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Cohomological Field Theories and First-Order Nonlinear PDEs

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Abstract: We introduce a formalism for constructing cohomological field theories (CohFTs), inspired by Witten's paradigm of "symmetries, fields, and equations". We apply this formalism to various first-order nonlinear PDEs and show that the resulting CohFTs agree with those previously proposed by physicists. In particular, applying it to the generalized Seiberg–Witten equations provides a unified perspective on the supersymmetric action functionals of the Donaldson–Witten, Seiberg–Witten, and Kapustin–Witten theories.

1. Introduction

Gauge field theory has led to spectacular advances in geometry. It started with the groundbreaking work of Donaldson [1] who used moduli spaces of anti-selfdual connections as invariants for smooth structures on 4-manifolds. See also [2] for a systematic exposition of the theory. The underlying gauge group here is non-abelian, for instance SU(2), and the gauge functional is the Yang–Mills functional. Later, the Seiberg–Witten equations were introduced [3–5]:

$$(F_A)_+ - \frac{1}{2}\mu_+(\sigma) = 0, \quad \not{D}_A^+\sigma = 0. \tag{1.1}$$

Here A is a connection on the determinant line bundle L of a spin^c structure on a Riemannian 4-manifold Σ, σ is a section of the positive half-spinor bundle $\not{S}^+$ of the spin^c-structure, $(F_A)_+$ is the self-dual part of the curvature of A, $\not{D}_A^+$ is the twisted Dirac operator, and $\mu_+ : \not{S}^+ \to i\Lambda_+^2 T^\vee\Sigma$ is a fiber-wise quadratic bundle morphism.

The resulting moduli space enabled a simpler approach than the original Donaldson theory. Still another such functional, the Kapustin–Witten functional [6] relates its first-order field equations to the geometric Langlands program and to non-abelian Hodge theory. For some recent development in the latter direction, see, for instance [7]. Both Seiberg–Witten and Kapustin–Witten equations/functionals can be derived from the

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supersymmetric Yang–Mills theories and can be unified by the generalized Seiberg–Witten (GSW) equations/functionals [8–10]. The GSW equations are of the same form as (1.1), except that one should interpret A as a connection on a principal $G/\mathbb{Z}_2$-bundle associated with a spin^G structure on Σ, and σ as a section of some generalized Dirac bundle of positive chirality on Σ.

In this situation, it seems desirable to develop a unified perspective within which to situate these and possibly other gauge functionals. Naturally, this should be inspired by the underlying supersymmetric physics. Let

$$\mathcal{H} \hookrightarrow \mathcal{E}_{tot} \longrightarrow \mathcal{E}$$

be a trivial $\mathcal{G}$-equivariant Fréchet vector bundle, where $\mathcal{G}$ denotes the group of gauge transformations, $\mathcal{E}$ denotes the space of smooth sections of some fiber bundle over Σ, and $\mathcal{H}$ denotes the space of smooth sections of some vector bundle over Σ. A first-order nonlinear PDE on Σ can be often interpreted as a $\mathcal{G}$-equivariant section

$$\mathcal{E} \xrightarrow{\mathcal{F}} \mathcal{E}_{tot}.$$

Denoting by $\mathrm{Sol}(\mathcal{F})$ the zero locus of $\mathcal{F}$, i.e., the solution space of the equation $\mathcal{F} = 0$, the moduli space $\mathcal{M}(\mathcal{F}) = \mathrm{Sol}(\mathcal{F})/\mathcal{G}$ encodes essential information about the smooth structure of Σ.

If $\mathcal{H}$ is equipped with a $\mathcal{G}$-invariant inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$, one can define the following action functional on $\mathcal{E}$:

$$S = \langle \mathcal{F}, \mathcal{F}\rangle_{\mathcal{H}} \in C^\infty(\mathcal{E}). \tag{1.2}$$

In certain cases, (1.2) admits a supersymmetric extension $\widetilde{S} \in C^\infty(\widetilde{\mathcal{E}})$ when $\Sigma = \mathbb{R}^n$. Here, $\widetilde{\mathcal{E}}$ is a Fréchet supermanifold with underlying Fréchet manifold $\mathcal{E}$. $\widetilde{S}$ can sometimes be "twisted" to remain well-defined on a general Riemannian manifold Σ, though this typically comes at the expense of breaking full supersymmetry. Specifically:

- the supermanifold $\widetilde{\mathcal{E}}$ can be equipped with a compatible $\mathbb{Z}$-grading;
- $\widetilde{S}$ has degree 0 and a remaining "scalar" supersymmetry Q of degree 1, $Q^2 = \frac{1}{2}[Q, Q] = 0$.

The triple $(\widetilde{\mathcal{E}}, Q, \widetilde{S})$ is known as a cohomological field theory (CohFT) [11], with the first example provided by Witten's topological twist of $N = 2$ supersymmetric Yang–Mills theory to give a physical interpretation of Donaldson theory [12].

In this paper, we argue that the fundamental input for defining a CohFT is not the Lagrangian action functional, but rather the equation of motion $\mathcal{F} = 0$ together with its symmetries. In particular, we show that a CohFT $(\widetilde{\mathcal{E}}, Q, \widetilde{S})$ can be constructed directly from this data—namely, the (gauge) symmetry group $\mathcal{G}$, the space $\mathcal{E}_{tot}$, and the first-order nonlinear PDE $\mathcal{F} = 0$—along with additional geometric structures (such as a Riemannian metric or a bundle metric), which are only needed to define the action functional $\widetilde{S}$. This perspective aligns with Witten's paradigm of "symmetries, fields, and equations" [11] and admits a natural mathematical interpretation: $(\widetilde{\mathcal{E}}, Q)$ provides an infinite-dimensional differential graded manifold that models the de Rham space of the moduli space $\mathcal{M}(\mathcal{F})$.

We also provide a universal expression for the supersymmetric action functional $\widetilde{S}$ within this framework. When applied to the GSW equations, our construction of $\widetilde{S}$ reproduces the full supersymmetric functionals of various CohFTs, including the Donaldson–Witten, Seiberg–Witten, and Kapustin–Witten theories. Our approach is closely related to

the Mathai–Quillen formalism for CohFTs [13,14]. For excellent reviews of the Mathai–Quillen formalism, we refer the reader to [15,16].

The organization of the paper is as follows. In Sect. 2, we review the geometric and algebraic preliminaries of differential graded geometry and algebraic equivariant cohomology. In Sect. 3, we introduce the minimal and standard constructions of CohFTs and explain how their variants relate to the Mathai–Quillen formalism and the Atiyah–Jeffrey formula. In Sect. 4, we apply these constructions to the trivial first-order PDE for maps between manifolds, as well as to the generalized Seiberg–Witten equations. In Sect. 5, we provide mathematical definitions of symmetries and observables of a CohFT within our framework. In particular, we discuss two classes of symmetries according to our definitions and compare them with those in the physics literature, as well as the traditional method of constructing CohFT observables via the topological descent equations. We also argue that the classical observables of a CohFT form a prefactorization algebra and establish a connection with the Costello–Gwilliam framework in the perturbative setting. In Sect. 6, we briefly discuss possible directions for future work.

2. Geometric and Algebraic Preliminaries

In this section, we introduce the notion of a differential graded manifold equipped with a compatible group action and discuss its algebraic equivariant cohomology in the framework of Guillemin and Sternberg.

2.1. Differential graded manifolds with group actions. For us, a graded vector space or algebra always means a $\mathbb{Z}$-graded vector space or algebra over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $V = \bigoplus_{i\in\mathbb{Z}}$ be a graded vector space of finite total rank. Let us consider the following descending filtration of the symmetric algebra $\mathrm{Sym}(V)$ of V:

$$\mathrm{Sym}(V) = F^0\mathrm{Sym}(V) \supset F^1\mathrm{Sym}(V) \supset F^2\mathrm{Sym}(V) \supset \cdots,$$

where $F^p\mathrm{Sym}(V) = \bigoplus_{q\geq p} \mathrm{Sym}^q(V)$, i.e., the ideal generated by elements of polynomial degree $\geq p$. Define $\widehat{\mathrm{Sym}}(V)$ as the inverse limit of $\mathrm{Sym}(V)/F^p\mathrm{Sym}(V)$ in the category of graded vector spaces. More precisely, $\widehat{\mathrm{Sym}}(V) = \bigoplus_{i\in\mathbb{Z}} \widehat{\mathrm{Sym}}(V)_i$, where

$$\widehat{\mathrm{Sym}}(V)_i = \varprojlim \mathrm{Sym}(V)_i/(\mathrm{Sym}(V)_i \cap F^p\mathrm{Sym}(V)).$$

Here, $\mathrm{Sym}(V)_i$ denotes the degree i subspace of $\mathrm{Sym}(V)$.

Lemma 2.1. *1. There exists an isomorphism of graded commutative algebras:*

$$\widehat{\mathrm{Sym}}(V) \cong \mathrm{Sym}(V^\vee)^\vee, \tag{2.1}$$

where $V^\vee$ denotes the dual space of V in the category of graded vector spaces.
2. Let V_{even} and V_{odd} denote the even and odd parts of V, respectively. Then

$$\widehat{\mathrm{Sym}}(V) \cong \widehat{\mathrm{Sym}}(V_{even}) \otimes \mathrm{Sym}(V_{odd}). \tag{2.2}$$

Proof. Let $F_p\mathrm{Sym}(V^\vee) = \bigoplus_{q<p} \mathrm{Sym}^q(V) \cong \mathrm{Sym}(V)/F^p\mathrm{Sym}(V)$. We have

$$\mathrm{Sym}(V^\vee)^\vee \cong \varprojlim (F_p\mathrm{Sym}(V^\vee))^\vee,$$

since $\mathrm{Sym}(V^\vee) = \varinjlim F_p\mathrm{Sym}(V^\vee)$. Furthermore, $(F_p\mathrm{Sym}(V^\vee))_i^\vee \cong F_p\mathrm{Sym}(V)_i$ because V is of finite total rank. This proves (2.1). It then follows that

$$\widehat{\mathrm{Sym}}(V) \cong \mathrm{Sym}(V_{even}^\vee \oplus V_{odd}^\vee)^\vee \cong (\mathrm{Sym}(V_{even}^\vee) \otimes \mathrm{Sym}(V_{odd}^\vee))^\vee \cong \widehat{\mathrm{Sym}}(V_{even}) \otimes \mathrm{Sym}(V_{odd}),$$

since V_{odd} is of finite total rank. □

Remark 2.1. If $V_i = \{0\}$ for all $i \leq 0$ or $i \geq 0$, then $\mathrm{Sym}(V)_i$ is of finite rank. (2.1) then implies that

$$\widehat{\mathrm{Sym}}(V) \cong \mathrm{Sym}(V) \cong \mathrm{Sym}(V_{even}) \otimes \mathrm{Sym}(V_{odd}).$$

If there exists $(i, j) \in \mathbb{Z} \times \mathbb{Z}$ such that $i + j = 0$, and V_i and V_j are of positive ranks, then $\mathrm{Sym}(V_{even}) \subsetneq \widehat{\mathrm{Sym}}(V_{even})$. In general,

$$\mathrm{Sym}(V_{odd}) \cong \Lambda(V_{odd}), \quad \mathrm{Sym}(V_{even}) \subseteq \widehat{\mathrm{Sym}}(V_{even}) \subseteq \mathbb{K}[[V_{even}]], \tag{2.3}$$

where $\Lambda(V_{odd})$ denote the exterior algebra of V_{odd} viewed as an ordinary vector space. $\mathbb{K}[[V_{even}]]$ denotes the algebra of formal power series over V_{even}, also regarded as an ordinary vector space.

Let R be a graded commutative algebra. Let V be a graded finitely generated projective R-module. Let $\mathrm{Sym}_R(V)$ denote the symmetric algebra of V over R. One can define the completion $\widehat{\mathrm{Sym}}_R(V)$ in a similar manner. If V is free over R, (2.1) still holds since $\mathrm{Sym}_R^p(V)$ is a graded free R-module of finite total rank for all $p \geq 0$.

Definition 2.1. A graded domain $\mathcal{U} = (U, \mathcal{O}_\mathcal{U})$ is a topological space U together with a sheaf of graded commutative algebras $\mathcal{O}_\mathcal{U}$, such that there exists a natural number n, a graded vector space V of finite total dimension, and a graded ringed space isomorphism

$$(U, \mathcal{O}_\mathcal{U}) \cong (\mathbb{R}^n, \mathcal{O}_V),$$

where $\mathcal{O}_V$ assigns an open subset U' of $\mathbb{R}^n$ the graded commutative algebra[1]

$$C^\infty(U', \mathbb{K}) \otimes \widehat{\mathrm{Sym}}(V^\vee).$$

Let m_1 and m_2 be two integers with $m_1 \leq m_2$. We say that $\mathcal{U}$ is of amplitude $[m_1, m_2]$ if $V_i = \{0\}$ for all $i < m_1$ and $i > m_2$.

Definition 2.2. A graded manifold $\mathcal{M} = (M, \mathcal{O}_\mathcal{M})$ of amplitude $[m_1, m_2]$ is a manifold M together with a sheaf of graded commutative algebras $\mathcal{O}_\mathcal{M}$, which is locally a graded domain of amplitude $[m_1, m_2]$. A graded manifold morphism is simply a morphism between the corresponding graded ringed spaces.

We denote the space of global sections $\mathcal{O}_\mathcal{M}(M)$ by $C^\infty(\mathcal{M})$. We say that $\mathcal{M}$ is of positive (resp., negative) amplitude, if $m_1 > 0$ (resp., $m_2 < 0$).

Example 2.1. A manifold M with the sheaf of smooth functions on it is a graded manifold.

[1] Our notion of graded domains is equivalent to the one in [17,18] if $V_0 = \{0\}$. See [18, Lemma 3].

Example 2.2. Let $E = \bigoplus_{i\in\mathbb{Z}} E_i$ be a graded vector bundle over M of finite total rank. E induces a graded manifold $(M, \mathcal{O}_E)$, where $\mathcal{O}_E$ assigns an open subset U of M the graded commutative algebra $\widehat{\mathrm{Sym}}_{C^\infty(U)}(\Gamma(E^\vee|_U))$. Batchelor's theorem tells us that every graded manifold is non-canonically of this form.

Let $\mathcal{M} = (M, \mathcal{O}_\mathcal{M})$ be a graded manifold. Let $\mathfrak{X}_\mathcal{M}(U)$ denote the graded Lie algebra of derivations of $\mathcal{O}_\mathcal{M}(U)$, where U is some open subset of M. The assignment $U \mapsto \mathfrak{X}_\mathcal{M}(U)$ defines a locally free graded $\mathcal{O}$-module over M. We denote the space of global sections $\mathfrak{X}_\mathcal{M}(M)$ by $\mathfrak{X}(\mathcal{M})$. An element $X \in \mathfrak{X}(\mathcal{M})$ is called a vector field over $\mathcal{M}$.

Definition 2.3. A differential graded (dg) manifold is a graded manifold $\mathcal{M}$ together with a vector field $Q_\mathcal{M}$ of degree 1 such that $Q^2_\mathcal{M} = \frac{1}{2}[Q_\mathcal{M}, Q_\mathcal{M}] = 0$. $Q_\mathcal{M}$ is called the cohomological vector field over $\mathcal{M}$.

For $p \in \mathbb{Z}$, we denote by $\mathbb{K}[p]$ the graded vector space with $\mathbb{K}$ in degree $-p$ and zero elsewhere. Let E be a (graded) vector bundle over M. We set $E[p] := E \otimes \mathbb{K}[p]$, where $\mathbb{K}[p]$ is interpreted as a trivial graded vector bundle over M.

Example 2.3. The shifted tangent bundle $T[1]M = (M, \mathcal{O}_{T[1]M})$ of a manifold M is a graded manifold of amplitude $[-1, -1]$. $C^\infty(T[1]M)$ can be identified with the space $\Omega(M)$ of differential forms on M. The de Rham differential d_{dR} of M can then be viewed as a cohomological vector field over $T[1]M$.

Example 2.4. Let $\mathfrak{g}$ be a Lie algebra, viewed as a vector bundle over a point. $\mathfrak{g}[1]$ is a graded manifold of amplitude $[-1, -1]$, whose algebra of functions $C^\infty(\mathfrak{g}[1])$ can be identified with $\Lambda(\mathfrak{g}^\vee)$. The Chevalley–Eilenberg differential d_{CE} of $\mathfrak{g}$ can then be viewed as a cohomological vector field over $\mathfrak{g}[1]$.

Example 2.5. Let E be an ordinary vector bundle over a manifold M. The shifted vector bundle $E[-1]$ is a graded manifold of amplitude $[1, 1]$, whose algebra of functions $C^\infty(E[-1])$ can be identified with $\Gamma(\Lambda(E^\vee))$, where $\Lambda(E^\vee)$ is the exterior bundle of $E^\vee$. Let $s \in \Gamma(E)$ be a section of E. The left contraction by s, denoted ι_s, is a derivation of $C^\infty(E[-1])$ of degree 1. The pair $(E[-1], \iota_s)$ forms a dg manifold because $\iota_s^2 = 0$.

In general, dg manifolds of negative amplitude correspond to higher Lie algebroids, while those of positive amplitude correspond to derived manifolds [19–23]. To study CohFTs, we must consider dg manifolds with mixed amplitudes, where the structure becomes more intricate yet interesting.

Example 2.6. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let E be a G-equivariant vector bundle over M. Consider the graded manifold

$$\mathcal{M} = E[-1] \oplus \mathfrak{g}[1],$$

where we view $\mathfrak{g}$ as a trivial vector bundle over M. Let $\{\xi_a\}$ be a set of basis vector of $\mathfrak{g}[1]$. Let $\{\theta^a\}$ denote the basis of $(\mathfrak{g}[1])^\vee$ induced by $\{\xi_a\}$, and let f^a_{bc} denote the structure constants of $\mathfrak{g}$ with respect to θ_a. Let (x^μ, χ^α) be local coordinates on $E[-1]$. Let X_a denote the fundamental vector field on $E[1]$ corresponding to θ_a. Locally, X_a takes the form

$$X_a = \rho^\mu_a(x)\frac{\partial}{\partial x^\mu} + \rho^\alpha_a(x, \chi)\frac{\partial}{\partial \chi^\alpha},$$

where $\rho_a^\mu(x)$ and $\rho_a^\alpha(x,\chi)$ are local functions of degrees 0 and -1, respectively. We consider the following vector field of degree 1 over $\mathcal{M}$:

$$\begin{aligned} Q_{\mathcal{M}} &= \iota_s + \theta^a X_a + d_{CE} \\ &= (\theta^a \rho_a^\alpha(x,\chi) + s^\alpha(x))\frac{\partial}{\partial \chi^\alpha} + \theta^a \rho_a^\mu(x)\frac{\partial}{\partial x^\mu} - \frac{1}{2} f_{bc}^a \theta^b \theta^c \frac{\partial}{\partial \theta^a}, \end{aligned}$$

where s is a G-equivariant section of E. One can check that

$$Q_{\mathcal{M}}^2 = \theta^a(\rho_a^\mu(x)\frac{\partial s^\alpha(x)}{\partial x^\mu} - \frac{\partial \rho_a^\alpha(x,\chi)}{\partial \chi^\beta} s^\beta(x))\frac{\partial}{\partial \chi^\alpha} + \frac{1}{2}\theta^a\theta^b[X_a, X_b] - \frac{1}{2} f_{ab}^c \theta^a \theta^b X_c.$$

The second and third terms cancel due to the group action property, while the first term vanishes because s is G-equivariant. Hence, $Q_{\mathcal{M}}^2 = 0$ and $(\mathcal{M}, Q_{\mathcal{M}})$ is a dg manifold of amplitude $[-1, 1]$.

Remark 2.2. Without loss of generality, let us assume that $E = M \times \mathbb{K}^r$ is a trivial G-equivariant vector bundle over M.[2] Let $s = (s_1, \ldots, s_r)$ be a G-equivariant section of E. From a physical perspective, one can interpret M as the phase space, G as the gauge group, and $s_1, \ldots, s_r$ as the constraints imposed on the phase space. The corresponding complex $(C^\infty(\mathcal{M}), Q_{\mathcal{M}})$ associated with the dg manifold in Example 2.6 can then be understood as the BRST complex of the system. Specifically, it is the total complex of the double complex (C, δ_K, d_{CE}):

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \\ & & \uparrow{\scriptstyle d_{CE}} & & \uparrow{\scriptstyle d_{CE}} & & \\ \cdots & \xrightarrow{\delta_K} & \Lambda^{p+1}\mathfrak{g}^\vee \otimes \Lambda^q F^\vee \otimes C^\infty(M) & \xrightarrow{\delta_K} & \Lambda^{p+1}\mathfrak{g}^\vee \otimes \Lambda^{q-1} F^\vee \otimes C^\infty(M) & \xrightarrow{\delta_K} & \cdots \\ & & \uparrow{\scriptstyle d_{CE}} & & \uparrow{\scriptstyle d_{CE}} & & \\ \cdots & \xrightarrow{\delta_K} & \Lambda^{p}\mathfrak{g}^\vee \otimes \Lambda^q F^\vee \otimes C^\infty(M) & \xrightarrow{\delta_K} & \Lambda^{p}\mathfrak{g}^\vee \otimes \Lambda^{q-1} F^\vee \otimes C^\infty(M) & \xrightarrow{\delta_K} & \cdots \\ & & \uparrow{\scriptstyle d_{CE}} & & \uparrow{\scriptstyle d_{CE}} & & \\ & & \vdots & & \vdots & & \end{array}$$

where δ_K is the Koszul differential, and d_{CE} is the Chevalley–Eilenberg differential. In particular, if $F = \mathfrak{g}$, M is a symplectic manifold with a Hamiltonian G-action, and $s : M \to F^\vee$ is the moment map, then we reduces to the cases considered in [24]. It is well known (see [25,26]) that if

$$0 \text{ is a regular value of } s, \tag{2.4}$$

then $(C^{p,\bullet}, \delta_K)$ is acyclic for all p and

$$H^\bullet_{Q_{\mathcal{M}}}(C^\infty(\mathcal{M})) \cong H^\bullet_{d_{CE}}(\Lambda\mathfrak{g}^\vee \otimes C^\infty(s^{-1}(0))).$$

[2] If not, we can replace it with a trivial $(G \times H)$-equivariant vector bundle over a principal H-bundle $P \to M$, where H is the structure group of E.

Moreover, if

$$G \text{ is connected, and acts freely and properly on } s^{-1}(0), \tag{2.5}$$

one can deduce that $H^\bullet_{Q_\mathcal{M}}(C^\infty(\mathcal{M})) \cong C^\infty(s^{-1}(0)/G) \otimes H^\bullet(G)$, where $H^\bullet(G)$ denotes the de Rham cohomology of G.

Let G be a Lie group with Lie algebra $\mathfrak{g}$, and let $\mathcal{M}$ be a graded manifold. A G-action on $\mathcal{M}$ is a G-action on the corresponding ringed space that preserves the grading of the structure sheaf.

Definition 2.4. Let $(\mathcal{M}, Q_\mathcal{M})$ be a dg manifold.

1. A G-action on $\mathcal{M}$ is said to be compatible with $Q_\mathcal{M}$ if, for each $\xi \in \mathfrak{g}$, the corresponding fundamental vector field X_ξ on $\mathcal{M}$ is $[Q_\mathcal{M}, \cdot]$-closed.
2. A G-action on $\mathcal{M}$ is called homotopically trivial if, for each $\xi \in \mathfrak{g}$, the fundamental vector field X_ξ is $[Q_\mathcal{M}, \cdot]$-exact.

Example 2.7. Let $(\mathfrak{g}[1], d_{CE})$ be the dg manifold defined in Example 2.4. The G-action on $\mathfrak{g}[1]$, induced by the coadjoint action of G on $\mathfrak{g}^\vee$, is homotopically trivial. Indeed, let Lie_a denote fundamental vector field on $\mathfrak{g}[1]$ corresponding to ξ_a, we have

$$\mathrm{Lie}_a = -f^c_{ab}\theta^b \frac{\partial}{\partial \theta^c} = [d_{CE}, \iota_a],$$

where $\iota_a = \delta^b_a \frac{\partial}{\partial \theta^b}$.

Let E be a graded vector bundle over M, viewed as a graded manifold $(M, \mathcal{O}_E)$. It is easy to see that a G-equivariant bundle structure of E induces a G-action on $(M, \mathcal{O}_E)$ via the universal property of the symmetric algebra.

Example 2.8. Let $(E[-1], \iota_s)$ be the dg manifold defined in Example 2.5. Supposing that E is a G-equivariant vector bundle, then the induced G-action on $E[-1]$ is compatible with ι_s if and only if s is a G-equivariant section of E.

The construction in Example 2.6 is a special case of the following proposition.

Proposition 2.1. *Let $(\mathcal{M}, Q_\mathcal{M})$ be a dg manifold (of positive amplitude) equipped with a G-action. Consider the graded manifold*

$$\mathcal{M}_G := \mathcal{M} \times \mathfrak{g}[1],$$

endowed with the degree 1 *vector field*

$$Q_{\mathcal{M}_G} := Q_\mathcal{M} \otimes 1 + X_a \otimes \theta^a + 1 \otimes d_{CE},$$

where $d_{CE} = -\frac{1}{2} f^a_{bc}\theta^b\theta^c \frac{\partial}{\partial \theta^a}$ is the Chevalley–Eilenberg differential of $\mathfrak{g}$, $\{\theta^a\}$ is the dual basis associated with a chosen basis $\{\xi_a\}$ of $\mathfrak{g}$, and X_a is the fundamental vector field on $\mathcal{M}$ corresponding to ξ_a. Then the followings are equivalent:

1. $Q^2_{\mathcal{M}_G} = 0$;
2. *The G-action on $\mathcal{M}$ is compatible with $Q_\mathcal{M}$;*
3. $Q^2_{\mathcal{M}_G} = 0$, *and the induced G-action on $\mathcal{M}_G$ is compatible with $Q_{\mathcal{M}_G}$;*
4. $Q^2_{\mathcal{M}_G} = 0$, *and the induced G-action on $\mathcal{M}_G$ is homotopically trivial.*

Proof. First, observe that $C^\infty(\mathcal{M}_G) \cong C^\infty(\mathcal{M}) \otimes \Lambda(\mathfrak{g}^\vee)$ as dgcas, so $Q_{\mathcal{M}_G}$ is well-defined. Let X_{Ga} denote the fundamental vector field over $\mathcal{M}_G$ corresponding to ξ_a. We have

$$X_{Ga} = X_a \otimes 1 + 1 \otimes \mathrm{Lie}_a,$$

where $\mathrm{Lie}_a = -f_{ab}^c \theta^b \frac{\partial}{\partial \theta^c}$. A direct computation shows that

$$[Q_{\mathcal{M}_G}, X_{Ga}] = [Q_{\mathcal{M}}, X_a] \otimes 1, \quad Q_{\mathcal{M}_G}^2 = [Q_{\mathcal{M}}, X_a] \otimes \theta^a,$$

both of which vanish if and only if $Q_{\mathcal{M}}$ commutes with X_a for all a. The proof is completed by noting that

$$X_{Ga} = [Q_{\mathcal{M}_G}, 1 \otimes \iota_a],$$

where $\iota_a = \delta_a^b \frac{\partial}{\partial \theta^b}$. □

The shifted tangent bundle construction of Example 2.3 can also be generalized. Let $\mathcal{M} = (M, \mathcal{O}_\mathcal{M})$ be a graded manifold. We set

$$\mathcal{M}_{dR} := T[1]\mathcal{M} := (M, \mathcal{O}_{\mathcal{M}_{dR}}),$$

where $\mathcal{O}_{\mathcal{M}_{dR}}$ assigns an open subset U of M the graded commutative algebra

$$\widehat{\mathrm{Sym}}_{\mathcal{O}_\mathcal{M}(U)}((\mathfrak{X}_\mathcal{M}(U)[1])^\vee),$$

where $\mathfrak{X}_\mathcal{M}(U)[1] = \mathfrak{X}_\mathcal{M}(U) \otimes \mathbb{K}[1]$, and $(\mathfrak{X}_\mathcal{M}(U)[1])^\vee$ is the dual of $\mathfrak{X}_\mathcal{M}(U)[1]$ in the category of graded $\mathcal{O}_\mathcal{M}(U)$-modules. If $(U, \mathcal{O}_\mathcal{M}|_U)$ is a graded domain which is isomorphic to $(\mathbb{R}^n, \mathcal{O}_V)$, one can show that

$$\widehat{\mathrm{Sym}}_{\mathcal{O}_\mathcal{M}(U)}(\mathfrak{X}_\mathcal{M}(U)[1]) \cong C^\infty(\mathbb{R}^n, \mathbb{K}) \otimes \widehat{\mathrm{Sym}}(\mathbb{K}^n[1] \oplus V \oplus V[1]).$$

Therefore, $\mathcal{M}_{dR}$ is also a graded manifold. Moreover, it carries a canonical cohomological vector field d_{dR}. Let $\{x^\mu\}_{\mu=1}^n$ be a set of coordinate functions on U, and let $\{\theta^a\}_{a=1}^{\dim V}$ be a basis of $V^\vee$. The functions x^μ and θ^a serve as local coordinates on $\mathcal{M}$, inducing local coordinates ψ^μ and ϕ^a on $T[1]\mathcal{M}$ of degrees 1 and $\deg(\theta^a) + 1$, respectively. In these coordinates, d_{dR} is locally given by

$$d_{dR} = \psi^\mu \frac{\partial}{\partial x^\mu} + \phi^a \frac{\partial}{\partial \theta^a}.$$

Let $X = X^\mu \frac{\partial}{\partial x^\mu} + X^a \frac{\partial}{\partial \theta^a}$ be a vector field of degree p over $\mathcal{M}$. X induces the following two vector fields over $\mathcal{M}_{dR}$ of degrees $p-1$ and p respectively:

$$\begin{aligned}
\iota_\mathcal{M}(X) &:= X^\mu \frac{\partial}{\partial \psi^\mu} + X^a \frac{\partial}{\partial \phi^a}, \\
\mathrm{Lie}_\mathcal{M}(X) &:= [\iota_\mathcal{M}(X), d_{dR}] \\
&= X^\mu \frac{\partial}{\partial x^\mu} + X^a \frac{\partial}{\partial \theta^a} \\
&\quad + (-1)^p \left(\psi^\mu \frac{\partial X^\nu}{\partial x^\mu} \frac{\partial}{\partial \psi^\nu} + \psi^\mu \frac{\partial X^b}{\partial x^\mu} \frac{\partial}{\partial \phi^b} + \phi^a \frac{\partial X^\nu}{\partial \theta^a} \frac{\partial}{\partial \psi^\nu} + \phi^a \frac{\partial X^b}{\partial \theta^a} \frac{\partial}{\partial \phi^b} \right).
\end{aligned}$$

We refer to $\iota_{\mathcal{M}}(X)$ as the contraction by X and to $\mathrm{Lie}_{\mathcal{M}}(X)$ as the Lie derivative along X. By Jacobi's identity, we have

$$[d_{dR}, \mathrm{Lie}_{\mathcal{M}}(X)] = 0, \tag{2.6}$$

for all $X \in \mathfrak{X}_{\mathcal{M}}(M)$. It can also be verified that

$$[\iota_{\mathcal{M}}(X), \iota_{\mathcal{M}}(Y)] = 0, \quad [\iota_{\mathcal{M}}(X), \mathrm{Lie}_{\mathcal{M}}(Y)] = (-1)^{\deg(Y)}\iota_{\mathcal{M}}([X, Y]), \tag{2.7}$$

which implies the desired relation

$$\begin{aligned}[\mathrm{Lie}_{\mathcal{M}}(X), \mathrm{Lie}_{\mathcal{M}}(Y)] &= (-1)^{\deg(X)}[[d_{dR}, \iota_{\mathcal{M}}(X)], \mathrm{Lie}_{\mathcal{M}}(Y)] \\ &= (-1)^{\deg(X)}[d_{dR}, [\iota_{\mathcal{M}}(X), \mathrm{Lie}_{\mathcal{M}}(Y)]] \\ &= (-1)^{\deg(X)+\deg(Y)}[d_{dR}, \iota_{\mathcal{M}}([X, Y])] = \mathrm{Lie}_{\mathcal{M}}([X, Y]),\end{aligned} \tag{2.8}$$

for all $X, Y \in \mathfrak{X}_{\mathcal{M}}(M)$. In particular, if $Q_{\mathcal{M}}$ is a cohomological vector field over $\mathcal{M}$, then

$$Q_{\mathcal{M}_{dR}} := \mathrm{Lie}_{\mathcal{M}}(Q_{\mathcal{M}})$$

is a cohomological vector field over $\mathcal{M}_{dR}$, and $[d_{dR}, Q_{\mathcal{M}_{dR}}] = 0$.

Definition 2.5. Let $(\mathcal{M}, Q_{\mathcal{M}})$ be a dg manifold. The Weil algebra $(W(\mathcal{M}), d_{W(\mathcal{M})})$ of $(\mathcal{M}, Q_{\mathcal{M}})$ is defined as the dgca $(C^\infty(\mathcal{M}_{dR}), d_{dR} + Q_{\mathcal{M}_{dR}})$.

Example 2.9. The Weil algebra of the dg manifold $(\mathfrak{g}[1], d_{CE})$ is the Weil algebra $(W(\mathfrak{g}), d_W)$ of the Lie algebra $\mathfrak{g}$, where $W(\mathfrak{g}) \cong \mathrm{Sym}(\mathfrak{g}^\vee) \otimes \Lambda(\mathfrak{g}^\vee)$ and

$$d_W = (\phi^a - \frac{1}{2} f^a_{bc}\theta^b\theta^c)\frac{\partial}{\partial\theta^a} - f^a_{bc}\theta^b\phi^c\frac{\partial}{\partial\phi^a}.$$

Note that $\mathfrak{g}_{dR} = T[1]\mathfrak{g} = \mathfrak{g} \oplus \mathfrak{g}[1]$ has a differential graded Lie algebra (dgLa) structure. The graded Lie bracket of $\mathfrak{g}_{dR}$ is induced by the bracket of $\mathfrak{g}$ and the adjoint action of $\mathfrak{g}$ on $\mathfrak{g}[1]$. The differential of $\mathfrak{g}_{dR}$ is given by

$$0 \to \mathfrak{g}[1] \xrightarrow{\mathrm{Id}} \mathfrak{g} \to 0.$$

The pair $(G, \mathfrak{g}_{dR})$, equipped with the adjoint action of G on $\mathfrak{g}_{dR}$, forms a super Harish–Chandra pair and can therefore be regarded as a graded Lie group, denoted by G_{dR}. However, in general, G_{dR} is not a differential graded Lie group.

Definition 2.6. Let $(\mathcal{M}, Q_{\mathcal{M}})$ be a dg manifold with a homotopically trivial G-action. $(\mathcal{M}, Q_{\mathcal{M}})$ is called a G_{dR}-manifold if its G-action can be extend to a G_{dR}-action, that is, one can find a vector field ι_a of degree -1 for each fundamental vector field X_a, such that

$$X_a = [Q_{\mathcal{M}}, \iota_a], \quad [\iota_a, \iota_b] = 0, \quad [\iota_a, X_b] = f^c_{ab}\iota_c.$$

One can easily verify that the dg manifold $(\mathcal{M}_G, Q_{\mathcal{M}_G})$ constructed in Proposition 2.1 is a G_{dR}-manifold. In particular, $(\mathfrak{g}[1], d_{CE})$ is a G_{dR}-manifold.

Proposition 2.2. *Let $(\mathcal{M}, Q_{\mathcal{M}})$ be a dg manifold equipped with a compatible G-action.*

1. *$(\mathcal{M}_{dR}, d_{dR}+Q_{\mathcal{M}_{dR}})$ carries a G_{dR}-manifold structure, where the G-action on $\mathcal{M}_{dR}$ is determined by the G-actions on $C^\infty(\mathcal{M})$ and $\mathfrak{X}(\mathcal{M})$, and the $\mathfrak{g}_{dR}$-action on $\mathcal{M}_{dR}$ is given by*

$$\begin{aligned} \mathfrak{g}_{dR} = \mathfrak{g} \oplus \mathfrak{g}[1] &\to \mathfrak{X}(\mathcal{M}_{dR}) \\ (\xi_1, \xi_2) &\mapsto \mathrm{Lie}_{\mathcal{M}}(X_{\xi_1}) + \iota_{\mathcal{M}}(X_{\xi_2}), \end{aligned} \tag{2.9}$$

   where X_ξ denotes the fundamental vector field on $\mathcal{M}$ associated with $\xi \in \mathfrak{g}$.
2. *If $(\mathcal{M}, Q_{\mathcal{M}})$ is itself a G_{dR}-manifold, then $(\mathcal{M}_{dR}, d_{dR} + Q_{\mathcal{M}_{dR}})$ admits an alternative G_{dR}-manifold structure. In this case, the G-action remains unchanged, while the $\mathfrak{g}_{dR}$-action on $\mathcal{M}_{dR}$ is instead induced from the $\mathfrak{g}_{dR}$-action on $\mathcal{M}$ via Lie derivatives.*

Proof. This follows directly from applying (2.6) to (2.8). □

2.2. Algebraic equivariant cohomology. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let us recall the definition of a G_{dR}-algebra. Such algebras are defined in [27, Definition 2.3.1], where they are referred to as $G^\star$-algebras.

Definition 2.7. A G_{dR}-algebra is a dgca (A, d_A), together with an action ρ_G of G as automorphisms of (A, d_A) and an action $\rho_{\mathfrak{g}_{dR}}$ of $\mathfrak{g}_{dR}$ as derivations of A which are compatible in the sense that

$$\frac{d}{dt}\rho_G(\exp(t\xi))|_{t=0} = \rho_{\mathfrak{g}_{dR}}(\xi), \tag{2.10}$$

$$\mathrm{Ad}_{\rho_G(g)}(\rho_{\mathfrak{g}_{dR}}(\cdot)) = \rho_{\mathfrak{g}_{dR}}(\mathrm{Ad}_g(\cdot)), \tag{2.11}$$

$$\mathrm{Ad}_{\rho_G(g)}(d_A) = d_A, \tag{2.12}$$

$$[\rho_{\mathfrak{g}_{dR}}(\cdot), d_A] = \rho_{\mathfrak{g}_{dR}}(d_{dR}(\cdot)), \tag{2.13}$$

for all $\xi \in \mathfrak{g} \subset \mathfrak{g}_{dR}$ and $g \in G$. The conditions (2.10) and (2.11) imply that the actions of G and $\mathfrak{g}_{dR}$ on A induces a graded Lie group action of G_{dR} on A. The conditions (2.12) and (2.13) ensure that the induced G_{dR}-action is compatible with the differential d_A.

A G_{dR}-algebra (A, d_A) is called acyclic if it is an acyclic dgca with respect to d_A.

Remark 2.3. By definition, the dgca of functions on a G_{dR}-manifold $(\mathcal{M}, Q_{\mathcal{M}})$ is a G_{dR}-algebra.

Example 2.10. The de Rham complex $(\Omega(P), d_{dR})$ of a principal G-bundle is a G_{dR}-algebra. The $\mathfrak{g}_{dR}$-action on $\Omega(P)$ is determined by the usual Lie derivatives and contractions.

Example 2.11. The Weil algebra of $\mathfrak{g}$ is an acyclic G_{dR}-algebra. The $\mathfrak{g}_{dR}$-action on $W(\mathfrak{g})$ is induced by the $\mathfrak{g}_{dR}$-action on $(\mathfrak{g}[1], d_{CE})$. More explicitly, it is given by

$$\mathrm{Lie}_a = -f^c_{ab}\theta^b\frac{\partial}{\partial\theta^c} - f^c_{ab}\phi^b\frac{\partial}{\partial\phi^c}, \quad \iota_a = \delta^b_a\frac{\partial}{\partial\theta^b}.$$

Definition 2.8. A morphism between G_{dR}-algebras (A_1, d_{A_1}) and (A_2, d_{A_2}) is a dgca morphism $\phi : A_1 \to A_2$ which is compatible with the G_{dR}-actions, i.e.,

$$[\phi, \rho_G(g)] = 0, \quad [\phi, \rho_{\mathfrak{g}_{dR}}(\xi_1, \xi_2)] = 0,$$

for all $g \in G$ and $(\xi_1, \xi_2) \in \mathfrak{g}_{dR} = \mathfrak{g} \oplus \mathfrak{g}[1]$.

A G_{dR}-algebra (A, d_A) is called a W_{dR}-algebra if there exists a G_{dR}-algebra morphism $\phi_{CW} : W(\mathfrak{g}) \to A$.

Example 2.12. Let P be a principal G-bundle endowed with a connection 1-form $A = A^a \otimes \xi_a$. Let $F_A = F_A^a \otimes \xi_a$ denote the curvature 2-form of A. The Chern–Weil homomorphism

$$\phi_{CW} : W(\mathfrak{g}) \to \Omega(P)$$

which maps θ^a to A^a and ϕ^a to F_A^a, is a G_{dR}-algebra morphism.

An element $a \in A$ is said to be basic if it is invariant under the G_{dR}-action. Let A_{bas} denote the subalgebra of basic elements in A. The conditions (2.12) and (2.13) imply that A_{bas} is preserved by the differential d_A. The corresponding cohomology is denoted by $H_{bas}(A)$ and is called the basic cohomology of A.

Definition 2.9. The (algebraic) equivariant cohomology $H_G(A)$ of a G_{dR}-algebra (A, d_A) is defined as $H_{bas}(A \otimes E)$, where E is an acyclic W_{dR}-algebra.

The equivariant cohomology $H_G(A)$ of (A, d_A) has the following properties [27]:

1. If A is a W_{dR}-algebra, then

$$H_{bas}(A) \cong H_{bas}(A \otimes E) = H_G(A). \tag{2.14}$$

2. $H_G(A)$ is independent of the choice of E. Indeed, let F be another acyclic W_{dR}-algebra. Then, using (2.14), we obtain

$$H_{bas}(A \otimes E) \cong H_{bas}(A \otimes E \otimes F) \cong H_{bas}(A \otimes F)$$

since both $A \otimes E$ and $A \otimes F$ are W_{dR}-algebras.

3. Let M be a G-manifold and $A = \Omega(M)$. If both G and M are compact, then

$$H_G(\Omega(M)) \cong H_G(M), \tag{2.15}$$

where $H_G(M)$ denotes the topological equivariant cohomology of M.

Let $(\mathcal{M}, Q_{\mathcal{M}})$ be a dg manifold with a compatible G-action. By the first part of Proposition 2.2, the Weil algebra $(W(\mathcal{M}), d_{W(\mathcal{M})})$ of $(\mathcal{M}, Q_{\mathcal{M}})$ has a G_{dR}-algebra structure. We then define the equivariant cohomology of $(\mathcal{M}, Q_{\mathcal{M}})$ as

$$H_G(\mathcal{M}) := H_G(W(\mathcal{M})) \cong H_{bas}(W(\mathcal{M}) \otimes W(\mathfrak{g})).$$

The differential of $W(\mathcal{M}) \otimes W(\mathfrak{g})$ is

$$d_{W(\mathcal{M})} \otimes 1 + 1 \otimes d_W.$$

The $\mathfrak{g}_{dR}$-action on $W(\mathcal{M}) \otimes W(\mathfrak{g})$ is given by

$$\begin{aligned} \rho_{\mathfrak{g}_{dR}}((\xi_a, 0)) &= \mathrm{Lie}_{\mathcal{M}}(X_a) \otimes 1 + 1 \otimes \mathrm{Lie}_a, \\ \rho_{\mathfrak{g}_{dR}}((0, \xi_a)) &= \iota_{\mathcal{M}}(X_a) \otimes 1 + 1 \otimes \iota_a, \end{aligned}$$

where ι_a and Lie_a are as defined in Example 2.11.

Remark 2.4. On the other hand, the dg manifold $(\mathcal{M}_G, Q_{\mathcal{M}_G})$ defined in Proposition 2.1 is a G_{dR}-manifold. By the second part of Proposition 2.2, $(\mathcal{M}_{G_{dR}}, d_{dR} + Q_{\mathcal{M}_{G_{dR}}})$ is a G_{dR}-manifold, where

$$\mathcal{M}_{G_{dR}} := (\mathcal{M}_G)_{dR} = T[1]\mathcal{M}_G, \quad Q_{\mathcal{M}_{G_{dR}}} := Q_{(\mathcal{M}_G)_{dR}} = \mathrm{Lie}_{\mathcal{M}_G}(Q_{\mathcal{M}_G}).$$

The corresponding Weil algebra $(W(\mathcal{M}_G), d_{W(\mathcal{M}_G)})$, where

$$W(\mathcal{M}_G) = C^\infty(\mathcal{M}_{G_{dR}}), \quad d_{W(\mathcal{M}_G)} = d_{dR} + Q_{\mathcal{M}_{G_{dR}}},$$

has a corresponding G_{dR}-algebra structure. Note that $W(\mathcal{M}_G) \cong W(\mathcal{M}) \otimes W(\mathfrak{g})$ as dgcas. It is easy to show that

$$d_{W(\mathcal{M}_G)} = d_{W(\mathcal{M})} \otimes 1 + \mathrm{Lie}_{\mathcal{M}}(X_a) \otimes \theta^a - \iota_{\mathcal{M}}(X_a) \otimes \phi^a + 1 \otimes d_W. \tag{2.16}$$

When $\mathcal{M}$ is an ordinary G-manifold, $d_{W(\mathcal{M}_G)}$ reduces to the usual Kalkman differential [28]. The $\mathfrak{g}_{dR}$-action on $W(\mathcal{M}_G)$ is given by

$$\begin{aligned}\rho_{\mathfrak{g}_{dR}}((\xi_a, 0)) &= \mathrm{Lie}_{\mathcal{M}}(X_a) \otimes 1 + 1 \otimes \mathrm{Lie}_a,\\ \rho_{\mathfrak{g}_{dR}}((0, \xi_a)) &= 1 \otimes \iota_a,\end{aligned}$$

where ι_a and Lie_a are as defined in Example 2.11.

Proposition 2.3. *There exists a G_{dR}-algebra isomorphism:*

$$\begin{aligned}\phi_{MQ} : W(\mathcal{M}_G) &\to W(\mathcal{M}) \otimes W(\mathfrak{g})\\ f &\mapsto \exp(-\iota_{\mathcal{M}}(X_a) \otimes \theta^a) f.\end{aligned}$$

ϕ_{MQ} is called the Mathai–Quillen isomorphism.

Proof. The proof is essentially the same as in [28], where $\mathcal{M}$ is a G-manifold. □

It follows that $H_G(\mathcal{M}) \cong H_{bas}(W(\mathcal{M}_G))$.

Example 2.13. Let E be a G-equivariant vector bundle over M, equipped with an G-equivariant section s. Let $(\mathcal{M}, Q_{\mathcal{M}}) = (E[-1], \iota_s)$. Then $(\mathcal{M}_{G_{dR}}, d_{dR} + Q_{\mathcal{M}_{G_{dR}}})$ is a G_{dR}-manifold, where

$$\mathcal{M}_{G_{dR}} = T[1](E[-1] \oplus \mathfrak{g}[1]), \quad C^\infty(\mathcal{M}_{G_{dR}}) \cong C^\infty(T[1]E[-1]) \otimes W(\mathfrak{g}),$$

and

$$d_{dR} + Q_{\mathcal{M}_{G_{dR}}} = (d_{dR} + \mathrm{Lie}_{E[-1]}(\iota_s)) \otimes 1 + \mathrm{Lie}_{E[-1]}(X_a) \otimes \theta^a - \iota_{E[-1]}(X_a) \otimes \phi^a + 1 \otimes d_W.$$

This construction can be interpreted as a de Rham version of the construction in Example 2.6. Under the same assumptions 2.4 and 2.5, it is natural to expect that

$$H_G(\mathcal{M}) \cong H_{bas}(W(\mathcal{M}_G)) \cong H^\bullet(s^{-1}(0)/G),$$

where $H^\bullet(s^{-1}(0)/G)$ denotes the de Rham cohomology of $s^{-1}(0)/G$.

3. Finite Dimensional Geometric Constructions

In this section, we define a 0-dimensional cohomological field theory in the language of G_{dR}-manifolds. We present two constructions for 0-dimensional CohFTs: the minimal construction and the standard construction, and discuss how they relate to the Mathai–Quillen formalism of CohFTs.

3.1. 0-dimensional cohomological field theories. Let $\mathcal{G}$ be a connected (compact) Lie group with Lie algebra $\mathrm{Lie}(\mathcal{G})$.

Definition 3.1. A 0-dimensional cohomological field theory of amplitude $[m_1, m_2]$ is a triple $(\widetilde{\mathcal{E}}, Q, \Psi)$, where:

1. $(\widetilde{\mathcal{E}}, Q)$ is a $\mathcal{G}_{dR}$-manifold of amplitude $[m_1, m_2]$, $m_1 < 0$;
2. Ψ is a degree -1 basic function on $\widetilde{\mathcal{E}}$, i.e., it is $\mathcal{G}_{dR}$-invariant.

We refer to $\widetilde{\mathcal{E}}$ as the configuration space, Q as the scalar supersymmetry, and Ψ as the gauge fixing fermion of the CohFT. The degree 0 basic function

$$\widetilde{S} = Q(\Psi)$$

is called the action functional of the CohFT. In general, a (homogeneous) Q-closed basic function O on $\widetilde{\mathcal{E}}$ is called an observable of the CohFT.

We will only consider CohFTs $(\widetilde{\mathcal{E}}, Q, \Psi)$ of small amplitudes $[m_1, m_2] \subseteq [-2, 2]$. To further simplify our discussion, we assume that

$$\widetilde{\mathcal{E}} = (\mathcal{E} \times V)_{dR} = T[1](\mathcal{E} \times V) = T[1]\mathcal{E} \times (V \oplus V[1]),$$

where $\mathcal{E}$ is an ordinary $\mathcal{G}$-manifold and V is a graded $\mathcal{G}$-representation (viewed as a graded manifold) with $V_0 = \{0\}$. Let

$$\widetilde{\mathcal{E}}_{even} := \mathcal{E} \times (V \oplus V[1])_{even} \quad \text{and} \quad \mathcal{E}_{tot} := \mathcal{E} \times \mathcal{H},$$

where $\mathcal{H} = V_1[1]$. We refer to $\mathcal{E}_{tot}$ and $\widetilde{\mathcal{E}}_{even}$ as the body and even body of $\widetilde{\mathcal{E}}$, respectively. Note that there are canonical inclusions

$$\iota : \mathcal{E}_{tot} \hookrightarrow \widetilde{\mathcal{E}}, \quad \iota_{even} : \widetilde{\mathcal{E}}_{even} \hookrightarrow \widetilde{\mathcal{E}}.$$

Let us regard $\widetilde{\mathcal{E}}$ as a $\mathbb{Z}$-graded supermanifold and equip it with a Berezinian μ. (We refer the reader to [29] for the notion of Berezinians on a ($\mathbb{Z}$-graded) supermanifold.) Let O be an observable of the CohFT $(\widetilde{\mathcal{E}}, Q, \Psi)$. We define

$$Z_t(O) := \int_{\widetilde{\mathcal{E}}} \mu \exp(it\widetilde{S})O,$$

where $\int_{\widetilde{\mathcal{E}}} \mu(\cdot)$ denotes the Berezin integral over $\mathcal{E}$ with respect to μ. $Z_t(O)$ is called the expectation value of O, and $Z_t(1)$ is called the partition function of the theory.

Remark 3.1. If μ is compatible with Q, i.e.,

$$\mathrm{div}_\mu(Q) = 0,$$

where $\mathrm{div}_\mu(Q)$ denotes the divergence of Q with respect to μ, then $Z_t(O)$ is independent of the parameter t and of the data (such as inner products on V or Riemannian metrics on $\mathcal{E}$) used to define the gauge fixing fermion Ψ, since $\widetilde{S}$ is Q-exact. One can evaluate the integral by taking either the limit $t \to 0$ or $t \to \infty$. The latter typically leads to localization on

$$\mathrm{Crit}(\widetilde{S}_{even}) \cap \widetilde{S}_{even}^{-1}(0) \subset \widetilde{\mathcal{E}}_{even}$$

via the stationary phase approximation, where

$$\widetilde{S}_{even} := \iota_{even}^* \widetilde{S} \in C^\infty(\widetilde{\mathcal{E}}_{even})$$

is the bosonic part of the action functional, and $\mathrm{Crit}(\widetilde{S}_{even})$ denotes its critical locus.

3.2. The minimal and standard constructions. Let $\mathcal{F}$ be a $\mathcal{G}$-equivariant section of $\mathcal{E}_{tot}$, i.e., a $\mathcal{G}$-equivariant map $\mathcal{E} \to \mathcal{H}$.

Construction 3.1. Consider the following CohFT $(\mathcal{E}_{min}, Q, \Psi_{min})$ of amplitude $[-1, 2]$:

- The configuration space $\mathcal{E}_{min}$ is given by

$$\mathcal{E}_{min} = \mathcal{E}_{tot}[-1]_{\mathcal{G}_{dR}} = T[1](\mathcal{E}_{tot}[-1] \oplus \mathrm{Lie}(\mathcal{G})[1]),$$

  where the cohomological vector field on $\mathcal{E}_{tot}[-1]$ is given by the contraction $\iota_{\mathcal{F}}$.
- The scalar supersymmetry Q is defined by (2.16). Let $(x^\mu, \chi^\alpha, \theta^a)$ be local coordinates on $\mathcal{E}_{tot}[-1] \oplus \mathrm{Lie}(\mathcal{G})[1]$, and let $(\psi^\mu, b^\alpha, \phi^a)$ denote the corresponding local coordinates of the shifted tangent space. Then Q acts as follows:

$$\begin{aligned}
&Q\theta^a = \phi^a - \frac{1}{2} f^a_{bc}\theta^b\theta^c, && Q\phi^a = -f^a_{bc}\theta^b\phi^c,\\
&Qx^\mu = \psi^\mu - \theta^a X^\mu_a(x), && Q\psi^\mu = -\theta^a \frac{\partial X^\mu_a(x)}{\partial x^\nu}\psi^\nu + \phi^a X^\mu_a(x),\\
&Q\chi^\alpha = b^\alpha - \theta^a X^\alpha_a(x,\chi) + \mathcal{F}^\alpha(x), && Qb^\alpha = -\theta^a \frac{\partial X^\alpha_a(x,\chi)}{\partial \chi^\beta} b^\beta + \phi^a X^\alpha_a(x,\chi) - \frac{\partial \mathcal{F}^\alpha(x)}{\partial x^\mu}\psi^\mu.
\end{aligned}$$

  Here, the fundamental vector field X_a on $\mathcal{E}_{tot}[-1] \oplus \mathrm{Lie}(\mathcal{G})[1]$ is given by

$$X_a = X^\mu_a(x)\frac{\partial}{\partial x^\mu} + X^\alpha_a(x,\chi)\frac{\partial}{\partial \chi^\alpha}.$$

  The $\mathrm{Lie}(\mathcal{G})_{dR}$-action is explicitly given by

$$\iota_a\theta^b = \delta^b_a, \quad \iota_a\phi^b = \iota_a x^\mu = \iota_a\psi^\mu = \iota_a\chi^\alpha = \iota_a b^\alpha = 0,$$

  and $\mathrm{Lie}_a = [Q, \iota_a]$.
- Let $h = \langle\cdot,\cdot\rangle$ be a $\mathcal{G}$-invariant inner product on $\mathcal{H}$. We define the gauge fixing fermion as

$$\Psi_{min} = h_{\alpha\beta}\chi^\alpha b^\beta.$$

  Since h is G-invariant and Ψ_{min} does not depend on θ^a, it follows that Ψ_{min} is $\mathcal{G}_{dR}$-invariant.

We refer to this construction as the minimal construction for CohFTs, as it involves the essential data needed to define a CohFT.

The dg manifold $(\mathcal{E}_{tot}[-1], \iota_{\mathcal{F}})$ provides a concrete model for the derived zero locus $\mathrm{Sol}(\mathcal{F})$ of the section $\mathcal{F}$ [30]. By our previous discussion (see Examples 2.6 and 2.13), an observable O in the minimal CohFT may be interpreted as a "closed differential form" on the moduli space $\mathcal{M}(\mathcal{F}) := \mathrm{Sol}(\mathcal{F})/\mathcal{G}$. We use quotation marks here because $\mathcal{M}(\mathcal{F})$ is not necessarily smooth. However, under the following assumptions:

1. The intersection of $\mathcal{F}$ with the zero section is transverse;
2. The action of $\mathcal{G}$ on $\mathrm{Sol}(\mathcal{F})$ is free and admits local slices,

the moduli space $\mathcal{M}(\mathcal{F})$ is a smooth manifold. In the infinite-dimensional setting, one needs to impose an additional condition: for each $x \in \mathrm{Sol}(\mathcal{F})$, the deformation complex

$$0 \to \mathrm{Lie}(\mathcal{G}) \xrightarrow{R_x} T_x\mathcal{E} \xrightarrow{\mathrm{Lin}_x(\mathcal{F})} \mathcal{H} \to 0$$

is elliptic. Here, R_x is the infinitesimal action of $\mathrm{Lie}(\mathcal{G})$ on $T_x\mathcal{E}$, and $\mathrm{Lin}_x(\mathcal{F})$ is the linearization of $\mathcal{F}$ at x, i.e., the tangent map of $\mathcal{F}$ at x. This condition, together with the previous assumptions, follows the standard approach in mathematical gauge theory (see e.g., [31, Theorem 183]).

In short, one may say that: In favorable cases, the observables of the minimal CohFT $(\mathcal{E}_{min}, Q, S_{min})$ correspond to representatives of de Rham cohomology classes of the moduli space $\mathcal{M}(\mathcal{F})$.

Let us omit the indices of the coordinates and introduce the following shorthand notations:

$$\begin{aligned} Q\theta &= \phi - \frac{1}{2}[\theta,\theta], & Q\phi &= -[\theta,\phi], \\ Qx &= \psi - \theta x, & Q\psi &= -\theta\psi + \phi x, \\ Q\chi &= b - \theta\chi + \mathcal{F}, & Qb &= -\theta b + \phi\chi - \mathrm{Lin}_x(\mathcal{F})\psi, \end{aligned}$$

and $\Psi_{min} = \langle \chi, b \rangle$. Henceforth, we will work with this new index-free notation, as it is more convenient for computations. The reader should find it not difficult to recover the original notation from the new one.

Definition 3.2. We define the first-order formulation of the action functional $S = |\mathcal{F}|^2$ to be the smooth function

$$S_{fo} := |b|^2 + \langle \mathcal{F}, b \rangle \in C^\infty(\mathcal{E}_{tot}), \tag{3.1}$$

where $|b|^2 = \langle b, b \rangle = h^{\alpha\beta} b_\alpha b_\beta$.

It is straightforward to verify that

$$\mathrm{crit}(S_{fo}) \cong \mathrm{crit}(S), \quad \mathrm{crit}(S_{fo}) \cap S_{fo}^{-1}(0) \cong \mathrm{Sol}(\mathcal{F}).$$

A direct computation shows that

$$S_{min} := Q(\Psi_{min}) = |b|^2 + \langle \mathcal{F}, b \rangle + \langle \chi, \mathrm{Lin}_x(\mathcal{F})\rangle - \langle \chi, \phi\chi \rangle, \tag{3.2}$$

It follows that the pullback of S_{min} to $\mathcal{E}_{tot}$ is simply the first-order formulation S_{fo} of S.

Construction 3.2. Consider the following CohFT $(\mathcal{E}_{std}, Q, \Psi_{std})$ of amplitude $[-2, 2]$:

- The configuration space $\mathcal{E}_{std}$ is given by

$$\mathcal{E}_{std} = (\mathrm{Lie}(\mathcal{G})[-2] \oplus \mathcal{E}_{tot}[-1])_{\mathcal{G}_{dR}} = T[1](\mathrm{Lie}(\mathcal{G})[-2] \oplus \mathcal{E}_{tot}[-1] \oplus \mathrm{Lie}(\mathcal{G})[1]),$$

  where $\mathrm{Lie}(\mathcal{G})[-2] \oplus \mathcal{E}_{tot}[-1]$ is equipped with the cohomological vector field $1 \otimes \iota_{\mathcal{F}}$.
- The scalar supersymmetry Q is again defined by (2.16). Let $(\lambda, x, \chi, \theta)$ be local coordinates on $\mathrm{Lie}(\mathcal{G})[-2] \oplus \mathcal{E}_{tot}[-1] \oplus \mathrm{Lie}(\mathcal{G})[1]$, and let (η, ψ, b, ϕ) denote the corresponding local coordinates of the shifted tangent space. The Q-action on $(x, \chi, \theta, \psi, b, \phi)$ is the same as in the Minimal Construction 3.1, and the Q-action on (λ, η) is

$$Q\lambda = \eta - [\theta, \lambda], \quad Q\eta = -[\theta, \eta] + [\phi, \lambda].$$

  The $\mathrm{Lie}(\mathcal{G})_{dR}$-action is also the same as in the Minimal Construction 3.1. It is determined by setting

$$\iota_\xi \theta = \xi,$$

  with ι_ξ acting trivially on all other coordinates, $[Q, \iota_\xi] = \mathrm{Lie}_\xi$, $\xi \in \mathrm{Lie}(\mathcal{G})$.
- We define the gauge fixing fermion as

$$\Psi_{std} = \langle \chi, b \rangle + \langle \psi, \lambda x \rangle + \langle \eta, [\phi, \lambda] \rangle. \tag{3.3}$$

  Here, $\langle \cdot, \cdot \rangle$ represents either an $\mathcal{G}$-invariant inner product h on $\mathcal{H}$, an $\mathcal{G}$-invariant Riemannian metric g on $\mathcal{E}$, or an $\mathcal{G}$-invariant inner product κ on $\mathrm{Lie}(\mathcal{G})$. Since the index-free notation here may cause confusion, let us reexpress (3.3) using the original notation:

$$\Psi_{std} = h_{\alpha\beta} \chi^\alpha b^\beta + g_{\mu\nu}(x) \psi^\mu \lambda^a X_a^\nu(x) + \kappa_{ab} \eta^a f_{cd}^b \phi^c \phi^d.$$

  Since $\langle \cdot, \cdot \rangle$ is $\mathcal{G}$-invariant and Ψ_{std} does not depend on θ, it follows that Ψ_{std} is $\mathcal{G}_{dR}$-invariant.

We refer to this construction as the standard construction for CohFTs, as it recovers the usual form of CohFT obtained by applying twisting to supersymmetric gauge theories.

A direct computation shows that

$$S_{std} := Q(\Psi_{std}) = S_{min} + \langle \phi x, \lambda x \rangle + |[\phi, \lambda]|^2 - \langle \psi, \eta x \rangle - \langle \psi, \lambda x \rangle - \langle \eta, [\phi, \eta] \rangle, \tag{3.4}$$

where S_{min} is given by (3.2). The underlying even manifold of $\mathcal{E}_{std}$ is $\mathrm{Lie}(\mathcal{G})[2] \times \mathcal{E}_{tot} \times \mathrm{Lie}(\mathcal{G})[-2]$. Since η and λ have nonzero degrees, $\iota^* S_{std}$ still takes the form (3.1), but $(S_{std})_{even}$ now has two more terms:

$$(S_{std})_{even} = S_{fo} + \langle \phi x, \lambda x \rangle + |[\phi, \lambda]|^2, \tag{3.5}$$

where S_{fo} is given by (3.1).

Let us drop the $\mathbb{Z}$ grading and view (3.5) as a function depending smoothly on x, ϕ, and λ. Let us denote by

$$\widetilde{\mathrm{Sol}(\mathcal{F})} := \mathrm{Crit}((S_{std})_{even}) \cap (S_{std})_{even}^{-1}(0)$$

the critical locus of (3.5) of value 0. It is easy to see that

$$\widetilde{\mathrm{Sol}(\mathcal{F})} \cong \{(x, \phi, \lambda) | \mathcal{F}(x) = 0,\ \lambda x = 0,\ \phi x = 0,\ [\phi, \lambda] = 0\}.$$

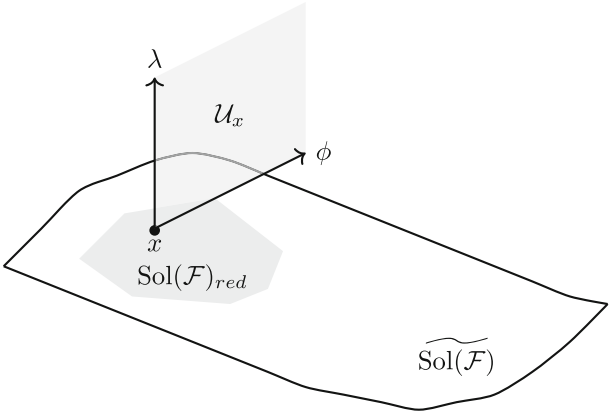


Fig. 1. A picture of $\widetilde{\mathrm{Sol}(\mathcal{F})}$

$\widetilde{\mathrm{Sol}(\mathcal{F})}$ has an interesting structure. (See Fig. 1.)

There exists a canonical projection

$$\mathrm{pr} : \widetilde{\mathrm{Sol}(\mathcal{F})} \to \mathrm{Sol}(\mathcal{F})$$

defined by sending ϕ and λ to 0. Let $\mathrm{Sol}(\mathcal{F})_{red}$ denote the subspace of $\mathrm{Sol}(\mathcal{F})$ on which $\mathcal{G}$ acts non-freely. The fiber $\mathcal{U}_x := \mathrm{pr}^{-1}(x)$ at $x \in \mathrm{Sol}(\mathcal{F})$ is nontrivial if and only if $x \in \mathrm{Sol}(\mathcal{F})_{red}$. Indeed, we have

$$\mathcal{U}_x \cong \{(\phi, \lambda) | \phi x = 0,\ \lambda x = 0,\ [\phi, \lambda] = 0\} = \ker[\cdot, \cdot]|_{\mathrm{Lie}(\mathcal{G}_x) \times \mathrm{Lie}(\mathcal{G}_x)},$$

where $\mathrm{Lie}(\mathcal{G}_x)$ denotes the Lie algebra of the isotropy group $\mathcal{G}_x$ of x, and $[\cdot, \cdot]$ denotes the Lie bracket of $\mathrm{Lie}(\mathcal{G})$.

3.3. Mathai–Quillen formalism and the Atiyah–Jeffrey formula. If the $\mathcal{G}$-action on $\mathcal{E}$ is free and proper, then $\mathcal{E}$ becomes a principal $\mathcal{G}$-bundle and can be equipped with a connection 1-form A_∇. In this case, the following construction provides a simplification of the Minimal Construction 3.1.

Construction 3.3. Consider the following CohFT $(\mathcal{E}_{MQ}, Q, \Psi_{MQ})$ of amplitude $[-1, 1]$:

- The configuration space $\mathcal{E}_{MQ}$ is given by

$$\mathcal{E}_{MQ} = \mathcal{E}_{tot}[-1]_{dR} = T[1]\mathcal{E}_{tot}[-1].$$

- Let (x, χ) be local coordinates of $\mathcal{E}_{tot}[-1]$. Let (ψ, b) be the corresponding coordinates of the shifted tangent space. The scalar supersymmetry Q takes the following form:

$$\begin{aligned} Qx &= \psi, & Q\psi &= 0, \\ Q\chi &= b - A_\nabla \chi + \mathcal{F}, & Qb &= -A_\nabla b + R_\nabla \chi - \nabla \mathcal{F}, \end{aligned}$$

where $\nabla = d + A_\nabla$, with d being the de Rham differential of $\mathcal{E}$, and R_∇ is the curvature 2-form of A_∇. The $\mathrm{Lie}(\mathcal{G})_{dR}$-action is explicitly given by

$$\iota_\xi x = 0, \quad \iota_\xi \psi = -\xi x,$$

$$\iota_\xi \chi = 0, \quad \iota_\xi b = 0,$$

and $\mathrm{Lie}_\xi = [Q, \iota_\xi]$, $\xi \in \mathrm{Lie}(\mathcal{G})$.

- Let $h = \langle \cdot, \cdot \rangle$ be a $\mathcal{G}$-invariant inner product on $\mathcal{H}$. We consider a gauge fixing fermion of the same form as in the Minimal Construction 3.1:

$$\Psi_{MQ} = \langle \chi, b \rangle.$$

It is easy to verify that Ψ_{MQ} is $\mathcal{G}_{dR}$-invariant. A direct computation shows that

$$S_{MQ} := Q(\Psi_{MQ}) = |b|^2 + \langle \mathcal{F}, b \rangle + \langle \chi, \nabla \mathcal{F} \rangle - \langle \chi, R_\nabla \chi \rangle. \tag{3.6}$$

We now explain how (3.6) recovers Mathai and Quillen's representative of the Euler class. Let $\mathcal{H} = \mathbb{R}^{2m}$, $\mathcal{G} = \mathrm{SO}(2m)$, $\mathcal{E}$ be the frame bundle $\mathrm{Fr}(N)$ over an $2m$-dimensional Riemannian manifold N, and ∇ be the Levi-Civita connection. The Fresnel type Berezin integral

$$\begin{aligned} e^{\mathcal{F}}_\nabla(t) &:= \frac{1}{(2\pi)^{2m}} \int d\chi db \exp(i S_{MQ}) \\ &= \frac{1}{(2\pi)^{2m}} \int db \exp(it(|b|^2 + \langle \mathcal{F}, b \rangle)) \int d\chi \exp(it(\langle \chi, \nabla \mathcal{F} \rangle - \langle \chi, R_\nabla \chi \rangle)) \\ &= \frac{1}{(2\pi)^m} \mathrm{Pf}(R_\nabla) \exp\left(-\frac{it}{4} \left(|\mathcal{F}|^2 + \langle \nabla \mathcal{F}, (R_\nabla)^{-1} \nabla \mathcal{F} \rangle \right)\right) \end{aligned}$$

defines a closed basic $2m$-form on $\mathrm{Fr}(N)$[13]. The partition function of our 0-dimensional minimal CohFT is then reduced to the integral

$$\int_{\mathrm{Fr}(N)} e^{\mathcal{F}}_\nabla(t).$$

However, there is an issue because

$$e^{\mathcal{F}}_\nabla(t) \text{ is not a top form on } \mathrm{Fr}(N).$$

There are two possible ways to address this. The first is to recognize that the "gauge group" $\mathcal{G}$ in this case is artificially introduced and represents redundant data. One can "quotient out" the $\mathcal{G}_{dR}$-action on $\mathcal{E}_{MQ}$ and instead work with the dg manifold $(T[1]T[-1]N, Q)$, where Q is locally given by

$$\begin{aligned} &Qx^\mu = \psi^\mu, && Q\psi^\mu = 0, \\ &Q\chi^\mu = b^\mu - \Gamma^\mu_{\nu\rho} \psi^\nu \chi^\rho + \mathcal{F}^\mu, && Qb^\mu = -\Gamma^\mu_{\nu\rho} \psi^\nu b^\rho + \frac{1}{2} R^\mu_{\nu\rho\sigma} \psi^\nu \psi^\rho \chi^\sigma - \frac{\partial \mathcal{F}^\mu}{\partial x^\nu} \psi^\nu - \Gamma^\mu_{\nu\rho} \psi^\nu \mathcal{F}^\rho, \end{aligned}$$

where $\Gamma^\mu_{\nu\rho}$ denotes the Christoffel symbols, $R^\mu_{\nu\rho\sigma}$ denotes the Riemann curvature tensor, and $\mathcal{F}$ is now regarded a vector field over N. Proceeding in this manner yields $e^{\mathcal{F}}_\nabla(t)$ in the same form, but now as a top form on N.

Since $\frac{d}{dt} e^{\mathcal{F}}_\nabla(t)$ is exact[3], and

$$\lim_{t \to 0} e^{\mathcal{F}}_\nabla(t) = \mathrm{Pf}\left(\frac{R_\nabla}{2\pi}\right),$$

[3] This can be proven either by using the fact that S_{MQ} is Q-exact, or simply by observing that $|\mathcal{F}|^2 + \langle \nabla \mathcal{F}, (R_\nabla)^{-1} \nabla \mathcal{F} \rangle = d\langle \mathcal{F}, \nabla^{-1} \mathcal{F} \rangle$.

$e^{\mathcal{F}}_{\nabla}(t)$ forms a representative of the Euler class of N. For a transversal $\mathcal{F}$, a proof of the Poincaré–Hopf theorem can be obtained by letting $t \to \infty$. More precisely, we have

$$\chi(N) = \lim_{t\to\infty} \int_N e^{\mathcal{F}}_{\nabla}(t) = \lim_{t\to\infty} \int_N \mathrm{Pf}\left(\frac{R_\nabla}{2\pi}\right) \exp\left(-\frac{it}{4}|\mathcal{F}|^2\right) = \sum_{x\in\mathcal{F}^{-1}(0)} \mathrm{ind}_x(\mathcal{F}), \tag{3.7}$$

where $\chi(N)$ is the Euler characteristic of N, and $\mathrm{ind}_x(\mathcal{F})$ is the degree of the vector field $\mathcal{F}$ at x. In the last equality of (3.7), we use the stationary phase approximation.

The second approach is to consider the following construction, which is a variant of the Standard Construction 3.2.

Construction 3.4. Consider the following CohFT $(\mathcal{E}_{AJ}, Q, \Psi_{AJ})$ of amplitude $[-2, 2]$:

- The configuration space $\mathcal{E}_{AJ}$ is the same as in the Standard Construction 3.2:

$$\mathcal{E}_{AJ} = T[1](\mathrm{Lie}(\mathcal{G})[-2] \oplus \mathcal{E}_{tot}[-1] \oplus \mathrm{Lie}(\mathcal{G})[1]).$$

- Let $(\lambda, x, \chi, \theta)$ be local coordinates on $\mathrm{Lie}(\mathcal{G})[-2]\oplus\mathcal{E}_{tot}[-1]\oplus\mathrm{Lie}(\mathcal{G})[1]$, and let (η, ψ, b, ϕ) denote the corresponding local coordinates of the shifted tangent space. The scalar supersymmetry Q takes the following form:

$$\begin{aligned}
&Q\theta = \phi - \frac{1}{2}[\theta,\theta], && Q\phi = -[\theta,\phi],\\
&Qx = \psi, && Q\psi = 0,\\
&Q\chi = b - \theta\chi + \mathcal{F}, && Qb = -\theta b + \phi\chi - (d+\theta)\mathcal{F},\\
&Q\lambda = \eta - [\theta,\lambda], && Q\eta = -[\theta,\eta] + [\phi,\lambda].
\end{aligned}$$

  The $\mathrm{Lie}(\mathcal{G})_{dR}$-action is explicitly given by

$$\iota_\xi\theta = \xi, \quad \iota_\xi\psi = -\xi x,$$

  with ι_ξ acting trivially on all other coordinates, and $\mathrm{Lie}_\xi = [Q, \iota_\xi]$, $\xi \in \mathrm{Lie}(\mathcal{G})$.
- The gauge fixing fermion is

$$\Psi_{AJ} = \langle\chi, b\rangle + \langle\psi + \theta x, \lambda x\rangle.$$

A direct computation shows that

$$S_{AJ} := Q(\Psi_{AJ}) = |b|^2 + \langle\mathcal{F}, b\rangle + \langle\chi, (d+\theta)\mathcal{F}\rangle - \langle\chi, \phi\chi\rangle + S_{proj}, \tag{3.8}$$

where

$$S_{proj} := \langle\phi x, \lambda x\rangle - \langle\psi, \lambda\psi\rangle - \langle\psi + \theta x, \eta x\rangle.$$

(3.8) recovers the Atiyah–Jeffrey formula [14, Equation (2.12)]. Let $C : \mathfrak{so}(2m) \to T_p\mathrm{Fr}(N)$ denote the tangent map of the $\mathrm{SO}(2m)$-action at $p \in \mathrm{Fr}(N)$. It is not hard to see that

$$\begin{aligned}
S_{proj} &= \langle C(\phi), C(\lambda)\rangle + \langle dC^*, \lambda\rangle - \langle C^*, \eta\rangle - \langle C(\theta), C(\eta)\rangle\\
&= \langle\phi + \frac{dC^*}{C^*C}, C^*C(\lambda)\rangle - \langle\theta + \frac{C^*}{C^*C}, C^*C(\eta)\rangle,
\end{aligned}$$

where $C^* : T_p\mathrm{Fr}(N) \to \mathfrak{g}$ is the adjoint of C. C^* can be viewed as a Lie algebra valued vertical 1-form on $\mathrm{Fr}(N)$. The Berezin integral

$$\begin{aligned}\gamma_\nabla(t) &:= \int d\phi d\lambda d\theta d\eta \exp(itS_{proj}) \\ &= \int d\phi d\lambda \exp(it\langle\phi + \frac{dC^*}{C^*C}, C^*C(\lambda)\rangle \int d\theta d\eta \exp(-it\langle\theta + \frac{C^*}{C^*C}, C^*C(\eta)\rangle)\end{aligned}$$

defines a vertical top form over $\mathrm{Fr}(N)$. $\gamma_\nabla(t)$ is independent of t and is called as the projection form in [32]. One can interpret it as

$$\gamma_\nabla(t) = \int d\phi d\theta \delta(\phi + \frac{dC^*}{C^*C})\delta(\theta + \frac{C^*}{C^*C})\mathrm{vol}_{\pi(p)},$$

where δ is the Dirac delta function, $\mathrm{vol}_{\pi(p)}$ is a volume form on the fiber of $\mathrm{Fr}(N)$ at $\pi(p)$ satisfying $\int_{\pi^{-1}(\pi(p))} \mathrm{vol}_{\pi(p)} = 1$. When being evaluated on horizontal vectors, one can replace A_∇ with $\frac{C^*}{C^*C}$, and R_∇ with $\frac{dC^*}{C^*C}$. It follows that

$$\chi(N) = \int_{\mathrm{Fr}(N)} e^{\mathcal{F}}_\nabla(t) \wedge \gamma_\nabla(t).$$

3.4. A perturbative perspective via L_∞ algebras. Let us recall the definition of an L_∞ algebra.

Definition 3.3. Let L be a graded vector space. An L_∞ algebra structure on L is specified by a collection of graded skew-symmetric multi-linear maps

$$l_n : \Lambda^n(L) \to L$$

of degree $2-n$, for $n \geq 1$, such that

$$\sum_{i=1}^{n}(-1)^{i(n-i)}\sum_{\epsilon}(-1)^{\epsilon} l_{n-i+1}(l_i(x_{\epsilon(1)}, \ldots, x_{\epsilon(i)}), x_{\epsilon(i+1)}, \ldots, x_{\epsilon(n)}) = 0, \qquad (3.9)$$

where $x_1, \ldots, x_n$ are homogenous elements in L, and ϵ ranges over $(i, n-i)$-unshuffles. (3.9) is called the Jacobi identity of L [33]. In particular, it implies that $l_1 \circ l_1 = 0$.

Let $\mathcal{M} = (M, \mathcal{O}_\mathcal{M})$ be a graded manifold. Let $Q_\mathcal{M}$ be a cohomological vector field on $\mathcal{M}$. Fix an point $p \in M$ and choose local coordinates z^i for $\mathcal{M}$ around p. The tangent space $T_p\mathcal{M}$ of $\mathcal{M}$ at p can be identified with the span of $\frac{\partial}{\partial z^i}$ over $\mathbb{R}$. Consider the Taylor expansion of Q at p:

$$Q = \sum_{k=0}^{\infty} \frac{1}{k!} Q^i_{i_1\ldots i_k}(p) z^{i_k} \cdots z^{i_1} \frac{\partial}{\partial z^i}$$

It is well known (see [34,35]) that if $Q^i_0(p) = 0$, then the graded symmetric multi-linear maps

$$Q^{(n)}_p : \mathrm{Sym}^n(T_p\mathcal{M}) \to T_p\mathcal{M}$$

defined by $Q_p^{(n)}(\frac{\partial}{\partial z^{i_1}}, \cdots, \frac{\partial}{\partial z^{i_n}}) = \pm Q^i_{i_1 \ldots i_n}(p)\frac{\partial}{\partial z^i}$, induce an L_∞-structure on $T_p\mathcal{M}[-1]$ via the décalage isomorphism:

$$\Lambda^n(T_p\mathcal{M}[-1]) \cong \mathrm{Sym}^n(T_p\mathcal{M})[-n].$$

Let us consider the L_∞ algebra associated with our Minimal Construction 3.1 of CohFTs. Recall that the cohomological vector field Q is locally given by

$$Q\theta^a = \phi^a - \frac{1}{2} f^a_{bc}\theta^b\theta^c, \qquad Q\phi^a = -f^a_{bc}\theta^b\phi^c,$$

$$Qx^\mu = \psi^\mu - \theta^a X^\mu_a(x), \qquad Q\psi^\mu = -\theta^a \frac{\partial X^\mu_a(x)}{\partial x^\nu}\psi^\nu + \phi^a X^\mu_a(x),$$

$$Q\chi^\alpha = b^\alpha - \theta^a X^\alpha_a(x,\chi) + \mathcal{F}^\alpha(x), \quad Qb^\alpha = -\theta^a \frac{\partial X^\alpha_a(x,\chi)}{\partial \chi^\beta} b^\beta + \phi^a X^\alpha_a(x,\chi) - \frac{\partial \mathcal{F}^\alpha(x)}{\partial x^\mu}\psi^\mu.$$

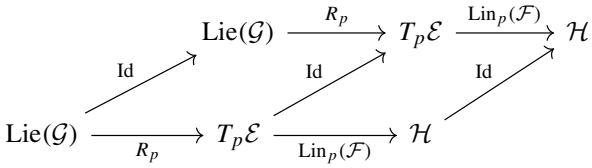
For any $p \in \mathrm{Sol}(\mathcal{F})$, the leading-order components $Q^i_0(p)$ in the Taylor expansion of Q vanish. Consequently, the shifted tangent space $T_p\mathcal{E}_{min}[-1]$ inherits an L_∞-structure. The presence of higher brackets l_k reflects the nonlinearity of $\mathcal{F}$ and the nontriviality of the $\mathcal{G}$-action on $\mathcal{E}$.

Note that

$$T_p\mathcal{E}_{min}[-1] \cong \mathrm{Lie}(\mathcal{G})[1] \oplus (T_p\mathcal{E} \oplus \mathrm{Lie}(\mathcal{G})) \oplus (\mathcal{H} \oplus T_p\mathcal{E})[-1] \oplus \mathcal{H}[-2].$$

The underlying complex of this L_∞ algebra is:

deg	-1	0	1	2



The two rows correspond precisely to the (shifted) deformation complex of $\mathcal{F}$ at p.

Remark 3.2. In particular, if $\mathcal{E}$ is an affine space with an affine $\mathcal{G}$-action and $\mathcal{F}$ is quadratic, the resulting structure is a dgLa. This will be the case later when we apply the minimal construction to generalized Seiberg–Witten equations.

4. Field Theory Applications

The geometric constructions for 0-dimensional CohFTs in the previous section can be extended to construct CohFTs in dimensions $d \geq 1$. The immediate complication is that both $\mathcal{E}$ and $\mathcal{H}$ become infinite-dimensional, and one must specify their topologies and the algebra of functions considered on them. Here, we proceed at a formal level and do not treat the mathematical details; these are supplied in [36] in terms of constructions involving infinite jet bundles, local forms, and evolutionary vector fields.

4.1. Nonlinear sigma models. Let (Σ, g) (the source) and (M, h) (the target) be Riemannian manifolds of dimensions d and $2m$, respectively. Define

$$\mathcal{E} = C^\infty(\Sigma, \mathrm{Fr}(M))$$

as the space of smooth maps from Σ to the frame bundle $\mathrm{Fr}(M)$ of M. Next, consider the space

$$\mathcal{E}_{tot} = C^\infty(T\Sigma, \mathrm{Fr}(M) \times \mathbb{R}^{2m}),$$

consisting of smooth bundle maps from the tangent bundle $T\Sigma$ over Σ to the trivial vector bundle $\mathrm{Fr}(M) \times \mathbb{R}^{2m}$ over $\mathrm{Fr}(M)$. The group $\mathcal{G} = C^\infty(\Sigma, \mathrm{SO}(2m))$ acts freely on $\mathcal{E}$ and $\mathcal{E}_{\mathrm{tot}}$ in canonical ways. This makes $\mathcal{E}_{\mathrm{tot}}$ an infinite-dimensional $\mathcal{G}$-equivariant trivial vector bundle over $\mathcal{E}$. The fiber of $\mathcal{E}_{\mathrm{tot}}$ can be identified with

$$\mathcal{H} = \Gamma(T^\vee\Sigma \otimes \mathbb{R}^{2m}),$$

where $T^\vee\Sigma$ denotes the cotangent bundle of Σ. The space $\mathcal{H}$ is equipped with a canonical $\mathcal{G}$-invariant inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ given by

$$\sum_{i=1}^{2m} \int_\Sigma \mathrm{vol}_g \, g^{\mu\nu}(x)\, s^i_\mu(x)\, t^i_\nu(x),$$

where s^i and t^i are sections of $T^\vee\Sigma$, and vol_g denotes the volume form associated with the metric g on Σ. The $\mathcal{G}$-equivariant section $\mathcal{F}$ is defined by sending $\phi \in \mathcal{E}$ to $\theta_M \circ d\phi \in \mathcal{E}_{tot}$, where:

- $d\phi : T\Sigma \to T\mathrm{Fr}(M)$ is the tangent map of ϕ,
- θ_M is the solder form of $\mathrm{Fr}(M)$, interpreted as a bundle map:

$$\theta_M : T\mathrm{Fr}(M) \to \mathrm{Fr}(M) \times \mathbb{R}^{2m}.$$

The CohFT construction is of the same form as in Construction 3.3. We set

$$\mathcal{E}_{NSM} = T[1]\mathcal{E}_{tot}[-1].$$

The field content of the theory consists of

$$\phi \in C^\infty(\Sigma, \mathrm{Fr}(M)), \qquad \psi \in \Gamma(\phi^* T\mathrm{Fr}(M))[-1],$$
$$\chi \in \Gamma(T^\vee\Sigma \otimes \mathbb{R}^{2m})[1], \qquad b \in \Gamma(T^\vee\Sigma \otimes \mathbb{R}^{2m}).$$

The scalar supersymmetry Q is given by

$$Q\phi = \psi, \qquad Q\psi = 0,$$
$$Q\chi = b - (\phi^* A_\nabla)(\psi)\chi + \mathcal{F}(\phi), \qquad Qb = -(\phi^* A_\nabla)(\psi)b + (\phi^* R_\nabla)(\psi, \psi)\chi - \phi^*\nabla\mathcal{F}(\phi).$$

Let us explain the notation and the objects involved. Here, A_∇ is the connection 1-form on $\mathrm{Fr}(M)$ corresponding to the Levi-Civita connection, and R_∇ is the curvature 2-form of A_∇. The pullback $\phi^* A_\nabla$ can be regarded as a section of $\phi^* T^\vee\mathrm{Fr}(M) \otimes \mathfrak{so}(2m)$. Composing it with ψ yields a section $(\phi^* A_\nabla)(\psi)$ of the trivial bundle $\Sigma \times \mathfrak{so}(2m)$. This section acts canonically on sections of $T^\vee\Sigma \otimes \mathbb{R}^{2m}$, allowing us to write expressions like $(\phi^* A_\nabla)(\psi)\chi$. Similarly, the notation $(\phi^* R_\nabla)(\psi, \psi)$ should now be self-explanatory.

Finally, $\nabla = d_{\mathrm{Fr}(M)} + A_\nabla$ is a connection on the trivial bundle $\mathrm{Fr}(M) \times \mathbb{R}^{2m}$, where $d_{\mathrm{Fr}(M)}$ is the de Rham differential of $\mathrm{Fr}(M)$. Its pullback via ϕ can be expressed as:

$$\phi^* \nabla = d_\Sigma + \phi^* A_\nabla,$$

where d_Σ is the de Rham differential on Σ. The term $\phi^* \nabla \mathcal{F}(\phi)$ should be understood as:

$$\phi^* \nabla \mathcal{F}(\phi) = d_\Sigma(\phi^* \theta_M(\psi)) + \phi^* A_\nabla(\psi) \mathcal{F}(\phi) \in \Gamma(T^\vee \Sigma \otimes \mathbb{R}^{2m})[-1],$$

where $\phi^* \theta_M$ is a section of $\phi^* T^\vee \mathrm{Fr}(M) \otimes \mathbb{R}^{2m}$. Consequently, $\phi^* \theta_M(\psi)$ is a section of the trivial bundle $\Sigma \times \mathbb{R}^{2m}$.

The $\mathrm{Lie}(\mathcal{G})_{dR}$-action is given by

$$\iota_\xi \psi(x) = \xi(x)(\phi(x)), \quad x \in \Sigma,$$

with ι_ξ acting trivially on all other fields, $[Q, \iota_\xi] = \mathrm{Lie}_\xi$, $\xi \in \mathrm{Lie}(\mathcal{G})$. Here, we identify $\mathrm{Lie}(\mathcal{G})$ with $C^\infty(\Sigma, \mathfrak{so}(2m))$ and $\xi(x) \in \mathfrak{so}(2m)$ with its corresponding fundamental vector fields over $\mathrm{Fr}(M)$.

The CohFT gauge fixing fermion is also of the same form as in Construction 3.3:

$$\Psi_{NSM} = \langle \chi, b \rangle_{\mathcal{H}} = \sum_{i=1}^{2m} \int_\Sigma \mathrm{vol}_g \, g^{\mu\nu}(x) \, \chi^i_\mu(x) \, b^i_\nu(x),$$

A direct computation shows that

$$S_{NSM} = Q(\Psi_{NSM}) = \langle b + \mathcal{F}(\phi), b \rangle_{\mathcal{H}} + \langle \chi, \phi^* \nabla \mathcal{F}(\phi) \rangle_{\mathcal{H}} - \langle \chi, \phi^* R_\nabla(\psi, \psi) \chi \rangle_{\mathcal{H}}.$$

As in the finite-dimensional case, the gauge symmetry group $\mathcal{G}$ introduces redundant data. We can simplify the whole discussion by "quotienting out" $\mathcal{G}_{dR}$ and working with $\mathcal{E} = C^\infty(\Sigma, M)$, $\mathcal{E}_{tot} = T\mathcal{E} = C^\infty(T\Sigma, TM)$, and the section:

$$\begin{aligned} \mathcal{F} : C^\infty(\Sigma, M) &\to C^\infty(T\Sigma, TM) \\ \phi &\mapsto d\phi. \end{aligned}$$

The CohFT configuration space is:

$$T[1]T[-1](C^\infty(\Sigma, M)).$$

The field content of the new theory consists of:

$$\begin{aligned} &\phi \in C^\infty(\Sigma, M), && \psi \in \Gamma(\phi^* TM)[-1], \\ &\chi \in \Gamma(T^\vee \Sigma \otimes \phi^* TM)[1], && b \in \Gamma(T^\vee \Sigma \otimes \phi^* TM). \end{aligned}$$

Fix a point $x \in \Sigma$. After choosing local coordinates x^μ around x and y^i around $y = \phi(x)$, the scalar supersymmetry Q is given by:

$$\begin{aligned} &Q\phi^i(x) = \psi^i(x), \quad Q\psi^i(x) = 0, \\ &Q\chi^i_\mu(x) = b^i_\mu(x) - \Gamma^i_{jk}(y)\psi^j(y)\chi^k_\mu(x) + \frac{\partial \phi^i}{\partial x^\mu}(x), \\ &Qb^i_\mu(x) = -\Gamma^i_{jk}(y)\psi^j(y)b^k_\mu(x) + \frac{1}{2} R^i_{jkl}(y)\psi^j(y)\psi^k(y)\chi^l_\mu(x) - \nabla_{\frac{\partial}{\partial x^\mu}} \psi^i(x), \end{aligned}$$

where Γ^i_{jk} are the Christoffel symbols of the metric h on M, R^i_{jkl} are the components of the Riemann curvature tensor of h, and $\nabla_{\frac{\partial}{\partial x^\mu}}\psi^i(x) := \frac{\partial \psi^i}{\partial x^\mu}(y) + \Gamma^i_{jk}(y)\psi^j(y)\frac{\partial \phi^k}{\partial x^\mu}(x)$.

The gauge fixing fermion is

$$\Psi_{NSM} = \int_\Sigma \mathrm{vol}_g g^{\mu\nu}(x) h_{ij}(\phi(x))\chi^i_\mu(x) b^j_\nu(x).$$

And the action functional is

$$\begin{aligned} S_{NSM} = Q(\Psi_{NSM}) = \int_\Sigma \mathrm{vol}_g\, g^{\mu\nu}(x) h_{ij}(\phi(x)) \Big(& (b^i_\mu(x) + \frac{\partial \phi^i}{\partial x^\mu}(x)) b^j_\nu(x) + \\ & \chi^i_\mu(x)(\frac{\partial \psi^j}{\partial x^\nu}(\phi(x)) + \Gamma^j_{kl}(\phi(x))\psi^k(\phi(x))\frac{\partial \phi^l}{\partial x^\nu}(x)) \\ & -\frac{1}{2}\chi^i_\mu(x) R^j_{klm}(\phi(x))\psi^k(\phi(x))\psi^l(\phi(x))\chi^m_\nu(x)\Big). \end{aligned}$$

Remark 4.1. Note that

$$\mathrm{Sol}(\mathcal{F}) = \{\text{Constant maps from } \Sigma \text{ to } M\} \cong M.$$

To obtain a more interesting moduli space than the space of constant maps, one can consider the following section of $\mathcal{E}_{tot}$:

$$\mathcal{F}_{\alpha_\Sigma, X_M}(\phi) = d\phi - \alpha_\Sigma \otimes \phi^* X_M,$$

where α_Σ is a 1-form on Σ and X_M is a vector field over M. Note that $\phi^* X_M$ is not a vector field over Σ, but a section of the pullback bundle $\phi^* TM$. It is straightforward to write down the scalar supersymmetry and action functional for the corresponding CohFT, which we leave as an exercise for the reader.

4.1.1. N=2 supersymmetric quantum mechanics Restricting the previous discussion to the $d = 1$ case yields $N = 2$ supersymmetric quantum mechanics. Here, the choice of source manifold Σ is limited: if Σ is connected, it must be either the circle S^1 in the compact case or the real line $\mathbb{R}$ in the non-compact case. Let us assume that $\Sigma = S^1$ and equip it with the standard Riemannian metric. Since $T^\vee S^1$ is a trivial line bundle, the field content of the theory consists of:

$$\begin{aligned} &\phi \in C^\infty(S^1, M), && \psi \in \Gamma(\phi^* TM)[-1], \\ &\chi \in \Gamma(\phi^* TM)[1], && b \in \Gamma(\phi^* TM). \end{aligned}$$

The section $\mathcal{F}$ can be identified as the map sending ϕ to $\dot\phi := \frac{d\phi}{dt}$. The scalar supersymmetry Q acts on the fields as

$$\begin{aligned} &Q\phi^i = \psi^i, && Q\psi^i = 0, \\ &Q\chi^i = b^i - \Gamma^i_{jk}\psi^j\chi^k + \dot\phi^i, && Qb^i = -\Gamma^i_{jk}\psi^j b^k + \frac{1}{2}R^i_{jkl}\psi^j\psi^k\chi^l - \nabla_{\frac{d}{dt}}\psi^i, \end{aligned}$$

where $\nabla_{\frac{d}{dt}}\psi^i := \dot\psi^i + \Gamma^i_{jk}\psi^j\dot\phi^k$.

The gauge fixing fermion is

$$\Psi_{SQM} = \int_{S^1} dt\, h_{ij}\chi^i b^j.$$

The supersymmetric action functional is

$$S_{SQM} = Q(\Psi_{SQM}) = \int_{S^1} dt\, h_{ij}\left((b^i + \dot{\phi}^i)b^j + \chi^i \nabla_{\frac{d}{dt}}\psi^j - \frac{1}{2}\chi^i R^j_{klm}\psi^k\psi^l\chi^m\right).$$

Remark 4.2. One can replace Σ with $\mathbb{R}$ and $\mathcal{F}$ with

$$\mathcal{F}_f(\phi) = d\phi - dt \otimes \mathrm{grad}(f),$$

where f is a Morse function on M and $\mathrm{grad}\, f$ is the gradient of f with respect to the Riemannian metric h on M. The corresponding CohFT recovers $N = 2$ supersymmetric quantum mechanics with a potential.

4.2. Generalized Seiberg–Witten theory. Let $\mathrm{Cl}(n)$ denote the Clifford algebra of $\mathbb{R}^n$ and $\mathrm{End}(m)$ denote the algebra of $m \times m$ matrices over $\mathbb{R}$ or $\mathbb{C}$. Let $G \subset \mathrm{End}(m)$ be a (compact) matrix group containing -1. Let $\mathrm{Spin}(n)$ denote the spin group of $\mathbb{R}^n$.

Definition 4.1. The spin^G-group $\mathrm{Spin}^G(n)$ is the subgroup of the unit group of $\mathrm{Cl}(n) \otimes \mathrm{End}(m)$ generated by $\mathrm{Spin}(n) \subset \mathrm{Cl}(n)$ and $G \subset \mathrm{End}(m)$.

There exists a canonical group homomorphism

$$\mathrm{Spin}(n) \times G \to \mathrm{Spin}^G(n) \subset \mathrm{Cl}(n) \otimes \mathrm{End}(m),$$

explicitly given by sending $(a, g) \in \mathrm{Spin}(n) \times G$ to $a \otimes g \in \mathrm{Spin}^G(n)$, whose kernel consists of $(1, 1)$ and $(-1, -1)$. It follows that there exists a short exact sequence

$$\mathrm{Id} \to \mathbb{Z}_2 \to \mathrm{Spin}^G(n) \to \mathrm{SO}(n) \times (G/\mathbb{Z}_2) \to \mathrm{Id}.$$

At the Lie algebra level, one has the following Lie algebra isomorphism

$$\mathfrak{spin}^G(n) = \mathfrak{spin}(n) \oplus \mathfrak{g} \cong \mathfrak{so}(n) \oplus \mathrm{Lie}(G/\mathbb{Z}_2),$$

where $\mathfrak{spin}^G(n)$, $\mathfrak{spin}(n)$, $\mathfrak{g}$, $\mathfrak{so}(n)$, and $\mathrm{Lie}(G/\mathbb{Z}_2)$ denote the Lie algebras of $\mathrm{Spin}^G(n)$, $\mathrm{Spin}(n)$, G, $\mathrm{SO}(n)$, and $G/\mathbb{Z}_2$, respectively.

Remark 4.3. Recall that there exists an explicit Lie algebra identification:

$$\Lambda^2(\mathbb{R}^n) \cong \mathfrak{spin}(n) \to \mathfrak{so}(n) \cong \Lambda^2(\mathbb{R}^n), \quad \varphi \mapsto 2\varphi.$$

The Lie algebra $\mathrm{Lie}(G/\mathbb{Z}_2)$ of $G/\mathbb{Z}_2$, however, does not have an explicit identification with the Lie algebra $\mathfrak{g}$ of G in general. Let $\pi : G \to G/\mathbb{Z}_2$ denote the canonical quotient map. We have an abstract isomorphism $d\pi_{\mathrm{Id}} : \mathfrak{g} \to \mathrm{Lie}(G/\mathbb{Z}_2)$, which is the differential of π at the identity. If G is abelian, one can consider the isomorphism defined by sending $[g] \in G/\mathbb{Z}_2$ to $g^2 \in G$. $d\pi_{\mathrm{Id}}$ can then be identified with the automorphism of $\mathfrak{g}$ defined by multiplying the factor 2.

Let (Σ, g) be an (oriented) Riemannian manifold of dimension d. Let L be a vector bundle over M of rank m with structure group $G/\mathbb{Z}_2$. The frame bundle of $T\Sigma \otimes L$ admits a $\mathrm{SO}(d) \times (G/\mathbb{Z}_2)$-subbundle

$$P(L, \Sigma) = P_{\mathrm{SO}(d)} \times_{\Sigma} P_{G/\mathbb{Z}_2},$$

where $P_{\mathrm{SO}(d)}$ is the special orthogonal frame bundle of Σ and $P_{G/\mathbb{Z}_2}$ is the principal $G/\mathbb{Z}_2$-bundle encoding the $G/\mathbb{Z}_2$-structure of L.

Definition 4.2. A spin^G-structure on Σ is a $\mathbb{Z}_2$-equivariant lift of $P(L, \Sigma)$ to a principal $\mathrm{Spin}^G(d)$-bundle $P_{\mathrm{Spin}^G(d)}$ with respect to the double covering $\mathrm{Spin}^G(d) \to \mathrm{SO}(d) \times (G/\mathbb{Z}_2)$.

Remark 4.4. We require the vector bundle L to be part of the data for defining a spin^G-structure on Σ. For $G = \mathrm{U}(1) \subset \mathrm{End}(2)$, a spin^G-structure reduces to a spin^c-structure. In this case, L is a hermitian line bundle, which is known as the determinant line bundle of the spin^c-structure.

We define the Clifford bundle of the pair (L, Σ) to be the algebra bundle

$$\mathrm{Cl}(L, \Sigma) = P(L, \Sigma) \times_{\rho_{d,G}} (\mathrm{Cl}(d) \otimes \mathrm{End}(m)),$$

where $\rho_{d,G}$ is the $\mathrm{SO}(d) \times (G/\mathbb{Z}_2)$-action on $\mathrm{Cl}(d) \otimes \mathrm{End}(m)$ induced by the canonical action of $\mathrm{SO}(d)$ on $\mathbb{R}^d$ and the adjoint action of G on $\mathrm{End}(m)$. Note that there exists a vector bundle isomorphism:

$$\gamma : \Lambda(\Sigma) \otimes \mathrm{End}(L) \cong \mathrm{Cl}(L, \Sigma). \tag{4.1}$$

The Levi-Civita connection ∇ of Σ together with a connection 1-form A on $P_{G/\mathbb{Z}_2}$ determines a connection on $\mathrm{Cl}(L, \Sigma)$, which we denote by ∇_A.

Let $\not{S}$ be a vector bundle over Σ equipped with a bundle metric $\langle \cdot, \cdot \rangle$. With a slight abuse of notation, we use ∇_A once again to denote a metric connection on $\not{S}$.

Definition 4.3. $\not{S}$ is called a (twisted) Dirac bundle of the pair (L, Σ) if it is a bundle of left modules over $\mathrm{Cl}(L, \Sigma)$, such that at each $x \in \Sigma$,

$$\langle (e \otimes g)\sigma_1, (e \otimes g)\sigma_2 \rangle = \langle \sigma_1, \sigma_2 \rangle \tag{4.2}$$

for all $\sigma_1, \sigma_2 \in \not{S}_x$ and all unit vectors $e \in T_x\Sigma$ and $g \in G \subset \mathrm{End}(L_x)$, and

$$\nabla_A(\varphi\sigma) = (\nabla_A\varphi)\sigma + \varphi(\nabla_A\sigma) \tag{4.3}$$

for all $\varphi \in \Gamma(\mathrm{Cl}(L, \Sigma))$ and $\sigma \in \Gamma(\not{S})$. The (twisted) Dirac operator of a Dirac bundle is a first-order differential operator $\not{D}_A : \Gamma(\not{S}) \to \Gamma(\not{S})$ defined by

$$\not{D}_A\sigma := \sum_{\mu=1}^{n} (e_\mu \otimes 1)\nabla_{A,e_\mu}\sigma$$

at $x \in M$, where $\{e_\mu\}_{\mu=1}^n$ is an orthonormal basis of $T_x\Sigma$.

Example 4.1. Let Σ be equipped with a spin^G-structure $P_{\mathrm{Spin}^G(d)}$. We define

$$S(L,\Sigma) = P_{\mathrm{Spin}^G(d)} \times_{\mu_{d,G}} S,$$

where S is a left module of $\mathrm{Cl}(d) \otimes \mathrm{End}(m)$, and $\mu_{d,G}$ is the induced left action of $\mathrm{Spin}^G(d)$ on S. Similarly, we define

$$V(L,\Sigma) = P_{\mathrm{Spin}^G(d)} \times_{\mathrm{Ad}} V,$$

where V is a bimodule of $\mathrm{Cl}(d) \otimes \mathrm{End}(m)$, and Ad is the induced adjoint action of $\mathrm{Spin}^G(d)$ on V. Both $S(L,\Sigma)$ and $V(L,\Sigma)$ are bundles of left modules over $\mathrm{Cl}(L,\Sigma)$.

The Levi-Civita connection 1-form on $P_{\mathrm{SO}(n)}$, together with a connection 1-form A on $P_{G/\mathbb{Z}_2}$, determines a connection 1-form on $P_{\mathrm{Spin}^G(d)}$. This induces a covariant derivative ∇_A on $S(L,\Sigma)$ or $V(L,\Sigma)$. If G is compact, S or V can be equipped with a $\mathrm{Spin}^G(d)$-invariant inner product, defining a bundle metric $\langle\cdot,\cdot\rangle$ on $S(L,\Sigma)$ or $V(L,\Sigma)$ that satisfies (4.2). Moreover, ∇_A is a metric connection with respect to $\langle\cdot,\cdot\rangle$ and satisfies (4.3) by construction. Thus, both $S(L,\Sigma)$ and $V(L,\Sigma)$ are Dirac bundles of (L,Σ).

From now on, we consider Dirac bundles $\not{S}$ constructed as either $S(L,\Sigma)$ or $V(L,\Sigma)$. We define the following quadratic bundle map

$$\mu : \not{S} \to \Lambda^2(\Sigma) \otimes \mathrm{ad}\, L, \tag{4.4}$$

where ad L is the adjoint bundle of $P_{G/\mathbb{Z}_2}$, by setting

$$\mu(\sigma) = \sum_a \langle (e_\mu e_\nu \otimes \xi_a)\sigma, \sigma\rangle (e^\mu \wedge e^\nu) \otimes \xi_a,$$

for $\sigma \in \not{S}_x$, where $\{e^\mu\}$ is an (orthonormal) basis of $T_x^\vee\Sigma$, $\{\xi_a\}$ is an orthonormal basis of $(\mathrm{ad}\, L)_x$[4], and the action of $(e_\mu e_\nu \otimes \xi_a)$ on σ is given by the $\mathrm{Cl}(L,\Sigma)$-module structure of $\not{S}$.

Let A be a $G/\mathbb{Z}_2$ connection on L. Note that curvature F_A of A is also a section of $\Lambda^2(\Sigma) \otimes \mathrm{ad}\, L$. We define the generalized Seiberg–Witten (GSW) equations on Σ as

$$F_A - \frac{1}{2}\mu(\sigma) = 0, \quad \not{D}_A\sigma = 0, \tag{4.5}$$

where $\sigma \in \Gamma(\not{S})$. If $\not{S}$ is complex, we define the chirality operator $\Gamma \in \mathrm{End}(\not{S})$ as

$$\Gamma = i^{d/2}\gamma(\mathrm{vol}_g \otimes 1),$$

where $\mathrm{vol}_g \in \Gamma(\Lambda^d(T^\vee\Sigma))$ is the volume form of Σ. Γ has the following properties:

1. $\Gamma^2 = 1$;
2. $\not{D}_A \circ \Gamma = -\Gamma \circ \not{D}_A$.

[4] We put the standard inner product $\langle\xi_1,\xi_2\rangle = \mathrm{Tr}(\xi_1\xi_2)$ on $(\mathrm{ad}\, L)_x \cong \mathrm{Lie}(G/\mathbb{Z}_2)$.

This implies that $\not{S}$ admits a decomposition $\not{S} = \not{S}^+ \oplus \not{S}^-$ such that $\Gamma|_{\not{S}^\pm} = \pm 1$, and the Dirac operator $\not{D}_A$ decomposes as $\not{D}_A = (\not{D}_A^+, \not{D}_A^-)$, mapping sections of $\not{S}^+ \oplus \not{S}^-$ to sections of $\not{S}^- \oplus \not{S}^+$.

In dimension 4, Γ can also be defined for a real $\not{S}$, and it induces a decomposition of the quadratic bundle map (4.4):

$$\mu_\pm : \not{S}^\pm \to \Lambda^2_\pm(\Sigma) \otimes \mathrm{ad}\, L,$$

where $\Lambda^2_\pm(\Sigma)$ denote the selfdual and anti-selfdual subbundles of $\Lambda^2(\Sigma)$ with respect to the Hodge star operator of g.

From now on, we fix Σ to be a compact oriented Riemannian 4-manifold. And we consider a selfdual version of the GSW equations (4.5):

$$(F_A)_+ - \frac{1}{2}\mu_+(\sigma) = 0, \quad \not{D}_A^+\sigma = 0, \tag{4.6}$$

where σ is a section of $\not{S}^+$. To apply our CohFT construction to (4.6), we define

$$\mathcal{E} = \mathcal{A}(L) \times \Gamma(\not{S}^+),$$

where $\mathcal{A}(L)$ denotes the affine space of $G/\mathbb{Z}_2$ connections on L, and define

$$\mathcal{E}_{tot} = \mathcal{E} \times \mathcal{H},$$

with $\mathcal{H} = \Omega^2_+(\Sigma, \mathrm{ad}\, L) \times \Gamma(\not{S}^-)$. The gauge symmetry group $\mathcal{G} = \mathrm{Aut}(L)$ acts on $\mathcal{E}$ and $\mathcal{E}_{tot}$, making $\mathcal{E}_{tot}$ an infinite-dimensional $\mathcal{G}$-equivariant trivial vector bundle over $\mathcal{E}$. The space $\mathcal{H}$ is equipped with a canonical $\mathcal{G}$-invariant inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ given by

$$\langle(\alpha_1 \otimes \xi_1, s_1), (\alpha_2 \otimes \xi_2, s_2)\rangle_{\mathcal{H}} = \int_\Sigma \mathrm{vol}_g \left(\langle\alpha_1, \alpha_2\rangle_{\Lambda^2_+(\Sigma)} \langle\xi_1, \xi_2\rangle_{\mathrm{ad}\, L} + \langle s_1, s_2\rangle_{\not{S}^-} \right),$$

where $\alpha_1, \alpha_2 \in \Omega^2_+(\Sigma)$, $\xi_1, \xi_2 \in \Gamma(\mathrm{ad}\, L)$, $s_1, s_2 \in \Gamma(\not{S}^-)$, $\langle\cdot,\cdot\rangle_{\Lambda^2_+(\Sigma)}$ is the bundle metric on $\Lambda^2_+(\Sigma)$ induced by the Riemannian metric g on Σ, $\langle\cdot,\cdot\rangle_{\mathrm{ad}\, L}$ is the canonical bundle metric on $\mathrm{ad}\, L$ induced by taking trace, and $\langle\cdot,\cdot\rangle_{\not{S}^-}$ is a bundle metric on $\not{S}^-$, which is part of data defining the Dirac bundle structure of $\not{S}$. Finally, the $\mathcal{G}$-equivariant map $\mathcal{F} : \mathcal{E} \to \mathcal{H}$ is given by

$$\mathcal{F}(A, \sigma) = ((F_A)_+ - \frac{1}{2}\mu_+(\sigma), \not{D}_A^+\sigma).$$

The CohFT construction is of the same form as in Standard Construction 3.2. We set

$$\mathcal{E}_{GSW} = T[1](\mathrm{Lie}(\mathcal{G})[-2] \oplus \mathcal{E}_{tot}[-1] \oplus \mathrm{Lie}(\mathcal{G})[2]),$$

where $\mathrm{Lie}(\mathcal{G})$ can be identified with $\Omega^0(\Sigma, \mathrm{ad}\, L)$. The field content of the theory consists of

$$\begin{aligned}
&\theta \in \Omega^0(\Sigma, \mathrm{ad}\, L)[-1], && \phi \in \Omega^0(\Sigma, \mathrm{ad}\, L)[-2],\\
&A \in \mathcal{A}(L), && \psi \in \Omega^1(\Sigma, \mathrm{ad}\, L)[-1],\\
&\sigma \in \Gamma(\not{S}^+), && \rho \in \Gamma(\not{S}^+)[-1],\\
&\chi \in \Omega^2_+(\Sigma, \mathrm{ad}\, L)[1], && b \in \Omega^2_+(\Sigma, \mathrm{ad}\, L),
\end{aligned}$$

$$\zeta \in \Gamma(\not{S}^-)[1], \qquad h \in \Gamma(\not{S}^-),$$
$$\lambda \in \Omega^0(\Sigma, \mathrm{ad}\, L)[2], \qquad \eta \in \Omega^0(\Sigma, \mathrm{ad}\, L)[1].$$

The scalar supersymmetry Q is given by

$$\begin{aligned}
&Q\theta = \phi - \frac{1}{2}[\theta, \theta], && Q\phi = -[\theta, \phi],\\
&QA = \psi + d_A\theta, && Q\psi = -[\theta, \psi] - d_A\phi,\\
&Q\sigma = \rho - \theta\sigma, && Q\rho = -\theta\rho + \phi\sigma,\\
&Q\chi = b - [\theta, \chi] + (F_A)_+ - \frac{1}{2}\mu_+(\sigma), && Qb = -[\theta, b] + [\phi, \chi] - d_A^+\psi + \mu_+(\sigma, \rho),\\
&Q\zeta = h - \theta\zeta + \not{D}_A^+\sigma, && Qh = -\theta h + \phi\zeta - \not{D}_A^+\rho - \psi\sigma,\\
&Q\lambda = \eta - [\theta, \lambda], && Q\eta = -[\theta, \eta] + [\phi, \lambda].
\end{aligned}$$

Let us explain the notation and objects involved. Here, d_A is the exterior covariant derivative corresponding to the connection 1-form A on L. Locally, we can write $d_A = d + A$, where d is the exterior derivative of Σ. The term $d_A^+\psi$ denotes the projection of $d_A\psi$ onto its selfdual part. Since $\not{S}$ is a left $\mathrm{Cl}(L, \Sigma)$-module, we use $\alpha\sigma$ to denote the action of an ad L-valued differential form α on σ via the identification (4.1). Finally, $\mu_+(\cdot, \cdot)$ represents the bilinear form associated with $\mu_+(\cdot)$.

The $\mathrm{Lie}(\mathcal{G})_{dR}$-action is given by

$$\iota_\xi\theta = \xi,$$

with ι_ξ acting trivially on all other fields, $[Q, \iota_\xi] = \mathrm{Lie}_\xi$, $\xi \in \mathrm{Lie}(\mathcal{G})$.

The minimal gauge fixing fermion is

$$\Psi_{min}^{GSW} = \langle(\chi, \zeta), (b, h)\rangle_{\mathcal{H}} = \int_\Sigma \mathrm{vol}_g\langle(\chi, \zeta), (b, h)\rangle,$$

where, for simplicity, we omit the subscripts of the bundle metrics involved.

The minimal CohFT action functional is

$$\begin{aligned}
S_{min}^{GSW} &= Q(\Psi_{min}^{GSW})\\
&= \int_\Sigma \mathrm{vol}_g\Big(\langle b + (F_A)_+ - \frac{1}{2}\mu_+(\sigma), b\rangle + \langle h + \not{D}_A^+\sigma, h\rangle\\
&\qquad + \langle\chi, d_A^+\psi - \mu_+(\sigma, \rho) - [\phi, \chi]\rangle + \langle\zeta, \not{D}_A^+\rho + \psi\sigma - \phi\zeta\rangle\Big). \qquad (4.7)
\end{aligned}$$

The standard gauge fixing fermion is

$$\Psi_{GSW} = \Psi_{min}^{GSW} + \int_\Sigma \mathrm{vol}_g\left(\langle(\psi, \rho), (d_A\lambda, -\lambda\sigma)\rangle + \langle\eta, [\phi, \lambda]\rangle\right).$$

The standard CohFT action functional is

$$\begin{aligned}
S_{GSW} &= Q(\Psi_{GSW})\\
&= \int_\Sigma \mathrm{vol}_g\Big(\langle b + (F_A)_+ - \frac{1}{2}\mu_+(\sigma), b\rangle + \langle h + \not{D}_A^+\sigma, h\rangle - \langle d_A\phi, d_A\lambda\rangle
\end{aligned}$$

$$
\begin{aligned}
&- \langle \phi\sigma, \lambda\sigma\rangle + |[\phi, \lambda]|^2 \\
&+ \langle \chi, d_A^+\psi - \mu_+(\sigma, \rho) - [\phi, \chi]\rangle + \langle \zeta, \not{D}_A^+\rho + \psi\sigma - \phi\zeta\rangle \\
&- \langle \psi, d_A\eta + [\psi, \lambda]\rangle + \langle \rho, \eta\sigma + \lambda\rho\rangle - \langle \eta, [\phi, \eta]\rangle\Big).
\end{aligned} \tag{4.8}
$$

Remark 4.5. Let $G = \mathrm{U}(2)$ and $S = \mathbb{C}^4 \otimes \mathbb{C}^2$ be an irreducible left-module of $\mathrm{Cl}(4) \otimes \mathrm{End}(2)$. We have

$$\mathrm{U}(2)/\mathbb{Z}_2 \cong \mathrm{U}(1) \times \mathrm{PU}(2) \cong \mathrm{S}^1 \times \mathrm{SO}(3),$$

and the connection A on L splits as

$$A = a + A_0.$$

Fixing a, the resulting GSW equations of (A_0, σ) are known as the SO(3) monopole equations [37]. Its moduli space contains both the moduli space of SO(3) anti-selfdual connections and the moduli space of U(1) monopoles. In this case, the CohFT action functional (4.8) coincides with the supersymmetric action functional of the twisted $N = 2$ SO(3) supersymmetric Yang–Mills theory with massless matter fields [38, Subsection 5.5.].

4.2.1. Donaldson–Witten theory Restricting the previous discussion to the case $G = \mathrm{SU}(2)$ and $\$ = \{0\}$ yields Donaldson–Witten theory. In this case, $\mathcal{F} = 0$ yields the anti-selfdual equations:

$$(F_A)_+ = 0.$$

The field content of the theory consists of

$$
\begin{aligned}
&\theta \in \Omega^0(\Sigma, \mathrm{ad}\, L)[-1], && \phi \in \Omega^0(\Sigma, \mathrm{ad}\, L)[-2], \\
&A \in \mathcal{A}(L), && \psi \in \Omega^1(\Sigma, \mathrm{ad}\, L)[-1], \\
&\chi \in \Omega^2_+(\Sigma, \mathrm{ad}\, L)[1], && b \in \Omega^2_+(\Sigma, \mathrm{ad}\, L), \\
&\lambda \in \Omega^0(\Sigma, \mathrm{ad}\, L)[2], && \eta \in \Omega^0(\Sigma, \mathrm{ad}\, L)[1].
\end{aligned}
$$

The scalar supersymmetry Q is given by:

$$
\begin{aligned}
&Q\theta = \phi - \frac{1}{2}[\theta, \theta], && Q\phi = -[\theta, \phi], \\
&QA = \psi + d_A\theta, && Q\psi = -[\theta, \psi] - d_A\phi, \\
&Q\chi = b - [\theta, \chi] + (F_A)_+, && Qb = -[\theta, b] + [\phi, \chi] - d_A^+\psi, \\
&Q\lambda = \eta - [\theta, \lambda], && Q\eta = -[\theta, \eta] + [\phi, \lambda].
\end{aligned}
$$

The gauge fixing fermion is

$$\Psi_{DW} = \int_\Sigma \mathrm{vol}_g(\langle \chi, b\rangle + \langle \psi, d_A\lambda\rangle + \langle \eta, [\phi, \lambda]\rangle).$$

The CohFT action functional is

$$S_{DW} = \int_\Sigma \mathrm{vol}_g\Big(\langle b + (F_A)_+, b\rangle - \langle d_A\phi, d_A\lambda\rangle + |[\phi, \lambda]|^2$$

$$+\langle\chi, d_A^+\psi - [\phi,\chi]\rangle - \langle\psi, d_A\eta - [\lambda,\psi]\rangle - \langle\eta,[\phi,\eta]\rangle\Big),$$

which coincides with the supersymmetric action functional of the twisted pure $N = 2$ supersymmetric Yang–Mills theory. See [12, Equation 2.13.] and [39].

4.2.2. Seiberg–Witten theory Restricting the previous discussion to the case $G = \mathrm{U}(1)$ and $\not{S}$ being the hermitian vector bundle associated with the irreducible complex representation $S = \mathbb{C}^4$ of $\mathrm{Cl}(4) \otimes \mathbb{C}$ yields the ordinary Seiberg–Witten theory. In this case, μ_+ is explicitly given by

$$\begin{aligned}\mu_+ : \not{S}^+ &\to i\Lambda^2_+(\Sigma)\\ \sigma &\mapsto 2\sigma\sigma^* - |\sigma|^2,\end{aligned}$$

where i denotes the imaginary unit, σ^* denotes the hermitian dual of σ, and we identify the skew-hermitian part of $\mathrm{End}(\not{S}^+)$ with $i\Lambda^2_+(\Sigma)$. $\mathcal{F} = 0$ yields the ordinary Seiberg–Witten equations

$$(F_A)_+ = \sigma\sigma^* - \frac{1}{2}|\sigma|^2, \quad \not{D}_A^+\sigma = 0.$$

The field content of the theory consists of

$$\begin{aligned}&\theta \in i\Omega^0(\Sigma)[-1], && \phi \in i\Omega^0(\Sigma)[-2],\\ &A \in \mathcal{A}(L), && \psi \in i\Omega^1(\Sigma)[-1],\\ &\sigma \in \Gamma(\not{S}^+), && \rho \in \Gamma(\not{S}^+)[-1],\\ &\chi \in i\Omega^2_+(\Sigma)[1], && b \in i\Omega^2_+(\Sigma),\\ &\zeta \in \Gamma(\not{S}^-)[1], && h \in \Gamma(\not{S}^-),\\ &\lambda \in i\Omega^0(\Sigma)[2], && \eta \in i\Omega^0(\Sigma)[1].\end{aligned}$$

The scalar supersymmetry Q is given by:

$$\begin{aligned}&Q\theta = \phi, && Q\phi = 0,\\ &QA = \psi + d\theta, && Q\psi = -d\phi,\\ &Q\sigma = \rho - \frac{1}{2}\theta\sigma, && Q\rho = -\frac{1}{2}\theta\rho + \frac{1}{2}\phi\sigma,\\ &Q\chi = b + (F_A)_+ - \frac{1}{2}\mu_+(\sigma), && Qb = -d^+\psi + \mu_+(\sigma,\rho),\\ &Q\zeta = h - \frac{1}{2}\theta\zeta + \not{D}_A^+\sigma, && Qh = -\frac{1}{2}\theta h + \frac{1}{2}\phi\zeta - \not{D}_A^+\rho - \psi\sigma,\\ &Q\lambda = \eta, && Q\eta = 0.\end{aligned}$$

Here, $\theta\sigma$ simply denotes the scalar multiplication of σ by θ. See Remark 4.3 for an explanation of the appearance of the factor $1/2$.

The gauge fixing fermion is

$$\Psi_{SW} = \int_\Sigma \mathrm{vol}_g\left(\langle(\chi,\zeta),(b,h)\rangle + \langle(\psi,\rho),(d\lambda,-\lambda\sigma)\rangle\right).$$

The CohFT action functional is

$$S_{SW} = \int_\Sigma \mathrm{vol}_g \Big(\langle b + (F_A)_+ - \frac{1}{2}\mu_+(\sigma), b\rangle + \langle h + \not{D}_A^+\sigma, h\rangle - \langle d\phi, d\lambda\rangle - \langle \phi\sigma, \lambda\sigma\rangle$$
$$+ \langle \chi, d^+\psi - \mu_+(\sigma,\rho)\rangle + \langle \zeta, \not{D}_A^+\rho + \psi\sigma - \phi\zeta\rangle - \langle \psi, d\eta\rangle + \langle \rho, \eta\sigma + \lambda\rho\rangle \Big),$$

which coincides with the supersymmetric action functional of the twisted $N = 2$ U(1) supersymmetric Yang–Mills theory with massless matter fields [38, Subsection 5.5.].

4.2.3. Kapustin–Witten theory Let $G = \mathrm{SU}(2)$ and $\not{S} = \Lambda T^\vee\Sigma \otimes \mathrm{ad}\, L$. The Dirac operator on $\not{S}$ is given by

$$\not{D}_A = d_A + d_A^*,$$

where d_A^* is the formal adjoint operator of d_A. For our purpose, we restrict σ to be an ad L-valued homogeneous 1-form on Σ. One can check that

$$\mu(\sigma) = [\sigma, \sigma].$$

Consequently, we consider the equivariant map $\mathcal{F}$ defined by

$$\mathcal{F}(A,\sigma) = ((F_A - \frac{1}{2}[\sigma,\sigma])_+, d_A^-\sigma, d_A^*\sigma) \in \Omega^2_+(\Sigma, \mathrm{ad}\, L) \oplus \Omega^2_-(\Sigma, \mathrm{ad}\, L) \oplus \Omega^0(\Sigma, \mathrm{ad}\, L).$$

Restricting the previous discussion to this case yields the Kapustin–Witten theory. The filed content of the theory consists of

$$\theta \in \Omega^0(\Sigma, \mathrm{ad}\, L)[-1], \qquad \phi \in \Omega^0(\Sigma, \mathrm{ad}\, L)[-2],$$
$$A \in \mathcal{A}(L), \qquad \psi \in \Omega^1(\Sigma, \mathrm{ad}\, L)[-1],$$
$$\sigma \in \Omega^1(\Sigma, \mathrm{ad}\, L) \qquad \widetilde{\psi} \in \Omega^1(\Sigma, \mathrm{ad}\, L)[-1],$$
$$\chi \in \Omega^2_+(\Sigma, \mathrm{ad}\, L)[1], \qquad b \in \Omega^2_+(\Sigma, \mathrm{ad}\, L),$$
$$\widetilde{\chi} \in \Omega^2_-(\Sigma, \mathrm{ad}\, L)[1], \qquad \widetilde{b} \in \Omega^2_-(\Sigma, \mathrm{ad}\, L),$$
$$\widetilde{\eta} \in \Omega^0(\Sigma, \mathrm{ad}\, L)[1], \qquad \widetilde{\lambda} \in \Omega^0(\Sigma, \mathrm{ad}\, L),$$
$$\lambda \in \Omega^0(\Sigma, \mathrm{ad}\, L)[2], \qquad \eta \in \Omega^0(\Sigma, \mathrm{ad}\, L)[1].$$

(Note that the fields λ and $\widetilde{\lambda}$ have different ghost degrees.)

The scalar supersymmetry Q of the theory takes the following form.

$$Q\theta = \phi - \frac{1}{2}[\theta,\theta], \qquad Q\phi = -[\theta,\phi],$$
$$QA = \psi + d_A\theta, \qquad Q\psi = -[\theta,\psi] - d_A\phi,$$
$$Q\sigma = \widetilde{\psi} - [\theta,\sigma], \qquad Q\widetilde{\psi} = -[\theta,\widetilde{\psi}] + [\phi,\sigma],$$
$$Q\chi = b - [\theta,\chi] + (F_A - \frac{1}{2}[\sigma,\sigma])_+, \qquad Qb = -[\theta,b] + [\phi,\chi] - d_A^+\psi + [\widetilde{\psi},\sigma]_+,$$

$$
\begin{aligned}
Q\widetilde{\chi} &= \widetilde{b} - [\theta, \widetilde{\chi}] + d_A^- \sigma, & Q\widetilde{b} &= -[\theta, \widetilde{b}] + [\phi, \widetilde{\chi}] - d_A^- \widetilde{\psi} - [\psi, \sigma]_-, \\
Q\widetilde{\eta} &= \widetilde{\lambda} - [\theta, \widetilde{\eta}] + d_A^* \sigma, & Q\widetilde{\lambda} &= -[\theta, \widetilde{\lambda}] + [\phi, \widetilde{\eta}] - d_A^* \widetilde{\psi} + \langle [\psi, \sigma] \rangle, \\
Q\lambda &= \eta - [\theta, \lambda], & Q\eta &= -[\theta, \eta] + [\phi, \lambda],
\end{aligned}
$$

where

$$
\langle [\psi, \sigma] \rangle := \sum_{\mu=1}^{4} [\psi_\mu, \sigma_\mu],
$$

with $\psi = \psi_\mu e^\mu$, $\sigma = \sigma_\mu e^\mu$, and $\{e^\mu\}_{\mu=1}^4$ being a local orthonormal coframe.

The gauge fixing fermion is

$$
\Psi_{KW} = \int_\Sigma \mathrm{vol}_g (\langle (\chi, \widetilde{\chi}, \widetilde{\eta}), (b, \widetilde{b}, \widetilde{\lambda}) \rangle + \langle (\psi, \widetilde{\psi}), (d_A \lambda, -[\lambda, \sigma]) \rangle + \langle \eta, [\phi, \lambda] \rangle).
$$

The CohFT action functional is

$$
\begin{aligned}
S_{KW} = \int_\Sigma \mathrm{vol}_g \Big(& \langle b + (F_A - \frac{1}{2}[\sigma, \sigma])_+, b \rangle + \langle \widetilde{b} + d_A^- \sigma, \widetilde{b} \rangle + \langle \widetilde{\lambda} + d_A^* \sigma, \widetilde{\lambda} \rangle \\
& - \langle d_A \phi, d_A \lambda \rangle - \langle [\phi, \sigma], [\lambda, \sigma] \rangle + |[\phi, \lambda]|^2 \\
& + \langle \chi, d_A^+ \psi - [\widetilde{\psi}, \sigma]_+ - [\phi, \chi] \rangle + \langle \widetilde{\chi}, d_A^- \widetilde{\psi} + [\psi, \sigma]_- - [\phi, \widetilde{\chi}] \rangle \\
& - \langle \psi, d_A \eta - [\sigma, \widetilde{\eta}] - [\lambda, \psi] \rangle - \langle \widetilde{\psi}, d_A \widetilde{\eta} + [\sigma, \eta] - [\lambda, \widetilde{\psi}] \rangle \\
& - \langle \widetilde{\eta}, [\phi, \widetilde{\eta}] \rangle - \langle \eta, [\phi, \eta] \rangle \Big).
\end{aligned}
$$

which coincides with the supersymmetric action functional of the twisted pure $N = 4$ supersymmetric Yang–Mills theory [6].

Noting that on $\mathbb{R}^4$, S_{KW} is invariant under the following operation:

$$
\chi \to -\widetilde{\chi}, \ \widetilde{\chi} \to \chi, \quad \psi \to -\widetilde{\psi}, \ \widetilde{\psi} \mapsto \psi, \quad \eta \to -\widetilde{\eta}, \ \widetilde{\eta} \mapsto \eta.
$$

This suggests that the CohFT is the complexification of another CohFT. Let us define the complexified fields as follows:

$$
A_c = A + i\sigma, \quad \psi_c = \psi + i\widetilde{\psi}, \quad \chi_c = \chi + i\widetilde{\chi}, \quad \eta_c = \eta + i\widetilde{\eta}, \quad b_c = b + i\widetilde{b},
$$

and define

$$
\lambda_c = i\widetilde{\lambda} + [\phi, \lambda].
$$

(Note that λ_c is of degree 0 and should not be confused with the complexification of the field λ.) A_c can be interpreted as a $\mathrm{SL}(2, \mathbb{C})$ connection on $L \otimes \mathbb{C}$. Its curvature F_{A_c} can be decomposed as

$$
F_{A_c} = F_A - \frac{1}{2}[\sigma, \sigma] + i d_A \sigma.
$$

It is easy to check that χ_c and b_c are both complex selfdual 2-forms.[5] Moreover, the $\mathcal{G}$-equivariant section $\mathcal{F}$ can be reformulated as

[5] A complex 2-form T over a 4-dimensional manifold is said to be selfdual if $T_- = \frac{1}{2}(T - \star\overline{T}) = 0$, where $\overline{T}$ is the complex conjugate of T.

$$\mathcal{F}(A_c) = ((F_{A_c})_+, id^*_{A_c}\mathrm{Im}(A_c)) \in \Omega^2_+(\Sigma, \mathrm{ad}\, L \otimes \mathbb{C}) \oplus \Omega^0(\Sigma, \mathrm{ad}\, L \otimes \mathbb{C}),$$

where $d^*_{A_c}\mathrm{Im}(A_c) = d^*_A\sigma + i\langle[\sigma, \sigma]\rangle = d^*_A\sigma$.

The expression of Q can be simplified as follows.

$$\begin{aligned}
&Q\theta = \phi - \frac{1}{2}[\theta, \theta], && Q\phi = -[\theta, \phi],\\
&QA_c = \psi_c + d_{A_c}\theta, && Q\psi_c = -[\theta, \psi_c] - d_{A_c}\phi,\\
&Q\chi_c = b_c - [\theta, \chi_c] + (F_{A_c})_+, && Qb_c = -[\theta, b_c] + [\phi, \chi_c] - d^+_{A_c}\psi_c,\\
&Q\eta_c = \lambda_c - [\theta, \eta_c] + id^*_{A_c}\mathrm{Im}(A_c), && Q\lambda_c = -[\theta, \lambda_c] + [\phi, \eta_c] - i\mathrm{Im}(d^*_{A_c}\psi_c),
\end{aligned}$$

and $Q\lambda = \mathrm{Re}(\eta_c) - [\theta, \lambda]$.

The expressions of Ψ_{KW} and S_{KW} can be simplified as follows.

$$\begin{aligned}
\Psi_{KW} &= \int_\Sigma \mathrm{vol}_g(\langle\chi_c, b_c\rangle + \langle\psi_c, d_{A_c}\eta_c\rangle + \langle\eta_c, \lambda_c\rangle),\\
S_{KW} &= \int_\Sigma \mathrm{vol}_g\Big(\langle b_c + (F_{A_c})_+, b_c\rangle + \langle\widetilde{\lambda} + d^*_A\sigma, \widetilde{\lambda}\rangle - \langle d_{A_c}\phi, d_{A_c}\lambda\rangle + |[\phi, \lambda]|^2\\
&\qquad + \langle\chi_c, d^+_{A_c}\psi_c - [\phi, \chi_c]\rangle - \langle\psi_c, d_{A_c}\eta_c - [\lambda, \psi_c]\rangle - \langle\eta_c, [\phi, \eta_c]\rangle\Big).
\end{aligned}$$

This suggests that one may interpret the Kapustin–Witten theory as the complexification of the Donaldson–Witten theory with an imaginary gauge fixing condition [40].

Remark 4.6. The CohFT configuration space has a U(1)-action, given by

$$\begin{aligned}
&\theta \to \theta,\ \phi \to \phi,\ A_c \to A_c,\ \psi_c \to e^{i\varphi}\psi_c,\ \chi_c \to e^{i\varphi}\chi_c,\ b_c \to e^{i\varphi}b_c,\\
&\eta_c \to e^{i\varphi}\eta_c,\ \lambda_c \to \lambda_c,\ \lambda \to \lambda.
\end{aligned}$$

One can check that the action functional S_{KW} is invariant under this action, along with the replacement

$$\mathcal{F} \to e^{i\varphi}\mathcal{F} := ((e^{i\varphi}F_{A_c})_+, id^*_{A_c}\mathrm{Im}(A_c)).$$

Note that $e^{i\varphi}\mathcal{F} = 0$ yields exactly the family of Kapustin–Witten equations [41]:

$$\begin{aligned}
(\cos(\varphi)(F_A - [\sigma, \sigma]/2) - \sin(\varphi)d_A\sigma)_+ = 0,\\
(\sin(\varphi)(F_A - [\sigma, \sigma]/2) + \cos(\varphi)d_A\sigma)_- = 0,
\end{aligned}$$

together with the imaginary gauge fixing condition: $d^*_A\sigma = 0$.

5. Symmetries and Observables

In this section, we discuss the spacetime symmetries and observables of CohFTs in the context of our Minimal Construction 3.1. The term 'spacetime symmetries' is used to distinguish these symmetries from the gauge symmetries described by the group $\mathcal{G}$. We show that minimal CohFTs are de Rham translation invariant. Additionally, we argue that the classical observables of a (minimal) CohFT form a prefactorization algebra over the spacetime manifold. This can be made precise perturbatively using the language of elliptic L_∞ algebras.

5.1. Spacetime symmetries. Let us assume that $\mathrm{Lie}(\mathcal{G})$, $\mathcal{E}$, and the fiber of $\mathcal{E}_{tot}$ can be identified with the spaces of sections of some (gauge) natural bundles over Σ. (We refer the reader to [42] for the notion of (gauge) natural bundles.) Since the diffeomorphism group $\mathrm{Diff}(\Sigma)$ has a natural action on the space of sections of a natural bundle, we have a $\mathfrak{X}(\Sigma)$-action on the CohFT configuration space $\mathcal{E}_{min} = T[1](\mathcal{E}_{tot}[-1] \oplus \mathrm{Lie}(\mathcal{G})[1])$:

$$\begin{aligned} \mathrm{Lie} : \mathfrak{X}(\Sigma) &\to \mathfrak{X}(\mathcal{E}_{min}) \\ X &\mapsto \mathrm{Lie}_X \end{aligned}$$

where $\mathfrak{X}(\mathcal{E}_{min})$ denotes the graded Lie superalgebra of derivations of $C^\infty(\mathcal{E}_{min})$, Lie_X denotes the Lie derivative of $X \in \mathfrak{X}(\Sigma)$ on $\mathcal{E}_{min}$ determined by the natural bundle structures.

Definition 5.1. An element $X \in \mathfrak{X}(\Sigma)$ is said to be an (infinitesimal) spacetime symmetry of the CohFT $(\mathcal{E}_{min}, Q, S_{min})$ if

1. $\mathrm{Lie}_X(S_{min}) = 0$;
2. Lie_X is homotopically trivial, i.e., there exists a degree -1 derivation $\iota_X \in \mathfrak{X}(\mathcal{E}_{min})$ such that $\mathrm{Lie}_X = [Q, \iota_X]$.

If X_1 and X_2 are both spacetime symmetries of the theory, then there exists ι_{X_1} and ι_{X_2} in $\mathfrak{X}(\mathcal{E}_{min})_{-1}$, such that

$$\mathrm{Lie}_{[X_1,X_2]} = [\mathrm{Lie}_{X_1}, \mathrm{Lie}_{X_2}] = [[Q, \iota_{X_1}], \mathrm{Lie}_{X_2}] = [Q, [\iota_{X_1}, \mathrm{Lie}_{X_2}]].$$

Moreover, we have

$$\mathrm{Lie}_{[X_1,X_2]}(S_{min}) = [\mathrm{Lie}_{X_1}, \mathrm{Lie}_{X_2}](S_{min}) = 0.$$

Thus, $[X_1, X_2]$ is also a spacetime symmetry of the theory. We conclude that

Proposition 5.1. *The spacetime symmetries of* $(\mathcal{E}_{min}, Q, S_{min})$ *form a subalgebra* $\mathfrak{s}$ *of* $\mathfrak{X}(\Sigma)$.

Definition 5.2. We say that the CohFT $(\mathcal{E}_{min}, Q, S_{min})$ is de Rham $\mathfrak{s}$-invariant if the $\mathfrak{s}$-action on the CohFT configuration space $\mathcal{E}_{min}$ can be extended to a $\mathfrak{s}_{dR}$-action. That is,

$$[\iota_{X_1}, \iota_{X_2}] = 0, \quad [\iota_{X_1}, \mathrm{Lie}_{X_2}] = \iota_{[X_1,X_2]},$$

for all $X_1, X_2 \in \mathfrak{s}$, and

$$\iota_X(S_{min}) = 0,$$

for all $X \in \mathfrak{s}$.

Example 5.1 (*Vector supersymmetries*). Let us assume that $\Sigma = \mathbb{R}^d$, $\mathfrak{s}$ is the Lie algebra of infinitesimal translations of Σ, and the CohFT is de Rham $\mathfrak{s}$-invariant. This is often the case if the CohFT comes from a topological twisting of a supersymmetric field theory on $\mathbb{R}^d$. Fixing a basis $\{\partial_\mu\}_{\mu=1}^d$ of $\mathfrak{s}$, we denote ι_{∂_μ} by K_μ. (K_μ are called as the vector supersymmetries of the theory by physicists [43–45].) By definition, we have

$$QK_\mu + K_\mu Q = \partial_\mu, \quad K_\mu K_\nu + K_\nu K_\mu = 0, \quad K_\mu \partial_\nu - \partial_\nu K_\mu = 0. \tag{5.1}$$

One can also organize K_μ into a single 1-form symmetry $K := K_\mu dx^\mu$. It follows that $QK + KQ = d$, where d is the de Rham differential of Σ. Under some technical assumptions, one can show that K forms a homotopy operator of the $\mathcal{G}_{dR}$-algebra $\Omega(M) \otimes C^\infty(\mathcal{E}_{min})$ [36]. For a modern mathematical treatment of the vector supersymmetries, we refer the reader to [46,47].

Example 5.2. Let us assume that $\Sigma = \mathbb{CP}^1$, $\mathfrak{s}$ is the Witt algebra, and the CohFT is de Rham $\mathfrak{s}$-invariant. This is often the case if the CohFT comes from a topological twisting of a superconformal field theory. Fixing a basis $\{L_n\}_{n\in\mathbb{Z}}$ of $\mathfrak{s}$, we denote ι_{L_n} by G_n. By definition, we have

$$QG_n + G_nQ = L_n, \quad G_nG_m + G_mG_n = 0, \quad G_nL_m - L_mG_n = (m-n)G_{m+n}.$$

The generator G_{-1} together with its complex conjugate can be interpreted as the vector supersymmetries of the theory. For an application of the extended Witt algebra $\mathfrak{s}_{dR}$, we refer the reader to [48].

Indeed, due to the shifted tangent bundle structure of our CohFT configuration space $\mathcal{E}_{min}$ and the specific form chosen for the minimal gauge fixing fermion Ψ_{min}, one can always construct a $(\mathbb{R}^d)_{dR}$-action on $\mathcal{E}_{min}$ to make the theory de Rham translation invariant. Let us illustrate this idea through the minimal construction of generalized Seiberg–Witten theory. We define

$$\begin{aligned} K_\mu\theta &= 0, & K_\mu\phi &= \partial_\mu\theta,\\ K_\mu A &= 0, & K_\mu\psi &= \partial_\mu A,\\ K_\mu\sigma &= 0, & K_\mu\rho &= \partial_\mu\sigma,\\ K_\mu\chi &= 0, & K_\mu b &= \partial_\mu\chi,\\ K_\mu\zeta &= 0, & K_\mu h &= \partial_\mu\zeta. \end{aligned}$$

It is straightforward to verify that Equations 5.1 hold, and that

$$K_\mu(\Psi_{min}^{GSW}) = K_\mu \int_{\mathbb{R}^4} d^4x(\langle\chi, b\rangle + \langle b, h\rangle) = -\int_{\mathbb{R}^4} d^4x(\langle\chi, \partial_\mu\chi\rangle + \langle\zeta, \partial_\mu\zeta\rangle = 0,$$

where we use the assumption that all fields are compactly supported on $\mathbb{R}^4$. It follows that

$$K_\mu(S_{min}^{GSW}) = [Q, K_\mu](\Psi_{min}^{GSW}) = 0.$$

However, there is a minor issue with this $(\mathbb{R}^d)_{dR}$-action: it is not compatible with the $\mathcal{G}_{dR}$-action [36]. That is,

$$[K_\mu, \iota_\xi] \neq 0. \tag{5.2}$$

In fact, one can verify that the condition (5.2) fails only for the ϕ field:

$$[K_\mu, \iota_\xi]\phi = \partial_\mu\xi \neq 0.$$

To resolve this issue, note that K_μ is not necessarily unique. One can always perturb it to $K_\mu + \delta K_\mu$, provided that δK_μ satisfies

$$[Q, \delta K_\mu] = 0, \quad [\delta K_\mu, K_\nu] + [K_\mu, \delta K_\nu] + [\delta K_\mu, \delta K_\nu] = 0, \quad [\delta K_\mu, \partial_\nu] = 0.$$

In our case, we can perturb the old vector supersymmetries to obtain the following:

$$\begin{aligned} K\theta &= A, & K\phi &= -\psi, \\ KA &= 0, & K\psi &= 2F_A, \\ K\sigma &= 0, & K\rho &= d_A\sigma, \\ K\chi &= 0, & Kb &= d_A\chi, \\ K\zeta &= 0, & Kh &= d_A\zeta, \end{aligned}$$

which is compatible with the $\mathcal{G}_{dR}$-action. There also exists other more complicated perturbations. The following one is closely related to the physics literature, as it arises from the twisting of the original supersymmetries:

$$\begin{aligned} K\theta &= A, & K\phi &= -\psi, \\ KA &= 2\chi, & K\psi &= 2(F_A)_- - 2b + \mu_+(\sigma), \\ K\sigma &= -e^\mu \wedge (e_\mu\zeta), & K\rho &= e^\mu \wedge (e_\mu h), \\ K\chi &= 0, & Kb &= 3d_A\chi - e^\mu \wedge \mu_+(e_\mu\zeta, \sigma), \\ K\zeta &= 0, & Kh &= -e^\mu \wedge \chi_{\mu\nu}(e^\nu\sigma), \end{aligned}$$

where $\{e_\mu\}_{\mu=1}^4$ denotes an orthonormal basis of $\mathbb{R}^4$. One can check that the desired formula $[Q, K] = d$ holds. Indeed, we have

$$\begin{aligned} [Q, K]\theta &= Q(A) + K(\phi - \frac{1}{2}[\theta, \theta]) \\ &= d_A\theta - [A, \theta] = d\theta, \\ [Q, K]A &= Q(2\chi) + K(\psi + d_A\theta) \\ &= 2(b - [\theta, \chi] + (F_A)_+ - \frac{1}{2}\mu_+(\sigma)) + 2(F_A)_- - 2b + \mu_+(\sigma) - dA - [A, A] \\ &= 2F_A - dA - [A, A] = dA, \\ [Q, K]\sigma &= Q(-e^\mu \wedge (e_\mu\zeta)) + K(\rho - \theta\sigma) \\ &= -e^\mu \wedge (e_\mu(h - \theta\zeta + \not{D}_A\sigma)) + e^\mu \wedge (e_\mu h) - A\sigma - e^\mu \wedge (e_\mu\theta\zeta) \\ &= -e^\mu \wedge (e_\mu \not{D}_A\sigma) - A\sigma \\ &= d_A\sigma - A\sigma = d\sigma, \\ [Q, K]\chi &= K(b - [\theta, \chi] + (F_A)_+ - \frac{1}{2}\mu_+(\sigma)) \\ &= 3d_A\chi - e^\mu \wedge \mu_+(e_\mu\zeta, \sigma) - [A, \chi] - 2d_A\chi - e^\mu \wedge \mu_+(-e_\mu\zeta, \sigma) \\ &= d_A\chi - [A, \chi] = d\chi, \\ [Q, K]\zeta &= K(h - \theta\zeta + \not{D}_A\sigma) \\ &= -e^\mu \wedge \chi_{\mu\nu}(e^\nu\sigma) - A\zeta + e^\mu \wedge \chi_{\mu\nu}(e^\nu\sigma) - e^\mu \wedge \not{D}_A(e_\mu\zeta) \\ &= d_A\zeta - A\zeta = d\zeta. \end{aligned}$$

The computation for the superpartners of the fields above follows similarly. Moreover, one can check that

$$K_\mu(\Psi_{min}^{GSW}) = K_\mu\left(\int_{\mathbb{R}^4} dx^4\,(\langle b, \chi\rangle + \langle h, \zeta\rangle)\right)$$

$$= \int_{\mathbb{R}^4} dx^4 \langle D_\mu \chi, -\langle \mu_+(e_\mu \zeta, \sigma), \chi \rangle - \langle \chi_{\mu\nu}(e^\nu \sigma), \zeta \rangle$$
$$= \int_{\mathbb{R}^4} dx^4 \langle D_\mu \chi, \chi \rangle = 0,$$

where $D_\mu := \iota_{e_\mu} d_A$, and we use the definition (4.4) of μ to get to the third line. It follows immediately that $K_\mu(S_{min}^{GSW}) = 0$, which is not surprising, because the original untwisted action functional is supersymmetric, and the twisting has no effect on $\mathbb{R}^4$.

Remark 5.1. Another wonderful consequence of these K_μ is that they can be used to give an elegant reformulation (see (5.3) below) of the minimal CohFT action functional S_{min}^{DW} of Donaldson–Witten theory on $\mathbb{R}^4$. To see this, let

$$\theta_K := \exp(K)\theta = \sum_{p=0}^{4} \frac{1}{p!} K^p \theta, \quad \phi_K := \exp(K)\phi = \sum_{p=0}^{4} \frac{1}{p!} K^p \phi.$$

For Donaldson–Witten theory, a direct computation shows that

$$\theta_K = \theta + A + \chi, \quad \phi_K = \phi - \psi + (b - (F_A)_-) + d_A \chi + \frac{1}{2}[\chi, \chi].$$

We define [36]

$$\int_{\mathbb{R}^4} \frac{1}{2} (\exp(K)\mathrm{Tr}(\phi^2))^{top} = \int_{\mathbb{R}^4} \mathrm{Tr}(\phi_K^2)^{top}$$
$$= \int_{\mathbb{R}^4} \mathrm{Tr} \left(\frac{1}{2} \phi [\chi, \chi] - \psi \wedge d_A \chi + \frac{1}{2} b \wedge b + \frac{1}{2} (F_A)_- \wedge (F_A)_- \right).$$

where Tr is the negative Killing form of the Lie algebra $\mathfrak{su}(2)$. Recall that the minimal CohFT action functional of the Donaldson–Witten theory is

$$S_{min}^{DW} = \int_{\mathbb{R}^4} d^4x \left(|b|^2 + \langle (F_A)_+, b \rangle + \langle \chi, d_A^+ \psi \rangle - \langle \chi, [\phi, \chi] \rangle \right).$$

We have

$$S_{min}^{DW} - \int_{\mathbb{R}^4} \mathrm{Tr}(\phi_K^2)^{top} = \int_{\mathbb{R}^4} d^4x \left(\frac{1}{2} |(F_A)_-|^2 + \frac{1}{2} |b|^2 + \langle b, (F_A)_+ \rangle \right).$$

"Integrating out" the auxiliary field b, or equivalently, setting $b = -(F_A)_+$, we get

$$S_{min}^{DW} \Big|_{b=-(F_A)_+} = \frac{1}{2} \int_{\mathbb{R}^4} \mathrm{Tr}(F_A \wedge F_A) + \int_{\mathbb{R}^4} \mathrm{Tr}(\phi_K^2)^{top} \Big|_{b=-(F_A)_+}. \tag{5.3}$$

(5.3) can be explained by the holomorphicity of the supersymmetric gauge theories involving only the vector multiplet. For more details, we refer the reader to [38, Chapter 4.].

5.2. Observables. Let us move forward to discuss the algebra of classical observables within our minimal construction of CohFTs. Since both $\mathrm{Lie}(\mathcal{G})$ and $\mathcal{E}_{tot}$ are defined as spaces of smooth sections of bundles over Σ, it is natural to consider their restrictions, $\mathrm{Lie}(\mathcal{G}|_U)$ and $\mathcal{E}_{tot}|_U$, to an open subset $U \subset \Sigma$, and accordingly, the restriction of $\mathcal{E}_{min}$ to $\mathcal{E}_{min}|_U$. In other words, we obtain a contravariant functor from the category $\mathbf{Open}_\Sigma$ of open subsets in Σ, to the category **GraFréchetMan** of graded Fréchet manifolds. On the other hand, taking the complex of functions on $\mathcal{E}_{min}$ yields another contravariant functor from **GraFréchetMan** to the category **DGCA** of differential graded commutative algebras. Composing these two functors gives us a covariant functor

$$\mathbf{Obs}^{cl}_{\mathcal{F}} : \mathbf{Open}_\Sigma \to \mathbf{DGCA},$$

which assigns to each open subset U of Σ the differential graded commutative algebra

$$(C^\infty(\mathcal{E}_{min}|_U), Q|_{\mathcal{E}_{min}|_U}).$$

Under some reasonable locality assumptions on $\mathcal{F}$, one can expect that $\mathbf{Obs}^{cl}_{\mathcal{F}}$ forms a prefactorization algebra on Σ. We call it the algebra of classical observables of the minimal CohFT.

A standard method for constructing CohFT observables is to integrate local forms over submanifolds of the spacetime manifold. Consider the prefactorization algebra $\mathbf{PreObs}^{cl}_{\mathcal{F}}$ defined by

$$U \mapsto (\Omega_{cp}(U) \otimes C^\infty(\mathcal{E}_{min}|_U), d + (-1)^\bullet Q|_{\mathcal{E}_{min}|_U})$$

where $\Omega_{cp}(U)$ is the de Rham complex of compactly supported differential forms on $U \subset \Sigma$, d is the de Rham differential of $\Omega_{cp}(U)$. Let $\mathcal{O}$ be a closed element in $\mathbf{PreObs}^{cl}_{\mathcal{F}}(U)$ of total degree $\dim \Sigma$. One can decompose $\mathcal{O}$ as $\mathcal{O} = \sum_{p=0}^{\dim \Sigma} \mathcal{O}^{(p)}$, where $\mathcal{O}^{(p)}$ is of bidegree $(p, \dim \Sigma - p)$ and satisfies

$$Q\mathcal{O}^{(p)} = d\mathcal{O}^{(p-1)}, \quad p = 1, \ldots, \dim \Sigma, \tag{5.4}$$

and $Q\mathcal{O}^{(0)} = 0$. The equations (5.4) are called as the topological descent equations [12]. Picking a k-dimensional closed submanifold γ_k inside $U \subset M$, one can define a map

$$\begin{aligned} \mathrm{ev}_{\gamma_k}|_U : H^{\dim \Sigma}(\mathbf{PreObs}^{cl}_{\mathcal{F}}(U)) &\to H^{\dim \Sigma - k}(\mathbf{Obs}^{cl}_{\mathcal{F}}(U)) \\ [\mathcal{O}] &\mapsto [\int_{\gamma_k} \mathcal{O}^{(k)}], \end{aligned}$$

where we use $H^\bullet(A)$ to denote the cohomology groups of a dgca A, and use $[a]$ to denote the cohomology class of a closed element a in A. It is not hard to see that this map only depends on the homology class of γ_k.

Example 5.3. In Donaldson–Witten theory, one starts with

$$\mathcal{O}^{(0)} = \mathrm{Tr}(\phi^2).$$

$\mathcal{O}^{(0)}$ is Q-closed and is of the correct bidegree $(4, 0)$. To solve the descent equations (5.4), we define

$$\mathbb{A}(\theta) = \theta + A, \quad \mathbb{F}(\phi) = \phi - \psi + F_A.$$

One can check that [39]

$$\mathbb{F}(\phi) = (d + (-1)^\bullet Q)\mathbb{A}(\theta) + \frac{1}{2}[\mathbb{A}(\theta), \mathbb{A}(\theta)], \quad (d + (-1)^\bullet Q)\mathbb{F}(\phi) + [\mathbb{A}(\theta), \mathbb{F}(\phi)] = 0.$$

Let $\mathcal{O} = \mathrm{Tr}(\mathbb{F}(\phi)^2)$, which satisfies

$$(d + (-1)^\bullet Q)\mathcal{O} = 0$$

by virtue of the second Bianchi identity: $(d+(-1)^\bullet Q)\mathbb{F}(\phi)+[\mathbb{A}(\theta), \mathbb{F}(\phi)] = 0$. Expanding $\mathcal{O}$ and integrating its components over the submanifolds of M gives the standard CohFTs observables in the physics literature.

In order to make the discussion related to the algebra of functions on an infinite-dimensional graded manifold mathematically precise, we shift from a nonperturbative to a perturbative perspective, where the framework developed by Costello and Gwilliam [49–51] provides a rigorous foundation for classical field theories and their perturbative quantization. The construction in this subsection can be seen as an infinite-dimensional generalization of the one in Subsection 3.4.

Let us first recall the definition of an elliptic L_∞ algebra.

Definition 5.3. Let Σ be a manifold. A local L_∞ algebra on Σ consists of the following data:

1. A graded vector bundle L on Σ, whose space of smooth sections will be denoted $\mathcal{L}$.
2. A differential $D : \mathcal{L} \to \mathcal{L}$ of cohomological degree 1, satisfying $D^2 = 0$.
3. A collection of poly-differential operators:

$$l^n : \mathcal{L}^{\otimes n} \to \mathcal{L}$$

for $n \geq 2$, which are alternating, are of cohomological degree $2 - n$, and endow $\mathcal{L}$ with the structure of $\mathcal{L}_\infty$ algebra.

A local L_∞ algebra is called an elliptic L_∞ algebra if $(\mathcal{L}, D)$ is an elliptic complex.

The definition of an elliptic L_∞ algebra on Σ is formulated to study the sheaf of derived moduli problems on Σ associated with the field equations of certain classical Lagrangian field theory $\mathcal{T}$ on Σ. For more details, see [51, Subection 3.1]. One can define the observables with support in the open subset U as the dgca:

$$\mathbf{Obs}^{cl}_{\mathcal{T}}(U) = \mathrm{CE}^\bullet(\mathcal{L}(U)),$$

where $\mathrm{CE}^\bullet(\mathcal{L}(U))$ denote the Chevalley-Eilenberg cochains of $\mathcal{L}(U)$. This assignment defines a factorization algebra on Σ [51, Subsection 5.1]. $\mathbf{Obs}^{cl}_{\mathcal{T}}$ is called the factorization algebra of classical observables of $\mathcal{T}$.

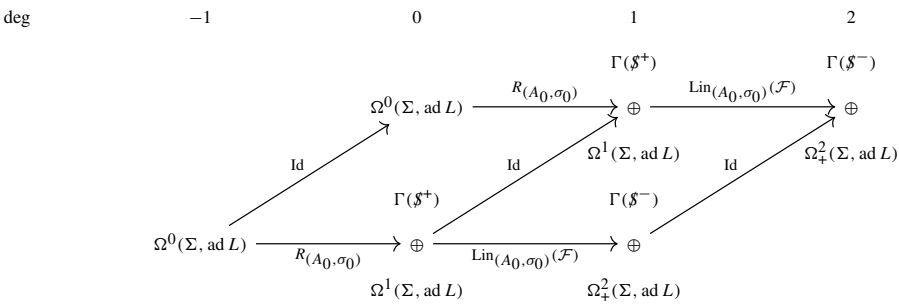
Remark 5.2. In the case of CohFTs, we have argued that the Lagrangian action functional should not be treated as the fundamental object. Instead, we begin with a first-order nonlinear PDE $\mathcal{F} = 0$ and provide various constructions of a Lagrangian action functional whose first-order field equation coincides with the given equation.

Example 5.4 (*Perturbative generalized Seiberg–Witten theory*). Fix an solution (A_0, σ_0) to the GSW equation. Let us construct an elliptic dgLa $\mathcal{L}_{GSW}$. The local graded Lie algebra structure on $\mathcal{L}_{GSW}$ is given by[6]

$$\mathcal{L}_{GSW} := \Omega^0(\Sigma, \operatorname{ad} L)_{dR} \ltimes (\Omega^1(\Sigma, \operatorname{ad} L)[-1] \oplus \Gamma(\not{S}^+)[-1] \oplus \Omega^2_+(\Sigma, \operatorname{ad} L)[-2] \oplus \Gamma(\not{S}^-)[-2])_{dR},$$

where $\ltimes$ denotes the semidirect product.

The underlying elliptic complex is



where

$$R_{(A_0,\sigma_0)}(\xi) = (d_{A_0}\xi, -\xi\sigma_0), \quad \xi \in \Omega^0(\Sigma, \operatorname{ad} L),$$

and

$$\mathrm{Lin}_{(A_0,\sigma_0)}(\mathcal{F})(\dot{A}, \dot{\sigma}) = (d^+_{A_0}\dot{A} - \mu_+(\sigma_0, \dot{\sigma}), \not{D}_{A_0}\dot{\sigma} + \dot{A}\sigma_0), \quad (\dot{A}, \dot{\sigma}) \in \Omega^1(\Sigma, \operatorname{ad} L) \oplus \Gamma(\not{S}^+).$$

Since the zero-order piece of the differential does not matter, the ellipticity of this complex follows from the ellipticity of both of its rows. We define the factorization algebra of classical observables of the GSW theory $\mathbf{Obs}^{cl}_{\mathcal{F}}$ by setting

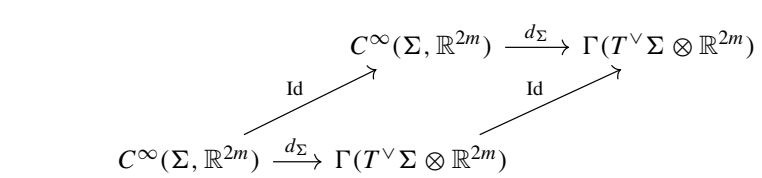
$$\mathbf{Obs}^{cl}_{\mathcal{F}}(U) = \mathrm{CE}^\bullet(\mathcal{L}_{GSW}(U)).$$

Example 5.5. (*Perturbative nonlinear sigma model*) Fix a constant map ϕ_0 from Σ to M. One can construct a local L_∞ algebra $\mathcal{L}_{NSM}$. As a graded vector space,

$$\mathcal{L}_{NSM} = (C^\infty(\Sigma, \mathbb{R}^{2m})[-1] \oplus \Gamma(T^\vee\Sigma \otimes \mathbb{R}^{2m})[-2])_{dR}.$$

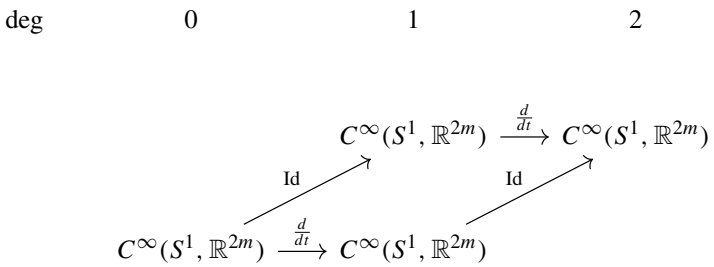
The higher brackets of $\mathcal{L}_{NSM}$ are determined by the Taylor expansion of the Christoffel symbols and the components of the Riemann curvature tensor in local coordinates around $\phi_0 \in M$. (If one chooses normal coordinates, the first nontrivial higher bracket is l_3.)

The underlying complex of $\mathcal{L}_{NSM}$ is



[6] Recall that for a graded vector space V, V_{dR} denotes $V \oplus V[1]$. A Lie algebra action of $\mathfrak{g}$ on V extends naturally to a graded Lie algebra action of $\mathfrak{g}_{dR}$ on V_{dR}.

Here, d_Σ denotes the de Rham differential on Σ. Unfortunately, this is not an elliptic complex for $\dim \Sigma > 1$. In dimension one, it reduces to the following elliptic complex



The corresponding elliptic L_∞ algebra fully encodes the structure of perturbative $N = 2$ supersymmetric quantum mechanics.

6. Concluding Remarks

We have proposed a mathematical framework for cohomological field theories, which provides a uniform description of various theories in the physics literature. Our primary focus is the application of this framework to the generalized Seiberg–Witten equations. However, as demonstrated in Subsection 4.1, the framework can also offer a consistent treatment of sigma-model-type theories. In dimension 1, this application recovers $N = 2$ supersymmetric quantum mechanics.

In this context, it would be interesting to explore the framework's applications to first-order nonlinear PDEs in higher dimensions. For instance, in dimension 2, applying the framework to the J-holomorphic curve equation should recover the A-model [52, 53], while applying it to the Euler–Lagrange equations of the Poisson sigma model [54,55] could yield new insights. For further applications, there is no reason for one to restrict his or her attention to ordinary first-order nonlinear PDEs – he or she may also consider supersymmetric first-order nonlinear PDEs, such as the super J-holomorphic curve equations, which were introduced in [56] with the aim to define a theory of super quantum cohomology [57].

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Declarations

Conflict of interest The authors declare that there is no Conflict of interest.

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