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ELEMENTARY PARTICLES AND FIELDS
Theory

# Galactic Center Shadows: Beyond the Standard Model

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**Abstract**—In 2005 Zakharov et al. predicted an opportunity to reconstruct a shadow in Sgr A* with ground based or space—ground interferometer acting in mm or sub-mm band (the Millimetron was mentioned for such needs). The prediction was confirmed in May 2022 since the Event Horizon Telescope (EHT) Collaboration presented results of a shadow reconstruction for our Galactic Center (the shadow around the supermassive black hole in M87 was reconstructed in 2019). These reconstructions were based on EHT observations done in 2017. In 2005 Zakharov et al. also derived analytical expressions for shadow size as a function of charge for Reissner–Nordström metric and later these results were generalized for a tidal charge case. We discuss opportunities to evaluate parameters of alternative theories of gravity with shadow size estimates done by the EHT Collaboration, in particular, a tidal charge could be estimated from these observations. We also discuss opportunities to use Millimetron facilities for shadow reconstructions in M87* and Sgr A*. In our recent studies we discuss shadow formations for cases where naked singularities, wormholes or more exotic models substitute conventional black holes in galactic centers.

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## 1. INTRODUCTION. ON SHOULDERS OF GIANTS

One of the most famous theorists Steven Weinberg wrote several years ago [1]: "I am a physicist, not a historian, but over the years I have become increasingly fascinated by the history of science. It is an extraordinary story, one of the most interesting in human history. It is also a story in which scientists like myself have a personal stake. Today's research can be aided and illuminated by a knowledge of its past, and for some scientists knowledge of the history of science helps to motivate present work. We hope that our research may turn out to be a part, however small, of the grand historical tradition of natural science." I believe that these sentences reflect feelings of many other researchers even in the cases if they are not so famous and they are not authors of the Standard Model.

Very often it is very hard to recognize the author name when we mention a theorem, definition or principle. As a well-known example is the following one. People mention Columbus who discovered America but it is not named Columbia. Famous Russian mathematician V.I. Arnold formulated a principle which was generalized by M. Berry and as a result Arnold noted [2] that "Prof. M. Berry once formulated the following two principles: The Arnold Principle: If a notion bears a personal name, then this name is not the name of the discoverer. The Berry Principle: The Arnold Principle is applicable to itself."

Currently the phrase "on the shoulders of giants" is associated with the book which is a compilation of works edited and with commentary by Stephen Hawking [3]; however, more often people reminded the following letter of I. Newton to R. Hooke [4]: "If I have seen further it is by standing on the shoulders of Giants". The letter was signed such as "Your humble Servant Is. Newton" while the correspondence was sent to "For his honoured Friend Mr Robert Hooke". Commenting this famous phrase V. Arnold noted that R. Hooke was rather small, while I. Newton was tall. Very similar ideas were expressed much earlier, really John of Salisbury wrote in 1159 in *Metalogicon* [5]: "Bernard of Chartres used to say that we are like dwarfs sitting on the shoulders of giants so that we are able to see more and further than they, not indeed by reason of the sharpness of our own vision or the height of our bodies, but because we are lifted up on high and raised aloft by the greatness of giants." I. Newton knew claims of a famous British monk and he slightly re-phrased similar ideas. Perhaps, it would be reasonable to remind relations between these two great scientists (Robert Hooke and Isaac Newton). Initially, they have rather friendly conversations concerning different problems [6]. On January 6, 1680 R. Hooke wrote a letter to I. Newton where he assumed that attraction is inversely proportional to the square distance [7] and I. Newton discovered universal gravity law using Hooke's assumption.

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After this, the relationship between these two great scientists became very strained, since each of them considered himself the discoverer of the law of gravity. Indeed, discussing the authorship of a particular result is a very delicate issue. In 1990 V.L. Ginzburg published a review [8] where, in part, he discussed priority issues in applications of synchrotron emission for different astrophysical problems and he reminded that on August 3, 1987, in his plenary talk at the 20th International Conference of Cosmic Rays he showed a transparency with two sentences "Priority questions are a dirty business. Priority mania or supersensitivity is a disease." However, in this paper V.L. Ginzburg very emotionally evaluated contributions of different people in the discussed field. We have to note here that there were priority disputes between V.L. Ginzburg and I.S. Shklovsky concerning their personal contributions in applications of synchrotron emission to explain astronomical phenomena.

In 2024 a scientific community celebrated jubilees of many Soviet and Russian physicists, such as the E.S. Fradkin's Centenary, 110 years since Ya.B. Zeldovich birthday, however, as far as I know, 120 years since Abram Isaakovich Alikhanov and George Gamow birthdays were not widely celebrated (actually it was the only one birthday for these two famous physicists). Two great physicists (George Gamow and Abram Alikhanov) were born at the same day (March 4, 1904) in Odessa and Elizavetpol (today Ganja, Azerbaijan). Both these scientists were brilliant representatives of Leningrad School of physics and they created a glory of Soviet science. Both of these scientists had an unusually powerful start to their scientific careers, so Gamow became a corresponding member of the USSR Academy of Sciences at the age of 28 (i.e. one of the youngest), and Alikhanov became a corresponding member at the age of 35, and a full member of the Academy of Sciences at the age of 39, which is also considered a fairly young age for an experimental physicist [9, 10]. G. Gamow presented his scientific biography in a nice book [11] firstly published in 1970 and written in his fascinating style, where real events are intricately intertwined with beautiful fictions, so in particular, Einstein's phrase that the introduction of the $\Lambda$—term is the biggest blunder of his entire life seems like a beautiful fiction (see also the discussion of this issue in Livio's book [12]) and we will not discuss Gamow achievements in the paper.

A.I. Alikhanov was an active member of the Soviet Atomic project. According to the special decree of the State Committee of Defence it was established the Special Committee led by L.P. Berya. This committee consists of 9 members (including two academicians, I.V. Kurchatov and P.L. Kapitsa). For preliminary considerations of related issues it was established the Technical Council led by B.L. Vannikov. A.I. Alikhanov was appointed as the scientific secretary of the Council. The Council had extraordinary powers. Besides A.I. Alikhanov the following academicians were included in the Council: A.F. Ioffe, P.L. Kapitsa, I.V. Kurchatov, Yu.B. Khariton, V.G. Khlopin. A.I. Alikhanov was the founder and the first director of the Institute of Theoretical and Experimental Physics since 1945 until 1968. On March 5, 1964 the Presidium of Soviet Academy of Sciences sent a letter to congratulate A.I. Alikhanov. Alikhanov's contribution to the development of atomic science and technology was highly appreciated by the Soviet government and the leadership of the USSR Academy of Sciences. In the letter signed by President of Academy of Sciences M.V. Keldysh, Vice-president of Academy of Sciences M.D. Millionshchikov, The Chief Scientific Secretary I.M. Sisakyan, Academician—secretary of Nuclear Physics Division of Academy of Sciences V.I. Veksler it was noted that A.I. Alikhanov made a significant contribution to the realization of the Soviet atomic project [9]. In particular, it was written in the letter "When life presented new major problems for Soviet nuclear physics, you honorably solved the most important national organizational tasks of the postwar period and led the creation of a new branch of science and technology. A special place in your activities is occupied by work at the Institute of Theoretical and Experimental Physics, to which you devoted a significant part of your life and which you still head. The work of this outstanding center for research in the field of nuclei and elementary particles has received well-deserved recognition...".

There are Dzhelepov's recollections on the late 1940s in the ROSATOM library [13]. V.P. Dzhelepov remembered that he had to report about a new accelerator in Dubna in summer 1949. It means the meeting was organized just before the first Soviet atomic test on 29 August 1949. A majority of participants of the meeting were involved in the Soviet Atomic Project and they very nervous due to these circumstances. Physicists planned to convinced officials that they need this accelerator to understand laws of nature to be ready to solve further tasks formulated by the Soviet government. It looked that these plans could not be realized but A.I. Alikhanov remarkably solved the problem.

Dzhelepov remembered [13]: "...Suddenly Vannikov gets up and says: Comrade Dzhelepov, what the hell are you telling us here? Tell me, will this make oats? (and then by "oats" they meant the problems that had to be solved to create nuclear weapons).

I answer: In your understanding, Boris Lvovich, there will be no "oats" in the near future. But in the future, what we are going to study may prove useful for a variety of purposes.

Vannikov turns to Kurchatov: Igor Vasilyevich! Why are we building this accelerator? I.V. is silent, as if the question does not apply to him. Vannikov asks it again. Kurchatov does not answer.

Alikhanov gets up: Allow me to say. Everything that Dzhelepov reports is correct. The fact that scientists at one time studied the corresponding nuclear processes at low particle energies made it possible, when necessary, to create atomic weapons. And thus, we solved the problems set before us and before you by the government. But when after some time you need to solve some other problem and you ask us how to do it, we will answer that, unfortunately, we do not have an accelerator with which you can obtain the necessary new knowledge. And we will not be able to help you.

Stubborn Vannikov again turns to Kurchatov: —Igor Vasilyevich, how is this possible?

Kurchatov stood up, stroked his beard from top to bottom and said: That's it, Boris Lvovich! Everything that Dzhelepov and Alikhanov said is correct. The accelerator has already been built, we are planning to launch it in the fall. Dzhelepov will now tell us what research they plan to carry out on this accelerator, how and what equipment is being prepared. Our task is to listen to him, and if everything is reasonable, then approve and ratify the program. After that Vannikov calmed down, and I was able to continue the report...".

So, A.I. Alikhanov solved an unsolvable problem, namely, he convinced a top official that fundamental studies are needed even in difficult times.

In 1965 ITEP together with the Academy of Sciences of the Soviet Union established new journal *Yadernaya Fizika* (Nuclear Physics) and currently its name for English translations of Russian original papers is *Physics of Atomic Nuclei*. Slightly earlier (in 1956) Elsevier established *Nuclear Physics*. For decades these two journals were among leading journals in nuclear physics.

The head of theoretical department in ITEP in 1950s–1960s was I.Ya. Pomeranchuk. Brief essays on the Pomeranchuk life and his scientific achievements are presented in [14, 15]. He made a number of remarkable discoveries including his predictions of electromagnetic emission of electrons moving in magnetic fields [16–20] (this physical phenomenon was analyzed earlier by Schott [21], however, in 1940s his studies were forgotten). Soon after publications of the papers by Pomeranchuk, Iwanenko and Artsimovich the emission was discovered at a synchrotron and it is started to name it a synchrotron emission, really the first detection of X-ray radiation from electrons in 70-MeV synchrotron has been done by Elder et al. [22–24]. The history of the discovery is presented in [25–27]. According to the I.I. Gurevich's opinion this Pomeranchuk's work on magnetic bremsstrahlung (synchrotron) radiation should have been crowned with a Nobel Prize. Pomeranchuk's other works were at the same high level [28].

In 1998 ITEP was celebrating 85 years since the I. Pomeranchuk birthday, the Institute established the Pomeranchuk prize. This prize is awarded annually to one foreign theoretical physicist and to domestic one. The list of laureates of this prize is quite impressive (especially taking into account that Nobel prize winner can not be nominated for the Pomeranchuk prize), see Pomeranchuk prize winners at https://en.wikipedia.org/wiki/Pomeranchuk_Prize. Three Nobel Prize winners (Y. Nambu, G. Parisi, R. Penrose) and one recipient of the Breakthrough prize in Fundamental Physics in 2024 (Alexander Zamolodchikov) were the Pomeranchuk prize winners. Probably, it would be reasonable to mention here that ITEP Professor Karen Ter-Martyrosyan was a supervisor of Alexander Zamolodchikov, it means that Zamolodchikov began his successful scientific career at ITEP.

One of the strongest divisions at ITEP was the Mathematics Department, headed by A.S. Kronrod[1]. A.S. Kronrod participated in Soviet Atomic Project and got State (Stalin) prize for these works. He was one of the leaders among mathematicians in the Project and he was responsible for numerical solutions of equations which must be solved for a successful realization of the Project. Later, researchers of the mathematical department worked with big data from physical experiments, with numerical solutions to help ITEP physicists and provided consulting in mathematical and programming issues. In 1960s A.S. Kronrod proposed main principles of structural programming (unfortunately, his book [31] was published 40 years after it was written). Many brilliant researchers worked in this laboratory, for instance, Lev Davidovich Landau worked part-time at the Kronrod's department in ITEP since Landau was a senior

[1] A.S. Kronrod (1921–1986) was the leader of the Mathematical laboratory in ITEP in 1950s and 1960s and the founder of AI studies in USSR. He introduced concepts of structured programming before E. Dijkstra [29] (the structured programming was re-discovered and started to be popular in Soviet Union in 1979 after publication of Russian translation of the E. Yourdon book [30]). He started programs of mathematical education in School N 7 (later, many famous mathematicians taught mathematics in this school, including N.N. Konstantinov, I.N. Bernstein, A.A. Bolibrukh, V.K. Beloshapka and others). At the end of his life he cured people with severe cases of cancer.

researcher there (until 1958 when part-time work was officially forbidden even for academicians [10]) and as it is well-known L. Landau had a full position in Institute of Physical Problems led by P.L. Kapitsa. In his book [31] A.S. Kronrod remembered the case when in 1962 researchers from some institute requested mathematicians from ITEP to help them to solve a problem of optimal project organization (it was a typical problem for PERT)[2]. G.M. Adelson-Velsky and F.M. Filler solved the requested problem in a few weeks using some fragments from their chess computer program. Kronrod ironically noted [31]: "It turned out that the computers centers and research institutes that were assigned such a work, didn't cope with it in a year (maybe because they weren't involved in games, but only in real work) and they even asked to send someone to USA to study. The poor guys were unlucky here. Quite by chance, they learned from their superiors that the PERT program had been in place at the Institute of Theoretical and Experimental Physics for six months. And they sent them to study not in America, but at ITEP. That is, much closer." Perhaps, in these sentences A.S. Kronrod meant not the original PERT but its alternative which was probably even better than the original. In the ITEP mathematical department in 1960s it was decided to choose a some intellectual game to compare skills of programmers working in different countries or in other words, it was necessary to select criteria to compare AI teams. A.S. Kronrod recalled that physicists complained that chess computer programs took up too much computer time, which was urgently needed to solve problems of processing data from physics experiments. In 1960s Alexander Kronrod coined slogan "Chess is drosophila of Artificial Intelligence" and ITEP programmers started to develop chess programs. After conversations of ITEP mathematicians with US programmers they started to developed also their own chess programs in 1960s (it was done in times when "the iron curtain" was rather high). In 1967 it was organized a chess tournament between Adelson-Velskij (ITEP) and Kotok–McCarthy[3] (Stanford, USA) programs and the Soviet program beat the American one with score 3–1. A description of the ITEP program was given in [32]. Soviet newspapers wrote about a sound success of Soviet programmers and mathematicians. It is clear that it is much more pleasant to remember your successes, but if you do not remember your defeats and do not draw conclusions from them, then victories may not come. The failure of US computer program in the chess tournament was so disappointed for American AI experts that they often remembered results of the tournament and the American failure in the competition in their books and articles starting since one of the first monograph on the subject [34] which was translated into Russian in 1973 and perhaps it was among the first books on AI in Russian. After the tournament between Soviet and American Chess programs in 1967 the Kronrod's slogan was posted in the AI Laboratory of Stanford University. American experts drew conclusions from the defeat in the first tournament of chess programs and set the task of writing a program that would play better than not only other computer program but better than the world chess champion. Such a task was realized in 1997 when the IBM Deep Blue program beat the world chess champion in the six-game match with score 3.5–2.5 with a time control as it was organized between humans. The summary of the activity in computer chess was written by John McCarthy in his short essay [35] where he wrote "In 1965 the Russian mathematician Alexander Kronrod said, "Chess is the Drosophila of artificial intelligence."" In his article [36] published earlier McCarthy chose the title "Chess is the Drosophila of AI". In 1974 at the first world chess championship among computer programs, the world champion was Kaissa, which was created by Donskoy and Arlazorov, who moved from ITEP to the Institute of Control Sciences. After that, in these championships success accompanied American AI specialists. Now everyone (from physicists to politicians) is speculating on AI problems and ways to solve them but this early success of Soviet (ITEP) mathematicians and programmers is almost completely forgotten. It is known that in 2024, the Nobel Prizes in Physics and Chemistry were awarded to scientists for their work in developing and actively using AI. Many physicists and chemists have criticized these decisions of the Nobel Committee, arguing that this is neither physics nor chemistry. I am not one of the critics of these decisions, since the adoption of these decisions demonstrated a recognition of the need to expand approaches to solving problems challenging physicists and chemists. There is no point in debating whether the discussed applications of AI to physics and chemistry are really the subject of physics or chemistry, since it is now clear to everyone that AI is a necessary tool in the natural sciences, whereas

[2] Program evaluation and review technique (PERT) was initially developed in 1958 to organize the U.S. Navy's Polaris nuclear submarine project in an optimal way. Later, algorithms of PERT were applied for many other projects.

[3] In 1955, John McCarthy who was an assistant professor of Mathematics at Dartmouth College in New Hapshire, decided to develop ideas on thinking machines. He chose the name 'Artificial Intelligence' for the new field. McCarthy and his co-authors wrote a proposal for 2-month 10-man Workshop for studies of artificial intelligence [33]. Really this Workshop was organized in Dartmouth in summer 1956. Usually, this Workshop is associated with the start of AI studies in the world and currently John McCarthy is called the father of AI.

60–70 years ago this was clear to a few, such as John McCarthy and A.S. Kronrod. Further developments in AI have confirmed their insights, for instance, several years ago, DeepMind company presented AlphaZero neutral network program which can efficiently play chess, Go, Shogi and this program was trained via self-play [37]. It is regrettable that this area of research (AI), which was at the highest scientific level in the past in our country, did not receive a proper support and development in the future. To paraphrase a well-known saying, we can say that a country that does not want to feed its AI experts will feed foreign ones.

The second ITEP director was Ivan Vasilievich Chuvilo (October 9, 1924–March 16, 2001). In the battles near Stalingrad I.V. Chuvilo was seriously wounded and lost his right hand. After demobilization, he began studying at the Physics Department of Moscow State University. Since I.V. Chuvilo was seriously injured, according to the rules of that time, he could not be accepted as a student at the physics department, physicians offered him the mechanics and mathematics department of MSU, but he convinced them that he can be physicist and became an outstanding experimentalist and organizer of science. He was the second ITEP director for almost 30 years (1968–1997). In 1954, on the invitation of Vladimir Veksler, Ivan Chuvilo became his deputy for scientific work at the Electrophysical Laboratory of the Soviet Academy of Sciences, which in 1956, with the formation of the Joint Institute for Nuclear Research, was reorganized into the High Energy Laboratory of JINR (therefore, JINR celebrated Chivilo's centenary in October 2024, https://www.jinr.ru/posts/pamyati-soosnovatelya-lve-oiyai-ivana-chuvilo/). During the Chuvilo's directorship ITEP did not lose its high position not only among physical institutes, but more generally among Soviet scientific institutions. As I.M. Khalatnikov noted [38] that in 1990 ITEP had the second place among all Soviet ordered according to their citation impacts CI (the number of citations divided by number of papers). The Landau ITP was at the first place and it had CI = 15.86, while ITEP had CI = 13.31, other Soviet institutes had CI less than 10 (see, also Table 3 in the paper by Eugene Garfield in [39]). Even in the last years ITEP positions was rather high in respect to other Government Research Institutions, for instance, according to https://www.scimagoir.com in 2019 ITEP was at the forth place among other Russian Governmental Institutions, see https://www.scimagoir.com/rankings.php?sector=Government&country=RUS&year=2013 and Fig. 1.

A. Strumia and R. Torre proposed an original algorithm to rank to order all institutions in world working in fundamental physics [40], in Table 11 of the paper ITEP was at the tenth place (it was the highest place among Russian Institutions), JINR (Dubna) was at the thirteen place (in the updated version of the same paper published on April 6, 2021 in arxiv.org the ITEP place was 15, while JINR place was the same). In the column of the Table 11 in considerations of publications after 2010 ITEP was at the twenty seventh place (the second position among all Russian Institutions), while JINR is at the seventh place (it was the highest position among Russian research institutions). In the updated version of column published on April 6, 2021, the ITEP place was 41st. It is possible to note a rapid deterioration of ITEP positions in 2019–2021 as one can see also from Fig. 1.

The reputation of an Institute and its scientific staff is determined by their results presented in the publications and the number of citations to their work. Usually, the informal opinion about the high level of a scientific institution is supported by its high positions in various scientific rankings.

In year of the celebrations of the centenary of Alikhanov's birth the Chief Editor of *Physics of Atomic Nuclei* Yu.G. Abov wrote in [41, 42]: "Not one monument was erected for Alikhanov, but he created his monument himself—it is the Institute of Theoretical and Experimental Physics, which was organized owing to his efforts and which was recently named after him."

Summarizing, I would like to note that we must remember our domestic giants since we "stay on their shoulders."

## 2. SHADOWS AND RESPECTIVE DISCUSSIONS

After the shadow reconstruction for M87* thousands of papers have been published and sometimes references and presentation of the subject are not fully adequate to discussed issues. For instance, in paper [43] published a few years ago it was discussed the possibility of observing black hole shadows using ground-space interferometers. In this paper, it is not mentioned that the idea of observing the shadow of a black hole in the Galactic Center using ground-space interferometers was first put forward in [44], in spite of the fact that this prediction and its realization were well-known in the literature, see for instance, [45] where the authors wrote "The idea of reconstructing the shadow of a black hole in the Galactic Center using global interferometers operating in the millimeter wavelength was initially suggested in [1]." (In [45] our paper [44] is cited as [1]). We also note that the method used by the authors of the work under discussion for obtaining isotropic geodesics in the

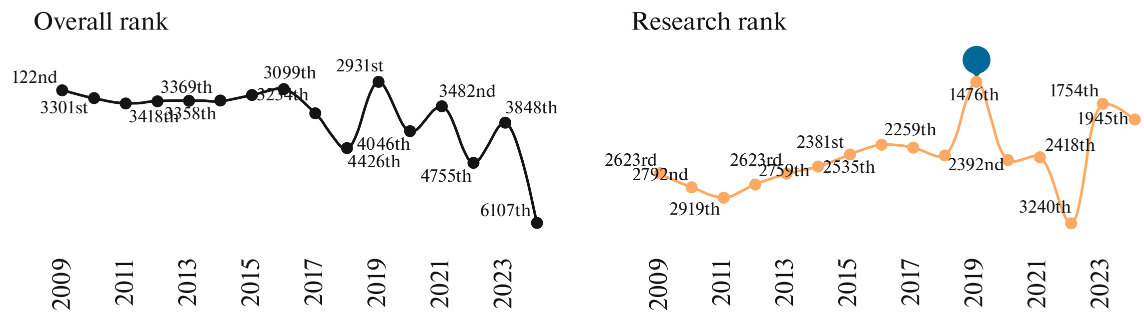


**Fig. 1.** ITEP positions according to https://www.scimagoir.com for overall and research indicators as dependencies on year. As one can see in 2019 ITEP had its highest position among research institutes in the world rating.

black hole metric was first proposed in [46], subsequently published in *Soviet Astronomy* [47] (later this journal became known as *Astronomy Reports*) and was used in a number of works, for example, in article [48]. However, later, without any changes, this method of integration of isotropic geodesics was used in a series of papers by Zakharov and Repin, published in 1999–2005. Recently, this approach was used in [43], but it was incorrectly attributed, since paper [49] was indicated as a reference, despite the fact that the authors are well aware that this approach was previously used in the papers indicated above, since there are corresponding references in [49]. The introduction of the concept of *Mino time* will be also discussed, which is also encountered in the discussion of obtaining geodesics in the Kerr metric, which corresponds to the variable $\sigma$ in papers [46, 47].

The idea of reconstructing the shadow of a black hole in the Galactic Center using global ground-space (and ground based) interferometers operating in the mm range was firstly proposed in [44], in particular, the possibility of using the planned Millimetron space telescope, as a space element of the ground–space interferometer, was mentioned. This prediction was remarkably fully confirmed when the Event Horizon Telescope (EHT) collaboration reconstructed the size of the shadow in the Galactic Center in [50] using a ground-based global interferometer operating in the millimeter range. It is known that the prediction of what object should be observed, by what means, and what should be the result of the observations is rarely exactly realized, as happened in the case of the prediction of the possibility of reconstructing the shadow in the GC, realized by the EHT collaboration. Thus, it is extremely important to note the fact that the prediction about the possibility of reconstructing the shadow of a black hole in the Galactic Center has been realized. It also seems unacceptable to discuss the possibility of reconstructing shadows in the vicinity of supermassive black holes without pointing out that this proposal had already been made in paper [44] many years before the publication of paper [43].

The size of the shadow in the Galactic Center is approximately 50 arc microseconds, while the angular resolution of the ground-based interferometer at a wavelength of 1.3 mm is about 25 arc microseconds (despite this, it turned out to be possible for the EHT collaboration to reconstruct the shadow for this black hole). Undoubtedly, the lack of understanding (or desire) of the authors of the article to note that, based on the results of VLBI observations in the mm range of the Event Horizon Telescope Collaboration, the size of the shadow in the Galactic Center was determined in accordance with the predictions of paper [44], indicates an inadequate discussion of the material on this issue. Next, we recall the equations that were previously given in papers [46–48], used in paper [43] without adequate attribution of the sources of the approach used.

The Kerr metric can be written in Boyer–Lindquist coordinates and natural units ($G = c = 1$) as

$$\begin{aligned} ds^2 = &-\frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta d\phi\right)^2 \\ &+ \frac{\sin^2\theta}{\rho^2}\left[\left(r^2+a^2\right)d\phi - a dt\right]^2 \\ &+ \frac{\rho^2}{\Delta}\left(dr\right)^2 + \rho^2\left(d\theta\right)^2, \end{aligned} \tag{1}$$

$$\Delta = r^2 - 2Mr + a^2, \quad \rho^2 = r^2 + a^2\cos^2\theta, \tag{2}$$

where $M$ is the mass of the black hole, $a = S/M$ ($S$ is the angular momentum of the black hole). As shown by B. Carter [51], the equations of geodesics reduce to the integration of the following equations

$$\rho^4\left(\frac{d\theta}{d\lambda}\right)^2 = \Theta\left(\theta\right), \tag{3}$$

$$\rho^4\left(\frac{dr}{d\lambda}\right)^2 = R\left(r\right), \tag{4}$$

$$\rho^2\frac{d\phi}{d\lambda} = -\left(aE - \frac{L_z}{\sin^2\theta}\right) + \frac{a}{\Delta}P\left(r\right), \tag{5}$$

$$\rho^2 \frac{dt}{d\lambda} = -a\left(aE\sin^2\theta - L_z\right) + \frac{\left(r^2+a^2\right)}{\Delta} P\left(r\right), \qquad (6)$$

where

$$\Theta\left(\theta\right) = \mathcal{Q} - \cos^2\theta\left[a^2\left(\mu^2 - E^2\right) + L_z^2/\sin^2\theta\right], \quad (7)$$

$$P\left(r\right) = E\left(r^2+a^2\right) - L_z a, \qquad (8)$$

$$R\left(r\right) = P^2\left(r\right) - \Delta\left[\mu^2 r^2 + \left(L_z - aE\right)^2 + \mathcal{Q}\right]. \quad (9)$$

The constants $E$, $\mu$, $L_z$, and $\mathcal{Q}$ are related to the particle, $E$ is its energy, $\mu$ is its mass, $L_z$ is its angular momentum, $\mathcal{Q}$ is the Carter constant of separation of variables, namely,

$$\mathcal{Q} = p_\theta^2 + \cos^2\theta\left[a^2\left(\mu^2 - E^2\right) + L_z^2/\sin^2\theta\right]. \quad (10)$$

We recall the corresponding equations proposed earlier in the works [46–48], used for the rapid numerical derivation of geodesics. If the motion of massive particles in the Kerr metric ($\mu \neq 0$) is considered. Let us reduce Eq. (4) to dimensionless form,

$$\rho^4 \frac{1}{\mu^2 M^2}\left(\frac{dr}{d\lambda}\right)^2 = \hat{R}\left(\hat{r}\right), \qquad (11)$$

where

$$\hat{R}\left(\hat{r}\right) = \left(\hat{E}^2 - 1\right)\hat{r}^4 + 2\hat{r}^3 + \left[\hat{a}^2\left(\hat{E}^2-1\right) - \hat{L}_z^2 - \hat{\mathcal{Q}}\right]\hat{r}^2 + 2\left[\left(\hat{a}\hat{E} - \hat{L}_z\right)^2 + \hat{\mathcal{Q}}\right]\hat{r} - \hat{a}^2\hat{\mathcal{Q}}, \qquad (12)$$

and $\hat{E} = E/\mu$, $\hat{a} = a/M$, $\hat{L}_z = L_z/\left(M\mu\right)$, $\hat{\mathcal{Q}} = \mathcal{Q}/\left(M\mu\right)^2$, $\hat{r} = r/M$. Thus, we obtain

$$M^2\left(\hat{r}^2 + \hat{a}^2\cos^2\theta\right)^2\left(\frac{d\hat{r}}{d\tau}\right)^2 = \hat{R}\left(\hat{r}\right). \qquad (13)$$

In this case, we can introduce a new independent variable $\sigma$ as was done in papers [46–48][4]

$$\left(\frac{d\hat{r}}{d\sigma}\right)^2 = \hat{R}\left(\hat{r}\right), \qquad (14)$$

where

$$\frac{d\tau}{d\sigma} = M\left(\hat{r}^2 + \hat{a}^2\cos^2\theta\right), \qquad (15)$$

[4] Later, a similar change of variables was made by J. Mino in the work [52], although some authors know that a similar change was made earlier, in particular, in the work [48], as indicated in the dissertation of E. Hackmann [53], this replacement of the variable similar to the one we use $\sigma$ in the works of many authors is called Mino time, which can be seen from the works of Hackmann and her co-authors cited in the mentioned dissertation.

or

$$\frac{d\hat{\tau}}{d\sigma} = \hat{r}^2 + \hat{a}^2\cos^2\theta, \qquad (16)$$

where $\hat{\tau} = \tau/M$. Similarly, from Eq. (3) we obtain

$$\left(\frac{d\theta}{d\sigma}\right)^2 = \hat{\Theta}\left(\theta\right), \qquad (17)$$

where

$$\hat{\Theta}\left(\theta\right) = \hat{\mathcal{Q}} - \cos^2\theta\left[\hat{a}^2\left(1 - \hat{E}^2\right) + L_z^2/\sin^2\theta\right]. \qquad (18)$$

Similarly, we transform Eq. (5) and obtain

$$\frac{d\phi}{d\sigma} = -\left(\hat{a}\hat{E} - \hat{L}_z/\sin^2\theta\right) + \frac{\hat{a}}{\hat{\Delta}}\left[\hat{E}\left(\hat{r}^2+\hat{a}^2\right) - \hat{L}_z\right], \qquad (19)$$

where

$$\hat{\Delta} = \hat{r}^2 - 2\hat{r} + \hat{a}^2. \qquad (20)$$

And instead of Eq. (6) we obtain the following equation

$$\frac{d\hat{t}}{d\sigma} = -\hat{a}\left(\hat{a}\hat{E}\sin^2\theta - \hat{L}_z\right) + \frac{\hat{r}^2+\hat{a}^2}{\hat{\Delta}}\left[\hat{E}\left(\hat{r}^2+\hat{a}^2\right) - \hat{L}_z\right], \qquad (21)$$

where $\hat{t} = t/M$. Thus, the problem of finding geodesics for massive particles is reduced to quadratures (calculations of integrals corresponding to Eq. (38), (17), however, these integrals have singularities at the roots of the functions $\hat{R}\left(\hat{r}\right)$ and $\hat{\Theta}\left(\theta\right)$ (in the case where the derivatives at the roots of these functions are non-zero—these singularities correspond to turning points in the coordinates $r$ and $\theta$, respectively).

Let us consider the corresponding equations for isotropic geodesics ($\mu = 0$). In this case, similarly to Chandrasekhar [54] we get

$$\frac{M^2}{E^2}\left(\hat{r}^2 + \hat{a}^2\cos^2\theta\right)^2\left(\frac{dr}{d\lambda}\right)^2 = \hat{R}_{ph}\left(\hat{r}\right), \qquad (22)$$

where

$$\hat{R}_{ph}\left(\hat{r}\right) = \hat{r}^4 + 2\hat{r}^3 + \left[\hat{a}^2 - \xi^2 - \eta\right]\hat{r}^2 + 2\left[\left(\hat{a}-\xi\right)^2 + \eta\right]\hat{r} - \hat{a}^2\eta, \qquad (23)$$

where $\eta = \mathcal{Q}/\left(M^2E^2\right)$ and $\xi = L_z/(ME)$ are Chandrasekhar constants.

Similar to the equations describing the motion of massive particles, the affine parameter $\lambda$ can be replaced by the parameter $\sigma$ in accordance with the relation

$$\frac{d\lambda}{d\sigma} = \frac{M}{E}\left(\hat{r}^2 + \cos^2\theta\right). \quad (24)$$

Similarly, we obtain an equation for the motion of a photon along the coordinate $\theta$

$$\frac{d\theta}{d\sigma} = \eta + \cos^2\theta\left(\hat{a}^2 - \xi^2/\sin^2\theta\right), \quad (25)$$

coordinate $\phi$

$$\frac{d\phi}{d\sigma} = -\left(\hat{a} - \xi/\sin^2\theta\right) + \left(\hat{a}/\hat{\Delta}\right)\left(\hat{r}^2 + \hat{a}^2 - \xi\hat{a}\right), \quad (26)$$

and coordinate $\hat{t} = t/M$

$$\frac{d\hat{t}}{d\sigma} = -\hat{a}\left(\hat{a}\sin^2\theta\right) + \left[\left(\hat{r}^2 + \hat{a}^2\right)/\hat{\Delta}\right]\left(\hat{r}^2 + \hat{a}^2 - \xi\hat{a}\right). \quad (27)$$

For that, to avoid the peculiarities associated with calculating the integral in Eq. (22), we replace this equation with two

$$\frac{d\hat{r}}{d\sigma} = r_1, \quad (28)$$

$$\frac{dr_1}{d\sigma} = 2\hat{r}^3 + \left[\hat{a}^2 - \xi^2 - \eta\right]\hat{r} + \left[(\hat{a} - \xi)^2 + \eta\right], \quad (29)$$

and similarly transform the equation for the coordinate $\theta$

$$\frac{d\theta}{d\sigma} = \theta_1, \quad (30)$$

$$\frac{d\theta_1}{d\sigma} = -\hat{a}^2\sin\theta\cos\theta + \xi^2\left(\cos\theta/\sin\theta\right)\left[1 + \left(\cos^2\theta/\sin\theta\right)^2\right]. \quad (31)$$

In [46–48] it was proposed to integrate Eqs. (26)–(31) to obtain isotropic geodesics in the Kerr metric and the same approach was used in [43] without proper references.

As noted earlier, direct integration of the relations (38), (17) can be reduced to elliptic integrals or to integrals of other special functions, but their calculation of the integral (38) near the turning points of the radial geodesics $\hat{r}_t$

$$\hat{R}(\hat{r}_t) = 0, \quad \left.\frac{\partial\hat{R}}{\partial\hat{r}}\right|_{\hat{r}=\hat{r}_t} > 0 \quad (32)$$

is reduced to calculating an improper integral, which converges, although it is necessary to keep in mind that the value of $\sqrt{\hat{R}(\hat{r})}$ at the point $\hat{r}_t$ changes sign, i.e. the decrease of the radial coordinate changes to an increase or vice versa. When approaching the stationary points $\hat{r}_s$, which correspond to the double roots of the polynomial $\hat{R}(\hat{r})$ and for which

$$\hat{R}(\hat{r}_s) = 0, \quad \left.\frac{\partial\hat{R}}{\partial\hat{r}}\right|_{\hat{r}=\hat{r}_s} = 0 \quad (33)$$

logarithmic divergence takes place. As shown in [55], if the motion of particles from infinity to a black hole is considered, for a fixed value of the particle energy $\hat{E}$, the parameters $\hat{\mathcal{Q}}$ and $\hat{L}_z$satisfying relations (33) separate the set of the parameters $\hat{\mathcal{Q}}$ and $\hat{L}_z$corresponding to the capture of particles and the set corresponding to their scattering. Similar remarks can also be made regarding the integration of isotropic geodesics in the Kerr metric.

## 3. BLACK HOLE SHADOWS AND PHOTON CIRCULAR ORBITS

About half a century ago, J.M. Bardeen considered a thought experiment in which a black hole with extreme rotation ($\hat{a} = 1$) at infinity has a flat radiating screen [57] and then a distant observer located in the equatorial plane can observe a dark spot since it is assumed that photons move along geodesics and are not scattered (for example, by electrons) in the vicinity of the black hole. A similar picture was later reproduced in the book [54]. It is clear that the region of impact parameters corresponding to the shadow is obtained from the region of impact parameters corresponding to the capture of photons by mirror reflection of this region relative to the axis of rotation. However, neither Bardeen nor Chandrasekhar discussed the possibility of observational detection of such a dark spot (shadow), since (a) even supermassive black holes have very small shadows[5]; (b) as a rule, there is no luminous screen behind an astrophysical black hole, however, we understood that secondary images of distant sources must be concentrated around shadows as it was discussed, for instance in [56]; (c) it is quite difficult to distinguish a

[5] As we know, it is now possible to reconstruct the shadow in the vicinity of a black hole in the Galactic Center and in the center of the galaxy M87 and the size of the shadow for these objects is several tens of arc microseconds, thus for black holes of stellar mass in our Galaxy the size of the shadow is approximately a million times smaller. In the 1970s and 1980s there was an intense discussion about the nature of the object in the center of our Galaxy and a number of researchers believed that in the case of the presence of a black hole in the Galactic center, its mass should not exceed several hundred solar masses.

dark region (shadow) from an area with low luminosity. Answers to questions about how to reconstruct a shadow from observations were given in the work [44]. Indeed, in the 2000s, the ground-space RadioAstron (whose angular resolution at the shortest wavelength of 1.3 cm was about 7 arc microseconds) and Millimetron (with an angular resolution several orders of magnitude better) were already discussed, as well as ground-based global VLBI systems with an angular resolution of about several tens (or several units) of arc microseconds, which is comparable to the size of the shadow in the Galactic center and M87*. In the presence of radiation sources outside photon circular orbits, secondary images of these sources should be near the shadow and this makes it possible to reconstruct the shadow of a black hole, as predicted in [44] and implemented in [50] when reconstructing the shadow of a black hole located in the Galactic center.

Recall the definition of photon circular orbits. Consider the case of a photon moving from infinity to a black hole. As shown in [55], if the polynomial $\hat{R}_{ph}(\hat{r})$ has no roots at the value $\hat{r} > \hat{r}_+ = 1 + \sqrt{1 - \hat{a}^2}$, then the photon is captured by the black hole, if the polynomial $\hat{R}_{ph}(\hat{r})$ has a single root $\hat{r}_t > \hat{r}_+ \left(\hat{R}_{ph}(\hat{r}_t) = 0\right) \left(\text{and } \frac{\partial \hat{R}_{ph}}{\partial \hat{r}}\Big|_{\hat{r}=\hat{r}_t} > 0\right)$, then we have scattering of a photon on a black hole and approach of the photon to the black hole is replaced by removal and in case there is a double root of the polynomial $\hat{R}_{ph}(\hat{r}_s) = \frac{\partial \hat{R}_{ph}}{\partial \hat{r}}\Big|_{\hat{r}=\hat{r}_s} = 0$ then the photon moving from infinity approaches an orbit with a constant value of the radial coordinate $\hat{r}_s = \text{const}$ and these orbits are often called *circular* in the literature, although they are exactly circular only in the case of equatorial motion at $\eta = 0$.

## 4. SIMPLEST EXAMPLE

Let us consider the formation of a shadow for the simplest model of a Schwarzschild black hole. In this case, Eq. (12) is of the form

$$\hat{R}_{ph}(\hat{r}) = \hat{r}^4 - L^2\hat{r}^2 + 2L^2\hat{r}, \tag{34}$$

where $L^2 = \xi^2 + \eta$. Relation (34) can also be obtained directly from the expression for the Schwarzschild metric, rather than by reduction from the Kerr metric. From relations (22), (34) it is clear that for the motion to be possible for a given value of $\hat{r}_*$ it is necessary that $\hat{R}_{ph}(\hat{r}_*) \geq 0$, moreover, the single root of the polynomial corresponds to the turning points of the geodesic, where the derivative changes sign, and the double root corresponds to an unstable circular orbit. Using notations from [58] we introduce the following function

$$B^{-2}(\hat{r}) = \hat{r}^{-2}\left(1 - \frac{2}{\hat{r}}\right), \tag{35}$$

then the right hand side of the Eq. (34) can be written in the following form

$$\hat{R}_{ph}(\hat{r}) = \hat{r}^4\left[1 - L^2 B^{-2}(\hat{r})\right], \tag{36}$$

therefore, requirement for a positivity of $\hat{R}_{ph}(\hat{r})$ is equivalent to

$$B^{-2}(\hat{r}) \leq \frac{1}{L^2}. \tag{37}$$

It is easy to see that maximal value of function $B^{-2}(\hat{r})$ is 1/27 and corresponding radial coordinate $\hat{r} = 3$. We can see that if we consider the motion of a photon from infinity to a Schwarzschild black hole, there is a critical value of the impact parameter $L_{\text{crit}} = 3\sqrt{3}$, which corresponds to an unstable circular orbit, onto which the orbit of a photon is moving from infinity. If, similarly to J. Bardeen [57], we consider a screen at infinity behind a Schwarzschild black hole, then due to the capture of photons with impact parameter values $L < L_{\text{crit}}$, a sufficiently distant observer can see a dark spot with a radius of $L_{\text{crit}}$ (the length is expressed in units of the black hole mass), i.e. a shadow from the black hole of the specified size appears (in the case of a Schwarzschild black hole, the shape of the shadow is a circle). If the impact parameter $L > L_{\text{crit}}$, then this photon can be detected by the observer. The graph for $B^{-2}(\hat{r})$ is shown in Fig. 2. As noted above, photons with impact parameters less than critical one are captured by the black hole, which is true for the Schwarzschild black hole and has been known for a long time [58], however, for other metrics used for the description, this statement may be incorrect, as shown below for the case of the naked Reissner–Nordström singularity[6].

## 5. SHADOWS OF KERR BLACK HOLES

Let us consider the motion of a photon from infinity to a Kerr black hole. In [55], a set of values of the

[6] Despite the principle of cosmic censorship declared many years ago by R. Penrose, which consists in the fact that there are no objects in nature that would be described by a metric corresponding to a naked singularity (see, for example, the article [59]), in recent years, solutions containing a naked singularity have been actively considered in theoretical astrophysics both within the framework of the standard GR and within the framework of alternative theories of gravity, however, there are no convincing observational or theoretical arguments in favor of considering naked singularities in those cases when astrophysical black holes conventionally were considered, usually not given.

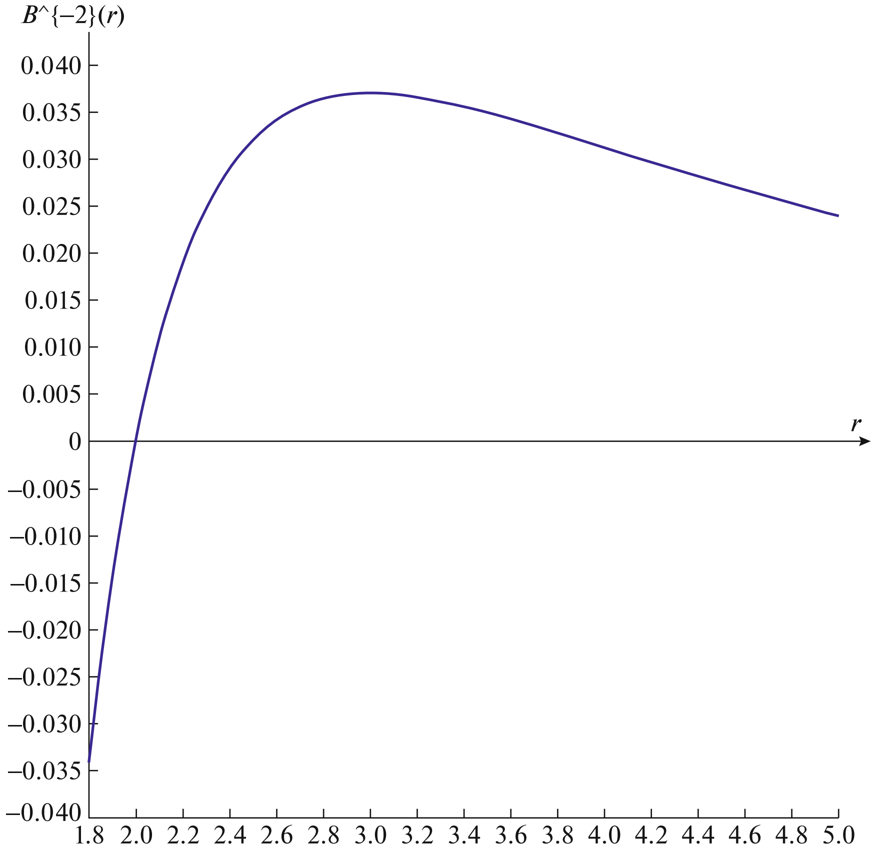


**Fig. 2.** Graph of the function $B^{-2}(\hat{r})$. The function reaches its maximum $1/27$ at $\hat{r} = 3$. Since function $B^{-2}(\hat{r})$ must be less than $1/L^2$, then, for photon capture we have $L^2 < 27$, so, the critical value $L^2 = 27$ corresponds to the case when photon approaches to the circular photon orbit, while for $L^2 > 27$ the photon after approaching the black hole again goes to infinity.

constants of photon motion $(\xi, \eta)$ for three different types of motion (capture, scattering, and asymptotic approximation of the photon trajectory to a surface with a constant value of the Boyer–Lindquist radial coordinate ($r = \text{const}$) is considered. This paper also presents a classification of unlimited orbits with arbitrary energy at infinity. A classification of particle motion occurring in a limited region of the radial coordinate is given in [60]. Rewriting Eqs. (22), (23) we obtain

$$\left(\frac{d\hat{r}}{d\sigma}\right)^2 = \hat{R}_{ph}(\hat{r}), \tag{38}$$

where

$$\hat{R}_{ph}(\hat{r}) = \hat{r}^4 + \left[\hat{a}^2 - \xi^2 - \eta\right]\hat{r}^2 + 2\left[(\hat{a} - \xi)^2 + \eta\right]\hat{r} - \hat{a}^2\eta. \tag{39}$$

In the paper [55] it is shown that the set of values of the parameters $(\xi, \eta)$, determining the capture of a photon by a black hole, corresponds to the absence of roots of the polynomial $\hat{R}(\hat{r})$ for $\hat{r} > \hat{r}_g$, where $r_g = 1 + \sqrt{1 - \hat{a}^2}$ is the outer event horizon of the Kerr black hole. In the case of a single root of the polynomial, i.e. $\hat{R}(\hat{r}_t) = 0$ for $\hat{r}_t > \hat{r}_g$ and $\hat{r}_t$ is the minimum value of the radial coordinate and thus the derivative of the polynomial $\hat{R}(\hat{r}_t)$ at the point $\hat{r}_t$ changes sign from minus to plus. In the case of a double root of the polynomial, i.e. $\hat{R}(\hat{r}_{ph}) = \left.\frac{\partial \hat{R}(\hat{r})}{\partial \hat{r}}\right|_{\hat{r}_t = \hat{r}_{ph}} = 0$ for $\hat{r}_{ph} > \hat{r}_g$ and the set defines the function $\eta(\xi)$. In the work [55] it is shown that the function $\eta(\xi) \geq 0$. It is known that negative values of $\eta$ correspond to the so-called vortex motions of photons, when photons move in the region $\theta_{\min} < \theta < \theta_{\max}$ [61].

## 6. SHADOWS FOR REISSNER–NORDSTRÖM BLACK HOLES

An analytical expression for the photon capture cross section of the Reissner–Nordström black hole was found 35 years ago [62] and these relations were subsequently given in papers [63, 64]. Later, equivalent relations were rediscovered in papers of other authors. In paper [65] these relations were used to estimate the charge from the size of the shadow for the black hole in the Galactic center, i.e. it was indicated that such a problem can be solved, which was done by the EHT collaboration after 5 years of processing the observational data obtained in April 2017, and the electric charge for the black hole in the Galactic center was estimated [50]. Soon after, in [66] these observational estimates of the shadow size were used to constrain the tidal charge and the Kazakov—Solodukhin charge, discussed in [67], for the black hole at the Galactic Center.

Let us recall the expression for the Reissner—Nordström metric in natural units ($G = c = 1$)

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \quad (40)$$

where $M$ is the mass of the black hole and $Q$ is its charge. The constants $E$ and $L$ are related to the photon, namely $E$ is its energy, $L$ is the angular momentum of the photon. If we introduce the normalized distance, impact parameter and charge, we have $\hat{r} = r/M, \xi = L/(ME), \hat{Q} = Q/M$. If we introduce the quantities $l = \xi^2, q = \hat{Q}^2$, then, as is easy to see from relation (23), the equation for isotropic geodesics in the Reissner—Nordström metric has the form

$$\left(\frac{d\hat{r}}{d\sigma}\right)^2 = R_{RN}(\hat{r}), \quad (41)$$

where

$$R_{RN}(\hat{r}) = \hat{r}^4 - l\hat{r}^2 + 2l\hat{r}^4 - ql, \quad (42)$$

and the critical value of the impact parameter, corresponding to a multiple root of the right-hand side of Eq. (41), determines the square of the radius shadows of a black hole

$$l_{\rm crit} = \frac{\left(8q^2 - 36q + 27\right) + \sqrt{D}}{2(1-q)}, \quad (43)$$

where $D = -512\left(q - \frac{9}{8}\right)^3$. In the case of a tidal charge [68, 69] (or a scalar-tensor theory of the Horndeski type [70, 71]), the value of $q$ can be negative.

Similarly to the Schwarzschild metric case we write the right-hand side of Eq. (41) in the form

$$R_{RN}(\hat{r}) = \hat{r}^4\left[1 - \frac{l}{\hat{r}^2}\left(1 - \frac{2}{\hat{r}} + \frac{q}{\hat{r}^2}\right)\right]. \quad (44)$$

We introduce the function $B^{-2}(\hat{r})$ which is a generalization of the Schwarzschild metric function $B^{-2}(\hat{r})$ for a Reissner—Nordström metric, namely

$$B^{-2}(r) = \frac{1}{\hat{r}^2}\left(1 - \frac{2}{\hat{r}} + \frac{q}{\hat{r}^2}\right). \quad (45)$$

We can find a local maximum for function $B^{-2}(r)$ from a critical point condition

$$\frac{dB^{-2}(\hat{r})}{d\hat{r}} = 0, \quad (46)$$

or

$$\hat{r}^2 - 3\hat{r} + 2q = 0, \quad (47)$$

and we obtain a radius for circular orbit of photon

$$\hat{r}_{sp} = \frac{3 + \sqrt{9 - 8q}}{2}. \quad (48)$$

Substituting $\hat{r}_{sp}$ in $B^{-2}(\hat{r})$ we obtain an expression for $l_{\rm crit}$

$$l_{\rm crit} = B^2(\hat{r}_{sp}), \quad (49)$$

which is equivalent to Eq. (43).

## 7. AN EXAMPLE OF CIRCULAR PHOTON ORBITS IN THE ABSENCE OF SHADOWS FOR NAKED REISSNER—NORDSTRÖM SINGULARITIES

The right-hand side of the Eq. (45) or in other words, function $B^{-2}(\hat{r})$ is given for some values of the parameters in Fig. 3. Thus, the green curve describes the case of the metric with ($q = 0.9$). For $0 \le q \le 1$ it is possible shadow formation since photons are captured by black hole for $\xi < \sqrt{l_{\rm crit}}$, photons are deflected by black hole for $\xi > \sqrt{l_{\rm crit}}$ (we would like to stress here that in spite of an existence of a turning point $r_t(R_{RN}(\hat{r}_t) = 0$ these photons are captured since the turning points are under horizon ($\hat{r}_t = 1 + \sqrt{1-q}$)), while photons are approaching circular photon orbits for $\xi = \sqrt{l_{\rm crit}}$. The black curve corresponds to the extreme Reissner—Nordström metric $q = 1$, other curves correspond to super-extreme metrics, namely, for $q = 1.0625$ (the blue curve), for $q = 1.125$ (the red curve) and for $q = 1.2$ (the violet curve). From inspection of the curves, we conclude that for $0 \le q \le 1$ and critical impact parameter $\xi = \sqrt{l_{\rm crit}}$ we have circular photon orbits and shadows corresponding to them, while for $1 \le q \le 9/8$ photons with critical impact parameters $\xi = \sqrt{l_{\rm crit}}$ moving from infinity are approaching circular photon orbits with $r = r_{sp}$ but there are no shadows while for $q > 9/8$ there are no neither shadows nor limiting circular photon orbits.

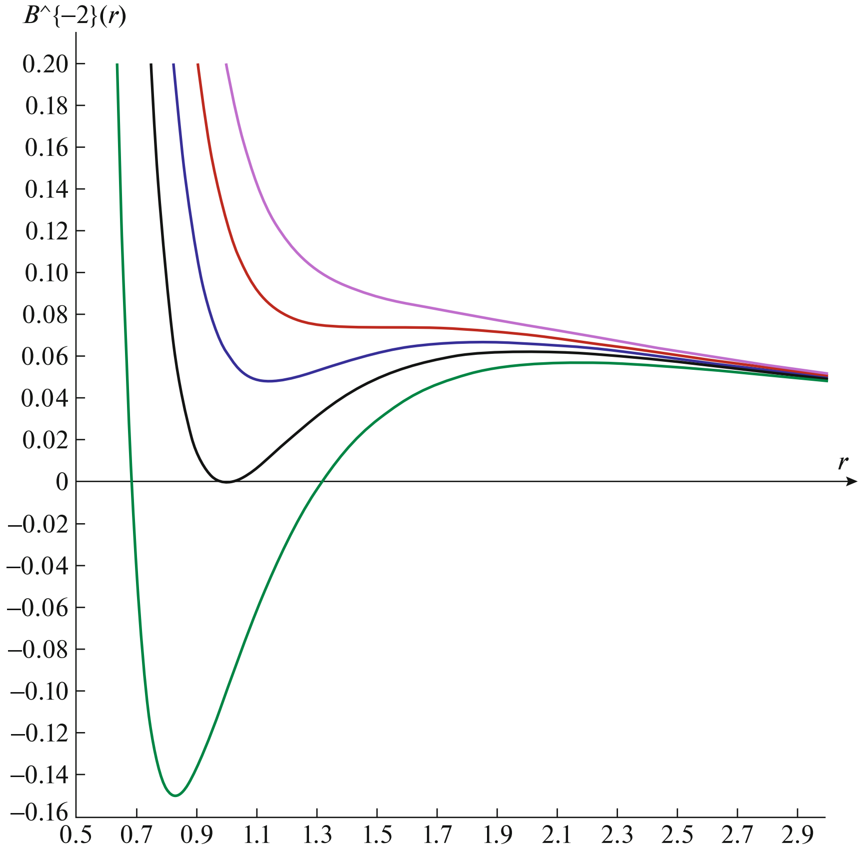


**Fig. 3.** Graph of the function $B^{-2}(\hat{r})$ for different $q$, namely for the green curve $q = 0.9$, for the black curve $q = 1$, for the blue curve $q = 1.0625$, for the red curve $q = 1.125$ and for the violet curve $q = 1.2$. From inspection of the curves we conclude that there are shadows for Reissner–Nordström metrics with $q \le 1$, there are photon circular orbits (but shadows do not exist) for $1 < q \le 9/8$ while for $q > 9/8$ there are no shadows or circular photon orbits in the vicinity of such a naked singularity.

Thus, it can be noted that for charge values $q \le 1$ black holes have shadows, while for $1 < q \le 9/8$ (there are no shadows, but there are circular photon orbits), and for $q > 9/8$ there are neither circular photon orbits nor shadows. Thus, there is structural instability in the sense discussed in the works of Poincaré–Pontryagin–Andronov–Anosov and other authors. The need to study structural stability was written about in many of their works by, for example, R. Thom and V.I. Arnold (see [72–75] and references in these works). In the case of discussing the presence of shadows of the Reissner–Nordström black hole, there is a structural instability in the vicinity of the value $q = 1$, since with any small change in the parameters, the properties of the system qualitatively change (in our case, this is the presence of shadows in the vicinity of black holes), since at the values of the limiting charge $q = 1 - \epsilon$ there are shadows, while at $q = 1 + \epsilon$ the shadows disappear, i.e. the shadows disappear at supercritical values of charge, as does the horizon of the Reissner–Nordström black hole.

However, the above example does not allow us to assert that black holes have shadows, while naked singularities do not, since the authors of [76] showed that there may be shadows in the vicinity of naked singularities.

## 8. SHADOWS FOR PERTURBED KERR BLACK HOLES

In this section we describe a sketch for a proof that in some cases conditions for circular photon orbits determine shadows as it was done for Kerr metric in [55]. We consider small perturbations of the Kerr metric $\tilde{g}_{\mu\nu}$ that could arise as a result of considering alternative theories of gravity. Moreover, we assume that when considering geodesics in such metrics it is possible to apply Carter's approach, in which the Hamilton–Jacobi equation allows separation of variables. In this case, it is necessary to introduce Chandrasekhar constants of motion for photons $(\xi, \eta)$ and therefore, for radial motion we have the following equation

$$\left(\frac{d\hat{r}}{d\sigma}\right)^2 = \hat{\mathcal{R}}\left(\hat{r}, \xi, \eta\right), \qquad (50)$$

where $\hat{\mathcal{R}}(\hat{r},\xi,\eta)$ is a generalization of the function $\hat{R}(\hat{r})$ in Eq. (38) for the perturbed metric $\tilde{g}_{\mu\nu}$. After that we have try to prove that the region with a double root of the function $\hat{\mathcal{R}}(\hat{r},\xi,\eta)$ in respect to variable $r$ really separates the regions of capture and scattering by a generalized black hole. If black hole spin is rather high ($a \to 1$) for a photon moving from infinity to a black hole near the equatorial plane, the position of the turning point approaches the event horizon, but for a Kerr black hole, the turning point is above the horizon, as shown in the work, but if a perturbed Kerr metric is considered, then the turning point can go below the horizon or the horizon can disappear, just as happened for the case of a naked Reissner–Nordström singularity, therefore the case of capture of a photon with a parameter $\xi$ close to critical and a parameters $\eta$ close to zero should be considered in more detail. The simplest and most direct way to check which set of parameter ($\xi$, $\eta$) corresponds to capture and which set correspond to scattering is the following. Using the Monte Carlo method, we determine which range of parameter values corresponds to capture and which to scattering (after which the results are transferred to the parameters ($\alpha$, $\beta$) which are dependent on the position angle $\theta$). Even for an arbitrary metric using MC simulation method we could numerically calculate propagations of photon in a given metric which substitutes Kerr metric in our case.

## 9. CONCLUSIONS

Following J. Wheeler and S. Chandrasekhar, it seems natural to consider that the most general solution of the black hole type for astrophysical applications is determined by the Kerr–Newman metric, and their various generalizations or alternatives, at least for now, are mainly of theoretical significance and are unlikely to dominate the interpretation of observational data in the near future. As noted earlier for Schwarzschild, Kerr, and Reissner–Nordström black holes, the impact parameters corresponding to circular photon orbits do indeed separate the regions of photon capture and scattering (i.e., the impact parameters determining the circular unstable photon orbits also determine the shadows for the corresponding metrics).

It is known that the diffraction limit of the EHT interferometer using 1.3 mm wavelengths is about 20 arc microseconds. In 2000 Falcke et al. [77] considered an opportunity to observe a shadow at the Galactic Center with ground based VLBI network but the authors estimated the shadow size as 30 arc microseconds based on an understated estimate of the black hole mass $M = 2.6 \times 10^6\ M_\odot$ given in [78, 79]. Now, we know that the black hole mass $M = 4.3 \times 10^6\ M_\odot$ [80], therefore that shadow size is larger than it was expected in 2000.

Despite the fact that the general relativity (GR) successfully describes all observational effects, there are problems of dark matter and dark energy within the framework of GR. In addition, despite significant efforts, a quantum theory of gravity has not been created. For these reasons, quite a lot alternative theories of gravity have been proposed recently, one of which is the so-called theory of massive gravity, in which the graviton is considered massive. Constraints on the graviton mass from observations of bright stars near GC were obtained in [81–85]. A comprehensive review of modified gravity theories (including massive gravity theories) is given in a new book [86].

The next generation interferometer (New Generation Event Horizon Telescope (ngEHT)) https://www.ngeht.org/ will use a wavelength of 0.87 mm, i.e. approximately one and a half times smaller, which will accordingly improve the angular resolution, however, even in this case, as shown in paper [44] even with the ngEHT interferometer it will be very problematic to estimate the black hole spin for M87* and Sgr A*, assuming that the Kerr black hole model can be used for these objects. Perhaps, further development of ngEHT and decreasing wavelength will be done in a more distant future. However, BH metric in Sgr A* may be tested in another way. Observations and detailed analysis done by the GRAVITY collaboration for orbital motion of Sgr A*'s near-infrared flares showed that the flares are fit by hot spots moving in Kerr black hole metric [87].

The idea of using ground-based and ground-based space VLBI operating in the millimeter or submillimeter range to reconstruct the shadow in the vicinity of the Galactic Center was proposed in [44] (which can naturally be generalized to other supermassive black holes, such as the black hole in the center of the galaxy M87). The method for obtaining isotropic geodesics used in [43] was proposed in [46–48] and therefore the reference to [49] seems incorrect, since there is a deliberately distorted indication of the sources of the approach used (in [49] it is directly stated that this method for calculating geodesics was proposed in [47] and subsequently used in [48]). Similar incorrect citation of mainly their own works (in which this approach was used later) is also present in a number of other works by the discussed authors, in articles published in the *Astronomy Reports* in last years, as well as in electronic preprints archive. Thus, the aforementioned authors have already become accustomed to the demonstratively inadequate lack of citation of original works in which the methods used in their work and proposed earlier in works not cited by them were previously used. Such deliberately distorted citation should be recognized as unacceptable, since it does not correspond to the actual state of affairs.

Recently, a number of works (see, for example, [88]) consider metrics that are proposed to replace the Kerr metric for astronomical objects with a compact mass distribution, in particular, for galactic centers. For these metrics, it is claimed without evidence that the impact parameters for unstable circular photon orbits determine the shape and size of the shadow, although as shown above, the presence of unstable circular photon orbits does not mean the presence of shadows, thus, the presence of unstable circular photon orbits in the theory does not mean that shadows corresponding to the metrics under consideration can be detected in observations even from a purely theoretical point of view, when an isolated naked singularity (or a "quantum" black hole) is considered in the absence of surrounding matter in the form of dust, gas, plasma, stars and possibly dark matter, as is usually the case in the centers of galaxies, the observation of shadows in which is usually discussed in the literature. In galactic centers, the presence of an additional gravitational potential caused by spatial mass distributions outside the black hole and the absence of a "standard model" of accretion onto a black hole makes the issue of estimating the parameters of the perturbation of the standard Kerr–Newman black hole due to the consideration of alternative models or theories very difficult, despite the expected (although, unfortunately, rather slow) progress in the observational capabilities of ngEHT interferometers.

At the end we say a few words on unattributed coincidences of figures and statements between current authors and previous ones. If we inspect Fig. 2 from the article [88], however, a comparison of this figure with Fig. 2 from article [44] shows that these figures largely coincide and differ only in that the article [88] presents the spin values ($a = 0$, 0.3, 0.5, 0.9, 0.98), while in article [44] ($a = 0$, 0.5, 1), i.e. the differences in these figures are very insignificant, and the same properties of the dependence of the shadow shape on the spin are discussed when the observer is located in the equatorial plane. The Fig. 2 from [88] was reproduced without proper attribution in slide 14 of O. Zenin's presentation https://indico.quarks.ru/event/2024/contributions /908/attachments/891/1082/Zenin.pdf at the Quarks-2024 conference and earlier in page 13 (or in slide 11) of Zenin's presentation at the AYSS-2023 conference). In a recent publication in the journal *Nature* [90] numerous examples of commissioned purchased authorship, plagiarism and other cases of violation of publication ethics are considered, in particular, in this publication a similar case of repeated reproduction of the figures of the article by other authors is discussed in detail, and it is noted that the article with the repeated reproduction of a previously published figure was withdrawn by the editorial board of the journal *Proteomics* due to "significant unattributed coincidence of the figures". As noted above, there is a significant unattributed coincidence between the figure from article [88] and the same figures from the aforementioned presentations by O. Zenin on the one hand, and Fig. 2 from article [44] on the other hand, however, the editors and reviewers of domestic journals are usually more tolerant when considering such cases, see, for example, the discussion of this issue in comment [89].

Shadows and related issues were discussed in recent papers [91–108].

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## CONFLICT OF INTEREST

The author of this work declares that he has no conflicts of interest.

## OPEN ACCESS

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