BRST and supermanifolds

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Abstract

In this paper the subject of the BRST symmetry representation by means of superfields is resumed and extended to $N = 1$ supersymmetric gauge theories. Then a new extension to diffeomorphisms is presented. Finally some speculations of a possible global representation of BRST symmetric theories by means of non-trivial supermanifolds are outlined.

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1. Introduction

The BRST symmetry, [1,2], has been a fundamental breakthrough in quantum field theory. Not only did it provide an elegant and rigorous framework for the renormalization program of quantum gauge theories, but it also opened the way to an amazing number of applications in related fields. Suffice it to mention topological field theories and sigma models, string and superstring theories and string field theories. Whenever there is a gauge symmetry in a classical theory, at the quantum level a BRST symmetry appears that governs the quantum behavior of the theory. It is not exaggerated to say that its basic property is nilpotence: a twice repeated BRST transformation vanishes. This in turn, at least historically, has two origins: the first is the group theoretical nature of the BRST transform, the second the anticommuting property of the ghost and antighost fields, i.e. their wrong spin-statistic connection. The latter property is a con-

sequence of the Faddeev–Popov quantization of gauge theories. If we perform two (classical) gauge transformations on a row and then repeat the operation in reverse order, the results of the two operations are related by a precise group theoretical rule. When, in the process of quantization, the classical gauge parameters are replaced by ghost fields, this rule becomes the nilpotence of the BRST transform.

In this paper in memory of Raymond Stora I would like to elaborate on these basic properties. The geometrical nature of the BRST transform was immediately sensed, [3,4]. However, the geometry in question is not that of a classical gauge theory, that is the geometry of a principal and associated fiber bundles, but rather that of a fiber bundle whose structure group is the infinite dimensional Lie group of gauge transformations. This was explained in [5]. But such a geometrical approach does not (or not yet) fit in the formalism of quantum field theory. A useful and simple synthesis of geometry and quantum field theory language seems to be provided by the supermanifold approach, [6,7]. One adds to spacetime additional anticommuting coordinates which faithfully mimic the above mentioned infinite dimensional geometry For instance, for a gauge field theory with Lie algebra valued potential form A, curvature F and ghost field c, one introduces an additional anticommuting coordinate ϑ, a superconnection $\mathcal{A} = \phi + \vartheta \psi$, where ϕ and ψ are superfields, and a super-exterior derivative $\tilde{d} = d + d\vartheta \frac{\partial}{\partial \vartheta}$. Then the condition that the supercurvature $\mathcal{F} = \tilde{d}\mathcal{A} + \mathcal{A}\mathcal{A}$ equals F, *horizontality condition*, completely determines the BRST 'geometry'. It is enough to identify A with the lowest component of ϕ and c with the lowest component of ψ.

Such a superfield formalism has been subsequently enriched and applied to various models, see [8,9] and references therein. But altogether one can say that the superfield formalism for BRST has been regarded so far as an elegant decoration, whereas, for instance, superspace has been an important tool in the development of supersymmetric theories. In time supermanifolds have grown in importance and, especially in superstring theories, they have become an essential tool for the integration over the (super)moduli space, [10]. Therefore it is not beside the point to wonder if also for BRST supermanifolds can play an analogous role. In this paper I resume this subject. I would like to show that the superfield formalism is very flexible and applies also to symmetries for which the formalism has not been implemented so far: to this end I will develop the formalism for supersymmetric gauge theories in superspace formulation and, then, for diffeomorphisms. Secondly I would like to discuss the prospects of BRST supermanifolds.

2. The superfield formalism in supersymmetric gauge theories. A proposal

In this section I will formulate the 'BRST supergeometry' of an $N = 1$ supersymmetric gauge theory formulated in the superspace. To start with let me summarize the superspace presentation of this theory.

2.1. The supermanifold formulation of SYM

From ch. XIII of [11], a supersymmetric gauge theory can be introduced as follows. One starts from a torsionful (but flat) superspace with supercoordinates $z^M = (z^m, \theta^\mu, \theta^{\dot{\mu}})$ and introduces a supervielbein basis

$$e^A(z) = dz^M \, e_M{}^A(z)$$

where $A = (a, \alpha, \dot{\alpha})$ are flat indices. The vielbein satisfy

$$
e_A{}^M e_M{}^B = \delta_A{}^B, \qquad e_M{}^A e_A{}^N = \delta_M{}^N, \qquad \delta_M{}^N = \begin{pmatrix} \delta_m{}^n & 0 & 0 \\ 0 & \delta_\mu{}^\nu & 0 \\ 0 & 0 & \delta^{\dot\mu}{}_{\dot\nu} \end{pmatrix}.
$$

The vielbein are *chosen* to be

$$
e_A{}^M = \begin{pmatrix} \delta_a{}^m & 0 & 0 \\ i\sigma^m_{\alpha\dot\alpha}\theta^{\dot\alpha} & \delta_\alpha{}^\mu & 0 \\ i\theta^\alpha \sigma^m_{\alpha\dot\beta}\epsilon^{\dot\beta\dot\alpha} & 0 & \delta^{\dot\alpha}{}_{\dot\mu} \end{pmatrix}, \qquad e_M{}^A = \begin{pmatrix} \delta_m{}^a & 0 & 0 \\ -i\sigma^a_{\mu\dot\mu}\bar\theta^{\dot\mu} & \delta_\mu{}^\alpha & 0 \\ -i\theta^\nu \sigma^a_{\nu\dot\nu}\epsilon^{\dot\nu\dot\mu} & 0 & \delta^{\dot\mu}{}_{\dot\alpha} \end{pmatrix}.
$$

In such a type of supergeometry one has

$$
de^A = dz^M dz^N \frac{\partial}{\partial z^N} e_M{}^A(z), \qquad \text{i.e.}
$$
$$
de^a = -2ie^\alpha \sigma^a{}_{\alpha\dot\alpha} e^{\dot\alpha},
$$
$$
de^\alpha = 0,
$$
$$
de^{\dot\alpha} = 0. \tag{1}
$$

The flat indices derivatives $D_A = e_A{}^M \partial_M$ correspond to

$$
D_a = e_a{}^m \partial_m = \partial_a, \qquad D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\sigma^m{}_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_m, \qquad D_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i\theta^\alpha \sigma^m{}_{\alpha\dot\alpha}\partial_m,
$$

because in flat space $e_a{}^m = \delta_m^a$. Moreover

$$
\{D_\alpha, \bar D_{\dot\alpha}\} = -2i\sigma^m{}_{\alpha\dot\alpha}\partial_m, \qquad \{D_\alpha, D_\beta\} = \{\bar D_{\dot\alpha}, \bar D_{\dot\beta}\} = 0.
$$

The superconnection is defined by

$$
\phi = e^A \phi_A, \qquad \phi_A = iT^r \phi_A^r, \qquad \phi_m^r\big|_{\theta=\bar\theta=0} = v_m^r, \tag{2}
$$

where v_m^r is the ordinary non-Abelian potential and T^r are the Hermitean generators of the gauge Lie algebra.

The gauge curvature is given by the superform

$$
F = d\phi - \phi\phi = \frac{1}{2}e^A e^B F_{BA}.
$$

On the flat basis this becomes

$$
F = de^A \phi_A + \frac{1}{2}e^A e^B \left(D_B\phi_A - (-)^{ab} D_A\phi_B - \phi_B\phi_A + (-)^{ab}\phi_A\phi_B \right), \tag{3}
$$

where the torsion term is the first on the RHS. We have

$$
F_{ab}\big|_{\theta=\bar\theta=0} = iT^r v_{ab}^r.
$$

The dynamics is determined by the super-Bianchi identity, $\mathcal{D}F = dF - [\phi, F] = 0$. They are solved by

$$
F_{\alpha\beta} = F_{\dot\alpha\dot\beta} = F_{\alpha\dot\beta} = 0, \tag{4}
$$

with further restrictions coming from:

$$
\sigma^a{}_{\alpha\dot\gamma}F_{a\beta} + \sigma^a{}_{\beta\dot\gamma}F_{a\alpha} = 0, \qquad \sigma^a{}_{\gamma\dot\beta}F_{a\dot\alpha} + \sigma^a{}_{\gamma\dot\alpha}F_{a\dot\beta} = 0. \tag{5}
$$

This allows us to write

$$F_{a\alpha} = -i\sigma_{a\alpha\dot\beta}\bar{W}^{\dot\beta}, \qquad \bar{W}^{\dot\alpha} = -\frac{i}{4}\bar\sigma^{a\dot\alpha\alpha}F_{a\alpha}. \tag{6}$$

Similarly

$$F_{a\dot\alpha} = -i W^\beta \sigma_{a\beta\dot\alpha}, \qquad W^\alpha = -\frac{i}{4}F_{a\dot\alpha}\bar\sigma^{a\dot\alpha\alpha}. \tag{7}$$

Moreover the W's must satisfy

$$\bar{\mathcal{D}}\bar{W} - \mathcal{D}W = \bar{\mathcal{D}}_{\dot\alpha}\bar{W}^{\dot\alpha} - \mathcal{D}^\alpha W_\alpha = 0, \qquad \mathcal{D}_\alpha \bar{W}_{\dot\alpha} = 0, \qquad \bar{\mathcal{D}}_{\dot\alpha}W_\alpha = 0. \tag{8}$$

2.2. The ϑ superfield formalism

We introduce the coordinates $\tilde{Z}^{\tilde{M}} = (x^m, \theta^\mu, \theta^{\dot\mu}, \vartheta^{\hat\mu}) = (z^M, \vartheta^{\hat\mu})$, with world indices $\tilde{M} = (m, \mu, \dot\mu, \hat\mu)$ together with flat ones $\tilde{A} = (a, \alpha, \dot\alpha, \hat\alpha)$. The new coordinates $\vartheta^{\hat\mu}$ will correspond to new 'ghost' coordinates $\vartheta, \bar\vartheta$. These new coordinates commute with z^M but anticommute with each other: $\vartheta^2 = 0$, $\bar\vartheta^2 = 0$ and $\vartheta\bar\vartheta + \bar\vartheta\vartheta = 0$.

For pedagogical reasons I will discard for the moment $\bar\vartheta$ and consider only ϑ and postpone to the next subsection the more complete treatment.

Notation. In this section the square bracket notation [,] denotes a *graded commutator*, with grading according to total Grassmannality ϵ of the two entries

$$[A, B] = AB - (-1)^{\epsilon(A)\epsilon B}BA. \tag{9}$$

The total Grassmannality ϵ includes both the one related to supersymmetry and to the BRST symmetry.

I will call supersuperfield (ss-field) a supersymmetric superfield that is a function also of the coordinate ϑ. In terms of $\tilde{Z}^{\tilde{M}} = (x^m, \theta^\mu, \theta^{\dot\mu}, \vartheta) = (z^M, \vartheta)$ we have

$$\tilde{f}(\tilde{Z}) = \tilde{f}(z, \vartheta) = f(z) + \vartheta g(z),$$

where $f(z)$ and $g(z)$ are ordinary supersymmetric superfields. We introduce also $\tilde{d} = d\tilde{Z}^{\tilde{M}}\frac{\partial}{\partial \tilde{Z}^{\tilde{M}}} = d + d\vartheta \frac{\partial}{\partial\vartheta}$. Next we can introduce the super–super-connection (ss-connection)

$$\tilde\Phi = \tilde{e}^{\tilde{A}}\tilde\Phi_{\tilde{A}}. \tag{10}$$

We *choose*

$$\tilde{e}^{\tilde{A}}(\tilde{Z}) = \begin{pmatrix} e^A(z) & 0 \\ 0 & d\vartheta \end{pmatrix}.$$

So

$$\tilde\Phi = \tilde\phi + d\vartheta\tilde\phi_\vartheta,$$

where

$$\tilde\phi = e^A\tilde\phi_A = e^A(\phi_A + \vartheta\psi_A), \qquad \tilde\phi_\vartheta = \varphi_\vartheta + \vartheta\psi_\vartheta. \tag{11}$$

ϕ_A, ψ_A, φ_ϑ and ψ_ϑ are ordinary superfields valued in the gauge Lie algebra with generators T^r. The BRST interpretation is:

$$g = \delta_B f, \qquad \psi_A = \delta_B \phi_A, \qquad \psi_\vartheta = \delta_B \varphi_\vartheta. \tag{12}$$

The ss-curvature can be written

$$\tilde{\mathcal{F}} = \tilde{d}\tilde{\Phi} - \tilde{\Phi}\tilde{\Phi}$$
$$= \tilde{F} + d\vartheta \left((\partial_\vartheta - \tilde{\phi}_\vartheta)\tilde{\phi} - (d - \tilde{\phi})\tilde{\phi}_\vartheta \right) + d\vartheta^2 (\partial_\vartheta \tilde{\phi}_\vartheta - \tilde{\phi}_\vartheta \tilde{\phi}_\vartheta), \tag{13}$$

the horizontality condition is $\tilde{\mathcal{F}} = F$. Thus the terms proportional to $d\vartheta$ and $d\vartheta^2$ must vanish. From

$$\partial_\vartheta \tilde{\phi}_\vartheta - \tilde{\phi}_\vartheta \tilde{\phi}_\vartheta = 0,$$

we get

$$\psi_\vartheta - \varphi_\vartheta \varphi_\vartheta = 0, \qquad \psi_\vartheta \varphi_\vartheta - \varphi_\vartheta \psi_\vartheta = 0. \tag{14}$$

The second equation is identically satisfied once we satisfy the first

$$\psi_\vartheta = \varphi_\vartheta \varphi_\vartheta. \tag{15}$$

This is a definition for ψ_ϑ. The lowest component of φ_ϑ is an anticommuting scalar valued in the gauge Lie algebra, and is to be identified with the ghost field $c = c^r(x) T^r$.

From the term proportional to $d\vartheta$ we get two conditions

$$\psi_A = D_A \varphi_\vartheta - [\phi_A, \varphi_\vartheta] \equiv \mathcal{D}_A \varphi_\vartheta, \tag{16}$$
$$\mathcal{D}_A \psi_\vartheta - [\psi_A, \varphi_\vartheta] = 0, \tag{17}$$

where $\mathcal{D}_A$ denotes the super-gauge-covariant derivative. Inserting the first into the LHS of the second we get

$$\mathcal{D}_A \left(\psi_\vartheta - \varphi_\vartheta \varphi_\vartheta \right),$$

which is the covariant derivative of (15) and thus vanishes. Therefore the independent relations are the first of (14) and (16). They define the BRST transform of φ_ϑ and ϕ_A, respectively.

The surviving term in (13) is

$$\tilde{F} = F + \vartheta \left((de^A)\psi_A - e^A \phi_A e^B \psi_B + e^A \psi_A e^B \phi_B \right). \tag{18}$$

The term de^A is the superspace (super)torsion. The term linear in ϑ is the BRST transform of the supercurvature. But some of the components of F actually vanish, so we have to verify that their BRST transform also vanish. At this point we could easily write the Bianchi identity $\mathcal{D}F = 0$ in a BRST-covariant form. Since this is a very cumbersome procedure we prefer to BRST-covariantize the constraints extracted from it in chapter XIII of [11].

For instance

$$\tilde{\mathcal{F}}_{\alpha\beta} = F_{\alpha\beta} + \vartheta \left(\mathcal{D}_\alpha \psi_\beta - \mathcal{D}_\beta \psi_\alpha \right).$$

Since $F_{\alpha\beta} = 0$ due to (4), it must be that

$$\mathcal{D}_\alpha \psi_\beta + \mathcal{D}_\beta \psi_\alpha = 0. \tag{19}$$

This can be verified using (16),

$$\mathcal{D}_\alpha \psi_\beta + \mathcal{D}_\beta \psi_\alpha = -[F_{\alpha\beta}, \varphi_\vartheta] = 0,$$

thanks again to (4).

In the $\alpha, \dot\beta$ case we must have instead

$$\tilde{\mathcal{F}}_{\alpha\dot\beta} = F_{\alpha\dot\beta} + \vartheta \left(2i\sigma^a_{\alpha\dot\beta} \psi_a + \mathcal{D}_\alpha \bar\psi_{\dot\beta} + \bar{\mathcal{D}}_{\dot\beta} \psi_\alpha \right) = 0. \tag{20}$$

We notice that

$$\mathcal{D}_\alpha \bar\psi_{\dot\beta} + \bar{\mathcal{D}}_{\dot\beta} \psi_\alpha = [F_{\alpha\dot\beta} - 2i\sigma^a_{\alpha\dot\beta} \mathcal{D}_a, \varphi_\vartheta].$$

Thus

$$\tilde{\mathcal{F}}_{\alpha\dot\beta} = F_{\alpha\dot\beta} - \vartheta [F_{\alpha\dot\beta}, \varphi_\vartheta]. \tag{21}$$

Therefore the BRST transform is consistent with the constraint $F_{\alpha\dot\beta} = 0$. In fact one must notice that, for all indices,

$$\tilde{\mathcal{F}}_{AB} = F_{AB} + \vartheta [F_{AB}, \varphi_\vartheta]. \tag{22}$$

Thanks to this equation also the constraints (5) can be written in a BRST-covariant form (that is, for instance, with F replaced by $\tilde{F}$).

So far the constraints (4) are consistently satisfied. Next we have to satisfy the analog of (8). We can introduce the BRST covariant definitions for W^α and $W^{\dot\alpha}$:

$$\tilde{W}^\alpha = -\frac{i}{4} \tilde{F}_{a\dot\alpha} \bar\sigma^{a\dot\alpha\alpha}, \qquad \tilde{\bar{W}}^{\dot\alpha} = -\frac{i}{4} \bar\sigma^{a\dot\alpha\alpha} F_{a\alpha}.$$

The BRST covariant analogs of (8) can be written as follows:

$$\tilde{\mathcal{D}}_\alpha \tilde{\bar{W}}_{\dot\alpha}, \qquad \tilde{\bar{\mathcal{D}}}_{\dot\alpha} \tilde{W}_\alpha = 0, \dots, \tag{23}$$

where we define

$$\tilde{\mathcal{D}}_A = D_A + [\tilde\phi_A, \quad]. \tag{24}$$

Using these definitions we find

$$\tilde{\bar{\mathcal{D}}}_{\dot\alpha} \tilde{W}_\alpha = \frac{i}{4} \sigma_{a\alpha\dot\beta} \left(\bar{D}_{\dot\alpha} \left(F^{a\dot\beta} + \vartheta [F^{a\dot\beta}, \varphi_\vartheta] \right) + \vartheta [\psi_{\dot\alpha}, F^{a\dot\beta}] \right)$$
$$= \bar{\mathcal{D}}_{\dot\alpha} W_\alpha - \vartheta [\bar{\mathcal{D}}_{\dot\alpha} W_\alpha, \varphi_\vartheta] = 0, \tag{25}$$

consistently. The other constraints can be rewritten in the same way.

2.3. The $\vartheta, \bar\vartheta$ superfield formalism

Now I will switch on the second 'ghost' coordinate $\bar\vartheta$ and I will call supersuperfield (ss-field) a superfield that is a function also of the coordinate $\bar\vartheta$. In terms of $\tilde{Z}^{\tilde{M}} = (x^m, \theta^\mu, \theta^{\dot\mu}, \vartheta, \bar\vartheta) = (z^M, \vartheta, \bar\vartheta)$ we have

$$\tilde{f}(\tilde{Z}) = \tilde{f}(z, \vartheta, \bar\vartheta) = f(z) + \vartheta \bar{g}(z) + \bar\vartheta g(z) + \vartheta \bar\vartheta h(z),$$

where $f(z)$, $g(z)$, $\bar{g}(z)$ and $h(z)$ are ordinary supersymmetric superfields. The BRST–anti-BRST interpretation is

$$g = \delta_B f, \qquad \bar{g} = \bar{\delta}_B f, \qquad h = \bar{\delta}_B g = -\delta_B \bar{g}. \tag{26}$$

We introduce also the ss-exterior derivative $\tilde{d} = d\tilde{Z}^{\tilde{M}} \frac{\partial}{\partial \tilde{Z}^{\tilde{M}}} = d + d\vartheta \frac{\partial}{\partial \vartheta} + d\bar{\vartheta} \frac{\partial}{\partial \bar{\vartheta}}$.

The super–super-connection (ss-connection) is

$$\tilde{\Phi} = \tilde{e}^{\tilde{A}} \tilde{\Phi}_{\tilde{A}}. \tag{27}$$

We choose

$$\tilde{e}^{\tilde{A}}(\tilde{Z}) = \begin{pmatrix} e^A(z) & 0 & 0 \\ 0 & d\vartheta & 0 \\ 0 & 0 & d\bar{\vartheta} \end{pmatrix}.$$

So

$$\tilde{\Phi} = \tilde{\phi} + d\vartheta\, \tilde{\phi}_\vartheta + d\bar{\vartheta}\, \tilde{\phi}_{\bar{\vartheta}},$$

where

$$\tilde{\phi} = e^A \tilde{\phi}_A = e^A \left(\phi_A + \vartheta \bar{\psi}_A + \bar{\vartheta} \psi_A + \vartheta \bar{\vartheta} \pi_A \right),$$
$$\tilde{\phi}_\vartheta = \varphi_\vartheta + \vartheta \bar{\psi}_\vartheta + \bar{\vartheta} \psi_\vartheta + \vartheta \bar{\vartheta} \varpi_\vartheta,$$
$$\tilde{\phi}_{\bar{\vartheta}} = \varphi_{\bar{\vartheta}} + \vartheta \bar{\psi}_{\bar{\vartheta}} + \bar{\vartheta} \psi_{\bar{\vartheta}} + \vartheta \bar{\vartheta} \varpi_{\bar{\vartheta}}. \tag{28}$$

$\phi_A, \psi_A, \ldots, \varpi_{\bar{\vartheta}}$ are ordinary superfields valued in the gauge Lie algebra with generators T^r.

The ss-curvature can be written

$$\tilde{\mathcal{F}} = \tilde{d}\tilde{\Phi} - \tilde{\Phi}\tilde{\Phi}$$
$$= \tilde{F} + d\vartheta \left((\partial_\vartheta - \tilde{\phi}_\vartheta)\tilde{\phi} - (d - \tilde{\phi})\tilde{\phi}_\vartheta \right) + d\bar{\vartheta} \left((\partial_{\bar{\vartheta}} - \tilde{\phi}_{\bar{\vartheta}})\tilde{\phi} - (d - \tilde{\phi})\tilde{\phi}_{\bar{\vartheta}} \right)$$
$$+ d\vartheta d\vartheta \left(\partial_\vartheta \tilde{\phi}_\vartheta - \phi_\vartheta \tilde{\phi}_\vartheta \right) + d\bar{\vartheta} d\bar{\vartheta} \left(\partial_{\bar{\vartheta}} \tilde{\phi}_{\bar{\vartheta}} - \phi_{\bar{\vartheta}} \tilde{\phi}_{\bar{\vartheta}} \right)$$
$$+ d\vartheta d\bar{\vartheta} \left(\partial_\vartheta \tilde{\phi}_{\bar{\vartheta}} + \partial_{\bar{\vartheta}} \tilde{\phi}_\vartheta - \phi_\vartheta \tilde{\phi}_{\bar{\vartheta}} - \tilde{\phi}_{\bar{\vartheta}} \tilde{\phi}_\vartheta \right), \tag{29}$$

and the horizontality condition is

$$\tilde{d}\tilde{\Phi} - \tilde{\Phi}\tilde{\Phi} = F. \tag{30}$$

It gives rise to the following set of equations

$$\tilde{F} = F, \tag{31}$$
$$(\partial_\vartheta - \tilde{\phi}_\vartheta)\tilde{\phi} - (d - \tilde{\phi})\tilde{\phi}_\vartheta = 0, \tag{32}$$
$$(\partial_{\bar{\vartheta}} - \tilde{\phi}_{\bar{\vartheta}})\tilde{\phi} - (d - \tilde{\phi})\tilde{\phi}_{\bar{\vartheta}} = 0, \tag{33}$$
$$\partial_\vartheta \tilde{\phi}_\vartheta - \phi_\vartheta \phi_\vartheta = 0, \tag{34}$$
$$\partial_{\bar{\vartheta}} \tilde{\phi}_{\bar{\vartheta}} - \phi_{\bar{\vartheta}} \phi_{\bar{\vartheta}} = 0, \tag{35}$$
$$\partial_\vartheta \tilde{\phi}_{\bar{\vartheta}} + \partial_{\bar{\vartheta}} \tilde{\phi}_\vartheta - \phi_\vartheta \phi_{\bar{\vartheta}} - \phi_{\bar{\vartheta}} \phi_\vartheta = 0. \tag{36}$$

Eqs. (34), (35) yield the identification

$$\bar{\psi}_\vartheta = \varphi_\vartheta \varphi_\vartheta, \qquad \psi_{\bar\vartheta} = \varphi_{\bar\vartheta} \varphi_{\bar\vartheta}, \tag{37}$$

$$\varpi_\vartheta = [\psi_\vartheta, \varphi_\vartheta], \qquad \varpi_{\bar\vartheta} = -[\bar{\psi}_{\bar\vartheta}, \varphi_{\bar\vartheta}]. \tag{38}$$

The remaining equations

$$\bar{\psi}_\vartheta \varphi_\vartheta = \varphi_\vartheta \bar{\psi}_\vartheta, \qquad \psi_{\bar\vartheta} \varphi_{\bar\vartheta} = \varphi_{\bar\vartheta} \psi_{\bar\vartheta}$$

$$\varpi_\vartheta \varphi_\vartheta + \varphi_\vartheta \varpi_\vartheta + \bar{\psi}_\vartheta \psi_\vartheta - \psi_\vartheta \bar{\psi}_\vartheta = 0, \qquad \varpi_{\bar\vartheta} \varphi_{\bar\vartheta} + \varphi_{\bar\vartheta} \varpi_{\bar\vartheta} + \bar{\psi}_{\bar\vartheta} \psi_{\bar\vartheta} - \psi_{\bar\vartheta} \bar{\psi}_{\bar\vartheta} = 0, \tag{39}$$

are identically satisfied.

The lowest component of φ_ϑ is an anticommuting scalar valued in the gauge Lie algebra, and is to be identified with the ghost field $c = c^r(x) T^r$. Its BRST transform is $\bar{\psi}_\vartheta$. φ_ϑ is the BRST transform parameter. The lowest component of $\varphi_{\bar\vartheta}$ is to be identified with the dual ghost field $\bar{c} = \bar{c}^r(x) T^r$. Its anti-BRST transform is $\psi_{\bar\vartheta}$.

Eq. (36) gives the relation

$$\psi_\vartheta + \bar{\psi}_{\bar\vartheta} = \varphi_\vartheta \varphi_{\bar\vartheta} + \varphi_{\bar\vartheta} \varphi_\vartheta, \tag{40}$$

which is to be interpreted as the Curci–Ferrari relation, [12], and $\psi_\vartheta + \bar{\psi}_{\bar\vartheta}$ are the Lautrup–Nakanishi superfields. Using (40) the remaining relations

$$\varpi_\vartheta = [\varphi_\vartheta, \bar{\psi}_{\bar\vartheta}] + [\varphi_{\bar\vartheta}, \bar{\psi}_\vartheta],$$

$$\varpi_{\bar\vartheta} = -[\varphi_\vartheta, \psi_{\bar\vartheta}] - [\varphi_{\bar\vartheta}, \psi_\vartheta],$$

$$[\bar{\psi}_\vartheta, \psi_{\bar\vartheta}] + [\bar{\psi}_{\bar\vartheta}, \psi_\vartheta] + [\varpi_\vartheta, \varphi_{\bar\vartheta}] + [\varpi_{\bar\vartheta}, \varphi_\vartheta] = 0, \tag{41}$$

turn out to be identically verified.

Let us come next to the constraint (32). It implies the definitions

$$\bar{\psi}_A = D_A \varphi_\vartheta - [\phi_A, \varphi_\vartheta] = \mathcal{D}_A \varphi_\vartheta, \tag{42}$$

$$\pi_A = \mathcal{D}_A \psi_\vartheta - [\psi_A, \varphi_\vartheta], \tag{43}$$

and the identities

$$\mathcal{D}_A \bar{\psi}_\vartheta - [\bar{\psi}_A, , \varphi_\vartheta] = 0$$

$$\mathcal{D}_A \varpi_\vartheta - [\pi_A, \varphi_\vartheta] + [\bar{\psi}_A, , \psi_\vartheta] - [\psi_A, , \bar{\psi}_\vartheta] = 0, \tag{44}$$

while from (33) we get the definitions

$$\psi_A = D_A \varphi_{\bar\vartheta} - [\phi_A, \varphi_{\bar\vartheta}] = \mathcal{D}_A \varphi_{\bar\vartheta}, \tag{45}$$

$$\pi_A = -\mathcal{D}_A \bar{\psi}_{\bar\vartheta} + [\bar{\psi}_A, \varphi_{\bar\vartheta}], \tag{46}$$

and the identities

$$\mathcal{D}_A \psi_{\bar\vartheta} - [\psi_A, \varphi_{\bar\vartheta}] = 0$$

$$\mathcal{D}_A \varpi_{\bar\vartheta} - [\pi_A, \varphi_{\bar\vartheta}] + [\bar{\psi}_A, , \psi_{\bar\vartheta}] - [\psi_A, , \bar{\psi}_{\bar\vartheta}] = 0. \tag{47}$$

The superfield $\psi_A, \bar{\psi}_A, \pi_A$ are easily recognized (anti)BRST transform. The equivalence of (43) and (46) can be proven by means of the CF condition.

Next let us come to (31). In general, using (3), one can show that

$$\tilde{F}_{AB} = F_{AB} - \vartheta[F_{AB}, \varphi_\vartheta] - \bar{\vartheta}[F_{AB}, \varphi_{\bar\vartheta}] - \vartheta\bar{\vartheta}\left([F_{AB}, \psi_\vartheta] - [[F_{AB}, \varphi_{\bar\vartheta}], \varphi_\vartheta]\right). \tag{48}$$

In proving this a particular attention must be paid to the $(A, B) = (\alpha, \dot{\beta})$ case. The definition (3) includes in this case also a contribution from the supertorsion; but this contribution is exactly canceled by an analogous term coming from the first commutator (2).

From (48) it this evident that the constraints (4) can be covariantly implemented in the BRST formalism. Also here instead of solving the ss-Bianchi identity, we prefer to covariantize the constraints extracted from it in chapter XIII of [11]. In the same way as (4) also (5) can be covariantly implemented in the BRST formalism. Moreover, using (6), (7), we can introduce the ss-field expressions for $\tilde{W}_\alpha$

$$\tilde{W}_\alpha = W_\alpha - \vartheta[W_\alpha, \varphi_\vartheta] - \bar{\vartheta}[W_\alpha, \varphi_{\bar{\vartheta}}] - \vartheta\bar{\vartheta}\left([W_\alpha, \psi_\vartheta] - [[W_\alpha, \varphi_{\bar{\vartheta}}], \varphi_\vartheta]\right), \tag{49}$$

and an analogous one for $\tilde{W}_{\dot{\alpha}}$. The next issue is now to BRST-covariantize the constraints (8).

Let us use the compact notation $\underline{\alpha}$ to denote both α and $\dot{\alpha}$ and introduce the BRST supercovariant derivative

$$\tilde{\mathcal{D}}_{\underline{\alpha}}\tilde{W}_{\underline{\beta}} = D_{\underline{\alpha}}\tilde{W}_{\underline{\beta}} - [\tilde{\phi}_{\underline{\alpha}}, \tilde{W}_{\underline{\beta}}], \tag{50}$$

then it is lengthy but straightforward to prove that

$$\begin{aligned}
\tilde{\mathcal{D}}_{\underline{\alpha}}\tilde{W}_{\underline{\beta}} = {}& \mathcal{D}_{\underline{\alpha}}W_{\underline{\beta}} - \vartheta[\mathcal{D}_{\underline{\alpha}}W_{\underline{\beta}}, \varphi_\vartheta] - \bar{\vartheta}[\mathcal{D}_{\underline{\alpha}}W_{\underline{\beta}}, \varphi_{\bar{\vartheta}}] \\
& - \vartheta\bar{\vartheta}\left([\mathcal{D}_{\underline{\alpha}}W_{\underline{\beta}}, \psi_\vartheta] - [[\mathcal{D}_{\underline{\alpha}}W_{\underline{\beta}}, \varphi_{\bar{\vartheta}}], \varphi_\vartheta]\right).
\end{aligned} \tag{51}$$

This allows us to write down the constraints (8) in a BRST covariant form.

Of course this is only the beginning of the story when quantizing an $N = 1$ supersymmetric Yang–Mills theory. Then one has to fix the gauge and show that the overall action can be written in an invariant way in the enlarged superspace. Finally one should not forget that concrete calculations are carried out in the Wess–Zumino gauge. But at least this section shows that the BRST formalism can be consistently embedded in a supermanifold that encompasses also the supersymmetric spinorial directions.

3. Diffeomorphisms and the superfield formalism

A first proposal of a superfield formalism for diffeomorphisms was made by [7]. Here I present another approach to the same problem, closer in spirit to the standard (commutative) geometrical approach and to the discussion in the next section.

Diffeomorphisms, or general coordinate transformations, are defined by means of a local parameter $\xi^\mu(x)$: $x^\mu \to x^\mu + \xi^\mu(x)$. In a quantized theory this is promoted to an anticommuting field. The BRST transformation are

$$\delta_\xi \varphi = \xi^\lambda \partial_\lambda, \varphi \tag{52}$$

$$\delta_\xi A_\mu = \xi^\lambda \partial_\lambda A_\mu + \partial_\mu \xi^\lambda A_\lambda, \tag{53}$$

$$\delta_\xi g_{\mu\nu} = \xi^\lambda \partial_\lambda g_{\mu\nu} + \partial_\mu \xi^\lambda g_{\lambda\nu} + \partial_\nu \xi^\lambda g_{\mu\lambda}, \tag{54}$$

$$\delta_\xi \xi^\mu = \xi^\lambda \partial_\lambda \xi^\mu, \tag{55}$$

for a scalar field, a vector field, the metric and ξ. These transformations are nilpotent. We will introduce another anticommuting field, $\bar{\xi}$, and a $\delta_{\bar{\xi}}$ transformation, which transforms a scalar, vector, the metric and $\bar{\xi}$ in the same way as δ_ξ, and, in addition,

$$\delta_\xi \bar{\xi}^\mu = \xi^\nu \partial_\nu \bar{\xi}^\mu, \qquad \delta_{\bar{\xi}} \xi^\mu = \bar{\xi}^\nu \partial_\nu \xi^\mu. \tag{56}$$

The overall transformation $\delta_\xi + \delta_{\bar\xi}$ is nilpotent:

$$(\delta_\xi + \delta_{\bar\xi})^2 = 0.$$

Introducing this additional field seems to be completely unmotivated, but in fact it is needed if we want an invertible supermetric.

3.1. The superfield formalism

One introduces the superspace $X^M = (x^\mu, \vartheta, \bar\vartheta)$, where $\vartheta, \bar\vartheta$ are anticommuting. A diffeomorphism is represented by a superspace transformation $X^M = (x^\mu, \vartheta) \to \tilde X^M = (x^\mu - \vartheta\bar\xi^\mu - \bar\vartheta\xi^\mu, \vartheta, \bar\vartheta)$, where $\xi, \bar\xi$ and $\vartheta\bar\vartheta$, anticommute. The horizontality condition is formulated by selecting appropriate *covariant expressions* in ordinary spacetime and identifying them with the same expressions extended to the superspace. Below we work out explicitly the case of a scalar, a vector field and the metric.

3.1.1. The scalar
For instance, a scalar field φ is embedded in the superfield

$$\Phi(X) = \varphi(x) + \vartheta\bar b(x) + \bar\vartheta b(x) + \vartheta\bar\vartheta c(x),$$

and gets transformed into

$$\begin{aligned}
\Phi(\tilde X) &= \varphi(x) - (\vartheta\bar\xi + \bar\vartheta\xi) \cdot \partial\varphi(x) + \vartheta\left(\bar b(x) - \bar\vartheta\xi(x) \cdot \bar b(x)\right) + \bar\vartheta\left(b(x) - \vartheta\bar b(x)\right) \\
&\quad + \vartheta\bar\vartheta\left(c(x) - \bar\xi^\mu\xi^\nu\partial_\mu\partial_\nu\varphi(x)\right) \\
&= \varphi(x) + \vartheta\left(\bar b(x) - \bar\xi \cdot \partial\varphi(x)\right) + \bar\vartheta\left(b(x) - \xi \cdot \partial\varphi(x)\right) \\
&\quad + \vartheta\bar\vartheta\left(c(x) - \xi \cdot \partial\bar b(x) + \bar\xi \cdot \partial b(x)\right).
\end{aligned} \tag{57}$$

Horizontality means

$$\Phi(\tilde X^M) = \varphi(x^\mu), \tag{58}$$

which implies

$$\begin{aligned}
b(x) &= \xi \cdot \partial\varphi(x), \qquad \bar b(x) = \bar\xi \cdot \partial\varphi(x) \\
c(x) &= \xi \cdot \partial\bar b(x) - \bar\xi \cdot \partial b(x) + \bar\xi^\mu\xi^\nu\partial_\mu\partial_\nu\varphi(x).
\end{aligned} \tag{59}$$

It is easy to prove that

$$\delta_{\bar\xi}\varphi(x) = \bar b(x), \qquad \delta_\xi\varphi(x) = b(x), \qquad \delta_\xi\delta_{\bar\xi}\varphi = -\delta_{\bar\xi}\delta_\xi\varphi = c(x). \tag{60}$$

That is the diffeomorphism transforms of φ are its superpartner in the superfield.

3.1.2. The vector
Let us extend this to a vector field. In order to apply the horizontality condition we have to identify the appropriate expression. This is a 1-superform:

$$\mathcal{A}_M(X)dX^M = A_\mu(X)dx^\mu + A_\vartheta(X)d\vartheta + A_{\bar\vartheta}(X)d\bar\vartheta, \tag{61}$$

and

$$\mathcal{A}_\mu(X) = A_\mu(x) + \vartheta\,\bar\phi_\mu(x) + \bar\vartheta\phi_\mu(x) + \vartheta\,\bar\vartheta\,B_\mu(x), \tag{62}$$

$$\mathcal{A}_\vartheta(X) = \chi(x) + \vartheta\bar{C}(x) + \bar\vartheta C(x) + \vartheta\bar\vartheta\psi(x), \tag{63}$$

$$\mathcal{A}_{\bar\vartheta}(X) = \lambda(x) + \vartheta\bar{D}(x) + \bar\vartheta D(x) + \vartheta\bar\vartheta\rho(x). \tag{64}$$

Horizontality means that

$$\mathcal{A}_M(\tilde{X})\tilde{d}\tilde{X}^M = A_\mu(x)dx^\mu, \tag{65}$$

where $\tilde{d} = \frac{\partial}{\partial x^\mu}dx^\mu + \frac{\partial}{\partial\vartheta}d\vartheta + \frac{\partial}{\partial\bar\vartheta}d\bar\vartheta$. Thus

$$\tilde{d}\tilde{X}^M = \left(dx^\mu - \vartheta\,\partial_\lambda\bar\xi^\mu dx^\lambda - \bar\vartheta\,\partial_\lambda\xi^\mu dx^\lambda - \xi^\mu d\vartheta - \bar\xi^\mu d\bar\vartheta, d\vartheta, d\bar\vartheta\right).$$

Expanding we find

$$\begin{aligned}
\tilde{A}_M(\tilde{X})\tilde{d}\tilde{X}^M = {}& \Bigg(A_\mu - \left(\vartheta\bar\xi + \bar\vartheta\xi\right)\cdot\partial A_\mu + \vartheta\bar\vartheta\,\xi\bar\xi\cdot\partial^2 A_\mu + \vartheta\left(\bar\phi_\mu - \bar\vartheta\xi\cdot\partial\bar\phi_\mu\right) \\
&+ \bar\vartheta\left(\phi_\mu - \vartheta\bar\xi\cdot\partial\phi_\mu\right) + \vartheta\bar\vartheta\,B_\mu(x)\Bigg)\left(dx^\mu - \vartheta\,\partial_\lambda\bar\xi^\mu dx^\lambda - \bar\vartheta\,\partial_\lambda\xi^\mu dx^\lambda - \bar\xi^\mu d\vartheta - \xi^\mu d\bar\vartheta\right) \\
&+ \Bigg(\chi(x) - \left(\vartheta\bar\xi + \bar\vartheta\xi\right)\cdot\partial\chi(x) + \vartheta\bar\vartheta\cdot\partial^2\chi(x) \\
&+ \vartheta\left(\bar{C}(x) - \bar\vartheta\xi\cdot\partial\bar{C}(x)\right) + \bar\vartheta\left(C(x) - \vartheta\bar\xi\cdot\partial C(x)\right) + \vartheta\bar\vartheta\psi(x)\Bigg)d\vartheta \\
&+ \Bigg(\lambda(x) - \left(\vartheta\bar\xi + \bar\vartheta\xi\right)\cdot\partial\lambda(x) + \vartheta\bar\vartheta\cdot\partial^2\lambda(x) \\
&+ \vartheta\left(\bar{D}(x) - \bar\vartheta\xi\cdot\partial\bar{D}(x)\right) + \bar\vartheta\left(D(x) - \vartheta\bar\xi\cdot\partial D(x)\right) + \vartheta\bar\vartheta\rho(x)\Bigg)d\bar\vartheta = A_\mu(x)dx^\mu, \tag{66}
\end{aligned}$$

where $\xi\bar\xi\cdot\partial^2 = \xi^\mu\bar\xi^\nu\partial_\mu\partial_\nu$. The commutation prescriptions are: $x^\mu, \vartheta, \bar\vartheta, \xi^\mu$ commute with dx^μ, $d\vartheta$; ξ^μ anticommute with $\vartheta\bar\vartheta$. Working out (66) we obtain the following identifications:

$$\phi_\mu = \xi\cdot\partial A_\mu + \partial_\mu\xi^\lambda A_\lambda, \tag{67}$$

$$\bar\phi_\mu = \bar\xi\cdot\partial A_\mu + \partial_\mu\bar\xi^\lambda A_\lambda, \tag{68}$$

$$\begin{aligned}
B_\mu = {}& \xi\cdot\partial\bar\phi_\mu - \bar\xi\cdot\partial\phi_\mu - \xi\bar\xi\cdot\partial^2 A_\mu + \partial_\mu\bar\xi^\lambda\xi\cdot\partial A_\lambda - \partial_\mu\xi^\lambda\bar\xi\cdot\partial A_\lambda \\
&+ \partial_\mu\bar\xi^\lambda\phi_\lambda - \partial_\mu\xi^\lambda\bar\phi_\lambda, \tag{69}
\end{aligned}$$

$$\chi = A_\mu\bar\xi^\mu, \tag{70}$$

$$C = -\xi\cdot\partial A_\mu\bar\xi^\mu + \phi_\mu\bar\xi^\mu + \xi\cdot\partial\chi, \tag{71}$$

$$\bar{C} = -\bar\xi\cdot\partial A_\mu\bar\xi^\mu + \bar\phi_\mu\bar\xi^\mu + \bar\xi\cdot\partial\chi, \tag{72}$$

$$\psi = \xi\bar\xi\cdot\partial^2 A_\mu\bar\xi^\mu - \xi\cdot\partial\bar\phi_\mu\bar\xi^\mu + \bar\xi\cdot\partial\phi_\mu\bar\xi^\mu + B_\mu\bar\xi^\mu - \xi\bar\xi\cdot\partial^2\chi + \xi\cdot\partial\bar{C} - \bar\xi\cdot\partial C, \tag{73}$$

and

$$\lambda = A_\mu\xi^\mu \tag{74}$$

$$D = -\xi\cdot\partial A_\mu\xi^\mu + \phi_\mu\xi^\mu + \xi\cdot\partial\lambda, \tag{75}$$

$$\bar{D} = -\bar\xi\cdot\partial A_\mu\xi^\mu + \bar\phi_\mu\xi^\mu + \bar\xi\cdot\partial\lambda, \tag{76}$$

$$\rho = \xi\bar\xi\cdot\partial^2 A_\mu\xi^\mu - \xi\cdot\partial\bar\phi_\mu\xi^\mu + \bar\xi\cdot\partial\phi_\mu\xi^\mu + B_\mu\xi^\mu - \xi\bar\xi\cdot\partial^2\lambda + \xi\cdot\partial\bar{D} - \bar\xi\cdot\partial D. \tag{77}$$

One can prove that

$$\phi_\mu = \delta_\xi A_\mu, \qquad \bar\phi_\mu = \delta_{\bar\xi} A_\mu, \qquad B_\mu = \delta_{\bar\xi}\phi_\mu = -\delta_\xi \bar\phi_\mu. \tag{78}$$

$$D = \delta_\xi \lambda, \qquad \bar D = \delta_{\bar\xi}\lambda, \qquad \rho = -\delta_{\bar\xi} D = \delta_\xi \bar D. \tag{79}$$

In particular

$$\rho = \xi \cdot \partial\bar\xi \cdot \partial A_\mu \xi^\mu - \bar\xi \cdot \partial\xi \cdot \partial A_\mu \xi^\mu + \bar\xi \cdot \partial\xi^\mu \xi \cdot \partial A_\mu \xi^\mu$$
$$- \xi \cdot \partial\bar\xi^\mu \xi \cdot \partial A_\mu \xi^\mu + \xi\bar\xi \cdot \partial^2 A_\mu \xi^\mu. \tag{80}$$

Analogously

$$C = \delta_\xi \chi, \qquad \bar C = \delta_{\bar\xi}\chi, \qquad \psi = -\delta_{\bar\xi} C = \delta_\xi \bar C. \tag{81}$$

3.1.3. The metric

In the same way we can treat the metric $g_{\mu\nu}(x)$. We embed it in a supermetric $G_{MN}(X)$ and form the symmetric 2-superform

$$G_{MN}(X)\tilde d X^M \vee \tilde d X^N, \tag{82}$$

where $\vee$ denotes the symmetric tensor product and

$$G_{\mu\nu}(X) = g_{\mu\nu}(x) + \vartheta\,\bar\Gamma_{\mu\nu}(x) + \bar\vartheta\,\Gamma_{\mu\nu}(x) + \vartheta\,\bar\vartheta\,V_{\mu\nu}(x),$$
$$G_{\mu\vartheta}(X) = \gamma_\mu(x) + \vartheta\,\bar g_\mu(x) + \bar\vartheta\,g_\mu(x) + \vartheta\,\bar\vartheta\,\Gamma_\mu(x) = G_{\vartheta\mu}(X),$$
$$G_{\mu\bar\vartheta}(X) = \bar\gamma_\mu(x) + \vartheta\,\bar f_\mu(x) + \bar\vartheta\,f_\mu(x) + \vartheta\,\bar\vartheta\,\bar\Gamma_\mu(x) = G_{\bar\vartheta\mu}(X), \tag{83}$$

$$G_{\vartheta\bar\vartheta}(X) = g(x) + \vartheta\,\bar\gamma(x) + \bar\vartheta\,\gamma(x) + \vartheta\,\bar\vartheta\,G(x) = -G_{\bar\vartheta\vartheta}(X), \tag{84}$$

while $G_{\vartheta\vartheta}(X) = 0 = G_{\bar\vartheta\bar\vartheta}(X)$, because the symmetric tensor product becomes antisymmetric for anticommuting variables: $d\vartheta \vee d\vartheta = 0 = d\bar\vartheta \vee d\bar\vartheta, d\vartheta \vee d\bar\vartheta = -d\bar\vartheta \vee d\vartheta$.

The horizontality condition is obtained by requiring

$$G_{MN}(\tilde X)\tilde d \tilde X^M \vee \tilde d \tilde X^N = g_{\mu\nu}(x)dx^\mu \vee dx^\nu, \tag{85}$$

which is explicitly rewritten as

$$g_{\mu\nu}(x)dx^\mu \vee dx^\nu$$
$$= G_{MN}(\tilde X)\tilde d \tilde X^M \vee \tilde d \tilde X^N$$
$$= \Big(g_{\mu\nu} - (\vartheta\bar\xi + \bar\vartheta\xi)\cdot\partial g_{\mu\nu} + \vartheta\,\bar\vartheta\,\xi\bar\xi\cdot\partial^2 g_{\mu\nu}$$
$$\quad + \vartheta\left(\bar\Gamma_{\mu\nu} - \bar\vartheta\xi\cdot\partial\bar\Gamma_{\mu\nu}\right) + \bar\vartheta\left(\Gamma_{\mu\nu} - \vartheta\xi\cdot\partial\Gamma_{\mu\nu}\right)$$
$$\quad + \vartheta\,\bar\vartheta\,V_{\mu\nu}(x)\Big)\left(dx^\mu - \vartheta\,\partial_\lambda\bar\xi^\mu dx^\lambda - \bar\vartheta\,\partial_\lambda\xi^\mu dx^\lambda - \bar\xi^\mu d\vartheta - \xi^\mu d\bar\vartheta\right)$$
$$\vee \left(dx^\nu - \vartheta\,\partial_\rho\bar\xi^\nu dx^\rho - \bar\vartheta\,\partial_\rho\xi^\nu dx^\rho - \bar\xi^\nu d\vartheta - \xi^\nu d\bar\vartheta\right)$$
$$+ 2\Big(\gamma_\mu - (\vartheta\bar\xi + \bar\vartheta\xi)\cdot\partial\gamma_\mu + \vartheta\,\bar\vartheta\,\xi\bar\xi\cdot\partial^2\gamma_\mu + \vartheta\left(\bar g_\mu - \bar\vartheta\xi\cdot\partial\bar g_\mu\right)$$
$$\quad + \bar\vartheta\left(g_\mu - \vartheta\xi\cdot\partial g_\mu\right) + \vartheta\,\bar\vartheta\,\Gamma_\mu(x)\Big)$$
$$\times \left(dx^\mu - \vartheta\,\partial_\lambda\bar\xi^\mu dx^\lambda - \bar\vartheta\,\partial_\lambda\xi^\mu dx^\lambda - \bar\xi^\mu d\vartheta - \xi^\mu d\bar\vartheta\right) \vee d\vartheta$$
$$+ 2\Big(\bar\gamma_\mu - (\vartheta\bar\xi + \bar\vartheta\xi)\cdot\partial\bar\gamma_\mu + \vartheta\,\bar\vartheta\,\xi\bar\xi\cdot\partial^2\bar\gamma_\mu + \vartheta\left(\bar f_\mu - \bar\vartheta\xi\cdot\partial\bar f_\mu\right)$$

$$
+ \bar{\vartheta} \left(f_\mu - \vartheta \xi \cdot \partial f_\mu \right) + \vartheta \bar{\vartheta} \, \bar{\Gamma}_\mu(x) \Big)
$$

$$
\times \left(dx^\nu - \vartheta \, \partial_\lambda \bar{\xi}^\nu dx^\lambda - \bar{\vartheta} \, \partial_\lambda \xi^\nu dx^\lambda - \bar{\xi}^\nu d\vartheta - \xi^\nu d\bar{\vartheta} \right) \vee d\vartheta
$$

$$
+ \left(g - \left(\vartheta \bar{\xi} + \bar{\vartheta} \xi \right) \cdot \partial g + \vartheta \bar{\vartheta} \, \xi \bar{\xi} \cdot \partial^2 g + \vartheta \left(\bar{\gamma} - \bar{\vartheta} \xi \cdot \partial \bar{\gamma} \right) \right.
$$

$$
\left. + \bar{\vartheta} \left(\gamma - \vartheta \xi \cdot \gamma \right) + \vartheta \bar{\vartheta} G \right) d\vartheta \vee d\bar{\vartheta}, \tag{86}
$$

where all the fields in the RHS are function of x. From this it follows that

$$
\Gamma_{\mu\nu} = \xi \cdot \partial g_{\mu\nu} + \partial_\mu \xi^\lambda g_{\lambda\nu} + \partial_\nu \xi^\lambda g_{\mu\lambda} = \delta_\xi g_{\mu\nu}, \tag{87}
$$

$$
\bar{\Gamma}_{\mu\nu} = \bar{\xi} \cdot \partial g_{\mu\nu} + \partial_\mu \bar{\xi}^\lambda g_{\lambda\nu} + \partial_\nu \bar{\xi}^\lambda g_{\mu\lambda} = \delta_{\bar{\xi}} g_{\mu\nu}, \tag{88}
$$

$$
\begin{aligned}
V_{\mu\nu} &= -\xi \bar{\xi} \cdot \partial^2 g_{\mu\nu} + \xi \cdot \partial \bar{\Gamma}_{\mu\nu} + \partial_\mu \xi^\lambda \bar{\Gamma}_{\lambda\nu} + \partial_\nu \xi^\lambda \bar{\Gamma}_{\mu\lambda} - \bar{\xi} \cdot \partial \Gamma_{\mu\nu} - \partial_\mu \bar{\xi}^\lambda \Gamma_{\lambda\nu} - \partial_\nu \bar{\xi}^\lambda \Gamma_{\mu\lambda} \\
&\quad + \partial_\mu \bar{\xi}^\lambda \xi \cdot \partial g_{\lambda\nu} + \partial_\nu \bar{\xi}^\lambda \xi \cdot \partial g_{\mu\lambda} - \partial_\mu \xi^\lambda \bar{\xi} \cdot \partial g_{\lambda\nu} - \partial_\nu \xi^\lambda \bar{\xi} \cdot \partial g_{\mu\lambda} \\
&\quad + \partial_\mu \bar{\xi}^\lambda \partial_\nu \xi^\rho g_{\lambda\rho} + \partial_\nu \bar{\xi}^\lambda \partial_\mu \xi^\rho g_{\lambda\rho} \\
&= \partial_\lambda \xi^\rho \partial_\nu \bar{\xi}^\lambda g_{\mu\rho} + \partial_\nu \xi^\lambda \partial_\lambda \bar{\xi}^\rho g_{\mu\rho} \\
&= \delta_\xi \bar{\Gamma}_{\mu\nu} = -\delta_{\bar{\xi}} \Gamma_{\mu\nu}. \tag{89}
\end{aligned}
$$

Moreover,

$$
\gamma_\mu = g_{\mu\nu} \bar{\xi}^\nu, \tag{90}
$$

$$
g_\mu = \partial_\mu \xi^\lambda g_{\lambda\nu} \bar{\xi}^\nu + \partial_\nu \xi^\lambda g_{\mu\lambda} \bar{\xi}^\nu + \xi \cdot \partial g_{\mu\nu} \bar{\xi}^\nu + g_{\mu\nu} \xi \cdot \partial \bar{\xi}^\nu = \delta_\xi \gamma_\mu, \tag{91}
$$

$$
\bar{g}_\mu = \bar{\xi} \cdot \partial g_{\mu\nu} \bar{\xi}^\nu + \partial_\mu \bar{\xi}^\lambda g_{\lambda\nu} \bar{\xi}^\nu = \delta_{\bar{\xi}} \gamma_\mu, \tag{92}
$$

$$
\begin{aligned}
\Gamma_\mu &= -\xi \bar{\xi} \cdot \partial^2 \gamma_\mu + \xi \cdot \partial \bar{g}_\mu - \partial_\mu \bar{\xi}^\lambda g_\lambda - \bar{\xi} \cdot \partial g_\mu + \partial_\mu \xi^\lambda \bar{g}_\lambda + \partial_\mu \bar{\xi}^\lambda \xi \cdot \partial \gamma_\lambda - \partial_\mu \xi^\lambda \bar{\xi} \cdot \partial \gamma_\lambda \\
&\quad + \partial_\lambda \xi^\rho \partial_\nu \bar{\xi}^\lambda g_{\mu\rho} \bar{\xi}^\nu + \partial_\nu \xi^\lambda \partial_\lambda \bar{\xi}^\rho g_{\mu\rho} \xi^\nu \\
&= -\delta_{\bar{\xi}} g_\mu \ = \ \delta_\xi \bar{g}_\mu, \tag{93}
\end{aligned}
$$

and

$$
\bar{\gamma}_\mu = g_{\mu\nu} \xi^\nu, \tag{94}
$$

$$
\bar{f}_\mu = \partial_\mu \bar{\xi}^\lambda g_{\lambda\nu} \xi^\nu + \partial_\nu \bar{\xi}^\lambda g_{\mu\lambda} \xi^\nu + \bar{\xi} \cdot \partial g_{\mu\nu} \xi^\nu + g_{\mu\nu} \bar{\xi} \cdot \partial \xi^\nu = \delta_{\bar{\xi}} \bar{\gamma}_\mu, \tag{95}
$$

$$
f_\mu = \xi \cdot \partial g_{\mu\nu} \xi^\nu + \partial_\mu \xi^\lambda g_{\lambda\nu} \bar{\xi}^\nu = \delta_\xi \bar{\gamma}_\mu, \tag{96}
$$

$$
\bar{\Gamma}_\mu = \delta_\xi \bar{f}_\mu \ = \ -\delta_{\bar{\xi}} f_\mu. \tag{97}
$$

Finally

$$
g = g_{\mu\nu} \left(\bar{\xi}^\mu \xi^\nu - \xi^\mu \bar{\xi}^\nu \right) = 2 g_{\mu\nu} \bar{\xi}^\mu \xi^\nu, \tag{98}
$$

$$
\gamma = 2\xi \cdot \partial g_{\mu\nu} \bar{\xi}^\mu \xi^\nu + 2 g_{\mu\nu} \left(\xi \cdot \partial \bar{\xi}^\mu - \bar{\xi} \cdot \partial \xi^\mu \right) \xi^\nu = \delta_\xi g, \tag{99}
$$

$$
\bar{\gamma} = 2\bar{\xi} \cdot \partial g_{\mu\nu} \bar{\xi}^\mu \xi^\nu + 2 g_{\mu\nu} \left(\xi \cdot \partial \bar{\xi}^\mu - \bar{\xi} \cdot \partial \xi^\mu \right) \bar{\xi}^\nu = \delta_{\bar{\xi}} g, \tag{100}
$$

$$
G = \delta_\xi \bar{\gamma} = -\delta_{\bar{\xi}} \gamma. \tag{101}
$$

This completes the verification that the horizontality condition leads to identifying the ϑ- and $\bar{\vartheta}$-superpartners of the metric as BRST transforms with the rule (26). It is easy to see that if we switch off $\bar{\vartheta}$, the supermetric cannot be inverted.

4. Speculations

The two previous sections are meant to make evident the fact that the superfield formalism can encompass all the BRST symmetries which characterize various different field theories. On the other hand, apart from the details which are sometimes thorny, there is a priori very little doubt that this should be the case given the isomorphic algebraic structures of BRST symmetry and the geometry of the supermanifolds considered above. The two previous sections, on the other hand, concern the general algebraic structure of the BRST symmetry, and are independent of the particular gauge fixing which one may choose (and which, when inserted in the action, with possible appropriate auxiliary terms, must respect this symmetry). In terms of supermanifolds this corresponds to the fact that all our previous supergeometries were in fact local, that is limited to a local (super)patch. To make this point more clear it is worth summarizing a few definitions concerning supermanifolds, see for instance [13] and references therein.

Supermanifolds are defined, in analogy with manifolds, by atlases of local patches. For simplicity let me start from a simple-minded superspace spanned by even and odd (anticommuting) coordinates $x^\mu, \vartheta^a, \mu = 1, \dots, d$ and $a = 1, 2$, where $\vartheta^1 = \vartheta, \vartheta^2 = \bar\vartheta$. Then a supermanifold M is given by local patches U_α with supercoordinates $x_\alpha^\mu, \vartheta_\alpha^a$, each one being a local copy of the previous superspace, with transition functions in the overlap $U_\alpha \cap U_\beta$

$$x_\alpha^\mu = f_{\alpha\beta}^\mu(x_\beta^1, \dots | \vartheta_\beta^1, \vartheta_\beta^2), \qquad \vartheta_\alpha^a = \psi_{\alpha\beta}^a(x_\beta^1, \dots | \vartheta_\beta^1, \vartheta_\beta^2), \tag{102}$$

and compatibility conditions on triple overlaps. This is the generic definition of a supermanifold. Notice that since the transition functions $\psi_{\alpha\beta}^a$ are odd, if we set $\vartheta^a = 0$ we get $\psi_{\alpha\beta}^a = 0$. As a consequence what remains are the transition functions $f_{\alpha\beta}^\mu(x_\beta^1, \dots | 0, 0)$ of an ordinary manifold M_{red}. To be more precise I should specify whether M is complex or real. But for my needs here the notion of *split supermanifold* is going to be enough. M is split if the even transition functions $f_{\alpha\beta}^\mu$ do not depend on the odd variable and the odd transition functions depend linearly on ϑ^a, i.e.

$$\psi_{\alpha\beta}^a(x_\beta^1, \dots | \vartheta_\beta^1, \vartheta_\beta^2) = \varphi_{\alpha\beta}^{a1}(x_\beta^1, \dots x_\beta^d)\vartheta_\beta^1 + \varphi_{\alpha\beta}^{a2}(x_\beta^1, \dots x_\beta^d)\vartheta_\beta^2. \tag{103}$$

In this case M is a vector bundle over M_{red} with a fiber determined by the odd directions ϑ^1 and ϑ^2. A split supermanifold seems to be the notion of supermanifold instrumental to the BRST superfield formalism in the case of a gauge theory. In this case in fact one uses the condition of 'horizontality', i.e. the fact that the gauge supercurvature in the supermanifold must equal the curvature in M_{red}. This idea comes from the Weil horizontality in the theory of fiber bundles, whereby the geometric curvature of the bundle is actually a basic form, that is it depends only on the points in the base space, not on the fiber. It would seem at first sensible to associate the notion of horizontality for the BRST formalism in gauge theories to the fact that the relevant supermanifold is split. But, in fact, the solution may not be so simple (see below).

In the case of gauge supersymmetry in superfield formulation the relevant coordinates are $(x^m, \theta^\mu, \theta^{\dot\mu}, \vartheta, \bar\vartheta)$, so M_{red} is itself a supermanifold.[1] So M in this case is an odd vector bundle over a supersymmetric supermanifold. If diffeomorphisms are concerned the notion of horizontality I have used may not be globally valid. It is a recent acquisition that, in the case of

---

[1] The spinorial variables $\theta^{\underline\mu}$ and ϑ^a, although all odd, are of different nature and should not be mixed. For instance we cannot sum the former with the latter ones.

superstring theory in the RNS formulation, the relevant supermanifold is generally not holomorphically split (a refinement of the previous notion of split supermanifold).

But let us return to the previous case of a (non-supersymmetric) gauge theory, and let us suppose that the gauge has been fixed and a BRST invariant action is at hand. In order to compute a given amplitude one has further to integrate over the moduli space $\mathcal{M}$, i.e. over the space of gauge orbits. In this case the relevant geometry is that of a family of superspaces, that is a fibered superspace over $\mathcal{M}$ where each fiber is a copy of M. The problem is well-known and not yet satisfactorily solved for 4d gauge theories, see for instance [14,15]. The difficulty is related to the nonexistence of a continuous section over the moduli space, i.e. of a continuous choice of a representative for each orbit. When fixing the gauge-slice locally, for instance the Landau gauge, the problem manifests itself with the appearances of Gribov copies at the horizon. These are global aspects of the problem and they certainly signal the lack of a global gauge-slice, but also the failure of BRST symmetry in foliating the space of gauge connections in a regular way. At this point it is illuminating to look at Ref. [16]. In this paper the authors analyze the observable of 2d topological gravity. The local correlators they compute come from integrating over the supermoduli space of the Riemann surfaces with punctures, and turn out not to be globally defined and simultaneously to violate the BRST Ward identities. It is by using these BRST anomalies (cocycles) and the *Čeck–De Rham* complex that it is possible to construct globally defined correlators which locally reduce to the previous ones. Although this problem is different (in some sense more complicated but in general far simpler) than the gauge theory one, it teaches us an important lesson. The appearance of Gribov copies at the horizon can be seen as a breakdown of the local BRST symmetry. This symmetry should be 'bent' or 'deformed' in some way in order to provide a regular foliation of the connection space and avoid copies. But this means that the linear vector space structure of the odd fibers of M breaks down, and, as a consequence, M cannot be anymore split. On the other hand, the example of [16] tells us that it may be possible to 'repair' these fractures of the gauge theory texture by a more complicated construction, hopefully a nontrivial supermanifold with non-split structure. If the above conjecture makes any sense it would mean that the superfield formulation (of any theory) can store very significant information about its quantum structure.

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