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A NON-RESONANT PERTURBATION THEORY

I. Introduction

This paper provides a theory for a non-resonant perturbation technique for measuring the electric field at various points within a device. It is adapted specifically to a study-state field with a sinusoidal time variation. Its value will be found primarily in microwave devices. The device can be a transmission line, waveguide, or, in fact, any object that has the following properties.

(1) Basically, the device consists of a cavity that contains an electromagnetic field.

(2) Electromagnetic power is permitted to enter the cavity only at a single port while perturbation measurements are being made. (This is the port at which reflection coefficient measurements are made.)

(3) In that port of the input waveguide where reflection coefficient measurements are made, a single waveguide mode is contained.

(4) The cavity walls very greatly attenuate the electromagnetic fields at the operating frequency; the fields are impressed upon them from either the inside or the outside. (In most practical cases these walls are made of highly conducting metals.)

(5) The cavity walls and the medium inside the cavity are assumed to have electrical parameters that are linear and isotropic.

Mallory[1] has also presented a non-resonant perturbation technique. This paper is presented in addition to the Mallory paper, since it provides a more rigorous justification for the measurement technique. Non-resonant perturbation measurements have been made by a number of workers for several years. The principal value of this paper is that it provides a rigorous, general, underlying theory.

II. Theory

Figure 1 shows a cross sectional view of the cavity. It has just one waveguide (or transmission line) port through which electromagnetic energy is permitted to pass into its interior. It can have any size or shape. The

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[1] K. Mallory, "A perturbation technique for impedance measurements," Conference on Microwave Measurement Technique, Institute of Electrical Engineering (London, September 1961).

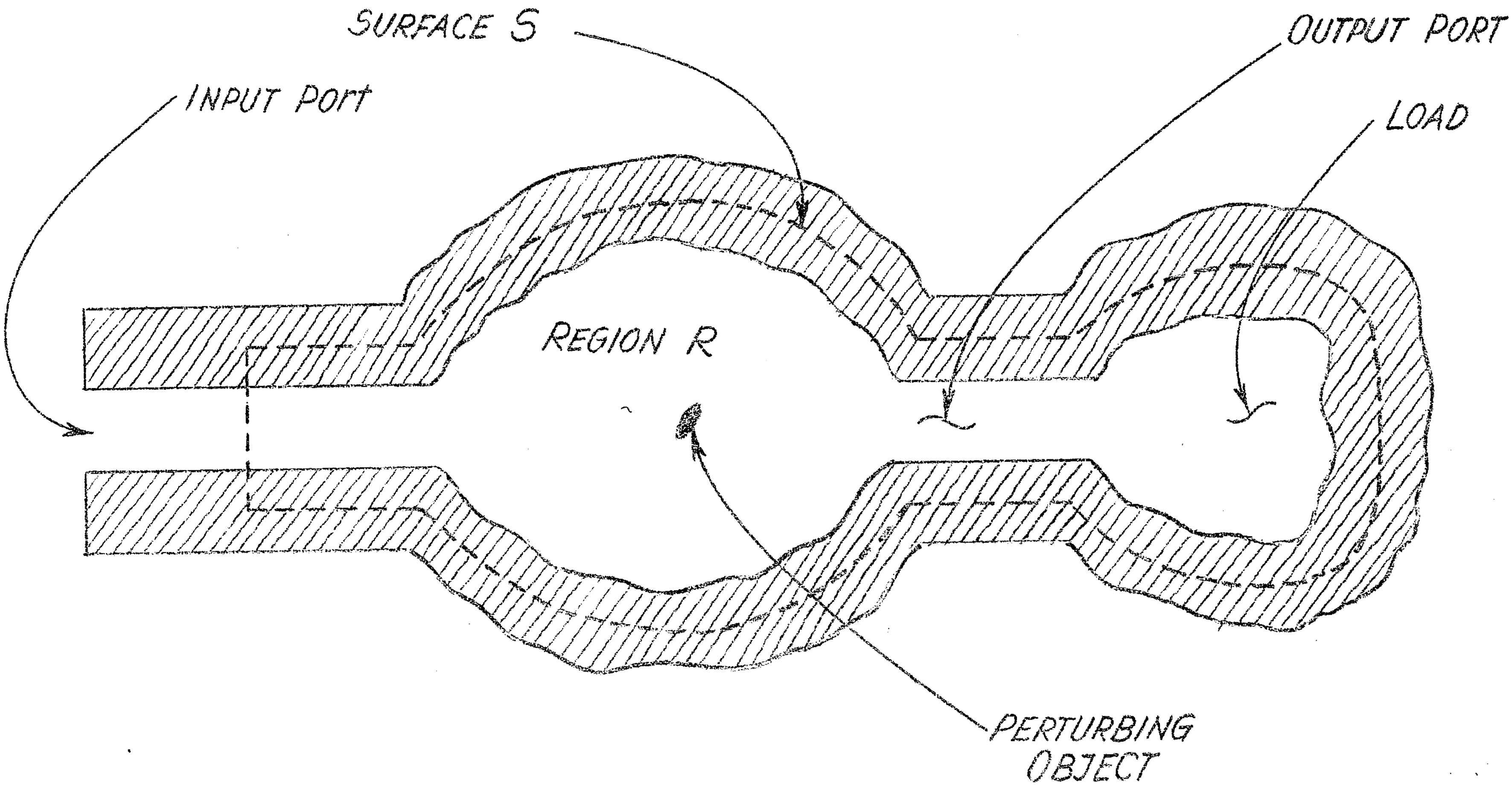

FIG. 1--Cavity in which perturbation measurements of field strength are made.

cavity can be either lossy (in its walls, its interior, or both) or lossless.

Suppose, now, that field measurements are required within a device (such as an accelerator section or a wavequide valve) that has one or more output ports. The cavity, as defined in the introduction and in the theoretical development given below, is considered to include these output ports, that is, the output waveguides and the loads to which they connect. Since the cavity wall then includes the walls of the output wavequides and loads, these walls must very greatly attenuate the electromagnetic fields at the operating frequency. This concept is illustrated with one output waveguide and load in Fig. 1.

Consider now the region R, of volume V, inside the closed surface S in Fig. 1. As shown, the surface S lies entirely within the cavity walls, except where it crosses the input waveguide in a plane normal to the waveguide axis.

The basic formulation for this theory is similar to that of the Lorentz Reciprocity Theorem.[2] Two different electromagnetic fields are considered within region R. One field, in the absence of a perturbing object, is designated by the electric and magnetic field components E_a and H_a respectively. The other field, in the presence of a perturbing object within region R, is designated by the electric and magnetic field components E_p and H_p, respectively. These two fields have the same frequency. We employ the vector $\vec{p}$ defined by

$$\vec{p} = \vec{E}_a \times \vec{H}_p - \vec{E}_p \times \vec{H}_a \qquad (1)$$

throughout region R and over the surface S. The first step in the derivation is to relate $\vec{p}$ over the surface S to $\vec{p}$ throughout volume V by the divergence theorem,[3]

$$\int_S (\vec{n} \cdot \vec{p})ds = \int_V (\vec{\nabla} \cdot \vec{p})dv \qquad (2)$$

[2]J.A. Stratton, Electromagnetic Theory (McGraw-Hill Book Co., 1941); p. 479.
[3]H.B. Phillips, Vector Analysis (John Wiley and Sons, Inc., 1933); p. 68.

where $\vec{n}$ is the unit vector, normally outward from surface S. In Eq. (2) the integral on the left is over the entire closed surface, S, and the integral on the right is throughout all of the volume V, contained in region R. In the paragraphs to follow, the integrals in Eq. (2) are developed into forms suitable for use in perturbation measurements.

Consider first the integral on the left of Eq. (2). Suppose that surface S consists entirely of two parts: S_1, the part that crosses the waveguide input port; and S_2, the part contained within the cavity wall. We assume that the cavity wall attenuates electromagnetic waves so effectively that, over surface S_2 (which lies between the inner and outer cavity walls),

$$\vec{E}_a = \vec{H}_a = \vec{E}_p = \vec{H}_p = \vec{p} = 0$$

Thus

$$\int_S \vec{n} \cdot \vec{p} \, ds = \int_{S_1} \vec{n} \cdot \vec{p} \, ds \tag{3}$$

Now over surface S_1, using Eq. (1),

$$\begin{aligned} \vec{n} \cdot \vec{p} &= \vec{n} \cdot (\vec{E}_a \times \vec{H}_p) - \vec{n} \cdot (\vec{E}_p \times \vec{H}_a) \\ \vec{n} \cdot \vec{p} &= (\vec{n} \times \vec{E}_a) \cdot \vec{H}_p - (\vec{n} \times \vec{E}_p) \cdot \vec{H}_a \\ \vec{n} \cdot \vec{p} &= (\vec{n} \times \vec{E}_{as}) \cdot \vec{H}_{ps} - (\vec{n} \times \vec{E}_{ps}) \cdot \vec{H}_{as} \end{aligned} \tag{4}$$

In Eq. (4), the subscript s denotes those components of the fields that lie in the plane surface S_1. Suppose, now, that over S_1, $\vec{E}_a$ and $\vec{H}_a$ are composed entirely of a single waveguide mode, and that $\vec{E}_p$ and $\vec{H}_p$ are composed entirely of the same waveguide mode. At each point on S_1, then,

$\vec{E}_{as}$ and $\vec{E}_{ps}$ must lie in the same direction and $\vec{H}_{as}$ and $\vec{H}_{ps}$ must lie in the same direction. In a single waveguide mode, the components of E and H that lie in a cross sectional plane must be perpendicular to each other. In Eq. (4), the vectors $(\vec{n} \times \vec{E}_{as})$ and $(\vec{n} \times \vec{E}_{ps})$ are perpendicular to $\vec{E}_{as}$ and $\vec{E}_{ps}$, but are parallel to $\vec{H}_{as}$ and $\vec{H}_{ps}$. Thus, Eq. (4) becomes

$$\vec{n} \cdot \vec{p} = E_{as} H_{ps} - E_{ps} H_{as} \tag{5}$$

where E_{as}, H_{as}, E_{ps}, and H_{ps} are all scalars. In general, these fields contain incident and reflected waves within the waveguide, and can be expressed as

$$E_{as} = (1 + \Gamma_a) E_{asi} \tag{6}$$

$$H_{as} = (1 - \Gamma_a) H_{asi} \tag{7}$$

$$E_{ps} = (1 + \Gamma_p) E_{psi} \tag{8}$$

$$H_{ps} = (1 - \Gamma_p) H_{psi} \tag{9}$$

where Γ_a and Γ_p are the reflection coefficients at S_1, (the input port) in the absence of the perturbing object, and in its presence, respectively. In these equations, the subscript i denotes the incident wave. When Eq. (5) is combined with Eq. (6) through Eq. (9), and one notes that

$$\frac{E_{psi}}{H_{psi}} = \frac{E_{asi}}{H_{asi}}$$

the result is

$$\vec{n} \cdot \vec{p} = (\Gamma_a - \Gamma_p)\,(E_{asi}H_{psi} + E_{psi}H_{asi}) \qquad (10)$$

From Poynting's Theorem,[4] and the fact that the E and H field components in Eq. (10) are perpendicular to each other, it is apparent that

$$\left| \int_{S_1} (E_{asi}H_{psi} + E_{psi}H_{asi})ds \right| = 2\sqrt{P_{ai}P_{pi}} \qquad (11)$$

In Eq. (11) the brackets around the integral denote taking the magnitude, and P_{ai} and P_{pi} are the power levels in the incident waves that pass through S_1, in the absence of, and in the presence, of the perturbing object, respectively. When Eqs. (3), (10), and (11) are combined, the result is

$$\left| \int_{S} (\vec{n} \cdot \vec{p})ds \right| = 2\sqrt{P_{ai}P_{pi}}\,\left| \Gamma_a - \Gamma_p \right| \qquad (12)$$

Since, in common practice, the incident wave power levels are equal in the presence and absence of the perturbing object,

$$P_{ai} = P_{pi} = P_i$$

and Eq. (12) becomes

$$\left| \int_{S} (\vec{n} \cdot \vec{p})ds \right| = 2\,P_i\,\left| \Gamma_a - \Gamma_p \right| \qquad (13)$$

[4] W.R. Smythe, Static and Dynamic Electricity (McGraw-Hill, Second Edition, 1950); p. 443.

Consider now the term on the right of Eq. (2). From Eq. (1),

$$\nabla \cdot \vec{p} = \nabla \cdot (\vec{E}_a \times \vec{H}_p) - \nabla \cdot (\vec{E}_p \times \vec{H}_a) \tag{14}$$

By means of a vector identity, Eq. **(14)** becomes

$$\nabla \cdot \vec{p} = (\nabla \times \vec{E}_a) \cdot \vec{H}_p - (\nabla \times \vec{H}_p) \cdot \vec{E}_a - (\nabla \times E_p) \cdot \vec{H}_a + (\nabla \times H_a) \cdot \vec{E}_p \tag{15}$$

Maxwell's Equations are now substituted into Eq. (15). For this purpose, these equations are written

$$\nabla \times \vec{E} = - j\omega\mu\vec{H} \tag{16}$$

and

$$\nabla \times \vec{H} = \vec{i}_c + j\omega\varepsilon\vec{E} = \vec{i}_c + \vec{i}_d = \vec{i}_t \tag{17}$$

where $\vec{i}_c$, $\vec{i}_d$ and $\vec{i}_t$ are the conduction, displacement, and total current densities, respectively. With these substitutions, Eq. (15) becomes

$$\nabla \cdot \vec{p} = - j\omega(\mu_a - \mu_p)\vec{H}_a \cdot \vec{H}_p + \vec{E}_p \cdot \vec{i}_{ta} - \vec{E}_a \cdot \vec{i}_{tp} \tag{18}$$

In Eq. (18), μ_a and μ_p are the magnetic permeabilities in region R in the absence of, and in the presence of, the perturbing object. Within this region, but outside the perturbing object,

$$\mu_a = \mu_p \tag{19}$$

If the perturbing object is a simple dielectric material, it is paramagnetic, and these permeabilities are also equal inside the space it occupies. If the perturbing object is a conductor, the two permeabilities may not be equal inside the space it occupies. However, with a good conductor, $\vec{H}_p$ is very small within this space. On the assumption that the perturbing object is either paramagnetic or a good conductor or both, the first term on the right side of Eq. (18) is neglected. This equation then becomes

$$\nabla \cdot \vec{p} = \vec{E}_p \cdot \vec{i}_{ta} - \vec{E}_a \cdot \vec{i}_{tp} \tag{20}$$

Now

$$\vec{i}_{ta} = (\sigma_a + j\omega\epsilon_a)\vec{E}_a$$

and

$$\vec{i}_{tp} = (\sigma_p + j\omega\epsilon_p)\vec{E}_a$$

where σ_a and σ_p are the conductances, and ϵ_a and ϵ_p are the respective permittivities in the absence of and in the presence of the perturbing object. With these equations, Eq. (20) becomes

$$\vec{\nabla} \cdot \vec{p} = \vec{E}_p \cdot \vec{E}_a \left[(\sigma_a - \sigma_p) + j\omega(\epsilon_a - \epsilon_p) \right] \tag{21}$$

From Eq. (21) one can see that in region R, outside the perturbing object,

$$\vec{\nabla} \cdot \vec{p} = 0 \tag{22}$$

and inside the space occupied by the perturbing object,

$$\nabla \cdot \vec{p} \neq 0 \tag{23}$$

From Eqs. (22) and (23), one can see that

$$\int_V (\vec{\nabla} \cdot \vec{p})dv = \int_{V_p} (\vec{\nabla} \cdot \vec{p})dv \qquad (24)$$

where V is the volume throughout region R, and V_p is only the volume occupied by the perturbing object.

When Eqs. (2), (13), (20), and (24) are combined, the result is

$$2P_i \left| \Gamma_a - \Gamma_p \right| = \left| \int_{V_p} (\vec{E}_p \cdot \vec{i}_{ta} - \vec{E}_a \cdot \vec{i}_{tp})dv \right| \qquad (25)$$

When Eqs. (2), (13), (21), and (24) are combined, the result is

$$2P_i \left| \Gamma_a - \Gamma_p \right| = \left| \int_{V_p} \vec{E}_p \cdot \vec{E}_a \left[(\sigma_a - \sigma_p) + j\omega(\epsilon_a - \epsilon_p) \quad dv \right] \right| \qquad (26)$$

For the purposes of the discussion below, it is convenient to define a quantity K by

$$2P_i \left| \Gamma_a - \Gamma_p \right| = KE_a^2 \qquad (27)$$

where E_a is taken at the center of the space to be occupied by the perturbing object. K has the dimensions of admittance times area. When one combines Eqs. (25) and (26) with Eq. (27), the results are

$$K = \frac{1}{E_a^2} \left| \int_{V_p} (\vec{E}_p \cdot \vec{i}_{ta} - \vec{E}_a \cdot \vec{i}_{tp})dv \right| \qquad (28)$$

and

$$K = \frac{1}{E_a^2} \left| \int_{V_p} \vec{E}_p \cdot \vec{E}_a \left[(\sigma_a - \sigma_p) + j\omega(\epsilon_a - \epsilon_p) \right] dv \right| \qquad (29)$$

Equations (27), (28), and (29) are those to be employed below in the development of the measurement technique.

III. Measurement Technique

For use in perturbation measurements, Eq. (27) is written in the form

$$\frac{E_a^2}{P_i} = \frac{2\left|\Gamma_p - \Gamma_a\right|}{K} \qquad (30)$$

Usually one wishes to evaluate the term on the left of Eq. (30) at a number of points in region R. It is only necessary, then, to measure $\left|\Gamma_p - \Gamma_a\right|$ for the perturbing object placed at these points, and to evaluate K. Often it is necessary to obtain not only the magnitude of E_a, but its direction as well. This is done by using a perturbing object that is much longer than it is wide. For such an object, K is a function of the angle between its axis and $\vec{E}_a$ (as well as being a function of its size, shape, and composition). Thus, when the perturbing object is rotated about a given point, the measured value of $\left|\Gamma_a - \Gamma_p\right|$ is greatest when its axis is parallel to $\vec{E}_a$.

Generally, there are three circumstances in which $\left|\Gamma_p - \Gamma_a\right|$ is measured.

1. If the cavity is lossy, one may choose to match the input impedance in the absence of the perturbing object. In this case,

$$\Gamma_a = 0$$

and

$$\left|\Gamma_p - \Gamma_a\right| = \left|\Gamma_p\right|$$

2. If the cavity is lossless, then Γ_a and Γ_p are both unity in magnitude. In this situation we measure the phase angle, Φ, between them, and

$$\left| \Gamma_p - \Gamma_a \right| = 2 \sin \frac{\Phi}{2}$$

3. Finally, the cavity may be lossy, but both Γ_p and Γ_a may be greater than zero. In this case, a straightforward way to evaluate $\left| \Gamma_p - \Gamma_a \right|$ is to plot the two reflection coefficients on a Smith Chart. The desired difference is then the ratio of the distance between the two points on the chart, to the chart radius.

The quantity K is usually a function only of the size, shape, and composition of the perturbing object, its orientation relative to the electric field, the electrical properties of the medium in region R, and the operating frequency. That is, K is usually independent of the location and orientation of the perturbing object. This is true if the perturbing object is kept far enough from the cavity walls to prevent cross coupling. In most applications this is true, and the electrical medium (outside the perturbing object) is uniform throughout R.

By placing the perturbing object in a cavity for which the term on the left of Eq. (30) is known, K can be measured. Such a cavity might be a waveguide of uniform cross section, in which the standing-wave pattern is known. The input reflection coefficients would be measured with and without the perturbing object in the cavity. The term on the left side of Eq. (30) would be calculated from the waveguide mode and standing-wave pattern. Finally, K would be calculated from Eq. (30).

For perturbing objects of certain simple shapes and compos tions, K can be calculated. Formulas for K for small conducting and dielectric prolate spheroids and spheres are presented below. The derivations of these formulas are outlined in the Appendix. These derivations assume that the perturbing object is introduced into a uniform electric field. In addition, they employ the quasi-static approximations; that is, the interaction between the electric and magnetic fields is neglected. As a result, these formulas pertain only to perturbing objects that are small compared to a wavelength. For the conducting prolate spheroid*

$$K_c = \frac{4\pi\omega\epsilon_a a^3}{3\eta_1^2}\left[\frac{1}{Q_1(\eta_1)}\cos^2\theta - \frac{2\sqrt{\eta_1^2 - 1}}{\eta_1 Q_1^1(\eta_1)}\sin^2\theta\right] \tag{31}$$

where

$Q_1(\eta_1)$ is a Legendre function of the second kind

$Q_1^1(\eta_1)$ is an associated Legendre function of the second kind

$$Q_1(\eta) = \tfrac{1}{2}\eta \ln \frac{\eta + 1}{\eta - 1} - 1$$

$$Q_1^1(\eta) = \sqrt{\eta^2 - 1}\left[\tfrac{1}{2}\ln \frac{\eta + 1}{\eta - 1} - \frac{\eta}{\eta^2 - 1}\right]$$

$$\eta_1 = \frac{1}{\sqrt{1 - (b/a)^2}}$$

2a is the length of the prolate spheroid

b is the radius of the prolate spheroid

θ is the angle between the axis of the prolate spheroid and $\vec{E}_a$

The prolate spheroid degenerates to a sphere as η_1 approaches infinity. By taking this limit with Eq. (31), one finds the formula for a conducting sphere to be

$$K_c = 4\pi\omega\epsilon_a a^3 \tag{32}$$

where a is its radius. For a small dielectric prolate spheroid*

*Note that in Eqs. (31) and (33) the sign is negative for terms that contain $Q_1^1(\eta_1)$. This results from the fact that $Q_1^1(\eta_1)$ is negative for values of η greater than unity.

$$K_d = \frac{4}{3}\pi\omega(\epsilon_p - \epsilon_a)ab^2 \left[\frac{\cos^2\theta}{1 + \left(\frac{\epsilon_p}{\epsilon_a} - 1\right)\left(\eta_1^2 - 1\right)Q_1(\eta_1)} + \frac{\sin^2\theta}{1 - \left(\frac{\epsilon_p}{\epsilon_a} - 1\right)\sqrt{\eta_1^2 - 1}\ \frac{\eta_1}{2}\ Q_1^1(\eta_1)}\right] \tag{33}$$

and for a small dielectric sphere

$$K_d = \frac{4\pi\omega(\epsilon_p - \epsilon_a)a^3}{3 + \left(\frac{\epsilon_p}{\epsilon_a} - 1\right)} \tag{34}$$

It is interesting to compare dielectric and conducting prolate spheroids by examining Eqs. (31) and (33). The following observations can be made:

1. For a dielectric prolate spheroid of a given size, shape, and angle with respect to the electric field, K_d increases with its dielectric constant.

2. For a dielectric prolate spheroid of a given shape, the ratio of maximum to minimum values of K_d (with variation of θ) increases with increasing dielectric constant.

3. As the dielectric constant of the prolate spheroid approaches infinity, the limiting value of K_d is just equal to K_s, for a conducting prolate spheroid of the same size, shape, etc.

These observations tend to indicate that, other factors being equal, conducting prolate spheroids are better than dielectric prolate spheroids for use as perturbing objects.

APPENDIX A

1. Conducting Prolate Spheroid

To derive Eq. (31), the components of K_c, for electric fields oriented parallel to, and perpendicular to, the axis of the conducting prolate spheroid were calculated separately. The derivation of the component of K for the electric field parallel to the axis of the prolate spheroid is presented below. The derivation of the other component follows an identical approach. It is most convenient to start with Eq. (28). Since $\vec{E}_p$ is effectively zero throughout the volume occupied by the prolate spheroid, V_p, Eq. (28) becomes

$$K = \frac{1}{E_a^2} \left| \int_{V_p} (\vec{E}_a \cdot \vec{i}_{tp}) dv \right| \qquad (35)$$

For this calculation, let the coordinate in the axial direction be Z. Then, by symmetry, one can see that Eq. (35) reduces to

$$K = \frac{1}{E_a} \left| \int_{-a}^{+a} I_Z dZ \right| \qquad (36)$$

where I_Z is the total current that flows over the surface of the prolate spheroid, in the Z direction.

The quasi-static approximations are used for this calculation, since the prolate spheroid is assumed to be small compared to a wavelength. That is, the interaction between changing electric and magnetic fields is neglected, and the electric field is assumed to be conservative. The electric field can then be represented as the negative of the gradient of an electric potential, V. Calculations are carried out in the prolate spheroidal coordinate system, using the convention of Smythe[5] First, the potential

[5]Smythe, op. cit., p. 157

field, V, outside the prolate spheroid is found, by choosing the appropriate solutions to Laplace's Equation and matching the boundary conditions, to be

$$V = E_a a\xi \left[\frac{\eta}{\eta_1} - \frac{Q_1(\eta)}{Q_1'(\eta_1)} \right] \tag{37}$$

In Eq. (33),

η is the dimensionless coordinate that defines the confocal prolate spheroids

η_1 is the value of η that defines the shape of the conducting prolate spheroid

ξ is the dimensionless coordinate that defines the confocal hyperboloids

and all other terms have the same definitions as given above.

Next, the surface charge density on the surface of the conducting prolate spheroid, σ, is calculated from the equation

$$\sigma = \epsilon_a E_\eta \Big|_{\eta_1}$$

where the electric field E_η, is calculated from Eq. (37) and the result is

$$\sigma = -\frac{\epsilon_a E_a \xi}{Q_1(\eta_1)\sqrt{(\eta_1^2 - \xi^2)(\eta_1^2 - 1)}} \tag{39}$$

The total charge Q is calculated over that portion of the prolate spheroid between one of its ends and a plane $Z = Z_1$. This is done by integration of

Eq. (39) over the required area, with the result that

$$Q = \frac{-\pi\epsilon_a E_a}{\eta_1^2 Q_1(\eta_1)} (a^2 - Z_1^2) \tag{40}$$

The total current that flows axially along the length of the prolate spheroid, I_Z, is then

$$I_Z = j\omega Q = \frac{-j\pi\omega\epsilon_a E_a}{\eta_1^2 Q_1(\eta_1)} (a^2 - Z_1^2) \tag{41}$$

Finally, when Eq. (41) is combined with Eq. (36), and the integration is performed, the result is

$$K_c = \frac{4\pi\omega\epsilon_a a^3}{3\eta_1^2 Q_1(\eta_1)} \tag{42}$$

2. Dielectric Prolate Spheroid

To derive Eq. (33), the components of K_d, for electric fields oriented parallel and perpendicular to the axis of the prolate spheroid, were calculated separately. The derivation of the components of K_d for the electric field parallel to this axis is presented below. The derivation of the other component follows an identical approach. Again, the quasi-static approximations are used.

It is most convenient to start with Eq. (29). In this case, the conductivity will be assumed to be zero, both in region R and in the perturbing object. The permittivities in the absence of, and in the presence of, the perturbing object are assumed to be uniform over the space occupied by the perturbing object. Equation (29) then becomes

$$K_d = \frac{1}{E_a^2} \left| j\omega(\epsilon_a - \epsilon_p) \int_{V_p} \vec{E}_p \cdot \vec{E}_a \, dv \right| \tag{43}$$

Now when this dielectric prolate spheroid is placed in a uniform electric field that is oriented either parallel to, or perpendicular to, its axis, the electric field inside the prolate spheroid is also uniform, and is oriented in the same direction as the external field. In this case, Eq. (43) becomes

$$K_d = \omega(\epsilon_p - \epsilon_a) \frac{E_p}{E_a} V_p \tag{44}$$

where V_p is the volume of the prolate spheroid, given by

$$V_p = \frac{4}{3} \pi a b^2 \tag{45}$$

The ratio E_p/E_a is obtained simply by solving the electrostatic problem of this prolate spheroid immersed in a uniform electric field. The result is

$$\frac{E_p}{E_a} = \frac{1}{1 + \left(\epsilon_p/\epsilon_a - 1 \right)(\eta_1^2 - 1) Q_1(\eta_1)} \tag{46}$$

Now when Eqs. (44), (45), and (46) are combined, the result is

$$K_d = \frac{\omega(\epsilon_p - \epsilon_a) \frac{4}{3} \pi a b^2}{1 + \left(\frac{\epsilon_p}{\epsilon_a} - 1 \right) (\eta_1^2 - 1) Q_1(\eta_1)} \tag{47}$$