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Unitary matrix integral for two-color QCD and the GSE-GUE crossover in random matrix theory

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ABSTRACT

We analytically evaluate a unitary matrix integral which appears in the low-energy limit of two-color QCD at finite chemical potential. The result is expressed as a pfaffian. We illustrate its application to the GSE-GUE crossover in random matrix theory.

1. Introduction

There are miscellaneous applications of matrix integrals in diverse fields of mathematics and theoretical physics (see [1,2] for reviews). In particular, they frequently appear in random matrix theory (RMT) [3–5]. Typically, when one wants to derive a joint probability distribution function of eigenvalues of a random matrix, one has to integrate out angular variables. This is done with the help of matrix integration formulae known in the literature. The list of most popular unitary matrix integrals includes the Brezin-Gross-Witten integral [6,7] (a.k.a. the Leutwyler-Smilga integral [8]), the Harish-Chandra-Itzykson-Zuber integral [9,10], and the Berezin-Karpelevich integral [11–13]. These integrals play deep roles in quantum chaos, disordered mesoscopic systems, Quantum Chromodynamics (QCD), quantum gravity, and lattice gauge theory.

In QCD, low-energy physics is governed by light pions owing to the spontaneous breaking of chiral symmetry. In the so-called ε-regime [14,8], exact zero modes of pions dominate the partition function and the path integral reduces to a finite-dimensional integral over a coset space. As is well known, there exist three classes of chiral symmetry breaking in QCD, depending on the representation of quarks [15]. In two-color QCD (and more generally, in QCD with quarks in a pseudoreal representation of the gauge group), the symmetry breaking pattern is $\mathrm{SU}(2N_f) \to \mathrm{Sp}(2N_f)$ where N_f is the number of Dirac fermions and $\mathrm{Sp}(N)$ is the unitary symplectic group [16,17]. The relevant degrees of freedom at low energy are expressed through a coset variable $U^T I U$ where $U \in \mathrm{U}(2N_f)$ and I is an antisymmetric matrix. When a quark chemical potential μ is added, the low-energy effective Lagrangian attains additional terms [17]. In this paper, we aim to evaluate the partition function of two-color QCD in the ε-regime when the chemical potential of each flavor is different, i.e., $(\mu_1, \cdots, \mu_{N_f})$ are all distinct. This case was not considered in [17]. We will analytically show that the partition function has a pfaffian form. We will also show that our integral formula has a useful application to the symmetry crossover between the Gaussian Symplectic Ensemble (GSE) and the Gaussian Unitary Ensemble (GUE) in RMT. Moreover, since $-IU^T IU$ is an element of the Circular Symplectic Ensemble in RMT [18], our result may have an application in this direction as well. The present paper can be viewed as a sequel to [19].

This paper is structured as follows. In Sec. 2 we summarize the main analytical result. In Sec. 3 some applications are illustrated. In Sec. 4 we give a derivation of our main integral formula. In Sec. 5 we discuss localization of the integral on critical points, in connection with the Duistermaat-Heckman theorem [20]. Finally in Sec. 6 we conclude. Some technical remarks on the volume of Lie groups are given in Appendix A.

2. Main result

Let N be an even positive integer and define an $N \times N$ antisymmetric matrix

$$I \equiv \mathrm{diag}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right). \tag{1}$$

Then, for an arbitrary Hermitian $N \times N$ matrix H with mutually distinct eigenvalues $\{e_k\}$ and an arbitrary nonzero $\beta \in \mathbb{C}$, we have

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$$\int_{\mathrm{U}(N)} \mathrm{d}U \exp\left[\beta \operatorname{Tr}(U^T I U H (U^T I U)^\dagger H^*)\right]$$
$$= \frac{\prod_{k=1}^{N/2}\Gamma(2k-1)}{(2\beta)^{N(N-2)/4}\Delta_N(e)} \operatorname*{Pf}_{1\le i,j\le N}\left[(e_i-e_j)\,\mathrm{e}^{2\beta e_i e_j}\right], \tag{2}$$

where $\mathrm{d}U$ denotes the normalized Haar measure, Pf denotes a pfaffian, and $\Delta_N(e) \equiv \prod_{i<j}(e_i - e_j)$ is the Vandermonde determinant.

We performed an intensive numerical check of this formula for $N \in \{2, 4, 6, 8, 10\}$ by estimating the left hand side of the formula with Monte Carlo methods. We used a Python library pfapack [21] for an efficient computation of a pfaffian.

As a side remark we note that similar pfaffian formulae are known for unitary matrix integrals considered in [22,19].

3. Applications

3.1. Two-color QCD

With a slight generalization of [17], we find that the leading static part of the partition function of low-energy massless two-color QCD with N_f flavors with nonzero chemical potential $\{\mu_f\}_{f=1}^{N_f}$ is given by

$$Z = \int_{\mathrm{U}(2N_f)} \mathrm{d}U \exp\left\{V_4 F^2 \operatorname{Tr}[U^T I U B (U^T I U)^\dagger B + B^2]\right\}, \tag{3}$$

where I is defined by (1), V_4 is the Euclidean spacetime volume, F is the pion decay constant, and

$$B \equiv \operatorname{diag}(\mu_1, \cdots, \mu_{N_f}, -\mu_1, \cdots, -\mu_{N_f}) \tag{4}$$

is the chemical potential matrix. The same coset integral also arises in the large-N limit of the chiral real Ginibre ensemble [23,24] which has exactly the same symmetry as two-color QCD. We remark that two-color QCD with non-degenerate chemical potentials has been investigated with chiral perturbation theory in p-regime in [25]. They considered a massive theory, while we focus on the strictly massless limit.

We introduce dimensionless variables

$$\begin{cases} \widehat{\mu}_f & \equiv \sqrt{V_4} F \mu_f \\ \widehat{\mu}_{N_f+f} & \equiv -\widehat{\mu}_f \end{cases} \quad \text{for } f = 1, 2, \cdots, N_f\,. \tag{5}$$

Then it is quite straightforward to apply (2) to (3), which yields

$$Z \propto \frac{\exp\left(2\sum_{f=1}^{N_f}\widehat{\mu}_f^2\right) \operatorname*{Pf}_{1\le f,g\le 2N_f}\left[(\widehat{\mu}_f-\widehat{\mu}_g)\,\mathrm{e}^{2\widehat{\mu}_f\widehat{\mu}_g}\right]}{\left[\prod_{1\le f<g\le N_f}(\widehat{\mu}_f-\widehat{\mu}_g)^2(\widehat{\mu}_f+\widehat{\mu}_g)^2\right]\prod_{f=1}^{N_f}\widehat{\mu}_f}. \tag{6}$$

This expression completely fixes the chemical potential dependence of the finite-volume partition function in the ε-regime.

3.2. GSE-GUE crossover

In the classical paper [26], a random matrix ensemble intermediate between GSE and GUE was solved by Mehta and Pandey. Their original approach was to study the random matrix

$$H = S + \alpha T\,, \tag{7}$$

where S is a Gaussian Hermitian real quaternion matrix and T is a Gaussian Hermitian matrix. As α grows from zero, the level statistics evolve from GSE to GUE, and the transition occurs at the scale $\alpha^2 \sim 1/N$ with N the matrix size. They derived the joint probability distribution function of eigenvalues of H by making use of the celebrated Harish-Chandra-Itzykson-Zuber integral [9,10].

An alternative approach to the GSE-GUE transition would be to investigate the random matrix

$$H = S + i\alpha A\,, \tag{8}$$

where A is a Gaussian anti-Hermitian real quaternion matrix.[1] Then we have

$$S^T = -ISI \quad \text{and} \quad A^T = IAI\,, \tag{9}$$

with I defined by (1). Hence the Gaussian weights of S and A may be cast into the form

$$\exp(-\operatorname{Tr} S^2 + \operatorname{Tr} A^2)$$
$$= \exp\left[-\frac{1+\alpha^2}{2\alpha^2}\operatorname{Tr}(H^2) - \frac{1-\alpha^2}{2\alpha^2}\operatorname{Tr}(HIH^T I)\right]. \tag{10}$$

Upon diagonalization $H = UEU^\dagger$ we end up with the unitary integral which exactly coincides with (2). By carrying out this integral we immediately arrive at the joint eigenvalue density derived by Mehta and Pandey [26]. Our integral thus provides a way to handle the transitive random matrix ensemble without recourse to the Harish-Chandra-Itzykson-Zuber integral.

4. Derivation of the formula

The derivation of our integration formula (2) proceeds in three steps. Throughout this section we will assume that N is even.

4.1. Step 1: the heat equation

We adopt the heat equation method, which has a long history [10]. Let us assume $t > 0$ and consider a function

$$z_N(t, H, U) \equiv \frac{1}{t^\alpha}\exp\left[-\frac{1}{t}\operatorname{Tr}(H - P^\dagger H^T P)^2\right] \tag{11}$$

where we wrote $P = U^T I U$ for brevity. Then

$$\frac{\partial}{\partial t} z_N(t,H,U) = \left[-\frac{\alpha}{t} + \frac{1}{t^2}\operatorname{Tr}(H - P^\dagger H^T P)^2\right] z_N(t,H,U)\,. \tag{12}$$

The Laplacian over Hermitian matrices is given by

$$\Delta_H \equiv \sum_i \frac{\partial^2}{\partial H_{ii}^2} + \frac{1}{2}\sum_{i<j}\left[\frac{\partial^2}{\partial(\operatorname{Re} H_{ij})^2} + \frac{\partial^2}{\partial(\operatorname{Im} H_{ij})^2}\right] \tag{13}$$
$$= \sum_{i,j}\frac{\partial}{\partial H_{ij}}\frac{\partial}{\partial H_{ji}}\,. \tag{14}$$

Then we readily obtain

$$\Delta_H z_N(t,H,U)$$
$$= \frac{1}{t^\alpha}\frac{\partial}{\partial H_{ij}}\frac{\partial}{\partial H_{ji}}\exp\left[-\frac{1}{t}\operatorname{Tr}(H - P^\dagger H^T P)^2\right] \tag{15}$$
$$= \left[-\frac{4N(N+1)}{t} + \frac{16}{t^2}\operatorname{Tr}(H - P^\dagger H^T P)^2\right] z_N(t,H,U)\,. \tag{16}$$

[1] Actually the ensemble (7) is essentially equivalent to (8), because the sum of two Gaussian self-dual matrices is again a Gaussian self-dual matrix with a modified variance.

Comparison of (12) and (16) tells that, if we set $\alpha = N(N+1)/4$, then

$$\left(\frac{\partial}{\partial t} - \frac{1}{16}\Delta_H\right) z_N(t,H,U) = 0\,. \tag{17}$$

It follows that

$$Z_N(t,H) \equiv \int_{\mathrm{U}(N)} \mathrm{d}U\, z_N(t,H,U) \tag{18}$$

$$= \frac{1}{t^\alpha}\exp\left(-\frac{2}{t}\operatorname{Tr} H^2\right)\int_{\mathrm{U}(N)} \mathrm{d}U \exp\left[\frac{2}{t}\operatorname{Tr}(PHP^\dagger H^T)\right] \tag{19}$$

also satisfies the same differential equation as $z_N(t,H,U)$. It is easy to confirm that $Z_N(t,H)$ depends on H only through its eigenvalues $\{e_1,\cdots,e_N\}$. It is very important to note that $Z_N(t,H)$ has translational invariance, i.e., $Z_N(t,H) = Z_N(t,H+a\mathbb{1}_N)$ for an arbitrary $a \in \mathbb{R}$, which can be easily shown based on the definition (19). Therefore $Z_N(t,H)$ depends on $\{e_k\}$ only through the differences $\{e_k - e_\ell\}$.

Let us transform the Laplacian into the polar coordinate [27]

$$\Delta_H = \frac{1}{\Delta_N(e)}\sum_{k=1}^{N}\frac{\partial^2}{\partial e_k^2}\Delta_N(e) + \Delta_O \tag{20}$$

where O denote the angular variables. Then

$$\left(\frac{\partial}{\partial t} - \frac{1}{16}\sum_{k=1}^{N}\frac{\partial^2}{\partial e_k^2}\right)[\Delta_N(e)Z_N(t,H)] = 0\,. \tag{21}$$

To find out the basic building block of $Z_N(t,H)$, we wish to look at the $N=2$ case in the next subsection.

4.2. Step 2: the $N=2$ case

For $N=2$, using the fact that $U^T I U = I$ for any $U \in \mathrm{SU}(2)$, we readily find

$$Z_2(t,H) = t^{-3/2}\exp\left[-\frac{2}{t}(e_1-e_2)^2\right] \tag{22}$$

$$=: Z_2(t,e_1,e_2)\,. \tag{23}$$

From (21), we know that (23) fulfills the equation

$$\left[\frac{\partial}{\partial t} - \frac{1}{16}\left(\frac{\partial^2}{\partial e_1^2} + \frac{\partial^2}{\partial e_2^2}\right)\right](e_1-e_2)Z_2(t,e_1,e_2) = 0\,. \tag{24}$$

Then one can easily find a solution for $N=4$ by treating Z_2 as a building block: for example,

$$\left[\frac{\partial}{\partial t} - \frac{1}{16}\sum_{k=1}^{4}\frac{\partial^2}{\partial e_k^2}\right](e_1-e_2)Z_2(t,e_1,e_2)(e_3-e_4)Z_2(t,e_3,e_4) = 0\,. \tag{25}$$

However, the solution $\Delta_N(e)Z_N(t,H)$ that we wish to obtain must be antisymmetric under the exchange of any two e_k's. Hence we naturally arrive at the Pfaffian

$$\Delta_N(e)Z_N(t,H) = \widetilde{C}_N \operatorname*{Pf}_{1\le i,j\le N}\left[(e_i-e_j)Z_2(t,e_i,e_j)\right], \tag{26}$$

where $\widetilde{C}_N$ is a not yet determined constant. By construction, $\widetilde{C}_2 = 1$. Notice that this Pfaffian form is consistent with the translational invariance of $Z_N(t,H)$. To fix the normalization uniquely, one has to examine the boundary condition imposed at $t=+0$.

4.3. Step 3: saddle point analysis

To determine the normalization of (26), let us perform the saddle point approximation of

$$Z_N(t,H) = \frac{1}{t^\alpha}\exp\left(-\frac{2}{t}\operatorname{Tr}H^2\right) \times \int_{\mathrm{U}(N)} \mathrm{d}U \exp\left[-\frac{2}{t}\operatorname{Tr}(U^T I U E U^\dagger I U^* E)\right] \tag{27}$$

for $t \to +0$, with $E = \mathrm{diag}(e_1,e_2,\cdots,e_N)$. For technical simplicity, let us assume

$$e_1 > e_2 > \cdots > e_N\,. \tag{28}$$

Then, apparently $U = \mathbb{1}_N$ is the dominant saddle point. A closer inspection of the exponent of (27) shows that, in fact, any U of the form SV with $S \in \mathrm{Sp}(N)$ and $V = \mathrm{diag}(e^{i\theta_1}\mathbb{1}_2,\cdots,e^{i\theta_{N/2}}\mathbb{1}_2) \in \mathrm{U}(1)^{N/2}$ is also a saddle point,[2] where $\mathrm{Sp}(N)$ is the unitary symplectic group. [Our convention is such that $\mathrm{Sp}(2) \cong \mathrm{SU}(2)$.] Hence the saddle point manifold appears to be given by $\mathrm{U}(1)^{N/2}\times\mathrm{Sp}(N)$. However there is a subtlety: the $2^{N/2}$ elements $\mathrm{diag}(\pm\mathbb{1}_2,\cdots,\pm\mathbb{1}_2)$ belong to both $\mathrm{U}(1)^{N/2}$ and $\mathrm{Sp}(N)$. To avoid duplication, the correct saddle point manifold must be considered as $[\mathrm{U}(1)^{N/2}\times\mathrm{Sp}(N)]/\{\pm1\}^{N/2}$. Thus the manifold of massive modes is given by

$$\mathcal{M} = \frac{\mathrm{U}(N)}{[\mathrm{U}(1)^{N/2}\times\mathrm{Sp}(N)]/\{\pm1\}^{N/2}}\,. \tag{29}$$

Using $\dim[\mathrm{Sp}(N)] = N(N+1)/2$ we find

$$\dim(\mathcal{M}) = N^2 - \frac{N}{2} - \frac{N(N+1)}{2} = \frac{N(N-2)}{2}\,. \tag{30}$$

Let X be a generator of $\mathrm{U}(N)$ that corresponds to a massive mode. Then X is Hermitian and satisfy $X^T I = IX$. Such a matrix can be most conveniently expressed through a quaternion defined as

$$q^a = (\mathbb{1}_2, i\sigma_1, i\sigma_2, i\sigma_3)\,. \tag{31}$$

Let us use indices with a bar $(\overline{p},\overline{q},\cdots)$ that run from 1 to $N/2$. Then the generators for massive modes are given by $\left\{X^a_{\overline{p}\,\overline{q}}\right\}^{a=0,1,2,3}_{1\le\overline{p}<\overline{q}\le N/2}$, where

$$(X^a_{\overline{p}\,\overline{q}})_{\overline{r}\,\overline{s}} = \frac{1}{\sqrt{2}}\left[\delta_{\overline{p}\,\overline{r}}\delta_{\overline{q}\,\overline{s}}q^a + \delta_{\overline{p}\,\overline{s}}\delta_{\overline{q}\,\overline{r}}(q^a)^\dagger\right]. \tag{32}$$

They are normalized as $\operatorname{Tr}[X^a_{\overline{p}\,\overline{q}}X^b_{\overline{r}\,\overline{s}}] = 2\delta_{ab}\delta_{\overline{p}\,\overline{r}}\delta_{\overline{q}\,\overline{s}}$. A quick counting of degrees of freedom shows

$$\dim(X) = \frac{N}{2}\left(\frac{N}{2}-1\right)\frac{1}{2}\times 4 = \frac{N(N-2)}{2}\,, \tag{33}$$

which matches (30) as it should.

For later use, we also define

$$\widetilde{E} \equiv -IEI \tag{34}$$

$$= \mathrm{diag}(e_2,e_1,e_4,e_3,\cdots,e_N,e_{N-1})\,. \tag{35}$$

An element of the massive manifold $U \in \mathcal{M}$ can be parametrized as

[2] Note that another diagonal Abelian group $\mathrm{diag}(e^{i\theta_1\sigma_3},\cdots,e^{i\theta_{N/2}\sigma_3}) \cong \mathrm{U}(1)^{N/2}$ is a subgroup of $\mathrm{Sp}(N)$.

$$U=\exp\left(i\sum_{a=0}^{3}\sum_{\overline{p}<\overline{q}}\theta^a_{\overline{p}\,\overline{q}}X^a_{\overline{p}\,\overline{q}}\right)\equiv e^{i\theta X}\,. \tag{36}$$

Then, using $U^TI=IU$, we obtain

$$\begin{aligned}&\mathrm{Tr}(U^TIUEU^\dagger IU^*E)\\ =&-\mathrm{Tr}[U^2E(U^\dagger)^2\widetilde{E}] \qquad (37)\\ =&-\mathrm{Tr}(E\widetilde{E})+4\Big\{\mathrm{Tr}[(\theta X)^2E\widetilde{E}]-\mathrm{Tr}[(\theta X)E(\theta X)\widetilde{E}]\Big\}+O(\theta^3)\,. \qquad (38)\end{aligned}$$

A tedious but straightforward calculation yields

$$\mathrm{Tr}[(\theta X)^2E\widetilde{E}]=\sum_a\sum_{\overline{p}<\overline{q}}(e_{2\overline{p}-1}e_{2\overline{p}}+e_{2\overline{q}-1}e_{2\overline{q}})(\theta^a_{\overline{p}\,\overline{q}})^2 \tag{39}$$

and

$$\begin{aligned}\mathrm{Tr}[(\theta X)E(\theta X)\widetilde{E}]=\sum_{\overline{p}<\overline{q}}\Big\{&\left(e_{2\overline{p}-1}e_{2\overline{q}}+e_{2\overline{p}}e_{2\overline{q}-1}\right)\left[(\theta^0_{\overline{p}\,\overline{q}})^2+(\theta^3_{\overline{p}\,\overline{q}})^2\right]\\ &+\left(e_{2\overline{p}-1}e_{2\overline{q}-1}+e_{2\overline{p}}e_{2\overline{q}}\right)\left[(\theta^1_{\overline{p}\,\overline{q}})^2+(\theta^2_{\overline{p}\,\overline{q}})^2\right]\Big\}\,. \end{aligned}\tag{40}$$

Therefore

$$\begin{aligned}Z_N(t,H)\approx&\frac{1}{t^\alpha}\frac{1}{\mathrm{Vol}(\mathcal{M})}\exp\left(-\frac{2}{t}\sum_{\overline{p}=1}^{N/2}(e_{2\overline{p}-1}-e_{2\overline{p}})^2\right)\\ &\times\int_{-\infty}^{\infty}\prod_a\prod_{\overline{p}<\overline{q}}\mathrm{d}\theta^a_{\overline{p}\,\overline{q}}\exp\left(-\theta\cdot\mathcal{G}\cdot\theta\right)\,,\end{aligned}\tag{41}$$

where

$$\begin{aligned}\theta\cdot\mathcal{G}\cdot\theta=\frac{8}{t}\sum_{\overline{p}<\overline{q}}\Big\{&(e_{2\overline{p}-1}-e_{2\overline{q}-1})(e_{2\overline{p}}-e_{2\overline{q}})\left[(\theta^0_{\overline{p}\,\overline{q}})^2+(\theta^3_{\overline{p}\,\overline{q}})^2\right]\\ &+(e_{2\overline{p}-1}-e_{2\overline{q}})(e_{2\overline{p}}-e_{2\overline{q}-1})\left[(\theta^1_{\overline{p}\,\overline{q}})^2+(\theta^2_{\overline{p}\,\overline{q}})^2\right]\Big\}.\end{aligned}\tag{42}$$

Note that the coefficients of θ^2 are positive definite, given (28). The Gaussian integral in the second line of (41) then yields

$$\begin{aligned}&\frac{\left(\frac{\pi t}{8}\right)^{\frac{\dim(\mathcal{M})}{2}}}{\prod_{\overline{p}<\overline{q}}(e_{2\overline{p}-1}-e_{2\overline{q}-1})(e_{2\overline{p}}-e_{2\overline{q}})(e_{2\overline{p}-1}-e_{2\overline{q}})(e_{2\overline{p}}-e_{2\overline{q}-1})}\\ &=\left(\frac{\pi t}{8}\right)^{\frac{\dim(\mathcal{M})}{2}}\frac{\prod_{\overline{p}=1}^{N/2}(e_{2\overline{p}-1}-e_{2\overline{p}})}{\Delta_N(e)}\,.\end{aligned}\tag{43}$$

Next, we compute the volume $\mathrm{Vol}(\mathcal{M})$. Using the volume formulae [28–31] (see Appendix for detailed comments on these formulae)

$$\mathrm{Vol}[\mathrm{U}(N)]=\frac{2^{N/2}\pi^{N(N+1)/2}}{\prod_{k=1}^{N}\Gamma(k)} \tag{44}$$

$$\mathrm{Vol}[\mathrm{Sp}(N)]=\frac{2^{N^2/4}\pi^{N(N+2)/4}}{\prod_{k=1}^{N/2}\Gamma(2k)} \tag{45}$$

we get

$$\begin{aligned}\mathrm{Vol}(\mathcal{M})&=\frac{\mathrm{Vol}[\mathrm{U}(N)]}{(2\pi)^{N/2}\mathrm{Vol}[\mathrm{Sp}(N)]\times 2^{-N/2}} \qquad (46)\\ &=\frac{2^{-N(N-2)/4}\pi^{N(N-2)/4}}{\prod_{k=1}^{N/2}\Gamma(2k-1)}\,. \qquad (47)\end{aligned}$$

Substituting (43) and (47) into (41) and recalling $\alpha=N(N+1)/4$, we obtain

$$\begin{aligned}Z_N(t,H)\approx&\,2^{-N(N-2)/2}\left\{\prod_{k=1}^{N/2}\Gamma(2k-1)\right\}t^{-3N/4}\\ &\times\frac{\prod_{\overline{p}=1}^{N/2}\left\{(e_{2\overline{p}-1}-e_{2\overline{p}})\exp\left[-\frac{2}{t}(e_{2\overline{p}-1}-e_{2\overline{p}})^2\right]\right\}}{\Delta_N(e)}\,.\end{aligned}\tag{48}$$

Next we turn to (26). Due to the ordering (28), we observe the asymptotic behavior

$$\begin{aligned}Z_N(t,H)\approx&\,\widetilde{C}_Nt^{-3N/4}\\ &\times\frac{\prod_{\overline{p}=1}^{N/2}\left\{(e_{2\overline{p}-1}-e_{2\overline{p}})\exp\left[-\frac{2}{t}(e_{2\overline{p}-1}-e_{2\overline{p}})^2\right]\right\}}{\Delta_N(e)}\,.\end{aligned}\tag{49}$$

Matching (48) and (49) yields

$$\widetilde{C}_N=2^{-N(N-2)/2}\prod_{k=1}^{N/2}\Gamma\left(2k-1\right)\,. \tag{50}$$

Finally, introducing $\beta\equiv 2/t$, we find that (26) together with (50) is equivalent to (2). Thus we have proved (2) for $\beta>0$. Since both sides of (2) are analytic in β, we conclude that (2) is valid for complex β, too. This completes the proof.

5. Exactness of the saddle-point approximation

The Duistermaat-Heckman theorem [20] states that there are special integrals that receive contributions only from critical points of the integrand; in other words, the saddle-point approximation becomes *exact* for such integrals. A trivial example is given by

$$\int_0^\pi \mathrm{d}\theta\ \sin\theta\ e^{a\cos\theta}=\frac{e^a-e^{-a}}{a}\,, \tag{51}$$

where the first term is associated with $\theta=0$ and the second term with $\theta=\pi$. The localization of the integral at these critical points can be understood from the fact that the integrand is a *total derivative*: $\mathrm{d}\theta\ \sin\theta\ e^{a\cos\theta}=-\frac{1}{a}\mathrm{d}(e^{a\cos\theta})$. A far more nontrivial and beautiful example for which localization holds is the Harish-Chandra-Itzykson-Zuber integral, as pointed out by [32].

Derivation of (2) in the last section clearly illustrates that the localization property does hold for our unitary integral as well. The purpose of this section is to work out explicitly, for the simplest nontrivial case of $N=4$, how such localization can happen.

To begin with, let us recall the fact that a 4×4 skew-symmetric unitary matrix U^TIU can be expressed as [33]

$$U^TIU=\sum_{i=1}^{6}n_i\Sigma_i\,, \tag{52}$$

where Σ_i are the basis matrices that are explicitly given in Sec. 3.1 of [33], while $\vec{n}$ is a six-dimensional real unit vector ($\vec{n}^2=1$). Substituting (52) and $H=\mathrm{diag}(e_1,e_2,e_3,e_4)$ into the left hand side of (2) we obtain

$$\int_{S^5} \mathrm{d}\vec{n}\ \mathrm{e}^{2\beta[(e_1e_2+e_3e_4)(n_2^2+n_4^2)+(e_1e_4+e_2e_3)(n_3^2+n_5^2)+(e_1e_3+e_2e_4)(n_1^2+n_6^2)]}\,. \tag{53}$$

Next, we introduce three sets of polar coordinates for (n_2,n_4), (n_3,n_5) and (n_1,n_6) respectively, with radial variables

$$a\equiv\sqrt{n_2^2+n_4^2},\ \ b\equiv\sqrt{n_3^2+n_5^2},\ \ c\equiv\sqrt{n_1^2+n_6^2}. \tag{54}$$

The angular variables can be integrated out trivially, yielding

$$\int_{-\infty}^{\infty}\mathrm{d}a\int_{-\infty}^{\infty}\mathrm{d}b\int_{-\infty}^{\infty}\mathrm{d}c\ |abc|\ \delta(a^2+b^2+c^2-1)\ \mathrm{e}^{\gamma_1a^2+\gamma_2b^2+\gamma_3c^2} \tag{55}$$

with $\gamma_1=2\beta(e_1e_2+e_3e_4)$, $\gamma_2=2\beta(e_1e_4+e_2e_3)$, and $\gamma_3=2\beta(e_1e_3+e_2e_4)$. Now the domain of integration is S^2 and it helps to introduce standard polar coordinates on S^2, which leads to

$$\begin{aligned}&\int_0^{\pi}\mathrm{d}\theta\ \sin\theta\int_{-\pi}^{\pi}\mathrm{d}\phi\ \left|\sin^2\theta\cos\theta\sin\phi\cos\phi\right|\ \mathrm{e}^{\gamma_1\cos^2\theta+\gamma_2\sin^2\theta\cos^2\phi+\gamma_3\sin^2\theta\sin^2\phi}\\ &=8\int_0^{\pi/2}\mathrm{d}\theta\int_0^{\pi/2}\mathrm{d}\phi\ \sin^3\theta\cos\theta\sin\phi\cos\phi\ \mathrm{e}^{\gamma_1\cos^2\theta+\gamma_2\sin^2\theta\cos^2\phi+\gamma_3\sin^2\theta\sin^2\phi}\,.\end{aligned} \tag{56}$$

A careful inspection of the integrand reveals that it is a total derivative:

$$\begin{aligned}&\sin^3\theta\cos\theta\sin\phi\cos\phi\ \mathrm{e}^{\gamma_1\cos^2\theta+\gamma_2\sin^2\theta\cos^2\phi+\gamma_3\sin^2\theta\sin^2\phi}\\ &=-\frac{1}{2}\frac{\partial}{\partial\theta}\frac{\partial}{\partial\phi}\left(\frac{\mathrm{e}^{\gamma_1\cos^2\theta+\gamma_2\sin^2\theta\cos^2\phi+\gamma_3\sin^2\theta\sin^2\phi}}{(\gamma_2-\gamma_3)[-2\gamma_1+\gamma_2+\gamma_3+(\gamma_2-\gamma_3)\cos(2\phi)]}\right).\end{aligned} \tag{57}$$

As a result, the integral localizes on the edges of the interval, leading to a relatively simple expression

$$\begin{aligned}(56)=2\bigg[&\frac{\mathrm{e}^{\gamma_1}}{(\gamma_1-\gamma_2)(\gamma_1-\gamma_3)}+\frac{\mathrm{e}^{\gamma_2}}{(\gamma_1-\gamma_2)(\gamma_3-\gamma_2)}\\ &+\frac{\mathrm{e}^{\gamma_3}}{(\gamma_1-\gamma_3)(\gamma_2-\gamma_3)}\bigg].\end{aligned} \tag{58}$$

The interpretation of each term is straightforward. The first term is associated with $\theta=0$, the second term with $(\theta,\phi)=(\pi/2,0)$, and the third term with $(\theta,\phi)=(\pi/2,\pi/2)$. As expected, (58) is consistent with (2). This elementary derivation clearly shows why our unitary group integral is localized on the critical points of the integrand. The key to this argument is the total derivative formula (57). We believe that this total derivative property extends to higher N, although showing this explicitly is beyond the scope of this section.

By contrast, for integrals over *symmetric* unitary matrices [19], localization does not take place, which can be confirmed explicitly for e.g., $N=2$.

6. Conclusions and outlook

In this paper, we obtained a pfaffian formula for a unitary matrix integral, and used it to evaluate the ε-regime partition function of two-color QCD. It was also shown that the classical result for the transitive ensemble between GSE and GUE can be reproduced with our formula.

The present work can be expanded in several directions.

- ✓ Our unitary matrix integral (2) is a special case of

$$\int_{\mathrm{U}(N)}\mathrm{d}U\exp\left[\mathrm{Tr}(U^TAUHU^\dagger AU^*H^*)\right] \tag{59}$$

  where H is a Hermitian matrix and A is a real antisymmetric matrix. Putting $A=I$ reproduces our integral. It is easy to see that this integral is a function of the eigenvalues of A and H only. This integral (59) deserves further study.
- ✓ In Sec. 3.1 we have only studied the partition function of massless two-color QCD. The extension of our formula to the massive case is an interesting open problem.
- ✓ An extension of our formula to supergroups may be possible, by following the methods of [34–36].
- ✓ Our formula is valid only for Hermitian H. It is tempting to ask whether one can relax this constraint and generalize the formula to an arbitrary complex matrix. We speculate that methods developed in [12,37] may be helpful in this regard.
- ✓ Our unitary matrix integral (2) is a special case of

$$\int_{\mathrm{U}(N)}\mathrm{d}U\exp\left[\mathrm{Tr}(U^TIUH_1(U^TIU)^\dagger H_2)\right] \tag{60}$$

  where H_1 and H_2 are Hermitian matrices. Extension of (2) to (60) is an open problem.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A. Volumes of the unitary and symplectic groups

There are many conventions on the volume of Lie groups in the literature, and one has to be extremely careful about their usage. In this paper we performed a saddle point analysis of integrals using generators normalized as $\mathrm{Tr}(X_AX_B)=2\delta_{AB}$. Thus it was mandatory to calculate volumes of groups in a way consistent with this normalization. We argue that (44) and (45) used in the main text are indeed the correct formulae that conform to our normalization. In Appendix A.1 we test our formulae by comparing them with exact integration formulae over Lie groups. In Appendix A.2 the derivation of (44) and (45) is given.

A.1. Comparison with exact integration formulae

A.1.1. Unitary group

There is a famous integration formula [8]

$$\mathcal{I}_N^{(\mathrm{U})}(x):=\int_{\mathrm{U}(N)}\mathrm{d}U\ \exp\left[\frac{x}{2}\mathrm{Tr}(U+U^\dagger)\right] \tag{A.1}$$

$$=\det_{1\le i,j\le N}\left[I_{i-j}(x)\right] \tag{A.2}$$

where $I_n(x)$ is the modified Bessel function of the first kind. Using asymptotic expansions of $I_n(x)$ at $x\gg1$ we find

$$\mathcal{I}_N^{(\mathrm{U})}(x)\approx\xi_N\frac{\mathrm{e}^{Nx}}{x^{N^2/2}} \tag{A.3}$$

with

$$\xi_1=\frac{1}{\sqrt{2\pi}},\quad \xi_2=\frac{1}{2\pi},\quad \xi_3=\frac{1}{\sqrt{2}\pi^{3/2}},\\ \xi_4=\frac{3}{\pi^2},\quad \xi_5=\frac{36\sqrt{2}}{\pi^{5/2}}\,. \tag{A.4}$$

Next, let us try the saddle-point approximation of (A.1) directly. Assuming the normalization $\mathrm{Tr}(X_AX_B)=2\delta_{AB}$ for the generators of U(N), we substitute $U=\mathrm{e}^{i\theta X}$ into (A.1) and get, for $x\gg 1$,

$$\mathcal{I}_N^{(\mathrm{U})}(x)\approx \mathrm{e}^{Nx}\int_{\mathrm{U}(N)}\mathrm{d}U\;\mathrm{e}^{-\frac{x}{2}\theta^2\,\mathrm{Tr}(X^2)} \tag{A.5}$$

$$\approx \frac{\mathrm{e}^{Nx}}{\mathrm{Vol}[\mathrm{U}(N)]}\int_{-\infty}^{\infty}\prod_A \mathrm{d}\theta_A\;\mathrm{e}^{-x\theta_A^2} \tag{A.6}$$

$$=\frac{\prod_{k=1}^N\Gamma(k)}{(2\pi)^{N/2}}\frac{\mathrm{e}^{Nx}}{x^{N^2/2}}\,, \tag{A.7}$$

where (44) was used in the last step. For $N\in\{1,2,3,4,5\}$, (A.7) reproduces (A.4).

A.1.2. Unitary symplectic group

There is an integration formula over a unitary symplectic group [38] (for even N)

$$\mathcal{I}_N^{(\mathrm{Sp})}(x):=\int_{\mathrm{Sp}(N)}\mathrm{d}U\;\mathrm{e}^{x\,\mathrm{Tr}\,U} \tag{A.8}$$

$$=\det_{1\le i,j\le N/2}\left[I_{i-j}(2x)-I_{N+2-i-j}(2x)\right]. \tag{A.9}$$

Using asymptotic expansions of $I_n(x)$ at $x\gg 1$, we find

$$\mathcal{I}_N^{(\mathrm{Sp})}(x)\approx \zeta_N\frac{\mathrm{e}^{Nx}}{x^{N(N+1)/4}} \tag{A.10}$$

with

$$\zeta_2=\frac{1}{2\sqrt{\pi}},\quad \zeta_4=\frac{3}{8\pi},\quad \zeta_6=\frac{45}{32\pi^{3/2}},\quad \zeta_8=\frac{14175}{256\pi^2}\,. \tag{A.11}$$

Next, let us try the saddle-point approximation of (A.8) directly. Assuming the normalization $\mathrm{Tr}(X_AX_B)=2\delta_{AB}$ for the generators of Sp(N), we substitute $U=\mathrm{e}^{i\theta X}$ into (A.8) and get, for $x\gg 1$,

$$\mathcal{I}_N^{(\mathrm{Sp})}(x)\approx\frac{\mathrm{e}^{Nx}}{\mathrm{Vol}[\mathrm{Sp}(N)]}\int_{-\infty}^{\infty}\prod_A\mathrm{d}\theta_A\;\mathrm{e}^{-x\theta_A^2} \tag{A.12}$$

$$=\frac{\mathrm{e}^{Nx}}{\mathrm{Vol}[\mathrm{Sp}(N)]}\left(\sqrt{\frac{\pi}{x}}\right)^{N(N+1)/2} \tag{A.13}$$

$$=\frac{\prod_{k=1}^{N/2}\Gamma(2k)}{2^{N^2/4}\pi^{N/4}}\frac{\mathrm{e}^{Nx}}{x^{N(N+1)/4}}\,, \tag{A.14}$$

where (45) was used in the last step. For $N\in\{2,4,6,8\}$, (A.14) reproduces (A.11).

These explicit computations for unitary and symplectic groups vindicate the validity of (44) and (45) under the normalization condition $\mathrm{Tr}(X_AX_B)=2\delta_{AB}$.

A.2. Derivation of the volume formulae

Let us begin with a brief review of the derivation of group volume formulae in Sec. V of [28]. Since the classical groups SO(N), U(N) and Sp(N) can all be viewed as a unitary group acting on the real/complex/quaternion vector space, we denote (following [28]) them as U(n; f) where f is the number of degrees of freedom per matrix element. For $f=1,2$ and 4 it is SO(n), U(n) and Sp($2n$), respectively.

An important observation is that there is a one-to-one correspondence between the coset

$$\frac{\mathrm{U}(n;f)}{\mathrm{U}(n-1;f)} \tag{A.15}$$

and an $(nf-1)$-dimensional sphere embedded in $\mathbb{R}^{nf}$. This can be seen from the fact that the coset acts on the polar vector $(0,\cdots,0,1)$ as

$$\left(\begin{array}{ccc|c} & & & \theta_1\\ \bullet & & & \theta_2\\ & \bullet & & \vdots\\ & & \bullet & \theta_{n-1}\\ \hline \bullet & \bullet & \bullet & \theta_n\end{array}\right)\begin{pmatrix}0\\0\\\vdots\\0\\\hline 1\end{pmatrix}=\begin{pmatrix}\theta_1\\\theta_2\\\vdots\\\theta_{n-1}\\\hline\theta_n\end{pmatrix} \tag{A.16}$$

$$\theta\in\begin{cases}\mathbb{R}: & \mathrm{SO}(n)\\ \mathbb{C}: & \mathrm{U}(n)\\ \mathbb{H}: & \mathrm{Sp}(2n)\end{cases} \tag{A.17}$$

$$\theta_1^*\theta_1+\theta_2^*\theta_2+\cdots+\theta_{n-1}^*\theta_{n-1}+\theta_n^*\theta_n=1\,. \tag{A.18}$$

Thus

$$\mathrm{Vol}[\mathrm{U}(n;f)]=\prod_{k=1}^n\mathrm{Vol}\left[\frac{\mathrm{U}(k;f)}{\mathrm{U}(k-1;f)}\right] \tag{A.19}$$

$$=\prod_{k=1}^n\mathrm{Vol}(S^{kf-1}\subset\mathbb{R}^{kf}) \tag{A.20}$$

$$=\prod_{k=1}^n\frac{2\pi^{kf/2}}{\Gamma(kf/2)} \tag{A.21}$$

$$=\frac{2^n\pi^{n(n+1)f/4}}{\prod_{k=1}^n\Gamma(kf/2)}\,. \tag{A.22}$$

Unfortunately this formula cannot be immediately used for our saddle-point analysis because it does not conform to our normalization condition $\mathrm{Tr}(X_AX_B)=2\delta_{AB}$ of generators. Let us explain how to amend (A.22) for $f=2$ and 4, which are the cases of interest.

We begin with $f=2$. In real basis, the map (A.16) for $n=2$ reads, in the vicinity of an identity map, as

$$\begin{pmatrix}* & \theta_1+i\theta_2\\ * & i\theta_3+\theta_0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}\theta_1+i\theta_2\\ i\theta_3+\theta_0\end{pmatrix} \tag{A.23}$$

where $\theta_k\in\mathbb{R}$ and $\theta_0=\sqrt{1-\sum_{k=1}^3\theta_k^2}$. This amounts to the parametrization of a group element $U=\mathrm{e}^{i\theta X}$ with generators

$$X_1=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\; X_2=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\; X_3=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}. \tag{A.24}$$

X_3 does not satisfy our normalization condition. We should instead use the generators $\{X_1,X_2,\sqrt{2}X_3\}$, or use the coordinates $\{\widetilde{\theta}\}:=\{\theta_0,\theta_1,\theta_2,\theta_3/\sqrt{2}\}$ that satisfy $\widetilde{\theta}_0^2+\widetilde{\theta}_1^2+\widetilde{\theta}_2^2+(\sqrt{2}\widetilde{\theta}_3)^2=1$. The volume then shrinks by a factor $1/\sqrt{2}$. This factor appears n times in the product (A.19), so we must multiply (A.22) by $2^{-n/2}$. This way we arrive at (44).

Next we turn to $f=4$. We again use the notation (31) for quaternions. In real basis, the map (A.16) for $n=2$ reads, in the vicinity of an identity map, as

$$\begin{pmatrix} * & \theta_a q^a \\ * & \varphi_a q^a \end{pmatrix} \begin{pmatrix} 0 \\ q^0 \end{pmatrix} = \begin{pmatrix} \theta_a q^a \\ \varphi_a q^a \end{pmatrix} \tag{A.25}$$

with $\theta_a^2 + \varphi_a^2 = 1$ and $\varphi_0 \approx 1$. This amounts to the parametrization $U = e^{i\theta_a X_a + i\varphi_k Y_k}$ ($a \in \{0,1,2,3\}$ and $k \in \{1,2,3\}$) with

$$X_a = \begin{pmatrix} 0 & -iq^a \\ (-iq^a)^\dagger & 0 \end{pmatrix}, \tag{A.26}$$

$$Y_k = \begin{pmatrix} 0 & 0 \\ 0 & -iq_k \end{pmatrix}. \tag{A.27}$$

$\{Y_k\}$ satisfy the normalization $\mathrm{Tr}(Y_k Y_\ell) = 2\delta_{k\ell}$, so they need no rescaling. On the other hand, $\{X_a\}$ satisfy $\mathrm{Tr}(X_a X_b) = 4\delta_{ab}$, which means we should use $\{X_a/\sqrt{2}\}$ as generators, or use the coordinates

$$(\widetilde{\theta}, \varphi) := (\sqrt{2}\theta_0, \sqrt{2}\theta_1, \sqrt{2}\theta_2, \sqrt{2}\theta_3, \varphi_0, \varphi_1, \varphi_2, \varphi_3) \tag{A.28}$$

that satisfy $(\widetilde{\theta}_a/\sqrt{2})^2 + \varphi_a^2 = 1$. Thus the volume swells by a factor $(\sqrt{2})^4$. This observation easily generalizes to arbitrary n: for the map (A.16) with general n, the number of generators that must be rescaled is obviously $4(n-1)$, so the volume swells by $(\sqrt{2})^{4(n-1)}$. This means that (A.22) for $f = 4$ must be multiplied by a factor

$$\prod_{k=1}^{n} (\sqrt{2})^{4(k-1)} = 2^{n(n-1)}. \tag{A.29}$$

This way we arrive at (45), with $N = 2n$.

This completes the derivation of the volume formulae consistent with our normalization convention for Lie algebra.

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