tau dx\, h(x) \sinh^2 x \,, \tag{1.54}$$

where $c \sim \varepsilon^{4/3}$. We find that $f_2 \sim \tau$ for small τ, and $f_2 \sim \tau e^{-2\tau/3}$ for large τ.

As we have remarked above, the U(1) baryon number symmetry acts as $A_k \to e^{i\alpha} A_k$, $B_j \to e^{-i\alpha} B_j$. The massless gauge field in AdS_5 dual to the baryon number current originates from the RR 4-form potential $\delta C_4 \sim \omega_3 \wedge \tilde{A}$ [44, 55].

The zero-mass pseudoscalar glueball arises from the spontaneous breaking of the global $\mathrm{U}(1)_B$ symmetry [56], as seen from the form of $\delta\tilde{F}_5$ in (1.52), which contains a term $\sim \omega_3 \wedge dp \wedge d\tau$ that leads us to identify $\tilde{A} \sim dp$.

If N is an integer multiple of M, the last step of the cascade leads to a $\mathrm{SU}(2M) \times \mathrm{SU}(M)$ gauge theory coupled to bifundamental fields A_i, B_j (with $i, j = 1, 2$). If the $\mathrm{SU}(M)$ gauge coupling were turned off, we would find an $\mathrm{SU}(2M)$ gauge theory coupled to $2M$ flavors. In this $N_f = N_c$ case, in addition to the usual mesonic branch, there exists a baryonic branch of the quantum moduli space [57]. This is important for the gauge theory interpretation of the KS background [49, 56]. Indeed, in addition to mesonic operators $(N_{ij})^\alpha_\beta \sim (A_i B_j)^\alpha_\beta$, the IR gauge theory has baryonic operators invariant under the $\mathrm{SU}(2M) \times \mathrm{SU}(M)$ gauge symmetry, as well as the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ global symmetry rotating A_i, B_j:

$$\mathcal{A} \sim \epsilon_{\alpha_1\alpha_2\ldots\alpha_{2M}} (A_1)^{\alpha_1}_1 (A_1)^{\alpha_2}_2 \ldots (A_1)^{\alpha_M}_M (A_2)^{\alpha_{M+1}}_1 (A_2)^{\alpha_{M+2}}_2 \ldots (A_1)^{\alpha_{2M}}_M \ , \tag{1.55}$$

$$\mathcal{B} \sim \epsilon_{\alpha_1\alpha_2\ldots\alpha_{2M}} (B_1)^{\alpha_1}_1 (B_1)^{\alpha_2}_2 \ldots (B_1)^{\alpha_M}_M (B_2)^{\alpha_{M+1}}_1 (B_2)^{\alpha_{M+2}}_2 \ldots (B_1)^{\alpha_{2M}}_M \ . \tag{1.56}$$

These operators contribute an additional term to the usual mesonic superpotential:

$$W = \lambda (N_{ij})^\alpha_\beta (N_{k\ell})^\beta_\alpha \epsilon^{ik} \epsilon^{j\ell} + X \left(\det[(N_{ij})^\alpha_\beta] - \mathcal{A}\mathcal{B} - \Lambda^{4M}_{2M} \right) \ , \tag{1.57}$$

where X can be understood as a Lagrange multiplier. The supersymmetry-preserving vacua include the baryonic branch:

$$X = 0 \ ; \quad N_{ij} = 0 \ ; \quad \mathcal{A}\mathcal{B} = -\Lambda^{4M}_{2M} \ , \tag{1.58}$$

where the SO(4) global symmetry rotating A_i, B_j is unbroken. In contrast, this global symmetry is broken along the mesonic branch $N_{ij} \neq 0$. Since the supergravity background of [49] is SO(4) symmetric, it is natural to assume that the dual of this background lies on the baryonic branch of the cascading theory. The expectation values of the baryonic operators spontaneously break the U(1) baryon number symmetry $A_k \to e^{i\alpha} A_k$, $B_j \to e^{-i\alpha} B_j$. The KS background corresponds to a vacuum where $|\mathcal{A}| = |\mathcal{B}| = \Lambda_{2M}^{2M}$, which is invariant under the exchange of the A's with the B's accompanied by charge conjugation in both gauge groups. This gives a field theory interpretation to the $\mathcal{I}$ symmetry of the warped deformed conifold background. As noted in [56], the baryonic branch has complex dimension one, and it can be parametrized by ζ as follows

$$\mathcal{A} = i\zeta \Lambda_{2M}^{2M} , \qquad \mathcal{B} = \frac{i}{\zeta} \Lambda_{2M}^{2M} . \tag{1.59}$$

The pseudoscalar Goldstone mode must correspond to changing ζ by a phase, since this is precisely what a $\mathrm{U}(1)_B$ symmetry transformation does.

Thus the non-compact warped deformed conifold exhibits a supergravity dual of the Goldstone mechanism due to breaking of the global $\mathrm{U}(1)_B$ symmetry [52, 53, 56]. On the other hand, if one considered a warped deformed conifold throat embedded in a flux compactification, $\mathrm{U}(1)_B$ would be gauged, the Goldstone boson $p(x)$ would combine with the U(1) gauge field to form a massive vector, and therefore in this situation we would find a manifestation of the supersymmetric Higgs mechanism.

The scalar zero-mode. By supersymmetry the massless pseudoscalar is part of a massless $\mathcal{N} = 1$ chiral multiplet. and therefore there must also be a massless scalar mode and corresponding Weyl fermion, with the scalar corresponding to changing ζ by a positive real factor. This scalar zero-mode comes from a metric perturbation that mixes with the NSNS 2-form potential.

The warped deformed conifold preserves the $\mathbb{Z}_2$ interchange symmetry $\mathcal{I}$. However, the pseudoscalar mode we found breaks this symmetry: from the form of the perturbations (1.52) we see that δF_3 is even under the interchange of (θ_1, ϕ_1) with (θ_2, ϕ_2), while F_3 is odd; similarly $\delta \tilde{F}_5$ is odd while $\tilde{F}_5$ is even. Therefore, the scalar mode must also break the $\mathcal{I}$ symmetry because in the field theory it breaks the symmetry between the expectation values of $|\mathcal{A}|$ and of $|\mathcal{B}|$. The necessary translationally invariant perturbation that preserves the SO(4) but breaks the $\mathcal{I}$ symmetry is given by the following variation of the NSNS 2-form and the metric:

$$\delta B_2 = \chi(\tau)\, dg^5 \ , \qquad \delta G_{13} = \delta G_{24} = \lambda(\tau) \ , \tag{1.60}$$

where, for example $\delta G_{13} = \lambda(\tau)$ means adding $2\lambda(\tau)\, g^{(1}g^{3)}$ to ds^2_{10}. To see that these components of the metric break the $\mathcal{I}$ symmetry, we note that

$$(e_1)^2 + (e_2)^2 - (\epsilon_1)^2 - (\epsilon_2)^2 = g^1g^3 + g^3g^1 + g^2g^4 + g^4g^2 \ . \tag{1.61}$$

Defining $\lambda(\tau) = h^{1/2}K \sinh\tau\, z(\tau)$ one finds [52, 53] that all the linearized supergravity equations are satisfied provided that

$$\frac{\left((K\sinh\tau)^2\, z'\right)'}{(K\sinh\tau)^2} = \left(2 + \frac{8}{9}\frac{1}{K^6} - \frac{4}{3}\frac{\cosh\tau}{K^3}\right)\frac{z}{\sinh^2\tau} \ , \tag{1.62}$$

and

$$\chi' = \frac{1}{2} g_s M z(\tau)\, \frac{\sinh 2\tau - 2\tau}{\sinh^2\tau} \ . \tag{1.63}$$

The solution of (1.62) for the zero-mode profile is remarkably simple:

$$z(\tau) = s\, \frac{(\tau\coth\tau - 1)}{(\sinh 2\tau - 2\tau)^{1/3}} \ , \tag{1.64}$$

where s is a constant. Like the pseudoscalar perturbation, the large τ asymptotic is again $z \sim \tau e^{-2\tau/3}$. We note that the metric perturbation has the very simple form $\delta G_{13} \sim h^{1/2}(\tau\coth\tau - 1)$. The perturbed metric $d\tilde{s}^2_6$ differs from the metric of the deformed conifold (1.28) by terms $\sim (\tau\coth\tau - 1)(g^1g^3 + g^3g^1 + g^2g^4 + g^4g^2)$, which grow as $\ln r$ in the asymptotic radial variable r.

The scalar zero-mode is actually an exact modulus; there is a one-parameter family of supersymmetric solutions which break the $\mathcal{I}$ symmetry but preserve the SO(4) (an ansatz with these properties was found in [54], and its linearization agrees with (1.60)). These backgrounds, the resolved warped deformed conifolds, will be reviewed below. We add the word resolved because both the resolution of the conifold, which is a Kähler deformation, and these resolved warped deformed conifolds break the $\mathcal{I}$ symmetry. In the dual gauge theory turning on this mode corresponds to the transformation $\mathcal{A} \to (1+s)\mathcal{A}$, $\mathcal{B} \to (1+s)^{-1}\mathcal{B}$ on the baryonic branch. Therefore, s is dual to the $\mathcal{I}$-breaking parameter of the resolved warped deformed conifold.

The presence of these massless modes is a further indication that the infrared dynamics of the cascading $\mathrm{SU}(M(k+1)) \times \mathrm{SU}(Mk)$ gauge theory, whose supergravity dual is the warped deformed conifold, is richer than that of the pure glue $\mathcal{N} = 1$ supersymmetric $\mathrm{SU}(M)$ theory. The former incorporates a Goldstone supermultiplet, which appears due to the $\mathrm{U}(1)_B$ symmetry breaking, as well as solitonic strings dual to the D-strings placed at $\tau = 0$ in the supergravity background.

In Chapter 3 we will discuss massive modes of the warped throat, which arise from generalizing the massless scalar and pseudoscalar modes by adding 4-dimensional momentum, as well as some massive vector excitations related to them by supersymmetry.

1.2.4 The Baryonic Branch

Since the global baryon number symmetry $\mathrm{U}(1)_B$ is broken by expectation values of baryonic operators, the spectrum contains the Goldstone boson found above. The zero-momentum mode of the scalar superpartner of the Goldstone mode leads to a Lorentz-invariant deformation of the background which describes a small motion along the baryonic branch. In this subsection we shall extend the discussion from linearized perturbations around the warped deformed conifold solution to finite deformations, and describe the supergravity backgrounds dual to the complete baryonic branch. These are the resolved warped deformed conifolds, which preserve the SO(4) global symmetry but break the discrete $\mathcal{I}$ symmetry of the warped deformed conifold.

The full set of first-order equations necessary to describe the entire moduli space of supergravity backgrounds dual to the baryonic branch was derived and solved numerically in [58] (for further discussion, see [59]). This continuous family of supergravity solutions is parameterized by the modulus of ζ (the phase of ζ is not manifest in these backgrounds). The corresponding metric can be written in the form of the Papadopoulos-Tseytlin ansatz [54] in the string frame:

$$ds^2 = h^{-1/2}\eta_{\mu\nu}dx^\mu dx^\nu + e^x ds^2_{\mathcal{M}} = h^{-1/2}dx^2_{1,3} + \sum_{i=1}^{6} G_i^2 \ , \tag{1.65}$$

where

$$G_1 \equiv e^{(x+g)/2}\, e_1 \ , \qquad G_2 \equiv \frac{\cosh\tau + a}{\sinh\tau}\, e^{(x+g)/2}\, e_2 + \frac{e^g}{\sinh\tau}\, e^{(x-g)/2}\,(\epsilon_2 - a e_2) \ , \tag{1.66}$$

$$G_3 \equiv e^{(x-g)/2}\,(\epsilon_1 - a e_1) \ , \quad G_4 \equiv \frac{e^g}{\sinh\tau}\, e^{(x+g)/2}\, e_2 - \frac{\cosh\tau + a}{\sinh\tau}\, e^{(x-g)/2}\,(\epsilon_2 - a e_2) \ , \tag{1.67}$$

$$G_5 \equiv e^{x/2}\, v^{-1/2} d\tau \ , \qquad G_6 \equiv e^{x/2}\, v^{-1/2} g^5 \ . \tag{1.68}$$

These one-forms describe a basis that rotates as we move along the radial direction, and are particularly convenient since they allow us to write down very simple expressions for the holomorphic $(3,0)$ form

$$\Omega = (G_1 + iG_2) \wedge (G_3 + iG_4) \wedge (G_5 + iG_6) \ , \tag{1.69}$$

and the fundamental $(1,1)$ form

$$J = \frac{i}{2}\Big[(G_1+iG_2)\wedge(G_1-iG_2)+(G_3+iG_4)\wedge(G_3-iG_4)+(G_5+iG_6)\wedge(G_5-iG_6)\Big] \ . \tag{1.70}$$

While in the warped deformed conifold case there was a single warp factor $h(\tau)$, now we find several additional functions $x(\tau), g(\tau), a(\tau), v(\tau)$. The warp factor $h(\tau)$ is deformed away from (1.41) when $|\zeta| \neq 1$.

The background also contains the fluxes

$$B_2 = h_1\,(\epsilon_1 \wedge \epsilon_2 + e_1 \wedge e_2) + \chi\,(e_1 \wedge e_2 - \epsilon_1 \wedge \epsilon_2) + \; h_2\,(\epsilon_1 \wedge e_2 - \epsilon_2 \wedge e_1)\ , \quad (1.71)$$

$$F_3 = -\frac{1}{2} g_5 \wedge \left[\epsilon_1 \wedge \epsilon_2 + e_1 \wedge e_2 - b\,(\epsilon_1 \wedge e_2 - \epsilon_2 \wedge e_1)\right] - \frac{1}{2}\, d\tau \wedge \left[b'\,(\epsilon_1 \wedge e_1 + \epsilon_2 \wedge e_2)\right]\ , \quad (1.72)$$

$$\tilde{F}_5 = \mathcal{F}_5 + *_{10}\mathcal{F}_5\ , \qquad \text{with} \quad \mathcal{F}_5 = -(h_1 + b\,h_2)\, e_1 \wedge e_2 \wedge \epsilon_1 \wedge \epsilon_2 \wedge \epsilon_3\ , \quad (1.73)$$

parameterized by functions $h_1(\tau), h_2(\tau), b(\tau)$ and $\chi(\tau)$. In addition, since the 3-form flux is not imaginary self-dual for $|\zeta| \neq 1$ (i.e. $*_6 G_3 \neq i G_3$), the dilaton ϕ now also depends on the radial coordinate τ.

The functions a and v satisfy a system of coupled first order differential equations [58] whose solutions are known in closed form only in the warped deformed conifold and the Chamseddine-Volkov-Maldacena-Nunez (CVMN) [60, 61, 62] limits. All other functions $h, x, g, h_1, h_2, b, \chi, \phi$ are unambiguously determined by $a(\tau)$ and $v(\tau)$ through the relations

$$h = \gamma\, U^{-2}\left(e^{-2\phi} - 1\right)\ , \qquad \gamma = 2^{10/3}(g_s M \alpha')^2 \varepsilon^{-8/3}\ , \quad (1.74)$$

$$e^{2x} = \frac{(b\,C - 1)^2}{4(a\,C - 1)^2}\; e^{2g + 2\phi}(1 - e^{2\phi})\ , \quad (1.75)$$

$$e^{2g} = -1 - a^2 + 2a\,C\ , \qquad b = \frac{\tau}{S}\ , \quad (1.76)$$

$$h_2 = \frac{e^{2\phi}(b\,C - 1)}{2S}\ , \qquad h_1 = -h_2\,C\ , \quad (1.77)$$

$$\chi' = a(b - C)(a\,C - 1)\, e^{2(\phi - g)}\ , \quad (1.78)$$

$$\phi' = \frac{(C - b)\,(a\,C - 1)^2}{(b\,C - 1)\,S}\; e^{-2g}\ , \quad (1.79)$$

where $C \equiv -\cosh\tau,\ S \equiv -\sinh\tau$, and we require $\phi(\infty) = 0$. In writing these equations we have specialized to the baryonic branch of the cascading gauge theory by imposing appropriate boundary conditions at infinity [59], which guarantee that the background asymptotes to the warped deformed conifold solution [51]. The full two-parameter family of SU(3) structure backgrounds discussed in [58] also includes the CVMN solution [60, 61, 62], which however is characterized by linear dilaton asymptotics that are qualitatively different from the backgrounds discussed here. The baryonic branch family of supergravity solutions is labelled by one real resolution parameter U [59]. While the leading asymptotics of all supergravity backgrounds dual to the baryonic branch are identical to those of the warped deformed conifold, terms subleading at large τ depend on U. As required, this family of supergravity solutions preserves the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ symmetry, but for $U \neq 0$ breaks the $\mathbb{Z}_2$ symmetry $\mathcal{I}$.

On the baryonic branch we can consider a transformation that takes ζ into ζ^{-1}, or equivalently U into $-U$. This transformation leaves $v(\tau)$ invariant and changes $a(\tau)$ as follows

$$a \rightarrow -\frac{a}{1+2a\cosh\tau} \ . \tag{1.80}$$

It is straightforward to check that $a\, e^{-g}$ is invariant while $(1+a\cosh\tau)\, e^{-g}$ changes sign. This transformation also exchanges $e^{g}+a^2e^{-g}$ with e^{-g} and therefore it is equivalent to the exchange of (θ_1,ϕ_1) and (θ_2,ϕ_2) involved in the $\mathcal{I}$ symmetry.

The baryonic condensates can be calculated on the string theory side of the duality by identifying the Euclidean D5-branes wrapped over the resolved warped deformed conifold, with appropriate gauge fields turned on, as the objects dual to the baryonic operators in the sense of gauge/string duality. The details of this identification will be the subject of Chapter 4.

1.3 Superconformal Chern-Simons Theories and an AdS_4/CFT_3 Duality

Let us now turn to an example of the AdS/CFT correspondence involving 2+1 dimensional field theory. A long-standing open problem in this area had been to find the gauge theory dual to the near-horizon geometry of a stack of M2-branes, the $AdS_4 \times S^7$ background of M-theory.

Significant progress towards this goal was made in the work of Bagger and Lambert [63, 64, 65], and the closely related work of Gustavsson [66], who succeeded in finding a 2+1 dimensional superconformal Chern-Simons theory with the maximal $\mathcal{N} = 8$ supersymmetry and manifest SO(8) R-symmetry. These papers were inspired in part by the ideas of [67, 68], whose original motivation was the search for a theory describing coincident M2-branes. An interesting clue emerged in [69, 70] where it was shown that, for a specially chosen level of the Chern-Simons gauge theory, its moduli space coincides with that of a pair of M2-branes at the $\mathbb{R}^8/\mathbb{Z}_2$ singularity. The $\mathbb{Z}_2$ acts by reflection of all 8 coordinates and therefore does not spoil the SO(8) symmetry. However, initial attempts to match the moduli space of the Chern-Simons gauge theory for arbitrary quantized level k with that of M2-branes led to a number of puzzles [69, 70, 71]. These puzzles were resolved by a very interesting modification of the Bagger-Lambert-Gustavsson (BLG) theory proposed by Aharony, Bergman, Jafferis and Maldacena (ABJM) [72] which, in particular, allows for a generalization to an arbitrary number of M2-branes.

The original BLG theory is a particular example of a Chern-Simons gauge theory with gauge group SO(4), but the Chern-Simons term has a somewhat unconventional form. However, van Raamsdonk [71] rewrote the BLG theory as an $SU(2)\times SU(2)$ gauge theory coupled to bifundamental matter. He found conventional Chern-Simons terms for each of the SU(2) gauge fields although with opposite signs, as noted already in [73]. A more general class of gauge theories of this type was introduced by Gaiotto and Witten (GW) [74] following [75]. In this formulation the opposite signs for the two SU(N) Chern-Simons terms are related to the SU($N|N$) supergroup structure. Although the GW formulation generally has only $\mathcal{N} = 4$ supersymmetry, it was recently shown how to enlarge the supersymmetry by adding another hypermultiplet [72, 76]. In particular, the maximally supersymmetric BLG theory emerges in the SU(2) × SU(2) case when the matter consists of two bifundamental hypermultiplets. Furthermore, the brane constructions presented in [72, 74] indicate that the relevant gauge theories are actually U(N) × U(N). The presence of the extra interacting U(1) compared to the original BLG formulation is crucial for the complete M-theory interpretation [72].

Below we will present the BLG theory using $\mathcal{N} = 2$ superspace formulation in 2+1 dimensions, which is quite similar to the familiar $\mathcal{N} = 1$ superspace in 3+1 dimensions. In such a formulation only the $U(1)_R$ symmetry is manifest, while the quartic superpotential has an additional SU(4) global symmetry. For a specially chosen normalization of the superpotential, the full scalar potential is manifestly SO(8) invariant. In Section 1.3.1, where we establish the superspace formulation of the BLG theory after briefly reviewing its components formulation, we demonstrate how this happens through a special cancellation involving the F- and D-terms.[2]

In Section 1.3.2 we study its generalizations to U(N) × U(N) gauge theory found by ABJM [72]. The quartic superpotential of this 2+1 dimensional theory has exactly the same form as in the 3+1 dimensional theory on N D3-branes at the conifold singularity [44]. For general N, its global symmetry is SU(2) × SU(2) but for $N = 2$ it becomes enhanced to SU(4) [77] (in this case the theory becomes equivalent to the BLG theory with an extra gauged U(1) [72]). For $N > 2$ ABJM showed that this theory possesses $\mathcal{N} = 6$ supersymmetry [72]. In the $\mathcal{N} = 2$ superspace formulation, this means that, for a specially chosen normalization of the superpotential, the global symmetry is enhanced

[2]This phenomenon is analogous to what happens when the $\mathcal{N} = 4$ SYM theory in 3+1 dimensions is written in terms of an $\mathcal{N} = 1$ gauge theory coupled to three chiral superfields. While only the $U(1)_R \times SU(3)$ symmetry is manifest in such a formulation, the full SU(4) $\sim$ SO(6) symmetry is found in the potential as a result of a specific cancellation between the F- and D-terms.

to $\mathrm{SU}(4)_R$. We demonstrate explicitly how this symmetry enhancement happens in terms of the component fields, once again due to a special cancellation involving F- and D-terms.

1.3.1 BLG Theory in Component and $\mathcal{N}=2$ Superspace Formulation

Here we review the BLG theory in van Raamsdonk's product gauge group formulation [71], which rewrites it as a superconformal Chern-Simons theory with $\mathrm{SU}(2)\times\mathrm{SU}(2)$ gauge group and bifundamental matter. It has a manifest global SO(8) R-symmetry which shows that it is $\mathcal{N}=8$ supersymmetric.

We use the following notation. Indices transforming under the first SU(2) factor of the gauge group are $a, b, \ldots$, and for the second factor we use $\hat{a}, \hat{b}, \ldots$. Fundamental indices are written as superscript and anti-fundamental indices as subscript. Thus, the gauge and matter fields are $A^a{}_b$, $\hat{A}^{\hat{a}}{}_{\hat{b}}$, $X^a{}_{\hat{b}}$, and $\Psi^a{}_{\hat{b}}$. The conjugate fields have indices $(X^\dagger)^{\hat{a}}{}_b$ and $(\Psi^\dagger)^{\hat{a}}{}_b$. Most of the time, however, we will use matrix notation and suppress gauge indices. Lorentz indices are $\mu=0,1,2$ and the metric on the world volume is $g_{\mu\nu}=\mathrm{diag}(-1,+1,+1)$. SO(8) vector indices are $I, J, \ldots$. The fermions a represented by 32-component Majorana spinors of $\mathrm{SO}(1,10)$ subject to a chirality condition on the world-volume which leaves 16 real degrees of freedom. The $\mathrm{SO}(1,10)$ spinor indices are generally omitted.

The action is then given by [71]

$$\begin{aligned}\mathcal{S}=\int d^3x\,\mathrm{tr}\Big[&-(\mathcal{D}^\mu X^I)^\dagger\mathcal{D}_\mu X^I+i\bar{\Psi}^\dagger\Gamma^\mu\mathcal{D}_\mu\Psi \\ &-\frac{2if}{3}\bar{\Psi}^\dagger\Gamma^{IJ}\big(X^I X^{J\dagger}\Psi+X^J\Psi^\dagger X^I+\Psi X^{I\dagger}X^J\big)-\frac{8f^2}{3}\,\mathrm{tr}\,X^{[I}X^{\dagger J}X^{K]}X^{\dagger[K}X^J X^{\dagger I]} \\ &+\frac{1}{2f}\epsilon^{\mu\nu\lambda}(A_\mu\partial_\nu A_\lambda+\frac{2i}{3}A_\mu A_\nu A_\lambda)-\frac{1}{2f}\epsilon^{\mu\nu\lambda}(\hat{A}_\mu\partial_\nu\hat{A}_\lambda+\frac{2i}{3}\hat{A}_\mu\hat{A}_\nu\hat{A}_\lambda)\Big]\,,\end{aligned} \tag{1.81}$$

where the covariant derivative is

$$\mathcal{D}_\mu X = \partial_\mu X + iA_\mu X - iX\hat{A}_\mu \ . \tag{1.82}$$

The Chern-Simons level k is contained in

$$f = \frac{2\pi}{k} \ . \tag{1.83}$$

The bifundamental scalars X^I are related to the original BLG variables x^I_a with SO(4) index a through

$$X^I = \frac{1}{2}(x^I_4 \mathbb{1} + ix^I_i \sigma^i) \ , \tag{1.84}$$

where σ^i are the Pauli matrices. It is important to note that the scalars satisfy the reality condition

$$X^* = -\varepsilon X \varepsilon \ , \tag{1.85}$$

where $\varepsilon = i\sigma_2$. This condition can only be imposed for the gauge group SU(2) × SU(2), which might seem to present an obstacle for generalizing the theory to rank $N > 2$. This obstacle was overcome by using complex bifundamental superfields [72], and this will be reviewed in Section 1.3.2.

Finally we note the form of the SU(2) × SU(2) gauge transformations

$$\begin{aligned} A_\mu &\to UA_\mu U^\dagger - iU\partial_\mu U^\dagger \ , & X &\to UX\hat{U}^\dagger \ , \\ \hat{A}_\mu &\to \hat{U}\hat{A}_\mu \hat{U}^\dagger - i\hat{U}\partial_\mu \hat{U}^\dagger \ , & X^\dagger &\to \hat{U}X^\dagger U^\dagger \ , \end{aligned} \tag{1.86}$$

where $U, \hat{U} \in$ SU(2).

Superfield formulation. Let us now write the BLG theory (1.81) in $\mathcal{N}=2$ superspace. Of the $SO(8)_R$ symmetry this formalism leaves only the subgroup $U(1)_R \times SU(4)$ manifest. However, we will demonstrate how the SO(8) R-symmetry is recovered when the action is expressed in terms of component fields. Our notations and many useful superspace identities are summarized in Appendix A.2.

The gauge fields A and $\hat{A}$ become components of two gauge vector superfields $\mathcal{V}$ and $\hat{\mathcal{V}}$. Their component expansions in Wess-Zumino gauge are

$$\mathcal{V} = 2i\,\theta\bar{\theta}\,\sigma(x) - 2\,\theta\gamma^m\bar{\theta}\,A_m(x) + \sqrt{2}i\,\theta^2\,\bar{\theta}\chi_\sigma^\dagger(x) - \sqrt{2}i\,\bar{\theta}^2\,\theta\chi_\sigma(x) + \theta^2\,\bar{\theta}^2\,\mathsf{D}(x)\ , \quad (1.87)$$

and correspondingly for $\hat{\mathcal{V}}$. Here σ and D are auxiliary scalars, and χ_σ and $\chi_\sigma^\dagger$ are auxiliary fermions. The matter fields X and Ψ are accommodated in chiral superfields $\mathcal{Z}$ and anti-chiral superfields $\bar{\mathcal{Z}}$ which transform in the fundamental and anti-fundamental representation of SU(4), respectively. Their SU(4) indices $\mathcal{Z}^A$ and $\bar{\mathcal{Z}}_A$ will often be suppressed. The component expansions are

$$\mathcal{Z} = Z(x_L) + \sqrt{2}\theta\zeta(x_L) + \theta^2\,F(x_L)\ , \quad (1.88)$$

$$\bar{\mathcal{Z}} = Z^\dagger(x_R) - \sqrt{2}\bar{\theta}\zeta^\dagger(x_R) - \bar{\theta}^2\,F^\dagger(x_R)\ . \quad (1.89)$$

The scalars Z are complex combinations of the BLG scalars

$$Z^A = X^A + iX^{A+4} \qquad \text{for } A = 1,\ldots,4\ . \quad (1.90)$$

We define two operations which conjugate the SU(2) representations and the SU(4) representation, respectively, as[3]

$$Z^{\ddagger A} := -\varepsilon(Z^A)^{\mathrm{T}}\varepsilon = X^{\dagger A} + iX^{\dagger A+4} \ , \tag{1.91}$$

$$\bar{Z}_A := -\varepsilon(Z^A)^*\varepsilon = X^A - iX^{A+4} \ . \tag{1.92}$$

Separating these two operations in possible only for gauge group $\mathrm{SU}(2)\times\mathrm{SU}(2)$, since for gauge groups of higher rank there is no reality condition analogous to (1.85). In these cases only the combined action, which is the hermitian conjugate $Z^\dagger \equiv \bar{Z}^\ddagger$, makes sense. The possibility to conjugate the SU(4) representation independently from the $\mathrm{SU}(2)\times\mathrm{SU}(2)$ representation allows us to invert (1.90):

$$X^A = \tfrac{1}{2}\big(Z^A + \bar{Z}_A\big) \quad , \quad X^{A+4} = \tfrac{1}{2i}\big(Z^A - \bar{Z}_A\big) \ . \tag{1.93}$$

The superspace action $\mathcal{S} = \mathcal{S}_{\mathrm{CS}} + \mathcal{S}_{\mathrm{mat}} + \mathcal{S}_{\mathrm{pot}}$ consists of a Chern-Simons part, a matter part and a superpotential given by

$$\mathcal{S}_{\mathrm{CS}} = -iK\int d^3x\, d^4\theta \int_0^1 dt\ \mathrm{tr}\Big[\mathcal{V}\bar{D}^\alpha\Big(e^{t\mathcal{V}}D_\alpha e^{-t\mathcal{V}}\Big) - \hat{\mathcal{V}}\bar{D}^\alpha\Big(e^{t\hat{\mathcal{V}}}D_\alpha e^{-t\hat{\mathcal{V}}}\Big)\Big] \ , \tag{1.94}$$

$$\mathcal{S}_{\mathrm{mat}} = -\int d^3x\, d^4\theta\ \mathrm{tr}\, \bar{\mathcal{Z}}_A e^{-\mathcal{V}}\mathcal{Z}^A e^{\hat{\mathcal{V}}} \ , \tag{1.95}$$

$$\mathcal{S}_{\mathrm{pot}} = L\int d^3x\, d^2\theta\, \mathrm{W}(\mathcal{Z}) + L\int d^3x\, d^2\bar{\theta}\, \bar{\mathrm{W}}(\bar{\mathcal{Z}}) \ , \tag{1.96}$$

with

$$\mathrm{W} = \frac{1}{4!}\epsilon_{ABCD}\,\mathrm{tr}\,\mathcal{Z}^A\mathcal{Z}^{\ddagger B}\mathcal{Z}^C\mathcal{Z}^{\ddagger D} \quad , \quad \bar{\mathrm{W}} = \frac{1}{4!}\epsilon^{ABCD}\,\mathrm{tr}\,\bar{\mathcal{Z}}_A\bar{\mathcal{Z}}^\ddagger_B\bar{\mathcal{Z}}_C\bar{\mathcal{Z}}^\ddagger_D \ . \tag{1.97}$$

In terms of SO(4) variables $\mathcal{Z}_a$, which are related to the $\mathrm{SU}(2)\times\mathrm{SU}(2)$ fields according

[3]We should caution that the bar denoting the anti-chiral superfield $\bar{\mathcal{Z}}$ is just a label and does *not* mean that the component fields are conjugated by (1.92). In fact, the components of $\bar{\mathcal{Z}}$ are the hermitian conjugates, see (1.89).

to (1.84), it assumes the form

$$W = -\frac{1}{8\cdot 4!}\,\epsilon_{ABCD}\epsilon^{abcd}\mathcal{Z}_a^A\mathcal{Z}_b^B\mathcal{Z}_c^C\mathcal{Z}_d^D\ . \tag{1.98}$$

This superpotential possesses only a $U(1)_R \times SU(4)$ global symmetry as opposed to the $SO(8)_R$ symmetry of the BLG theory. We will show in the following that when the normalization constants K and L are related as $K = \frac{1}{L}$, then the R-symmetry of the model is enhanced to SO(8). If we furthermore set $L = 4f$, we recover precisely the action (1.81).

The gauge transformations are given by [78]

$$e^{t\mathcal{V}} \to e^{i\Lambda}e^{t\mathcal{V}}e^{-i\bar{\Lambda}}\ ,\ \ e^{t\hat{\mathcal{V}}} \to e^{i\hat{\Lambda}}e^{t\hat{\mathcal{V}}}e^{-i\hat{\bar{\Lambda}}}\ ,\ \ \mathcal{Z} \to e^{i\Lambda}\mathcal{Z}e^{-i\hat{\Lambda}}\ ,\ \ \bar{\mathcal{Z}} \to e^{i\hat{\bar{\Lambda}}}\bar{\mathcal{Z}}e^{-i\bar{\Lambda}}\ , \tag{1.99}$$

where the parameters $\Lambda, \hat{\Lambda}$ and $\bar{\Lambda}, \hat{\bar{\Lambda}}$ are chiral and anti-chiral superfields, respectively. Their t dependence is determined by consistency of the transformation law for $\mathcal{V}$ and $\hat{\mathcal{V}}$. In order to preserve the WZ gauge, these fields have to be simply

$$\Lambda = \lambda(x_L) \quad , \quad \bar{\Lambda} = \lambda(x_R) \quad , \quad \hat{\Lambda} = \hat{\lambda}(x_L) \quad , \quad \hat{\bar{\Lambda}} = \hat{\lambda}(x_R)\ . \tag{1.100}$$

with λ and $\hat{\lambda}$ real. These transformations reduce to the ones given in (1.86) when we set $U(x) \equiv e^{i\lambda(x)}$ and $\hat{U}(x) \equiv e^{i\hat{\lambda}(x)}$.

Expressions in components. We will now show that the above superspace action describes the BLG theory by expanding it into component fields. The Chern-Simons action then reads

$$\begin{aligned}\mathcal{S}_{\rm CS} = K\int d^3x\ {\rm tr}\Big[\ \ & 2\epsilon^{\mu\nu\lambda}(A_\mu\partial_\nu A_\lambda + \frac{2i}{3}A_\mu A_\nu A_\lambda) - 2\epsilon^{\mu\nu\lambda}(\hat{A}_\mu\partial_\nu\hat{A}_\lambda + \frac{2i}{3}\hat{A}_\mu\hat{A}_\nu\hat{A}_\lambda) \\ & + 2i\chi_\sigma^\dagger\chi_\sigma - 2i\hat{\chi}_\sigma^\dagger\hat{\chi}_\sigma - 4\mathsf{D}\sigma + 4\hat{\mathsf{D}}\hat{\sigma}\Big]\ ,\end{aligned} \tag{1.101}$$

and the matter action becomes[4]

$$\mathcal{S}_{\rm mat} = \int d^3x\ {\rm tr}\Big[-(\mathcal{D}_\mu Z)^\dagger \mathcal{D}^\mu Z + i\zeta^\dagger \not{\mathcal{D}}\zeta + F^\dagger F + Z^\dagger \mathsf{D} Z - Z^\dagger Z \hat{\mathsf{D}}$$
$$+ iZ^\dagger \chi_\sigma \zeta + i\zeta^\dagger \chi_\sigma^\dagger Z - iZ^\dagger \zeta \hat{\chi}_\sigma - i\zeta^\dagger Z \hat{\chi}_\sigma^\dagger$$
$$- Z^\dagger \sigma^2 Z - Z^\dagger Z \hat{\sigma}^2 + 2Z^\dagger \sigma Z \hat{\sigma} - i\zeta^\dagger \sigma \zeta + i\zeta^\dagger \zeta \hat{\sigma} \Big] . \qquad (1.102)$$

The gauge covariant derivative is defined in (1.82). The superpotential contains the following interactions of the component fields

$$\mathcal{S}_{\rm pot} = -\frac{L}{12}\int d^3x\ {\rm tr}\Big[\quad \epsilon_{ABCD}\big(\zeta^A \zeta^{\ddagger B} Z^C Z^{\ddagger D} - \zeta^{\ddagger A}\zeta^B Z^{\ddagger C} Z^D + \zeta^A Z^{\ddagger B} \zeta^C Z^{\ddagger D}\big)$$
$$+ \epsilon^{ABCD}\big(\bar{\zeta}_A^\ddagger \bar{\zeta}_B \bar{Z}_C^\ddagger \bar{Z}_D - \bar{\zeta}_A \bar{\zeta}_B^\ddagger \bar{Z}_C \bar{Z}_D^\ddagger + \bar{\zeta}_A^\ddagger \bar{Z}_B \bar{\zeta}_C^\ddagger \bar{Z}_D\big)$$
$$+ 2\epsilon_{ABCD} F^A Z^{\ddagger B} Z^C Z^{\ddagger D} - 2\epsilon^{ABCD} \bar{F}_A^\ddagger \bar{Z}_B \bar{Z}_C^\ddagger \bar{Z}_D \Big] . \qquad (1.103)$$

Integrating out auxiliary fields. The fields D and $\hat{\mathsf{D}}$ are Lagrange multipliers for the constraints

$$\sigma^n = \frac{1}{4K}\,{\rm tr}\, t^n Z Z^\dagger \quad , \quad \hat{\sigma}^n = \frac{1}{4K}\,{\rm tr}\, t^n Z^\dagger Z \ , \qquad (1.104)$$

where t^n are the generators of SU(2). The equations of motion for the χ_σ's are

$$\chi_\sigma^n = -\frac{1}{2K}\,{\rm tr}\, t^n Z \zeta^\dagger \ , \qquad (\chi_\sigma^\dagger)^n = -\frac{1}{2K}\,{\rm tr}\, t^n \zeta Z^\dagger \ , \qquad (1.105)$$
$$\hat{\chi}_\sigma^n = -\frac{1}{2K}\,{\rm tr}\, t^n \zeta^\dagger Z \ , \qquad (\hat{\chi}_\sigma^\dagger)^n = -\frac{1}{2K}\,{\rm tr}\, t^n Z^\dagger \zeta \ , \qquad (1.106)$$

[4]Let us remind the reader that our notation suppresses indices in standard positions, e.g.

$${\rm tr}\, Z^\dagger \chi_\sigma \zeta \ \equiv\ {\rm tr}\, Z_A^\dagger \chi_\sigma^\alpha \zeta_\alpha^A \ \equiv\ (Z_A^\dagger)^{\hat{a}}{}_b (\chi_\sigma^\alpha)^b{}_c (\zeta_\alpha^A)^c{}_{\hat{a}} \ .$$

The standard position of an index is defined when the field is introduced, and those for spinor indices are explained in Appendix A.2.

and the ones for F are

$$F^A = -\frac{L}{6}\epsilon^{ABCD}\bar{Z}_B\bar{Z}^\ddagger_C\bar{Z}_D \quad , \quad F^\dagger_A = +\frac{L}{6}\epsilon_{ABCD}Z^{\ddagger B}Z^C Z^{\ddagger D} \,. \tag{1.107}$$

Using these relations one finds the following action

$$\begin{aligned}\mathcal{S} = \int d^3x \Big[& 2K\,\epsilon^{\mu\nu\lambda}\,\mathrm{tr}\big(A_\mu\partial_\nu A_\lambda + \tfrac{2i}{3}A_\mu A_\nu A_\lambda - \hat{A}_\mu\partial_\nu\hat{A}_\lambda - \tfrac{2i}{3}\hat{A}_\mu\hat{A}_\nu\hat{A}_\lambda\big) \\ & - \mathrm{tr}(\mathcal{D}_\mu Z)^\dagger\mathcal{D}^\mu Z + i\,\mathrm{tr}\,\zeta^\dagger\not{\mathcal{D}}\zeta - V_{\mathrm{ferm}} - V_{\mathrm{bos}}\Big] \,. \end{aligned} \tag{1.108}$$

The quartic terms V_{ferm} are interactions between fermions and bosons, and the sextic terms V_{bos} are interactions between bosons only. Separated according to their origin we have

$$V^{\mathrm{ferm}}_D = \frac{i}{4K}\,\mathrm{tr}\Big[\zeta^A\zeta^\dagger_A Z^B Z^\dagger_B - \zeta^\dagger_A\zeta^A Z^\dagger_B Z^B + 2\zeta^A Z^\dagger_A Z^B\zeta^\dagger_B - 2Z^\dagger_A\zeta^A\zeta^\dagger_B Z^B\Big] \,, \tag{1.109}$$

$$\begin{aligned} V^{\mathrm{ferm}}_F = & \frac{L}{12}\epsilon_{ABCD}\,\mathrm{tr}\Big[\zeta^A\zeta^{\ddagger B}Z^C Z^{\ddagger D} - \zeta^{\ddagger A}\zeta^B Z^{\ddagger C}Z^D + \zeta^A Z^{\ddagger B}\zeta^C Z^{\ddagger D}\Big] \\ & + \frac{L}{12}\epsilon^{ABCD}\,\mathrm{tr}\Big[\bar{\zeta}^\ddagger_A\bar{\zeta}_B\bar{Z}^\ddagger_C\bar{Z}_D - \bar{\zeta}_A\bar{\zeta}^\ddagger_B\bar{Z}_C\bar{Z}^\ddagger_D + \bar{\zeta}^\ddagger_A\bar{Z}_B\bar{\zeta}^\ddagger_C\bar{Z}_D\Big] \,, \end{aligned} \tag{1.110}$$

and

$$V^{\mathrm{bos}}_D = \frac{1}{16K^2}\,\mathrm{tr}\Big[Z^A Z^\dagger_A Z^B Z^\dagger_B Z^C Z^\dagger_C + Z^\dagger_A Z^A Z^\dagger_B Z^B Z^\dagger_C Z^C - 2Z^\dagger_A Z^B Z^\dagger_B Z^A Z^\dagger_C Z^C\Big] \,, \tag{1.111}$$

$$V^{\mathrm{bos}}_F = -\frac{L^2}{36}\epsilon_{ABCG}\epsilon^{DEFG}\,\mathrm{tr}\,Z^{\ddagger A}Z^B Z^{\ddagger C}\bar{Z}_D\bar{Z}^\ddagger_E\bar{Z}_F \,. \tag{1.112}$$

When substituting in (1.90) we find that for $K = \frac{1}{L}$ all sextic interactions can be joined together to

$$V^{\mathrm{bos}} = \frac{L^2}{6}\,\mathrm{tr}\,X^{[I}X^{\dagger J}X^{K]}X^{\dagger[K}X^J X^{\dagger I]} \,. \tag{1.113}$$

Furthermore, setting $L = 4f$, this is precisely the scalar potential of the BLG theory (1.81). With this choice also the other coefficients match exactly.

1.3.2 ABJM $\mathrm{U}(N) \times \mathrm{U}(N)$ Gauge Theory in Superspace

As remarked above, it is not immediately obvious how to generalize van Raamsdonk's formulation of the BLG theory to higher rank gauge groups. This difficulty is also evident in our superspace formulation, since the manifestly SU(4) invariant superpotential is gauge invariant only for $\mathrm{SU}(2) \times \mathrm{SU}(2)$ gauge theory.

The way to circumvent this difficulty is the generalization proposed by ABJM [72]. Their key idea is to give up the manifest global SU(4) invariance by forming the following complex combinations of the bifundamental fields:

$$Z^1 = X^1 + iX^5 , \qquad W_1 = X^{3\dagger} + iX^{7\dagger} , \tag{1.114}$$

$$Z^2 = X^2 + iX^6 , \qquad W_2 = X^{4\dagger} + iX^{8\dagger} . \tag{1.115}$$

Promoting these fields to chiral superfields, the superpotential of the BLG theory (1.97) may be written as [72]

$$\mathcal{S}_{\mathrm{pot}} = L \int d^3x\, d^2\theta\, \mathrm{W}(\mathcal{Z}, \mathcal{W}) + L \int d^3x\, d^2\bar{\theta}\, \bar{\mathrm{W}}(\bar{\mathcal{Z}}, \bar{\mathcal{W}}) , \tag{1.116}$$

with

$$\mathrm{W} = \frac{1}{4}\epsilon_{AC}\epsilon^{BD}\, \mathrm{tr}\, \mathcal{Z}^A \mathcal{W}_B \mathcal{Z}^C \mathcal{W}_D \quad , \quad \bar{\mathrm{W}} = \frac{1}{4}\epsilon^{AC}\epsilon_{BD}\, \mathrm{tr}\, \bar{\mathcal{Z}}_A \bar{\mathcal{W}}^B \bar{\mathcal{Z}}_C \bar{\mathcal{W}}^D . \tag{1.117}$$

This form of the superpotential is exactly the same as for the theory of D3-branes on the conifold [44] and it generalizes readily to $\mathrm{SU}(N) \times \mathrm{SU}(N)$ gauge group. This superpotential has a global symmetry $\mathrm{SU}(2) \times \mathrm{SU}(2)$ and also a "baryonic" U(1) symmetry

$$\mathcal{Z}^A \rightarrow e^{i\alpha}\mathcal{Z}^A \quad , \quad \mathcal{W}_B \rightarrow e^{-i\alpha}\mathcal{W}_B . \tag{1.118}$$

In the 3+1 dimensional case this symmetry is originally gauged, but far in the IR it becomes global [44]. However, in the present 2+1 dimensional example this does not happen, so it is natural to add it to the gauge symmetry [72]. Including also the trivial neutral U(1), we thus find the $\mathrm{U}(N) \times \mathrm{U}(N)$ Chern-Simons gauge theory at level k. The gauging of the symmetry (1.118) seems important for obtaining the correct M-theory interpretation for arbitrary k and N. Since this symmetry corresponds to simultaneous rotation of the 4 complex coordinates of $\mathbb{C}^4$ transverse to the M2-branes, this space actually turns into an orbifold $\mathbb{C}^4/\mathbb{Z}_k$ [72]. Because of this gauging, even for $N = 2$ the ABJM theory is slightly different from the BLG theory.

Let us summarize the properties of the ABJM theory [72] and explicitly prove that its $\mathrm{U}(1)_R \times \mathrm{SU}(2) \times \mathrm{SU}(2)$ global symmetry becomes enhanced to $\mathrm{SU}(4)_R$. The fields $\mathcal{Z}$ and $\mathcal{W}$ transform in the $(\mathbf{2}, \mathbf{1})$ and the $(\mathbf{1}, \bar{\mathbf{2}})$ of the global $\mathrm{SU}(2) \times \mathrm{SU}(2)$ and in the $(\mathbf{N}, \bar{\mathbf{N}})$ and the $(\bar{\mathbf{N}}, \mathbf{N})$ of the gauge group $\mathrm{U}(N) \times \mathrm{U}(N)$, respectively. We use the following conventions for $\mathrm{SU}(2) \times \mathrm{SU}(2)$ indices: $\mathcal{Z}^A$, $\bar{\mathcal{Z}}_A$, $\mathcal{W}_A$, $\bar{\mathcal{W}}^A$ and for $\mathrm{U}(N) \times \mathrm{U}(N)$ indices: $\mathcal{Z}^a{}_{\hat{a}}$, $\bar{\mathcal{Z}}^{\hat{a}}{}_a$, $\mathcal{W}^{\hat{a}}{}_a$, $\bar{\mathcal{W}}^a{}_{\hat{a}}$. The gauge superfields have indices $\mathcal{V}^a{}_b$ and $\hat{\mathcal{V}}^{\hat{a}}{}_{\hat{b}}$. The component fields for $\mathcal{Z}$, $\bar{\mathcal{Z}}$ and $\mathcal{V}$ are as previously in (1.88), (1.89) and (1.87). The components of $\mathcal{W}$ and $\bar{\mathcal{W}}$ will be denoted by

$$\mathcal{W} = W(x_L) + \sqrt{2}\theta\omega(x_L) + \theta^2\, G(x_L)\ , \tag{1.119}$$

$$\bar{\mathcal{W}} = W^\dagger(x_R) - \sqrt{2}\bar{\theta}\omega^\dagger(x_R) - \bar{\theta}^2\, G^\dagger(x_R)\ . \tag{1.120}$$

The Chern-Simons action is formally unaltered (1.94), the matter part (1.95) splits into

$$\mathcal{S}_{\mathrm{mat}} = \int d^3x\, d^4\theta\ \mathrm{tr}\Big[-\bar{\mathcal{Z}}_A e^{-\mathcal{V}} \mathcal{Z}^A e^{\hat{\mathcal{V}}} - \bar{\mathcal{W}}^A e^{-\hat{\mathcal{V}}} \mathcal{W}_A e^{\mathcal{V}}\Big]\ , \tag{1.121}$$

and the superpotential is given by (1.116). The symmetry enhancement to $\mathrm{SU}(4)_R$ requires the normalization constants in (1.94) and (1.116) to be related as $K = \frac{1}{L}$.

Expressions in components. The component form of the Chern-Simons action has been computed in (1.101) and the matter action involving $\mathcal{Z}$ looks identical to (1.102) where now Z, ζ, F have only two components. The matter action for $\mathcal{W}$ is analogously given by

$$\begin{aligned}\mathcal{S}^{\mathcal{W}}_{\text{mat}} = \int d^3x \ \text{tr}\Big[&- (\mathcal{D}_\mu W)^\dagger \mathcal{D}^\mu W + i\omega^\dagger \not{\mathcal{D}}\omega + G^\dagger G + W^\dagger \hat{\mathsf{D}} W - W^\dagger W \mathsf{D} \\ &+ iW^\dagger \hat{\chi}_\sigma \omega + i\omega^\dagger \hat{\chi}^\dagger_\sigma W - iW^\dagger \omega \chi_\sigma - i\omega^\dagger W \chi^\dagger_\sigma \\ &- W^\dagger \hat{\sigma}^2 W - W^\dagger W \sigma^2 + 2W^\dagger \hat{\sigma} W \sigma - i\omega^\dagger \hat{\sigma}\omega + i\omega^\dagger \omega \sigma \Big] , \end{aligned} \tag{1.122}$$

where $\mathcal{D}_\mu W = \partial_\mu W + i\hat{A}_\mu W - iWA_\mu$. The superpotential expands to

$$\begin{aligned}\mathcal{S}_{\text{pot}} = \frac{L}{4}\int d^3x \ \text{tr}\Big[&\epsilon_{AC}\,\epsilon^{BD}\Big(2F^A W_B Z^C W_D + 2Z^A W_B Z^C G_D - 2\zeta^A W_B Z^C \omega_D \\ &-2\zeta^A \omega_B Z^C W_D - Z^A \omega_B Z^C \omega_D - \zeta^A W_B \zeta^C W_D\Big) \\ - \epsilon^{AC}\,\epsilon_{BD}&\Big(2F^\dagger_A W^{\dagger B} Z^\dagger_C W^{\dagger D} + 2Z^\dagger_A W^{\dagger B} Z^\dagger_C G^{\dagger D} + 2\zeta^\dagger_A W^{\dagger B} Z^\dagger_C \omega^{\dagger D} \\ &+2\zeta^\dagger_A \omega^{\dagger B} Z^\dagger_C W^{\dagger D} + Z^\dagger_A \omega^{\dagger B} Z^\dagger_C \omega^{\dagger D} + \zeta^\dagger_A W^{\dagger B} \zeta^\dagger_C W^{\dagger D}\Big)\Big] . \end{aligned} \tag{1.123}$$

Integrating out auxiliary fields. The auxiliary fields can be replaced by means of the following equations:

$$\sigma^n = \frac{1}{4K}\,\text{tr}\, T^n\big(ZZ^\dagger - W^\dagger W\big) , \qquad \hat{\sigma}^n = \frac{1}{4K}\,\text{tr}\, T^n\big(Z^\dagger Z - WW^\dagger\big) , \tag{1.124}$$

$$\chi^n_\sigma = -\frac{1}{2K}\,\text{tr}\, T^n\big(Z\zeta^\dagger - \omega^\dagger W\big) , \qquad (\chi^\dagger_\sigma)^n = -\frac{1}{2K}\,\text{tr}\, T^n\big(\zeta Z^\dagger - W^\dagger \omega\big) , \tag{1.125}$$

$$\hat{\chi}^n_\sigma = -\frac{1}{2K}\,\text{tr}\, T^n\big(\zeta^\dagger Z - W\omega^\dagger\big) , \qquad (\hat{\chi}^\dagger_\sigma)^n = -\frac{1}{2K}\,\text{tr}\, T^n\big(Z^\dagger \zeta - \omega W^\dagger\big) , \tag{1.126}$$

$$F^A = +\frac{L}{2}\epsilon^{AC}\epsilon_{BD} W^{\dagger B} Z^\dagger_C W^{\dagger D} , \qquad G_A = -\frac{L}{2}\epsilon_{AC}\epsilon^{BD} Z^\dagger_B W^{\dagger C} Z^\dagger_D , \tag{1.127}$$

$$F^\dagger_A = -\frac{L}{2}\epsilon_{AC}\epsilon^{BD} W_B Z^C W_D , \qquad G^{\dagger A} = +\frac{L}{2}\epsilon^{AC}\epsilon_{BD} Z^B W_C Z^D . \tag{1.128}$$

Then the complete action reads

$$\mathcal{S} = \int d^3x \Big[2K\, \epsilon^{\mu\nu\lambda}\, \mathrm{tr}\big(A_\mu \partial_\nu A_\lambda + \tfrac{2i}{3} A_\mu A_\nu A_\lambda - \hat{A}_\mu \partial_\nu \hat{A}_\lambda - \tfrac{2i}{3} \hat{A}_\mu \hat{A}_\nu \hat{A}_\lambda\big)$$
$$- \ \mathrm{tr}(\mathcal{D}_\mu Z)^\dagger \mathcal{D}^\mu Z - \mathrm{tr}(\mathcal{D}_\mu W)^\dagger \mathcal{D}^\mu W + i\, \mathrm{tr}\, \zeta^\dagger \not{\mathcal{D}} \zeta + i\, \mathrm{tr}\, \omega^\dagger \not{\mathcal{D}} \omega$$
$$- \ V_{\mathrm{ferm}} - V_{\mathrm{bos}} \Big] \,, \tag{1.129}$$

with the potentials

$$V_D^{\mathrm{ferm}} = \tag{1.130}$$
$$\frac{i}{4K}\, \mathrm{tr}\Big[\big(\zeta^A \zeta_A^\dagger - \omega^{\dagger A} \omega_A\big)\big(Z^B Z_B^\dagger - W^{\dagger B} W_B\big) - \big(\zeta_A^\dagger \zeta^A - \omega_A \omega^{\dagger A}\big)\big(Z_B^\dagger Z^B - W_B W^{\dagger B}\big)\Big] +$$
$$\frac{i}{2K}\, \mathrm{tr}\Big[\big(\zeta^A Z_A^\dagger - W^{\dagger A} \omega_A\big)\big(Z^B \zeta_B^\dagger - \omega^{\dagger B} W_B\big) - \big(Z_A^\dagger \zeta^A - \omega_A W^{\dagger A}\big)\big(\zeta_B^\dagger Z^B - W_B \omega^{\dagger B}\big)\Big] \,,$$
$$V_F^{\mathrm{ferm}} = \tag{1.131}$$
$$\frac{L}{4} \epsilon_{AC} \epsilon^{BD}\, \mathrm{tr}\Big[2\zeta^A W_B Z^C \omega_D + 2\zeta^A \omega_B Z^C W_D + Z^A \omega_B Z^C \omega_D + \zeta^A W_B \zeta^C W_D\Big] +$$
$$\frac{L}{4} \epsilon^{AC} \epsilon_{BD}\, \mathrm{tr}\Big[2\zeta_A^\dagger W^{\dagger B} Z_C^\dagger \omega^{\dagger D} + 2\zeta_A^\dagger \omega^{\dagger B} Z_C^\dagger W^{\dagger D} + Z_A^\dagger \omega^{\dagger B} Z_C^\dagger \omega^{\dagger D} + \zeta_A^\dagger W^{\dagger B} \zeta_C^\dagger W^{\dagger D}\Big] \,,$$

and

$$V_D^{\mathrm{bos}} = \frac{1}{16K^2}\, \mathrm{tr}\Big[\big(Z^A Z_A^\dagger + W^{\dagger A} W_A\big)\big(Z^B Z_B^\dagger - W^{\dagger B} W_B\big)\big(Z^C Z_C^\dagger - W^{\dagger C} W_C\big) \tag{1.132}$$
$$+\big(Z_A^\dagger Z^A + W_A W^{\dagger A}\big)\big(Z_B^\dagger Z^B - W_B W^{\dagger B}\big)\big(Z_C^\dagger Z^C - W_C W^{\dagger C}\big)$$
$$-2 Z_A^\dagger \big(Z^B Z_B^\dagger - W^{\dagger B} W_B\big) Z^A \big(Z_C^\dagger Z^C - W_C W^{\dagger C}\big)$$
$$-2 W^{\dagger A} \big(Z_B^\dagger Z^B - W_B W^{\dagger B}\big) W_A \big(Z^C Z_C^\dagger - W^{\dagger C} W_C\big)\Big] \,,$$
$$V_F^{\mathrm{bos}} = -\frac{L^2}{4}\, \mathrm{tr}\Big[W^{\dagger A} Z_B^\dagger W^{\dagger C} W_A Z^B W_C - W^{\dagger A} Z_B^\dagger W^{\dagger C} W_C Z^B W_A \tag{1.133}$$
$$+ Z_A^\dagger W^{\dagger B} Z_C^\dagger Z^A W_B Z^C - Z_A^\dagger W^{\dagger B} Z_C^\dagger Z^C W_B Z^A\Big] \,.$$

Let us note that V_F^{bos} and V_D^{bos} are separately non-negative. Indeed, the F-term contribution is related to the superpotential W through

$$V_F^{\rm bos} = \left|\frac{\partial {\rm W}}{\partial Z^A}\right|^2 + \left|\frac{\partial {\rm W}}{\partial W_A}\right|^2 = {\rm tr}\big[F_A^\dagger F^A + G^{\dagger A} G_A\big] \ , \tag{1.134}$$

with F^A and G_A from (1.127) and (1.128). The D-term contribution may be written as

$$V_D^{\rm bos} = {\rm tr}\big[N_A^\dagger N^A + M^{\dagger A} M_A\big] \ , \tag{1.135}$$

where $N^A = \sigma Z^A - Z^A \hat{\sigma}$ and $M_A = \hat{\sigma} W_A - W_A \sigma$. Thus, the total bosonic potential vanishes if and only if

$$F^A = G_A = N^A = M_A = 0 \ . \tag{1.136}$$

SU(4) invariance. If the coefficients of the Chern-Simons action and the superpotential are related by $K = \frac{1}{L}$, then the R-symmetry of the theory is enhanced to SU(4).[5] In order to make this symmetry manifest we combine the SU(2) fields Z and W into fundamental and anti-fundamental representations of SU(4) as

$$Y^A = \{Z^A, W^{\dagger A}\} \quad , \quad Y_A^\dagger = \{Z_A^\dagger, W_A\} \ , \tag{1.137}$$

where the index A on the left hand side now runs from 1 to 4. Then the potential can be written as [72]

$$\begin{aligned} V^{\rm bos} = -\frac{L^2}{48}\,{\rm tr}\Big[& Y^A Y_A^\dagger Y^B Y_B^\dagger Y^C Y_C^\dagger + Y_A^\dagger Y^A Y_B^\dagger Y^B Y_C^\dagger Y^C \\ & + 4 Y^A Y_B^\dagger Y^C Y_A^\dagger Y^B Y_C^\dagger - 6 Y^A Y_B^\dagger Y^B Y_A^\dagger Y^C Y_C^\dagger \Big] \ . \end{aligned} \tag{1.138}$$

The fermions have to be combined as follows

$$\psi_A = \{\epsilon_{AB}\zeta^B\, e^{-i\pi/4}, -\epsilon_{AB}\omega^{\dagger B}\, e^{i\pi/4}\} \quad , \quad \psi^{A\dagger} = \{-\epsilon^{AB}\zeta_B^\dagger\, e^{i\pi/4}, \epsilon^{AB}\omega_B\, e^{-i\pi/4}\} \ , \tag{1.139}$$

[5] This $SU(4)_R$ symmetry should not be confused with the global SU(4) of the BLG theory. The latter is not manifest in the ABJM theory, but should nevertheless be present for $k = 1$ and $k = 2$ [72].

and we can write fermionic interactions in the manifestly SU(4) invariant way

$$V^{\text{ferm}} = \frac{iL}{4}\,\mathrm{tr}\Big[Y_A^\dagger Y^A \psi^{B\dagger}\psi_B - Y^A Y_A^\dagger \psi_B \psi^{B\dagger} + 2Y^A Y_B^\dagger \psi_A \psi^{B\dagger} - 2Y_A^\dagger Y^B \psi^{A\dagger}\psi_B \\ - \epsilon^{ABCD} Y_A^\dagger \psi_B Y_C^\dagger \psi_D + \epsilon_{ABCD} Y^A \psi^{B\dagger} Y^C \psi^{D\dagger}\Big]\,. \tag{1.140}$$

Thus, the $U(1)_R \times SU(2) \times SU(2)$ global symmetry is enhanced to $SU(4)_R$ symmetry, with the $U(1)_R$ corresponding to the generator $\frac{1}{2}\,\mathrm{diag}(1, 1, -1, -1)$. This shows that the theory in general possesses $\mathcal{N} = 6$ supersymmetry.

In [72] it was proposed that this $U(N) \times U(N)$ Chern-Simons theory at level k describes the world volume of N coincident M2-branes placed at the $\mathbb{Z}_k$ orbifold of $\mathbb{C}^4$ where the action on the 4 complex coordinates[6] is $y^A \to e^{2\pi i/k} y^A$. This action preserves the SU(4) symmetry that rotates them, which in the gauge theory is realized as the R-symmetry. The $\mathcal{N} = 6$ supersymmetry of this orbifold can be checked as follows. The generator of $\mathbb{Z}_k$ acts on the spinors of SO(8) as

$$\Psi \to e^{2\pi i(s_1+s_2+s_3+s_4)/k}\Psi\ , \tag{1.141}$$

where $s_i = \pm 1/2$ are the spinor weights. The chirality projection implies that the sum of all s_i must be even, producing an 8-dimensional representation. The spinors that are left invariant by the orbifold have $\sum_{i=1}^4 s_i = 0\ (\mathrm{mod}\, k)$. This selects 6 out of the 8 spinors; therefore, the theory on M2-branes has 12 supercharges in perfect agreement with the Chern-Simons gauge theory with general level k. This constitutes just one of many pieces of evidence that strongly suggest that the theory reviewed in this section is indeed dual to M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$ with N units of flux. For $k = 1$ or 2 there is further enhancement to $\mathcal{N} = 8$ supersymmetry, which is subtle in the gauge theory [72] and requires the introduction of monopole operators, as we will discuss in Chapter 5.

[6] Let us note that these coordinates are not the same as the complex coordinates z^A natural for the superspace formulation of BLG theory in Section 1.3.1. They are related through $y^1 = z^1, y^2 = z^2, y^3 = \bar{z}^3, y^4 = \bar{z}^4$.

Chapter 2

On the Strong Coupling Scaling Dimension of High Spin Operators

2.1 Introduction

The dimensions of high-spin operators are important observables in gauge theories. It is well-known that the anomalous dimension of a twist-2 operator grows logarithmically for large spin S,

$$\Delta - S = f(g) \ln S \ + \mathcal{O}(S^0) \ , \qquad \text{with} \ \ g = \frac{\sqrt{g_{YM}^2 N}}{4\pi} \ . \tag{2.1}$$

This was demonstrated early on at one-loop order [24, 25] and at two loops [79] where a cancellation of $\ln^3 S$ terms occurs. There are solid arguments that (2.1) holds to all orders in perturbation theory [80, 81, 82], and that it also applies to high-spin operators of twist greater than two [29]. The function of coupling $f(g)$ also measures the anomalous dimension of a cusp in a light-like Wilson loop, and is of definite physical interest in QCD.

There has been significant interest in determining $f(g)$ in the $\mathcal{N} = 4$ SYM theory. This is partly due to the fact that the AdS/CFT correspondence [1, 2, 3] relates the large g behavior of f to the energy of a folded string spinning around the center of a weakly curved AdS_5 space [20]. This gives the prediction that $f(g) \to 4g$ at strong coupling. The same result was obtained from studying the cusp anomaly using string theory methods [83]. Furthermore, the semi-classical expansion for the spinning string energy predicts the following correction [28]:

$$f(g) = 4g - \frac{3\ln 2}{\pi} + O(1/g) \ . \tag{2.2}$$

It is of obvious interest to confirm these explicit predictions of string theory using extrapolation of the perturbative expansion for $f(g)$ provided by the gauge theory.

Explicit perturbative calculations are quite formidable, and until a few years ago were available only up to three-loop order [35, 84]:

$$f(g) = 8g^2 - \frac{8}{3}\pi^2 g^4 + \frac{88}{45}\pi^4 g^6 + O(g^8) \ . \tag{2.3}$$

Kotikov, Lipatov, Onishchenko and Velizhanin [35] extracted the $\mathcal{N} = 4$ answer from the QCD calculation of [85] using their proposed transcendentality principle stating that each expansion coefficient has terms of the same degree of transcendentality.

Since then, the methods of integrability in AdS/CFT[1] [36, 37, 38], prompted in part by [19, 20], have led to dramatic progress in studying the weak coupling expansion. In the beautiful paper by Beisert, Eden and Staudacher [40], which followed closely the important earlier work in [30, 39], an integral equation that determines $f(g)$ was proposed, yielding an expansion of $f(g)$ to an arbitrary desired order. The expansion coefficients obey the KLOV transcendentality principle. In an independent remarkable paper by Bern, Czakon, Dixon, Kosower and Smirnov [89], an explicit calculation led to

[1]For earlier work on integrability in gauge theories, see [86, 87, 88], for reviews see [22, 23].

a value of the four-loop term,

$$-16\left(\frac{73}{630}\pi^6 + 4\zeta(3)^2\right) g^8 \ , \tag{2.4}$$

which agrees with the idea advanced in [40, 89] that the exact expansion of $f(g)$ is related to that found in [30] simply by multiplying each ζ-function of an odd argument by an i, $\zeta(2n+1) \to i\zeta(2n+1)$. The integral equation of [40] generates precisely this perturbative expansion for $f(g)$.

A crucial property of this integral equation is that it is related through integrability to the "dressing phase" in the magnon S-matrix, whose general form was deduced in [32, 90]. In [40] a perturbative expansion of the phase was given, which starts at four-loop order, and at strong coupling coincides with the earlier results from string theory [39, 32, 91, 92, 93]. An important requirement of crossing symmetry [94] is satisfied by this phase, and it also satisfies the KLOV transcendentality priciple. Therefore, this phase is very likely to describe the exact magnon S-matrix at any coupling [40], which constitutes remarkable progress in the understanding of the $\mathcal{N}=4$ SYM theory, and of the AdS/CFT correspondence.

The papers [40, 89] thoroughly studied the perturbative expansion of $f(g)$ which follows from the integral equation. Although the expansion has a finite radius of convergence, as is customary in certain planar theories (see, for example, [95]), it is expected to determine the function completely. Solving the integral equation of [40] is an efficient tool for attacking this problem. In this chapter we solve the integral equation numerically at intermediate and strong coupling, and show that $f(g)$ is a smooth function that approaches the asymptotic form (2.2) predicted by string theory for $g > 1$. The two leading strong coupling terms match those in (2.2) with high accuracy. This constitutes a remarkable confirmation of the AdS/CFT correspondence for this non-supersymmetric observable.

This chapter is based on the papers [96, 97] coauthored with L. F. Alday, G. Arutyunov, S. Benvenuti, B. Eden, I. R. Klebanov and A. Scardicchio, and its structure is as follows. The BES integral equation is reviewed and solved numerically in Section 2.2. An interpretation of these results and their implications for the AdS/CFT correspondence are given in Section 2.3, where we also summarize further investigations which are the subject of the remainder of this chapter. In Section 2.4 we derive the exact fluctuation density that solves the integral equation in the strong coupling limit, before we conclude in Section 2.5.

2.2 Numerical Study of the Integral Equation

The cusp anomalous dimension $f(g)$ can be written as [30, 40, 98]

$$f(g) = 16g^2 \hat{\sigma}(0) \ , \tag{2.5}$$

where $\hat{\sigma}(t)$ obeys the integral equation

$$\hat{\sigma}(t) = \frac{t}{e^t - 1} \left(K(2gt, 0) - 4g^2 \int_0^\infty dt' K(2gt, 2gt') \hat{\sigma}(t') \right) \ , \tag{2.6}$$

with the kernel given by [40]

$$K(t, t') = K^{(m)}(t, t') + 2K^{(c)}(t, t') \ . \tag{2.7}$$

The main scattering kernel $K^{(m)}$ of [30] is

$$K^{(m)}(t, t') = \frac{J_1(t) J_0(t') - J_0(t) J_1(t')}{t - t'} \ , \tag{2.8}$$

and the dressing kernel $K^{(c)}$ is defined as the convolution

$$K^{(c)}(t,t') = 4g^2 \int_0^\infty dt'' K_1(t, 2gt'') \frac{t''}{e^{t''}-1} K_0(2gt'', t') \ , \tag{2.9}$$

where K_0 and K_1 denote the parts of the kernel that are even and odd, respectively, under change of sign of t and t':

$$K_0(t,t') = \frac{t J_1(t) J_0(t') - t' J_0(t) J_1(t')}{t^2 - t'^2} = \frac{2}{t\,t'} \sum_{n=1}^{\infty} (2n-1) J_{2n-1}(t) J_{2n-1}(t') \ , \tag{2.10}$$

$$K_1(t,t') = \frac{t' J_1(t) J_0(t') - t J_0(t) J_1(t')}{t^2 - t'^2} = \frac{2}{t\,t'} \sum_{n=1}^{\infty} (2n) J_{2n}(t) J_{2n}(t') \ . \tag{2.11}$$

We find it useful to introduce the function

$$s(t) = \frac{e^t - 1}{t} \, \hat{\sigma}(t) \ , \tag{2.12}$$

in terms of which the integral equation becomes

$$s(t) = K(2gt, 0) - 4g^2 \int_0^\infty dt' K(2gt, 2gt') \frac{t'}{e^{t'}-1} s(t') \ . \tag{2.13}$$

Again, $f(g) = 16g^2 s(0)$.

Both $K^{(m)}$ and $K^{(c)}$ can conveniently be expanded as sums of products of functions of t and functions of t':

$$K^{(m)}(t,t') = K_0(t,t') + K_1(t,t') = \frac{2}{t\,t'} \sum_{n=1}^{\infty} n J_n(t) J_n(t') \ , \tag{2.14}$$

and

$$K^{(c)}(t,t') = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{8n(2m-1)}{t\,t'} Z_{2n,2m-1} J_{2n}(t) J_{2m-1}(t') \ . \tag{2.15}$$

This suggests writing the solution in terms of linearly independent functions as

$$s(t) = \sum_{n\geq 1} s_n \frac{J_n(2gt)}{2gt} \ , \tag{2.16}$$

so that the integral equation becomes a matrix equation for the coefficients s_n. The desired function $f(g)$ is now

$$f(g) = 8g^2 s_1 \ . \tag{2.17}$$

It is convenient to define the matrix Z_{mn} as

$$Z_{mn} \equiv \int_0^\infty dt \frac{J_m(2gt)J_n(2gt)}{t(e^t-1)} \ . \tag{2.18}$$

Using the representations (2.14) and (2.15) of the kernels and (2.16) for $s(t)$, the integral equation above is now of the schematic form

$$s_n = h_n - \sum_{m\geq 1} \left(K^{(m)}_{nm} + 2K^{(c)}_{nm}\right) s_m \ , \tag{2.19}$$

whose solution formally is

$$s = \frac{1}{1+K^{(m)}+2K^{(c)}} h \ . \tag{2.20}$$

The matrices appearing in (2.19) are

$$K^{(m)}_{nm} = 2(NZ)_{nm} \ , \tag{2.21}$$

$$K^{(c)}_{nm} = 2(CZ)_{nm} \ , \tag{2.22}$$

$$C_{nm} = 2(PNZQN)_{nm} \ , \tag{2.23}$$

where $Q = \text{diag}(1,0,1,0,...)$, $P = \text{diag}(0,1,0,1,...)$, $N = \text{diag}(1,2,3,...)$ and the vector h can be written as $h = (1+2C)e$, where $e = (1,0,0,...)^{\rm T}$. The crucial point for the numerics to work is that the matrix elements of Z decay sufficiently fast with increasing

m, n (they decay like $e^{-\max(m,n)/g}$). For intermediate g (say $g < 20$) we can work with moderate size d by d matrices, where d does not have to be much larger than g. The integrals in Z_{nm} can be obtained numerically without much effort and so we can solve for the s_n. We find that the results are stable with respect to increasing d.

Even though at strong coupling all elements of Z_{nm} are of the same order in $1/g$, those far from the upper left corner are numerically small (the leading terms in $1/g$ are suppressed by a factor $[(m-n-1)(m-n+1)(m+n-1)(m+n+1)]^{-1}$ for $m \neq n \pm 1$). This last fact makes the numerics surprisingly convergent even at large g and moreover gives some hope that the analytic form of the strong coupling expansion of $f(g)$ could be obtained from a perturbation theory for the matrix equation.

Therefore, when formulated in terms of the Z_{mn}, the problem becomes amenable to numerical study at all values of the coupling. We find that the numerical procedure converges rather rapidly, and truncate the series expansions of $s(t)$ and of the kernel after the first 30 orders of Bessel functions.

The function $f(g)$ is the lowest curve plotted in Figure 1. For comparison, we also plot $f_m(g)$ which solves the integral equation with kernel $K^{(m)}$ [30], and $f_0(g)$ which solves the integral equation with kernel $K^{(m)} + K^{(c)}$. Clearly, these functions differ at strong coupling. The function $f(g)$ is monotonic and reaches the asymptotic, linear form quite early, for $g \simeq 1$. We can then study the asymptotic, large g form easily and compare it with the prediction from string theory. The best fit result (using the range $2 < g < 20$) is

$$f(g) = (4.000000 \pm 0.000001)g - (0.661907 \pm 0.000002) - \frac{0.0232 \pm 0.0001}{g} + \dots \quad (2.24)$$

The first two terms are in remarkable agreement with the string theory result (2.2), while the third term is a numerical prediction for the $1/g$ term in the strong coupling expansion.

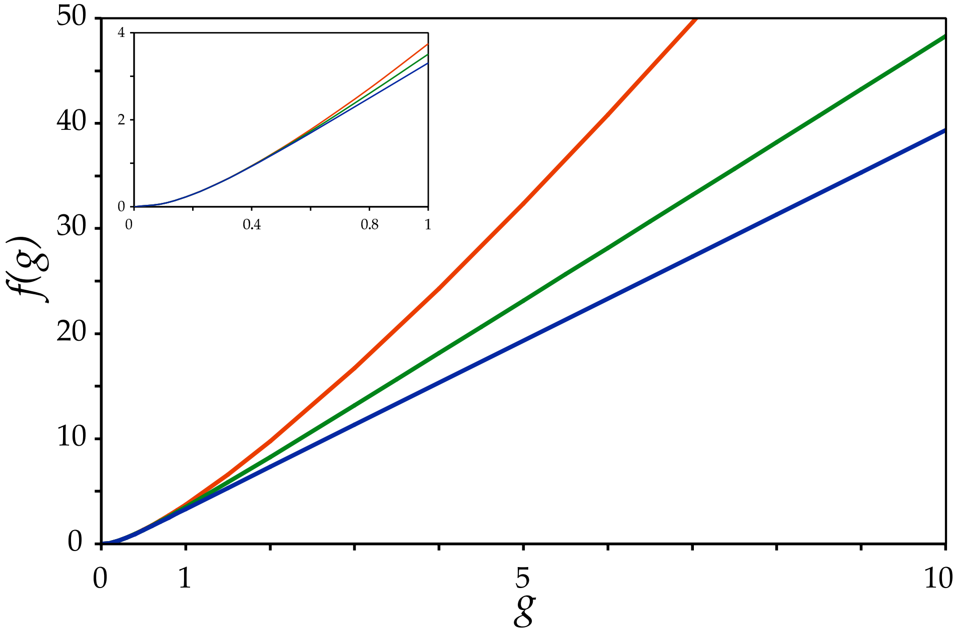


Figure 2.1: Plot of the solutions of the integral equations: $f_m(g)$ for the ES kernel $K^{(m)}$ (upper curve, red), $f_0(g)$ for the kernel $K^{(m)} + K^{(c)}$ (middle curve, green), and $f(g)$ for the BES kernel $K^{(m)} + 2K^{(c)}$ (lower curve, blue). Notice the different asymptotic behaviors. The inset shows the three functions in the crossover region $0 < g < 1$.

2.3 Discussion of Results and Further Investigations

A very satisfying result of this investigation is that the BES integral equation yields a smooth universal function $f(g)$ whose strong coupling expansion is in excellent numerical agreement with the spinning string predictions of [20, 28]. This provides a highly non-trivial confirmation of the AdS/CFT correspondence.

The agreement of this strong coupling expansion was anticipated in [40] based on a similar agreement of the dressing phase. However, some concerns about this argument were raised in [89] based on the slow convergence of the numerical extrapolations. Luckily, our numerical methods employed in solving the integral equation converge rapidly and produce a smooth function that approaches the asymptotics (2.2). The cross-over region of $f(g)$ where it changes from the perturbative to the linear behavior lies right around the radius of convergence, $g_c = 1/4$, corresponding to $g_{YM}^2 N = \pi^2$.

Analytic Structure. The qualitative structure of the interpolating function $f(g)$ is quite similar to that involved in the circular Wilson loop, where the conjectured exact result [99, 100] is

$$\ln\left(\frac{I_1(4\pi g)}{2\pi g}\right) . \tag{2.25}$$

The function (2.25) is analytic on the complex plane, with a series of branch cuts along the imaginary axis, and an essential singularity at infinity. The function $f(g)$ is also expected to have an infinite number of branch cuts along the imaginary axis, and an essential singularity at infinity [40].

Let us compare this with the exact anomalous dimension of a single giant magnon of momentum p [31, 93, 101, 102, 103]:

$$-1+\sqrt{1+16g^2\sin^2\left(\frac{p}{2}\right)} . \tag{2.26}$$

This function has a single branch cut along the imaginary axis, going from $g = \frac{i}{4\sin\left(\frac{p}{2}\right)}$ to infinity, and no essential singularity at infinity. Observables of the gauge theory are composites of giant magnons with various momenta $p \in (0, 2\pi)$. We thus expect the anomalous dimension of a generic unprotected operator to have a superposition of many cuts along the imaginary axis. This should endow a generic multi-magnon state with an analytic structure similar to that of $f(g)$.

We found numerically the presence of the first two branch cuts of $f(g)$ on the imaginary axis, starting at $g = \pm\frac{ni}{4}$, $n = 1, 2$. The first of them, which also occurs for the giant magnon with maximal momentum $p = \pi$, agrees with the summation of the perturbative series [40]. The full structure of $f(g)$ is expected to contain an infinite number of branch cuts accumulating at infinity, where an essential singularity is present.

Fluctuation Density. It is remarkable that the integral equation of [40] allows the universal scaling function $f(g)$, which is not a BPS quantity, to be solved for. This amounts to evaluating the fluctuation density $\hat{\sigma}(t)$ at the origin, $t = 0$, but of course the full function $\hat{\sigma}(t)$ is of interest in its own right, and in the following section we shall derive an exact analytic expression for it in the strong coupling limit.

If one insists on truncating the BES equation at leading order for large g the kernel becomes degenerate and therefore a unique solution for $\hat{\sigma}(t)$ cannot be found without additional assumptions. However, for finite values of g the solution is unique and can be found numerically with high precision. These two statements are consistent because expanding the BES equation in a power series in $1/g$ we find a second equation at subleading order that removes any ambiguity in the strong coupling solution $\hat{\sigma}(t)$.

To gain some intuition our approach will be to first investigate the solution numerically for finite values of the coupling, which will suggest a simple additional requirement that should be imposed on the underdetermined, truncated BES equation to single out a unique solution in the strong coupling limit which is the convergence point of our numerics. After this additional requirement is identified, we find an analytic solution for $\hat{\sigma}(t)$ in the strong coupling limit. We then show that this solution is in fact completely determined by taking into account the subleading contributions to the BES equation, thus confirming analytically the validity of the auxiliary condition suggested by the numerics and of our strong coupling solution $\hat{\sigma}(t)$.

Upon performing the Fourier transform to the rapidity u-plane the corresponding density $\hat{\sigma}(u)$ reads

$$\hat{\sigma}(u) = \frac{1}{4\pi g^2}\left(1 - \frac{\theta(|u|-1)}{\sqrt{2}}\sqrt{1 + \frac{1}{\sqrt{1-\frac{1}{u^2}}}}\right), \tag{2.27}$$

where $\theta(u)$ is the step function. Thus, on the rapidity plane the leading density is an algebraic function which is constant in the interval $|u| < 1$. We see that, in contrast to

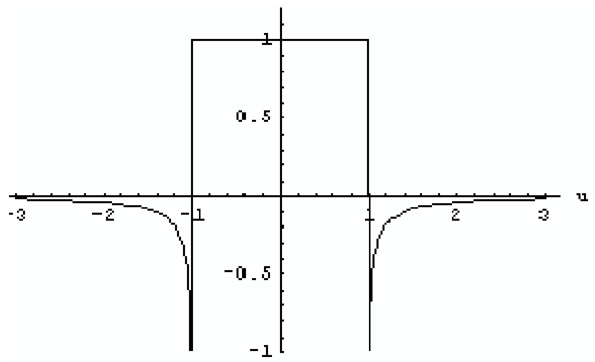


Figure 2.2: Plot of the analytic solution for the leading density $4\pi g^2\hat{\sigma}(u)$.

the weak coupling solution of the BES equation [40], the strong coupling density exhibits a gap between $[-\infty, -1]$ and $[1, \infty]$, see Figure 2.2. Remarkably, this is reminiscent of the behavior of the corresponding solutions describing classical spinning strings.

Below we will analyze the matrix form of the BES equation at strong coupling, first numerically and then analytically. We compute the coefficients in the expansion of $\hat{\sigma}(t)$ in terms of Bessel functions and based on this numerical analysis we make a guess of how these coefficients could be (partially) related to each other at strong coupling. In the strong coupling limit we obtain an analytic equation for the coefficients which turns out to have a degenerate kernel. Supplementing this equation with the proposed relation among the coefficients allows us to find an exact analytical solution for $\hat{\sigma}(t)$. We then show that in fact no guess is necessary if one expands the BES equation to higher order in $1/g$, which leads to a second equation that singles out our solution as the correct one.

We proceed to find an analogous pair of equations for the subleading coefficients at strong coupling. Again they appear at different orders in the inverse coupling constant, each of them individually being a degenerate, half-rank equation. We identify some constraints on the subleading value of $\hat{\sigma}(t)$, but as we shall see this case is more subtle than the leading solution.

2.4 Fluctuation Density at Strong Coupling

In this section we study the density of fluctuations $\hat{\sigma}(t)$ for large values of the coupling constant. First we argue, by performing a numerical analysis, that $\hat{\sigma}(t)$ obeys a certain additional requirement. Then we consider the matrix BES equation at leading order in the large g expansion, use the previously found requirement to solve for the density and confirm by expanding the BES equation further that this solution is in fact complete determined analytically. Finally, we derive and briefly discuss constraints on the subleading value of $\hat{\sigma}(t)$.

As discussed in Section 2.2, the key observation is that for intermediate values of g one can approximate the infinite-dimensional matrices entering the BES equation by matrices of finite rank d, with d not much larger than g. With matrices of finite rank it is possible to solve numerically for the coefficients s_k for different values of the coupling constant and to find the best fit result for an expansion of the type

$$s_k = \frac{1}{g} s_k^{\ell} + \frac{1}{g^2} s_k^{s\ell} + \dots \ . \tag{2.28}$$

As the numerical analysis indicates, the finite rank approximation is valid for computing the coefficients s_k with $k \ll d$. In the table below the values for a few leading coefficients s_k^{ℓ} are exhibited.

k	s_{2k-1}^{ℓ}	s_{2k}^{ℓ}	$100\lvert(s_{2k-1}^{\ell} - s_{2k}^{\ell})/s_{2k}^{\ell}\rvert$
1	0.500006	0.499993	0.003
2	-0.75005	-0.74977	0.038
3	0.93727	0.93676	0.055
4	-1.09281	-1.09415	0.12
5	1.2239	1.2333	0.77

Here we have solved numerically equation (2.19) for numerous points in the range $2 < g < 20$, using $d = 50$. Some comments are in order. First, we stress again that the value for s_1^ℓ is in perfect agreement with the value predicted from string theory, $s_1^\ell = 1/2$, and confirmed numerically above (with a precision higher than the one presented here). Second, notice that the difference between s_{2k-1}^ℓ and s_{2k}^ℓ is in all the cases smaller than 1%, and gets bigger as k increases; this is related to the fact that the rank of the matrices used is finite.

Thus, the numerical analysis suggests that in the limit of infinite rank matrices the following relation holds for the leading coefficients in the strong coupling expansion

$$s_{2k-1}^\ell = s_{2k}^\ell \,. \tag{2.29}$$

As we will see later on, this condition will allow one to solve analytically for the coefficients s_k^ℓ and the values obtained will be in perfect agreement with the ones computed numerically.

Further evidence comes from the fact that one can approximate the matrix elements Z_{mn} by their analytic values at strong coupling (see next subsection). Therefore, one can consider matrices of much higher rank, fix a sufficiently large value of g and compute (numerically) the coefficients s_k^ℓ. Below we present the results for $g = 10000$ and $d = 250$.

k	s_{2k-1}^ℓ	s_{2k}^ℓ	$100\lvert(s_{2k-1}^\ell - s_{2k}^\ell)/s_{2k}^\ell\rvert$
1	0.49993	0.49991	0.0049
2	-0.74943	-0.74938	0.0073
3	0.93585	0.93577	0.0085
4	-1.09033	-1.09023	0.0089
5	1.22455	1.22444	0.0087

As d increases, we see that the difference between s^ℓ_{2k-1} and s^ℓ_{2k} (for $k = 2, 3, \ldots$) decreases considerably. Also, the difference is approximately constant for small values of k. We should stress, however, that a priory there is no reason to expect the results obtained by keeping in the BES equation only the leading term for Z to be valid, since as we will see the subleading terms in the matrix elements Z_{mn} are necessary to fix uniquely the leading order solution for s_k. Surprisingly, one still obtains a good approximation to the large g solution in this fashion. Nevertheless, we regard the present computation as less robust.

To conclude, requiring continuity in g, the leading coefficients s^ℓ_k in the strong coupling expansion of the function $\hat{\sigma}(t)$ exhibit the relation (2.29), which constraints the form of $\hat{\sigma}(t)$.

2.4.1 Analytic Solution at Strong Coupling

For finite real values of g the matrix element Z_{mn} is given by a convergent integral. However, it is not obvious if it is possible to express the result of integration as a power series in $1/g$. Indeed, expanding the integrand in (2.18) as

$$Z_{mn} = \int_0^\infty \mathrm{d}t \, \frac{J_m(t) J_n(t)}{t} \left(\frac{2g}{t} - \frac{1}{2} + \frac{t}{24g} + \ldots \right) , \tag{2.30}$$

leads to a power series (with coefficients given essentially by the Bernoulli numbers) which converges only for $|t/(2g)| < 2\pi$. We see that only the first two leading terms in this expansion can be integrated, while already in the third term divergent integrals appear. It is therefore natural to assume that the expansion of Z develops as

$$Z = gZ^\ell + Z^{s\ell} + \ldots , \tag{2.31}$$

where the first two terms, Z^ℓ and $Z^{s\ell}$, are given by the convergent integrals mentioned above. Explicitly, they are

$$Z^\ell_{mn} = -\frac{8}{\pi}\,\frac{\cos((m-n)\pi/2)}{(m+n+1)(m+n-1)(m-n+1)(m-n-1)}\,, \tag{2.32}$$

$$Z^{s\ell}_{mn} = -\frac{1}{\pi}\,\frac{\sin((m-n)\pi/2)}{m^2-n^2}\,. \tag{2.33}$$

Assumption (2.31) is well supported by the numerics. We have checked that numerical values for Z_{mn} for large g are in a very good agreement with the analytic expressions (2.32) and (2.33). On the other hand, the status of the higher order terms in (2.31) is not obvious. In what follows, we largely restrict our investigation of the BES equation to the first two leading terms.

With this word of caution we proceed to investigate (2.19) in the strong coupling limit. It is not hard to see that it implies the following equation for the leading vector s^ℓ

$$K^{(c)\,\ell} s^\ell = C^\ell e\,, \tag{2.34}$$

where the leading matrices $K^{(c)\,\ell}$ and C^ℓ are obtained by keeping the leading contribution Z^ℓ only. It turns out that for even values of d the kernel $K^{(c)\,\ell}$ has rank $d/2$, hence from the equation above it is possible to solve only for half of the components of s^ℓ. However, imposing the condition (2.29) together with (2.34) allows one to uniquely determine the vector s^ℓ.

In order to find the solution in the limit of infinite d it is convenient to express the leading equation in the following way

$$K^o s^o + K^e s^e = \frac{1}{2}e\,, \tag{2.35}$$

where s^o, s^e are vectors of length $d/2$ comprising the odd and even components of s^ℓ respectively: $s^o = (s^\ell_1, s^\ell_3, s^\ell_5, ...)^{\mathrm{T}}$, $s^e = (s^\ell_2, s^\ell_4, s^\ell_6, ...)^{\mathrm{T}}$ and we recall that $e = (1, 0, 0, ...)^{\mathrm{T}}$.

It is easy to check that equations (2.34) and (2.35) are equivalent provided that

$$(K^o)_{mn} = Z^\ell_{2m-1,2n-1} + Z^\ell_{2m+1,2n-1}\,, \qquad (K^e)_{mn} = Z^\ell_{2m-1,2n} + Z^\ell_{2m+1,2n}\,. \tag{2.36}$$

Then the unique solution of (2.35) satisfying the relation $s^o = s^e$ turns out to be

$$s^\ell_{2k-1} = s^\ell_{2k} = (-1)^{k+1}\frac{\Gamma(k+\frac{1}{2})}{\Gamma(k)\Gamma(\frac{1}{2})}\,. \tag{2.37}$$

This remarkably simple expression for the coefficients s^ℓ_k is the main result of this section. By using the following identities (true in the limit of infinite rank)

$$(K^{e\ -1})_{mn} = -4(-1)^{m+n+1}mn^2, \quad \text{for } n \le m\ , \tag{2.38}$$

$$(K^{e\ -1})_{mn} = -4(-1)^{m+n+1}m^3, \quad \text{for } n > m\ , \tag{2.39}$$

$$(K^{e\ -1}K^o)_{mn} = (-1)^{m-n}\frac{32}{\pi}\frac{m^3}{(4m^2-(1-2n)^2)(1-2n)^2}\ , \tag{2.40}$$

one can check explicitly that the coefficients (2.37) indeed solve (2.35). Thus, we found that restricting the coefficients s_k to the leading order expressions s^ℓ_k the function $s(t)$ is

$$s(t) = \frac{1}{g}\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\Gamma(k+\frac{1}{2})}{\Gamma(k)\Gamma(\frac{1}{2})}\frac{J_{2k}(2gt)+J_{2k-1}(2gt)}{2gt} + \dots\,, \tag{2.41}$$

where we have omitted the subleading contributions. This expression can be considered the leading term in the large g expansion of the density $s(t,g)$ with $g\,t$ kept finite. As is clear from (2.6), we are only interested in values of $s(t)$ for $t \ge 0$. The series can be summed and for this range of t the result expressed in terms of the confluent hypergeometric function of the second kind $U(a,b,x)$:

$$\begin{aligned} s(t) = &-\frac{i}{8\pi g^2 t}\,e^{2igt}\ \Big(\Gamma(\tfrac{3}{4})\,U(-\tfrac{1}{4},0,-4igt)\ +\ \Gamma(\tfrac{5}{4})\,U(\tfrac{1}{4},0,-4igt)\Big) \\ &+\frac{i}{8\pi g^2 t}\,e^{-2igt}\Big(\Gamma(\tfrac{3}{4})\,U(-\tfrac{1}{4},0,4igt)\ +\ \Gamma(\tfrac{5}{4})\,U(\tfrac{1}{4},0,4igt)\Big)\,. \end{aligned} \tag{2.42}$$

The leading density (2.42) has a rather complicated profile. For $g\,t \to 0$ it perfectly reproduces the desired result $s(t) \to 1/(4g)$. On the other hand, the asymptotics of $s(t)$ for $g\,t \to \infty$ exhibit a highly oscillatory behavior.

2.4.2 An Alternative Derivation

Let us now show that the result (2.37) can in fact be derived without resorting to any auxiliary conditions obtained from numerical arguments. The degenerate equation appearing at leading order, which can be used to express one half of the coefficients s_n in terms of the other, can be supplemented by another half-rank equation from subleading terms in the BES equation. Together they determine a unique solution.

Writing out the integral equation (2.6) in the basis (2.16) with explicit matrix indices we find

$$\begin{aligned} s_n \frac{J_n(2gt)}{2gt} = \frac{J_1(2gt)}{2gt} + 8n Z_{2n,1} \frac{J_{2n}(2gt)}{2gt} - 2n Z_{nm} s_m \frac{J_n(2gt)}{2gt} \\ -16n(2m-1) Z_{2n,2m-1} Z_{2m-1,r} s_r \frac{J_{2n}(2gt)}{2gt} \ , \end{aligned} \tag{2.43}$$

where all indices are summed over from 1 to ∞. Now we split up the integral equation according to powers of g and into odd and even rows (indices of Bessel functions).

At $\mathcal{O}(g)$ the odd equation is trivial and the even one reads

$$2(2m-1) Z^{\ell}_{2n,2m-1} Z^{\ell}_{2m-1,r} s^{\ell}_r = Z^{\ell}_{2n,1} = \frac{1}{4}\delta_{n,1} \ . \tag{2.44}$$

This is precisely equation (2.34) employed in the previous subsection. At $\mathcal{O}(1)$ the odd rows lead to the condition

$$Z^{\ell}_{2m-1,r} s^{\ell}_r = \frac{1}{2}\delta_{m,1} \ . \tag{2.45}$$

Actually this equation implies the previous one. It determines one half of the coefficients

s_n^ℓ in terms of the other. Expanding further, at $\mathcal{O}(1)$ the even equation is given by

$$\begin{aligned}&8nZ^{s\ell}_{2n,1} - 4nZ^\ell_{2n,m}s^\ell_m - 16n(2m-1)Z^\ell_{2n,2m-1}Z^\ell_{2m-1,r}s^{s\ell}_r\\&-16n(2m-1)Z^\ell_{2n,2m-1}Z^{s\ell}_{2m-1,r}s^\ell_r - 16n(2m-1)Z^{s\ell}_{2n,2m-1}Z^\ell_{2m-1,r}s^\ell_r = 0\ .\end{aligned} \tag{2.46}$$

To determine the other half of the coefficients s_n^ℓ we need to eliminate $s_n^{s\ell}$ from this equation. To do this we examine the odd equation at $\mathcal{O}(1/g)$

$$-2(2m-1)Z^\ell_{2m-1,r}s^{s\ell}_r - 2(2m-1)Z^{s\ell}_{2m-1,r}s^\ell_r = s^\ell_{2m-1}\ . \tag{2.47}$$

Now we use (2.45) and (2.47) to simplify (2.46), which gives

$$Z^\ell_{2n,m}s^\ell_m - 2Z^\ell_{2n,2m-1}s^\ell_{2m-1} = Z^\ell_{2n,m}(-1)^m s^\ell_m = 0\ . \tag{2.48}$$

This is the second equation we were looking for, which together with (2.45) completely determines s_n^ℓ. If we define a matrix $\tilde{Z}_{nm}$ which is identical to Z_{nm} except for a sign flip when both n is even and m is odd, we can combine the two conditions into the full-rank equation for the strong coupling solution

$$\tilde{Z}^\ell_{nm}s^\ell_m = \frac{1}{2}\delta_{n,1}\ . \tag{2.49}$$

Note that the additional minus signs in the definition of $\tilde{Z}_{nm}$ arise precisely because of the introduction of the dressing kernel, and would be absent if the kernel consisted solely of the main scattering part. Writing out (2.49) explicitly shows that the s_n^ℓ have to satisfy

$$\begin{aligned}&\sum_{k=1}^{\infty}\frac{4(-1)^{n+k+1}s^\ell_{2k-1}}{(2n-2k-1)(2n-2k+1)(2n+2k-3)(2n+2k-1)\pi}\\&\qquad = -\frac{s^\ell_{2n}}{8n(2n-1)} - \frac{s^\ell_{2n-2}}{8(n-1)(2n-1)} + \frac{1}{4}\delta_{n,1}\ ,\end{aligned} \tag{2.50}$$

$$\sum_{k=1}^{\infty} \frac{4(-1)^{n+k+1} s^{\ell}_{2k}}{(2n-2k-1)(2n-2k+1)(2n+2k-1)(2n+2k+1)\pi}$$
$$= \frac{s^{\ell}_{2n+1}}{8n(2n+1)} + \frac{s^{\ell}_{2n-1}}{8n(2n-1)} \,, \tag{2.51}$$

for all $n \geq 1$ (where it is understood that the second term on the right hand side of (2.50) is absent for $n = 1$). Indeed these equations are obeyed by

$$s^{\ell}_{2n-1} = s^{\ell}_{2n} = \frac{(-1)^{n-1}(2n-1)!!}{2^n (n-1)!} \,, \tag{2.52}$$

which is precisely the solution (2.37) found in the previous subsection. To show this, note that the coefficients s^{ℓ}_{2n-1} are generated by the Taylor expansion of $(1+x)^{-3/2}$, which makes it easy to perform the above sums as integrals over functions of the form $x^m(1-x^2)^{-3/2}$ for some appropriate power m chosen to generate the necessary terms in the denominators of the left hand sides of (2.50) and (2.51).

2.4.3 Fluctuation Density in the Rapidity Plane

To get more insight into the structure of the leading solution, we find it convenient to perform the (inverse) Fourier transform of the density $\hat{\sigma}(t) \to \hat{\sigma}(u)$:

$$\hat{\sigma}(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}t \; e^{i\,2gtu} e^{|t|/2} \hat{\sigma}(|t|) \,. \tag{2.53}$$

We recall that u is a rapidity variable originally used to parameterize the Bethe root distributions of gauge and string theory Bethe ansätze [31, 32], which gives another reason for studying the density of fluctuations on the u-plane. Thus, substituting into (2.53) the power series expansion for $\hat{\sigma}(t)$ we get

$$\hat{\sigma}(u) = \frac{1}{2\pi} \sum_{n=1}^{\infty} s_n \int_{-\infty}^{\infty} \mathrm{d}t \; e^{i\,2gtu} e^{-|t|/2} \frac{|t|}{1-e^{-|t|}} \frac{J_n(2g|t|)}{2g|t|} + \dots \,. \tag{2.54}$$

For large values of g this expression can be well approximated as

$$\hat{\sigma}(u) = \frac{1}{2\pi} \sum_{n=1}^{\infty} s_n \int_{-\infty}^{\infty} \mathrm{d}t \; e^{i\,2gtu} e^{-|t|/2} \frac{J_n(2g|t|)}{2g|t|} + \dots \, . \tag{2.55}$$

The last integral is computed by using the following formula [30]

$$\int_0^{\infty} \mathrm{d}t \; e^{\pm 2giut} e^{-t/2} \frac{J_n(2gt)}{2gt} = \frac{(2g)^{n-1}}{n} \left(u^{\pm} \left(1 + \sqrt{1 + 4g^2/(u^{\pm})^2} \right) \right)^{-n} , \tag{2.56}$$

with $u^{\pm} = 1/2 \mp 2igu$.

In this way we obtain the following series representation for the density $\hat{\sigma}(u)$

$$\hat{\sigma}(u) = \frac{1}{2\pi g} \sum_{n=1}^{\infty} s_n f_n + \dots \, , \tag{2.57}$$

where

$$f_n = \frac{(2g)^n}{2n} \left[\left(u^+ (1 + \sqrt{1 + 4g^2/(u^+)^2}) \right)^{-n} + \left(u^- (1 + \sqrt{1 + 4g^2/(u^-)^2}) \right)^{-n} \right] . \tag{2.58}$$

In what follows it is convenient to introduce the expansion parameter $\epsilon = 1/(2g)$. Our considerations above suggest that the density $\hat{\sigma}(u)$ expands starting from the second order in ϵ:

$$\hat{\sigma}(u) \; = \; \epsilon^2 \hat{\sigma}^{\ell}(u) + \epsilon^3 \hat{\sigma}^{s\ell}(u) + \dots \, . \tag{2.59}$$

To find the leading contribution $\hat{\sigma}^{\ell}(u)$ we have to develop the large g expansion of the functions f_n. The result is not uniform, it depends on whether n is even or odd and also on the value of u . For n even we find

$$f_{2k} = \begin{cases} \frac{(-1)^k}{2k} T_{2k}(u) + \mathcal{O}(\epsilon) & \text{for} \quad |u| < 1 \, , \\ \frac{(-1)^k}{2k} \left(u \left(1 + \sqrt{1 - \frac{1}{u^2}} \right) \right)^{-2k} + \mathcal{O}(\epsilon) & \text{for} \quad |u| > 1 \, . \end{cases} \tag{2.60}$$

Here $T_{2k}(u)$ are the Chebyshev polynomials of the first kind. For n odd we obtain

$$f_{2k-1} = \begin{cases} -\frac{(-1)^k}{2k-1}\sqrt{1-u^2}\; U_{2k-2}(u) + \mathcal{O}(\epsilon) & \text{for} \quad |u| < 1\,, \\ 0 + \mathcal{O}(\epsilon) & \text{for} \quad |u| > 1\,, \end{cases} \tag{2.61}$$

where $U_{2k-2}(u)$ are the Chebyshev polynomials of the second kind. We recall that the Chebyshev polynomials of the first and the second kind form a sequence of orthogonal polynomials on the interval $[-1,1]$ with the weights $(1-u^2)^{-1/2}$ and $(1-u^2)^{1/2}$, respectively.

To find the leading density $\hat{\sigma}^\ell(u)$ inside the interval $|u| < 1$ we can use the trigonometric definition of the Chebyshev polynomials which corresponds to choosing parametrization $u = \cos\theta$ with $0 \le \theta \le \pi$. Thus, taking the limit $g \to \infty$ we obtain for the leading density the following expression

$$\hat{\sigma}^\ell(u) = \frac{2}{\pi}\sum_{k=1}^{\infty}\Big(s^\ell_{2k-1} f_{2k-1}(\theta) + s^\ell_{2k} f_{2k}(\theta)\Big)\,, \tag{2.62}$$

where

$$f_{2k}(\theta) = (-1)^k\frac{\cos(2k\theta)}{2k}\,, \qquad f_{2k-1}(\theta) = -(-1)^k\frac{\sin((2k-1)\theta)}{2k-1}\,. \tag{2.63}$$

Given the result (2.37) for the coefficients s^ℓ_k, we can now sum the series for $\hat{\sigma}^\ell(u)$ and obtain

$$\hat{\sigma}^\ell(u) = \frac{1}{\pi}\,. \tag{2.64}$$

Thus, inside the interval $[-1,1]$ the density $\hat{\sigma}^\ell(u)$ is *constant*.

Further, it is easy to sum up the series defining the leading density for $|u| > 1$. The

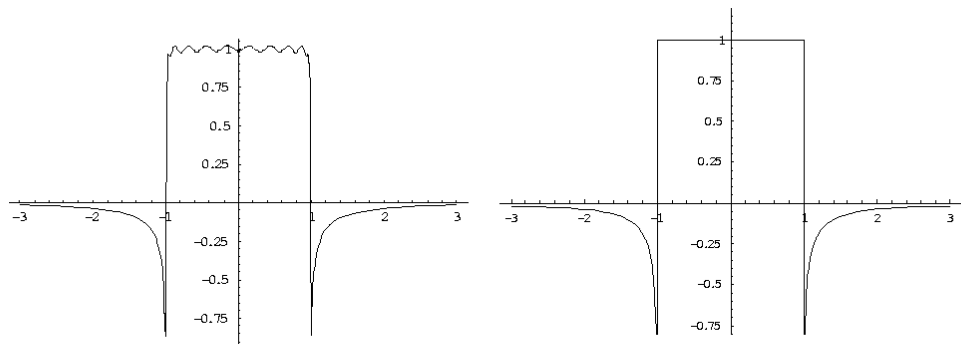


Figure 2.3: The left figure is a plot of a numerical solution for the leading density $\pi\hat{\sigma}^\ell(u)$. The right figure shows the exact analytic solution for the same quantity.

result is

$$\hat{\sigma}^\ell(u) = \frac{1}{2\pi}\left(2 - \frac{\sqrt{2}}{\sqrt{1 - u(u \mp \sqrt{u^2-1})}}\right). \tag{2.65}$$

Here the minus and plus signs in the denominator corresponds to the regions $u > 1$ and $u < -1$, respectively. The plot of the complete analytic solution for $\hat{\sigma}^\ell(u)$ is presented in Figure 2.3. It should be compared to the plot of the numerical solution obtained by using in the series representation for the density with the coefficients s_k^ℓ obtained from our numerical analysis. The numerical plot corresponds to taking $g = 40$ and truncating the series at $k = 20$.

Finally, we mention the expression for the scaling function $f(g)$ in terms of the density $\hat{\sigma}(u)$

$$f(g) = 32g^3 \int_{-\infty}^{\infty} \mathrm{d}u\, \hat{\sigma}(u) = 8g \int_{-\infty}^{\infty} \mathrm{d}u\, \hat{\sigma}^\ell(u) + \ldots = 4g + \ldots\ , \tag{2.66}$$

which confirms analytically the leading term of the numerical results of Section 2.2. This completes our discussion of the leading order analytic solution of the BES equation in the strong coupling limit.

2.4.4 Subleading Corrections

Here we will investigate the first subleading correction to the leading coefficients s_k^ℓ. By expanding the BES equation we have already obtained equation (2.47) which expresses the subleading density $s_n^{s\ell}$ in terms of the leading one:

$$Z^\ell_{2m-1,r} s^{s\ell}_r = -Z^{s\ell}_{2m-1,r} s^\ell_r - \frac{1}{2(2m-1)} s^\ell_{2m-1} \,. \tag{2.67}$$

Again this equation is degenerate and needs to be supplemented by a second one that appears at higher order in $1/g$ in the BES equation. Substituting the explicit form of the coefficients s_n^ℓ obtained above, the right hand side of (2.67) is seen to vanish (i.e. $K^{(c)\,\ell} s^{s\ell} = 0$), which already implies tight restrictions on the subleading corrections $s^{s\ell}$. As discussed above, this equation allows us to solve for half of the components. For instance, using (2.35) and (2.40) we can solve for the even components in terms of the odd ones

$$s^{s\ell}_{2m} = -\sum_{n=1}^{\infty} (-1)^{m-n} \frac{32}{\pi} \frac{m^3}{(4m^2-(1-2n)^2)(1-2n)^2}\, s^{s\ell}_{2n-1} \,. \tag{2.68}$$

To obtain another equation constraining the subleading solution, we examine corrections to the BES equation as follows. From the $\mathcal{O}(1/g)$ contribution to the even rows of the BES equation (2.43) we find

$$\begin{aligned} s^\ell_{2n} = {} & 8n Z^{ss\ell}_{2n,1} - 4n Z^\ell_{2n,m} s^{s\ell}_m - 4n Z^{s\ell}_{2n,m} s^\ell_m \\ & - 16n(2m-1)\Big[Z^\ell_{2n,2m-1} Z^\ell_{2m-1,r} s^{ss\ell}_r + Z^\ell_{2n,2m-1} Z^{s\ell}_{2m-1,r} s^{s\ell}_r + Z^\ell_{2n,2m-1} Z^{ss\ell}_{2m-1,r} s^\ell_r \\ & \qquad + Z^{s\ell}_{2n,2m-1} Z^\ell_{2m-1,r} s^{s\ell}_r + Z^{s\ell}_{2n,2m-1} Z^{s\ell}_{2m-1,r} s^\ell_r + Z^{ss\ell}_{2n,2m-1} Z^\ell_{2m-1,r} s^\ell_r \Big] \,. \end{aligned} \tag{2.69}$$

To eliminate the term in $s^{ss\ell}$ we turn to the odd rows of the $\mathcal{O}(1/g^2)$ equation

$$-2(2m-1)\Big[Z^\ell_{2m-1,r} s^{ss\ell}_r + Z^{s\ell}_{2m-1,r} s^{s\ell}_r + Z^{ss\ell}_{2m-1,r} s^\ell_r \Big] = s^{s\ell}_{2m-1} \,. \tag{2.70}$$

Using this together with (2.45) and (2.47) to simplify (2.69) we obtain

$$Z^{\ell}_{2n,m}(-1)^m s^{s\ell}_m = -Z^{s\ell}_{2n,m}(-1)^m s^{\ell}_m - \frac{1}{4n} s^{\ell}_{2n} \ . \tag{2.71}$$

This can be combined with (2.67) into the full rank equation

$$\tilde{Z}^{\ell}_{nm} s^{s\ell}_m = -\tilde{Z}^{s\ell}_{nm} s^{\ell}_m - \frac{1}{2n} s^{\ell}_n \ , \tag{2.72}$$

where again the right hand side vanishes when evaluated on the leading solution s^ℓ found above, i.e. the subleading corrections appear to satisfy a homogeneous equation.

Thus complications arise at higher order in g which suggest that the equations for the subleading terms in s_n have to be supplemented by additional constraints. Luckily, the equations for the leading solution derived in Subsection 2.4.2 are not affected by this complication. This clearly cannot be the whole story, which is not surprising in view of the divergence of the $\mathcal{O}(1/g)$ correction to Z_{nm} that we have ignored here by naively expanding to third order.

Subsequently this problem was solved, and the complete asymptotic expansion of $f(g)$ determined in an impressive paper [104] (for further work, see [105]). This expansion obeys its own transcendentality principle. In particular, the coefficient of the $1/g$ term in (2.2) was shown to be given by

$$-\frac{\mathrm{K}}{4\pi^2} \approx -0.0232 \ , \tag{2.73}$$

where K is the Catalan constant, in agreement with the numerical result (2.24). As a further check, this coefficient was reproduced analytically from a two-loop calculation in string sigma-model perturbation theory [106, 107].

2.5 Conclusions

In this chapter we have addressed the problem of extrapolating the perturbative expansion controlled by the BES equation, which describes the universal scaling function of high spin operators in $\mathcal{N} = 4$ gauge theory, to large values of the coupling constant and shown that the resulting expression is consistent with string theory predictions.

First, we devised a numerical method to solve the BES equation around the strong coupling point and demonstrated that the numerical solution is in perfect agreement with equation (2.2), providing a non-trivial test of the AdS/CFT correspondence. Thus, the cusp anomaly $f(g)$ is an example of an interpolation function for an observable not protected by supersymmetry that smoothly connects weak and strong coupling regimes. The final form of $f(g)$ was arrived at using inputs from string theory, perturbative gauge theory, and the conjectured exact integrability of planar $\mathcal{N} = 4$ SYM.

Secondly, we have analytically studied the strong coupling limit of the BES equation and demonstrated that expanding it in inverse powers of the coupling constant leads to two equations for the large g solution, one appearing at leading, the other at subleading order in $1/g$. Together they determine a unique solution in the $g \to \infty$ limit whose exact analytic form we present.[2]

As was shown in [40] (see also [109]), the coefficients of the perturbative series describing the solution of the BES equation at weak coupling admit an analytic continuation to strong coupling, where they coincide with those predicted by string theory. The approach we adopt here can be considered as another, complementary way to analytically continue from weak to strong coupling.

On the rapidity u-plane the leading fluctuation density $\hat{\sigma}^{\ell}(u)$ appears to be constant inside the unit interval $|u| < 1$. We could argue that this constant part of $\hat{\sigma}^{\ell}(u)$ is an artifact of the way the BES equation was derived: The non-vanishing constant part of

[2] The leading term in the strong coupling asymptotic expansion of the fluctuation density $\hat{\sigma}(t)$ was derived analytically also in [108], independently of the present work.

the leading density offsets the splitting of the weak-coupling density into a log-divergent one-loop part and a regular higher loop piece carrying $\log S$ as a coefficient. Further, it is not hard to show [97] that the subleading correction inside the unit interval is absent; in a manner of speaking a gap opens between $[-\infty, -1]$ and $[1, \infty]$. This can be qualitatively compared to the results obtained from string theory. Indeed, the solution of the integral equation describing the classical spinning strings in $\mathrm{AdS}_3 \times \mathrm{S}^1$ [110] in the limit $S \to \infty$ with angular momentum J along S^1 fixed has support only outside the interval $|u| < 1$. The same behavior is expected for the GKP solution which is obtained in the limit $J \to 0$. For finite S the solution is elliptic and it exhibits logarithmic singularities in the limit $S \to \infty$. On the other hand, our strong coupling density (2.65) is an algebraic function which carries $\log S$ as a normalization. Of course, this density leads to the same energy as for the GKP string. It is desirable to understand the detailed matching between the string density (higher conserved charges) and the density we found from the strong coupling limit of the BES equation.

Chapter 3

On Normal Modes of a Warped Throat

3.1 Introduction

Duality between the cascading $\mathrm{SU}(k(M+1)) \times \mathrm{SU}(kM)$ gauge theory and type IIB strings on the warped deformed conifold [49] provides a rich yet tractable example of gauge/string correspondence[1]. This background demonstrates in a geometrical language such features of the $\mathrm{SU}(M)$ supersymmetric gluodynamics as color confinement and the breaking of the $\mathbb{Z}_{2M}$ chiral R-symmetry down to $\mathbb{Z}_2$ via gluino condensation [49]. In fact, it has been argued [49] that by reducing the continuous parameter $g_s M$ one can interpolate between the cascading theory solvable in the supergravity limit and $\mathcal{N} = 1$ supersymmetric $\mathrm{SU}(M)$ gauge theory.

The problem of finding the spectra of bound states at large $g_s M$ can be mapped to finding normalizable fluctuations around the supergravity background. This problem is complicated by the presence of 3-form and 5-form fluxes, but some results on the spectra are already available in the literature [52, 53, 113, 114, 115, 116, 117, 118, 119].

[1] For earlier work leading up to this duality, see [44, 51, 111, 112], and for reviews [10, 11, 12]. The most pertinent facts were summarized above in Section 1.2.

A particularly impressive effort was made by Berg, Haack and Mück (BHM) who used a generalized PT ansatz [54] to derive and numerically solve a system of seven coupled scalar equations [116, 117]. Each of the resulting glueballs is even under the charge conjugation $\mathbb{Z}_2$ symmetry preserved by the KS solution (this symmetry was called the $\mathcal{I}$-symmetry in [52, 53]), and therefore has $S^{PC} = 0^{++}$. The present chapter will study three other families of glueballs, which are odd under the $\mathcal{I}$-symmetry. Two of them originate from a pair of coupled scalar equations, generalizing the zero momentum case studied in [52, 53], and have $S^{PC} = 0^{+-}$. The third, pseudoscalar family arises from a decoupled fluctuation of the RR two-form C_2 and has $S^{PC} = 0^{--}$.

An important aspect of the low-energy dynamics is that the baryonic $U(1)_B$ symmetry is broken spontaneously by the condensates of baryonic operators $\mathcal{A}$ and $\mathcal{B}$. This phenomenon, anticipated in the cascading gauge theory in [49, 56], was later demonstrated on the supergravity side where the fluctuations corresponding to the pseudoscalar Goldstone boson and its scalar superpartner [52, 53], as well as the fermionic superpartner [118], were identified. Furthermore, finite deformations along the scalar direction give rise to a continuous family of supergravity solutions [58, 59] dual to the baryonic branch, $\mathcal{A}\mathcal{B} = \text{const}$, of the gauge theory moduli space.

The main purpose of this chapter is to obtain a deeper understanding of the GHK scalar fluctuations [52, 53] and their radial excitations. Our motivation is two-fold. On the one hand, we seek an improved understanding of the glueball spectra and their supermultiplet structure. On the other, we would like to shed new light on the normal modes of the warped deformed conifold throat embedded into a string compactification, which has played a role in models of moduli stabilization [120] and D-brane inflation [121, 122, 123]. In such inflation models, the reheating of the universe involves emission of modes localized near the bottom of the throat, which are dual to glueballs in the gauge theory [124, 125, 126, 127].

This chapter is based on the paper [128] written in collaboration with A. Dymarsky,

I. R. Klebanov and A. Solovyov; it is structured as follows. In Section 3.2 we construct a generalization of the ansatz for the NSNS 2-form and metric perturbations that allows us to study radial excitations of the GHK scalar mode. We derive a system of coupled radial equations and determine their spectrum (by numerically solving those differential equations). In Section 3.3 we show that a similar ansatz for the RR 2-form perturbation decouples from the metric giving rise to a single decoupled equation for pseudoscalar glueballs. In Section 3.4 we argue that the scalar glueballs we find belong to massive axial vector multiplets, and the pseudoscalar glueballs belong to massive vector multiplets. Agreement of the corresponding equations is explicitly demonstrated in the large radius (KT) limit. In Section 3.5 we give a perturbative treatment of the coupled equations for small mass that allows us to study the scalar mass in models where the length of the throat is finite.

3.2 Radial Excitations of the GHK scalar

The ansatz that produced a normalizable scalar mode independent of the 4-dimensional coordinates x^μ was [52, 53]

$$\delta B_2 = \chi(\tau)\, dg^5 \ , \qquad \delta G_{13} = \delta G_{24} = \psi(\tau) \ . \tag{3.1}$$

Our first goal is to find a generalization of this ansatz that will allow us to study the radial excitations of this massless scalar, i.e. the series of modes that exist at non-vanishing $k_\mu^2 = -m_4^2$. Thus, we must include the dependence of all fields on x^μ. Such an ansatz that decouples from other fields at linear order is

$$\delta F_3 = 0\,, \tag{3.2}$$

$$\delta \tilde{F}_5 = 0\,, \tag{3.3}$$

$$\delta B_2 = \chi(x,\tau)\, dg^5 + \partial_\mu \sigma(x,\tau)\, dx^\mu \wedge g^5\,, \tag{3.4}$$

$$\delta H_3 \equiv d\delta B_2 = \chi'\, d\tau \wedge dg^5 + \partial_\mu(\chi - \sigma)\, dx^\mu \wedge dg^5 + \partial_\mu \sigma'\, d\tau \wedge dx^\mu \wedge g^5\,, \tag{3.5}$$

$$\delta G_{13} = \delta G_{24} = \psi(x,\tau)\,. \tag{3.6}$$

The ansatz for δB_2 originates from the longitudinal component of a 5-dimensional vector:

$$\delta B_2 = (A_\tau d\tau + A_\mu dx^\mu) \wedge g^5\ . \tag{3.7}$$

Requiring the 4-dimensional field strength to vanish, $F_{\mu\nu} = 0$, restricts A_μ to be of the form ∂_μ acting on a function. Then, choosing

$$A_\tau = -\chi'\ , \qquad A_\mu = \partial_\mu(\sigma - \chi)\ , \tag{3.8}$$

we recover the ansatz (3.4) up to a gauge transformation.

Yet another gauge equivalent way of writing (3.4) is

$$\delta B_2 = (\chi - \sigma)\, dg^5\ - \sigma'\, d\tau \wedge g^5\ . \tag{3.9}$$

The new feature of our ansatz compared to the generalized PT ansatz used in [116, 117] is the presence of the second function in δB_2 which multiplies $d\tau \wedge g^5$. Terms of this type, which are allowed by the 4-dimensional Lorentz symmetry, turn out to be crucial for studying the modes that are odd under the $\mathcal{I}$-symmetry.

Using $\delta(G^{-1}) = -G^{-1}\,\delta G\, G^{-1}$, we find that $\delta G^{13} = \delta G^{24} = -G^{11}\, G^{33}\,\psi$. The unperturbed metric components (see Subsection 1.2.2 for a review of the KS solution) are

$$G^{11} = G^{22} = \frac{2}{\varepsilon^{4/3} K(\tau) \sinh^2(\tau/2) h^{1/2}(\tau)} , \qquad (3.10)$$

$$G^{33} = G^{44} = \frac{2}{\varepsilon^{4/3} K(\tau) \cosh^2(\tau/2) h^{1/2}(\tau)} , \qquad (3.11)$$

$$G^{55} = G^{\tau\tau} = \frac{6\, K(\tau)^2}{\varepsilon^{4/3} h^{1/2}} , \qquad (3.12)$$

$$G^{\mu\nu} = h^{1/2}\, \eta^{\mu\nu} . \qquad (3.13)$$

In order to find the dynamic equations for the functions ψ, χ and σ in (3.4)-(3.6) we study the linearized supergravity equations below (the type IIB SUGRA equations are reviewed in Appendix A.1).

3.2.1 Equations of Motion for NSNS- and RR-Forms

All the Bianchi identities are automatically satisfied with the ansatz (3.2)-(3.6). Indeed, the relation $d\delta H_3 = 0$ is obvious, and consistent with vanishing $d\delta\tilde{F}_5$ we find that $\delta H_3 \wedge F_3 = 0$ (using (1.38) one can verify that $dg^5 \wedge F_3 = 0$ and $d\tau \wedge g^5 \wedge F_3 = 0$).

The self-duality equation for $\tilde{F}_5$ reads

$$\delta * \tilde{F}_5 = 0 . \qquad (3.14)$$

Given that $\tilde{F}_5$ has components along $g^1 \wedge g^2 \wedge g^3 \wedge g^4 \wedge g^5$ and along $d^4x \wedge d\tau$, our adopted deformation of the metric does not affect $*\tilde{F}_5$ to first order.

Even though the variations of the forms F_3 and $\tilde{F}_5$ are zero, the deformations of their Hodge duals $\delta * F_3$ and $\delta * \tilde{F}_5$ will in general be non-zero because of the deformations of metric components. In the equation for F_3

$$d\delta * F_3 = \tilde{F}_5 \wedge \delta H_3 , \qquad (3.15)$$

the product $\tilde{F}_5 \wedge \delta H_3$ on the right hand side vanishes identically. From the explicit form of F_3 given in (1.38) we see that the perturbation of its Hodge dual will be a closed form $\delta * F_3 = A(x,\tau)\, d^4x \wedge d\tau \wedge (\ldots)$. Note that the term in (1.38) proportional to $d\tau \wedge (g^1 \wedge g^3 + g^2 \wedge g^4)$ is not affected by the deformation of the metric, and thus $d\delta * F_3 = 0$ is satisfied identically.

The remaining equations are nontrivial. In particular

$$d\delta * H_3 = 0\,, \tag{3.16}$$

turns out to be more complicated than the equation for F_3. The variation

$$\delta * H_3 = *\delta H_3 + \delta_G * H_3 \tag{3.17}$$

consists of two parts: $*\delta H_3$ accounting for the deformation of the form H_3 itself, and $\delta_G * H_3$ arising from the deformation of the Hodge star. Explicit calculation shows that

$$\begin{aligned} *\delta H_3 =& -\sqrt{-G}\, G^{11}\, G^{33}\, G^{55}\, \chi'\, d^4x \wedge dg^5 \wedge g^5 \qquad (3.18)\\ & -\sqrt{-G}\, G^{11}\, G^{33}\, |G^{\mu\mu}|\, \partial_\mu(\chi - \sigma)\ *_4\, dx^\mu \wedge d\tau \wedge dg^5 \wedge g^5 \\ & +\tfrac{1}{2}\sqrt{-G}\, (G^{55})^2\, |G^{\mu\mu}|\, \partial_\mu \sigma'\ *_4\, dx^\mu \wedge dg^5 \wedge dg^5\,, \\ \delta_G * H_3 =& -\frac{g_s M \alpha'}{2}\, \sqrt{-G}\, G^{11}\, G^{33}\, G^{55} \big[f' G^{11} + k' G^{33}\big] \psi\, d^4x \wedge dg^5 \wedge g^5\,, \qquad (3.19) \end{aligned}$$

where in the last equation we have introduced the shorthands

$$f(\tau) \equiv \frac{\tau \coth\tau - 1}{2\sinh\tau}(\cosh\tau - 1)\ , \qquad k(\tau) \equiv \frac{\tau \coth\tau - 1}{2\sinh\tau}(\cosh\tau + 1)\ . \tag{3.20}$$

The four-dimensional Hodge star $*_4$ is taken with respect to the standard Minkowski metric. Differentiating the expression for $\delta * H_3$ and equating to zero the coefficients multiplying linearly independent forms gives three equations:

$$d^4x \wedge dg^5 \wedge dg^5 : \; 2\, G^{11} G^{33} \left[\frac{g_s M \alpha'}{2} \left[f' G^{11} + k' G^{33} \right] \psi + \chi' \right] = G^{55}\, h^{\frac{1}{2}}\, \Box_4 \sigma' \,, \tag{3.21}$$

$$d^4x \wedge d\tau \wedge dg^5 \wedge g^5 : \; \partial_\tau \left\{ \sqrt{-G}\, G^{11} G^{33} G^{55} \left[\frac{g_s M \alpha'}{2} \left[f' G^{11} + k' G^{33} \right] \psi + \chi' \right] \right\} + \sqrt{-G}\, G^{11} G^{33}\, h^{\frac{1}{2}}\, \Box_4 (\chi - \sigma) = 0 \,, \tag{3.22}$$

$$*_4 dx^\mu \wedge d\tau \wedge dg^5 \wedge dg^5 : \; 2 \sqrt{-G}\, G^{11} G^{33}\, h^{\frac{1}{2}}\, \partial_\mu (\chi - \sigma) + \partial_\tau \left\{ \sqrt{-G}\, (G^{55})^2\, h^{\frac{1}{2}}\, \partial_\mu \sigma' \right\} = 0 \,, \tag{3.23}$$

where we have substituted for the warp factor $|G^{\mu\mu}| = h^{1/2}$ (no summation over μ is implied). Not all of these equations are independent. Indeed, using (3.21) equation (3.22) simplifies to

$$\partial_\tau \left\{ \sqrt{-G}\, (G^{55})^2\, h^{1/2}\, \Box_4 \sigma' \right\} + 2 \sqrt{-G}\, G^{11}\, G^{33}\, h^{1/2}\, \Box_4 (\chi - \sigma) = 0 \,. \tag{3.24}$$

This is exactly what we obtain by acting on (3.23) with ∂^μ and contracting indices. Thus only (3.21) and (3.23) are independent. The coefficient functions in these equations are given by (we have dropped some inessential constant factor in $\sqrt{-G}$):

$$f' G^{11} + k' G^{33} = \frac{2\, (\sinh 2\tau - 2\tau)}{\varepsilon^{4/3}\, h^{1/2}\, K(\tau) \sinh^3 \tau} = \frac{4\, K(\tau)^2}{\varepsilon^{4/3}\, h^{1/2}} \,, \tag{3.25}$$

$$G^{11}\, G^{33} = \frac{16}{\varepsilon^{8/3}\, h\, K(\tau)^2 \sinh^2 \tau} \,, \tag{3.26}$$

$$G^{55}\, h^{1/2} = \frac{6\, K(\tau)^2}{\varepsilon^{4/3}} \,, \tag{3.27}$$

$$\sqrt{-G}\, G^{11}\, G^{33}\, h^{1/2} \sim \frac{4}{K(\tau)^2} \,, \tag{3.28}$$

$$\sqrt{-G}\, (G^{55})^2\, h^{1/2} \sim 9\, K(\tau)^4\, \sinh^2 \tau \,. \tag{3.29}$$

Taking into account these expressions, equations (3.21) and (3.23) read

$$2(g_s M\alpha')\,\frac{K(\tau)^2}{\varepsilon^{4/3}\,\sqrt{h(\tau)}}\,\psi + \chi' = \frac{3}{16}\,\varepsilon^{4/3}\,h(\tau)\,K(\tau)^4\,\sinh^2\tau\,\Box_4\sigma'\,, \tag{3.30}$$

$$\partial_\mu(\chi-\sigma) + \frac{9}{8}\,K(\tau)^2\,\partial_\tau\Big\{K^4\,\sinh^2\tau\,\partial_\mu\sigma'\Big\} = 0\,. \tag{3.31}$$

3.2.2 Einstein Equations

The first order perturbation of the Ricci curvature tensor is given by

$$\delta R_{ij} = \frac{1}{2}\left(-\delta G_a{}^a{}_{;ij} - \delta G_{ij;a}{}^a + \delta G_{ai;j}{}^a + \delta G_{aj;i}{}^a\right)\ , \tag{3.32}$$

where covariant derivatives and contractions of indices are performed using the unperturbed metric. The first term in this expression vanishes because the metric perturbation is traceless. The remaining three terms combine to give the only non-zero perturbations $\delta R_{13} = \delta R_{24}$:

$$\begin{aligned}\delta R_{13} =& -\frac{3}{\varepsilon^{4/3}}K^3\sinh\tau\ z\left[\frac{K''}{K} + \frac{1}{2}\frac{h''}{h} + \frac{z''}{z} + \frac{(K')^2}{K^2} - \frac{1}{2}\frac{(h')^2}{h^2} + \frac{K'}{K}\frac{h'}{h}\right.\\ &\left. + 2\,\frac{K'}{K}\frac{z'}{z} + \coth\tau\left(\frac{h'}{h} + 4\,\frac{K'}{K} + 2\,\frac{z'}{z}\right) + 2 - \frac{1}{\sinh^2\tau} - \frac{4}{9}\,\frac{1}{\sinh^2\tau\,K^6}\right]\\ &-\tfrac{1}{2}h(\tau)K\sinh\tau\ \Box_4 z\\ =& -\frac{3}{\varepsilon^{4/3}}K^3\sinh\tau\ z\left[\frac{1}{2}\frac{\big((K\sinh\tau)^2\,(\ln h)'\big)'}{(K\sinh\tau)^2} + \frac{\big((K\sinh\tau)^2\,z'\big)'}{(K\sinh\tau)^2 z}\right.\\ &\left. - \frac{2}{\sinh^2\tau} - \frac{8}{9}\frac{1}{K^6\sinh^2\tau} + \frac{4}{3}\frac{\cosh\tau}{K^3\sinh^2\tau}\right] - \tfrac{1}{2}h(\tau)K\sinh\tau\ \Box_4 z\,,\end{aligned} \tag{3.33}$$

where $z(x,\tau)$ is defined by

$$\psi(x,\tau) = h^{1/2}K\sinh\tau\ z(x,\tau) = 2^{-1/3}(\sinh 2\tau - 2\tau)^{1/3}h^{1/2}\,z(x,\tau)\,. \tag{3.34}$$

The source terms on the right hand side of the Einstein equation $R_{ij} = T_{ij}$, spelled out in (A.3), are due to the deformations of the metric and B_2 form. It turns out that the only nontrivial deformations are those with indices 13 or 24, with $\delta T_{13} = \delta T_{24}$. E.g. for the 13 component δT_{13} we have the following contributions:

$$
\begin{aligned}
\frac{1}{4}\,\delta_B(H_{1ab}\,{H_3}^{ab}) &= \frac{1}{4}\,[H_{1ab}\,\delta {H_3}^{ab} + \delta H_{1ab}\,{H_3}^{ab}] \\
&= \frac{1}{2}\,\Big[G^{11}\,H_{12\tau}\,\delta H_{32\tau} + G^{33}\,\delta H_{14\tau}\,H_{34\tau}\Big]\,G^{55} \\
&= -\frac{1}{4}\,(g_s M\alpha')\,G^{55}\,\Big[G^{11}\,f' + G^{33}\,k'\Big]\,\chi'\ , \qquad (3.35) \\
\frac{g_s^2}{96}\,\delta_G(\tilde{F}_{1abcd}{\tilde{F}_3}{}^{abcd}) &= \frac{g_s^2}{4}\,(G^{11})^2\,(G^{33})^2\,G^{55}\,(\tilde{F}_{12345})^2\,\psi\,, \qquad (3.36) \\
\frac{1}{4}\,\delta_G(H_{1ab}{H_3}^{ab}) &= \frac{1}{2}\,\Big[H_{135}\,H_{315}\,\delta G^{13}\,G^{55} + H_{12\tau}\,H_{34\tau}\,\delta G^{24}\,G^{\tau\tau}\Big] \\
&= \frac{1}{2}\Big[(H_{135})^2 - H_{12\tau}\,H_{34\tau}\Big]\,G^{11}\,G^{33}\,G^{55}\,\psi \\
&= \frac{1}{8}\,(g_s M\alpha')^2\,\left[\frac{1}{4}(k-f)^2 - f'k'\right]\,G^{11}\,G^{33}\,G^{55}\,\psi\,, \qquad (3.37) \\
\frac{g_s^2}{4}\,\delta_G(F_{1ab}{F_3}^{ab}) &= \frac{g_s^2}{2}\,\Big[F_{125}\,F_{345}\,\delta G^{24}\,G^{55} + F_{13\tau}\,F_{31\tau}\,\delta G^{31}\,G^{\tau\tau}\Big] \\
&= \frac{g_s^2}{2}\Big[(F_{13\tau})^2 - F_{125}\,F_{345}\Big]\,G^{11}\,G^{33}\,G^{55}\,\psi \\
&= \frac{1}{8}\,(g_s M\alpha')^2\,\Big[F'^2 - F(1-F)\Big]\,G^{11}\,G^{33}\,G^{55}\,\psi\,, \qquad (3.38)
\end{aligned}
$$

with $F(\tau) \equiv (\sinh\tau - \tau)/(2\sinh\tau)$ and

$$
\begin{aligned}
-\frac{1}{48}\,\delta_G\Big[G_{13}(H_{abc}H^{abc} + g_s^2 F_{abc}F^{abc})\Big] &= -\frac{1}{8}\,(H_3^2 + g_s^2 F_3^2)\,\psi \\
&= -\frac{1}{32}\,(g_s M\alpha')^2\,G^{55}\,\Big[(G^{11})^2\,f'^2 + (G^{33})^2\,k'^2 \\
&\quad + \tfrac{1}{2}\,G^{11}\,G^{33}\,(k-f)^2 + (G^{11})^2\,F^2 \\
&\quad + (G^{33})^2\,(1-F)^2 + 2\,G^{11}\,G^{33}\,F'^2\Big]\psi\,. \qquad (3.39)
\end{aligned}
$$

Denoting

$$
\delta T_{13} = \Big[A_1(\tau) + A_2(\tau)\Big]\,\psi(x,\tau) + B(\tau)\,\chi'(x,\tau)\,, \qquad (3.40)
$$

where A_1 stands for the contribution from $\tilde{F}_5$, we get

$$
\begin{aligned}
A_1(\tau) &= \frac{3(g_s M\alpha')^4}{2^{1/3}\varepsilon^{20/3}h^{5/2}} \frac{(\tau\coth\tau - 1)^2(\sinh 2\tau - 2\tau)^{4/3}}{\sinh^6\tau}\,, \\
A_2(\tau) &= -\frac{3\,(g_s M\alpha')^2}{8\varepsilon^4 h^{3/2}\sinh^6\tau}\Big[3\cosh 4\tau - 8\tau\sinh 2\tau - 8\tau^2\cosh 2\tau - 8\cosh 2\tau + 16\tau^2 + 5\Big]\,, \\
B(\tau) &= -3\,(g_s M\alpha')\,\frac{(\sinh 2\tau - 2\tau)\,K(\tau)}{\varepsilon^{8/3}\,h\,\sinh^3\tau}\,.
\end{aligned}
\tag{3.41}
$$

Then eliminating χ' with the help of (3.21) yields

$$
\begin{aligned}
\delta T_{13} &= \frac{3}{2^{2/3}}\,\frac{(g_s M\alpha')^4}{\varepsilon^{20/3}h^2}\,\frac{(\tau\coth\tau - 1)^2(\sinh 2\tau - 2\tau)^{5/3}}{\sinh^6\tau}\,z(x,\tau) \\
&\quad + \frac{3}{8\cdot 2^{1/3}}\,\frac{(g_s M\alpha')^2}{\varepsilon^4 h}\,\frac{(\sinh 2\tau - 2\tau)^{1/3}}{\sinh^6\tau}\times \\
&\qquad \times\Big[\cosh 4\tau + 8(1+\tau^2)\cosh 2\tau - 24\tau\sinh 2\tau + 16\tau^2 - 9\Big]\,z(x,\tau) \\
&\quad - \frac{9}{16}\,\frac{g_s M\alpha'}{\varepsilon^{4/3}}\,\frac{\sinh 2\tau - 2\tau}{\sinh\tau}\,K^5\,\Box_4\sigma'(x,\tau) \\
&= -\frac{3}{\varepsilon^{4/3}}K^3\sinh\tau\left[-\frac{1}{2}\,\frac{(h')^2}{h^2} + \frac{1}{2}\frac{h''}{h} + \frac{K'}{K}\frac{h'}{h} + \coth\tau\frac{h'}{h}\right]\,z \\
&\quad - \frac{9}{16}\,\frac{g_s M\alpha'}{\varepsilon^{4/3}}\,\frac{\sinh 2\tau - 2\tau}{\sinh\tau}\,K^5\,\Box_4\sigma'\,.
\end{aligned}
\tag{3.42}
$$

Equating (3.33) and (3.42) we obtain the final form of the linearized Einstein equation.

3.2.3 Two Coupled Scalars

Combining the equations for the field strengths and the Einstein equations we have the system

$$
(g_s M\alpha')\,\frac{\sinh 2\tau - 2\tau}{\varepsilon^{4/3}\sinh^2\tau}\,z + \chi' = \frac{3}{16}\,\varepsilon^{4/3}\,h(\tau)\,K(\tau)^4\,\sinh^2\tau\,\Box_4\sigma'\,,
\tag{3.43}
$$

$$
\partial_\mu(\chi - \sigma) = -\frac{9}{8}\,K(\tau)^2\,\partial_\tau\Big\{K^4\,\sinh^2\tau\,\partial_\mu\sigma'\Big\}\,,
\tag{3.44}
$$

$$
\begin{aligned}
\frac{\left((K\sinh\tau)^2\,z'\right)'}{(K\sinh\tau)^2} + \frac{\varepsilon^{4/3}h}{6\,K^2}\,\Box_4 z &= \left(\frac{2}{\sinh^2\tau} + \frac{8}{9}\frac{1}{K^6\sinh^2\tau} - \frac{4}{3}\frac{\cosh\tau}{K^3\sinh^2\tau}\right)z \\
&\quad + \frac{3}{16}\,(g_s M\alpha')\,\frac{\sinh 2\tau - 2\tau}{\sinh^2\tau}\,K^2\,\Box_4\sigma'\,.
\end{aligned}
\tag{3.45}
$$

Note that χ can be eliminated between (3.43) and (3.44). Further, a change of variables

$$\tilde{z} = zK \sinh\tau \ , \tag{3.46}$$

$$\tilde{w} = \frac{\varepsilon^{4/3}}{g_s M\alpha'} K^5 \sinh^2\tau \ \sigma' \ , \tag{3.47}$$

leads to a more symmetric pair of equations

$$\tilde{z}'' - \frac{2}{\sinh^2\tau}\,\tilde{z} + \frac{\varepsilon^{4/3}h}{6K^2}\,\Box_4\tilde{z} = \frac{3(g_sM\alpha')^2}{16\varepsilon^{4/3}}\,\frac{\sinh 2\tau - 2\tau}{K^2\sinh^3\tau}\,\Box_4\tilde{w} \ , \tag{3.48}$$

$$\tilde{w}'' - \frac{\cosh^2\tau + 1}{\sinh^2\tau}\,\tilde{w} + \frac{\varepsilon^{4/3}h}{6K^2}\,\Box_4\tilde{w} = \frac{8}{9}\,\frac{\sinh 2\tau - 2\tau}{K^2\sinh^3\tau}\,\tilde{z}\,. \tag{3.49}$$

Introducing the dimensionless mass-squared $\tilde{m}^2$ according to

$$\tilde{m}^2 = m_4^2\,\frac{2^{2/3}(g_sM\alpha')^2}{6\,\varepsilon^{4/3}}\,, \tag{3.50}$$

we can rewrite the equations for $\tilde{z}$ and $\tilde{w}$ as

$$\tilde{z}'' - \frac{2}{\sinh^2\tau}\,\tilde{z} + \tilde{m}^2\frac{I(\tau)}{K^2(\tau)}\,\tilde{z} = \tilde{m}^2\frac{9}{4\cdot 2^{2/3}}\,K(\tau)\,\tilde{w} \ , \tag{3.51}$$

$$\tilde{w}'' - \frac{\cosh^2\tau + 1}{\sinh^2\tau}\,\tilde{w} + \tilde{m}^2\frac{I(\tau)}{K^2(\tau)}\,\tilde{w} = \frac{16}{9}\,K(\tau)\,\tilde{z}\,. \tag{3.52}$$

This is a system of coupled equations which defines the mass spectrum of certain scalar glueballs with positive 4-dimensional parity. The natural charge conjugation symmetry of the KS background is the $\mathcal{I}$-symmetry, under which these modes are odd. Therefore, we assign $S^{PC} = 0^{+-}$ to this family of glueballs.[2]

In the massless case these equations lead to the GHK solution [52, 53]. If we assume that $\Box_4 = -k_\mu^2 = m_4^2 = 0$, then there are two solutions [52, 53], $\tilde{z}_1 = \coth\tau$ and

[2]For comparison, the glueballs found in [116, 117] are 0^{++}. The glueballs whose spectrum comes from the minimal scalar equation [113, 114] resulting from the analysis of graviton fluctuations are 2^{++}. The axial vector $U(1)_R$ fluctuations [119] give rise to 1^{++} glueballs whose masses are also determined by the minimal scalar equation.

$\tilde{z}_2 = \tau \coth\tau - 1$. The solution for $\tilde{z}$ which is non-singular at the origin is $\tilde{z} = \tau \coth\tau - 1$. Substituting it into the second equation, we find

$$\tilde{w}'' - \frac{\cosh^2\tau + 1}{\sinh^2\tau}\,\tilde{w} = \frac{16}{9}\,K(\tau)\,(\tau\coth\tau - 1) \equiv -\,\frac{2^{2/3}\,8}{9}\,I'(\tau)\sinh\tau\,. \tag{3.53}$$

The two solutions of the homogeneous equation are given by $\tilde{w}_1 = 1/\sinh\tau$ and by $\tilde{w}_2 = (\sinh 2\tau - 2\tau)/\sinh\tau$; both of them are singular either at zero or at infinity. This means that the regular solution of the inhomogeneous equation is uniquely fixed. With the Wronskian $W(\tilde{w}_1, \tilde{w}_2) = \tilde{w}_1\tilde{w}_2' - \tilde{w}_1'\tilde{w}_2 = 4$, we can find a general solution

$$\begin{aligned}\tilde{w}(\tau) = -\,\frac{2^{2/3}\,8}{9}\,\Big\{&\tilde{w}_1(\tau)\big[C_1 - \int^\tau dx\,\frac{\tilde{w}_2(x)}{W(x)}\,I'(x)\sinh x\big] \\ &+\tilde{w}_2(\tau)\big[C_2 + \int^\tau dx\,\frac{\tilde{w}_1(x)}{W(x)}\,I'(x)\sinh x\big]\Big\}\,.\end{aligned} \tag{3.54}$$

Integrating by parts and choosing the particular homogeneous solution to make $\tilde{w}$ well behaved at both zero and infinity we get

$$\tilde{w}(\tau) = -\,\frac{2^{2/3}\,8}{9}\,\frac{1}{\sinh\tau}\int_0^\tau dx\,I(x)\sinh^2 x\,. \tag{3.55}$$

Let us note that the non-vanishing $\tilde{w}$ in the zero momentum case $k_\mu = 0$ is not in contradiction with the GHK solution. This is because $\tilde{w}$ enters (3.2)-(3.6) only through $\partial_\mu\sigma$, which is zero as long as the momentum vanishes.

3.2.4 Numerical Analysis

To determine the spectrum of glueballs in the field theory, we need to solve the eigenvalue problem for $\tilde{m}^2$ in the infinite throat limit. This system of equations (3.51)-(3.52) does not seem amenable to analytical solution and we employ a numerical approach to find the spectrum of normalizable solutions. It is convenient to use the determinant method, which generalizes the usual shooting technique to a system of several coupled equations (see [117]). Let us briefly describe the details of the numerical analysis as well as the subtleties specific to the system (3.51)-(3.52).

A standard method of finding the spectrum of a single second-order differential equation is the shooting technique. For a system of several coupled linear equations the shooting method has to be generalized. The idea underlying the calculation scheme (called the determinant method [117]) is to set the initial conditions at infinity corresponding to the two solutions regular at infinity, $(\tilde{z}_1(\tau), \tilde{w}_1(\tau))^{\mathrm{T}}$ and $(\tilde{z}_2(\tau), \tilde{w}_2(\tau))^{\mathrm{T}}$, and extend them numerically to small τ. Then the matrix

$$\begin{pmatrix} \tilde{z}_1(0) & \tilde{z}_2(0) \\ \tilde{w}_1(0) & \tilde{w}_2(0) \end{pmatrix} \tag{3.56}$$

becomes degenerate at the critical points (eigenvalues) in the spectral parameter space.

Let us find the asymptotic behavior of regular and singular solutions near both zero and infinity. At small τ equations (3.51) and (3.52) decouple,

$$\tilde{z}'' - \frac{2}{\tau^2}\,\tilde{z} = 0\,, \tag{3.57}$$

$$\tilde{w}'' - \frac{2}{\tau^2}\,\tilde{w} = 0\,. \tag{3.58}$$

There are two regular solutions with $\tilde{z},\ \tilde{w} \sim \tau^2$ and also two singular solutions with $\tilde{z},\ \tilde{w} \sim 1/\tau$. For large τ we have

$$\tilde{z}'' = \tilde{m}^2 \frac{9}{4\cdot 2^{1/3}}\, e^{-\tau/3}\tilde{w}\,, \tag{3.59}$$

$$\tilde{w}'' - \tilde{w} = \frac{16\cdot 2^{1/3}}{9}\, e^{-\tau/3}\,\tilde{z}\,. \tag{3.60}$$

The asymptotic behavior of the two regular solutions is

$$\begin{pmatrix} \tilde{z}_1 \\ \tilde{w}_1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2^{4/3}\, e^{-\tau/3} \end{pmatrix}, \qquad \begin{pmatrix} \tilde{z}_2 \\ \tilde{w}_2 \end{pmatrix} = \begin{pmatrix} \frac{81}{64\cdot 2^{1/3}}\, \tilde{m}^2 e^{-4\tau/3} \\ e^{-\tau} \end{pmatrix}, \tag{3.61}$$

and the singular solutions are

$$\begin{pmatrix} \tilde{z}_3 \\ \tilde{w}_3 \end{pmatrix} = \begin{pmatrix} \tau \\ -2^{4/3}\left(\tau - \frac{3}{4}\right)e^{-\tau/3} \end{pmatrix}, \qquad \begin{pmatrix} \tilde{z}_4 \\ \tilde{w}_4 \end{pmatrix} = \begin{pmatrix} \frac{81}{16\cdot 2^{1/3}}\,\tilde{m}^2 e^{2\tau/3} \\ e^{\tau} \end{pmatrix}. \qquad (3.62)$$

A particular subtlety of this setup is that at large τ the two singular solutions don't diverge equally fast: one of them grows exponentially while the other is only linear in τ. This makes it difficult to start shooting from zero, since imposing the regularity condition at infinity would require vanishing of both linear and exponential terms. To cancel the linear term in the presence of the exponential one is difficult to achieve numerically. That is why for this particular system it is convenient to start shooting from large τ, since both singular solutions at zero share the same behavior ($\sim 1/\tau$).

The result is that the spectrum consists of two distinct series, each with a quadratic growth of $\tilde{m}_n^2$ for large n. These series are interpreted as the radial excitation spectra of two different particles. The lowest eigenvalues ($\tilde{m}^2 < 100$) for these spectra are shown in Table 3.1. The quadratic fit for spectrum I is

$$\tilde{m}^2_{In} = 2.31 + 1.91\,n + 0.294\,n^2\,. \qquad (3.63)$$

For spectrum II we drop the lowest eigenvalue when fitting, and find

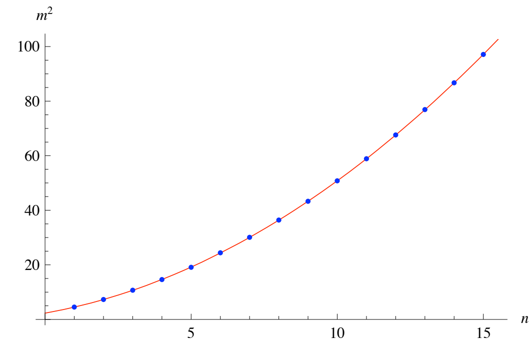
$$\tilde{m}^2_{IIn} = 0.36 + 0.14\,n + 0.279\,n^2\,. \qquad (3.64)$$

It is interesting to compare these results with those found for the 0^{++} modes by Berg, Haack and Mück (BHM) [117]. Their conventions correspond to a particular choice of the KS parameters, and the relation between the masses is

$$m^2_{BHM} = (3/2)^{2/3} I(0)\,\tilde{m}^2 \approx 0.9409\,\tilde{m}^2\,. \qquad (3.65)$$

Spectrum I:

n	$\tilde{m}_n^2$	n	$\tilde{m}_n^2$	n	$\tilde{m}_n^2$	n	$\tilde{m}_n^2$
1	4.53	5	19.1	9	43.3	13	76.9
2	7.30	6	24.4	10	50.8	14	86.7
3	10.7	7	30.1	11	58.9	15	97.1
4	14.6	8	36.4	12	67.6		



Spectrum II:

n	$\tilde{m}_n^2$	n	$\tilde{m}_n^2$	n	$\tilde{m}_n^2$	n	$\tilde{m}_n^2$
1	.129	6	8.06	11	30.1	16	65.1
2	.703	7	11.2	12	35.5	17	73.9
3	1.76	8	15.0	13	42.1	18	83.3
4	3.33	9	19.3	14	49.2	19	93.3
5	5.43	10	24.1	15	56.9		

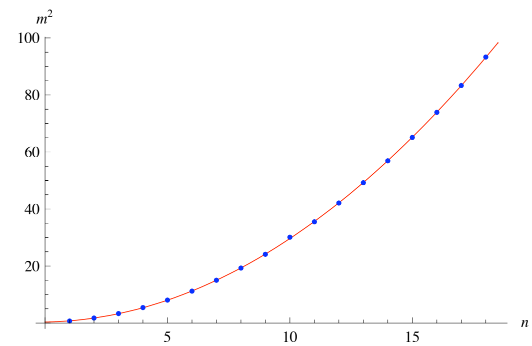


Table 3.1: Non-zero eigenvalues with $\tilde{m}^2 < 100$. There are the two distinct spectra. Both spectra can be fitted by quadratic polynomials in the eigenvalue number n (the red line in the plots).

Using this relation one can convert the mass eigenvalues to the BHM normalization. We note that the lightest glueball we find, the first entry of spectrum II in Table 3.1, has $m^2_{BHM} \approx 0.121$. For comparison, the lightest 0^{++} eigenvalue found in [117] has mass-squared $m^2_{BHM} \approx 0.185$. The fact that the 0^{+-} sector has the lightest glueballs may be qualitatively understood as follows. Roughly speaking, glueball masses increase with the dimensions of the operators that create them. The lowest dimension operator from the 0^{++} sector is the gluino bilinear $\mathrm{Tr}\lambda\lambda$ of dimension 3, but the 0^{+-} sector contains an operator of dimension 2, namely $\mathrm{Tr}(A^\dagger A - B^\dagger B)$.

Converting the asymptotics of the two spectra to BHM units, we find

$$m^2_{I\,BHM} \approx 2.17 + 1.79\,n + 0.277\,n^2\,, \tag{3.66}$$

$$m^2_{II\,BHM} \approx 0.34 + 0.13\,n + 0.262\,n^2\,. \tag{3.67}$$

The coefficients of the quadratic terms are close to those found in [117]. The quadratic dependence on n, which is characteristic of Kaluza-Klein theory, is a special feature of strongly coupled gauge theories that have weakly curved gravity duals (see [129] for a discussion). Note that m_4^2 is obtained from $\tilde{m}^2$ through multiplying by a factor $\sim T_s/(g_s M)$, where T_s is the confining string tension. Thus, for $n \ll \sqrt{g_s M}$ these modes are much lighter than the string tension scale, and therefore much lighter than all glueballs with spin > 2. Such anomalously light bound states appear to be typical of gauge theories that stay very strongly coupled in the UV, such as the cascading gauge theory; they do not appear in asymptotically free gauge theories. Therefore, the anomalously light glueballs could perhaps be used as a 'special signature' of gauge theories with gravity duals if they are realized in nature.

One may be puzzled why the spectrum in Table 3.1 does not include the GHK massless mode. This is because in solving the coupled equations (3.51)-(3.52) we required that both wave-functions $\tilde{z}$ and $\tilde{w}$ go to a constant as $\tau \to \infty$. This excludes the GHK zero mode which grows as $\tilde{z} \sim \tau$. On the other hand, this growth is a lot slower than the exponential growth found for generic solutions. The meaning of the GHK mode as the baryonic branch modulus seems to be well established since even the solutions at finite distance along this modulus are available [58, 59]. Thus, the GHK scalar zero-mode should be normalizable with a proper definition of norm. In fact, the GHK pseudoscalar and its fermionic superpartner are normalizable [52, 53, 118]; therefore, the supersymmetry of the problem implies that the GHK scalar is normalizable as well and is part of the spectrum.

3.3 Pseudoscalar Modes from the RR Sector

The type of ansatz used in Section 3.2 works even more simply for the RR 2-form field:

$$\delta H_3 = 0\,, \tag{3.68}$$

$$\delta \tilde{F}_5 = 0\,, \tag{3.69}$$

$$\delta C_2 = \chi(x,\tau)\, dg^5 + \partial_\mu \sigma(x,\tau)\, dx^\mu \wedge g^5\,, \tag{3.70}$$

$$\delta F_3 \equiv d\delta C_2 = \chi'\, d\tau \wedge dg^5 + \partial_\mu(\chi - \sigma)\, dx^\mu \wedge dg^5 + \partial_\mu \sigma'\, d\tau \wedge dx^\mu \wedge g^5\,. \tag{3.71}$$

This ansatz is similar to, but somewhat simpler than the GHK pseudoscalar ansatz [52, 53] which involved mixing with $\delta\tilde{F}_5$. Since $\delta F_3 \wedge H_3 = 0$, now it is consistent to set $\delta\tilde{F}_5 = 0$. We also have $\tilde{F}_5 \wedge \delta F_3 = 0$, so it is consistent to take $\delta H_3 = 0$. Finally, one needs to study mixing with metric fluctuations. At first glance it seems that δG_{12} and δG_{34} might need to be turned on, but a more detailed analysis shows that their sources vanish:

$$\delta T_{12} = F_{13\tau}\delta {F_2}^{3\tau} + \delta F_{14\tau} {F_2}^{4\tau} = \frac{M\alpha'}{2}\, G^{33}\, G^{55} \big[F'\chi' - F'\chi'\big] \;=\; 0\,, \tag{3.72}$$

$$\delta T_{34} = F_{31\tau}\delta {F_4}^{1\tau} + \delta F_{32\tau} {F_4}^{2\tau} = 0\,. \tag{3.73}$$

Thus, the perturbation (3.68)-(3.71) decouples from all other modes, and the only non-trivial linearized equation is

$$d * \delta F_3 = 0\,. \tag{3.74}$$

The calculation we need to perform is the same as above, except we now set $\psi = 0$ and find

$$\chi' = \frac{3}{16}\, \varepsilon^{4/3}\, h(\tau)\, K(\tau)^4\, \sinh^2\tau\, \Box_4 \sigma'\,, \tag{3.75}$$

$$\partial_\mu(\chi - \sigma) = -\frac{9}{8}\, K(\tau)^2\, \partial_\tau \Big\{ K^4\, \sinh^2\tau\, \partial_\mu \sigma' \Big\}\,. \tag{3.76}$$

Eliminating χ and changing variables as before,

$$\tilde{w} = \frac{\varepsilon^{4/3}}{g_s M \alpha'} K^5 \sinh^2 \tau \ \sigma' \,, \tag{3.77}$$

we find

$$\tilde{w}'' - \frac{\cosh^2 \tau + 1}{\sinh^2 \tau} \, \tilde{w} + \frac{\varepsilon^{4/3} h}{6K^2} \, \Box_4 \tilde{w} = 0 \,. \tag{3.78}$$

After introducing the dimensionless mass as in (3.50), we get a non-minimal scalar equation

$$\tilde{w}'' - \frac{\cosh^2 \tau + 1}{\sinh^2 \tau} \tilde{w} + \tilde{m}^2 \frac{I(\tau)}{K(\tau)^2} \, \tilde{w} = 0 \,. \tag{3.79}$$

Since the 4-dimensional parity operation includes sign reversal of RR fields, we identify the family of glueballs coming from this eigenvalue problem as pseudoscalars whose S^{PC} quantum numbers are 0^{--}.

If we set $\tilde{m} = 0$ the solution regular at small τ is $(\sinh 2\tau - 2\tau)/\sinh \tau$. Since this blows up at large τ we conclude that this equation does not contain a massless glueball. A simple numerical analysis using the shooting method allows one to find the spectrum. The lowest eigenvalues ($\tilde{m}^2 < 100$) are listed in Table 3.2. The quadratic fit is

$$\tilde{m}^2_{IIIn} = 0.996 + 1.15\, n + 0.289\, n^2 \,, \tag{3.80}$$

and in the BHM normalization it is given by

$$m^2_{III\,BHM} = 0.938 + 1.08\, n + 0.272\, n^2 \,. \tag{3.81}$$

The spectrum can be reproduced with good accuracy using a semiclassical (WKB) approximation. The effective potential in (3.79) is singular at $\tau = 0$ which does not allow us to use the conventional WKB approximation. Yet we can cast the equation (3.79) in the form $Q_1 Q_2 \tilde{w} = m^2 \tilde{w}$, where Q_i are first-order differential operators and

n	$\tilde{m}_n^2$	n	$\tilde{m}_n^2$	n	$\tilde{m}_n^2$	n	$\tilde{m}_n^2$
1	2.41	5	14.0	9	34.8	13	64.7
2	4.47	6	18.0	10	41.4	14	73.7
3	7.08	7	23.2	11	48.6	15	83.2
4	10.3	8	28.7	12	56.4	16	93.3

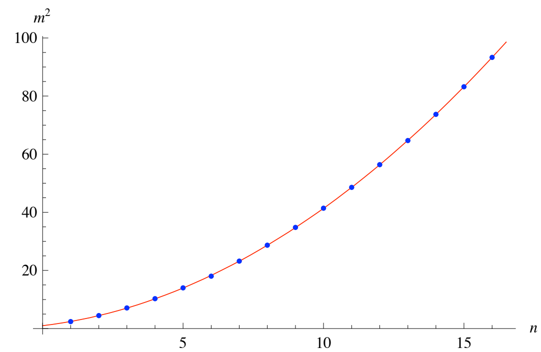


Table 3.2: Non-zero eigenvalues with $\tilde{m}^2 < 100$ in the RR sector. This spectrum can also be fitted by a quadratic polynomial (red line).

then consider an equation $Q_2 Q_1 \tilde{\tilde{w}} = m^2 \tilde{\tilde{w}}$, which must give rise to the same spectrum up to a zero mode. Namely, in our case this means that for A such that

$$A^2 + A' = \frac{\cosh^2 \tau + 1}{\sinh^2 \tau} \,, \tag{3.82}$$

equation (3.79) shares the spectrum with an equation

$$\tilde{\tilde{w}}'' - (B^2 + B')\tilde{\tilde{w}} + \tilde{m}^2 \frac{I(\tau)}{K(\tau)^2}\, \tilde{\tilde{w}} = 0 \,, \tag{3.83}$$

$$\text{where} \qquad B = -A - \frac{1}{2}\frac{d}{d\tau} \log \frac{I(\tau)}{K(\tau)^2} \,. \tag{3.84}$$

The general solution of (3.82) reads

$$A = -\coth \tau + \frac{2 \sinh^2 \tau}{\cosh \tau \sinh \tau - \tau + \mathfrak{C}} \,. \tag{3.85}$$

For (3.83) to be non-singular at the origin $\mathfrak{C}$ has to be non-zero. For finite $\mathfrak{C}$ the potential is regular everywhere but not monotonic and (3.83) admits a zero mode. The most convenient choice is to take infinite $\mathfrak{C}$, which leads to $A = -\coth \tau$. In this case the WKB approximation is applicable in its simplest form (see [113, 114]) and yields the same results as the shooting method up to the accuracy given above.

3.4 Organizing the Modes into Supermultiplets

The pseudoscalar Goldstone mode and the massless scalar found in [52, 53] belong to a 4-dimensional chiral multiplet. These fields appear as the phase and the modulus of the baryonic order parameters that vary along the baryonic branch. When a long KS throat is embedded into a Calabi-Yau compactification with fluxes, the baryonic U(1) symmetry becomes gauged and a supersymmetric version of the Higgs mechanism is expected to take place. The axial vector $\mathrm{U}(1)_B$ gauge field 'eats' the pseudoscalar mode and acquires a mass degenerate with the mass of a scalar Higgs. These fields constitute the bosonic content of a massive $\mathcal{N} = 1$ axial vector supermultiplet.

Above we explicitly constructed the massive modes that are radial excitations of the GHK scalar. It is, of course, interesting to find the supermultiplets they belong to. We will argue that each of these scalar radial excitations is also a member of a massive axial vector supermultiplet. Similarly, each pseudoscalar glueball found in Section 3.3 is a member of a massive vector multiplet. To prove these facts we would need to demonstrate the existence of the $S^{PC} = 1^{+-}$ glueballs degenerate with the 0^{+-} glueballs found in Section 3.2, as well as of 1^{--} glueballs degenerate with the 0^{--} glueballs found in Section 3.3. Unfortunately, constructing decoupled equations for vector supergravity fluctuations around the KS background is a difficult task. Instead, we will provide some evidence for our claims by studying axial vector and vector fluctuation equations in the large radius (KT) limit (setting $\alpha' = g_s = 1$, $N = 0$ and $M = 2$; see Subsection 1.2.2).

First we reconsider the simple decoupled pseudoscalar equation from the RR sector (3.78) and argue that its superpartner is given by the four-dimensional vector A_1 in

$$\delta B_2 = A_1 \wedge g^5 \ , \tag{3.86}$$

where we have chosen the ansatz so that the corresponding radial component $\sim dr \wedge g^5$ vanishes. The equation for $d * H_3$ implies (with primes denoting derivatives with respect to r)

$$\left[\frac{r^3}{6} *_4 A_1'\right]' - \frac{4r}{3} *_4 A_1 - \frac{hr^3}{6} d_4 *_4 d_4 A_1 = 0 \ , \tag{3.87}$$

and $d_4 *_4 A_1 = 0$, i.e. the vector is divergence-free. Since the Laplacian acting on such a vector is $\Box_4 = - *_4 d_4 *_4 d_4$ (note the Minkowski signature of the four dimensional metric), we find

$$\left[\frac{r^3}{6} A_1'\right]' - \frac{4r}{3} A_1 + \frac{hr^3}{6} \Box_4 A_1 = 0 \ . \tag{3.88}$$

Defining a new variable $\tilde{A}_1 = rA_1$, it is easy to see that its equation of motion,

$$\frac{r}{3}\left[\frac{r}{3} \tilde{A}_1'\right]' - \tilde{A}_1 + \frac{hr^2}{9} \Box_4 \tilde{A}_1 = 0 \ , \tag{3.89}$$

coincides with the KT limit of the equation for the decoupled pseudoscalar $\tilde{w}$, once we identify $r \sim \varepsilon^{2/3} e^{\tau/3}$. In fact, if we make the same ansatz (3.86) in the full KS background, the equation of motion for $\tilde{A}_1 = K^2 \sinh\tau A_1$ resulting from the terms $\sim d^3x \wedge d\tau \wedge \omega_2 \wedge \omega_2$ in $d * H_3 = 0$ is precisely as in (3.78):

$$\frac{d^2}{d\tau^2} \tilde{A}_1 - \frac{\cosh^2\tau + 1}{\sinh^2\tau} \tilde{A}_1 + \frac{\varepsilon^{4/3} h}{6K^2} \Box_4 \tilde{A}_1 = 0 \ . \tag{3.90}$$

However, this ansatz is not closed in the KS case. The Bianchi identity for $\tilde{F}_5$ is not satisfied, so this NSNS vector must mix with RR excitations of F_3 and/or $\tilde{F}_5$ in the KS background.

Let us now turn to the massive axial vector superpartners of the coupled scalars (3.48), (3.49) found above. We make the following ansatz, which is similar to the one studied in [130],

$$\delta C_4 = B_1 \wedge \omega_3 + F_2 \wedge \omega_2 + K_1 \wedge dr \wedge \omega_2 \, , \tag{3.91}$$

$$\delta C_2 = C_1 \wedge g^5 + D_2 + E_1 \wedge dr \, , \tag{3.92}$$

$$\delta B_2 = H_2 + J_1 \wedge dr \, ; \tag{3.93}$$

where B_1, C_1, E_1, J_1, K_1 are axial vectors and D_2, F_2, H_2 are two-forms in four dimensions. We choose to split the six degrees of freedom residing in the two-form into a vector and a dual vector, e.g. $D_2 = d_4(\ldots) + *_4 d_4 D_1$. The degrees of freedom contained in the first (exact) part are in fact the same as those in E_1, so we can simply write $D_2 = *_4 d_4 D_1$ without loss of generality. Similarly, $F_2 = *_4 d_4 F_1$ and $H_2 = *_4 d_4 H_1$, and the corresponding exact parts can be absorbed into the vectors K_1 and J_1, respectively.

The equations of motion then imply that B_1 and C_1 have to be divergence-free: $d_4 *_4 B_1 = d_4 *_4 C_1 = 0$. If this were not the case their divergences would simply couple to additional scalars $\delta C_4 \sim dr \wedge \omega_3$ and $\delta C_2 \sim dr \wedge g^5$, respectively, but we will not consider this here (i.e. as for A_1 above we choose as gauge in which these radial components vanish). In fact we will assume that all vectors in our ansatz are divergence-free, and that the terms appearing in the RR- and NSNS-potentials are eigenstates of the Laplacian $\Box_4 = - *_4 d_4 *_4 d_4$ with eigenvalue m^2.

Let us present some of the details of the derivation of the equations of motion. With the ansatz (3.91)–(3.93), the deformations of the field strengths are

$$\delta H_3 = - *_4 \Box_4 H_1 + \big(*_4 d_4 H_1' + d_4 J_1\big) \wedge dr\,, \tag{3.94}$$

$$*\delta H_3 = -\frac{h^2 r^5}{54} \Box_4 H_1 \wedge \omega_2 \wedge \omega_3 \wedge dr + \frac{h r^5}{54} \big(d_4 H_1' - *_4 d_4 J_1\big) \wedge \omega_2 \wedge \omega_3\,; \tag{3.95}$$

$$\delta F_3 = d_4 C_1 \wedge g^5 - C_1' \wedge dr \wedge g^5 - C_1 \wedge dg^5 + \big(*_4 d_4 D_1' + d_4 E_1\big) \wedge dr + d_4 *_4 d_4 D_1\,, \tag{3.96}$$

$$\begin{aligned} *\delta F_3 = {} & \frac{h r^3}{6} *_4 d_4 C_1 \wedge \omega_2 \wedge \omega_2 \wedge dr + \frac{r^3}{6} *_4 C_1' \wedge \omega_2 \wedge \omega_2 + \frac{r}{3} *_4 C_1 \wedge dg^5 \wedge g^5 \wedge dr \\ & + \frac{h r^5}{54} \big(d_4 D_1' - *_4 d_4 E_1\big) \wedge \omega_2 \wedge \omega_3 - \frac{h^2 r^5}{54} \Box_4 D_1 \wedge \omega_2 \wedge \omega_3 \wedge dr\,; \end{aligned} \tag{3.97}$$

$$\delta F_5 = \delta\mathcal{F}_5 + *\delta\mathcal{F}_5\,, \tag{3.98}$$

$$\begin{aligned} \delta\mathcal{F}_5 = {} & \big(d_4 B_1 - B_1' \wedge dr\big) \wedge \omega_3 + \big(- *_4 \Box_4 F_1 + *_4 d_4 F_1' \wedge dr\big) \wedge \omega_2 \\ & + d_4 K_1 \wedge dr \wedge \omega_2\,, \end{aligned} \tag{3.99}$$

$$\begin{aligned} *\delta\mathcal{F}_5 = {} & \frac{3}{r} *_4 d_4 B_1 \wedge \omega_2 \wedge dr + \frac{3}{hr} *_4 B_1' \wedge \omega_2 - \frac{hr}{3} \Box_4 F_1 \wedge \omega_3 \wedge dr \\ & + \frac{r}{3} \big(d_4 F_1' - *_4 d_4 K_1\big) \wedge \omega_3\,. \end{aligned} \tag{3.100}$$

Substituting these expressions into the Bianchi identity for $\tilde{F}_5$, and the form equations (A.2) for $*F_3$ and $*H_3$, we find the equations of motions. Splitting them into exact and coexact parts with respect to the 4-dimensional derivative operator d_4 shows that the vectors E_1, H_1 and K_1 decouple[3]. The resulting equations for the remaining vectors read

$$\left[\frac{3}{hr}\, B_1'\right]' + \frac{3}{r}\,\Box_4 B_1 = -\frac{3}{r}\,\Box_4 D_1\,, \tag{3.101}$$

$$\left[\frac{r}{3}\, F_1'\right]' + \frac{hr}{3}\,\Box_4 F_1 = J_1 + \frac{3}{r}\, C_1\,, \tag{3.102}$$

$$\left[\frac{r^3}{6}\, C_1'\right]' - \frac{4r}{3}\, C_1 + \frac{hr^3}{6}\,\Box_4 C_1 = \frac{3}{r}\Box_4 F_1 - \frac{9}{hr^2}\, B_1'\,, \tag{3.103}$$

$$\left[\frac{hr^5}{54}\, D_1'\right]' + \frac{h^2 r^5}{54}\,\Box_4 D_1 = -F_1' - \frac{3}{r}\, B_1 - 3\log\frac{r}{r_*} J_1\,, \tag{3.104}$$

$$\left[\frac{hr^5}{54}\, J_1\right]' + 3\log\frac{r}{r_*} D_1' = F_1' + \frac{3}{r}\, B_1\,, \tag{3.105}$$

$$\Box_4 F_1 - \frac{3}{hr}\, B_1' = \frac{hr^5}{54}\,\Box_4 J_1 + 3\log\frac{r}{r_*}\Box_4 D_1\,, \tag{3.106}$$

where (3.102), (3.104) and (3.105) hold modulo terms annihilated by d_4. It is easy to see that (3.105) and (3.106) imply (3.101), so the latter is not independent. We thus have the five coupled equations for the five vectors B_1, C_1, D_1, F_1 and J_1.

In the massless case, our ansatz includes the pseudoscalar found in [52, 53]. Putting

$$C_1 = -f_2(r)\, d_4 a(x)\ , \tag{3.107}$$

$$\Box_4 D_1 = f_1\, d_4 a(x)\ , \tag{3.108}$$

$$B_1' = -f_1 h\, r \log\frac{r}{r_*}\, d_4 a(x)\ , \tag{3.109}$$

$$F_1 = J_1 = 0\ , \tag{3.110}$$

for some constant f_1 and a four-dimensional massless pseudoscalar $a(x)$, all equations

[3] More precisely, we set $\frac{hr^5}{54}\, E_1 = -3\log\frac{r}{r_*}\, H_1 = r\log\frac{r}{r_*}\, K_1$, and find a single second order differential equation obeyed by these fields. Thus we have found another decoupled vector, but this is not the one we are looking for. Given this relation between them, E_1, H_1 and K_1 do not mix with the other vectors.

of motion are satisfied provided

$$\left[\frac{r^3}{6} f_2'\right]' - \frac{4r}{3} f_2 = -\frac{9}{r} f_1 \log\frac{r}{r_*} \ ; \tag{3.111}$$

in perfect agreement with the literature (see also Subsection 1.2.3).

Now we would like to consider massive excitations however, and find axial vector-like solutions to the equations (3.101)-(3.106) which give rise to the superpartners of the massive scalar excitations of (3.48) and (3.49). In particular, changing variables to $W_1 = rC_1$ equation (3.103) becomes

$$\frac{r}{3}\left[\frac{r}{3} W_1'\right]' - W_1 + \frac{hr^2}{9}\Box_4 W_1 = \frac{2}{r}\Box_4 F_1 - \frac{6}{hr^2} B_1' \ . \tag{3.112}$$

Thus we can identify W_1 with $\tilde{w}$ in (3.49), which suggests setting the right hand side of this equation proportional to the counterpart of $\tilde{z}/r$. Hence we define

$$Z_1 \equiv \Box_4 F_1 - \frac{3}{hr} B_1' \, . \tag{3.113}$$

Using (3.101) and (3.102) one can deduce that this new field obeys

$$\frac{r}{3}\left[\frac{r}{3} Z_1'\right]' + \frac{hr^2}{9}\Box_4 Z_1 = \frac{1}{r}\Box_4 W_1 + \frac{r}{3}\Box_4\big(J_1 + D_1'\big) \, . \tag{3.114}$$

Our reduced ansatz containing five axial vectors is still too general. In order to match the spectrum of the scalar particles found above, we need to impose an additional constraint to reduce the number of dynamical vectors obeying independent second order differential equations to two. The correct constraint for our purposes is given by

$$\Box_4\big(J_1 + D_1'\big) = \frac{3}{r^2}\Box_4 W_1. \tag{3.115}$$

In order to show that we can consistently impose this relation we need to examine the remaining equations. First of all, with this constraint (3.106) reads

$$Z_1 = -\frac{hr^5}{54}\,\Box_4 D_1' + \frac{hr^3}{18}\,\Box_4 W_1 + 3\log\frac{r}{r_*}\Box_4 D_1\,. \tag{3.116}$$

Adding (3.104) and (3.105), using the constraint and the fact that W_1 is a mass eigenstate we find

$$\left[\frac{hr^3}{18}\,W_1\right]' + \frac{9}{r^2}\log\frac{r}{r_*}W_1 + \frac{h^2r^5}{54}\,\Box_4 D_1 = 0\,. \tag{3.117}$$

Eliminating $\Box_4 D_1$ between the last two equations we obtain a second order differential equation containing only W_1 and Z_1. A non-trivial fact is that this equation is identical to (3.112). This relies heavily on the precise expression for the warp factor (1.48), and shows the consistency of the constraint equation with the equations of motion.

Finally, introducing a symbol for the other combination of the vectors F_1 and B_1 that appears in the equations of motion

$$Y_1 \equiv F_1' + \frac{3}{r}\,B_1\,, \tag{3.118}$$

equation (3.105) implies

$$\Box_4 Y_1 = Z_1' - \frac{3}{r}\,\Box_4 D_1\ . \tag{3.119}$$

In summary, we have the two coupled dynamical equations

$$\frac{r}{3}\,\left[\frac{r}{3}\,Z_1'\right]' + \frac{hr^2}{9}\,\Box_4 Z_1 = \frac{2}{r}\,\Box_4 W_1\ , \tag{3.120}$$

$$\frac{r}{3}\,\left[\frac{r}{3}\,W_1'\right]' - W_1 + \frac{hr^2}{9}\,\Box_4 W_1 = \frac{2}{r}Z_1\ , \tag{3.121}$$

which determine W_1 and Z_1. In terms of these $\Box_4 D_1$ is determined by (3.117), J_1 by (3.115), and $\Box_4 Y_1$ by (3.119). Equations (3.120), (3.121) are precisely the KT limit of the scalar equations (3.48), (3.49) up to a rescaling of the fields by a numerical factor.[4]

[4]Looking at (3.48) one might have expected the term $2\tilde{z}/\sinh^2\tau$ to give rise to a term proportional to Z_1/r^6 in (3.120), but in fact this is not the case because it is too small to be seen in the KT limit.

Since the KT limits of their equations of motion agree, we thus argue that the axial vectors Z_1 and W_1 are the superpartners of the coupled scalars $\tilde{z}$ and $\tilde{w}$ found above, and that their massive excitations combine into vector multiplets. Subsequently, this was shown rigorously by explicit construction of the vector fluctuations of the full KS solution in [131], where all $\mathcal{I}$-odd $SU(2) \times SU(2)$ invariant perturbations of the warped throat were found and categorized.

3.5 Effects of Compactification

Now we will embed the KS throat into a flux compactification, along the lines of [132], and estimate the mass of the Higgs scalar. Generally, glueballs are dual to the normalizable modes localized near the bottom of the throat, and one does not expect them to be strongly affected by the bulk of the Calabi-Yau. This is indeed the case for all the massive radial excitations found in Sections 3.2 and 3.3. We will see, however, that the case of the GHK scalar is more subtle and exhibits some UV sensitivity.

To model a compactification, we will introduce a UV cut-off on the radial coordinate, $\tau_{\max}$. We also need to include a deformation of the KS solution introduced by bulk effects. On the field theory side this corresponds to perturbing the Lagrangian of the cascading gauge theory by some irrelevant operators. Here we are not interested in classifying all of them but rather model the compactification effects in the simplest way by considering one perturbation which simulates the main features of the compactified solution. We consider a shift of the warp factor $\delta h = \text{const.}$, which corresponds to the dimension 8 operator on the field theory side [133, 134, 135]. This also has a simple geometrical meaning: the warp factor of the compactified solution is a finite constant in the bulk of the Calabi-Yau and therefore should not drop below a certain value along the throat.

In the KS background it arises from a subleading term in the variation of the Ricci tensor (3.33) but such terms that are asymptotically suppressed by powers of r compared to the leading terms are not taken into account in the KT metric. Indeed, if we write ansatz (3.2)-(3.6) in the KT background and follow the same strategy as we did for the full KS background, the term proportional to $\tilde{z}/r^6$ does not appear in the Einstein equations.

Let us introduce a small parameter δ which shifts the rescaled warp factor, $I(\tau) \to I(\tau) + \delta$, and consider the system (3.51)–(3.52) in perturbation theory near $\tilde{m}^2 = 0$:

$$\tilde{z} = \tilde{z}_0 + \tilde{m}^2 \tilde{z}_1 \ , \tag{3.122}$$

$$\tilde{w} = \tilde{w}_0 + \tilde{m}^2 \tilde{w}_1 \ , \tag{3.123}$$

$$\tilde{z}_0 = \tau \coth \tau - 1 \ , \tag{3.124}$$

$$\tilde{w}_0 = -\frac{2^{2/3}\, 8}{9} \frac{1}{\sinh \tau} \int_0^\tau dx\, I(x) \sinh^2 x \, . \tag{3.125}$$

At leading order in $\tilde{m}^2$ we find

$$\tilde{z}_1 = (\tau \coth \tau - 1) \int_0^\tau dx\, u(x) \coth x - \coth \tau \int_0^\tau dx\, u(x)\, (x \coth x - 1) \, , \tag{3.126}$$

$$\tilde{w}_1 = -\frac{1}{4\, \sinh \tau} \int_0^\tau dx\, v(x)\, \frac{\sinh 2x - 2x}{\sinh x} - \frac{\sinh 2\tau - 2\tau}{4\, \sinh \tau} \int_\tau^\infty dx\, v(x)\, \frac{1}{\sinh x} \, , \tag{3.127}$$

where

$$u(\tau) = -\frac{I(\tau)}{K^2(\tau)}\, \tilde{z}_0 + \frac{9}{4 \cdot 2^{2/3}}\, K(\tau)\, \tilde{w}_0 - \frac{\delta}{K^2}\, \tilde{z}_0 \, , \tag{3.128}$$

$$v(\tau) = -\frac{I(\tau)}{K^2(\tau)}\, \tilde{w}_0 + \frac{16}{9}\, K(\tau)\, \tilde{z}_1 - \frac{\delta}{K^2}\, \tilde{w}_0 \ . \tag{3.129}$$

Keeping in mind that for large τ we have $u \simeq -2^{-2/3} \delta\, \tau\, e^{2\tau/3}$ one finds the asymptotic behavior

$$\tilde{z}_1(\tau) \simeq -2^{-2/3} \delta \int_0^\tau dx\, (\tau - x)\, x\, e^{2x/3} \ \simeq \ -\frac{9\, \delta}{4\, 2^{2/3}}\, \tau\, e^{2\tau/3} \, . \tag{3.130}$$

This yields $v \simeq -2^{2/3} \delta\, \tau\, e^{\tau/3}$ and

$$\begin{aligned} \tilde{w}_1 &= -\frac{1}{4\, \sinh \tau} \int_0^\tau dx\, v_0(x)\, \frac{\sinh 2x - 2x}{\sinh x} - \frac{\sinh 2\tau - 2\tau}{4\, \sinh \tau} \int_\tau^\infty dx\, v_0(x)\, \frac{1}{\sinh x} \\ &\simeq \frac{9\, 2^{2/3} \delta}{8}\, \tau\, e^{\tau/3} \, . \end{aligned} \tag{3.131}$$

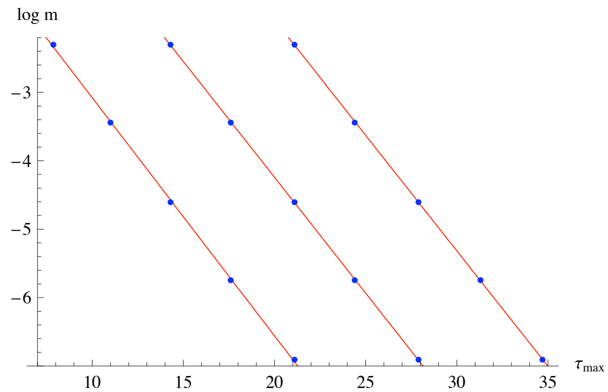


Figure 3.1: The dependence of $\log \tilde{m}$ on $\tau_{\rm max}$ is linear with the slope equal to -1/3. The three lines shown correspond to $\delta = 1$, $\delta = 0.01$ and $\delta = 0.0001$.

Finally, up to first order in the mass-squared and δ:

$$\tilde{z} \;\simeq\; \tau\,\big[1 - \frac{9\,\delta\,\tilde{m}^2}{4\,2^{2/3}}\,e^{2\tau/3}\big]\,, \qquad \tilde{w} \;\simeq\; -2^{4/3}\tau\,e^{-\tau/3}\,\big[1 - \frac{9\,\delta\,\tilde{m}^2}{8\,2^{2/3}}\,e^{2\tau/3}\big]\,. \tag{3.132}$$

This suggests that for generic boundary conditions the cut-off value

$$\tau_{\rm max} \simeq -\log\big(\delta^{3/2}\tilde{m}^3\big)\,. \tag{3.133}$$

This prediction can be tested numerically. In order to do so one can specify some small $\tilde{m}$ and plot the determinant

$$\det\begin{pmatrix} \tilde{z}_1(\tau) & \tilde{z}_2(\tau) \\ \tilde{w}_1(\tau) & \tilde{w}_2(\tau) \end{pmatrix}, \tag{3.134}$$

of the two linearly independent solutions regular at $\tau = 0$ as a function of τ. The first zero marks the point $\tau_{\rm max}$ such that there is a regular solution with $z(\tau_{\rm max}) = w(\tau_{\rm max}) = 0$. Hence $\tau_{\rm max}$ is the corresponding cut-off value. As Figure 3.1 shows, the relation (3.133) holds for $\tau_{\rm max}$ large enough that $\tilde{m}^2$ is small, where

$$\tilde{m}^2 \sim \delta^{-1} e^{-2\tau_{\rm max}/3}\,. \tag{3.135}$$

Let us consider a simple model of compactification where the throat is embedded into an asymptotically conical space that terminates at some large cut-off value $\tau_{\max}$. To calculate the mass from (3.135) we need to know δ as well as $\tau_{\max}$. The former is the asymptotic value of the (rescaled) warp factor. The point where the field theory warp factor approaches δ marks the UV cutoff of the field theory

$$I(\tau_{UV}) \sim \tau_{UV} e^{-4\tau_{UV}/3} \simeq \delta \ . \tag{3.136}$$

Using this in (3.135) we find $\tilde{m}^2 \sim e^{(4\tau_{UV}-2\tau_{\max})/3}$. This shows that the Higgs mass becomes parametrically small only for $\tau_{\max} \gg 2\tau_{UV}$. This is not satisfied in general; the geometry requires only that $\tau_{\max} > \tau_{UV}$ because τ_{UV} is the length of the throat embedded into a CY space. With the ratio between the UV and IR scales of the field theory around $4\cdot 10^3$ [121] we estimate that $\tau_{UV} \simeq 25$ [59]. The cut-off $\tau_{\max}$ can be related to the warped volume of the Calabi-Yau which, in a singular conifold approximation, is

$$V_6^w = \mathrm{Vol}(T^{1,1}) \int_0^{r_{\max}} dr\, h(r) \sqrt{\frac{\det\, g_6}{\det\, g_{T^{1,1}}}} \ , \tag{3.137}$$

where $r \sim \varepsilon^{2/3} e^{\tau/3}$. The integral from zero to r_{UV} is the warped volume of the throat, and from r_{UV} to $r_{\max}$ is the bulk volume. Assuming that the latter dominates,

$$V_6^w \simeq \frac{16\pi^3}{27} \varepsilon^{4/3} (g_s M \alpha')^2 \left[r_{\max}^6 - r_{UV}^6 \right] r_{UV}^{-4} \ . \tag{3.138}$$

Requiring $\tau_{\max} \gg 50$ leads to an enormous V_6^w, far larger than, for example, $V_6^w \simeq 5^6 \alpha'^3$ in [121]. Thus, while for $\tau_{\max} \gg 2\tau_{UV}$ the Higgs scalar becomes parametrically lighter than the other normal modes, in compactifications with realistic parameters it may actually be heavier. This is due to the special feature of its wave function $\tilde{z}$ which grows linearly with τ in the throat. The only conclusion we can draw from our simplified model of compactification is that this mode is rather UV sensitive, so to determine its mass we need to know the details of the compactification.

Chapter 4

Baryonic Condensates on the Conifold

4.1 Introduction

Consideration of a stack of N D3-branes leads to the duality of $\mathcal{N} = 4$ super Yang-Mills theory to type IIB string theory on $AdS_5 \times S^5$ [1, 2, 3]. A different, $\mathcal{N} = 1$ supersymmetric example of the AdS/CFT correspondence follows from placing the stack of D3-branes at the tip of the conifold [44, 45]. This suggests a duality between a certain $SU(N) \times SU(N)$ superconformal gauge theory and type IIB string theory on $AdS_5 \times T^{1,1}$. Addition of M D5-branes wrapped over the two-sphere near the tip of the conifold changes the gauge group to $SU(N + M) \times SU(N)$ [111, 112]. This theory is non-conformal; it undergoes a cascade of Seiberg dualities [50] $SU(N + M) \times SU(N) \to SU(N - M) \times SU(N)$ as it flows from the UV to the IR [49, 51] (for reviews, see [10, 11, 12] and Section 1.2).

The gauge theory contains two doublets of bifundamental, chiral superfields A_i, B_j (with $i, j = 1, 2$). In the conformal case, $M = 0$, it has continuous global symmetries $SU(2)_A \times SU(2)_B \times U(1)_R \times U(1)_B$. The two SU(2) groups rotate the doublets A_i and B_j, while one U(1) is an R-symmetry. The remaining U(1) factor corresponds to the baryon number symmetry which we will be most interested in. As argued in [49, 52, 53, 56, 59], in the cascading theory where N is an integer multiple of M, $N = kM$, this symmetry is spontaneously broken by condensates of baryonic operators. In this chapter we will provide a quantitative verification of this effect.

For $N = kM$ the last step of the cascade is an $\mathrm{SU}(2M)\times\mathrm{SU}(M)$ theory which admits two baryon operators (sometimes referred to as baryon and antibaryon)

$$\begin{aligned}
\mathcal{A} &\sim \epsilon_{\alpha_1\alpha_2\ldots\alpha_{2M}} (A_1)_1^{\alpha_1}(A_1)_2^{\alpha_2}\ldots(A_1)_M^{\alpha_M}(A_2)_1^{\alpha_{M+1}}(A_2)_2^{\alpha_{M+2}}\ldots(A_1)_M^{\alpha_{2M}} \ , \\
\mathcal{B} &\sim \epsilon_{\alpha_1\alpha_2\ldots\alpha_{2M}} (B_1)_1^{\alpha_1}(B_1)_2^{\alpha_2}\ldots(B_1)_M^{\alpha_M}(B_2)_1^{\alpha_{M+1}}(B_2)_2^{\alpha_{M+2}}\ldots(B_1)_M^{\alpha_{2M}} \ .
\end{aligned} \tag{4.1}$$

Baryon operators of the general $\mathrm{SU}(M(k+1)) \times \mathrm{SU}(Mk)$ theory have the schematic form $(A_1A_2)^{k(k+1)M/2}$ and $(B_1B_2)^{k(k+1)M/2}$, with appropriate contractions described in [56]. Unlike the dibaryon operators of the conformal $\mathrm{SU}(N) \times \mathrm{SU}(N)$ theory [111], $\mathcal{A}$ and $\mathcal{B}$ are singlets under the two global SU(2) symmetries. These operators acquire expectation values that spontaneously break the $\mathrm{U}(1)_B$ baryon number symmetry; this is why the gauge theory is said to be on the baryonic branch of its moduli space [57]. Supersymmetric vacua on the one complex dimensional baryonic branch are subject to the constraint $\mathcal{AB} = -\Lambda_{2M}^{4M}$, and thus we can parameterize it as follows

$$\mathcal{A} = i\zeta\Lambda_{2M}^{2M} \ , \qquad \mathcal{B} = \frac{i}{\zeta}\Lambda_{2M}^{2M} \ . \tag{4.2}$$

The non-singular supergravity dual of the theory with $|\zeta| = 1$ is the warped deformed conifold found in [49]. In [52, 53] the linearized scalar and pseudoscalar perturbations, corresponding to small deviations of ζ from 1, were constructed, and in the previous chapter we have generalized this construction to fluctuations with non-zero momentum. The full set of first-order equations necessary to describe the entire moduli space of supergravity backgrounds dual to the baryonic branch, called the resolved warped deformed conifolds, was derived and solved numerically in [58] (for further discussion of these solutions, see [59] and Subsection 1.2.4).

The construction of this moduli space of supergravity backgrounds, which have just the right symmetries to be identified with the baryonic branch in the cascading gauge theory, provides an excellent check on the gauge/string duality in this intricate setting. Yet, one question remains: how do we identify the baryonic expectation values on the string side of this duality? Among other things, this is needed to construct a map between the parameter U that labels the supergravity solutions, and the parameter $|\zeta|$ in the gauge theory.

The dual string theory description of the baryon operators (4.1) was first considered by Aharony [56]. He argued that the heavy particle dual to such an operator is described at large r by a D5-brane wrapped over the $T^{1,1}$, with some D3-branes dissolved in it (to account for this, the world volume gauge field needs to be turned on). To calculate the two-point function of baryon operators inserted at x_1 and x_2 we may use a semi-classical approach to the AdS/CFT correspondence. Then we need a (Euclidean) D5-brane whose world volume has two $T^{1,1}$ boundaries at large r, located at x_1 and x_2. Here we will be interested in a simpler embedding of the D5-brane: as suggested by Witten (unpublished, 2004), the object needed to calculate the baryonic expectation values is the Euclidean D5-brane that has the appearance of a pointlike instanton from the four-dimensional point of view, and wraps the remaining six (generalized Calabi-Yau) directions of the ten-dimensional spacetime. This object has a single $T^{1,1}$ boundary at large r, corresponding to insertion of just one baryon operator. As we will find, supersymmetry requires that the world volume gauge field is also turned on, so there are D3-branes dissolved in the D5. This identification will be corroborated by demonstrating that the D5-brane couples correctly to the pseudoscalar zero-mode of the theory that changes the phase of the baryon expectation value [52, 53].

Close to the boundary, a field φ dual to an operator of dimension Δ in the AdS/CFT correspondence behaves as

$$\varphi(x,r) = \varphi_0(x)\, r^{\Delta-4} + A_\varphi(x)\, r^{-\Delta}\ , \tag{4.3}$$

Here A_φ is the operator expectation value [48], and φ_0 is the source for it. In the cascading theory, which is near-AdS in the UV, the same formulae hold modulo powers of $\ln r$ [136, 137]. The field corresponding to a baryon will be identified, at a semi-classical level, with $e^{-\mathcal{S}_{D5}(r)}$, where $\mathcal{S}_{D5}(r)$ is the action of a D5-brane wrapping the Calabi-Yau coordinates up to the radial coordinate cut-off r. The different baryon operators $\mathcal{A}, \overline{\mathcal{A}}, \mathcal{B}, \overline{\mathcal{B}}$ will be distinguished by the two possible D5-brane orientations, and the two possible κ-symmetric choices for the world volume gauge field that has to be turned on inside the D5-brane. In the cascading gauge theory there is no source added for baryonic operators, hence we find that $\varphi_0 = 0$. On the other hand, the term scaling as $r^{-\Delta}$ is indeed revealed by our calculation of $e^{-\mathcal{S}_{D5}(r)}$ as a function of the radial cut-off, allowing us to find the dimensions of the baryon operators, and the values of their condensates.

This chapter relies heavily on material from the paper [138] coauthored with A. Dymarsky and I. R. Klebanov, and is subdivided as follows. In the remainder of Section 4.1 we discuss the Killing spinors of the warped supergravity backgrounds dual to the baryonic branch. We also review the κ-symmetry conditions for D-brane embeddings, and briefly discuss a number of brane configurations that satisfy them. Section 4.2 is devoted to the derivation of the first-order equation for the gauge field. We first discuss a Lorentzian D7-brane wrapping the warped deformed conifold directions, before presenting a parallel treatment for the more subtle case of the Euclidean D5-brane wrapping the conifold. In Section 4.3 we investigate the physics of the D5-instanton in the KS background. From the behavior of the D5-brane action as a function of the radial cut-off we extract the dimension of the baryon operator, and show that it matches the expectations from the dual cascading gauge theory. We also show that the D5-brane couples to the baryonic branch complex modulus in the way consistent with our identification of the condensates. In particular, we demonstrate that pseudoscalar perturbations of the backgrounds shift the phase of the baryon expectation value. We generalize to the complete baryonic branch in Section 4.4, where we compute the baryon expectation values as a function of the supergravity modulus U. The product of the expectation values calculated from the D5-brane action is shown to be independent of U in agreement with (4.2). Finally, we present an integral expression for their ratio and evaluate it numerically, which provides a relation between the baryonic branch modulus $|\zeta|$ in the gauge theory and the modulus U in the dual supergravity description, and show that they satisfy $\mathcal{AB} = \text{const}$. We conclude briefly in Section 4.5.

4.1.1 D-Branes, κ-Symmetry and Killing Spinors of the Conifold

A Dirichlet p-brane (with p spatially extended dimensions) in string theory is described by an action consisting of two terms [139, 140, 141]: the Dirac-Born-Infeld action, which is essentially a minimal area action including non-linear electrodynamics, and the Chern-Simons action, which describes the coupling to the RR background fields:

$$\mathcal{S} = \mathcal{S}_{DBI} + \mathcal{S}_{CS} = -\int_{\mathcal{W}} \mathrm{d}^{p+1}\sigma e^{-\phi}\sqrt{-\det(G+\mathcal{F})} + \int_{\mathcal{W}} e^{\mathcal{F}} \wedge C \ . \tag{4.4}$$

Here $\mathcal{W}$ is the worldvolume of the brane and we have set the brane tension to unity. Further, G is the induced metric on the worldvolume, $\mathcal{F} = F_2 + B_2$ is the sum of the gauge field strength $F_2 = dA_1$ and the pullback of the NSNS two-form field, and $C = \sum_i C_i$ is the formal sum of the RR potentials. In superstring theory all these fields should really be understood as superfields, but we shall ignore fermionic excitations here.

Wick rotation of this action to Euclidean space such that all $p+1$ directions become spatially extended (which leads to a Euclidean worldvolume D-instanton) effectively multiplies the action by a factor of i. This cancels the minus sign under the square root in the DBI term and leaves it real since the determinant is now positive. The CS term however is purely imaginary now. Consequently the equations of motion that follow from the DBI and CS terms now have be satisfied independently of each other if we insist on the gauge field being real.

The action (4.4) is invariant on shell under the so-called κ-symmetry [142, 143, 144]. This allows us to find first-order equations for supersymmetric configurations which are easier to solve than the second order equations of motion. The κ-symmetry condition can be written as

$$\Gamma_\kappa \varepsilon = \varepsilon \ , \tag{4.5}$$

where ε is a doublet of Majorana-Weyl spinors, and the operator Γ_κ is specified below. Satisfying this equation guarantees worldvolume supersymmetry in the probe brane approximation, and every solution for which ε is a Killing spinor corresponds to a supersymmetry compatible with those preserved by the background.

The decomposition of a Weyl spinor ε into a doublet of Majorana-Weyl spinors

$$\varepsilon = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix} \tag{4.6}$$

is achieved by projecting onto the eigenstates of charge conjugation[1] $\varepsilon_1 = (\varepsilon + \varepsilon^*)/2$ and $\varepsilon_2 = (\varepsilon - \varepsilon^*)/2i$.

In IIB superstring theory on a $(9,1)$ signature spacetime, the κ-symmetry operator Γ_κ for a Lorentzian D-brane extended along the time direction x_0 and p spatial directions

[1] Given any spinor ε we denote its charge conjugate by ε^*, which of course is represented by complex conjugation and left multiplication by a charge conjugation matrix B. We do not write B explicitly here, though its presence is understood.

is given by

$$\Gamma_\kappa = \frac{\sqrt{-\det G}}{\sqrt{-\det(G+\mathcal{F})}} \sum_{n=0}^{\infty} (-1)^n \not{\mathcal{F}}^n \Gamma_{(p+1)} \otimes (\sigma_3)^{n+\frac{p-3}{2}} i\sigma_2 \ , \tag{4.7}$$

$$\Gamma_{(p+1)} \equiv \frac{1}{(p+1)!\sqrt{-\det G}} \, \varepsilon^{\mu_1 \dots \mu_{p+1}} \Gamma_{\mu_1 \dots \mu_{p+1}} \ , \tag{4.8}$$

$$\not{\mathcal{F}}^n \equiv \frac{1}{2^n n!} \Gamma_{\nu_1 \dots \nu_{2n}} \mathcal{F}_{\sigma_1 \sigma_2} \dots \mathcal{F}_{\sigma_{2n-1}\sigma_{2n}} G^{\nu_1 \sigma_1} \dots G^{\nu_{2n}\sigma_{2n}} \ . \tag{4.9}$$

Here σ_i are the usual Pauli matrices. We use Greek labels for the worldvolume indices of the D-brane and consequentially the Γ_μ are induced Dirac matrices. In what follows we denote the Minkowski spacetime coordinates by $x_0 \dots x_3$ and label the tangent space of the internal manifold $\mathcal{M}$ by $1, 2 \dots 6$ in reference to the basis one-forms (1.66). The expression for Γ_κ can be significantly simplified for an embedding covering all six directions of the deformed conifold, in which case we simply align the worldvolume tangent space with that of $\mathcal{M}$.

The Killing spinor Ψ of the supergravity backgrounds dual to the baryonic branch is built out of a six-dimensional pure spinor η^- and an arbitrary spinor ζ^- of negative four-dimensional chirality,

$$\Psi = \alpha\, \zeta^- \otimes \eta^- + i\beta\, \zeta^+ \otimes \eta^+ \ , \tag{4.10}$$

$$(\Gamma_1 - i\Gamma_2)\zeta^- \otimes \eta^- = (\Gamma_3 - i\Gamma_4)\zeta^- \otimes \eta^- = (\Gamma_5 - i\Gamma_6)\zeta^- \otimes \eta^- = 0 \ , \tag{4.11}$$

where $\eta^+ = (\eta^-)^*$ and $\zeta^+ = (\zeta^-)^*$. The functions α and β are real [58, 59] and given by

$$\alpha = \frac{e^{\phi/4}(1+e^\phi)^{3/8}}{(1-e^\phi)^{1/8}} \ , \qquad \beta = \frac{e^{\phi/4}(1-e^\phi)^{3/8}}{(1+e^\phi)^{1/8}} \ . \tag{4.12}$$

(this expression for β is for $U > 0$; β changes sign when U does). The corresponding

Majorana-Weyl spinors Ψ_1 and Ψ_2 are

$$\Psi_1 = \frac{1}{2}\left((\alpha - i\beta)\zeta^- \otimes \eta^- + (\alpha + i\beta)\zeta^+ \otimes \eta^+\right) \ , \tag{4.13}$$

$$\Psi_2 = \frac{1}{2i}\left((\alpha + i\beta)\zeta^- \otimes \eta^- - (\alpha - i\beta)\zeta^+ \otimes \eta^+\right) \ . \tag{4.14}$$

4.1.2 Branes Wrapping the Angular Directions

In the context of the conifold, the closest analogue to the baryon vertex in $\mathrm{AdS}_5 \times \mathrm{S}^5$ that was discussed in [145, 146, 147], would be a D5-brane wrapping the five angular directions of the internal space, with worldvolume coordinates $\sigma^\mu = (x^0, \theta_1, \phi_1, \theta_2, \phi_2, \psi)$. The brane describing the baryon vertex in $\mathrm{AdS}_5 \times \mathrm{S}^5$ has "BI-on" spikes corresponding to fundamental strings attached to the brane and ending on the boundary of AdS, indicating that it is not a gauge-invariant object. Here however, we are interested in gauge-invariant, supersymmetric objects, that are candidate duals to chiral operators in the gauge theory, so we might try to consider a smooth embedding at constant radial coordinate (the difference between a "baryon" and a "baryon vertex" was already stressed in [145]).

To avoid having the BI-on spikes, it was proposed [56] that we should use an appropriate combination of D5-branes wrapping all the angular coordinates, and of D3-branes wrapping the S^3. This is equivalent to turning on a particular gauge field on the wrapped D5-brane. Unfortunately, it is not clear how to maintain the supersymmetry of such an object. It is not hard to see, for example from the appropriate κ-symmetry equations, that a (Lorentzian) D5-brane wrapping the five angular direction of the conifold and embedded at constant r cannot be a supersymmetric object. The κ-symmetry equation seems to call for an additional constraint of the form $\Gamma_{x^0\psi}\varepsilon^* = -i\varepsilon$ on the Killing spinors, which would imply also $\Gamma_{x^0r}\varepsilon^* = -\varepsilon$, i.e. precisely what we would expect for strings stretched in the radial direction. However, such a projection does not commute with the other conditions that the Killing spinors have to satisfy and thus is not consistent.

This was pointed out in [148] for the case of the singular conifold [44], and the argument carries over to the deformed conifold. Even with a worldvolume gauge field such a D5-brane cannot be a BPS object.

The same conclusion also follows from the equation of motion for the radial component of the embedding $X^M(\sigma^\mu)$. The leading term (as $r \to \infty$) in the D-brane Lagrangian arises from the B_2-field contribution to the DBI term and is proportional to $r\,(\ln r)^2$, so this brane is bound to contract and move to smaller r, until eventually it reaches the tip of the conifold, where the two-cycle collapses.

On the other hand, as suggested by Aharony [56], the D5-branes with D3-branes dissolved within them are the particles dual to the baryon operators. As proposed by Witten (unpublished, 2004), to find the baryonic condensates we need to consider a *Euclidean* D5-brane wrapping the deformed conifold directions, with a certain gauge field turned on. While there are no non-trivial two-cycles in this case, the worldvolume gauge field does modify the coupling of this D-instanton to the RR potential C_4. We will show that such a configuration can be made κ-symmetric and then yields the baryonic condensates consistent with the gauge theory expectations.

As a first example of a supersymmetric brane wrapping all the angular directions, we shall discuss a D7-brane wrapping the warped deformed conifold, with the remaining one space and one time directions extended in $\mathbb{R}^{3,1}$. The supersymmetry conditions for general D-branes in $\mathcal{N} = 1$ backgrounds were derived in [149, 150, 151], and our results will be consistent with theirs. We will show that the Lorentzian D7-brane configuration on the KS background is supersymmetric in the absence of a worldvolume gauge-field, though the κ-symmetry analysis will also reveal supersymmetric configurations with non-zero gauge field. The fact that switching on this field is not required for supersymmetry might have been guessed from a naive counting argument. This embedding of the D7-brane should be mutually supersymmetric with the D3-branes filling the $\mathbb{R}^{3,1}$, since the number of Neumann-Dirichlet directions for strings stretched between them equals eight.

The object we are most interested in is the Euclidean D5-brane completely wrapped on the conifold. In contrast to the case of the D7-brane, we will find that supersymmetry requires a non-trivial gauge field on the worldvolume. Again this is consistent with the naive count of Neumann-Dirichlet directions with the D3-branes, which gives ten in this case and thus indicates that these branes cannot be mutually supersymmetric if $F_2 = 0$.

4.2 Derivation of the First-Order Equation for the Worldvolume Gauge Bundle

In this section we derive the first-order equation of motion that the U(1) gauge field has to satisfy to obtain a supersymmetric configuration. Because the κ-symmetry of the Euclidean D5-brane is subtle, we will first discuss the closely related case of a Lorentzian D7-brane wrapping the six-dimensional deformed conifold, with non-zero gauge bundle only in these directions. This object is extended as a string in the $\mathbb{R}^{3,1}$ but in the case of a non-compact space dual to the cascading gauge theory the tension of such a string diverges with the cut-off as $e^{2\tau/3}$. Therefore, this string is not part of the gauge theory spectrum.

4.2.1 κ-Symmetry of the Lorentzian D7-Brane

The explicit form of the κ-symmetry equation for the D7-brane with non-trivial U(1) bundle on the six-dimensional internal space is given by

$$\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\end{pmatrix} = \Gamma_\kappa \begin{pmatrix}\varepsilon_1\\ \varepsilon_2\end{pmatrix} \sim \left[-(\not{\mathcal{F}} + \not{\mathcal{F}}^3)\,\sigma_3 + (1+\not{\mathcal{F}}^2)\right] i\sigma_2\,\Gamma_{x^0x^1123456} \begin{pmatrix}\varepsilon_1\\ \varepsilon_2\end{pmatrix}\,, \tag{4.15}$$

For the case of Euclidean D-branes wrapping certain cycles in Calabi-Yau manifolds, it was shown in [149] that the κ-symmetry condition (4.15) can be rewritten in more geometrical terms. This results in the conditions that $\mathcal{F}^{2,0} = 0$, and that

$$\frac{1}{2!} J \wedge J \wedge \mathcal{F} - \frac{1}{3!} \mathcal{F} \wedge \mathcal{F} \wedge \mathcal{F} = \mathfrak{g} \left(\frac{1}{3!} J \wedge J \wedge J - \frac{1}{2!} J \wedge \mathcal{F} \wedge \mathcal{F} \right). \tag{4.16}$$

The constant $\mathfrak{g}$ was found [149] to encode some information about the geometry, namely a relative phase between coefficients of the covariantly constant spinors in the expansion of the ε_i [149]. As we shall see below, the same equation holds in our case of a generalized Calabi-Yau with fluxes, except that $\mathfrak{g}$ becomes coordinate dependent.

With the SU(2) × SU(2) invariant ansatz for the gauge potential

$$A_1 = \xi(\tau) g_5 \ , \tag{4.17}$$

we find that the gauge-invariant two-form field strength is given by

$$\begin{aligned} \mathcal{F} = & \frac{i e^{-x}}{2 \sinh(\tau)} \times \qquad (4.18) \\ & \Big[e^{-g} \Big[\tilde{\xi}(\cosh(\tau) + 2a + a^2 \cosh(\tau)) + h_2 \sinh^2(\tau)(1 - a^2) \Big] (G_1 + iG_2) \wedge (G_1 - iG_2) \\ & + e^{g} \Big[\tilde{\xi} \cosh(\tau) - h_2 \sinh^2(\tau) \Big] (G_3 + iG_4) \wedge (G_3 - iG_4) \\ & + \xi' v \sinh(\tau)(G_5 + iG_6) \wedge (G_5 - iG_6) + \Big[\tilde{\xi}(1 + a \cosh(\tau)) - h_2 a \sinh^2(\tau) \Big] \\ & \big((G_1 + iG_2) \wedge (G_3 - iG_4) + (G_3 + iG_4) \wedge (G_1 - iG_2) \big) \Big] \ , \end{aligned}$$

where $\tilde{\xi} = \xi + \chi$. This explicitly shows that $\mathcal{F}$ is a $(1,1)$ form, which is one of the κ-symmetry conditions [149, 150, 151]. Now it is convenient to define

$$\begin{aligned} \mathfrak{a}(\xi, \tau) &\equiv e^{-2x} [e^{2x} + h_2^2 \sinh^2(\tau) - (\xi + \chi)^2] \ , \\ \mathfrak{b}(\xi, \tau) &\equiv 2 e^{-x-g} \sinh(\tau) [a(\xi + \chi) - h_2(1 + a \cosh(\tau))] \ . \end{aligned} \tag{4.19}$$

In terms of these expressions we find that

$$\begin{aligned}\frac{1}{3!}J\wedge J\wedge J-\frac{1}{2!}J\wedge\mathcal{F}\wedge\mathcal{F}&=(\mathfrak{a}+ve^{-x}\mathfrak{b}\,\xi')\,\mathrm{vol}_6\ ,\\ \frac{1}{2!}J\wedge J\wedge\mathcal{F}-\frac{1}{3!}\mathcal{F}\wedge\mathcal{F}\wedge\mathcal{F}&=(-\mathfrak{b}+ve^{-x}\mathfrak{a}\,\xi')\,\mathrm{vol}_6\ ,\end{aligned}\tag{4.20}$$

where $\mathrm{vol}_6=(J\wedge J\wedge J)/3!$. Thus (4.16) would lead to a differential equation of the form

$$\xi'=\frac{e^x(\mathfrak{g}\mathfrak{a}+\mathfrak{b})}{v(\mathfrak{a}-\mathfrak{g}\mathfrak{b})}\ ,\tag{4.21}$$

for some as yet undetermined $\mathfrak{g}$. In order to confirm the validity of this equation and determine the function $\mathfrak{g}$ we return to the full κ-symmetry equation (4.15) with the Majorana-Weyl spinors $\varepsilon_1=(\Psi+\Psi^*)/2$ and $\varepsilon_2=(\Psi-\Psi^*)/2i$ constructed from the Killing spinor. The analysis of this equation is much simplified by noting that $\Gamma_{1..6}\eta^\pm=\mp i\eta^\pm$ and that the spinors $\eta^\pm$ are in fact eigenspinors[2] of $\not{\mathcal{F}}^n$

$$\not{\mathcal{F}}\eta^\pm=\pm i\eta^\pm\left(\mathcal{F}_{12}+\mathcal{F}_{34}+\mathcal{F}_{56}\right)\ ,\tag{4.22}$$

$$\not{\mathcal{F}}^2\eta^\pm=-\eta^\pm\left(\mathcal{F}_{12}\mathcal{F}_{34}+\mathcal{F}_{14}\mathcal{F}_{23}+\mathcal{F}_{12}\mathcal{F}_{56}+\mathcal{F}_{34}\mathcal{F}_{56}\right)\ ,\tag{4.23}$$

$$\not{\mathcal{F}}^3\eta^\pm=\mp i\eta^\pm\left(\mathcal{F}_{12}\mathcal{F}_{34}\mathcal{F}_{56}+\mathcal{F}_{14}\mathcal{F}_{23}\mathcal{F}_{56}\right)\ ,\tag{4.24}$$

where the indices refer the basis one-forms (1.66). Then it follows from (4.20) that the two terms in the κ-symmetry equation act on the spinors in a rather simple fashion:

$$\begin{aligned}\left[1+\not{\mathcal{F}}^2\right]\eta^\pm&=\left[\mathfrak{a}+ve^{-x}\mathfrak{b}\xi'\right]\eta^\pm\ ,\\ \left[\not{\mathcal{F}}+\not{\mathcal{F}}^3\right]\eta^\pm&=\pm i\left[-\mathfrak{b}+ve^{-x}\mathfrak{a}\xi'\right]\eta^\pm\ .\end{aligned}\tag{4.25}$$

Using these relations it is easy to see that the Killing spinor (4.10) indeed solves (4.15) provided we impose the conditions that its four-dimensional parts $\zeta^\pm$ obey the condition

[2]For simplicity we drop the four-dimensional spinors $\zeta^\pm$ in $\zeta^\pm\otimes\eta^\pm$.

$\Gamma_{x^0x^1}\zeta^{\pm} \otimes \eta^{\pm} = \zeta^{\pm} \otimes \eta^{\pm}$, and that the gauge field $\xi(\tau)$ satisfies (4.21) with

$$\mathfrak{g}(\tau) = \mathfrak{g}_7(\tau) \equiv -\frac{2\alpha\beta}{\alpha^2 - \beta^2} = -e^{-\phi}\sqrt{1 - e^{2\phi}} \; . \tag{4.26}$$

Thus indeed (4.16) holds and (4.21) is the correct first order differential equation given this function $\mathfrak{g}(\tau)$.

The fact that the κ-symmetry condition (4.15) is satisfied implies worldvolume supersymmetry in the probe brane approximation. However, we also ask for the worldvolume supersymmetries to be compatible with those of the background. In order to check how many supersymmetries of the background are preserved by the brane we need to enumerate the solutions of (4.15) for which $\varepsilon_1 + i\varepsilon_2$ is not just any spinor, but a Killing spinor. For the particular case of the D7-brane with U(1) gauge bundle determined by the first-order equation (4.21) we saw that Killing spinors of the form (4.10) solve the κ-symmetry equation if $\Gamma_{x^0x^1}\zeta^{\pm} \otimes \eta^{\pm} = \zeta^{\pm} \otimes \eta^{\pm}$, and thus half of the supersymmetries of the background are preserved.

4.2.2 An Equivalent Derivation Starting from the Equation of Motion

Here we present an alternative derivation of the first-order equations for the gauge field $\xi(\tau)$, starting from the second-order equation of motion. This method has the advantage that it applies equally well to Lorentzian D7 and Euclidean D5-branes wrapping the conifold. The κ-symmetry argument we employed in the previous section for the D7-brane is somewhat complicated in the case of the D5-instanton by the fact that we are forced to Wick rotate to Euclidean spacetime signature where there are no Majorana-Weyl spinors. However, knowing that a first-order differential equation for the gauge field exists, as well as its general features, it is not hard to derive it directly from the second-order equation of motion.

Since with Euclidean signature the DBI action is real and the CS action pure imaginary, two sets of equations of motion have to be satisfied simultaneously if we insist on the gauge field being real. With the ansatz (4.17) for the gauge potential, the CS equations are automatically satisfied, as are five of the DBI equations; only the one for the g_5 component of the gauge field (or equivalently its ψ component) is non-trivial.

In terms of the (implicitly U-dependent) functions defined in [58] the determinant that appears in the DBI action is given by

$$\begin{aligned}\det_{\mathcal{M}}(G+\mathcal{F}) = v^{-2}e^{6x}(1+(\xi')^2v^2e^{-2x})\Big[1+e^{-4x}\left((\xi+\chi)^2-\sinh^2(\tau)h_2^2\right)^2 \\ -2e^{-2x}\left((\xi+\chi)^2+\sinh^2(\tau)h_2^2\right)\left(1-2e^{-2g}a^2\sinh^2(\tau)\right) \\ -8e^{-2x-2g}\sinh^2(\tau)\,a\,h_2(\xi+\chi)(1+a\cosh(\tau))\Big]\,,\end{aligned} \tag{4.27}$$

where we have omitted the angular dependence $\sim \sin^2\theta_1\sin^2\theta_2$. Here we have only taken into account the six-dimensional internal manifold $\mathcal{M}$. If the brane is also extended in the Minkowski directions (but carries zero gauge bundle in these directions) there are additional ξ-independent factors multiplying the DBI determinant that appears in the action (4.4). E.g. for the Lorentzian D7-brane this factor is equal to e^{4A}. Using the definitions (4.19), the term in square brackets in (4.27) can be written as a sum of squares $\mathfrak{a}^2+\mathfrak{b}^2$.

We know from the form of the κ-symmetry equation that the first-order differential equation we are looking for must

i) be polynomial (of at most third order) in ξ and its first derivative,

ii) contain ξ' only at linear order (i.e. no $(\xi')^2$ terms),

iii) be such that the determinant factorizes.

In particular the last condition means that when we eliminate ξ' from the action, the ξ-dependent term must be a perfect square, else the factor of $\sqrt{\det_{\mathcal{M}}(G+\mathcal{F})}$ in the denominator of (4.7) cannot be cancelled by the numerator to give unit eigenvalue. This implies that we must have

$$(1 + (\xi')^2 v^2 e^{-2x}) = \frac{\mathfrak{a}^2 + \mathfrak{b}^2}{\mathfrak{f}^2(\xi, \tau)} \ , \tag{4.28}$$

for some $\mathfrak{f}(\xi, \tau)$, so that

$$\xi' = \frac{e^x \sqrt{\mathfrak{a}^2 + \mathfrak{b}^2 - \mathfrak{f}^2(\xi, \tau)}}{v\, \mathfrak{f}(\xi, \tau)} \ . \tag{4.29}$$

Because we expect the equation to be polynomial in ξ one must be able to explicitly take the square root, and thus $\mathfrak{f}(\xi, \tau)$ can be written as

$$\mathfrak{f}(\xi, \tau) = \frac{\mathfrak{a} - \mathfrak{g}(\tau)\mathfrak{b}}{\sqrt{1 + \mathfrak{g}^2(\tau)}} \ , \tag{4.30}$$

for some function $\mathfrak{g}(\tau)$, where all the ξ-dependence is now implicit in $\mathfrak{a}$ and $\mathfrak{b}$. With this ansatz we have

$$\xi' = \frac{e^x(\mathfrak{g}\mathfrak{a} + \mathfrak{b})}{v(\mathfrak{a} - \mathfrak{g}\mathfrak{b})} \ , \tag{4.31}$$

which is of the same form as the first order differential equation we derived for the D7-brane in the previous section. The function $\mathfrak{g}$ follows by varying the action with respect to ξ and substituting for ξ' using (4.31). It is not difficult to check that the equations of motion that follow from the DBI action of the D7-brane $\int e^{2A-\phi}\sqrt{\det_{\mathcal{M}}(G + \mathcal{F})}$ are indeed implied by the first order equation (4.31) with

$$\mathfrak{g} = \mathfrak{g}_7 = \frac{e^{x-g}(1 + a\cosh(\tau))}{h_2 \sinh(\tau)} = -e^{-\phi}\sqrt{1 - e^{2\phi}} \ , \tag{4.32}$$

as we found above using a κ-symmetry argument.

Using the same method, we can now find the first-order equation for the gauge field on the Euclidean D5-brane. Having constrained the equation we are looking for to the form (4.31) we vary the DBI action $\int e^{-\phi}\sqrt{\det_{\mathcal{M}}(G + \mathcal{F})}$ using (4.27) and eliminate ξ'

to obtain

$$\frac{\delta}{\delta\xi}\left[e^{-\phi}\sqrt{\det(G+F+B)}\right] = 0 =$$
$$\frac{2e^{-\phi}e^{2x}\sqrt{1+\mathfrak{g}^2}}{v(\mathfrak{a}-\mathfrak{g}\mathfrak{b})}\left[-(\xi+\chi)e^{-x}\mathfrak{a}+e^{-g}a\sinh(\tau)\mathfrak{b}\right]-\frac{d}{d\tau}\left[\frac{e^{-\phi}e^{2x}(\mathfrak{g}\mathfrak{a}+\mathfrak{b})}{\sqrt{1+\mathfrak{g}^2}}\right] . \quad (4.33)$$

Collecting powers of ξ and equating their coefficients to zero we find differential equations for $\mathfrak{g}(\tau)$ which are solved simultaneously by

$$\mathfrak{g} = \mathfrak{g}_5 \equiv -\frac{e^{-x+g}h_2\sinh(\tau)}{(1+a\cosh(\tau))} = \frac{e^{\phi}}{\sqrt{1-e^{2\phi}}} . \quad (4.34)$$

Substituting this into (4.31), the first-order equation we were looking for, written out in full detail, is

$$\xi' = \Big[-h_2\sinh(\tau)e^{2g}[e^{2x}+h_2^2\sinh^2(\tau)-(\xi+\chi)^2]$$
$$+2e^{2x}\sinh(\tau)(1+a\cosh(\tau))[a(\xi+\chi)-h_2(1+a\cosh(\tau))]\Big]\times$$
$$\Big[ve^{g}\big[(1+a\cosh(\tau))[e^{2x}+h_2^2\sinh^2(\tau)-(\xi+\chi)^2]$$
$$+2h_2\sinh^2(\tau)[a(\xi+\chi)-h_2(1+a\cosh(\tau))]\big]\Big]^{-1} . \quad (4.35)$$

In spite of its complicated appearance, this equation can be integrated and can in fact be solved fairly explicitly. In the KS limit it reduces to a simpler equation (4.40) that will be discussed in Section 4.3.

Let us note here the interesting fact that the Euclidean D5-brane and the Lorentzian D7-brane are related by $\mathfrak{g}_5 = -1/\mathfrak{g}_7$. For the D7-brane we find $\mathfrak{g}_7 = 0$ for the KS background (since there $1+a\cosh(\tau)=0$), while $\mathfrak{g}_7$ diverges far along the baryonic branch where $h_2 \to 0$, and correspondingly for $\mathfrak{g}_5$ the situation is the other way around[3].

The first order equation for the gauge bundle we have derived is in fact more general

[3] As a curious aside, note that taking $\mathfrak{g}=0$ in (4.31) leads to an equation consistent with the action $\int e^{2A-2\phi}\sqrt{\det_{\mathcal{M}}(G+\mathcal{F})}$. This coincides with the D7 brane case for the KS solution (since here $\phi=0$), but in general it is not clear what (if anything) this corresponds to.

than we have made explicit, and when written in the form (4.35) applies to the whole two-parameter (η, U) family of SU(3) structure backgrounds discussed in [58]. The baryonic branch in particular corresponds to the choice of boundary condition $\eta = 1$ at $\tau = \infty$ in the notation of [58], but the above family of solutions also includes the CVMN background [60, 61, 62], which has the linear dilation boundary condition $\eta = 0$ at infinity.

4.2.3 κ-Symmetry of the Euclidean D5-Brane

Let us now reconsider the Euclidean D5-brane using the κ-symmetry approach. The κ-symmetry projection operator in [142, 144] was derived using the superspace formalism for Lorentzian worldvolume branes in (9,1) signature spacetimes, and thus it is not immediately clear if it is applicable to the case of a Euclidean worldvolume instanton which necessarily has to reside in a (10,0) signature spacetime. For now we shall nevertheless proceed by performing just a naive Wick-rotation of the κ-symmetry projector, which simply introduces a factor $-i$ in (4.7) such that $\Gamma_\kappa^2 = 1$ still holds.

The analog of the κ-symmetry condition (4.15) for the Euclidean D5-brane is then given by

$$\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix} = \Gamma_\kappa \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix} \sim \left[-(\not{\mathcal{F}} + \not{\mathcal{F}}^3) + (1 + \not{\mathcal{F}}^2)\,\sigma_3 \right] \sigma_2\, \Gamma_{123456} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix} . \qquad (4.36)$$

Re-expressing this in geometrical terms leads to an equation of the same form as (4.16), but now we expect $\mathfrak{g}(\tau)$ to be equal to $\mathfrak{g}_5(\tau)$. Using the same ansatz $A_1 = \xi(\tau)\, g_5$ as above it is clear that equations (4.20) and thus (4.21) still hold, and of course $\mathcal{F}$ is still a (1,1) form.

However, with the gauge bundle we derived in the previous subsection (i.e. with $\mathfrak{g} = \mathfrak{g}_5 = (\alpha^2 - \beta^2)/(2\alpha\beta))$ the κ-symmetry equation (4.36) does not have solutions for $\varepsilon_1 + i\varepsilon_2$ being equal to the Killing spinor (4.10). We can find solutions for other spinors by expanding the ε_i in terms of pure spinors:

$$\varepsilon_i = x_i(\tau)\,\zeta^- \otimes \eta^- + y_i(\tau)\,\zeta^+ \otimes \eta^+ \ , \tag{4.37}$$

where $i = 1, 2$. We find that with this ansatz (4.36) is solved if the coefficients satisfy

$$\frac{x_1}{x_2} = i\frac{(\alpha - i\beta)^2}{\alpha^2 + \beta^2} \ , \qquad \frac{y_1}{y_2} = i\frac{(\alpha + i\beta)^2}{\alpha^2 + \beta^2} \ . \tag{4.38}$$

Thus we have obtained a family of spinors (4.38) that solves the κ-symmetry equation with the correct gauge bundle, but this family does not seem to contain the Killing spinor (which differs by a sign in y_1/y_2). This would imply that even though for the gauge field configuration we have found there is worldvolume supersymmetry in the probe brane approximation, these supersymmetries would not be compatible with those of the background.

We believe that this difficulty is just an artefact of applying the κ-symmetry operator in a Euclidean spacetime to a Euclidean worldvolume brane without properly taking into account the subtleties of Wick-rotating the spinors and the projector itself, and that the D5-instanton does preserve the background supersymmetries. In fact it is known that for a Euclidean D5-brane wrapping six internal dimensions the correct κ-symmetry equations are not the ones obtained by the naive Wick rotation we performed above, but instead are identical to those for a Lorentzian D9-brane[4]. The κ-symmetry conditions for the Lorentzian D9-brane lead to equations identical to (4.38) except for a change of sign on the right hand side of the equation for y_1/y_2, so that they are now satisfied by the Killing spinor. This shows that the worldvolume gauge field found above is consistent with properly defined κ-symmetry.

In either case we consider the independent derivation of the first-order equation (4.35) in the previous subsection a compelling argument that this gauge bundle is in

[4]We would like to thank L. Martucci for pointing this out to us.

fact the correct one for our purposes, which will be corroborated below by the successful extraction of the baryon operator dimension from its large τ behaviour.

4.3 Euclidean D5-Brane on the KS Background

We will now specialize the discussion of the previous section to the case of a Euclidean D5-brane wrapping the deformed conifold in the KS background. Since this background is known analytically, the formulae are more explicit in this case. We interpret the Euclidean D5-brane (which has the appearance of a pointlike instanton in Minkowski space) as the dual of the baryon in the field theory, in the sense that its action captures information about the (scale-dependent) anomalous dimension of the baryon operator, as well as its expectation value.

4.3.1 The Gauge Field and the Integrated Form of the Action

For the KS background (where for simplicity we set $M\alpha' = 2$ and $g_s = \varepsilon = 1$), we have $a = -1/\cosh(\tau)$ and $\chi = 0$, so the first-order differential equation (4.35) simplifies to

$$\xi' = \frac{e^{2x} + h_2^2 \sinh^2(\tau) - \xi^2}{2\, v\, \xi} \ , \tag{4.39}$$

or more explicitly, substituting in the KS expressions for x, h_2 and v:

$$3\, \frac{\sinh(\tau)\cosh(\tau) - \tau}{\sinh^2(\tau)}\, \xi'\xi + \xi^2 = \frac{(\sinh(\tau)\cosh(\tau) - \tau)^{2/3} h}{16} + \frac{1}{4}\, (\tau \coth(\tau) - 1)^2 \ . \tag{4.40}$$

Note that there is no $\xi'\xi^2$ term. Thus we can multiply the equation by an integrating factor to turn the left hand side into the total derivative $[(\sinh(\tau)\cosh(\tau) - \tau)^{1/3}\xi^2]'$ and reduce the equation to the integral

$$\xi^2 = (\sinh(\tau)\cosh(\tau) - \tau)^{-1/3} J(\tau) \ , \tag{4.41}$$

where[5]

$$J(\tau) = \int_0^\tau \left(\frac{\sinh^2(x)\, h(x)}{24} + \frac{\sinh^2(x)(x\coth(x) - 1)^2}{6\,(\sinh(x)\cosh(x) - x)^{2/3}} \right) dx \ . \tag{4.43}$$

We have set the integration constant to zero by requiring regularity at $\tau = 0$.

Now consider the DBI action of the Euclidean D5-brane with this worldvolume gauge field. Neglecting the five angular integrals for the time being, and focussing on the radial integral, we see that the Lagrangian is in fact a total derivative, and thus the action is given by

$$\begin{aligned} \mathcal{S}_{DBI} &\sim \int \mathrm{d}\tau\, e^{-\phi}\sqrt{\det G + \mathcal{F}} \\ &= -\frac{1}{3(\sinh(\tau)\cosh(\tau) - \tau)^{1/2}} J^{3/2} \\ &\quad + \left[\frac{(\sinh(\tau)\cosh(\tau) - \tau)^{1/2} h}{16} + \frac{(\tau\coth(\tau) - 1)^2}{4(\sinh(\tau)\cosh(\tau) - \tau)^{1/6}} \right] J^{1/2} \ . \end{aligned} \tag{4.44}$$

We are particularly interested in the UV behaviour of these quantities. From (4.41) it is easy to find the asymptotic expansion of the gauge field as $\tau \to \infty$:

$$\xi^2 \to \frac{1}{4}\tau^2 - \frac{7}{8}\tau + \frac{47}{32} + \mathcal{O}(e^{-2\tau/3}) \ . \tag{4.45}$$

Note that to leading order this approximates $h_2^2 \sinh^2(\tau)$, so for large τ the coefficients of the F_2 and B_2 fields become equal and cancellations occur in the action. This is essential for obtaining the τ^3 behaviour of the action for large cut-off τ, which as we will see gives the correct τ^2 scaling of the baryon operator dimensions.

To extract the asymptotic behaviour of the action we will use the integrated form

[5]The integral looks "almost" like the explicitly computable one

$$\begin{aligned} &\int_0^\tau \left(\frac{\sinh^2(x)\, h(x)}{24} + \frac{\sinh^2(x)(x\coth(x) - 1)^2}{18\,(\sinh(x)\cosh(x) - x)^{2/3}} \right) dx \\ &= \frac{1}{48}(\sinh(\tau)\cosh(\tau) - \tau)\, h(\tau) + \frac{1}{12}\,(\tau\coth(\tau) - 1)^2 (\sinh(\tau)\cosh(\tau) - \tau)^{1/3} \ , \end{aligned} \tag{4.42}$$

but a relative factor of 3 in the second term of (4.43) prevents us from performing it in closed form.

(4.44). The leading terms in the expansion are easily found analytically, with the result

$$\mathcal{S}_{DBI} = \int \mathrm{d}\tau e^{-\phi}\sqrt{\det(G+\mathcal{F})} \to \frac{1}{6}(\tau^2+\tau-2)\left(\frac{1}{4}\tau^2 - \frac{7}{8}\tau + \frac{47}{32}\right)^{1/2} + \mathcal{O}(e^{-2\tau/3})$$
$$\to \frac{1}{12}\tau^3 - \frac{1}{16}\tau^2 - \frac{25}{128}\tau + \frac{943}{1536} + \mathcal{O}(1/\tau)\ . \qquad (4.46)$$

Below we will argue that the $\mathcal{O}(1)$ term in this expansion determines the expectation value of the baryon operator. Of particular interest is the variation of this expectation value along the baryonic branch; we will investigate it in the next section. First, however, we will give a field theoretic interpretation to the terms that increase with τ. As we will see, the coefficients of these divergent terms are universal for all backgrounds along the baryonic branch.

4.3.2 Scaling Dimension of Baryon Operator

We have seen that for large cut-off r (i.e. large τ), the DBI action of the Euclidean D5-brane will behave as $\mathcal{S}(r) \sim (\ln(r))^3$. Since this object corresponds to the baryon in the field theory, we expect that $\exp(-\mathcal{S})$ is related to $r^{-\Delta}$, where Δ is the scaling dimension of the baryon operator.

To make this statement more precise we consider the RG flow equation relating the operator dimension Δ to the boundary behavior of the dual field $\varphi(r)$:

$$-r\frac{\mathrm{d}\varphi(r)}{\mathrm{d}r} = \Delta(r)\varphi(r)\ . \qquad (4.47)$$

This equation obviously holds in the usual AdS/CFT case where all operator dimensions have a limit as the UV cut-off is removed. The case of cascading theories is more subtle, since there exist operators, such as the baryons, whose dimensions grow in the UV. As we will see, in these cases (4.47) is still applicable. Identifying the field dual to a baryon operator as

$$\varphi(r) \sim \exp(-\mathcal{S}(r)) \ , \tag{4.48}$$

we find

$$\Delta(r) = r\frac{\mathrm{d}\mathcal{S}(r)}{\mathrm{d}r} = \frac{\mathrm{d}\mathcal{S}(r)}{\mathrm{d}\ln(r)} \ . \tag{4.49}$$

To calculate the scaling dimension of the baryon in the gauge theory, we simply count the number of constituent fields required to build a baryon operator for a given gauge group $\mathrm{SU}(kM) \times \mathrm{SU}((k+1)M)$ and multiply by the dimension of the chiral superfield A or B; the latter approaches $3/4$ in the UV where the theory is quasi-conformal. This gives

$$\Delta(r) = \frac{3}{4}Mk(k+1) = \frac{27g_s^2M^3}{16\pi^2}(\ln(r))^2 + \mathcal{O}(\ln(r)) \ , \tag{4.50}$$

where k labels the cascade steps and we have used the asymptotic expression for the radius (energy scale) at which the kth Seiberg duality is performed:

$$r_k = r_0 \exp\left(\frac{2\pi k}{3g_sM}\right) . \tag{4.51}$$

Here and in the remainder of this subsection we keep factors of g_s, M, ε and α' explicit.

Let us now compare this to the scaling dimension we obtain from the action of the D5-instanton according to equation (4.49). The leading term in the action is $\tau^3/12$, which is multiplied by a factor $(g_sM\alpha'/2)^3$ that we had previously set to one, a factor $64\pi^3$ from the previously neglected five angular integrals and a factor of $T_5 = (2\pi)^{-5}\alpha'^{-3}g_s^{-1}$. Therefore, using (1.29) we have

$$\mathcal{S} = \frac{\tau^3}{12}\left(\frac{g_sM\alpha'}{2}\right)^3\frac{64\pi^3}{(2\pi)^5\alpha'^3g_s} + \mathcal{O}(\tau^2) = \frac{9g_s^2M^3}{16\pi^2}(\ln(r))^3 + \mathcal{O}((\ln(r))^2) \ . \tag{4.52}$$

From (4.49) we find that this string theoretic calculation gives

$$\Delta(r) = \frac{27 g_s^2 M^3}{16\pi^2} (\ln(r))^2 + \mathcal{O}(\ln(r)) \ . \tag{4.53}$$

The term of leading order in $\ln(r)$ is in perfect agreement with the gauge theory result (4.50). We consider this a strong argument that the relation (4.48) between the Euclidean D5-brane action and the field dual to the baryon is indeed correct.

4.3.3 The Pseudoscalar Mode and the Phase of the Baryonic Condensate

Let us now turn to a discussion of the Chern-Simons terms in the D-brane action. Given our conventions (1.39) for the gauge-invariant and self-dual five-form field strength $\tilde{F}_5$, there is a slight subtlety in the CS term of the action (4.4). Its standard form, given above, is valid with the choice of conventions where $\tilde{F}_5 = F_5 + H_3 \wedge C_2 = dC_4 + dB_2 \wedge C_2$. In these conventions dC_4 is invariant under B_2 gauge transformations $B_2 \to B_2 + d\lambda_1$, but transforms under C_2 gauge transformations $C_2 \to C_2 + d\Lambda_1$ such as to leave $\tilde{F}_5$ invariant. However, we work in different conventions where $\tilde{F}_5 = dC_4 + B_2 \wedge F_3$; here dC_4 changes under B_2 gauge transformations. This choice also alters the form of the CS term in the action. The new RR fields are obtained by $C_4 \to C_4 + B_2 \wedge C_2$ combined with $C_2 \to -C_2$ everywhere else, which modifies some of the terms in the CS action that will be relevant for us:

$$\frac{1}{2}\int C_2 \wedge \mathcal{F} \wedge \mathcal{F} + \int C_4 \wedge \mathcal{F} \to -\frac{1}{2}\int C_2 \wedge F \wedge F + \frac{1}{2}\int C_2 \wedge B \wedge B + \int C_4 \wedge \mathcal{F} \ . \tag{4.54}$$

For the KS background the CS action simply vanishes. However, it is interesting to consider small perturbations around it. The pseudoscalar glueball discovered in [52, 53] is the Goldstone boson of the broken $U(1)$ baryon number symmetry; it is associated with the phase of the baryon expectation value. This massless mode is a deformation of the RR fields (which is generated for example by a D1-string extended in $\mathbb{R}^{3,1}$) given by

$$\begin{aligned}\delta F_3 &= *_4 da + f_2(\tau)\, da \wedge dg^5 + f_2'(\tau)\, da \wedge d\tau \wedge g^5 \ , \\ \delta \tilde{F}_5 &= (1+*)\delta F_3 \wedge B_2 = \left(*_4 da - \frac{h(\tau)}{6K^2(\tau)}\, da \wedge d\tau \wedge g^5 \right) \wedge B_2 \ ,\end{aligned} \tag{4.55}$$

where $a(x^0, x^1, x^2, x^3)$ is a pseudoscalar field in four dimensions that satisfies $d *_4 da = 0$ and would experience monodromy around a D-string. This deformation solves the supergravity equations with

$$f_2(\tau) = \frac{1}{6\, K^2(\tau) \sinh^2 \tau} \int_0^\tau dx h(x) \sinh^2(x) \ . \tag{4.56}$$

If we wish to identify the exponential $\exp(-\mathcal{S}) = \exp(-\mathcal{S}_{DBI} - \mathcal{S}_{CS})$ of the brane action (or more precisely the constant term in its asymptotic expansion as $\tau \to \infty$) with the baryon expectation value, then the pseudoscalar massless mode has to shift the phase of this quantity, contained in the imaginary Chern-Simons term. The DBI action is obviously unaffected by this deformation of the background since the NSNS fields are unchanged. This is consistent with the magnitudes of the baryon expectation values being unaffected by the pseudoscalar mode; these magnitudes depend only on the scalar modulus U in supergravity, corresponding to $|\zeta|$ in the gauge theory.

The phase $\exp(-\mathcal{S}_{CS})$ by itself is not gauge invariant and thus not physical. Because our brane configuration has a boundary at $\tau = \infty$, only the difference in phase $\exp(-\Delta\mathcal{S}_{CS}) = \exp(-i\Delta\phi)$ between two Euclidean D5-branes displaced slightly in one of the transverse directions (i.e. between two instantons at different points in Minkowski space) is gauge-invariant. Taking into account the anomalous Bianchi identity for $\tilde{F}_5$ and the RR gauge transformations we see that this gauge-invariant phase difference is given by

$$\Delta\phi = \Delta\phi_B + \Delta\phi_F \ , \tag{4.57}$$

where

$$\Delta\phi_B = \int \left[\frac{1}{2}\delta F_3 \wedge B \wedge B + \delta F_5 \wedge B + *_{10}\delta F_3 \right] \ , \tag{4.58}$$

$$\Delta\phi_F = \int \left[-\frac{1}{2}\delta F_3 \wedge F \wedge F + \delta F_5 \wedge F \right] \ . \tag{4.59}$$

The integrals are performed over the six internal dimensions as well as a line in Minkowski space. Note that here $F_5 \equiv \mathrm{d}C_4 = \tilde{F}_5 - B_2 \wedge F_3$. For small perturbations around KS the contribution $\Delta\phi_F$ from the coupling to the gauge field vanishes (the first term in (4.59) is a total derivative with vanishing boundary terms, while the second term doesn't have the right angular structure to give a non-zero result). Substituting the explicit form of the RR deformations from (4.55) we find that the phase difference is

$$\Delta\phi_B = -\frac{1}{2}\int \left(\frac{h}{6K^2} + f_2' \right) (\tau\coth(\tau) - 1)^2 \, \mathrm{d}a \wedge \mathrm{d}\tau \wedge g^1 \wedge g^2 \wedge g^3 \wedge g^4 \wedge g^5 \ . \tag{4.60}$$

We can interpret $\Delta\phi$ as Δa times a baryon number. It is satisfying to see that the pseudoscalar Goldstone mode indeed shifts the phase of the baryon expectation value and not its magnitude. A more stringent test of our interpretation would be to carry out this computation for the whole baryonic branch and check whether the numerical value of the baryon number computed this way is independent of the modulus U. This is rather difficult, since the pseudoscalar mode at a general point along the baryonic branch is not explicitly known at present.

4.4 Euclidean D5-Brane on the Baryonic Branch

In this section we extend the discussion of the previous section from the KS solution to the entire baryonic branch. In particular we are interested in the dependence of the baryon expectation value on the modulus U of the supergravity solutions. All supergravity backgrounds dual to the baryonic branch have the same asymptotics [59] and we will see that the leading terms (cubic, quadratic and linear in τ) in the asymptotic expansion of the action (4.46) are universal. This implies that the leading scaling dimensions of the baryon operators do not depend on U, consistent with field theory expectations. However, the finite term in the asymptotic expansion of the brane action does depend on U. This provides a map from the one-parameter family of supergravity solutions labelled by U to the family of field theory vacua with different baryon expectation values (4.2), parameterized by ζ.

4.4.1 Solving for the Gauge Field and Integrating the Action

Having derived the differential equation that determines the gauge field in full generality in Section 4.2, let us now turn to a more detailed investigation of the first order equation (4.35). First of all we note that it can be rewritten as

$$\begin{aligned}&\frac{d}{d\tau}\left[-\frac{1}{3}\xi^3+\left(\frac{ah_2\sinh^2(\tau)}{1+a\cosh(\tau)}-\chi\right)\xi^2+\left(e^{2x}-h_2^2\sinh^2(\tau)-\chi^2+\frac{2ah_2\sinh^2(\tau)}{1+a\cosh(\tau)}\chi\right)\xi\right]\\&=-\frac{h_2\sinh(\tau)e^g}{v(1+a\cosh(\tau))}[e^{2x}+h_2^2\sinh^2(\tau)-\chi^2]\\&\quad+\frac{2e^{2x}\sinh(\tau)}{ve^g}[a\chi-h_2(1+a\cosh(\tau))]\ .\end{aligned} \tag{4.61}$$

For notational convenience we define $\tilde{\xi}\equiv\xi+\chi$,

$$\mathfrak{A}(\tau)\equiv\frac{ah_2\sinh^2(\tau)}{1+a\cosh(\tau)}\ , \tag{4.62}$$

$$\mathfrak{B}(\tau)\equiv e^{2x}-h_2^2\sinh^2(\tau)\ , \tag{4.63}$$

and

$$\rho(\tau) \equiv \int_0^\tau \left[\frac{h_2 \sinh(\tau) e^g}{v(1 + a\cosh(\tau))} [e^{2x} + h_2^2 \sinh^2(\tau)] \right.$$
$$\left. + \frac{2e^{2x} h_2 \sinh(\tau)(1 + a\cosh(\tau))}{ve^g} - [e^{2x} - h_2^2 \sinh^2(\tau)]\chi' \right] d\tau \ , \tag{4.64}$$

which allows us to write (4.61) more compactly as

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\Big[-\frac{1}{3}\tilde{\xi}^3 + \mathfrak{A}(\tau)\tilde{\xi}^2 + \mathfrak{B}(\tau)\tilde{\xi} + \rho(\tau) \Big] = 0 \ . \tag{4.65}$$

Thus the solutions for the shifted field $\tilde{\xi}$ are given by the roots of the third order polynomial

$$-\frac{1}{3}\tilde{\xi}^3 + \mathfrak{A}(\tau)\tilde{\xi}^2 + \mathfrak{B}(\tau)\tilde{\xi} + \rho(\tau) = C \ , \tag{4.66}$$

where C is the integration constant.[6] To fix it, we consider the small τ expansion, which is valid for any U

$$\mathfrak{A} \sim \tau + \mathcal{O}(\tau^3) \ , \tag{4.67}$$

$$\mathfrak{B} \sim \tau^2 + \mathcal{O}(\tau^4) \ , \tag{4.68}$$

$$\rho \sim \tau^3 + \mathcal{O}(\tau^4) \ . \tag{4.69}$$

Note that at $\tau = 0$ all coefficients in (4.66) vanish, except the first one; therefore, the integration constant C has to be zero for this cubic to admit more than one real solution. Then we find that $\tilde{\xi} = 0$ at $\tau = 0$ for any solution on the baryonic branch.

Let us examine the cubic equation (4.66) more closely in the KS limit ($U \to 0$) to see how our earlier result (4.41) is recovered. In the $U \to 0$ limit $a \to -1/\cosh(\tau)$ and

[6] This equation is quite general; it does not assume boundary conditions $\eta = 1$ that characterize the baryonic branch [59]. In particular this result is also valid for a brane embedded in the CVMN solution [60, 61, 62].

therefore $(1 + a\cosh(\tau))$ vanishes. For small U [52, 53, 58, 59]

$$(1 + a\cosh(\tau)) = 2^{-5/3} U Z(\tau) + \mathcal{O}(U^2) \ , \tag{4.70}$$

$$Z(\tau) \equiv \frac{(\tau - \tanh(\tau))}{(\sinh(\tau)\cosh(\tau) - 1)^{1/3}} \ . \tag{4.71}$$

In this case $\mathfrak{A}$ and the first term in ρ diverge as U^{-1}. All other terms can be dropped and we have instead of (4.65)

$$\tilde{\xi}^2 \frac{a h_2 \sinh^2(\tau)}{Z(\tau)} + \int_0^\tau d\tau \ \frac{h_2 \sinh(\tau) e^g}{v Z(\tau)} [e^{2x} + h_2^2 \sinh(\tau)^2] = 0 \ . \tag{4.72}$$

After substituting the KS values for a, v, h_2, x we recover (4.41).

While it would be desirable to obtain a closed form expression for the integral $\rho(\tau)$ in order to evaluate ξ explicitly, this appears to be impossible, since even in the KS case we cannot perform the corresponding integral $J(\tau)$.

Evaluating the DBI Lagrangian on-shell using (4.31) we find

$$e^{-\phi}\sqrt{\det(G + \mathcal{F})} = \frac{e^{-\phi} e^{3x} \sqrt{1 + \mathfrak{g}^2}\,(\mathfrak{a}^2 + \mathfrak{b}^2)}{v|\mathfrak{a} - \mathfrak{g}\mathfrak{b}|} \ , \tag{4.73}$$

where we have taken the absolute value since the sign of $\mathfrak{a} - \mathfrak{g}\mathfrak{b}$ will turn out to depend on which root of equation (4.66) we pick.

For the baryonic branch backgrounds we can show that the action is a total derivative. First note that the DBI Lagrangian (4.73) can be rewritten in the form

$$\begin{aligned} e^{-\phi}\sqrt{\det(G + \mathcal{F})} &= \frac{e^{-\phi} e^{3x}}{v\sqrt{1 + \mathfrak{g}^2}} \frac{(\mathfrak{g}\mathfrak{a} + \mathfrak{b})^2 + (\mathfrak{a} - \mathfrak{g}\mathfrak{b})^2}{|\mathfrak{a} - \mathfrak{g}\mathfrak{b}|} \\ &= \left| \frac{e^{4x}(1 + a\cosh(\tau))}{v h_2 \sinh(\tau) e^g} [v e^{-x} \xi' (\mathfrak{g}\mathfrak{a} + \mathfrak{b}) + (\mathfrak{a} - \mathfrak{g}\mathfrak{b})] \right| \ , \end{aligned} \tag{4.74}$$

where the right hand side is now cubic in ξ (and its derivative) much like the differential equation (4.31). In fact, substituting for $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{g} = \mathfrak{g}_5$ this equation can be integrated in the same manner, which results in the action

$$\mathcal{S} = \left| -\frac{1}{3}\tilde{\xi}^3 + \mathfrak{C}(\tau)\tilde{\xi}^2 + \mathfrak{D}(\tau)\tilde{\xi} + \sigma(\tau) \right| \,, \tag{4.75}$$

with $\mathfrak{C}, \mathfrak{D}, \sigma$ defined as

$$\mathfrak{C} = -\frac{e^{2x} a\,(1 + a\cosh(\tau))}{h_2 e^{2g}} \,, \tag{4.76}$$

$$\mathfrak{D} = [e^{2x} + h_2^2 \sinh^2(\tau) + 2e^{2x}(1 + a\cosh(\tau))^2 e^{-2g}] \,, \tag{4.77}$$

$$\sigma = -\int_0^\tau \left[\frac{e^{2x}(1 + a\cosh(\tau))}{v h_2 \sinh(\tau) e^g} [e^{2x} - h_2^2 \sinh^2(\tau)] + \right. \tag{4.78}$$

$$\left. [e^{2x} + h_2^2 \sinh^2(\tau) + 2e^{2x}(1 + a\cosh(\tau))^2 e^{-2g}]\chi' \right] d\tau \,. \tag{4.79}$$

Again the ξ-independent term is an integral, that we denoted by $\sigma(\tau)$. Thus we have a fairly explicit expression for the action involving two integrals: $\rho(\tau)$, which appears in the equation for $\tilde{\xi}$, and $\sigma(\tau)$.

To conclude this subsection we will demonstrate that the third solution of (4.65), which is absent (formally divergent for all τ) in the KS case (4.41), produces a badly divergent action and is therefore unacceptable for any point on the branch. Restoring the $-\tilde{\xi}^3/3$ term in (4.72) we see that in the GHK region $U \to 0$ the third solution is simply

$$\xi = -\frac{2^{2/3}3}{U} \left(\cosh(\tau)\sinh(\tau) - \tau\right)^{1/3} + \mathcal{O}(U) \,. \tag{4.80}$$

The value of the Lagrangian in this case is

$$\sqrt{\det(G + \mathcal{F})} = \frac{36}{U^3} \sinh^2(\tau) + \mathcal{O}(U^{-2}) \,. \tag{4.81}$$

This expression can be used to extract the leading UV asymptotics of the Lagrangian for any U as the UV behavior is universal for all U:

$$\sqrt{\det(G+\mathcal{F})} \to \frac{9}{U^3} e^{2\tau} \ . \tag{4.82}$$

Since the action for the third solution diverges exponentially at large τ it does not seem possible to interpret this solution as the dual of an operator in the same sense as we do for the other two solutions.

4.4.2 Baryonic Condensates

We shall now study the D5-brane action (4.75) in more detail. First we develop an asymptotic expansion of the action (4.75) as a function of the cut-off. This expansion is useful because the divergent terms give the scaling dimension of the baryon operator, while the finite term encodes its expectation value.[7] Then we present a perturbative treatment of small U region followed by a numerical analysis of the whole baryonic branch. The main result of this section will be an expression for the expectation value as a function of U which can be evaluated numerically. This leads to an explicit relation between the field theory modulus $|\zeta|$ and the string theory modulus U.

To calculate the baryonic condensates we need asymptotic the behavior of $\mathfrak{A}, \mathfrak{B}, \rho$ and $\mathfrak{C}, \mathfrak{D}$ for large τ. Notice that since for any U the solution approaches the KS solution at large τ, the terms divergent at $U = 0$ are UV divergent as well

$$\mathfrak{A} \to \frac{e^{2\tau/3}}{U} + \mathcal{O}(e^{-2\tau/3}) \ , \tag{4.83}$$

$$\mathfrak{B} \to \mathcal{O}(\tau^2) \ , \tag{4.84}$$

$$\rho \to -\frac{e^{2\tau/3}}{U}\left(\frac{1}{4}\tau^2 - \frac{7}{8}\tau + \frac{47}{32}\right) + \mathcal{O}(1) \ , \tag{4.85}$$

[7] A systematic procedure for isolating the finite terms is holographic renormalization [152, 153], but here we limit ourselves to a more heuristic approach.

and similarly

$$\mathfrak{C} \rightarrow \mathcal{O}(e^{-2\tau/3}) \ , \tag{4.86}$$

$$\mathfrak{D} \rightarrow \left(\frac{1}{4}\tau^2 - \frac{1}{8}\tau + \frac{5}{32}\right) + \mathcal{O}(e^{-4\tau/3}) \ . \tag{4.87}$$

From the expansion for $\mathfrak{A}, \mathfrak{B}, \rho$ we find that at large τ the gauge field $\tilde{\xi}$ grows linearly with τ and approaches the KS value with exponential precision

$$\tilde{\xi}(\tau, U) \rightarrow \pm \left(\frac{1}{4}\tau^2 - \frac{7}{8}\tau + \frac{47}{32}\right)^{1/2} + \mathcal{O}(e^{-2\tau/3}) \ . \tag{4.88}$$

It is crucial that the dependence on U in (4.88) is exponentially suppressed.

Since $\mathfrak{C}$ is exponentially small and the leading term in $\mathfrak{D}$ is U-independent we can explicitly express the action (4.75) in terms of σ:

$$\mathcal{S}_{\pm}(U, \tau) = \mathcal{S}_{\text{div}}(\tau) \pm \sigma(U, \tau) + \mathcal{O}(e^{-2\tau/3}) \ , \tag{4.89}$$

where the U-independent divergent part of the action is given by

$$\mathcal{S}_{\text{div}}(\tau) = \frac{1}{6}(\tau^2 + \tau - 2)\left(\frac{1}{4}\tau^2 - \frac{7}{8}\tau + \frac{47}{32}\right)^{1/2} \ , \tag{4.90}$$

Note that

$$\left| -\frac{1}{3}\tilde{\xi}^3 + \mathfrak{D}(\tau)\tilde{\xi} \right| = \mathcal{S}_{\text{div}}(\tau) + \mathcal{O}(e^{-2\tau/3}) \ . \tag{4.91}$$

The two signs stand for the two well-behaved solutions $\xi(\tau)$ corresponding to the two baryons $\mathcal{A}$ and $\mathcal{B}$. As we argued in Section 4.1, the $\mathcal{I}$-symmetry which exchanges the $\mathcal{A}$ and $\mathcal{B}$ baryons is equivalent to changing the sign of U. Our explicit expression (4.89) confirms that

$$\mathcal{S}_+(U,\tau) = \mathcal{S}_-(-U,\tau) \ , \tag{4.92}$$

$$\mathcal{S}_-(U,\tau) = \mathcal{S}_+(-U,\tau) \ , \tag{4.93}$$

since $\sigma(U,\tau)$ is antisymmetric in U according to the arguments presented around (1.80). In order to find the expectation value of the baryons we evaluate the action (4.75) on these solutions and remove the divergence by subtracting the KS value. The expectation values hence are given by $\exp[-\lim_{\tau\to\infty}\mathcal{S}_0(\xi_{1,2})]$, where by $\mathcal{S}_0$ we denote the finite part of the action. It is simplest to work with the product (normalized to the KS value) and ratio of the expectation values. The former is given by

$$\frac{\langle\mathcal{A}\rangle\langle\mathcal{B}\rangle}{\langle\mathcal{A}\rangle_{KS}\langle\mathcal{B}\rangle_{KS}} = \lim_{\tau\to\infty}\exp\left[\mathcal{S}_+(U,\tau) + \mathcal{S}_-(U,\tau) - 2\mathcal{S}(0,\tau)\right] \ , \tag{4.94}$$

where we have used the fact that the two solutions coincide in the KS case, where $\sigma = 0$. It follows from (4.94) that

$$\langle\mathcal{A}\rangle\langle\mathcal{B}\rangle = \langle\mathcal{A}\rangle_{KS}\langle\mathcal{B}\rangle_{KS} \ , \tag{4.95}$$

which corresponds to the constraint $\mathcal{AB} = -\Lambda_{2M}^{4M}$ in the gauge theory. The ratio of the baryon condensates is given by

$$\frac{\langle\mathcal{A}\rangle}{\langle\mathcal{B}\rangle} = \lim_{\tau\to\infty}\exp\left[\mathcal{S}_+(U,\tau) - \mathcal{S}_-(U,\tau)\right] = \lim_{\tau\to\infty} e^{2\sigma} \ , \tag{4.96}$$

or

$$\log\langle\mathcal{A}\rangle \simeq \lim_{\tau\to\infty}\sigma(\tau) \ . \tag{4.97}$$

Unfortunately we were not able to calculate σ analytically, since the U-dependent terms of order $\mathcal{O}(\tau^n)\exp(-2\tau/3)$ in the integrand are significant. However, we can evaluate the integral[8] to first order in U for small U:

$$\sigma \simeq 3.3773\, U + \mathcal{O}(U^3) \ , \tag{4.98}$$

and thus obtain the slope of the expectation values in the vicinity of KS. Even though we lack analytical arguments that would fix the behavior of the expectation values for large U, we can compute the integral $\sigma(\tau)$ numerically. Our results for the expectation value as a function of the modulus are shown in Figure 4.1. Since $\langle \mathcal{A} \rangle \sim \zeta$ this plot provides a mapping from the supergravity modulus U to the field theory modulus ζ (as we remarked before, careful holographic renormalization is needed to check this relation).

4.5 Conclusions

In previous work, increasingly convincing evidence has been emerging [49, 52, 53, 56, 59] that the warped deformed conifold background of [49] is dual to the cascading gauge theory with condensates of the baryon operators $\mathcal{A}$ and $\mathcal{B}$. Furthermore, a one-parameter family of more general warped deformed conifold backgrounds was constructed [58, 59] and argued to be dual to the entire baryonic branch of the moduli space, $\mathcal{A}\mathcal{B} = \text{const}$.

In this chapter we have presented additional, and more direct, evidence for this identification by calculating the baryonic condensates on the string theory side of the duality. Following [56], we have identified the Euclidean D5-branes wrapped over the deformed conifold, with appropriate gauge fields turned on, with the fields dual to the baryonic

[8]The coefficient of the term linear in U is given by

$$2^{-5/3}\int_0^\infty \mathrm{d}\tau \left[\frac{h\sinh^2(\tau)}{12(\sinh(\tau)\cosh(\tau)-\tau)^{2/3}} \left(\frac{h(\sinh(\tau)\cosh(\tau)-\tau)^{2/3}}{16} - \frac{(\tau\coth(\tau)-1)^2}{4} \right) \right.$$
$$\left. - \frac{(\tau\coth(\tau)-1)(\sinh(\tau)\cosh(\tau)-\tau)^{2/3}}{\sinh^2(\tau)} \left(\frac{h(\sinh(\tau)\cosh(\tau)-\tau)^{2/3}}{16} + \frac{(\tau\coth(\tau)-1)^2}{4} \right) \right] .$$

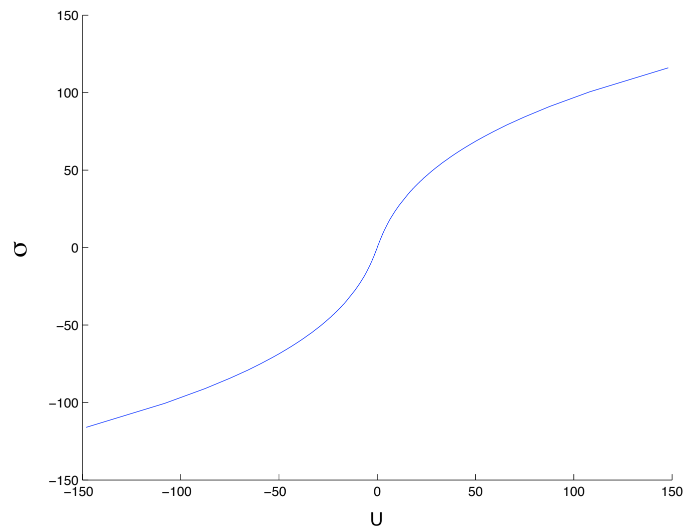


Figure 4.1: Plot of numerical results for the $\mathcal{O}(\tau^0)$ term in the asymptotic expansion of the action versus U. The slope at $U = 0$ matches the value from (4.98). The baryon expectation value $\langle \mathcal{A} \rangle \sim \langle \mathcal{B} \rangle^{-1}$ in units of Λ_{2M}^{2M} is given by the exponential of this function.

operators in the sense of gauge/string dualities. We derived the first order equations for the gauge fields and solved them explicitly. The solutions were subjected to a number of tests. From the behavior of the D5-brane action at large radial cut-off r we have deduced the r-dependence of the baryon operator dimensions and matched it with that in the cascading gauge theory. Furthermore, we used the D5-brane action to calculate the condensates as functions of the modulus U that is explicit in the supergravity backgrounds. We found that the product of the $\mathcal{A}$ and $\mathcal{B}$ condensates indeed does not depend on U.

This calculation also establishes a map between the parameterizations of the baryonic branch on the string theory and on the gauge theory sides of the duality, which should be useful for comparing other physical quantities along the baryonic branch.

Chapter 5

Charges of Monopole Operators in Chern-Simons Yang-Mills Theory

5.1 Introduction

Superconformal Chern-Simons gauge theories are excellent candidates for describing the dynamics of coincident M2-branes [68]. Bagger and Lambert [63, 64, 65], and Gustavsson [66] succeeded in constructing the first $\mathcal{N} = 8$ supersymmetric classical actions for Chern-Simons gauge fields coupled to matter. Requiring manifest unitarity restricts the gauge group to SO(4) [154, 155]; this model may be reformulated as $\mathrm{SU}(2) \times \mathrm{SU}(2)$ gauge theory with conventional Chern-Simons terms having opposite levels k and $-k$ [71, 73]. Aharony, Bergman, Jafferis, and Maldacena (ABJM) [72] proposed that a similar $\mathrm{U}(N) \times \mathrm{U}(N)$ Chern-Simons gauge theory with levels k and $-k$ arises on the world volume of N M2-branes placed at the singularity of $\mathbb{R}^8/\mathbb{Z}_k$, where $\mathbb{Z}_k$ acts by simultaneous rotation in the four planes. Therefore, the ABJM theory was conjectured to be dual, in the sense of AdS/CFT correspondence [1, 2, 3], to M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$. For $k > 2$ this orbifold preserves only $\mathcal{N} = 6$ supersymmetry, and so does the ABJM theory [5, 72, 156]. The conjectured duality predicts that for $k = 1, 2$ the supersymmetry of the gauge theory must be enhanced to $\mathcal{N} = 8$.

The mechanism for this symmetry enhancement in the quantum theory was suggested in [72]; it relies on the existence of certain monopole operators in 3-dimensional gauge theories [157, 158, 159, 160, 161, 162, 163][1]. Insertion of such an operator at some point creates quantized flux in a U(1) subgroup of the gauge group through a sphere surrounding this point. For example, in a U(1) gauge theory on $\mathbb{R}^3$, a monopole operator placed at the origin creates the Dirac monopole field

$$A = \frac{H}{2} \frac{\pm 1 - \cos\theta}{r} \, \mathrm{d}\varphi \ , \tag{5.1}$$

where the upper sign is for the northern hemisphere and the lower sign for the southern one. In addition, some scalar fields may need to be turned on as well; they are required for BPS monopoles that preserve supersymmetry. The fluctuations of fermionic matter fields can shift the dimension of such a monopole operator. These effects were studied in some simple models, mostly with U(1) gauge group, in [160, 161, 162]. In this chapter we will generalize these calculations to more complicated models. Our primary goal is to calculate R-charges and dimensions of the monopole operators in ABJM theory for any level k. Since for small values of k we cannot use perturbation theory in $1/k$, we will actually study the $\mathcal{N} = 3$ supersymmetric Yang-Mills Chern-Simons theory that provides a weakly coupled UV completion of the ABJM theory.[2]

In gauge theories with $\mathrm{U}(N)$ gauge group there exists a rich set of monopole operators labeled by the generator H that specifies the embedding of U(1) into $\mathrm{U}(N)$ [165] (see Appendix D of [166] for a brief discussion). The generalized Dirac quantization condition restricts, up to gauge equivalence, the background gauge field to be proportional to the Cartan generator $H = \mathrm{diag}(q_1, \ldots, q_N)$ where the integers q_i satisfy

[1]A more appropriate name may be "instanton operators" since they create instantons of a Euclidean 3-dimensional theory whose spacetime dependence resembles the spatial profile of monopoles in 3+1 dimensions.

[2]We are grateful to Juan Maldacena for this suggestion. A similar trick was used in [164].

$q_1 \geq q_2 \ldots \geq q_N$. If the action of the gauge theory includes a Chern-Simons term with level k, then the monopole operators are expected to transform non-trivially under the U(N) gauge group, in U(N) representations given by the Young tableaux with rows of length $kq_1, kq_2, \ldots, kq_N$ [165].

The monopole operators in the U(N) $\times$ U(N) ABJM theory have been a subject of several recent investigations [164, 166, 167, 168, 169]. In general the monopoles are described by two different generators, H and $\hat{H}$, which specify the form of the two gauge potentials, A and $\hat{A}$, subject to the constraint $\sum_i q_i = \sum_i \hat{q}_i$ [164]. However, the monopoles are believed to be BPS and thus not suffer renormalization of their dimensions for $H = \hat{H}$ [164]. A proposal for the U(1) R-charge of the monopole operator in gauge theories with $\mathcal{N} = 3$ supersymmetry was made in [170, 171] based on the results of [161] and group theoretic arguments. It states that the R-charge induced by fermionic fluctuations is

$$Q_R^{\text{mon}} = \frac{1}{2}\left(\sum_i |h_i| - \sum_j |v_i|\right) \;, \tag{5.2}$$

where h_i and v_j are, respectively, the R-charges of the fermions in hyper and vector multiplets weighted by the effective monopole charges appropriate for their gauge representations. We establish this formula through an explicit calculation in Section 5.4, and derive its non-abelian generalization in Section 5.5. In the ABJM theory, if we consider the diagonal U(N), we find two hyper multiplets and two vector multiplets with charges such that there is a desired cancellation of the anomalous dimension. Using the monopole operators $(\mathcal{M}^{-2})^{\hat{a}\hat{b}}_{ab}$ with $kq_1 = k\hat{q}_1 = 2$ which exist for $k = 1, 2$, we can form twelve real (or six complex) conserved currents of dimension 2,

$$J_\mu^{AB} = \mathcal{M}^{-2}\Big[Y^A \mathcal{D}_\mu Y^B - \mathcal{D}_\mu Y^A Y^B + i\psi^{\dagger A}\gamma^\mu \psi^{\dagger B}\Big] \;, \tag{5.3}$$

which are responsible for the symmetry enhancement from $\mathrm{SU}(4)_R$ to $\mathrm{SO}(8)_R$.

Monopoles are crucial not only for the supersymmetry enhancement at small level $k = 1, 2$ but also for matching the spectrum of the dual gravity theory at any level k. In fact, supergravity modes with momentum along the M-theory direction are dual to gauge invariant operators involving monopoles. The spectra match if there are monopole operators which can render a gauge theory operator gauge invariant without altering the global charges and dimensions of the matter fields. These monopole operator themselves have to be singlets under all global symmetries and have to have vanishing dimension. In $\mathrm{U}(N) \times \mathrm{U}(N)$ gauge theories coupled to N_f bifundamental hyper multiplets, this dimension is proportional to $N_f - 2$. Thus, the requisite monopole operators exist in the ABJM theory, which has $N_f = 2$.

The outline of this chapter, which is based on joint work with I. R. Klebanov and T. Klose [172], is as follows. Section 5.2 summarizes our reasoning and the results, and ties together the subsequent more technical sections. Section 5.3 contains the details of the theories under consideration. In Section 5.4 and Section 5.5 we present the computation of the $\mathrm{U}(1)_R$ and $\mathrm{SU}(2)_R$ charges of the monopole operators, respectively. Brief conclusions and an outlook are given in Section 5.6.

5.2 Summary

In this section we go through the logic of our arguments and present the results of our computations while omitting the technical details.

A monopole operator is defined by specifying the singular behavior of the gauge field, as in (5.1), and appropriate matter fields close to the insertion point. As such it cannot be written as a polynomial in the fundamental fields appearing in the Lagrangian of the theory. However, often the bare monopole operator has to be supplemented by a number of fundamental fields in order to construct a gauge invariant operator.

Our aim is to find the conformal dimensions of such monopole operators in the ABJM model. In particular we are interested in small Chern-Simons level k, as for $k = 1, 2$ we expect supersymmetry enhancement in this theory. For small k, however, ABJM is strongly coupled and thus we cannot employ perturbation theory since there is no small parameter. The key idea to circumvent this obstacle is to add a Yang-Mills term for the gauge fields in the action which introduces another coupling[3] g as a second parameter besides k. This also requires adding dynamical fields in the adjoint representation in order to preserve $\mathcal{N} = 3$ supersymmetry and an $\mathrm{SU}(2)_R$ subgroup of the $\mathrm{SU}(4)_R$ group of ABJM (which has $\mathcal{N} = 6$ supersymmetry).

The coupling g is dimensionful and not a parameter we can dial. Since the Yang-Mills term is irrelevant in three dimensions, there is a renormalization group (RG) flow from the UV, where the theory is free (vanishing g) to a conformal IR fixed point (divergent g). This RG flow appears naturally in the brane construction of the ABJM model and in fact is essential for understanding how a pure Chern-Simons theory such as ABJM can arise from D-branes that support Yang-Mills theories.

In the IR, the Yang-Mills terms as well as the kinetic terms for the adjoint matter fields drop out of the action and the equations of motion for the latter degenerate into constraint equations, allowing us to integrate them out and recover the ABJM theory.

Now the idea is to perform all relevant computations in the far UV where g is small and the theory weakly coupled, then flow to the IR. The scaling dimensions of monopole operators are of course only well-defined at the IR fixed point where the theory is conformal, so they cannot be determined directly in this fashion. However, we can instead compute a quantity that is preserved along the RG flow and related by supersymmetry to the scaling dimensions in the IR: the non-abelian R-charges of the monopole operators. Their one-loop value is exact and preserved along the RG flow because non-abelian representations cannot change continuously and therefore cannot depend (non-trivially) on g.

[3]The reader is cautioned that this is not the same as the coupling g used in Chapter 2. There the Yang-Mills coupling was denoted as g_{YM} but here we drop the subscript YM for notational convenience.

Let us describe the computation of the non-abelian R-charges. In the UV there is a separation of scales between the BPS background that inserts a flux at a spacetime point in accordance with (5.1) – recall that its magnitude is constrained by the Dirac quantization condition – and the typical size of quantum fluctuations of fields. Therefore we can treat the monopole operator as a classical background. For this background to satisfy the BPS condition the scalar fields ϕ_i and $\hat{\phi}_i$, which are in the same $\mathcal{N} = 3$ vector multiplets as the gauge fields A_μ and $\hat{A}_\mu$, respectively, need to be turned on. A possible choice of scalar background in radial quantization on $\mathbb{R} \times \mathrm{S}^2$ is

$$\phi_i = -\hat{\phi}_i = -\frac{H}{2}\delta_{i3} \ . \tag{5.4}$$

These scalar fields transform in the **3** of $\mathrm{SU}(2)_R$ and therefore a non-zero expectation value breaks $\mathrm{SU}(2)_R$ to $\mathrm{U}(1)_R$.

As a first step we will find the one-loop $\mathrm{U}(1)_R$ charge of the monopole operator described by this fixed background. This is done by computing the normal ordering constant for the $\mathrm{U}(1)_R$ charge operator. Such a calculation has been performed in a different context in [160, 161]. We will present the argument as applicable to our case in Section 5.4. The result will turn out to be expression (5.2) above or more concretely (5.69) for $\mathcal{N} = 3$ $\mathrm{U}(N)\times\mathrm{U}(N)$ gauge theory with hyper multiplets in the bifundamental.

These two formulas are related as follows. From Table 5.2, or equivalently from the R-current (5.38), one can read off the R-charges $y(\zeta^A) = y(\omega_A) = -\frac{1}{2}$ (for $A = 1, \ldots, N_f$) for the hyper multiplet fermions and $y(\chi_\sigma) = y(\hat{\chi}_\sigma) = 1$, $y(\chi_\phi) = y(\hat{\chi}_\phi) = 0$ for those in the vector multiplets. In the expression for the $\mathrm{U}(1)_R$ charge of the BPS monopole (5.2), these R-charges are weighted by $\sum_{r,s}|q_r - q_s|$ and the result is given by

$$Q_R^{\mathrm{mon}} = \frac{1}{2}\Big(2N_f \cdot \big|-\tfrac{1}{2}\big| - 2\cdot|+1|\Big)\sum_{r,s}|q_r - q_s| = \left(\frac{N_f}{2} - 1\right)\sum_{r,s}|q_r - q_s| \ , \tag{5.5}$$

as found in (5.69). For the ABJM model we have $N_f = 2$ and hence $Q_R^{\mathrm{mon}} = 0$.

These $\mathrm{U}(1)_R$ charges might in principle be renormalized as we flow to the IR, since abelian charges can vary continuously. To exclude this possibility, we need to find the non-abelian $\mathrm{SU}(2)_R$ charge of the monopole operator by taking into account the bosonic zero modes of the background (5.4). These zero modes are described by a unit vector $\vec{n}$ on the two-sphere $\mathrm{SU}(2)_R/\mathrm{U}(1)_R$. In other words, we will treat the $\mathrm{SU}(2)_R$ orientation $\vec{n}$ of the scalar field as a collective coordinate of the BPS background and quantize its motion. To this end we consider a more general background than (5.4), which is allowed to depend on (Euclidean) time τ

$$\phi_i = -\hat{\phi}_i = -\frac{H}{2} n_i(\tau) \ . \tag{5.6}$$

The rotation of the background is assumed to be adiabatic such that it cannot excite finite energy quantum fluctuations around the background. In Section 5.5 we will compute the quantum mechanical effective action for the collective coordinate $\vec{n}(\tau)$ and find the allowed $\mathrm{SU}(2)_R$ representations. These representations are precisely the $\mathrm{SU}(2)_R$ spectrum of BPS monopole operators.

If there were no interactions between the bosonic zero modes and other fields in the Lagrangian, the collective coordinate would simply be described by a free particle on a sphere. Such a particle – and therefore the monopole operator – could be in any representation of $\mathrm{SU}(2)_R$. The crucial point is, however, that the collective coordinate $\vec{n}$ is not free, but subject to interactions with the fermions of the theory. It turns out that the induced interaction term is a coupling of $\vec{n}$ to a Dirac monopole of magnetic charge $h \in \mathbb{Z}$ on the collective coordinate moduli space (not to be confused with the spacetime monopole background). The charge h depends on the background flux H and the field content of the theory. In Section 5.5 we derive

$$h = (N_f - 2)\, q_{\mathrm{tot}} = (N_f - 2) \sum_{r,s} |q_r - q_s| \ , \tag{5.7}$$

see (5.103), which is nothing but $h = 2Q_R^{\text{mon}}$.

The equation of motion for the collective coordinates $\vec{n}$ has the structure

$$M\,\ddot{\vec{n}} + M(\dot{\vec{n}})^2\,\vec{n} + \frac{h}{2}\,\vec{n}\times\dot{\vec{n}} = 0\ , \tag{5.8}$$

which allows for a simple particle interpretation. The first term is the usual kinetic term while the second term represents a constraining force that keeps the particle on the sphere $\vec{n}^2 = 1$. If we assign unit electric charge to the particle, then the third term is the Lorentz force $\dot{\vec{n}}\times\vec{B}$ due to a monopole field $\vec{B} = \frac{h}{2}\vec{n}$. The $\mathrm{SU}(2)_R$ charge is given by the conserved angular momentum

$$\vec{L} = M\,\vec{n}\times\dot{\vec{n}} - \frac{h}{2}\,\vec{n}\ . \tag{5.9}$$

The presence of the second term forces the quantized angular momentum to have an orbital quantum number of at least $l = \frac{|h|}{2}$, the possible representations being $\vec{L}^2 = l(l{+}1)$ with $l = \frac{|h|}{2}, \frac{|h|}{2}+1, \frac{|h|}{2}+2, \ldots$. Hence, a non-zero coupling h will necessarily give the monopole a non-trivial $\mathrm{SU}(2)_R$ charge. The ABJM theory is special (as compared to theories with different field content) in that all contributions to the effective h cancel each other, and thus it allows $\mathrm{SU}(2)_R$ singlet monopoles which do not contribute to the IR dimensions of gauge invariant operators.

5.3 $\mathcal{N} = 3$ Chern-Simons Yang-Mills Theory

We will study $\mathcal{N} = 3$ supersymmetric $\mathrm{U}(N)\times\hat{\mathrm{U}}(N)$ 3-dimensional gauge theory coupled to bifundamental matter fields and $\mathrm{SU}(N_f)_{\mathrm{fl}}$ flavor symmetry. For $N_f = 1$ and $N_f = 2$ this model is the UV completion of the $\mathcal{N} = 4$ model of GW [74] and the $\mathcal{N} = 6$ model of ABJM [72], respectively.

Manifest symmetry	Super fields	Component fields								
		dynamical in the IR				auxiliary in the IR				aux.
$\mathrm{U}(1)_R \times \mathrm{SU}(N_f)_{\mathrm{fl}}$	$\mathcal{V}$	A				σ		χ_σ, $\chi_\sigma^\dagger$		D
	$\hat{\mathcal{V}}$	$\hat{A}$					$\hat{\sigma}$		$\hat{\chi}_\sigma$, $\hat{\chi}_\sigma^\dagger$	$\hat{\mathsf{D}}$
	Φ					ϕ		χ_ϕ		F_ϕ
	$\bar{\Phi}$					$\phi^\dagger$		$\chi_\phi^\dagger$		$F_\phi^\dagger$
	$\hat{\Phi}$						$\hat{\phi}$		$\hat{\chi}_\phi$	$\hat{F}_\phi$
	$\hat{\bar{\Phi}}$						$\hat{\phi}^\dagger$		$\hat{\chi}_\phi^\dagger$	$\hat{F}_\phi^\dagger$
	$\mathcal{Z}^A$	Z^A		ζ^A						F^A
	$\bar{\mathcal{Z}}_A$		$Z_A^\dagger$		$\zeta_A^\dagger$					$F_A^\dagger$
	$\mathcal{W}_A$		W_A		ω_A					G_A
	$\bar{\mathcal{W}}^A$	$W^{\dagger A}$		$\omega^{\dagger A}$						$G^{\dagger A}$
$\mathrm{SU}(2)_R \times \mathrm{SU}(N_f)_{\mathrm{fl}}$		X^{Aa}	$X_{Aa}^\dagger$	ξ^{Aa}	$\xi_{Aa}^\dagger$	ϕ_i	$\hat{\phi}_i$	λ^{ab}	$\hat{\lambda}^{ab}$	

Table 5.1: Field content of $\mathcal{N} = 3$ Chern-Simons Yang-Mills theory. Each row (except the last one) shows the components of one $\mathcal{N} = 2$ superfield ordered into columns according to whether they are dynamical or auxiliary. Within each of these columns they have been arranged such that one can read off from the last row which components join to form $\mathrm{SU}(2)_R$ multiplets. $A = 1, \ldots, N_f$ is an $\mathrm{SU}(N_f)$ flavor index, $a = 1, 2$ is an $\mathrm{SU}(2)_R$ spinor index, and $i = 1, 2, 3$ is an $\mathrm{SU}(2)_R$ vector index.

5.3.1 Action and Supersymmetry Transformations

The complete field content is listed in Table 5.1. We use $\mathcal{N} = 2$ superfields and our notation is explained in Appendix A.2. In the gauge sector we have two vector superfields $\mathcal{V} = (\sigma, A_\mu, \chi_\sigma, \mathsf{D})$ and $\hat{\mathcal{V}} = (\hat{\sigma}, \hat{A}_\mu, \hat{\chi}_\sigma, \hat{\mathsf{D}})$ and two chiral superfields $\Phi = (\phi, \chi_\phi, F_\phi)$ and $\hat{\Phi} = (\hat{\phi}, \hat{\chi}_\phi, \hat{F}_\phi)$ in the adjoint of the two gauge groups $\mathrm{U}(N)$ and $\hat{\mathrm{U}}(N)$, respectively, which together comprise two $\mathcal{N} = 3$ gauge multiplets. In the matter sector we have N_f chiral superfields $\mathcal{Z}^A = (Z^A, \zeta^A, F^A)$ and $\mathcal{W}_A = (W_A, \omega_A, G_A)$ in the gauge representations $(\mathbf{N}, \bar{\mathbf{N}})$ and $(\bar{\mathbf{N}}, \mathbf{N})$, respectively.

We write down the $\mathcal{N} = 3$ action on $\mathbb{R}^{1,2}$ with signature $(-, +, +)$. It consists of five parts:

$$\mathcal{S} = \mathcal{S}_{\rm CS} + \mathcal{S}_{\rm YM} + \mathcal{S}_{\rm adj} + \mathcal{S}_{\rm mat} + \mathcal{S}_{\rm pot} \ . \tag{5.10}$$

First of all there are Chern-Simons terms

$$\mathcal{S}_{\rm CS} = -i\frac{k}{8\pi}\int d^3x\, d^4\theta \int_0^1 ds\ {\rm tr}\Big[\mathcal{V}\bar{D}^\alpha\Big(e^{s\mathcal{V}}D_\alpha e^{-s\mathcal{V}}\Big) - \hat{\mathcal{V}}\bar{D}^\alpha\Big(e^{s\hat{\mathcal{V}}}D_\alpha e^{-s\hat{\mathcal{V}}}\Big)\Big] \ , \tag{5.11}$$

with opposite levels k and $-k$ for the two gauge group factors. The s integral is nothing but a convenient way of writing the non-abelian Chern-Simons action. Secondly, there is a Yang-Mills term

$$\mathcal{S}_{\rm YM} = \frac{1}{4g^2}\int d^3x\, d^2\theta\ {\rm tr}\Big[\mathcal{U}^\alpha\mathcal{U}_\alpha + \hat{\mathcal{U}}^\alpha\hat{\mathcal{U}}_\alpha\Big] \ , \tag{5.12}$$

which introduces a coupling g of mass dimension $\frac{1}{2}$. The super field strength is given by $\mathcal{U}_\alpha = \frac{1}{4}\bar{D}^2 e^{\mathcal{V}} D_\alpha e^{-\mathcal{V}}$ and similarly for $\hat{\mathcal{U}}$. The last term in the gauge sector is given by the kinetic terms for the adjoint scalar fields

$$\mathcal{S}_{\rm adj} = \frac{1}{g^2}\int d^3x\, d^4\theta\ {\rm tr}\Big[-\bar{\Phi}e^{-\mathcal{V}}\Phi e^{\mathcal{V}} - \bar{\hat{\Phi}}e^{-\hat{\mathcal{V}}}\hat{\Phi}e^{\hat{\mathcal{V}}}\Big] \ . \tag{5.13}$$

In the matter sector we have the minimally coupled action for the bifundamental fields

$$\mathcal{S}_{\rm mat} = \int d^3x\, d^4\theta\ {\rm tr}\Big[-\bar{\mathcal{Z}}_A e^{-\mathcal{V}}\mathcal{Z}^A e^{\hat{\mathcal{V}}} - \bar{\mathcal{W}}^A e^{-\hat{\mathcal{V}}}\mathcal{W}_A e^{\mathcal{V}}\Big] \tag{5.14}$$

and a super potential term

$$\mathcal{S}_{\rm pot} = \int d^3x d^2\theta\, {\rm W} - \int d^3x d^2\bar{\theta}\, \bar{\rm W} \tag{5.15}$$

with

$$\mathrm{W} = \mathrm{tr}\big(\Phi \mathcal{Z}^A \mathcal{W}_A + \hat{\Phi} \mathcal{W}_A \mathcal{Z}^A\big) + \frac{k}{8\pi}\,\mathrm{tr}\big(\Phi\Phi - \hat{\Phi}\hat{\Phi}\big) \;, \tag{5.16}$$

$$\bar{\mathrm{W}} = \mathrm{tr}\big(\bar{\Phi} \bar{\mathcal{W}}^A \bar{\mathcal{Z}}_A + \bar{\hat{\Phi}} \bar{\mathcal{Z}}_A \bar{\mathcal{W}}^A\big) + \frac{k}{8\pi}\,\mathrm{tr}\big(\bar{\Phi}\bar{\Phi} - \bar{\hat{\Phi}}\bar{\hat{\Phi}}\big) \;. \tag{5.17}$$

This theory is not conformal and will flow to an IR fixed point which is strongly coupled unless k is large. At the fixed point g diverges, which renders the gauge fields and the adjoint scalars non-dynamical. If we integrate out Φ and $\hat{\Phi}$, we recover the ABJM superpotential for $N_f = 2$. This theory has enhanced flavor symmetry, $\mathrm{SU}(2)_{\mathrm{fl}} \times \mathrm{SU}(2)_{\mathrm{fl}}$, under which $\mathcal{Z}^A$ and $\mathcal{W}_A$ transform separately.

For our computation below we need the action in terms of component fields. To this end we perform the Grassmann integrals in the action and integrate out the auxiliary fields D, $\hat{\mathsf{D}}$, F^A, G_A, F_ϕ and $\hat{F}_\phi$. The remaining component fields can be arranged into $\mathrm{SU}(2)_R$ multiplets as follows.

The adjoint matter fields constitute two scalars (where the lower/upper index is the row/column index)

$$\phi^a_b = \phi_i(\sigma_i)^a_b = \begin{pmatrix} -\sigma & \phi^\dagger \\ \phi & \sigma \end{pmatrix} \quad , \quad \hat{\phi}^a_b = \hat{\phi}_i(\sigma_i)^a_b = \begin{pmatrix} \hat{\sigma} & \hat{\phi}^\dagger \\ \hat{\phi} & -\hat{\sigma} \end{pmatrix} \;, \tag{5.18}$$

transforming in the $\mathbf{3}$ of $\mathrm{SU}(2)_R$, and two fermions

$$\lambda^{ab} = \begin{pmatrix} \chi_\sigma\, e^{-i\pi/4} & \chi^\dagger_\phi\, e^{-i\pi/4} \\ \chi_\phi\, e^{+i\pi/4} & -\chi^\dagger_\sigma\, e^{+i\pi/4} \end{pmatrix} \quad , \quad \hat{\lambda}^{ab} = \begin{pmatrix} \hat{\chi}_\sigma\, e^{-i\pi/4} & -\hat{\chi}^\dagger_\phi\, e^{-i\pi/4} \\ -\hat{\chi}_\phi\, e^{+i\pi/4} & -\hat{\chi}^\dagger_\sigma\, e^{+i\pi/4} \end{pmatrix} \;, \tag{5.19}$$

transforming in the reducible representation $\mathbf{2}\times\mathbf{2} = \mathbf{3}+\mathbf{1}$ of $\mathrm{SU}(2)_R$, which in particular implies that λ^{ab} is neither symmetric nor anti-symmetric in its indices. These fields satisfy

$$(\lambda^{ab})^* = -\lambda_{ab} = -\epsilon_{ac}\epsilon_{bd}\lambda^{cd} \quad , \quad (\phi^a_b)^* = \phi^b_a = \epsilon_{ac}\epsilon^{bd}\phi^c_d \tag{5.20}$$

and the same for the hatted fields.

The bifundamental matter fields can be grouped into N_f doublets of the $\mathrm{SU}(2)_R$ symmetry group in the following way:

$$X^{Aa} = \begin{pmatrix} Z^A \\ W^{\dagger A} \end{pmatrix} \quad , \quad X^\dagger_{Aa} = \begin{pmatrix} Z^\dagger_A \\ W_A \end{pmatrix} \; , \tag{5.21}$$

and

$$\xi^{Aa} = \begin{pmatrix} \omega^{\dagger A}\, e^{i\pi/4} \\ \zeta^A\, e^{-i\pi/4} \end{pmatrix} \quad , \quad \xi^\dagger_{Aa} = \begin{pmatrix} \omega_A\, e^{-i\pi/4} \\ \zeta^\dagger_A\, e^{i\pi/4} \end{pmatrix} \; . \tag{5.22}$$

The component action with manifest $\mathrm{SU}(2)_R$ symmetry and the corresponding supersymmetry transformations are given in Appendix A.3. One observes that in the IR, where g becomes large, the Yang-Mills terms disappears together with the kinetic terms for ϕ, $\hat{\phi}$, λ and $\hat{\lambda}$, and we can integrate out these fields. Doing this for $N_f = 2$ we end up with ABJM theory. The ABJM Lagrangian with manifest $\mathrm{SU}(4)_R$ symmetry [5] is recovered when one defines $Y^A = \{X^{11}, X^{21}, X^{12}, X^{22}\}$ and $\psi_A = \{-\xi^{22}, \xi^{12}, \xi^{21}, -\xi^{11}\}$.

As outlined in the overview, Section 5.2, we will carry out our computations in the far UV region of the radially quantized theory. The point of going to the far UV is that the theory becomes perturbative in g and we can find the quantum R-charges from a one-loop computation. This one-loop result is in fact the exact answer because $\mathrm{SU}(2)_R$ is preserved along the flow from small to large g and non-abelian representations cannot change continuously. We will use radial quantization because we are interested in the spectrum of conformal dimensions in the IR theory, which is related to the energy spectrum by the operator state correspondence.

The steps required for deriving the action relevant for radial quantization are as follows. First we perform a Wick rotation from $\mathbb{R}^{1,2}$ to $\mathbb{R}^3$ by defining Euclidean coordinates $(x^1, x^2, x^3) = (x^1, x^2, ix^0)$, then we change to polar coordinates (r, θ, φ) and finally we introduce a new radial variable τ by setting $r = e^\tau$. The result is a theory on $\mathbb{R} \times \mathrm{S}^2$ described by the coordinates (τ, θ, φ), where $\tau \in \mathbb{R}$ is the "Euclidean time". The change of the coordinates is accompanied by a Weyl rescaling of the fields according to

$$\mathcal{A} = e^{-\dim(\mathcal{A})\tau} \tilde{\mathcal{A}} \,, \tag{5.23}$$

where $\mathcal{A}$ is a generic field of dimension $\dim(\mathcal{A})$ on $\mathbb{R}^3$ and $\tilde{\mathcal{A}}$ is the field we use on $\mathbb{R} \times \mathrm{S}^2$. After the transformation we will drop the tildes in order to avoid cluttered notation. Since the theory is not conformal, the action will change under these rescaling. In addition to the mass terms which are generated even in the conformal case, every factor of the coupling g will turn into

$$\tilde{g} = e^{\tau/2}\, g \,. \tag{5.24}$$

This relation makes the RG-flow explicit and relates it to the "time" on $\mathbb{R} \times \mathrm{S}^2$. In the infinite past the effective Yang-Mills coupling $\tilde{g}$ vanishes, and it grows without bound toward future infinity. To compute in the UV therefore means to work at $\tau \to -\infty$ and to flow to the IR means to send $\tau \to +\infty$.

Now we are ready to give the complete action of $\mathcal{N} = 3$ Chern-Simons Yang-Mills theory on $\mathbb{R} \times \mathrm{S}^2$. In addition to the manipulations just described we have rescaled $\lambda \to g\lambda$ and $\hat{\lambda} \to g\hat{\lambda}$ (before the Weyl rescaling) which is the appropriate scaling for fluctuations in the UV. We do *not* perform a similar rescaling of the other fields in the gauge sector, A, $\hat{A}$, ϕ, or $\hat{\phi}$, since these fields will later provide a large classical background. Introducing finally the rescaled Chern-Simons coupling $\kappa = \frac{k}{4\pi}$, the kinetic part of the action reads[4]

$$\begin{aligned}\mathcal{S}^E_{\text{kin}} = \int d\tau\, d\Omega\ \text{tr}\Big[&+ \tfrac{1}{2\tilde{g}^2} F^{mn} F_{mn} - \kappa\, i\epsilon^{mnk}\big(A_m \partial_n A_k + \tfrac{2i}{3} A_m A_n A_k\big) \\ &+ \tfrac{1}{2\tilde{g}^2} \hat{F}^{mn} \hat{F}_{mn} + \kappa\, i\epsilon^{mnk}\big(\hat{A}_m \partial_n \hat{A}_k + \tfrac{2i}{3} \hat{A}_m \hat{A}_n \hat{A}_k\big) \\ &+ \mathcal{D}_m X^\dagger \mathcal{D}^m X + \tfrac{1}{4} X^\dagger X - i\xi^\dagger \not{\mathcal{D}} \xi \\ &+ \tfrac{1}{2\tilde{g}^2} \mathcal{D}_m \phi^a_b \mathcal{D}^m \phi^b_a + \tfrac{\kappa^2 \tilde{g}^2}{2}\, \phi^a_b \phi^b_a + \tfrac{1}{2\tilde{g}^2} \mathcal{D}_m \hat{\phi}^a_b \mathcal{D}^m \hat{\phi}^b_a + \tfrac{\kappa^2 \tilde{g}^2}{2}\, \hat{\phi}^a_b \hat{\phi}^b_a \\ &+ \tfrac{i}{2} \lambda^{ab} \not{\mathcal{D}} \lambda_{ab} + \tfrac{\kappa \tilde{g}^2}{2}\, i\lambda^{ab} \lambda_{ba} + \tfrac{i}{2} \hat{\lambda}^{ab} \not{\mathcal{D}} \hat{\lambda}_{ab} - \tfrac{\kappa \tilde{g}^2}{2}\, i\hat{\lambda}^{ab} \hat{\lambda}_{ba} \Big]\end{aligned} \tag{5.25}$$

and the interaction terms are given by

$$\begin{aligned}\mathcal{S}^E_{\text{int}} = \int d\tau\, d\Omega\ \text{tr}\Big[&+ \kappa \tilde{g}^2\, X^\dagger_a \phi^a_b X^b - \kappa \tilde{g}^2\, X^a \hat{\phi}^b_a X^\dagger_b + i\xi^\dagger_a \phi^a_b \xi^b + i\xi^a \hat{\phi}^b_a \xi^\dagger_b \\ &- \tilde{g}\, \epsilon_{ac} \lambda^{cb} X^a \xi^\dagger_b + \tilde{g}\, \epsilon^{ac} \lambda_{cb} \xi^b X^\dagger_a + \tilde{g}\, \epsilon_{ac} \hat{\lambda}^{cb} \xi^\dagger_b X^a - \tilde{g}\, \epsilon^{ac} \hat{\lambda}_{cb} X^\dagger_a \xi^b \\ &- \tfrac{\kappa}{6} \phi^a_b [\phi^b_c, \phi^c_a] - \tfrac{\kappa}{6} \hat{\phi}^a_b [\hat{\phi}^b_c, \hat{\phi}^c_a] + \tfrac{i}{2} \lambda_{ab} [\phi^b_c, \lambda^{ac}] - \tfrac{i}{2} \hat{\lambda}_{ab} [\hat{\phi}^b_c, \hat{\lambda}^{ac}] \\ &+ \tfrac{\tilde{g}^2}{4} (X \sigma_i X^\dagger)(X \sigma_i X^\dagger) + \tfrac{\tilde{g}^2}{4} (X^\dagger \sigma_i X)(X^\dagger \sigma_i X) \\ &+ \tfrac{1}{2} (X X^\dagger) \phi^a_b \phi^b_a + \tfrac{1}{2} (X^\dagger X) \hat{\phi}^a_b \hat{\phi}^b_a + X^\dagger_{Aa} \phi^b_c X^{Aa} \hat{\phi}^c_b \\ &- \tfrac{1}{8\tilde{g}^2} [\phi^a_b, \phi^c_d][\phi^b_a, \phi^d_c] - \tfrac{1}{8\tilde{g}^2} [\hat{\phi}^a_b, \hat{\phi}^c_d][\hat{\phi}^b_a, \hat{\phi}^d_c] \Big]\ .\end{aligned} \tag{5.26}$$

The covariant derivatives are given by $\mathcal{D}_m X = \nabla_m X + iA_m X - iX\hat{A}_m$ etc. We also translate the supersymmetry variations from flat Lorentzian space as given in Appendix A.3 to Euclidean $\mathbb{R} \times \mathrm{S}^2$. They are conveniently expressed in terms of a rescaled parameter

$$\tilde{\varepsilon}_{ab}(\tau) = \varepsilon_{ab} e^{-\tau/2}\ . \tag{5.27}$$

In the following we will use the τ dependent parameter; however, we will drop the tilde

[4] Suppressed indices are assumed to be in the standard positions as defined in (5.21) and (5.22). The indices of the Pauli matrices are placed accordingly, e.g. $X\sigma_i X^\dagger \equiv X^{Aa} (\sigma_i)_a{}^b X^\dagger_{Ab}$ or $X^\dagger \sigma_i X \equiv X^\dagger_{Aa} (\sigma_i)^a{}_b X^{Ab}$. And by definition we have $(\sigma_i)_a{}^b = \sigma_i$ and $(\sigma_i)^a{}_b = \sigma_i^{\mathrm{T}}$.

for notational simplicity. This parameter satisfies the Killing spinor equation

$$\nabla_m \varepsilon = -\tfrac{1}{2}\gamma_m \gamma^\tau \varepsilon \,, \tag{5.28}$$

which is the curved spacetime generalization of the usual condition of (covariant) constancy that the supersymmetry variation parameter obeys in flat space. The $\mathcal{N} = 3$ supersymmetry transformations read

$$\begin{aligned}
\delta A_m &= -\tfrac{i\tilde{g}}{2}\varepsilon_{ab}\gamma_m \lambda^{ab} \,, &(5.29)\\
\delta\lambda^{ab} &= \tfrac{i}{2\tilde{g}}\epsilon^{mnk}F_{mn}\gamma_k\varepsilon^{ab} - \tfrac{i}{\tilde{g}}\not{D}\phi^b_c\varepsilon^{ac} - \tfrac{2i}{3\tilde{g}}\phi^b_c\not{\nabla}\varepsilon^{ac} + \tfrac{i}{2\tilde{g}}[\phi^b_c,\phi^c_d]\varepsilon^{ad} + \kappa\tilde{g}\, i\phi^b_c\varepsilon^{ac} &(5.30)\\
&\quad +\tilde{g}\, iX^aX^\dagger_c\varepsilon^{cb} - \tfrac{i\tilde{g}}{2}(XX^\dagger)\varepsilon^{ab} \,, \\
\delta\phi^a_b &= -\tilde{g}\varepsilon_{cb}\lambda^{ca} + \tfrac{\tilde{g}}{2}\delta^a_b\varepsilon_{cd}\lambda^{cd} \,, &(5.31)
\end{aligned}$$

$$\begin{aligned}
\delta\hat{A}_m &= -\tfrac{i\tilde{g}}{2}\varepsilon_{ab}\gamma_m \hat{\lambda}^{ab} \,, &(5.32)\\
\delta\hat{\lambda}^{ab} &= \tfrac{i}{2\tilde{g}}\epsilon^{mnk}\hat{F}_{mn}\gamma_k\varepsilon^{ab} + \tfrac{i}{\tilde{g}}\not{D}\hat{\phi}^b_c\varepsilon^{ac} + \tfrac{2i}{3\tilde{g}}\hat{\phi}^b_c\not{\nabla}\varepsilon^{ac} + \tfrac{i}{2\tilde{g}}[\hat{\phi}^b_c,\hat{\phi}^c_d]\varepsilon^{ad} + \kappa\tilde{g}\, i\hat{\phi}^b_c\varepsilon^{ac} &(5.33)\\
&\quad -\tilde{g}\, i\varepsilon^{bc}X^\dagger_cX^a + \tfrac{i\tilde{g}}{2}(X^\dagger X)\varepsilon^{ab} \,, \\
\delta\hat{\phi}^a_b &= -\tilde{g}\varepsilon_{cb}\hat{\lambda}^{ca} + \tfrac{\tilde{g}}{2}\delta^a_b\varepsilon_{cd}\hat{\lambda}^{cd} \,, &(5.34)
\end{aligned}$$

$$\begin{aligned}
\delta X^{Aa} &= -i\varepsilon^a_b\xi^{Ab} \,, \qquad \delta\xi^{Aa} = \not{D}X^{Ab}\varepsilon^a_b + \tfrac{1}{3}X^{Ab}\not{\nabla}\varepsilon^a_b + \phi^a_b\varepsilon^b_cX^{Ac} + X^{Ac}\varepsilon^b_c\hat{\phi}^a_b \,, &(5.35)\\
\delta X^\dagger_{Aa} &= -i\xi^\dagger_{Ab}\varepsilon^b_a \,, \qquad \delta\xi^\dagger_{Aa} = \not{D}X^\dagger_{Ab}\varepsilon^b_a + \tfrac{1}{3}X^\dagger_{Ab}\not{\nabla}\varepsilon^b_a + \hat{\phi}^b_a\varepsilon^c_bX^\dagger_{Ac} + X^\dagger_{Ac}\varepsilon^c_b\phi^b_a \,.
\end{aligned}$$

$\mathbf{SU(2)_R}$ and $\mathbf{U(1)_R}$ charge. Fundamental and anti-fundamental $\mathrm{SU}(2)_R$ indices a, b transform under infinitesimal rotations as

$$\delta\mathcal{A}^a = i\varepsilon^a_b\mathcal{A}^b \quad , \quad \delta\mathcal{A}_a = -i\varepsilon^b_a\mathcal{A}^\dagger_b \,, \tag{5.36}$$

y_i	-1	$-\frac{1}{2}$	0	$+\frac{1}{2}$	$+1$
field	$\chi_\sigma^\dagger$, $\hat{\chi}_\sigma^\dagger$	ζ^A, ω_A,	χ_ϕ, $\hat{\chi}_\phi$, $\chi_\phi^\dagger$, $\hat{\chi}_\phi^\dagger$	$\zeta_A^\dagger$, $\omega^{\dagger A}$	χ_σ, $\hat{\chi}_\sigma$

Table 5.2: $\mathrm{U}(1)_R$ charges of the fermion fields. These numbers show what the sum (chiral only) of R-charges vanishes precisely for two flavors, $A = 1, 2$.

where $\mathcal{A}$ represents a generic field. The Noether current is

$$J_a^{\mu b} \sim \mathrm{tr}\Big[iX_{Aa}^\dagger \overleftrightarrow{\mathcal{D}}^\mu X^{Ab} + i\phi_a^c \overleftrightarrow{\mathcal{D}}^\mu \phi_c^b + i\hat{\phi}_a^c \overleftrightarrow{\mathcal{D}}^\mu \hat{\phi}_c^b - \xi_{Aa}^\dagger \gamma^\mu \xi^{Ab} \\ + \tfrac{1}{2}\lambda_{ac}\gamma^\mu\lambda^{bc} + \tfrac{1}{2}\lambda_{ca}\gamma^\mu\lambda^{cb} + \tfrac{1}{2}\hat{\lambda}_{ac}\gamma^\mu\hat{\lambda}^{bc} + \tfrac{1}{2}\hat{\lambda}_{ca}\gamma^\mu\hat{\lambda}^{cb}\Big] . \tag{5.37}$$

In Section 5.4 we will be dealing with the $\mathrm{U}(1)_R$ component of this current which is related to the transformation (5.36) with $\varepsilon_b^a \sim (\sigma_3)_b^a$. Hence the current is the contraction of (5.37) with $(\sigma_3)_b^a$. The part due to the fermions is given by

$$J^\mu \sim \mathrm{tr}\Big[-\tfrac{1}{2}\,\zeta_A^\dagger\gamma^\mu\zeta^A - \tfrac{1}{2}\,\omega^{\dagger A}\gamma^\mu\omega_A + \chi_\sigma^\dagger\gamma^\mu\chi_\sigma + \hat{\chi}_\sigma^\dagger\gamma^\mu\hat{\chi}_\sigma \Big] , \tag{5.38}$$

where we have reverted back to the $\mathrm{U}(1)_R \times \mathrm{SU}(N_f)_{\mathrm{fl}}$ fields, see Table 5.1. From this expression we read off the $\mathrm{U}(1)_R$ charges of the fermions as given in Table 5.2.

5.3.2 Classical Monopole Solution

Since we are interested in BPS monopoles, our first task is to find a classical BPS solution with flux emanating from a point in spacetime. Starting from the gauge field configuration of a Dirac monopole in $\mathbb{R}^3$ given in (5.1) and performing the Weyl rescaling appropriate for fields on $\mathbb{R} \times \mathrm{S}^2$ as described above, we find that the dependence on the radial coordinate disappears from the gauge potential. In fact the Hodge dual of the corresponding field strength, $\epsilon^{mnk}F_{mn}$, is constant in magnitude and purely radial (i.e. only its τ component is non-vanishing).

In order to show that there is indeed a BPS solution with such a Dirac monopole potential, let us examine in detail the supersymmetry variations of λ^{ab}. It contains terms of order $\tilde{g}$ and ones of order $\tilde{g}^{-1}$, but since we are looking for a solution that is supersymmetric along the whole RG flow they should cancel separately. We will also assume that all background fields (in our choice of gauge) are valued in the Cartan subalgebras of the gauge group factors, so that all commutators vanish.

Focusing on the terms of order $\tilde{g}^{-1}$ for now, we thus want the sum of the first three terms in $\delta\lambda^{ab}$, as given in (5.29), to vanish for an appropriate, non-trivial choice of supersymmetry variation parameter. The fact that $\epsilon^{mnk}F_{mn}$ is constant suggests that ϕ_i should also be constant, in which case the second term vanishes by itself. Hence we simply have to balance the first term and the third one, which simplifies upon using the Killing spinor equation (5.28).

Recalling that $\varepsilon^{ab} = \varepsilon_i(\sigma_i)^{ab}$ is traceless Hermitian, we can take the $\mathrm{SU}(2)_R$ trace of $\delta\lambda^{ab}$ which implies that $\varepsilon_i\phi_i = 0$, i.e. the non-trivial supersymmetry variation parameter has to be orthogonal to the background scalar in the R-symmetry directions.

Given this restriction, we can now contract $\delta\lambda^{ab}$ with $(\sigma_i)_{ab}$ and find

$$\tfrac{i}{2}\epsilon^{mnl}F_{mn}\gamma_l\,\varepsilon_i + \epsilon_{ijk}\phi_j\gamma^\tau\varepsilon_k = 0\ . \tag{5.39}$$

Thus the magnitudes of the scalar background and gauge fields are related, and picking an $\mathrm{SU}(2)_R$ orientation we can choose e.g. $\frac{1}{2}\epsilon^{mn\tau}F_{mn} = -\eta\,\phi_3$, where $\eta = \pm 1$ distinguishes BPS from anti-BPS monopoles. In this case $\varepsilon_3 = 0$ and the remaining supersymmetry parameters have to satisfy[5] $\varepsilon_1 - i\eta\,\varepsilon_2 = 0$.

Let us now turn to the terms of order $\tilde{g}$. Following the same lines of reasoning, they imply

$$\phi_a^b = -\tfrac{1}{\kappa}\big(X^bX_a^\dagger - \tfrac{1}{2}\delta_a^b XX^\dagger\big) \quad\Leftrightarrow\quad \phi_i = -\tfrac{1}{2\kappa}X\sigma_iX^\dagger\ . \tag{5.40}$$

[5]In Euclidean space we treat $\varepsilon_1 \pm i\varepsilon_2$ as two independent supersymmetry parameters.

It is evident that if we choose $\hat{A} = A$ and $\hat{\phi}_i = -\phi_i$ the variations $\delta\hat{\lambda}^{ab}$ (5.32) will vanish also in the same manner, provided that

$$\hat{\phi}_a^b = \tfrac{1}{\kappa}\big(X_a^\dagger X^b - \tfrac{1}{2}\delta_a^b X^\dagger X\big) \quad \Leftrightarrow \quad \hat{\phi}_i = \tfrac{1}{2\kappa} X^\dagger \sigma_i X \; . \tag{5.41}$$

This leaves the supersymmetry transformations of the remaining fermions, $\delta\xi^{Aa}$ and its complex conjugate (5.35), to be verified. Given that $\hat{\phi}_i = -\phi_i$ causes the last two terms to cancel, they simply fix the functional dependence of the bifundamental scalars to be $X^{A1} \sim \exp(-\eta\,\tau/2)$ and $X^{A2} \sim \exp(\eta\,\tau/2)$.[6] Then all of the $\delta\xi^{Aa}$ and $\delta\xi^\dagger_{Aa}$ vanish either by virtue of this particular τ-dependence, which is consistent with the equation of motion for X

$$\mathcal{D}^2 X^{Aa} - \tfrac{1}{4} X^{Aa} = 0 \; , \tag{5.42}$$

or because of a vanishing variation parameter.

Fixing the coefficients of the X fields such that (5.40) and (5.41) are satisfied, the above BPS conditions are of course also consistent with the remaining equations of motion. In particular, those for the gauge fields, given that the Dirac monopole potential satisfies $\mathcal{D}_n\big(\frac{1}{\tilde{g}^2} F^{mn}\big) = \mathcal{D}_n\big(\frac{1}{\tilde{g}^2}\hat{F}^{mn}\big) = 0$, reduce to

$$\begin{aligned} \kappa\,\epsilon^{mnk} F_{nk} &= X\mathcal{D}^m X^\dagger - \mathcal{D}^m X X^\dagger \; , \\ \kappa\,\epsilon^{mnk} \hat{F}_{nk} &= \mathcal{D}^m X^\dagger X - X^\dagger \mathcal{D}^m X \; . \end{aligned} \tag{5.43}$$

In summary, a convenient choice of classical (anti-)BPS solution is given by

$$A = \hat{A} = \frac{H}{2}(\pm 1 - \cos\theta)\mathrm{d}\varphi \quad , \quad \phi_i = -\hat{\phi}_i = -\eta\frac{H}{2}\delta_{i3} \; , \tag{5.44}$$

(where the upper sign holds on the northern and the lower one on the southern hemi-

[6]And similarly $X^\dagger_{A1} \sim \exp(\eta\,\tau/2)$ and $X^\dagger_{A2} \sim \exp(-\eta\,\tau/2)$. While it may appear unusual that X and $X^\dagger$ are not complex conjugates of each other, this is simply an artefact of the Euclidean signature of spacetime. After a suitable Wick rotation they would evidently be conjugate in the usual sense.

sphere), supplemented by an appropriate X expectation value chosen to satisfy (5.40), (5.41) and (5.43), e.g. for positive semi-definite H

$$
\begin{aligned}
X^{11} = Z^1 &= \sqrt{H\kappa}\, e^{-\tau/2} \ , \qquad X^\dagger_{11} = Z^\dagger_1 = \sqrt{H\kappa}\, e^{\tau/2} \quad \text{for } \eta = 1 \text{ or} \\
X^{12} = W^{1\dagger} &= \sqrt{H\kappa}\, e^{-\tau/2} \ , \quad X^\dagger_{12} = W_1 = \sqrt{H\kappa}\, e^{\tau/2} \quad \text{for } \eta = -1 \ ,
\end{aligned}
\tag{5.45}
$$

with all other X's vanishing. Note that the choice of X expectation value breaks the flavor symmetry and here we have arbitrarily used the first flavor. This background is invariant under supersymmetry transformations with the parameter $\varepsilon_1 + i\eta\varepsilon_2$, and can be generalized to any preferred $\mathrm{SU}(2)_R$ orientation.

Our classical solution is BPS along the full RG flow for any value of $\tilde{g}$, which is an important prerequisite for our arguments. However, the actual calculation we wish to carry out will be performed in the far UV, and here the role of the X expectation value is quite different from the expectation values of the adjoint scalars and gauge fields. If we were to do perturbation theory in g around this background, we would be led to rescale the (quantum) fields A and ϕ (and their hatted analogues) by a factor of g (before carrying out the Weyl transformation), and thus g sets the scale of quantum fluctuations. Since the background values of A and ϕ are of order unity, they are parametrically larger than these fluctuations in the UV, and thus can be treated classically.

The fluctuations of the bifundamental field X do not suffer such a rescaling however, and both quantum excitations as well as the expectation value in our classical solution are of the same order of magnitude. Therefore, we shall not treat the X fields as a classical background in the UV theory. They are to be though of as quantum excitations which dress up the monopole background, and thus we shall drop them for the purpose of describing the bare monopole operator. We keep in mind however, that a bare monopole operator is not gauge invariant, and will eventually have to be contracted with a number of basic fields appearing in the Lagrangian in order to form a gauge invariant operator.

In this context it is interesting to note that the number of X excitations corresponding to our classical solution is of order Hk, which is related to the number of fundamental fields we expect to contract the bare monopole operator with.

Finally, we will need to generalize the monopole background to arbitrary, possibly τ-dependent $\mathrm{SU}(2)_R$ orientation n_i and thus, to conclude this discussion, we collect the basic properties of the semi-classical BPS monopoles background we will make use of below.

BPS monopole background. For the background to be BPS, we have to essentially identify the two gauge groups $\mathrm{U}(N)$ and $\hat{\mathrm{U}}(N)$, and turn on expectation values for the gauge fields and adjoint scalars ϕ_i and $\hat{\phi}_i$ given by

$$A = \hat{A} = \frac{H}{2}(\pm 1 - \cos\theta)\mathrm{d}\varphi \quad , \quad \phi_i = -\hat{\phi}_i = -\frac{H}{2}n_i(\tau)\ . \tag{5.46}$$

The monopole background is diagonal in the $\mathrm{U}(N)$ gauge indices $H = \mathrm{diag}(q_1, \ldots, q_N)$ where $q_r \in \mathbb{Z}$ are the $\mathrm{U}(1)^N$ gauge charges. They determine how the monopole transforms under gauge transformations. Furthermore, the background is labeled by a unit $\mathrm{SU}(2)_R$ vector n_i which gives the direction of the scalar fields. This vector is a collective coordinate of the background (5.46) and spans the moduli space S^2.

5.4 $\mathrm{U}(1)_R$ Charges from Normal Ordering

In this section we compute the quantum corrections to the $\mathrm{U}(1)_R$ charge which is preserved by the static background (5.46) with $n_i = \delta_{i3}$. These quantum corrections are due to fermions fluctuations and are encoded in the normal ordering constant of the $\mathrm{U}(1)_R$ charge operator. Before going into the specifics of ABJM theory, we will discuss the computation for a toy example which is then simple to generalize.

5.4.1 Prototype

Let us consider a single fermion $\psi(\tau,\Omega)$ in an abelian gauge theory subject to the equation of motion

$$\not{D}\psi + \frac{\eta}{2}q\psi = 0\ , \qquad\qquad (\eta = \pm 1) \tag{5.47}$$

and compute the oscillator expansion of the charge

$$Q = -i\int d\Omega\, \psi^\dagger\gamma^\tau\psi\ . \tag{5.48}$$

The Dirac operator $\not{D} = \not{\nabla} + i\not{A}$ in (5.47) includes a monopole background of the kind (5.46) with magnetic charge $H \to q$. The mass term in (5.47) whose magnitude is proportional to q plays the role of the coupling to the background scalar. The two signs, $\eta = \pm 1$, correspond to a BPS and an anti-BPS background, respectively.

The easiest way to solve (5.47) is to expand ψ in monopole spinor harmonics which are eigenfunctions of the monopole Dirac operator on the sphere, $\not{D}_S$. This operator is contained in the full operator in (5.47) simply as $\not{D} = \gamma^\tau\partial_\tau + \not{D}_S$, since the monopole potential does not have any component along τ. We note the explicit form of the monopole spinor harmonics in Appendix A.4. Here we only need their eigenvalues and multiplicities. For given magnetic charge q, the quantum numbers of the total angular momentum are given by

$$\begin{aligned} j &= \frac{|q|-1}{2} + p \qquad \text{with} \qquad p = 0,1,2,\dots\ , \\ m &= -j, -j+1, \dots, j\ , \end{aligned} \tag{5.49}$$

where the state $j = \frac{|q|-1}{2}$ is absent for $q = 0$.

We denote the eigenfunctions by

$$\not{\mathcal{D}}_{\rm S}\Upsilon^0_{qm} = 0 \qquad\qquad \text{for } j = \tfrac{|q|-1}{2} \tag{5.50}$$

$$\not{\mathcal{D}}_{\rm S}\Upsilon^{\pm}_{qjm} = i\Delta^{\pm}_{jq}\Upsilon^{\pm}_{qjm} \qquad\qquad \text{for } j = \tfrac{|q|+1}{2}, \tfrac{|q|+3}{2}, \ldots \tag{5.51}$$

with eigenvalues

$$\Delta^{\pm}_{jq} = \pm\tfrac{1}{2}\sqrt{(2j+1)^2 - q^2}\ . \tag{5.52}$$

This spectrum is plotted in Figure 5.1(a). The $|q|$ zero modes which exist for non-zero magnetic charge will be responsible for a shift of the quantized version of the charge (5.48).

Now we are ready to write the harmonic expansion of the wave function as

$$\psi(\tau,\Omega) = \sum_m \psi_m(\tau)\Upsilon^0_{qm}(\Omega) + \sum_{jm\varepsilon}\psi^{\varepsilon}_{jm}(\tau)\Upsilon^{\varepsilon}_{qjm}(\Omega)\ , \tag{5.53}$$

where $\varepsilon = \pm 1$. Plugging this expansion into the equation of motion (5.47) and using the properties (A.36),(A.38) and the orthogonality (A.39) one finds

$$\dot{\psi}_m = -\eta\frac{|q|}{2}\psi_m \quad , \quad \begin{pmatrix} \dot{\psi}^+_{jm} \\ \dot{\psi}^-_{jm} \end{pmatrix} = \begin{pmatrix} 0 & -i\Delta^- - \eta\frac{q}{2} \\ -i\Delta^+ - \eta\frac{q}{2} & 0 \end{pmatrix} \begin{pmatrix} \psi^+_{jm} \\ \psi^-_{jm} \end{pmatrix}\ . \tag{5.54}$$

The solution is

$$\psi = \sum_m \Big[c_m\, u^0\, e^{-\frac{|q|}{2}\tau} + d^{\dagger}_m\, v^0\, e^{\frac{|q|}{2}\tau}\Big]\Upsilon^0_m + \sum_{jm\varepsilon}\Big[c_{jm}\, u^{\varepsilon}_j\, e^{-E_j\tau} + d^{\dagger}_{jm}\, v^{\varepsilon}_j\, e^{E_j\tau}\Big]\Upsilon^{\varepsilon}_{jm}\ , \tag{5.55}$$

where the energies are given by $E_j = j+\frac{1}{2}$. The wavefunctions for the BPS case ($\eta = +1$)

are given by

$$u^0 = 1 \ , \quad v^0 = 0 \ , \quad u_j^+ = v_j^+ = 1 \ , \quad u_j^- = -v_j^- = \tfrac{1}{\sqrt{2}}\Big(\tfrac{q}{2j+1} + i\sqrt{1-\big(\tfrac{q}{2j+1}\big)^2}\Big) \ , \tag{5.56}$$

and the ones for the anti-BPS case ($\eta = -1$) are obtained from this by exchanging $u^0 \leftrightarrow v^0$ and $u_j^\pm \leftrightarrow v_j^\mp$. The normalization of the wave functions are such that we have

$$\{\psi_\alpha(\tau,\Omega), \Pi^\beta(t,\Omega')\} = \delta_\alpha^\beta \delta^{(2)}(\Omega - \Omega') \ , \tag{5.57}$$

$$\{c_{jm}, c^\dagger_{j'm'}\} = \delta_{jj'}\delta_{mm'} \ , \tag{5.58}$$

$$\{d_{jm}, d^\dagger_{j'm'}\} = \delta_{jj'}\delta_{mm'} \ , \tag{5.59}$$

where $\Pi = -i\psi^\dagger\gamma^\tau$ is the canonically conjugate momentum.

The energy spectra are plotted in Figure 5.1(b) and Figure 5.1(c). One observes that the zero modes of the Dirac operator have turned into "unpaired states", i.e. states for which there are no states with the opposite energy in the spectrum. There are $2j+1 = |q|$ unpaired states with energy $\eta\frac{|q|}{2}$, which is positive for the BPS and negative for the anti-BPS case.

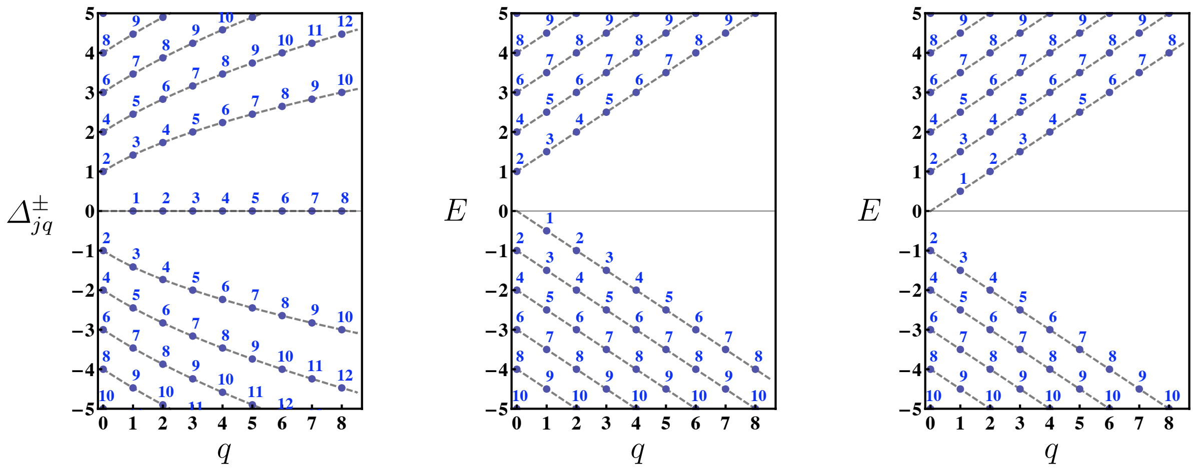
Now we compute the $\mathrm{U}(1)_R$ charge using point splitting regularization as in [161]:

$$Q(\beta) = -\frac{i}{2}\int d\Omega \left[\psi^\dagger\big(\tau + \tfrac{\beta}{2}\big)\,\gamma^\tau\,\psi\big(\tau - \tfrac{\beta}{2}\big) - \psi\big(\tau + \tfrac{\beta}{2}\big)\,\gamma^\tau\,\psi^\dagger\big(\tau - \tfrac{\beta}{2}\big)\right] . \tag{5.60}$$

where $\beta > 0$ will be taken to zero in the end. Inserting the oscillator expansion[7] and ordering the terms, we find the normal ordered piece

$$Q_1(\beta = 0) = \sum_{jm}\left[c^\dagger_{jm}c_{jm} - d^\dagger_{jm}d_{jm}\right] \tag{5.61}$$

[7]The oscillator expansion of $\psi^\dagger$ is the complex conjugate of that of ψ where in addition the sign of τ is reversed.



(a) Dirac operator (b) BPS (c) anti-BPS

Figure 5.1: Eigenvalues of the Dirac operator on the S^2 and energy spectra for BPS and anti-BPS backgrounds. The eigenvalues $\Delta^{\pm}_{jq} = \pm\frac{1}{2}\sqrt{(2j+1)^2 - q^2}$ and $E = \pm E_j = \pm(j+\frac{1}{2})$ are parametrized by $j = \frac{|q|-1}{2} + p$ for $p = 0, 1, 2, \ldots$. The dashed lines correspond to a fixed value p. The numbers next to the points denote the multiplicities of the corresponding eigenvalues. Note in particular that there are $|q|$ zero modes of the Dirac operator for monopole charge q, which are lifted to non-zero unpaired modes when the background scalar is turned on.

and a normal ordering constant

$$Q_0(\beta) = -\frac{1}{2}\sum_{jm\varepsilon}\Big[u^{\varepsilon\dagger}_j u^{\varepsilon}_j - v^{\varepsilon\dagger}_j v^{\varepsilon}_j\Big] e^{-\beta E_j} \ , \tag{5.62}$$

where in both sums we understand the zero modes with $j = \frac{|q|-1}{2}$ to be included. For that value of j there is no sum over $\varepsilon = \pm 1$ and $E_j = \frac{|q|}{2}$. The normal ordering constant can be written in a concise form by noticing that $\sum_\varepsilon u^{\varepsilon\dagger}_j u^{\varepsilon}_j = 1$ gives a contribution for every positive energy state and $\sum_\varepsilon v^{\varepsilon\dagger}_j v^{\varepsilon}_j = 1$ one for every negative energy state. Hence we can write

$$Q_0(\beta) = -\frac{1}{2}\sum_{\text{states}} \text{sign}(E)\, e^{-\beta|E|} \ , \tag{5.63}$$

where the sum extends over all states in the spectrum[8] and vanishes if the spectrum is symmetric with respect to $E = 0$. This is not the case due to the unpaired states. Instead we find

$$Q_0 = -\eta \frac{|q|}{2} \ , \tag{5.64}$$

where the factor $|q|$ is the number of unpaired states (the degeneracy of the mode $j = \frac{|q|-1}{2}$) and η the sign of their energy. This normal ordering constant gives the $\mathrm{U}(1)_R$ charge of the BPS or anti-BPS monopole background, in agreement with [161]. As it was also shown in [161], the bosonic fields do not contribute the induced charge because their spectrum is symmetric.

5.4.2 Application to $\mathcal{N} = 3$ Gauge Theory

The above discussion is readily applied to the $\mathcal{N} = 3$ gauge theories with N_f hyper multiplets. In place of the charge (5.48) we now have

$$Q = -i \int d\Omega \ \mathrm{tr}\Big[-\tfrac{1}{2}\, \zeta_A^\dagger \gamma^\tau \zeta^A - \tfrac{1}{2}\, \omega^{\dagger A} \gamma^\tau \omega_A + \chi_\sigma^\dagger \gamma^\tau \chi_\sigma + \hat{\chi}_\sigma^\dagger \gamma^\tau \hat{\chi}_\sigma \Big] \ , \tag{5.65}$$

see (5.38), and the equation of motion (5.47) is replaced by

$$\begin{aligned} &\not{D}\zeta^A + \tfrac{\eta}{2}[H, \zeta^A] = 0 \ , \qquad \not{D}\chi_\sigma + \tfrac{\eta}{2}[H, \chi_\sigma] = 0 \ , \qquad \not{D}\hat{\chi}_\sigma + \tfrac{\eta}{2}[H, \hat{\chi}_\sigma] = 0 \ , \\ &\not{D}\omega_A + \tfrac{\eta}{2}[H, \omega_A] = 0 \ , \qquad \not{D}\chi_\phi + \tfrac{\eta}{2}[H, \chi_\phi] = 0 \ , \qquad \not{D}\hat{\chi}_\phi + \tfrac{\eta}{2}[H, \hat{\chi}_\phi] = 0 \ , \end{aligned} \tag{5.66}$$

which hold in the static background (5.44) and in the far UV where $\tilde{g} \to 0$. The only qualitative difference is that we are now dealing with a non-abelian theory. Since the background fields have the same $\mathrm{U}(N)$ dependence for both gauge group factors,

[8] This formulae looks superficially different from that in [162] because we sum over *all* states including their degeneracy, not just over different energies levels.

namely $H = \mathrm{diag}(q_1, \ldots, q_N)$, all fields effectively transform in the adjoint of the diagonal $\mathrm{U}(N)_d \subset \mathrm{U}(N) \times \hat{\mathrm{U}}(N)$. This is apparent from the fact that we could write commutators in (5.66), and also the gauge field inside the Dirac operator acts via a commutator. Now the key observation is

$$[H, \psi]_{rs} = q_r \delta_{rt} \psi_{ts} - \psi_{rt} q_t \delta_{ts} = (q_r - q_s)\psi_{rs} \, , \tag{5.67}$$

where ψ_{rs} is one of the $N \times N$ matrix elements. This allows us to consider all matrix elements separately, if we use

$$q \to q_{rs} \equiv q_r - q_s \tag{5.68}$$

as the effective monopole charge. This immediately implies that the $\mathrm{U}(1)_R$ charge of the vacuum, i.e. the monopole background, is

$$\begin{aligned} Q_R^{\mathrm{mon}} &= \sum_{r,s} \left[-\tfrac{1}{2} \cdot N_f \cdot \left(-\eta \tfrac{|q_{rs}|}{2} \right) - \tfrac{1}{2} \cdot N_f \cdot \left(-\eta \tfrac{|q_{rs}|}{2} \right) + 1 \cdot \left(-\eta \tfrac{|q_{rs}|}{2} \right) + 1 \cdot \left(-\eta \tfrac{|q_{rs}|}{2} \right) \right] \\ &= \eta \left(\frac{N_f}{2} - 1 \right) \sum_{r,s=1}^{N} |q_r - q_s| \, . \end{aligned} \tag{5.69}$$

The sum produces an answer at least of order N unless all q_r are equal. For example, for the simplest monopole whose only non-vanishing label is $q_1 = 1$, and therefore transforms in the bifundamental of $\mathrm{U}(N) \times \mathrm{U}(N)$ for $k = 1$, we find that the induced R-charge is $(N_f - 2)(N - 1)$.

For the BPS background where $\eta = +1$, the R-charge of the monopole is positive if $N_f > 2$. For $N_f = 1$ the monopole R-charges are negative and the theory may not flow to a conformal limit [170, 171]. The BPS monopoles in ABJM theory have vanishing R-charge since there are two flavors, $N_f = 2$. However, this does not mean that there are gauge invariant operators with vanishing R-charge because such operators necessarily involve matter fields in addition to the monopoles. In fact, in a strongly coupled theory it is generally impossible to separate the monopole part from the matter part; nevertheless, this is possible in the weakly coupled UV limit where the monopole part is semi-classical.

We finally remark that from a similar computation, where we normal order the flavor charge operator instead of the R-charge operator, one can see that the monopole does not carry any induced flavor representation. Using the mode expansion in the flavor charge

$$Q^n_{\text{fl}} \sim \int d\Omega \ \operatorname{tr} \xi^\dagger_A (T^n)^A{}_B \gamma^\tau \xi^B = \int d\Omega \ \operatorname{tr} \left[\zeta^\dagger_A (T^n)^A{}_B \gamma^\tau \zeta^B - \omega^{\dagger B} (T^n)^A{}_B \gamma^\tau \omega_A \right] \tag{5.70}$$

yields a vanishing normal ordering constant simply because the generators $(T^i)^A{}_B$ of $\mathrm{SU}(N_f)$ are traceless. Note that considering the static background (5.44) which preserves only $\mathrm{U}(1)_R \subset \mathrm{SU}(2)_R$ is general enough for this argument, because flavor and R-charge are completely independent.

5.5 $\mathrm{SU}(2)_R$ Charges and Collective Coordinate Quantization

In this section we compute the $\mathrm{SU}(2)_R$ charges of the BPS monopole operators by quantizing the collective coordinate $\vec{n}(\tau)$ of the corresponding class of classical monopole backgrounds (5.46). The dynamics of the collective coordinate is governed by the interaction with the other fields of the theory. We take the influence of these interactions into account by computing the effective action $\Gamma(\vec{n})$. As explained in Section 5.2 it is sufficient to carry out this computation in the UV limit of the theory where the Yang-Mills coupling $\tilde{g}$ goes to zero. In this limit precisely the interactions with the fermions survive. Thus the effective action is obtained by integrating out the fermions:

$$e^{-\Gamma(\vec{n})} = \int [d\xi^\dagger][d\xi][d\lambda][d\hat{\lambda}]\, e^{-\mathcal{S}} \,, \tag{5.71}$$

where the relevant part of the action is

$$\mathcal{S} = \int d\tau\, d\Omega\ \mathrm{tr}\Big[-i\xi^\dagger_{Aa} \not{\mathcal{D}} \xi^{Aa} - \tfrac{i}{2}\, n_i\, \xi^\dagger_{Aa} (\sigma_i)^a{}_b [H, \xi^{Ab}] \tag{5.72}$$

$$+ \tfrac{i}{2} \lambda_{ab} \not{\mathcal{D}} \lambda^{ab} - \tfrac{i}{2}\, n_i\, \lambda_{ab} (\sigma_i)^b{}_c [H, \lambda^{ac}] \tag{5.73}$$

$$+ \tfrac{i}{2} \hat{\lambda}_{ab} \not{\mathcal{D}} \hat{\lambda}^{ab} - \tfrac{i}{2}\, n_i\, \hat{\lambda}_{ab} (\sigma_i)^b{}_c [H, \hat{\lambda}^{ac}] \Big] \,. \tag{5.74}$$

Since the action is quadratic the path integral is "simply" a determinant, albeit a determinant of a matrix operator with very many indices. We have displayed explicitly the $\mathrm{SU}(2)_{\mathrm{fl}}$ index (A) and the $\mathrm{SU}(2)_R$ indices (a, b, c). Besides those, there are implicit $\mathrm{U}(N)$ gauge indices which we denote by ξ_{rs}, λ_{rs}, etc. below. Then there is the spatial dependence of the fermions which we will trade for a set of mode indices by expanding the fields into harmonics on the sphere.

The good news is that the operator is fairly diagonal and couples only very few components together. For instance it is completely diagonal in the flavor and gauge indices, and couples at most two modes of the harmonic expansion. Thus we can perform the computation for a generic component and take the sum over the indices into account later.

5.5.1 Prototype

We start the discussion with a quantum mechanical model where there is only a (Euclidean) time coordinate τ. This example already exposes the essential point of the whole argument. The dependence of the fields on the angular coordinates on the sphere will be taken into account in the next paragraph.

Quantum mechanics. Let us consider one fermion $\psi^a(\tau)$ in the fundamental representation of $\mathrm{SU}(2)_R$, as indicated by the index a, with the action[9]

$$\mathcal{S} = \int d\tau \left[-i\psi^\dagger_a \partial_\tau \psi^a - \tfrac{i}{2}\, q n_i(\tau)\, \psi^\dagger_a (\sigma_i)^a{}_b \psi^b \right] . \tag{5.75}$$

Since we have not included any spatial dependence in this example, there is no monopole gauge field either. The coupling of ψ to the collective coordinate has been written as q in anticipation of it becoming the monopole charge later.

Formally we find for the effective action[10]

$$\Gamma(\vec{n}) = -\ln\det\Big(i\partial_\tau - i\tfrac{q}{2} n_i(\tau)\sigma_i \Big) . \tag{5.76}$$

Due to the unspecified τ-dependence of n_i, we cannot evaluate this determinant exactly. However, since the collective coordinate is considered as being quasi-static, it is legitimate to do a derivative expansion. The general form of the effective action then is

$$\Gamma(\vec{n}) = \int d\tau \Big[-V_{\text{eff}}(\vec{n}) + i\, \dot{n}_i A_i(\vec{n}) + \tfrac{1}{2}\dot{n}_i \dot{n}_j B_{ij}(\vec{n}) + \ldots \Big] . \tag{5.77}$$

Our aim is to find the function $A_i(\vec{n})$ by expanding (5.76). However, it is not immediately possible to expand (5.76) in $\dot{n}_i$, as it does not contain $\dot{n}_i$ explicitly. The trick due to [174] is to write

$$n_i(\tau) = \mathring{n}_i + \tilde{n}_i(\tau) , \tag{5.78}$$

where $\mathring{n}_i$ is constant with $\mathring{n}^2 = 1$ and $\tilde{n}_i(\tau)$ a small "fluctuation". Then the form of the

[9]A similar model with a fermion in the **3** of SU(2) was studied in [173].

[10]The relative sign between the two terms may superficially appear to have changed since $(\sigma_i)^a{}_b$ in (5.75) are transposed Pauli matrices, while in (5.76) we use non-transposed ones, $(\sigma_i)_a{}^b$.

effective action to second order in fluctuations reads

$$\Gamma(\vec{n}) = \int d\tau \Big[-V_{\text{eff}}(\mathring{\vec{n}}) - \tilde{n}_i \partial_i V_{\text{eff}}(\mathring{\vec{n}}) - \tfrac{1}{2}\tilde{n}_i\tilde{n}_j\partial_i\partial_j V_{\text{eff}}(\mathring{\vec{n}}) \\ + i\,\dot{\tilde{n}}_i A_i(\mathring{\vec{n}}) + i\,\dot{\tilde{n}}_i\tilde{n}_j\partial_j A_i(\mathring{\vec{n}}) + \tfrac{1}{2}\dot{\tilde{n}}_i\dot{\tilde{n}}_j B_{ij}(\mathring{\vec{n}}) + \ldots \Big] . \tag{5.79}$$

The point of (5.78) is that we can now expand (5.76) in powers of $\tilde{n}_i$, some of which will come with τ-derivatives, and compare the result with the general expression (5.79). We cannot determine A_i from the term $\dot{\tilde{n}}_i A_i(\mathring{\vec{n}})$ as this is a total derivative and hence will not show up in the expansion of (5.76). Therefore we will focus on the next term, the unique term with two powers of $\tilde{n}$ and one τ-derivative

$$\Gamma_{(2,1)}(\vec{n}) = i\int d\tau\, \dot{\tilde{n}}_i\tilde{n}_j\partial_j A_i(\mathring{\vec{n}}) = -\frac{i}{2}\int d\tau\, \dot{\tilde{n}}_i\tilde{n}_j \left(\partial_i A_j(\mathring{\vec{n}}) - \partial_j A_i(\mathring{\vec{n}})\right) . \tag{5.80}$$

Now we start from (5.76) using (5.78)

$$\Gamma(\vec{n}) = -\operatorname{tr}\ln\Big(i\partial_\tau - i\mathring{\not{m}} - i\tilde{\not{m}}\Big) = -\operatorname{tr}\ln\Big(i\partial_\tau - i\mathring{\not{m}}\Big) - \operatorname{tr}\ln\Big(\mathbb{1} - \tfrac{1}{\partial_\tau - \mathring{\not{m}}}\tilde{\not{m}}\Big) , \tag{5.81}$$

where we have introduced the shorthands $\mathring{m}_i = \frac{q}{2}\mathring{n}_i$, $\tilde{m}_i = \frac{q}{2}\tilde{n}_i$, and $\not{m} \equiv m_i\sigma_i$. We isolate the term with two powers of $\tilde{n}_i \sim \tilde{m}_i$ by expanding the logarithm:

$$\Gamma_{(2)}(\vec{n}) = \frac{1}{2}\operatorname{tr}\left[\frac{\partial_\tau + \mathring{\not{m}}}{\partial_\tau^2 - \mathring{m}^2}\,\tilde{\not{m}}\,\frac{\partial_\tau + \mathring{\not{m}}}{\partial_\tau^2 - \mathring{m}^2}\,\tilde{\not{m}}\right] . \tag{5.82}$$

Next we move all derivatives to the right using

$$\frac{1}{\partial^2 - \mathring{m}^2}\phi = \sum_{k=0}^{\infty}(-1)^k \underbrace{[\partial^2,[\partial^2,\ldots,[\partial^2}_{k},\phi]\ldots]]\frac{1}{(\partial^2 - \mathring{m}^2)^{k+1}} , \tag{5.83}$$

and perform the trace over $\mathrm{SU}(2)_R$ indices using

$$\operatorname{tr}\sigma_i\sigma_j = 2\delta_{ij} \quad , \quad \operatorname{tr}\sigma_i\sigma_j\sigma_k = 2i\epsilon_{ijk} \quad , \quad \operatorname{tr}\sigma_i\sigma_j\sigma_k\sigma_l = 2\delta_{ij}\delta_{kl} - 2\delta_{ik}\delta_{jl} + 2\delta_{il}\delta_{jk} . \tag{5.84}$$

We find for the terms which contain exactly one derivative

$$
\begin{aligned}
\Gamma_{(2,1)}(\vec{n}) =& -3\,\mathrm{tr}\,\dot{\tilde{m}}_i\tilde{m}_i\frac{\partial_\tau(\partial_\tau^2+\mathring{m}^2)}{(\partial_\tau^2-\mathring{m}^2)^3} - i\,\mathrm{tr}\,\epsilon_{ijk}\dot{\tilde{m}}_i\tilde{m}_j\mathring{m}_k\frac{1}{(\partial_\tau^2-\mathring{m}^2)^2} \\
&- 6\,\mathrm{tr}\big(2\dot{\tilde{m}}_i\tilde{m}_j\mathring{m}_i\mathring{m}_j - \dot{\tilde{m}}_j\tilde{m}_j\mathring{m}_i\mathring{m}_i\big)\frac{\partial_\tau}{(\partial_\tau^2-\mathring{m}^2)^3}\ .
\end{aligned}
\tag{5.85}
$$

Now that the coordinate dependent part is separated from the derivatives it is easy to evaluate the functional trace: it leads to one integral over τ and one over the energy ω, where $\partial_\tau \to -i\omega$. All but the second term will lead to an integral over a total τ-derivative and therefore can be dropped. We are left with only the second term which becomes

$$
\Gamma_{(2,1)}(\vec{n}) = -i\int d\tau\ \epsilon_{ijk}\dot{\tilde{m}}_i\tilde{m}_j\mathring{m}_k\int\frac{d\omega}{2\pi}\ \frac{1}{(\omega^2+\mathring{m}^2)^2} = -\frac{i}{4}\,\mathrm{sign}(q)\int d\tau\ \epsilon_{ijk}\dot{\tilde{n}}_i\tilde{n}_j\frac{\mathring{n}_k}{|\mathring{\vec{n}}|^3}\ . \tag{5.86}
$$

Comparing this to (5.80), we read off

$$
\partial_i A_j(\vec{n}) - \partial_j A_i(\vec{n}) = \frac{\mathrm{sign}(q)}{2}\,\epsilon_{ijk}\frac{n_k}{|\vec{n}|^3}\ . \tag{5.87}
$$

This expression is recognized as the field strength for a magnetic monopole with charge $\frac{\mathrm{sign}(q)}{2}$ and hence $A_i(\vec{n})$ is the corresponding gauge potential. We should stress that this monopole has nothing to do with the original monopole background of ABJM theory which we set out to study. In fact, the action (5.75) does not contain any monopole background for ψ. The monopole potential we are finding here lives on the space spanned by $\vec{n}$, which just happens to be the moduli space of supersymmetric monopoles in $\mathcal{N}=3$ gauge theory.

At any rate what we have found is that the effect of the fermion ψ is to induce a Wess-Zumino term in the effective action for the collective coordinate. In other words, the dynamics of the collective coordinate $\vec{n}$ is the same as that of a point particle on a sphere with a magnetic monopole at its center. The coefficient of the Wess-Zumino term is the product of the electric charge of the point particle and the magnetic charge of the monopole. This analogy immediately implies that the allowed $\mathrm{SU}(2)_R$ representations for the quantized collective coordinate, and hence the ABJM monopoles, are bounded from below by the Wess-Zumino coefficient. Thus, if we want to have monopole operators in the singlet representation, we will have to find that the Wess-Zumino term cancels from the effective action when all contributions are included. This is what will indeed happen for ABJM theory, but not in general.

Spatial dependence. We reinstate the dependence of ψ on the coordinates of the sphere and consider the case

$$\mathcal{S} = \int d\tau\, d\Omega \left[-i\psi_a^\dagger \not{D} \psi^a - \tfrac{i}{2}\, q n_i(\tau)\, \psi_a^\dagger (\sigma_i)^a{}_b \psi^b \right] , \tag{5.88}$$

which contains a monopole background with magnetic charge q but is still abelian. The easiest way to deal with the spatial dependence is to expand $\psi(\tau, \Omega)$ into monopole spinor harmonics and perform the S^2-integration in the action. All relevant properties of these harmonics have already been discussed in Subsection 5.4.1, see also Appendix A.4.

The expansion of ψ is the same as in Section 5.4, eq. (5.53). Since $\vec{n}(\tau)$ is constant on the sphere, the orthogonality of the monopole harmonics (A.39) implies that different (jm)-modes of ψ do not couple to each other. The only coupling is between the $\pm$-modes which is due to the property (A.36). Indeed the action for the modes becomes

$$\begin{aligned} \mathcal{S} = &\sum_m \int d\tau \left[-i \, \mathrm{sign}(q)\, \psi_m^\dagger \partial_\tau \psi_m - \tfrac{i}{2}\, q n_i\, \psi_m^\dagger \sigma_i \psi_m \right] \\ &+ \sum_{jm\varepsilon} \int d\tau \left[-i\psi_{jm}^{\varepsilon\dagger} \partial_\tau \psi_{jm}^\varepsilon + \Delta_{jq}^\varepsilon\, \psi_{jm}^{-\varepsilon\dagger} \psi_{jm}^\varepsilon - \tfrac{i}{2}\, q n_i\, \psi_{jm}^{-\varepsilon\dagger} \sigma_i \psi_{jm}^\varepsilon \right] . \end{aligned} \tag{5.89}$$

This decoupling means that we can compute the contribution to the effective action for each pair (jm) individually. One such term in the zero-mode sector, $j = \frac{|q|-1}{2}$, is essentially the previously considered case (5.75). The additional sign(q) in (5.89), which originates from (A.38), removes the sign(q) from the result (5.86). The sum over m introduces a factor of $2j+1 = |q|$. This is how the Wess-Zumino term acquires the dependence on the monopole charge. Thus the Wess-Zumino potential is

$$\partial_i A_j(\vec{n}) - \partial_j A_i(\vec{n}) = \frac{|q|}{2}\,\epsilon_{ijk}\frac{n_k}{|\vec{n}|^3}\ . \tag{5.90}$$

Now we turn to the non-zero-mode sector. It will turn out that there is no contribution to the effective action from this sector, and that (5.90) is the final result. In the following we think of j and m as fixed to some values and suppress these labels. Due to the coupling between the $\pm$-modes, we now have a 2×2 matrix in "mode space" on top of the matrix structure that mixes the two components of the $\mathrm{SU}(2)_R$ doublet:

$$\Gamma(\vec{n}) = -\ln\det\begin{pmatrix} i\partial_\tau & -\Delta^- - i\not{m} \\ -\Delta^+ - i\not{m} & i\partial_\tau \end{pmatrix}\ . \tag{5.91}$$

The generalization of (5.82) is

$$\Gamma_{(2)}(\vec{n}) = \frac{1}{2}\,\mathrm{tr}\left[\frac{\begin{pmatrix} -\partial_\tau & -i\Delta - \mathring{\not{m}} \\ i\Delta - \mathring{\not{m}} & -\partial_\tau \end{pmatrix}}{\partial_\tau^2 - \Delta^2 - \mathring{m}^2}\begin{pmatrix} 0 & \tilde{\not{m}} \\ \tilde{\not{m}} & 0 \end{pmatrix}\right]^2\ , \tag{5.92}$$

where $\Delta \equiv \Delta^+ = -\Delta^-$. Evaluating this expression analogously to the previous case yields that the term $\Gamma_{(2,1)}(\vec{n})$ is zero (up to surface terms), i.e. the non-zero modes do not contribute to the Wess-Zumino term.

Recall that in the computation of the $\mathrm{U}(1)_R$ charge of the monopole operator in Section 5.4, we also found that the non-zero modes did not contribute. There the cancellation occurred between states of equal but opposite energy. In order to demonstrate explicitly that the same mechanism is at work here, too, we pretend that Δ^+ and Δ^- are unrelated for the time being and repeat the computation. Without presenting any of the lengthy intermediate steps, we arrive at

$$\Gamma_{(2,1)} = \int d\tau \int \frac{d\omega}{2\pi} \frac{-4i\epsilon_{ijk}\dot{\tilde{m}}_i \tilde{m}_j \mathring{m}_k \,(\Delta^+ + \Delta^-)(\omega^2 + \Delta^+\Delta^-)\,\omega}{\left[\omega^4 + 2(\Delta^+\Delta^- - \mathring{m}^2)\omega^2 + (\Delta^+\Delta^-)^2 + \big((\Delta^+)^2 + (\Delta^-)^2\big)\mathring{m}^2 + \mathring{m}^4\right]^2} \; . \tag{5.93}$$

Indeed we see that the vanishing is due to the pairing of eigenvalues, $\Delta^+ = -\Delta^-$.

5.5.2 Application to $\mathcal{N} = 3$ Gauge Theory

Having obtained the result (5.90) for the prototype action (5.88) it is a simple matter to specialize to ABJM theory and its $\mathcal{N} = 3$ UV completion. All that needs to be done is to include the gauge indices and sum over the field content.

Gauge structure. The non-abelian nature of the theory is taken care of just as in Section 5.4.2. From (5.67) it is clear that the action is diagonal in gauge indices and therefore every matrix element contributes independently from the others to the effective action. The effective monopole charge that the matrix element ψ_{rs} experiences is given by

$$q_{rs} \equiv q_r - q_s \; . \tag{5.94}$$

Hence the non-abelian result is obtained from the abelian one by the replacement

$$\Gamma_q(\vec{n}) \to \sum_{r,s=1}^{N} \Gamma_{q_{rs}}(\vec{n}) \; . \tag{5.95}$$

We will see that the sum over r, s will factorize, because all fermions transform effectively in the same representation.

Therefore, it will be convenient to define the total charge

$$q_{\mathrm{tot}} = \sum_{r,s=1}^{N} |q_{rs}| = 2\sum_{r>s} |q_r - q_s| \ . \tag{5.96}$$

Hyper multiplet fermions. Upon using (5.67), the action for ξ^A is just N_f copies of the prototype (5.88). Therefore we can immediately write down the total contribution from the hyper multiplet fermions

$$\partial_i A_j(\vec{n}) - \partial_j A_i(\vec{n}) = N_f \, \frac{q_{\mathrm{tot}}}{2} \, \epsilon_{ijk} \frac{n_k}{|\vec{n}|^3} \ . \tag{5.97}$$

Vector multiplet fermions. The computation for λ and $\hat{\lambda}$ reduces to (5.88) as well, but we have to be careful not to over-count the degrees of freedom. Due to the relations (5.20) there are only two independent (complex) components. We choose $\lambda^{11} \sim \chi_\sigma$ and $\lambda^{12} \sim \chi_\phi^\dagger$ as the independent components and denote their complex conjugates by

$$\lambda^\dagger_{11} \equiv (\lambda^{11})^* \quad , \quad \lambda^\dagger_{12} \equiv (\lambda^{12})^* \ , \tag{5.98}$$

and similarly for $\hat{\lambda}$. When expressed in terms of these fields, the action reads

$$\begin{aligned} \mathcal{S} = \int d\tau \, d\Omega \ \mathrm{tr}\Big[& - i\lambda^\dagger_{1a} \not{D} \lambda^{1a} + \tfrac{i}{2} \, n_i \, \lambda^\dagger_{1a} (\sigma_i)^a{}_b [H, \lambda^{1b}] \\ & - i\hat{\lambda}^\dagger_{1a} \not{D} \hat{\lambda}^{1a} + \tfrac{i}{2} \, n_i \, \hat{\lambda}^\dagger_{1a} (\sigma_i)^a{}_b [H, \hat{\lambda}^{1b}] \Big] \ . \end{aligned} \tag{5.99}$$

This is nothing but the action for ξ with reversed sign of the interaction, cf. (5.72). Thus the contribution from the vector multiplet fermions is

$$\partial_i A_j(\vec{n}) - \partial_j A_i(\vec{n}) = -2 \times \frac{q_{\mathrm{tot}}}{2} \, \epsilon_{ijk} \frac{n_k}{|\vec{n}|^3} \ , \tag{5.100}$$

where the factor of 2 arises because we have λ and $\hat{\lambda}$.

Collective coordinate quantization. Adding all contributions together, the total effective action for the collective coordinate is given by

$$\Gamma(\vec{n}) = \int d\tau \left[\tfrac{1}{2} M \dot{\vec{n}}^2 + i\, \vec{A}(\vec{n}) \cdot \dot{\vec{n}} + \lambda(\vec{n}^2 - 1) \right] , \tag{5.101}$$

where the kinetic term is simply the sum of the kinetic terms for ϕ and $\hat{\phi}$. The last term is a Lagrange multiplier term which enforces the constraint that the modulus of $\vec{n}$ is fixed to one. This actions describes a particle with unit electric charge and large mass (as $\tilde{g} \to 0$ in the UV)

$$M = M(\tau) = \frac{1}{\tilde{g}^2} \operatorname{tr} H^2 = g^{-2} e^{-2\tau} \sum_r q_r^2 \tag{5.102}$$

on a sphere surrounding a magnetic monopole with charge

$$h = (N_f - 2)\, q_{\text{tot}} = (N_f - 2) \sum_{r,s} |q_r - q_s| \ , \tag{5.103}$$

and field strength $\vec{B} = \vec{\nabla} \times \vec{A} = \frac{h}{2} \vec{n}$.

This is the Euclidean version of the system discussed in Section 5.2 with the modification that now M depends on time. Still, the conserved angular momentum is given by

$$\vec{L} = M\, \vec{n} \times \dot{\vec{n}} - \frac{h}{2} \vec{n} \ , \tag{5.104}$$

and its quantized values are the $\mathrm{SU}(2)_R$ charges of the monopole operator described by the BPS background (5.46). Due to the second term in (5.104), the smallest possible $\mathrm{SU}(2)_R$ representation has spin

$$l = \frac{|h|}{2} = \left| \frac{N_f}{2} - 1 \right| \sum_{r,s} |q_r - q_s| \tag{5.105}$$

and dimension $2l + 1 = |h| + 1$.

A monopole in the singlet representation and hence of vanishing IR dimension, is only possible if $h = 0$. One way of achieving this is to set all fluxes q_r equal. This corresponds to a monopole in the diagonal $\mathrm{U}(1)_d \subset \mathrm{U}(N) \times \mathrm{U}(N)$ of the gauge group which decouples from the matter fields. Another way to have singlet monopoles is to consider $N_f = 2$ hyper multiplets, which is the field content of ABJM theory. This is true regardless of how the fluxes q_r are distributed inside the total $\mathrm{U}(N)_d$ flux H.

5.6 Conclusions and Outlook

In this chapter we have calculated the global charges and dimensions of monopole operators in certain three-dimensional $\mathcal{N} = 3$ supersymmetric Yang-Mills Chern-Simons theories. This is the smallest amount of supersymmetry leading to a non-abelian R-symmetry which was crucial for our argument, because the $\mathrm{SU}(2)_R$ spin of a monopole operator cannot change along an RG flow. This allowed us to find the exact charges from a one-loop calculation in the weakly coupled UV limit of the gauge theory.

In the far UV the monopole operator was adequately described by a classical Dirac monopole background for the gauge fields. For the description of BPS monopoles the background needs to be supersymmetric which required us to also turn on a classical background for the adjoint scalar fields.

In Section 5.4 we considered a static scalar background. Since such a background breaks the R-symmetry from $\mathrm{SU}(2)_R$ to $\mathrm{U}(1)_R$, we could only determine the abelian charge of the monopole in this case. Using the methods developed in [161], we found that fermionic fluctuations around the background induce a $\mathrm{U}(1)_R$ charge of the monopole proportional to the R-charges of the fermions times their magnetic coupling to the background. Our complete formula (5.69) for the R-charge in $\mathrm{U}(N) \times \mathrm{U}(N)$ gauge theory coupled to N_f hyper multiplets in the bifundamental representation is consistent with the proposal made in [170, 171].

However, knowing the $U(1)_R$ charges at small Yang-Mills coupling is in general not enough as this quantity is not protected and one cannot make any reliable statement about their IR values. In [161] the computation was performed directly in the IR limit (of SQED) which was possible by assuming a large number of flavors. Here we could not resort to this trick because we wanted to keep the number of flavors arbitrary. However, we can make use of the $\mathcal{N} = 3$ supersymmetry and compute non-abelian R-charges which *are* protected. They follow from quantization of the SU(2)/U(1) collective coordinate of the background. By calculating the fermionic determinants which induce a Wess-Zumino term in the effective action of the collective coordinate, we demonstrated in Section 5.5 that the smallest allowed $SU(2)_R$ representation is given by (5.105). The largest $U(1)_R$ charge within this representation coincides with our findings in Section 5.4.

Note that the induced R-charge of the monopole is entirely due to the fermions of the theory. In the normal ordering computation, Section 5.4, the reason why the bosons do not contribute is because their spectrum is symmetric with respect to zero and therefore the states with positive energy cancel the effect of those with negative energy. In the collective coordinate computation, Section 5.5, this follows from the fact that the coupling between the bosons and the collective coordinate goes to zero in the UV.

After the theory has flown to the superconformal Chern-Simons fixed point, we can use these results to argue that the contribution of the monopole operator to gauge-invariant operator dimensions is given by (5.69), as long as it is non-negative (when it is negative, a conventional fixed point does not exist). For ABJM theory, where $N_f = 2$, this contribution vanishes which is crucial for matching the spectrum with supergravity on $\mathrm{AdS}_4 \times \mathrm{S}^7/\mathbb{Z}_\mathrm{k}$ and for the supersymmetry enhancement to $\mathcal{N} = 8$.

Since the gauginos make a crucial negative contribution to the R-charge, and they are not even dynamical in the IR Chern-Simons theory, it is not clear how to carry out this calculation reliably without appealing to the UV theory containing the Yang-Mills term. Luckily, there are various other theories to which the method of starting with a weakly coupled UV theory can be applied. One obvious example is to consider quiver theories with more than two U(N) gauge groups and bifundamental hyper multiplets. These Yang-Mills Chern-Simons theories can flow in the IR to $\mathcal{N} = 3$ superconformal fixed points; some of them have M-theory AdS_4 duals found by Jafferis and Tomasiello [175].

Determination of monopole operator dimensions poses more of a challenge in $\mathcal{N} = 2$ superconformal Chern-Simons theories, since we cannot rely on a non-abelian R-symmetry. Nevertheless, it should again be possible to define some of these theories via flow from weakly coupled Yang-Mills Chern-Simons theories, where monopole operator R-charge can be computed semiclassically. It is conceivable that the $U(1)_R$ charge does not change under the RG flow, which would then determine it in the superconformal theory.

As we have noted, in some quiver theories the induced R-charge of the monopole in the UV is simply proportional to the sum over the R-charges of all the fermions. If a "parent" quiver Yang-Mills gauge theory can be written down in four dimensions with the same superpotential, then all the fermion R-charges in it are the same as in three dimensions. In this "parent theory" the sum over all fermion R-charges determines the $U(1)_R$ anomaly. In particular, if this quantity vanishes, then the 4-dimensional gauge theory is superconformal. Thus, it is tempting to conjecture a relation between $U(1)_R$ anomaly in a parent 4-dimensional gauge theory and the induced monopole operator R-charge in a descendant 3-dimensional gauge theory. For example, in the class of $U(N) \times U(N)$ theories we have considered in this paper, the induced monopole R-charge is $\sim (1 - N_f/2)$. In its parent 4-dimensional gauge theory, the $U(1)_R$ anomaly coefficient is $(1 - N_f/2)N$, with the first term due to an adjoint gluino of R-charge 1, and the second due to N_f bifundamentals of R-charge $-1/2$. The anomaly cancellation for

$N_f = 2$ singles out the 4-dimensional superconformal gauge theory describing D3-branes on the conifold [44].

This discussion suggests that, if a quiver gauge theory is superconformal in four dimensions, then at least some monopole operators in its 3-dimensional descendant (presumably the ones that correspond to turning on monopoles in all gauge groups) have vanishing monopole operator dimensions. Clearly, the possibility of a connection between monopole R-charges in three dimensions and anomaly coefficients in four dimensions requires a more detailed study.

Appendix: Useful Formulae, Notations and Conventions

A.1 The Type IIB Supergravity Equations

Here we succinctly list the equations of motion of IIB supergravity. They are used heavily in Chapter 3 to study 2-form potential and metric perturbations. For simplicity we have set the dilaton and RR scalar to zero.

Bianchi identities:

$$\begin{aligned} dF_3 &= 0\,, \\ dH_3 &= 0\,, \\ d\tilde{F}_5 &= H_3 \wedge F_3\,. \end{aligned} \tag{A.1}$$

Dynamic equations:

$$\begin{aligned} d * H_3 &= -g_s^2 \tilde{F}_5 \wedge F_3\,, \\ d * F_3 &= \tilde{F}_5 \wedge H_3\,, \\ \tilde{F}_5 &= *\tilde{F}_5\,. \end{aligned} \tag{A.2}$$

Einstein equation:

$$R_{ij} = T_{ij} = \frac{g_s^2}{96}\tilde{F}_{iabcd}\tilde{F}_j{}^{abcd} + \frac{1}{4}H_{iab}H_j{}^{ab} - \frac{1}{48}G_{ij}H_{abc}H^{abc} + \frac{g_s^2}{4}F_{iab}F_j{}^{ab} - \frac{g_s^2}{48}G_{ij}F_{abc}F^{abc}\,. \tag{A.3}$$

A.2 Chern-Simons Field Theory Notations and Conventions

Lorentzian ABJM model. The world-volume metric is $g^{\mu\nu} = \mathrm{diag}(-1,+1,+1)$ with index range $\mu = 0,1,2$. We use Dirac matrices $(\gamma^\mu)_\alpha{}^\beta = (i\sigma^2, \sigma^1, \sigma^3)$ satisfying $\gamma^\mu\gamma^\nu = g^{\mu\nu} + \epsilon^{\mu\nu\rho}\gamma_\rho$. The fermionic coordinate of superspace is a complex two-component spinor θ. Indices are raised, $\theta^\alpha = \epsilon^{\alpha\beta}\theta_\beta$, and lowered, $\theta_\alpha = \epsilon_{\alpha\beta}\theta^\beta$, with $\epsilon^{12} = -\epsilon_{12} = 1$. Note that lowering the spinor indices of the Dirac matrices makes them symmetric $\gamma^\mu_{\alpha\beta} = (-\mathbb{1}, -\sigma^3, \sigma^1)$. In products like $\theta^\alpha\theta_\alpha \equiv \theta^2$, $\theta^\alpha\bar{\theta}_\alpha \equiv \theta\bar{\theta}$ etc and $\theta^\alpha\gamma^\mu{}_\alpha{}^\beta\bar{\theta}_\beta \equiv \theta\gamma^\mu\bar{\theta}$ we suppress the indices. We have

$$\theta_\alpha\theta_\beta = \tfrac{1}{2}\epsilon_{\alpha\beta}\theta^2 \quad , \quad \theta^\alpha\theta^\beta = \tfrac{1}{2}\epsilon^{\alpha\beta}\theta^2 \tag{A.4}$$

and likewise for $\bar{\theta}$ and derivatives. The Fierz identities are[11]

$$\begin{aligned}
(\psi_1\psi_2)(\psi_3\psi_4) &= -\tfrac{1}{2}(\psi_1\psi_4)(\psi_3\psi_2) - \tfrac{1}{2}(\psi_1\gamma^\mu\psi_4)(\psi_3\gamma_\mu\psi_2)\ , \qquad (A.5)\\
(\psi_1\psi_2)(\psi_3\gamma^\mu\psi_4) &= -\tfrac{1}{2}(\psi_1\gamma^\mu\psi_4)(\psi_3\psi_2) - \tfrac{1}{2}(\psi_1\psi_4)(\psi_3\gamma^\mu\psi_2) - \tfrac{1}{2}\epsilon^{\mu\nu\rho}(\psi_1\gamma_\nu\psi_4)(\psi_3\gamma_\rho\psi_2)\ ,\\
(\psi_1\gamma^\mu\psi_2)(\psi_3\gamma^\nu\psi_4) &= -\tfrac{1}{2}g^{\mu\nu}(\psi_1\psi_4)(\psi_3\psi_2) + \tfrac{1}{2}g^{\mu\nu}(\psi_1\gamma^\rho\psi_4)(\psi_3\gamma_\rho\psi_2) - (\psi_1\gamma^{(\mu}\psi_4)(\psi_3\gamma^{\nu)}\psi_2)\\
&\quad -\tfrac{1}{2}\epsilon^{\mu\nu\rho}\big[(\psi_1\gamma_\rho\psi_4)(\psi_3\psi_2) - (\psi_1\psi_4)(\psi_3\gamma_\rho\psi_2)\big]\ ,
\end{aligned}$$

which imply in particular

$$(\theta\bar{\theta})^2 = -\tfrac{1}{2}\theta^2\bar{\theta}^2\ , \tag{A.6}$$

$$(\theta\bar{\theta})(\theta\gamma^\nu\bar{\theta}) = 0\ , \tag{A.7}$$

$$(\theta\gamma^\mu\bar{\theta})(\theta\gamma^\nu\bar{\theta}) = \tfrac{1}{2}g^{\mu\nu}\theta^2\bar{\theta}^2\ . \tag{A.8}$$

[11] Here and everywhere we use symmetrization and anti-symmetrization with weight one $X_{[a}Y_{b]} = \frac{1}{2}\big(X_aY_b - X_bY_a\big)$ and $X_{(a}Y_{b)} = \frac{1}{2}\big(X_aY_b + X_bY_a\big)$.

The supercovariant derivatives and supersymmetry generators are

$$D_\alpha = \partial_\alpha + i\gamma^\mu_{\alpha\beta}\bar{\theta}^\beta\partial_\mu \ , \qquad Q_\alpha = \partial_\alpha - i\gamma^\mu_{\alpha\beta}\bar{\theta}^\beta\partial_\mu \ , \tag{A.9}$$

$$\bar{D}_\alpha = -\bar{\partial}_\alpha - i\theta^\beta\gamma^\mu_{\alpha\beta}\partial_\mu \ , \qquad \bar{Q}_\alpha = -\bar{\partial}_\alpha + i\theta^\beta\gamma^\mu_{\alpha\beta}\partial_\mu \ , \tag{A.10}$$

where the only non-trivial anti-commutators are

$$\{D_\alpha, \bar{D}_\beta\} = -2i\gamma^\mu_{\alpha\beta}\partial_\mu \quad , \quad \{Q_\alpha, \bar{Q}_\beta\} = 2i\gamma^\mu_{\alpha\beta}\partial_\mu \ . \tag{A.11}$$

We use the following conventions for integration

$$d^2\theta \equiv -\tfrac{1}{4}d\theta^\alpha d\theta_\alpha \quad , \quad d^2\bar{\theta} \equiv -\tfrac{1}{4}d\bar{\theta}^\alpha d\bar{\theta}_\alpha \quad , \quad d^4\theta \equiv d^2\theta\, d^2\bar{\theta} \ , \tag{A.12}$$

such that

$$\int d^2\theta\, \theta^2 = 1 \quad , \quad \int d^2\bar{\theta}\, \bar{\theta}^2 = 1 \quad , \quad \int d^4\theta\, \theta^2\bar{\theta}^2 = 1 \ . \tag{A.13}$$

It is useful to note that up to a total derivative

$$\int d^4\theta\ \ldots = \frac{1}{16}\big(D^2\bar{D}^2 \ldots\big)|_{\theta=\bar{\theta}=0} \ . \tag{A.14}$$

We use the $N \times N$ hermitian matrix generators T^n ($n = 0, \ldots, N^2-1$) and t^n ($n = 1, \ldots, N^2-1$) for U(N) and SU(N) respectively. We have $T^n = (T^0, t^n)$ with $T^0 = \mathbb{1}/\sqrt{N}$.

The generators are normalized as $\operatorname{tr} T^n T^m = \delta^{nm}$. Their completeness implies that $\operatorname{tr} AT^n \operatorname{tr} BT^n = \operatorname{tr} AB$ and $\operatorname{tr} AT^n BT^n = \operatorname{tr} A \operatorname{tr} B$ for U(N). Similarly for SU(N) we have $\operatorname{tr} At^n \operatorname{tr} Bt^n = \operatorname{tr} AB - \frac{1}{N}\operatorname{tr} A \operatorname{tr} B$ and $\operatorname{tr} At^n Bt^n = \operatorname{tr} A \operatorname{tr} B - \frac{1}{N}\operatorname{tr} AB$.

Euclidean Chern-Simons Yang Mills theory in radial quantization. The main part of our computations in Chapter 5 are performed on $\mathbb{R} \times \mathrm{S}^2$ with the metric $ds^2 = g_{mn}dx^m dx^n = d\tau^2 + d\theta^2 + \sin^2\theta\, d\varphi^2$. As Dirac matrices in the tangent frame we use $(\gamma^a)_\alpha{}^\beta = (-\sigma^2, \sigma^1, \sigma^3)$, which satisfy $\gamma^a\gamma^b = \delta^{ab} + i\epsilon^{abc}\gamma^c$. As above all spinor indices are raised and lowered from the left, $\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta$ and $\psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta$, with $\epsilon^{12} = -\epsilon_{12} = 1$. Again, $(\gamma^a)_{\alpha\beta} = (-i\mathbb{1}, -\sigma^3, \sigma^1)$ are symmetric and we also have $(\gamma^a)_\alpha{}^\beta = (\gamma^a)^\beta{}_\alpha$. For contracting spinor indices we use the NW-SE convention, e.g $\psi\gamma^m\gamma^n\chi \equiv \psi^\alpha(\gamma^m)_\alpha{}^\beta(\gamma^n)_\beta{}^\gamma\chi_\gamma$ etc. The spin connection is

$$\nabla_m\psi = (\partial_m + \omega_m)\psi \quad , \quad \omega_m = \tfrac{1}{4}\omega_{mab}\gamma^{ab} \quad , \quad \gamma^{ab} = \tfrac{1}{2}[\gamma^a, \gamma^b] \tag{A.15}$$

with the only non-zero component being $\omega_{\varphi 21} = -\omega_{\varphi 12} = \cos\theta$. For raising and lowering $\mathrm{SU}(2)_R$ indices (also called $a, b, \ldots$) we use the same conventions as for spinor indices. The standard index position for Pauli matrices is $(\sigma_i)_a{}^b$.

$\mathcal{N} = 2$ superfields. The component expansion of the $\mathcal{N} = 2$ superfields are as follows. The vector superfield $\mathcal{V}(x, \theta, \bar\theta)$ in Wess-Zumino gauge contains the $\mathrm{U}(N)\times\hat{\mathrm{U}}(N)$ gauge field A_μ, a complex two-component fermion χ_σ, a real scalar σ and an auxiliary scalar D, such that

$$\mathcal{V} = 2i\,\theta\bar\theta\,\sigma(x) - 2\,\theta\gamma^m\bar\theta\,A_m(x) + \sqrt{2}i\,\theta^2\,\bar\theta\chi_\sigma^\dagger(x) - \sqrt{2}i\,\bar\theta^2\,\theta\chi_\sigma(x) + \theta^2\,\bar\theta^2\,\mathsf{D}(x)\ , \tag{A.16}$$

$$\hat{\mathcal{V}} = 2i\,\theta\bar\theta\,\hat\sigma(x) - 2\,\theta\gamma^m\bar\theta\,\hat A_m(x) + \sqrt{2}i\,\theta^2\,\bar\theta\hat\chi_\sigma^\dagger(x) - \sqrt{2}i\,\bar\theta^2\,\theta\hat\chi_\sigma(x) + \theta^2\,\bar\theta^2\,\hat{\mathsf{D}}(x)\ , \tag{A.17}$$

and similarly the chiral superfields in the adjoints of the two gauge group factors are

$$\Phi = \phi(x_L) + \sqrt{2}\,\theta\chi_\phi(x_L) + \theta^2\,F_\phi(x_L)\ , \quad \bar\Phi = \phi^\dagger(x_R) - \sqrt{2}\,\bar\theta\chi_\phi^\dagger(x_R) - \bar\theta^2\,F_\phi^\dagger(x_R)\ , \tag{A.18}$$

$$\hat\Phi = \hat\phi(x_L) + \sqrt{2}\,\theta\hat\chi_\phi(x_L) + \theta^2\,\hat F_\phi(x_L)\ , \quad \hat{\bar\Phi} = \hat\phi^\dagger(x_R) - \sqrt{2}\,\bar\theta\hat\chi_\phi^\dagger(x_R) - \bar\theta^2\,\hat F_\phi^\dagger(x_R)\ . \tag{A.19}$$

The components of the chiral and anti-chiral superfields, $\mathcal{Z}(x_L,\theta)$ and $\bar{\mathcal{Z}}(x_R,\bar{\theta})$, are a complex boson Z, a complex two-component fermion ζ as well as a complex auxiliary scalar F, and similarly for $\mathcal{W}$ and $\bar{\mathcal{W}}$. The expansions of these bifundamental matter fields are given by

$$\mathcal{Z} = Z(x_L) + \sqrt{2}\,\theta\zeta(x_L) + \theta^2\,F(x_L)\;, \tag{A.20}$$

$$\bar{\mathcal{Z}} = Z^\dagger(x_R) - \sqrt{2}\,\bar{\theta}\zeta^\dagger(x_R) - \bar{\theta}^2\,F^\dagger(x_R)\;, \tag{A.21}$$

$$\mathcal{W} = W(x_L) + \sqrt{2}\,\theta\omega(x_L) + \theta^2\,G(x_L)\;, \tag{A.22}$$

$$\bar{\mathcal{W}} = W^\dagger(x_R) - \sqrt{2}\,\bar{\theta}\omega^\dagger(x_R) - \bar{\theta}^2\,G^\dagger(x_R)\;, \tag{A.23}$$

where $x_L^m = x^m - i\theta\gamma^m\bar{\theta}$ and $x_R^m = x^m + i\theta\gamma^m\bar{\theta}$.

A.3 $\mathcal{N} = 3$ Chern-Simons Yang-Mills on $\mathbb{R}^{1,2}$

Here we present the action of the $\mathcal{N} = 3$ Chern-Simons Yang-Mills theory on $\mathbb{R}^{1,2}$ with signature $(-,+,+)$. For further explanations we refer the reader to Section 5.3. The kinetic and mass terms are

$$\begin{aligned}\mathcal{S}_{\text{kin}} = \int d^3x\ \text{tr}\Big[&-\tfrac{1}{2g^2}F^{\mu\nu}F_{\mu\nu} + \kappa\,\epsilon^{\mu\nu\lambda}\big(A_\mu\partial_\nu A_\lambda + \tfrac{2i}{3}A_\mu A_\nu A_\lambda\big)\\ &-\tfrac{1}{2g^2}\hat{F}^{\mu\nu}\hat{F}_{\mu\nu} - \kappa\,\epsilon^{\mu\nu\lambda}\big(\hat{A}_\mu\partial_\nu\hat{A}_\lambda + \tfrac{2i}{3}\hat{A}_\mu\hat{A}_\nu\hat{A}_\lambda\big)\\ &-\mathcal{D}_\mu X^\dagger\mathcal{D}^\mu X + i\xi^\dagger\not{\mathcal{D}}\xi\\ &-\tfrac{1}{2g^2}\mathcal{D}_\mu\phi^a_b\mathcal{D}^\mu\phi^b_a - \tfrac{1}{2}\kappa^2g^2\,\phi^a_b\phi^b_a - \tfrac{1}{2g^2}\mathcal{D}_\mu\hat{\phi}^a_b\mathcal{D}^\mu\hat{\phi}^b_a - \tfrac{1}{2}\kappa^2g^2\,\hat{\phi}^a_b\hat{\phi}^b_a\\ &-\tfrac{i}{2g^2}\lambda^{ab}\not{\mathcal{D}}\lambda_{ab} - \tfrac{\kappa}{2}\,i\lambda^{ab}\lambda_{ba} - \tfrac{i}{2g^2}\hat{\lambda}^{ab}\not{\mathcal{D}}\hat{\lambda}_{ab} + \tfrac{\kappa}{2}\,i\hat{\lambda}^{ab}\hat{\lambda}_{ba}\Big]\;,\end{aligned} \tag{A.24}$$

and the interactions, which involve cubic and quartic scalar vertices, as well as Yukawa terms, are given by

$$\begin{aligned}\mathcal{S}_{\text{int}} = \int d^3x\ \text{tr}\Big[&- \kappa g^2\, X_a^\dagger \phi_b^a X^b + \kappa g^2\, X^a \hat{\phi}_a^b X_b^\dagger - i\xi_a^\dagger \phi_b^a \xi^b - i\xi^a \hat{\phi}_a^b \xi_b^\dagger \\ &+ \epsilon_{ac}\lambda^{cb} X^a \xi_b^\dagger - \epsilon^{ac}\lambda_{cb}\xi^b X_a^\dagger - \epsilon_{ac}\hat{\lambda}^{cb}\xi_b^\dagger X^a + \epsilon^{ac}\hat{\lambda}_{cb} X_a^\dagger \xi^b \\ &+ \tfrac{\kappa}{6}\phi_b^a[\phi_c^b, \phi_a^c] + \tfrac{\kappa}{6}\hat{\phi}_b^a[\hat{\phi}_c^b, \hat{\phi}_a^c] - \tfrac{1}{2g^2} i\lambda_{ab}[\phi_c^b, \lambda^{ac}] + \tfrac{1}{2g^2} i\hat{\lambda}_{ab}[\hat{\phi}_c^b, \hat{\lambda}^{ac}] \\ &- \tfrac{g^2}{4}(X\sigma_i X^\dagger)(X\sigma_i X^\dagger) - \tfrac{g^2}{4}(X^\dagger \sigma_i X)(X^\dagger \sigma_i X) \\ &- \tfrac{1}{2}(XX^\dagger)\phi_b^a \phi_a^b - \tfrac{1}{2}(X^\dagger X)\hat{\phi}_b^a \hat{\phi}_a^b - X_{Aa}^\dagger \phi_c^b X^{Aa} \hat{\phi}_b^c \\ &+ \tfrac{1}{8g^2}[\phi_b^a, \phi_d^c][\phi_a^b, \phi_c^d] + \tfrac{1}{8g^2}[\hat{\phi}_b^a, \hat{\phi}_d^c][\hat{\phi}_a^b, \hat{\phi}_c^d]\Big]\ .\end{aligned} \tag{A.25}$$

The supersymmetry variations with parameter $\varepsilon_{ab} = \varepsilon_i(\sigma_i)_{ab}$ in the **3** of $\text{SU}(2)_R$ read

$$\delta A_\mu = -\tfrac{i}{2}\varepsilon_{ab}\gamma_\mu \lambda^{ab}\ , \tag{A.26}$$

$$\begin{aligned}\delta\lambda^{ab} = &\tfrac{1}{2}\epsilon^{\mu\nu\lambda}F_{\mu\nu}\gamma_\lambda \varepsilon^{ab} - i\not{D}\phi_c^b \varepsilon^{ac} + \tfrac{i}{2}[\phi_c^b, \phi_d^c]\varepsilon^{ad} \\ &+ \kappa g^2\, i\phi_c^b \varepsilon^{ac} + g^2\, iX^a X_c^\dagger \varepsilon^{cb} - \tfrac{ig^2}{2}(XX^\dagger)\varepsilon^{ab}\ ,\end{aligned} \tag{A.27}$$

$$\delta\phi_b^a = -\varepsilon_{cb}\lambda^{ca} + \tfrac{1}{2}\delta_b^a \varepsilon_{cd}\lambda^{cd}\ , \tag{A.28}$$

$$\delta\hat{A}_\mu = -\tfrac{i}{2}\varepsilon_{ab}\gamma_\mu \hat{\lambda}^{ab}\ , \tag{A.29}$$

$$\begin{aligned}\delta\hat{\lambda}^{ab} = &\tfrac{1}{2}\epsilon^{\mu\nu\lambda}\hat{F}_{\mu\nu}\gamma_\lambda \varepsilon^{ab} + i\not{D}\hat{\phi}_c^b \varepsilon^{ac} + \tfrac{i}{2}[\hat{\phi}_c^b, \hat{\phi}_d^c]\varepsilon^{ad} \\ &+ \kappa g^2\, i\hat{\phi}_c^b \varepsilon^{ac} - g^2\, i\varepsilon^{bc} X_c^\dagger X^a + \tfrac{ig^2}{2}(X^\dagger X)\varepsilon^{ab}\ ,\end{aligned} \tag{A.30}$$

$$\delta\hat{\phi}_b^a = -\varepsilon_{cb}\hat{\lambda}^{ca} + \tfrac{1}{2}\delta_b^a \varepsilon_{cd}\hat{\lambda}^{cd}\ , \tag{A.31}$$

$$\delta X^{Aa} = -i\varepsilon_b^a \xi^{Ab}\ , \qquad \delta\xi^{Aa} = \not{D}X^{Ab}\varepsilon_b^a + \phi_b^a \varepsilon_c^b X^{Ac} + X^{Ac}\varepsilon_c^b \hat{\phi}_b^a\ , \tag{A.32}$$

$$\delta X_{Aa}^\dagger = -i\xi_{Ab}^\dagger \varepsilon_a^b\ , \qquad \delta\xi_{Aa}^\dagger = \not{D}X_{Ab}^\dagger \varepsilon_a^b + \hat{\phi}_a^b \varepsilon_b^c X_{Ac}^\dagger + X_{Ac}^\dagger \varepsilon_b^c \phi_a^b\ . \tag{A.33}$$

A.4 Monopole Spinor Harmonics

We define monopole spinor harmonics as eigenspinors of the Dirac operator on the sphere in a monopole background with magnetic charge q:

$$-i\not{D}_{\mathrm{S}}\Upsilon^{\pm}_{qjm} = \Delta^{\pm}_{jq}\Upsilon^{\pm}_{qjm} \tag{A.34}$$

with eigenvalues

$$\Delta^{\pm}_{jq} = \pm\tfrac{1}{2}\sqrt{(2j+1)^2 - q^2} \tag{A.35}$$

for $j = \frac{|q|-1}{2}, \frac{|q|+1}{2}, \ldots$ and $m = -j, -j+1, \ldots, j$. The spectrum is drawn in Figure 5.1(a) on page 154. These spinors also satisfy

$$\gamma^{\tau}\,\Upsilon^{\pm}_{qjm} = \Upsilon^{\mp}_{qjm}\ , \tag{A.36}$$

which couples modes with positive and negative eigenvalue. The lowest modes, $j = \frac{|q|-1}{2}$, which only exists for $q \neq 0$ are zero-modes and the corresponding $\Upsilon^{\pm}$-spinors are not independent. We introduce a special notation for them

$$\Upsilon^{0}_{qm} \equiv \frac{1}{\sqrt{2}}\Big(\Upsilon^{+}_{qjm} + \mathrm{sign}(q)\,\Upsilon^{-}_{qjm}\Big)_{j=\frac{|q|-1}{2}}\ . \tag{A.37}$$

Then (A.36) implies

$$\gamma^{\tau}\Upsilon^{0}_{qm} = \mathrm{sign}(q)\,\Upsilon^{0}_{qm}\ . \tag{A.38}$$

Further properties are the orthogonality

$$\int d\Omega\,\Upsilon^{0\dagger}_{qm}\,\gamma^{\tau}\,\Upsilon^{0}_{qm'} = i\delta_{mm'} \quad , \quad \int d\Omega\,\Upsilon^{\varepsilon\dagger}_{qjm}\,\gamma^{\tau}\,\Upsilon^{\varepsilon'}_{qj'm'} = i\delta^{\varepsilon\varepsilon'}\,\delta_{jj'}\,\delta_{mm'}\ , \tag{A.39}$$

and completeness relations

$$\sum_{m} \Upsilon^0_{qm}(\Omega)\Upsilon^{0\dagger}_{qm}(\Omega') + \sum_{jm\varepsilon} \Upsilon^{\varepsilon}_{qjm}(\Omega)\Upsilon^{\varepsilon\dagger}_{qjm}(\Omega') = i\gamma^\tau \delta^2(\Omega - \Omega') \ . \tag{A.40}$$

We also note the explicit expressions. The generalization of the spinor harmonics in [176] to non-zero monopole background is

$$\tilde{\Upsilon}^{+}_{qjm} = \sqrt{\tfrac{1+r_{qj}}{2}}\, \Omega^{+}_{qjm} + i\,\mathrm{sign}(q)\, \sqrt{\tfrac{1-r_{qj}}{2}}\, \Omega^{-}_{qjm}\ , \tag{A.41}$$

$$\tilde{\Upsilon}^{-}_{qjm} = \mathrm{sign}(q)\sqrt{\tfrac{1-r_{qj}}{2}}\, \Omega^{+}_{qjm} + i\sqrt{\tfrac{1+r_{qj}}{2}}\, \Omega^{-}_{qjm}\ , \tag{A.42}$$

with $r_{qj} = \sqrt{1 - \frac{q^2}{(2j+1)^2}}$ and

$$\Omega^{\pm}_{qjm} = \frac{(-)^{j-m}\left(\frac{i}{2}\right)^{j+\frac{1}{2}}\left(j+\frac{1}{2}\right)}{\sqrt{\Gamma\left(j+\frac{3}{2}-\frac{q}{2}\right)\Gamma\left(j+\frac{3}{2}+\frac{q}{2}\right)}} \sqrt{\frac{(j-m)!}{(j+m)!}}\, \frac{e^{i\left(m+\frac{q}{2}\right)\varphi}}{\sqrt{2\pi}} \times \tag{A.43}$$

$$\times \begin{pmatrix} \mp\sqrt{\mp i}\,(1-x)^{m_{-+}/2}\,(1+x)^{m_{+-}/2}\,\dfrac{d^{j+m}}{dx^{j+m}}\left((1-x)^{j_{+-}}(1+x)^{j_{-+}}\right) \\[2ex] \pm\sqrt{\pm i}\,(1-x)^{m_{++}/2}\,(1+x)^{m_{--}/2}\,\dfrac{d^{j+m}}{dx^{j+m}}\left((1-x)^{j_{--}}(1+x)^{j_{++}}\right) \end{pmatrix}$$

where

$$j_{\varepsilon_1\varepsilon_2} \equiv j + \varepsilon_1 \tfrac{1}{2} + \varepsilon_2 \tfrac{q}{2} \quad , \quad m_{\varepsilon_1\varepsilon_2} \equiv m + \varepsilon_1 \tfrac{1}{2} + \varepsilon_2 \tfrac{q}{2}\ . \tag{A.44}$$

We rotate to our basis of Dirac matrices by defining $\Upsilon^{\pm}_{qjm} = \frac{1}{\sqrt{2}}(1 - i\sigma^1)\tilde{\Upsilon}^{\pm}_{qjm}$.

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