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Quantization, dequantization, and distinguished states

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Abstract
Geometric quantization is a natural way to construct quantum models starting from classical data. In this work, we start from a symplectic vector space with an inner product and—using techniques of geometric quantization—construct the quantum algebra and equip it with a distinguished state. We compare our result with the construction due to Sorkin—which starts from the same input data—and show that our distinguished state coincides with the Sorkin-Johnson state. Sorkin's construction was originally applied to the free scalar field over a causal set (locally finite, partially ordered set). Our perspective suggests a natural generalization to less linear examples, such as an interacting field.

Keywords: Sorkin–Johnston state, symplectic space, Kähler space, geometric quantization, dequantization, Berezin–Toeplitz quantization

1. Introduction

Geometric quantization provides a very elegant and natural way to quantize classical systems using geometrical data. Despite some limitations, it is still a very attractive approach and in this work we apply it in a new and maybe unexpected context, namely in quantum field theory on *causal sets*.

A few years ago, Afshordi *et al* [1] published a construction of a distinguished pure, quasi-free state for quantum field theory in curved spacetimes based on earlier works [2, 3]. It was later shown that this state fails to be Hadamard [4, 5], so its singularity structure does not have

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some desirable features. Modifications can recover the Hadamard condition but remove the uniqueness of the state [6, 7].

In causal set theory one replaces the spacetime manifold with a locally finite partially ordered set, referred to as *the causal set*, for which the Hadamard condition is not meaningful and the Sorkin–Johnston state is applicable without modifications. In this work, we want to study the Sorkin–Johnston state on causal sets from the perspective of geometric quantization.

Traditionally, the Sorkin–Johnston state is obtained as the unique state that satisfies certain natural requirements [8], referred to as *Sorkin–Johnston axioms*. Our construction takes a new perspective (without using the axioms) and delivers the same state through the application of geometric quantization for symplectic manifolds with a Riemannian metric. Our construction may later be generalized to an interacting field on causal sets or to quantum field theory on curved spacetime, as long as the required geometry conditions are met. The main results have also been part of the PhD Thesis [9].

We review deformation quantization and the Weyl algebra in section 2. Taking the algebraic perspective, we formulate the Sorkin–Johnston state as a quasi-free state on the Weyl algebra for scalar fields on a causal set. In section 3, we review geometric quantization and the Toeplitz quantization map, as well as the dual map, known as Berezin–Toeplitz dequantization. We apply geometric construction in the case of scalar field theory on a causal set in section 4 and show that the dequantization map gives rise to the Sorkin–Johnston state in section 5. Finally, we show how the Berezin–Toeplitz quantization and dequantization maps correspond to strict deformation quantizations.

2. Deformation quantization of classical field algebras, and states

First, we consider scalar field theory on a causal set to define a vector space with Poisson bracket and determine the corresponding symplectic form. For the review of deformation quantization, the Weyl relations as well as the Weyl algebra here, it suffice to know the Poisson structure. The symplectic form is required for the derivation of the Sorkin–Johnston state and the geometric construction later on.

2.1. Scalar fields on causal sets

In the algebraic formulation of classical scalar fields on a (finite subset of a) causal set $\mathcal{C}$ as considered in [10], we start with the *off-shell configuration space* (a vector space) $\mathcal{E}$ of real-valued functions over $\mathcal{C}$ that is equipped with an inner product $\langle\cdot,\cdot\rangle$. We are interested in those real-valued functions over $\mathcal{C}$ that obey a discretized version of the Klein–Gordon field equations.

Given the Pauli-Jordan operator $E_{\text{off}} \in \text{End}(\mathcal{E})$ as the difference of the retarded and advanced Green's operators for the field equations, the space of classical observables $\mathfrak{P}_{\text{off}}$ is the space of complex-valued functions over the configuration space $\mathcal{E}$ with a Poisson structure $\pi_{\text{off}} \in \bigwedge^2 \mathcal{E}$ so that for all $f_1, f_2 \in \mathfrak{P}_{\text{off}}$:

$$\{f_1, f_2\}_{\text{off}} = \pi_{\text{off}}(\mathrm{d}f_1, \mathrm{d}f_2)\,. \tag{1}$$

Note that this bracket is equivalently expressed as the map $\pi^{\sharp}_{\text{off}} : \mathcal{E}^* \to \mathcal{E}$. In general, $\pi^{\sharp}_{\text{off}}$ is degenerate but its image is an even dimensional sub-space $\mathcal{S} = \text{img}(\pi^{\sharp}_{\text{off}}) \subset \mathcal{E}$. The *on-shell Poisson algebra* $\mathfrak{P}$ is then the quotient of $\mathfrak{P}_{\text{off}}$ by the ideal generated by all observables that vanish on $\mathcal{S}$ [10]. We write the Poisson bracket on $\mathfrak{P}$ as $\{\cdot,\cdot\}$ and note that the corresponding map $\pi^{\sharp} : \mathcal{S}^* \to \mathcal{S}$ is now non-degenerate. The inner product on $\mathcal{E}$ restricts to an inner product

$\langle\cdot,\cdot\rangle$ on $\mathcal{S}$ and determines the metric $g^\flat : \mathcal{S} \to \mathcal{S}^*$. The inverse of the Poisson bracket is a symplectic form ω on $\mathcal{S}$, which we express with the inverse of the (restricted) Pauli-Jordan operator $E \in \mathrm{End}(\mathcal{S})$,

$$\forall v_1, v_2 \in \mathcal{S}: \qquad \omega(v_1, v_2) = \left\langle v_1, E^{-1} v_2 \right\rangle. \tag{2}$$

This structure on the vector space $\mathcal{S}$ is our starting point for the state construction. For a globally hyperbolic spacetime manifold, a symplectic vector space is similarly constructed from the configuration space of smooth functions. In that case, the symplectic vector space is infinite dimensional. However, our main focus in this work lies on the given structure for a finite dimensional vector space.

The operator E is anti-symmetric and anti-self-adjoint, $E^* = -E$. As we have constructed a symplectic vector space, the kernel of the Pauli–Jordan operator E (restricted to $\mathcal{S}$) is trivial and we have the polar decomposition

$$E = |E|U^*, \qquad E^* = U|E|. \tag{3}$$

where U is a unitary operator and $|E|$ is the strictly positive operator i.e. invertible

$$|E| := \sqrt{E^*E} = \sqrt{-E^2}. \tag{4}$$

We insert this decomposition into (2) to find

$$\forall v_1, v_2 \in \mathcal{S}: \qquad -\omega(v_1, Uv_2) = \left\langle v_1, |E|^{-1} v_2 \right\rangle. \tag{5}$$

Since $|E|^{-1}$ is a positive, self-adjoint operator, the right hand side of (5) is also a symmetric, bi-linear form. We denote this form by η,

$$\forall v_1, v_2 \in \mathcal{S}: \qquad \eta(v_1, v_2) := \left\langle v_1, |E|^{-1} v_2 \right\rangle. \tag{6}$$

The operator $J = -U$ is a complex structure on $\mathcal{S}$ (such that $J^2 = -\mathbf{1}$) and the relation between ω and η is Kähler.

Definition 1. A *Kähler vector space* is a quadruple $(\mathcal{S}, \omega, \eta, J)$ of a vector space $\mathcal{S}$ with a complex structure J, a symmetric bi-linear form η, and a symplectic form ω such that

$$\forall v_1, v_2 \in \mathcal{S}: \qquad \omega(v_1, Jv_2) = \eta(v_1, v_2). \tag{7}$$

Note that, on the one hand, in the presence of a complex structure J, the real vector space $\mathcal{S}$ turns into a complex vector space $\mathcal{S}_J$ by

$$\forall v \in \mathcal{S} : \forall a, b \in \mathbb{R}: \qquad (a + \mathrm{i}b)\, v := av + bJv. \tag{8}$$

with $\dim_{\mathbb{C}} \mathcal{S}_J = \frac{1}{2} \dim_{\mathbb{R}} \mathcal{S}$. On the other hand, the complexification of $\mathcal{S}$ yields a complex vector space $\mathcal{S}^{\mathbb{C}}$ such that $\dim_{\mathbb{C}} \mathcal{S}^{\mathbb{C}} = \dim_{\mathbb{R}} \mathcal{S}$. The complexified vector space has a holomorphic and an anti-holomorphic subspace, $\mathcal{S}^{\mathbb{C},+}$ and $\mathcal{S}^{\mathbb{C},-}$, respectively,

$$\mathcal{S}^{\mathbb{C},\pm} := \{ v \mp \mathrm{i}Jv \,|\, v \in \mathcal{S} \}. \tag{9}$$

For the introduction of the Sorkin–Johnston in section 2.4 state as well as for its geometric construction in sections 4 and 5, we use the complex vector space $\mathcal{S}_J$ (or equivalently the holomorphic subspace $\mathcal{S}^{\mathbb{C},+}$).

2.2. Formal deformation quantization

Formal deformation quantization is a deformation of the classical pointwise product of the Poisson algebra $C^\infty(\mathcal{M},\mathbb{C})$ to a star product (a series expansion in powers of $\hbar$). The additional properties of the definition are as in [11].

Definition 2. Let $(\mathcal{M},\{\cdot,\cdot\})$ be a Poisson manifold. A *star product* $\star$ is a product on the space of formal power series $C^\infty(\mathcal{M},\mathbb{C})[[\hbar]]$ such that for functions $f_1,f_2 \in C^\infty(\mathcal{M},\mathbb{C})$:

$$f_1 \star f_2 = \sum_{k=0}^{\infty} B_k(f_1,f_2)\,\hbar^k, \tag{10}$$

where B_k are bi-linear maps fulfilling the conditions, $\forall f_1,f_2,f_3 \in C^\infty(\mathcal{M},\mathbb{C})$:

$$\begin{aligned} &\text{associativity:} && (f_1\star f_2)\star f_3 = f_1\star(f_2\star f_3),\\ &\text{pointwise product:} && B_0(f_1,f_2) = f_1 f_2. \end{aligned} \tag{11}$$

The star product $\star$ is

$$\begin{aligned} &\text{Poisson compatible if} && B_1(f_1,f_2) - B_1(f_2,f_1) = \mathrm{i}\{f_1,f_2\},\\ &\text{unital if} && f_1\star 1 = 1\star f_1 = f_1,\\ &\text{self-adjoint if} && \overline{f_1\star f_2} = \overline{f_2}\star\overline{f_1}, \end{aligned}$$

and it is *differential* if all B_k are bi-differential maps.

As a standard example, consider the Weyl algebra over a real vector space $\mathcal{S}$ with a Poisson structure $\{\cdot,\cdot\}$. Let $\mathcal{S}^* = \mathrm{Hom}(\mathcal{S},\mathbb{R})$ be the dual vector space. The Weyl algebra $\mathfrak{W}_\hbar$ is generated by a map $W_\hbar$ on $\mathcal{S}^*$ that fulfills the Weyl relations (for all $\forall\phi,\phi' \in \mathcal{S}^*$)

$$W_\hbar(\phi)\,W_\hbar(\phi') = \exp\left(-\frac{\mathrm{i}\hbar}{2}\{\phi,\phi'\}\right)W_\hbar(\phi+\phi'), \tag{12a}$$

$$W_\hbar(\phi)^* = W_\hbar(-\phi), \tag{12b}$$

$$W_\hbar(0) = \mathbf{1}. \tag{12c}$$

These relations are realized by a deformation of $C^\infty(\mathcal{S},\mathbb{C})$ with the Moyal product (exponentiated Poisson bracket followed by pointwise multiplication m)

$$\begin{aligned} \mathrm{e}^{\mathrm{i}\phi}\star_W \mathrm{e}^{\mathrm{i}\phi'} &= m\circ\exp\left(\frac{\mathrm{i}\hbar}{2}\{\cdot,\cdot\}\right)\left(\mathrm{e}^{\mathrm{i}\phi}\otimes\mathrm{e}^{\mathrm{i}\phi'}\right),\\ &= \exp\left(-\frac{\mathrm{i}\hbar}{2}\{\phi,\phi'\}\right)\mathrm{e}^{\mathrm{i}(\phi+\phi')}. \end{aligned} \tag{13}$$

There exists a norm on the image of $W_\hbar$ that satisfies the C^*-property, so that the image can be completed to a C^*-algebra $\mathfrak{W}_\hbar$ (that is a subset of bounded operators on some Hilbert space), see [12].

In general, the power series (10) does not converge, hence the deformation quantization is called *formal*. Further below, we determine star products that correspond to Toeplitz quantization and Berezin–Toeplitz dequantization, for which we use the *strict* notion of deformation quantization.

2.3. Strict deformation quantization

Strict deformation quantization involves a family of C*-algebras $\mathfrak{A}_\hbar$ parametrized by $\hbar \in I$ for some parameter range $I \subseteq \mathbb{R}$ including the classical limit $\hbar = 0$. In the following, we will have $I = [0,\infty)$ and denote $I_* = I \setminus \{0\}$. For some constructions of geometric quantization, however, we will see that the quantization parameter has to take values in $I_* = \left\{p^{-1} | p \in \mathbb{N}_*\right\}$, where the classical limit $\hbar \to 0$ is equivalent to $p \to \infty$.

At the value $\hbar = 0$, take a C*-algebra $\mathfrak{A}_0$ between functions vanishing at infinity and bounded functions, $\mathrm{C}_0(\mathcal{M},\mathbb{C}) \subseteq \mathfrak{A}_0 \subseteq \mathrm{C}_\mathrm{b}(\mathcal{M},\mathbb{C})$ with the supremum norm (for any $f \in \mathfrak{A}_0$)

$$\|f\| := \sup_{x\in\mathcal{M}} |f(x)|. \tag{14}$$

On a dense Poisson subalgebra $\mathcal{A}_0 \subseteq \mathfrak{A}_0$, we have a Poisson bracket (either explicit or determined by a symplectic form ω) that is closed under complex conjugation.

Quantization is described as a deformation of this Poisson algebra by a family of linear maps.

Definition 3. A *quantization* Q is a family of linear maps from the classical algebra $\mathcal{A}_0 \subseteq \mathfrak{A}_0$ to the C*-algebra of quantum observables $\mathfrak{A}_\hbar$ parametrized by $\hbar$,

$$Q_\hbar : \mathcal{A}_0 \to \mathfrak{A}_\hbar \tag{15}$$

that respects the involution, $Q_\hbar(f)^* = Q_\hbar(\bar{f})$, and if there exists a unit $1 \in \mathcal{A}_0$, it is also unital, $Q_\hbar(1) = \mathbf{1}$.

It is easily seen that the map $W_\hbar$ that we considered in section 2.2 is a quantization map, known as Weyl quantization.

In the following, we write the commutator of two operators as $[A,B]_- := AB - BA$ and we use the little-o notation i.e. a continuous function $f(\hbar)$ is of order $\mathrm{o}(\hbar)$ if

$$\lim_{\hbar\to 0} \frac{1}{\hbar} f(\hbar) = 0. \tag{16}$$

For a (strict) quantization, one may require additional conditions on the quantization maps, see [13, chapter II, definition 1.1.1]. In particular, we would like to have a quantization that is compatible with the Poisson structure, meaning that for all differentiable functions $f_1, f_2 \in \mathcal{A}_0$:

$$[Q_\hbar(f_1), Q_\hbar(f_2)]_- = \mathrm{i}\hbar Q_\hbar(\{f_1, f_2\}) + \mathrm{o}(\hbar). \tag{17}$$

For some quantizations, there may also exist a family of dual maps, from the quantum C*-algebras to the C*-algebra of classical observables.

Definition 4. A *dequantization* Υ is a family of linear maps

$$\Upsilon_\hbar : \mathfrak{A}_\hbar \to \mathfrak{A}_0, \tag{18}$$

that respects involution, $\Upsilon_\hbar(A^*) = \overline{\Upsilon_\hbar(A)}$, and if there exists a unit $\mathbf{1} \in \mathfrak{A}_\hbar$, it is also unital, $\Upsilon_\hbar(\mathbf{1}) = 1$.

Note that for a quantization Q and a dequantization Υ, in general $\Upsilon_\hbar \circ Q_\hbar$ is not the identity map.

We will consider a continuous field for the family of C*-algebras $(\mathfrak{A}_\hbar)_{\hbar\in I}$ over the quantization range I as defined in [14, 15].

Definition 5. Let I be a topological space and let $(\mathfrak{A}_\hbar)_{\hbar\in I}$ be a family of C*-algebras. A *continuous field of C*-algebras* is a triple $\left(I,(\mathfrak{A}_\hbar)_{\hbar\in I},\varGamma\right)$ with $\varGamma\subseteq\prod_{\hbar\in I}\mathfrak{A}_\hbar$ such that

(i) $\varGamma$ is a linear subspace of $\prod_{\hbar\in I}\mathfrak{A}_\hbar$, closed under multiplication and involution,
(ii) for every $\hbar\in I$ the set $\{A(\hbar)\in\mathfrak{A}_\hbar \mid A\in\varGamma\}$ is dense in $\mathfrak{A}_\hbar$, and
(iii) for every element $A\in\varGamma$ the norm function $n_A: I\to\mathbb{R}$,

$$n_A(\hbar) := \|A(\hbar)\| \tag{19}$$

is continuous, $n_A\in \mathrm{C}(I,\mathbb{R})$, as well as
(iv) for any $A'\in\prod_{\hbar\in I}\mathfrak{A}_\hbar$, we have $A'\in\varGamma$ if the following condition is fulfilled. For all $\hbar\in I$ and for all real constants $\delta>0$ there exists a neighborhood $N_\hbar\subset I$ of $\hbar$ such that

$$\exists A\in\varGamma:\forall\hbar'\in N_\hbar: \qquad \|A'(\hbar')-A(\hbar')\|\leqslant\delta. \tag{20}$$

The elements of $\varGamma$ are called (continuous) *sections* of the field.

Products and the involution of any $A_1,A_2\in\prod_{\hbar\in I}\mathfrak{A}_\hbar$ are computed pointwise, $A_1A_2:\hbar\mapsto A_1(\hbar)A_2(\hbar)$ and $A_1^*:\hbar\mapsto A_1(\hbar)^*$, respectively.

If I is locally compact, then the subset of sections $\mathfrak{A}\subseteq\varGamma$ that have a continuous norm function vanishing at infinity, $n_A\in\mathrm{C}_0(I,\mathbb{R})$, is a C*-algebra with the supremum norm

$$\|A\| := \sup_{\hbar\in I}\|A(\hbar)\|. \tag{21}$$

The triple $(I,(\mathfrak{A}_\hbar)_{\hbar\in I},\mathfrak{A})$ is also referred to as a *C*-bundle* [16].

As an example for any $f\in\mathcal{A}_0$ define

$$Q(f):\hbar\mapsto\begin{cases} f & \hbar=0,\\ Q_\hbar(f) & \hbar\in I_*.\end{cases} \tag{22}$$

Our aim is to determine a quantization such that there exists a continuous field of C*-algebras where these are sections, as sketched in figure 1. Furthermore, we want the quantization to admit a star product in the following sense.

Definition 6. The *quantization star product* $\star_Q$ of a quantization—if it exists—is the star product (10) with operators $B_{Q,k}:\mathcal{A}_0\times\mathcal{A}_0\to\mathcal{A}_0$, such that for all $k\in\mathbb{N}$ and for all $f_1,f_2\in\mathcal{A}_0$, the kth order remainder

$$R_Q^k(f_1,f_2,\hbar) := \frac{1}{\hbar^k}\left\| Q_\hbar(f_1)\,Q_\hbar(f_2)-\sum_{j=0}^{k}Q_\hbar\left(B_{Q,j}(f_1,f_2)\right)\hbar^j\right\| \tag{23}$$

vanishes in the classical limit,

$$\lim_{\hbar\to0}R_Q^k(f_1,f_2,\hbar)=0. \tag{24}$$

Heuristically, the star product of two functions $f_1\star_Q f_2$ of an infinite order quantization Q is an asymptotic expansion of $Q_\hbar^{-1}\left(Q_\hbar(f_1)Q_\hbar(f_2)\right)$, though the inverse $Q_\hbar^{-1}$ usually does not exist even if the star product exists.

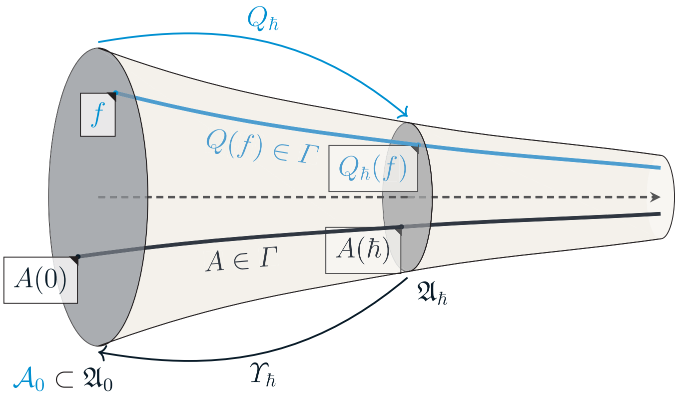

Figure 1. Continuous field of C*-algebras $\left(I, (\mathfrak{A}_\hbar)_{\hbar\in I}, \Gamma\right)$ with (continuous) sections $Q(f)$ (upper solid line, blue) and A (lower solid line, black), a family of quantization maps $(Q_\hbar)_{\hbar\in I}$ and dequantization maps $(\Upsilon_\hbar)_{\hbar\in I}$. At each value $\hbar \in I$ (horizontal axis), there is a quantum algebra $\mathfrak{A}_\hbar$ (ellipse, black) with a dense Poisson subalgebra $\mathcal{A}_\hbar$ (dotted patterns, blue/gray).

Definition 7. A quantization Q is an *infinite order strict deformation quantization* if

(i) there exists a continuous field of C*-algebras $\left(I, (\mathfrak{A}_\hbar)_{\hbar\in I}, \Gamma\right)$ such that

$$\forall f \in \mathcal{A}_0 \subseteq \mathfrak{A}_0 : \qquad Q(f) \in \Gamma, \tag{25}$$

(ii) the quantization star product $\star_Q$ exists, and
(iii) the star product is Poisson compatible.

We now consider sections that are well-behaved with respect to quantization and dequantization.

Definition 8. A section $A \in \Gamma$ of the continuous field of C*-algebras $\left(I, (\mathfrak{A}_\hbar)_{\hbar\in I}, \Gamma\right)$ is *Q-quantization expandable (quantization expandable or Q-expandable for short)* if for any $k \in \mathbb{N}$

$$\exists f_0, \ldots, f_k \in \mathcal{A}_0 : \qquad \lim_{\hbar\to 0} \frac{1}{\hbar^k} \left\| A(\hbar) - \sum_{j=0}^{k} Q_\hbar(f_j)\, \hbar^j \right\| = 0 \tag{26}$$

and it is *Υ-dequantization expandable (dequantization expandable or Υ-expandable for short)* if for any $k \in \mathbb{N}$

$$\exists f_0, \ldots, f_k \in \mathcal{A}_0 : \qquad \lim_{\hbar\to 0} \frac{1}{\hbar^k} \left\| \Upsilon_\hbar(A(\hbar)) - \sum_{j=0}^{k} f_j \hbar^j \right\| = 0. \tag{27}$$

Denote the space of Υ-expandable sections by $\Gamma_\Upsilon \subseteq \Gamma$ and let $\Sigma^k_\Upsilon(\cdot,\hbar) : \Gamma_\Upsilon \to \mathcal{A}_0[[\hbar]]$ map to the expansion of any section $A \in \Gamma_\Upsilon$ by the functions from (27) truncated at order k,

$$\Sigma^k_\Upsilon(A,\hbar) := \sum_{j=0}^{k} f_j \hbar^j. \tag{28}$$

On the one hand, it is immediately seen that the quantization section $Q(f) \subset \varGamma$ for any $f \in \mathcal{A}_0$ is Q-expandable with $f_0 = f$ and $f_k = 0$ for all $k > 0$. On the other hand, if the space of $\varUpsilon$-expandable sections forms a *-subalgebra of $\varGamma$, then dequantization also admits a star product (which is not necessarily Poisson compatible).

Definition 9. The *dequantization star product* $\star_\varUpsilon$ is the star product (10) with operators $B_{\varUpsilon,k} : \mathcal{A}_0 \times \mathcal{A}_0 \to \mathcal{A}_0$ such that for all $k \in \mathbb{N}$ and any dequantization expandable sections $A_1, A_2 \in \varGamma_\varUpsilon \subseteq \varGamma$, the expansion map intertwines $\star_\varUpsilon$ with the product on $\varGamma$, meaning

$$\Sigma^k_\varUpsilon (A_1 A_2, \hbar) = \sum_{j=0}^{k} B_{\varUpsilon,j} \left(\Sigma^k_\varUpsilon (A_1, \hbar), \Sigma^k_\varUpsilon (A_2, \hbar) \right) \hbar^j \mod \hbar^{k+1}. \tag{29}$$

Definition 10. Given a continuous field of C*-algebras $\left(I, (\mathfrak{A}_\hbar)_{\hbar \in I}, \varGamma\right)$, a dequantization $\varUpsilon$ is an *infinite order strict deformation dequantization* if

(i) the dequantization of any section $A \in \varGamma$,

$$\varUpsilon(A) : \mathcal{M} \times I \to \mathbb{C}, \qquad \varUpsilon(A)(x, \hbar) \mapsto \varUpsilon_\hbar (A(\hbar))(x), \tag{30}$$

is a continuous function $\varUpsilon(A) \in \mathrm{C}(\mathcal{M} \times I, \mathbb{C})$,

(ii) the dequantization star product $\star_\varUpsilon$ exists, and

(iii) the star product is Poisson compatible.

The construction of a strict deformation quantization, a corresponding continuous field of C*-algebras $\left(I, (\mathfrak{A}_\hbar)_{\hbar \in I}, \varGamma\right)$ and a corresponding strict deformation dequantization can be quite complicated in the general case. We consider strict deformation quantization associated to the (de)quantization maps that we obtain from geometric quantization in the case of a vector space in section 4. Before coming to a general review of the necessary aspects of geometric quantization in the next section, let us introduce the Sorkin–Johnston state that we reconstruct later on.

2.4. The Sorkin–Johnston state as an algebraic state

In algebraic quantum field theory, states are defined as functionals on a given *-algebra without requiring a Hilbert space.

Definition 11. A linear functional $\sigma : \mathcal{A} \to \mathbb{C}$ on an involutive algebra (*-algebra) $\mathcal{A}$ is a *state* if and only if it is positive,

$$\forall A \in \mathcal{A} : \qquad \sigma(A^* A) \geqslant 0, \tag{31}$$

and has unit norm.

For the definition of the Sorkin–Johnston state, in particular, consider the following class of states.

Definition 12. Let $\mathfrak{W}_\hbar$ be the Weyl algebra for a real vector space $\mathcal{S}$ (see also the example in section 2.2). A state σ on $\mathfrak{W}_\hbar$ is called *quasi-free* (or Gaussian) if there exists a symmetric, bi-linear form γ (called *covariance* of the state) on $\mathcal{S}^*$ such that

$$\sigma(W_\hbar(\phi)) = \exp\left(-\frac{\hbar}{4} \gamma(\phi, \phi) \right) \tag{32}$$

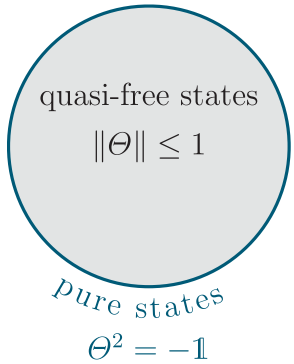

Figure 2. Operator norm of the operator Θ determined by the symplectic form and the inner product η_{G} as given in (36). For pure states, the operator norm lies on the boundary of this graphical representation and Θ is a complex structure.

holds for the Weyl generator $W_\hbar(\phi)$ of every element $\phi \in \mathcal{S}^*$.

The main argument for the Sorkin–Johnston state [3, 8, 17] is that there exists a Hermitian operator (as solution to the Sorkin–Johnston axioms)

$$A_{\mathrm{SJ}} = \frac{1}{2}(|E| + \mathrm{i}E) \tag{33}$$

that yields the 'positive eigenspace' of the (Pauli–Jordan) operator E, and determines a two-point function [17]. We call A_{SJ} the Sorkin–Johnston operator that acts on the complexified vector space $\mathcal{S}_J$. By the one-to-one correspondence between two-point functions and quasi-free states, the bi-linear form η, see (6), determines the covariance of a quasi-free state.

Definition 13. The *Sorkin–Johnston state* $\sigma_{\mathrm{SJ}} : \mathfrak{W}_\hbar \to \mathbb{C}$ is the quasi-free (or Gaussian) state with a covariance given by the inverse of the symmetric, bi-linear form η as defined in (6), so that for all $\phi \in \mathcal{S}^*$

$$\sigma_{\mathrm{SJ}}(W_\hbar(\phi)) = \exp\left(-\frac{\hbar}{4}\eta^{-1}(\phi,\phi)\right). \tag{34}$$

More generally, for a quasi-free state with covariance γ, the bi-linear form $\eta_{\mathrm{G}} = \gamma^{-1}$ and the symplectic form ω satisfy the Cauchy–Schwarz inequality, $\forall v_1, v_2 \in \mathcal{S}$:

$$|\omega(v_1, v_2)|^2 \leqslant \eta_{\mathrm{G}}(v_1, v_1)\,\eta_{\mathrm{G}}(v_2, v_2), \tag{35}$$

known as the domination condition [4]. Given a relation between the bi-linear forms ω and η_{G} such that for all $v_1, v_2 \in \mathcal{S}$:

$$\omega(v_1, \Theta v_2) = \eta_{\mathrm{G}}(v_1, v_2), \tag{36}$$

the domination condition implies that $\|\Theta\| \leqslant 1$. Figure 2 illustrates this condition on the operator norm, where the norm is induced by the inner product. For the proofs of the equivalences of the three statements, see [18].

For the structure that determines the Sorkin–Johnston state, the operator $\Theta = J$ is a complex structure, meaning that $\Theta^2 = -\mathbf{1}$, so that $\|\Theta\| = 1$ and (35) is saturated. This means that the Sorkin–Johnston state is *pure*—it cannot be written as a convex combination of two other states. By the one-to-one correspondence between quasi-free states and two-point functions, the Sorkin–Johnston state corresponds to a two-point function determined by the Sorkin–Johnston operator A_{SJ}.

3. Aspects of geometric quantization

For the geometric construction, we start with the given structure of the symplectic form ω and the inner product $\langle\cdot,\cdot\rangle$ as symmetric bi-linear form over $\mathcal{S}$ to construct a quantum algebra. Our construction will yield the Sorkin–Johnston state without imposing the Sorkin–Johnston axioms in section 5.2. Throughout the construction, the quantization parameter $\hbar$ is kept explicit to eventually define a field of algebras over all $\hbar \in I$ and discuss the classical limit.

3.1. Quantization bundle and polarizations

Definition 14. Let $(\mathcal{M},\omega)$ be a real, symplectic manifold. For some $\hbar$, a *quantization bundle* is a Hermitian line bundle $\mathcal{L}_\hbar \to \mathcal{M}$ with connection $\nabla_\hbar$ that preserves the inner product, such that its curvature $\mathrm{curv}(\nabla_\hbar)$ is proportional to the symplectic form,

$$\mathrm{curv}\left(\nabla_\hbar\right) = -\frac{\mathrm{i}}{\hbar}\omega. \tag{37}$$

Given a symplectic manifold, a quantization bundle does not necessarily exist and is not necessarily unique. A quantization bundle for $(\mathcal{M},\omega)$ exists if and only if the cohomology class of $\omega/2\pi\hbar$ in $H^2(\mathcal{M},\mathbb{R})$ is integral. This is known as the *prequantization (or integrality) condition*, see [19, section 3].

For geometric quantization, it is furthermore necessary to find a 'physical' Hilbert space $\mathcal{H}_\hbar$ as a subspace of square-integrable sections $\mathrm{L}^2(\mathcal{M},\mathcal{L}_\hbar)$ (or valued in $\mathcal{L}_\hbar$ tensored with some vector bundle). In some cases, the space $\mathcal{H}_\hbar$ is determined by a polarization [19].

Definition 15. Let $(\mathcal{M},\omega)$ be an $2N$-dimensional real symplectic manifold and $\mathcal{L}_\hbar \to \mathcal{M}$ a quantization bundle. A (complex) *polarization* is a subbundle $P \subset (\mathrm{T}\mathcal{M})^{\mathbb{C}}$ that is involutive, $X, Y \in \Gamma(P) \implies [X,Y] \in \Gamma(P)$, and maximally isotropic (Lagrangian), $\forall X,Y \in \Gamma(P)$: $\omega(X,Y) = 0$ and $\forall x \in \mathcal{M} : \dim_{\mathbb{C}} P_x = N$. We say that a section $\psi \in \Gamma(\mathcal{M},\mathcal{L}_\hbar)$ is *polarized* if $\forall X \in \Gamma(P) : \nabla_{\hbar,X}\psi = 0$. The *physical Hilbert space* $\mathcal{H}_\hbar$ is constructed from polarized sections of $\mathcal{L}_\hbar$.

For a Kähler manifold (a symplectic manifold with a compatible complex structure, J), the *Kähler polarization* is the subbundle on which J has eigenvalue $-\mathrm{i}$, and the polarized sections are precisely the holomorphic sections. Compact Kähler manifolds have been studied before, see [11, 20, 21].

For the more general case of a symplectic manifold without pre-defined complex structure, we will consider an alternative construction of the physical Hilbert space from the spectrum of a Laplace operator in section 4.

3.2. (Berezin-)Toeplitz quantization and dequantization

For a given symplectic manifold $(\mathcal{M},\omega)$, suppose that a Hilbert subspace $\mathcal{H}_\hbar \subset \mathrm{L}^2(\mathcal{M},\mathcal{L}_\hbar)$ has been constructed. Let

$$\Pi_\hbar : \mathrm{L}^2\left(\mathcal{M},\mathcal{L}_\hbar\right) \to \mathcal{H}_\hbar \tag{38}$$

be the projector to this Hilbert space. With the given Hilbert space $\mathcal{H}_\hbar$, we also obtain a quantization map [22].

Definition 16. The *(Berezin-)Toeplitz quantization* map $T_\hbar$ assigns a bounded operator on the Hilbert space $\mathcal{H}_\hbar$ to every classical observable,

$$T_\hbar : \mathcal{A}_0 \to \mathcal{B}(\mathcal{H}_\hbar), \tag{39}$$

using the projection $\Pi_\hbar$ from the space of square-integrable sections $\mathrm{L}^2(\mathcal{M}, \mathcal{L}_\hbar)$ such that

$$\forall \psi \in \mathcal{H}_\hbar : \qquad T_\hbar(f)\psi = \Pi_\hbar(f\psi). \tag{40}$$

For each $\hbar \in I$, we also choose an algebra $\mathfrak{A}_\hbar \subseteq \mathcal{B}(\mathcal{H}_\hbar)$ such that it contains the image of $T_\hbar$. The Toeplitz quantization map $T_\hbar$ is linear and respects involution. Its domain actually extends to all bounded functions $\mathrm{C_b}(\mathcal{M}, \mathbb{C})$ so that $1 \in \mathrm{C_b}(\mathcal{M}, \mathbb{C})$ is mapped to $\mathbf{1} \in \mathcal{B}(\mathcal{H}_\hbar)$. However, it will be easier to use the C^*-algebra of compact operators $\mathfrak{A}_\hbar = \mathcal{K}(\mathcal{H}_\hbar)$ in the construction of a continuous field of C^*-algebras. Note that $\mathfrak{A}_\hbar$ coincides with $\mathcal{B}(\mathcal{H}_\hbar)$ if $\dim \mathcal{H}_\hbar < \infty$. Furthermore, it will also be necessary to restrict to a dense subalgebra $\mathcal{A}_0 \subset \mathfrak{A}_0$ for the construction of formal deformation quantizations (star products).

If the Toeplitz operators of compactly supported functions $\mathrm{C_c}(\mathcal{M}, \mathbb{C})$ on the physical Hilbert space $\mathcal{H}_\hbar$ are of trace-class, we define a measure $\mu_\hbar$ such that

$$\mathrm{Tr}(T_\hbar(f)) = \int_\mathcal{M} f \, \mathrm{d}\mu_\hbar \tag{41}$$

holds for all $f \in \mathrm{C_c}(\mathcal{M}, \mathbb{C})$. When such a measure exists, we have an adjoint operation to Toeplitz quantization.

Definition 17. Suppose the measure $\mu_\hbar$ determined by (41) exists. The *(Berezin)-Toeplitz dequantization* is a family of linear maps

$$\Xi_\hbar : \mathfrak{A}_\hbar \to \mathfrak{A}_0, \tag{42}$$

such that for all complex-valued, compactly supported functions $f \in \mathrm{C_c}(\mathcal{M}, \mathbb{C})$ and all operators $A_\hbar \in \mathfrak{A}_\hbar$

$$\mathrm{Tr}(A_\hbar T_\hbar(f)) = \int_\mathcal{M} \Xi_\hbar(A_\hbar) f \, \mathrm{d}\mu_\hbar. \tag{43}$$

Consider the case of a symplectic manifold with a physical Hilbert space $\mathcal{H}_\hbar$ such that Toeplitz dequantization exists. If the algebras $\mathfrak{A}_\hbar$ are unital, then Toeplitz dequantization preserves the unit, $\Xi_\hbar(\mathbf{1}) = 1$ since the measure is normalized,

$$\forall f \in \mathrm{C_c}(\mathcal{M}, \mathbb{C}) : \qquad \int_\mathcal{S} \Xi_\hbar(\mathbf{1}) f \, \mathrm{d}\mu_\hbar = \int_\mathcal{S} f \, \mathrm{d}\mu_\hbar. \tag{44}$$

Applying dequantization to a Toeplitz operator $T_\hbar(f)$ yields a 'smearing' of the original function. For more details on Berezin–Toeplitz dequantization, see also [23].

Definition 18. The *Berezin transform* of a classical observable $f \in \mathcal{A}_0$ over the symplectic manifold $(\mathcal{M}, \omega)$ is the dequantization of its Toeplitz operator, $(\Xi_\hbar \circ T_\hbar)(f)$.

A function $f \in \mathcal{A}_0$ is sometimes referred to as the *contravariant* or *lower* symbol of the Toeplitz operator $T_\hbar(f)$, while the Berezin transform $(\Xi_\hbar \circ T_\hbar)(f)$ is also called the *covariant* or *upper* symbol of $T_\hbar(f)$ [24].

3.3. Laplacians on the quantization bundle

Given a symplectic manifold with Riemannian metric, we want to identify the physical Hilbert space as a subspace of quantization bundle sections that correspond to the lowest part of the spectrum of the Laplacian defined with the metric. In this section, we review some general arguments for symplectic manifolds to motivate a generalization of our results in section 4.

Definition 19. Let $(\mathcal{M},\omega,g)$ be a symplectic manifold with Riemannian metric, $\mathcal{L}_\hbar \to \mathcal{M}$ be a quantization bundle for some $\hbar \in I_*$. The *Bochner Laplacian* $\triangle_\hbar$ is an unbounded operator on square-integrable, smooth sections of the bundle. It is determined by the connection $\nabla_\hbar$ and metric g,

$$\triangle_\hbar = \nabla_\hbar^* \nabla_\hbar. \tag{45}$$

In the case of a $2N$-dimensional Kähler manifold, let ∇_i denote the holomorphic and $\nabla_{\bar{\jmath}}$ the anti-holomorphic components of the connection $\nabla_\hbar$, with $i \in [1,N]$, $\bar{\jmath} \in [\bar{1},\bar{N}]$. There is another, naturally defined Laplace operator, the Kodaira Laplacian

$$\triangle_\hbar^{\mathrm{K}} = -g^{i\bar{\jmath}}\nabla_i \nabla_{\bar{\jmath}} \tag{46}$$

using the summation convention. The Kodaira Laplacian is related to the Bochner Laplacian,

$$2\triangle_\hbar^{\mathrm{K}} = \triangle_\hbar - \frac{N}{\hbar}. \tag{47}$$

With the Kähler polarization, the physical Hilbert space is constructed from the space of holomorphically polarized sections with respect to the complex structure of the Kähler manifold. The kernel of the Kodaira Laplacian (46) contains the space of holomorphic sections. In fact, the kernel is precisely the space of holomorphic sections since the holomorphic components ∇_i are adjoint to the anti-holomorphic components $\nabla_{\bar{\imath}}$. The Kodaira Laplacian is positive, so the space of holomorphic sections is the eigenspace of the Bochner Laplacian corresponding to the lowest eigenvalue $\frac{N}{\hbar}$, see (47). Thus, the physical Hilbert space is equivalently determined from the spectrum of the Bochner Laplacian.

In [25], it was shown how to use a renormalized Bochner Laplacian for a natural generalization to almost Kähler manifolds (with a non-integrable, almost complex structure). The renormalized Bochner Laplacian is a generalization of the expression on the right hand side of (47) and coincides with $2\triangle_\hbar^{\mathrm{K}}$ in the Kähler case, see also [26]. A choice of a physical Hilbert space is again given by the eigenspace corresponding to the lowest part of the spectrum, even though the lowest part does not have to be a single eigenvalue anymore.

A further generalization starts with a symplectic manifold with Riemannian metric without pre-defined complex structure, but for bounded geometry at infinity, see [27–29]. For this, consider a $2N$-dimensional, compact, real, symplectic manifold $(\mathcal{M},\omega)$ with quantization bundle $\mathcal{L}_\hbar \to \mathcal{M}$ and Riemannian metric g. There exists an anti-self-adjoint linear map $E : \mathrm{T}\mathcal{M} \to \mathrm{T}\mathcal{M}$ such that for all $v_1, v_2 \in \mathrm{T}\mathcal{M}$

$$\omega(v_1,v_2) = g\left(v_1, E^{-1}v_2\right). \tag{48}$$

There exists an almost complex structure $J : \mathrm{T}\mathcal{M} \to \mathrm{T}\mathcal{M}$ such that $g(Jv_1,Jv_2) = g(v_1,v_2)$ and $\omega(Jv_1,Jv_2) = \omega(v_1,v_2)$ for all $v_1,v_2 \in \mathrm{T}\mathcal{M}$. We define a new metric η such that for all $v_1,v_2 \in \mathrm{T}\mathcal{M}$

$$\eta(v_1,v_2) := \omega(v_1,Jv_2). \tag{49}$$

The almost complex structure commutes with E^{-1}, so

$$J = -E^{-1}|E|. \tag{50}$$

At every point $x \in \mathcal{M}$, the operator E_x^{-1} is an endomorphism and we denote half the trace of $|E_x|^{-1}$ as

$$\lambda(x) := \frac{1}{2}\operatorname{tr}\left(J_x E_x^{-1}\right) = \frac{1}{2}\operatorname{tr}|E_x|^{-1}. \tag{51}$$

Note that in the special case of a Kähler manifold with a Kähler metric so that $E_x = \mathrm{id}$, this trace is N, half the real dimension of $\mathcal{M}$.

It was shown that the trace $\lambda(x)$ is positive for all $x \in \mathcal{M}$. A renormalized Bochner Laplacian $\triangle_{\hbar,\Phi}$ is then defined with λ and a smooth Hermitian section Φ on a tensor product of the quantization line bundle with a vector bundle, see [30]. They have shown—using Spin$^{\mathbb{C}}$ Dirac operators—that there exist two positive constants κ and μ that are independent of $\hbar$, such that the spectrum of the renormalized Bochner Laplacian fulfills

$$\mathrm{spec}(\triangle_{\hbar,\Phi}) \subset [-\kappa,\kappa] \cup \left[\frac{2\mu}{\hbar} - \kappa, \infty\right). \tag{52}$$

Given this spectrum condition for the renormalized Bochner Laplacian on a symplectic manifold with Riemannian metric $(\mathcal{M},\omega,g)$, the physical Hilbert space $\mathcal{H}_\hbar \subset \mathrm{L}^2(\mathcal{M},\mathcal{L}_\hbar)$ is spanned by the sections corresponding to the lower part of the spectrum i.e. the part contained in $[-\kappa,\kappa]$.

4. Geometric quantization for a symplectic vector space with inner product

For a symplectic vector space with inner product $(\mathcal{S},\omega,\langle\cdot,\cdot\rangle)$ as described in section 2.1, we now use the idea of the geometric construction to derive a physical Hilbert space $\mathcal{H}_\hbar$. The inner product corresponds to the metric g and we use a basis of holomorphic and anti-holomorphic vectors to express components with indices raised and lowered by g. This choice of a complex basis will allow us to write the sections of the Hilbert space as holomorphic sections with respect to the complex structure J given in (50). Further details on this construction are also given in [31, section 4.1.6] and [27, section 1.4].

4.1. The Bochner Laplacian and its spectrum

On the vector space $\mathcal{S}$, consider an exact symplectic form $\omega = -\mathrm{d}\theta$ such that we have a trivial line bundle $\mathcal{L}_\hbar := \mathcal{S} \times \mathbb{C}$ with non-trivial connection parametrized by $\hbar$,

$$\nabla_\hbar = \mathrm{d} + \frac{\mathrm{i}}{\hbar}\theta. \tag{53}$$

With the complex structure J given as in (50), we turn the real vector space $\mathcal{S}$ into a complex vector space $\mathcal{S}_J$ by the assignment (8). The operator $\nabla_\hbar$ increases the total degree $p+q$ of complex differential forms $\Omega^{p,q}$ by 1. We define operators $\mathcal{D}_\hbar^+ : \Omega^{p,q} \to \Omega^{p+1,q}$ raising the holomorphic degree and $\overline{\mathcal{D}}_\hbar^+ : \Omega^{p,q} \to \Omega^{p,q+1}$ raising the anti-holomorphic degree such that

$$\nabla_\hbar = \mathcal{D}_\hbar^+ + \overline{\mathcal{D}}_\hbar^+. \tag{54}$$

Use the Hodge dual operator $*:\Omega^{p,q}\to\Omega^{N-q,N-p}$ to define the adjoint operators

$$\mathcal{D}_\hbar^- := -*\overline{\mathcal{D}}_\hbar^+*, \qquad \overline{\mathcal{D}}_\hbar^- := -*\mathcal{D}_\hbar^+*. \tag{55}$$

Thus the Bochner Laplacian is given as

$$\triangle_\hbar = \mathcal{D}_\hbar^-\mathcal{D}_\hbar^+ + \mathcal{D}_\hbar^-\overline{\mathcal{D}}_\hbar^+ + \overline{\mathcal{D}}_\hbar^-\mathcal{D}_\hbar^+ + \overline{\mathcal{D}}_\hbar^-\overline{\mathcal{D}}_\hbar^+. \tag{56}$$

Note that we are want to act on $(0,0)$-forms, for which the two middle terms vanish.

For the following computation, we choose complex coordinates $z^i = x^i + \mathrm{i}y^i$ (with indices $i\in[1,N]$, $\bar\imath\in[\bar 1,\bar N]$) in which $|E|^{-1}$ is diagonal with diagonal components ϑ_i. The indices are raised with the inverse metric $g^{\bar\imath i}$ derived from the inner product on $\mathcal{S}$. Raising an index also changes it from holomorphic to anti-holomorphic and vice versa. For a slightly compacter notation, we omit the $\hbar$ subscript whenever the connection $\nabla_\hbar$ is expressed in coordinates.

The difference of the first and last operator pair in (56) is

$$\mathcal{D}_\hbar^-\mathcal{D}_\hbar^+ - \overline{\mathcal{D}}_\hbar^-\overline{\mathcal{D}}_\hbar^+ = g^{\bar\imath j}\left[\nabla_j,\nabla_{\bar\imath}\right]_- = -\frac{\mathrm{i}}{\hbar}g^{\bar\imath j}\omega_{j\bar\imath} =: \lambda_\hbar. \tag{57}$$

We use this identity to replace the first operator pair of the Bochner Laplacian (45) by the anti-holomorphic operators $\overline{\mathcal{D}}_\hbar^\pm$ along with the positive shift constant $\lambda_\hbar$, such that we obtain for $(0,0)$-forms

$$\triangle_\hbar = 2\overline{\mathcal{D}}_\hbar^-\overline{\mathcal{D}}_\hbar^+ + \lambda_\hbar\mathbf{1}, \tag{58}$$

which is relating the Bochner and Kodaira Laplacians as in the case of a Kähler space (47), see also [31, equation (1.4.31)]. The constant $\lambda_\hbar$ is (up to the quantization parameter $\hbar$) half the trace of $|E|^{-1}$,

$$\lambda_\hbar = \frac{1}{2\hbar}\operatorname{tr}|E|^{-1}. \tag{59}$$

In our choice of complex coordinates such that $|E|^{-1}$ is diagonal, this constant is half the sum over all the diagonal components divided by $\hbar$.

Combine the operators $\overline{\mathcal{D}}_\hbar^\pm:\Omega^{0,q}\to\Omega^{0,q\pm1}$ that increase and decrease the anti-holomorphic degree to a self-adjoint operator,

$$\overline{\mathcal{D}}_\hbar := \overline{\mathcal{D}}_\hbar^+ + \overline{\mathcal{D}}_\hbar^-. \tag{60}$$

So the Laplacian acting on $(0,0)$-forms $\psi\in\Omega^{0,0}$ becomes

$$\triangle_\hbar\psi = 2\overline{\mathcal{D}}_\hbar^2\psi + \lambda_\hbar\psi = -2g^{i\bar\jmath}\nabla_i\nabla_{\bar\jmath}\psi + \lambda_\hbar\psi. \tag{61}$$

Since by construction the Laplacian and the operator $\overline{\mathcal{D}}_\hbar^2$ are self-adjoint and positive, $\lambda_\hbar$ is the lower bound on the spectrum of the Laplacian. We express the components of the symmetric bilinear form $g^{i\bar\jmath}$ (corresponding to the inner product $\langle\cdot,\cdot\rangle$) in terms of the diagonal components ϑ_i of $|E|^{-1}$,

$$\begin{aligned}\left(g_{i\bar\jmath}\right)_{i\in[1,N],\bar\jmath\in[\bar 1,\bar N]} &= \operatorname{diag}\left(\frac{1}{\vartheta_1},\frac{1}{\vartheta_2},\dots,\frac{1}{\vartheta_N}\right),\\ \left(g^{i\bar\jmath}\right)_{i\in[1,N],\bar\jmath\in[\bar 1,\bar N]} &= \operatorname{diag}\left(\vartheta_1,\vartheta_2,\dots,\vartheta_N\right).\end{aligned} \tag{62}$$

The components of the covariant derivative fulfill the commutation relations (see also [31, equation (4.1.75)] and [27, equation (1.86)])

$$[\nabla_i, \nabla_j]_- = [\nabla_{\bar{\imath}}, \nabla_{\bar{\jmath}}]_- = 0, \qquad [\nabla_i, \nabla_{\bar{\jmath}}]_- = -\frac{\mathrm{i}}{\hbar}\omega_{i\bar{\jmath}} = \frac{1}{\hbar}\delta_{i\bar{\jmath}}. \tag{63}$$

With the component representation as given here, we find the full spectrum of the Bochner Laplacian, see [31, theorem 4.1.20] and [27, theorem 1.15]).

Theorem 1. *Let $\triangle_\hbar$ be the Bochner Laplacian for square-integrable sections of the quantization bundle $\mathcal{L}_\hbar \to \mathcal{S}$ over a real 2N-dimensional symplectic vector space $(\mathcal{S}, \omega, \langle\cdot,\cdot\rangle)$ with a non-degenerate symplectic form ω and an inner product $\langle\cdot,\cdot\rangle$. Let ϑ_i be the diagonal components as given in* (62)*. The spectrum of the Laplacian is given by*

$$\operatorname{spec}(\triangle_\hbar) = \left\{ \frac{1}{\hbar}\sum_{i=1}^{N}(2n_i+1)\,\vartheta_i \,\middle|\, n_i \in \mathbb{N} \right\}. \tag{64}$$

Figure 3 depicts the spectrum, which gets denser towards infinity. Note that the spectrum (64) does not depend on the complex structure J. We use the space of eigensections at the lowest spectral value as the physical Hilbert space $\mathcal{H}_\hbar$ on which bounded operators act as quantized observables. The complex structure J is determined by the operator E^{-1} in (50) such that the eigenspace of the lowest spectral value corresponds to the holomorphic sections (66).

4.2. The physical Hilbert space

The canonical and real-valued form of the symplectic potential corresponding to the chosen complex coordinates is

$$\theta = \frac{\mathrm{i}}{2}\delta_{i\bar{\jmath}}\left(\bar{z}^{\bar{\jmath}}\mathrm{d}z^i - z^i\mathrm{d}\bar{z}^{\bar{\jmath}}\right), \tag{65}$$

and we write $|z|^2 = \delta_{i\bar{\imath}}z^i\bar{z}^{\bar{\imath}}$. In this gauge, any holomorphic section ψ has an arbitrary, smooth, holomorphic function α as amplitude, so that for all $z \in \mathcal{S}$

$$\psi(z) = \frac{\alpha(z)}{\sqrt{2\pi\hbar}^N}\exp\left(-\frac{1}{2\hbar}|z|^2\right), \tag{66}$$

and it is a solution of the differential equation

$$\overline{\mathcal{D}}_\hbar^- \overline{\mathcal{D}}_\hbar^+ \psi = -g^{i\bar{\jmath}}\nabla_i\nabla_{\bar{\jmath}}\psi = 0. \tag{67}$$

The physical Hilbert space $\mathcal{H}_\hbar \subset \mathrm{L}^2(\mathcal{S}, \mathcal{L}_\hbar)$ is spanned by the sections corresponding to the lowest part of the spectrum, these are the *holomorphic* sections of the quantization bundle $\mathcal{L}_\hbar$ that take the form (66) in the complex coordinates.

The Hilbert space has the inner product

$$\langle\psi_1|\psi_2\rangle_\hbar = \int_{\mathcal{S}}\overline{\psi_1}\psi_2\,\mathrm{dvol}, \qquad \mathrm{dvol} = (-1)^{\frac{1}{2}N(N-1)}\frac{1}{N!}\bigwedge_{i=1}^{N}\omega. \tag{68}$$

Let $\mathfrak{A}_0$ be the C*-algebra $\mathrm{C}_0(\mathcal{S}, \mathbb{C})$ and $\mathfrak{A}_\hbar$ be the C*-algebra of compact operators $\mathcal{K}(\mathcal{H}_\hbar)$. Define Toeplitz quantization $T_\hbar : \mathrm{C}_{\mathrm{S}}^\infty(\mathcal{S}, \mathbb{C}) \to \mathfrak{A}_\hbar$ as in definition 16 with the projector $\Pi_\hbar$

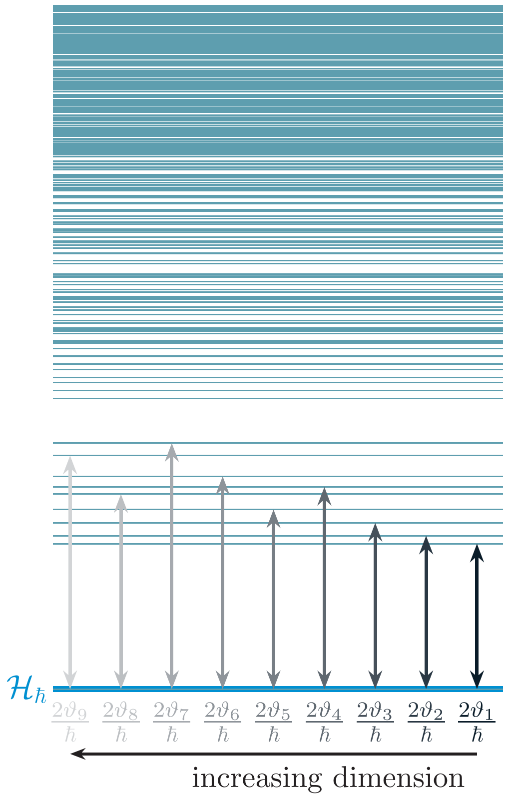

Figure 3. Illustration of the spectrum of the Bochner Laplacian $\triangle_\hbar = \nabla_\hbar^*\nabla_\hbar$ for a symplectic vector space with 18 dimensions ($N=9$) and diagonal metric components ϑ_i. The Hilbert space $\mathcal{H}_\hbar$ is constructed from the section (66) corresponding to the lowest spectral value (cyan) is the solution.

as defined in (38). Note that Toeplitz quantization actually extends to a map from bounded functions to bounded operators, $\mathrm{C_b}(\mathcal{S},\mathbb{C}) \to \mathcal{B}(\mathcal{H}_\hbar)$, but we will restrict to Schwartz functions to construct a continuous field of C*-algebras later on.

For any two holomorphic sections $\psi_{1,2}$ as in (66) with smooth, holomorphic functions $\alpha_{1,2}$ as amplitudes, the inner product reads

$$\langle\psi_1|\psi_2\rangle_\hbar = \frac{1}{(2\pi\hbar)^N}\int_{\mathcal{S}} \overline{\alpha_1}\alpha_2 \exp\left(-\frac{1}{\hbar}|z|^2\right) \mathrm{dvol}. \tag{69}$$

Use the section basis given in ket-notation for any $n_1,\ldots,n_N \in \mathbb{N}$, similar to [31, equation (4.1.83)] and [27, equation (1.89)],

$$|n_1,\ldots,n_N\rangle_\hbar := \left|\frac{1}{\sqrt{2\pi\hbar}^N}\left(\prod_{i=1}^N \frac{1}{\sqrt{n_i!}}\left(\frac{z^i}{\sqrt{\hbar}}\right)^{n_i}\right)\exp\left(-\frac{1}{2\hbar}|z|^2\right)\right\rangle_\hbar . \tag{70}$$

This basis is orthonormal,

$$\langle m_1,\ldots,m_N|n_1,\ldots,n_N\rangle_\hbar = \prod_{i=1}^{N}\delta_{m_in_i}. \tag{71}$$

We may now define unbounded ladder operators (in the summation convention), see also [31, equation (4.1.74)] and [27, equation (1.85)],

$$a_{\bar\imath}^{+} := \frac{1}{\sqrt{\hbar}}\delta_{\bar\imath i}z^{i} - \sqrt{\hbar}\nabla_{\bar\imath}, \qquad a_{i}^{-} := \frac{1}{\sqrt{\hbar}}\delta_{i\bar\imath}\bar z^{\bar\imath} + \sqrt{\hbar}\nabla_{i}, \tag{72}$$

which are adjoint to each other for each index pair $i=\bar\imath$. Using the commutators of the quantization bundle connection (63), we find the commutators for the ladder operators to be those known from an N-dimensional quantum mechanical harmonic oscillator,

$$\left[a_i^{\pm},a_j^{\pm}\right]_- = 0, \qquad \left[a_i^{-},a_{\bar\jmath}^{+}\right]_- = \delta_{i\bar\jmath}\mathbf{1}. \tag{73}$$

The action of the ladder operators on the Hilbert space basis yields

$$\begin{aligned} a_i^{+}\left|n_1,\ldots,n_i,\ldots,n_N\right\rangle_\hbar &= \sqrt{n_i+1}\left|n_1,\ldots,n_i+1,\ldots,n_N\right\rangle_\hbar, \\ a_i^{-}\left|n_1,\ldots,n_i,\ldots,n_N\right\rangle_\hbar &= \sqrt{n_i}\left|n_1,\ldots,n_i-1,\ldots,n_N\right\rangle_\hbar. \end{aligned} \tag{74}$$

The heuristic relation between the Toeplitz operators of the (unbounded) coordinate functions z^i and $\bar z^{\bar\imath}$ with the ladder operators as well as examples for anti-normal ordering of Toeplitz quantization are given in the appendix.

Before we come to the construction of a continuous field of C*-algebras from the family of Hilbert spaces parametrized in $\hbar$, we use dequantization to determine the explicit form of the Berezin transform in the following. Dequantization is the operation that leads us to the definition of a state—the Sorkin–Johnston state.

4.3. Dequantization and the Berezin transform

We defined the adjoint map $\Xi_\hbar : \mathfrak{A}_\hbar \to \mathfrak{A}_0$ to Toeplitz quantization by the relation (43), which becomes

$$\mathrm{Tr}\left(A_\hbar T_\hbar(f)\right) = \frac{1}{(2\pi\hbar)^N}\int_{\mathcal{S}} \Xi_\hbar(A_\hbar) f\,\mathrm{dvol}. \tag{75}$$

for the $2N$-dimensional vector space $\mathcal{S}$. Similarly to Toeplitz quantization, we may extend the domain of Berezin–Toeplitz dequantization to all bounded operators, however, the trace is only partially defined. For all trace-class operators $A_\hbar$ and all complex-valued Schwartz functions $f\in \mathrm{C}^\infty_{\mathrm{S}}(\mathcal{S},\mathbb{C})$, the trace on the left hand side written in the section basis (70) becomes

$$\mathrm{Tr}\left(A_\hbar T_\hbar(f)\right) = \sum_{n_1,\ldots,n_N=0}^{\infty} \langle n_1,\ldots,n_N|_\hbar A_\hbar T_\hbar(f)\left|n_1,\ldots,n_N\right\rangle_\hbar. \tag{76}$$

Dequantizing a projector $\left|j_1,\ldots,j_N\right\rangle_\hbar \langle j_1,\ldots,j_N|_\hbar$ for any $j_1,\ldots,j_N\in\mathbb{N}$ yields

$$\Xi_\hbar\left(|j_1,\ldots,j_N\rangle_\hbar\langle j_1,\ldots,j_N|_\hbar\right)(z)=\exp\left(-\frac{1}{\hbar}|z|^2\right)\prod_{k=1}^{N}\frac{1}{j_k!}\left(\frac{|z^k|^2}{\hbar}\right)^{j_k}. \tag{77}$$

We use this as a consistency check and notice that the identity operator $\mathbf{1}\in\mathcal{B}(\mathcal{H}_\hbar)$ dequantizes to the constant function 1 when we extend the dequantization map to the domain of bounded operators and codomain of bounded functions.

We define the Berezin transform kernel $b_\hbar$ from the exponential factor in (77) such that for any $z\in\mathcal{S}$:

$$b_\hbar(z):=\frac{1}{(2\pi\hbar)^N}\exp\left(-\frac{1}{\hbar}|z|^2\right). \tag{78}$$

Let $\circledast$ denote the convolution product between any pair of functions $f_1,f_2\in\mathrm{L}^1(\mathcal{S},\mathbb{C})$ such that

$$(f_1\circledast f_2)(z)=\int_{\mathcal{S}}f_1(z-z')f_2(z')\,\mathrm{dvol}(z'). \tag{79}$$

By expanding Toeplitz operators in terms of the projectors $|j_1,\ldots,j_N\rangle_\hbar\langle j_1,\ldots,j_N|_\hbar$, we derive the explicit expression for the Berezin transform of any Schwartz function $f\in\mathrm{C}^\infty_{\mathrm{S}}(\mathcal{S},\mathbb{C})$ (or even any bounded function) as a convolution with the Berezin kernel (78),

$$(\Xi_\hbar\circ T_\hbar)(f)=b_\hbar\circledast f, \tag{80}$$

which is also an element of $\mathrm{C}^\infty_{\mathrm{b}}(\mathcal{S},\mathbb{C})$. This follows also from the integral kernels of the projectors, see also [31, equation (4.1.84)] and [27, equation (1.91)], since for all $f,g\in\mathrm{C}^\infty_{\mathrm{S}}(\mathcal{S},\mathbb{C})$:

$$\begin{aligned}\mathrm{Tr}(T_\hbar(f)T_\hbar(g))&=\int_{\mathcal{S}}(\Xi_\hbar\circ T_\hbar)(f)(z)\,g(z)\,\mathrm{d}\mu_\hbar(z)\\&=\iint_{\mathcal{S}}\Pi_\hbar(z,z')f(z')\,\Pi_\hbar(z',z)\,g(z)\,\mathrm{d}\mu_\hbar(z)\,\mathrm{d}\mu_\hbar(z')\\(\Xi_\hbar\circ T_\hbar)(f)(z)&=\int_{\mathcal{S}}\underbrace{\Pi_\hbar(z',z)\,\Pi_\hbar(z,z')}_{b_\hbar(z-z')}f(z')\,\mathrm{d}\mu_\hbar(z').\end{aligned} \tag{81}$$

There is an example for a Berezin transform given in the appendix. Note that in the classical limit $\hbar\to0$, the Gaussian function (78) converges to a delta distribution—the identity with respect to convolution, thus

$$\lim_{\hbar\to0}(\Xi_\hbar\circ T_\hbar)(f)=f. \tag{82}$$

These observations are useful for the construction of a continuous field of C^*-algebras (including the classical limit $\hbar=0$) further below.

5. The Sorkin–Johnston state, and the star products of (Berezin-)Toeplitz (de)quantization

To define a state with dequantization and prove that it is the same quasi-free state as the Sorkin–Johnston state, we first need to understand the relationship between the Weyl quantization and the Berezin–Toeplitz dequantization map. We also use the Weyl generators in the proofs of strict deformation quantizations further below.

5.1. The Weyl algebra and its relation to (Berezin-)Toeplitz (de)quantization

Lemma 2. *For $\phi \in \mathcal{S}^* = \mathrm{Hom}(\mathcal{S},\mathbb{R})$, denote the complex components as $\phi_i \in \mathbb{C}$ such that (in the summation convention)*

$$\phi(z) = \phi_i z^i + \overline{\phi}_{\bar{\imath}} \bar{z}^{\bar{\imath}}. \tag{83}$$

With this notation, denote a corresponding linear combination of the creation $a^+_{\bar{\jmath}}$ and annihilation operators a^-_j by

$$\varPhi_\hbar(\phi) := \sqrt{\hbar}\, \delta^{i\bar{\imath}} \left(\phi_i a^+_{\bar{\imath}} + \overline{\phi}_{\bar{\imath}} a^-_i\right). \tag{84}$$

Define a function $W_\hbar : \mathcal{S}^* \to \mathcal{B}(\mathcal{H}_\hbar)$ by

$$W_\hbar(\phi) := \exp\left(\mathrm{i}\varPhi_\hbar(\phi)\right). \tag{85}$$

These operators $W_\hbar(\phi)$ fulfill the Weyl relations given in (12a).

Proof. The unit of the Weyl algebra (12*a*) is obviously given by $W_\hbar(0)$. For the involution, note that $(a^\pm_j)^* = a^\mp_{\bar{\jmath}}$ and thus (84) is self-adjoint. Hence

$$W_\hbar(\phi)^* = W_\hbar(-\phi). \tag{86}$$

For the product of two generators, compute the commutator

$$\begin{aligned}
[\varPhi_\hbar(\phi), \varPhi_\hbar(\phi')]_- &= \hbar\delta^{i\bar{\imath}}\delta^{j\bar{\jmath}} \left[\phi_i a^+_{\bar{\imath}} + \overline{\phi}_{\bar{\imath}} a^-_i, \phi'_j a^+_{\bar{\jmath}} + \overline{\phi'}_{\bar{\jmath}} a^-_j\right]_- \\
&= -\hbar\delta^{i\bar{\imath}}\delta^{j\bar{\jmath}} \left(\phi_i \overline{\phi'}_{\bar{\jmath}} - \overline{\phi}_{\bar{\imath}} \phi'_j\right) \delta_{\bar{\imath}j} \mathbf{1} \\
&= \mathrm{i}\hbar \{\phi, \phi'\} \mathbf{1}.
\end{aligned} \tag{87}$$

It is seen that the commutator of this expression with $\varPhi_\hbar(\phi)$ vanishes so that the Baker-Campbell-Hausdorff formula yields

$$\exp\left(\mathrm{i}\varPhi_\hbar(\phi)\right)\exp\left(\mathrm{i}\varPhi_\hbar(\phi')\right) = \exp\left(-\frac{1}{2}\left[\varPhi_\hbar(\phi), \varPhi_\hbar(\phi')\right]_-\right)\exp\left(\mathrm{i}\varPhi_\hbar(\phi+\phi')\right). \tag{88}$$

Replace the commutator in (88) with expression (87) to show that the generators (85) also fulfill (12*a*), and thus all Weyl relations (12). □

The Berezin–Toeplitz quantization respects anti-normal ordering and dequantization respects normal ordering (see also the appendix). To reorder the terms in the series expansion of a Weyl generator (85), we use the commutation relations (73) and derive commutators for powers of ladder operators in anti-normal or normal order. For an index pair $(i,\bar{\imath}) \in \{(1,\bar{1}),(2,\bar{2}),\dots,(N,\bar{N})\}$, the two orders of the commutators are, respectively,

$$\left[\left(a_i^-\right)^m, \left(a_{\bar{\imath}}^+\right)^n\right]_- = \sum_{l=1}^{\min(m,n)} (-1)^{l+1}\, l! \binom{m}{l}\binom{n}{l} \left(a_i^-\right)^{m-l}\left(a_{\bar{\imath}}^+\right)^{n-l},$$

$$\left[\left(a_i^-\right)^m, \left(a_{\bar{\imath}}^+\right)^n\right]_- = \sum_{l=1}^{\min(m,n)} l! \binom{m}{l}\binom{n}{l} \left(a_{\bar{\imath}}^+\right)^{n-l}\left(a_i^-\right)^{m-l}, \tag{89}$$

while for all index pairs $(i,\bar{\imath}) \notin \{(1,\bar{1}),(2,\bar{2}),\dots,(N,\bar{N})\}$ these commutators vanish. For the second order term, for example, the reorder yields one extra term that is the same for quantization and dequantization but with opposite sign,

$$\begin{aligned}\left(\mathrm{i}\Phi_\hbar(\phi)\right)^2 &= -\hbar\delta^{i\bar{\imath}}\delta^{j\bar{\jmath}}\Big(\underbrace{\phi_i\phi_j a_{\bar{\imath}}^+ a_{\bar{\jmath}}^+ + 2\overline{\phi}_{\bar{\imath}}\phi_j a_i^- a_{\bar{\jmath}}^+ + \overline{\phi}_{\bar{\imath}}\overline{\phi}_{\bar{\jmath}} a_i^- a_j^-}_{\text{anti-normal ordered, by } T\text{-quantization}} - \overline{\phi}_{\bar{\imath}}\phi_j\delta_{i\bar{\jmath}}\mathbf{1}\Big)\\ &= -\hbar\delta^{i\bar{\imath}}\delta^{j\bar{\jmath}}\Big(\underbrace{\phi_i\phi_j a_{\bar{\imath}}^+ a_{\bar{\jmath}}^+ + 2\phi_i\overline{\phi}_{\bar{\jmath}} a_{\bar{\imath}}^+ a_j^- + \overline{\phi}_{\bar{\imath}}\overline{\phi}_{\bar{\jmath}} a_i^- a_j^-}_{\text{normal ordered, for } \Xi\text{-dequantization}} + \phi_i\overline{\phi}_{\bar{\jmath}}\delta_{i\bar{\jmath}}\mathbf{1}\Big).\end{aligned} \tag{90}$$

The extra terms of all orders yield an exponential amplitude factor depending on $\hbar$ and $|\phi|^2 = \sum_{i=1}^N |\phi_i|^2$,

$$W_\hbar(\phi) = \exp\left(\frac{\hbar}{2}|\phi|^2\right) T_\hbar\left(\mathrm{e}^{\mathrm{i}\phi}\right). \tag{91}$$

Similarly, dequantization of the Weyl generators gives

$$\Xi_\hbar\left(W_\hbar(\phi)\right) = \exp\left(-\frac{\hbar}{2}|\phi|^2\right)\mathrm{e}^{\mathrm{i}\phi}. \tag{92}$$

Like the Toeplitz sections given in (22), every Weyl generator forms a vector field in $\prod_{\hbar\in I}\mathcal{A}_\hbar$.

Definition 20. The *Weyl section* of a covector $\phi \in \mathcal{S}^* = \mathrm{Hom}(\mathcal{S},\mathbb{R})$ is

$$W(\phi) : \hbar \mapsto \begin{cases} \mathrm{e}^{\mathrm{i}\phi} & \hbar = 0,\\ W_\hbar(\phi) & \hbar > 0.\end{cases} \tag{93}$$

In the limit $\hbar \to 0$, the quantization (91) and dequantization (92) coincide with $\exp(\mathrm{i}\phi)$, which is used in showing that the Weyl map is a strict deformation quantization and also continuous in the classical limit.

Any Toeplitz operator of a Schwartz function $f \in \mathrm{C}^\infty_{\mathrm{S}}(\mathcal{S},\mathbb{C})$ can also be written in terms of Weyl generators. For this, consider the Fourier transform, which is an automorphism on $\mathrm{C}^\infty_{\mathrm{S}}(\mathcal{S},\mathbb{C})$,

$$\hat{f}(\phi) = \frac{1}{(2\pi)^{2N}}\int_{\mathcal{S}} f(z)\,\mathrm{e}^{-\mathrm{i}\phi(z)}\,\mathrm{dvol}(z), \tag{94}$$

and its inverse transform (with the volume form dvol^* on $\mathcal{S}^*$),

$$f(z) = \int_{\mathcal{S}^*} \hat{f}(\phi)\, \mathrm{e}^{\mathrm{i}\phi(z)}\, \mathrm{dvol}^*(\phi)\,. \tag{95}$$

Recall that the result (92) is related to (91) by a Berezin transform. The Weyl quantization is related to 'half' a Berezin transform, so consider

$$\sqrt{\widehat{b_\hbar}}(\phi) = \frac{1}{(2\pi)^N} \exp\left(-\frac{\hbar}{2}|\phi|^2\right). \tag{96}$$

Note that the exponential function here is exactly the same as in the dequantization (92). The Toeplitz operator of f is then

$$\begin{aligned} T_\hbar(f) &= \int_{\mathcal{S}^*} \hat{f}(\phi)\, T_\hbar\left(\mathrm{e}^{\mathrm{i}\phi}\right) \mathrm{dvol}^*(\phi) \\ &= (2\pi)^N \int_{\mathcal{S}^*} \hat{f}(\phi) \sqrt{\widehat{b_\hbar}}(\phi)\, W_\hbar(\phi)\, \mathrm{dvol}^*(\phi)\,. \end{aligned} \tag{97}$$

Note that in the classical limit $\hbar \to 0, f_\hbar \to f$ similarly to the limit of the Berezin transform, the left hand side of (97) becomes f, and the right hand side of (97) becomes the inverse Fourier transform (95). More details on the Weyl algebra and its relation to Toeplitz operators are given in [13, chapter II].

5.2. *The Sorkin–Johnston state from dequantization*

Now we use dequantization to define a state and compare its properties to the Sorkin–Johnston state using the Weyl algebra and the relation to Berezin–Toeplitz dequantization.

Theorem 3. *The linear map $\sigma_\hbar : \mathfrak{A}_\hbar \to \mathbb{C}$ given by*

$$\sigma_\hbar(A) := \Xi_\hbar(A)(0) \tag{98}$$

is the Sorkin–Johnston state.

Proof. In order to show that this map is the Sorkin–Johnston state (34), we need to evaluate it on Weyl generators. Recall the result (92) when dequantizing the Weyl generator $W_\hbar(\phi)$ of any covector $\phi \in \mathcal{S}^* = \mathrm{Hom}(\mathcal{S}, \mathbb{R})$. Evaluation at $0 \in \mathcal{S}$ yields

$$\sigma_\hbar(W_\hbar(\phi)) = \exp\left(-\frac{\hbar}{2}|\phi|^2\right). \tag{99}$$

In order to compare it with (34), notice that

$$\eta^{-1}(\phi, \phi) = \delta^{i\bar{\imath}} \phi_i \overline{\phi}_{\bar{\imath}} + \delta^{\bar{\imath}i} \overline{\phi}_{\bar{\imath}} \phi_i = 2|\phi|^2 \tag{100}$$

so that

$$|\phi|^2 = \frac{1}{2}\eta^{-1}(\phi, \phi)\,. \tag{101}$$

We obtain the inverse of the bi-linear form η on $\mathcal{S}$, which is identical to the covariance of the Sorkin–Johnston state (34). The form η is compatible with the complex structure J yielding a Kähler vector space $(\mathcal{S}, \omega, \eta, J)$. □

For any Toeplitz operator $T_\hbar(f) \in \mathfrak{A}_\hbar$, the Sorkin–Johnston state $\sigma_\hbar$ is the Berezin transform of $f \in \mathcal{A}_0$ evaluated at 0,

$$\sigma_\hbar\left(T_\hbar(f)\right) = \int_{\mathcal{S}} b_\hbar(z) f(z)\, \mathrm{dvol}(z)\,. \tag{102}$$

Note that the dequantization state is parametrized by $\hbar$, so $(\sigma_\hbar)_{\hbar\in I}$ is a family of states with the classical limit $\sigma_0(f) := f(0)$. For any section A of the continuous field of C*-algebras $\left(I, (\mathfrak{A}_\hbar)_{\hbar\in I}, \varGamma\right)$, the map $\sigma(A): I \to \mathbb{C}$ given by

$$\sigma(A)(\hbar) := \sigma_\hbar(A(\hbar)) \tag{103}$$

is continuous since $\varXi(A): \mathcal{M}\times I \to \mathbb{C}$ defined as in (30) is continuous.

5.3. Infinite order strict deformation (de)quantization

Let $(\mathfrak{A}_\hbar)_{\hbar\in I}$ be the family of C*-algebras with $\mathfrak{A}_0 = \mathrm{C}_0(\mathcal{S},\mathbb{C})$ — as the closure of $\mathcal{A}_0 = \mathrm{C}^\infty_{\mathrm{S}}(\mathcal{S},\mathbb{C})$ — and compact operators $\mathfrak{A}_\hbar = \mathcal{K}(\mathcal{H}_\hbar)$ for $\hbar \in I_*$. There exists a continuous field of C*-algebras $\left(I, (\mathfrak{A}_\hbar)_{\hbar\in I}, \varGamma\right)$ equivalently determined by the Weyl sections and the Toeplitz sections [13, chapter II, section 2.6]. As an instance of our general discussion of dequantization-expandibility in definition 8, we will show that Toeplitz sections of Schwartz functions are $\varXi$-dequantization expandable sections.

In the proofs below, we have to bound kth order remainders $\mathrm{er}_k \in \mathrm{C}^\infty(\mathbb{C},\mathbb{C})$ for the Taylor expansion of the exponential function,

$$\mathrm{er}_k(\zeta) = \mathrm{e}^\zeta - \sum_{i=0}^{k} \frac{\zeta^j}{j!}. \tag{104}$$

Lemma 4. *For every $k \in \mathbb{N}$, there exists a real constant $C_k > 0$ such that for all $\zeta \in \mathbb{C}$*

$$\left|\mathrm{er}_k(\zeta)\right| \leqslant C_k \left(1 + \mathrm{e}^{\mathrm{Re}\,\zeta}\right) |\zeta|^{k+1}. \tag{105}$$

Proof. Taylor's theorem states that

$$\left|\mathrm{er}_k(\zeta)\right| = \mathcal{O}\left(|\zeta|^{k+1}\right) \tag{106}$$

as $|\zeta|$ becomes small. To find a bound for $|\zeta| \to \infty$, use the triangle inequality,

$$\left|\mathrm{er}_k(\zeta)\right| \leqslant \mathrm{e}^{\mathrm{Re}\,\zeta} + \sum_{j=0}^{k} \frac{|\zeta|^k}{j!}. \tag{107}$$

We add the two bounds (106) and (107) together to obtain (105). □

Proposition 5. *Given any Schwartz function $f \in \mathrm{C}^\infty_{\mathrm{S}}(\mathcal{S},\mathbb{C})$, the corresponding Toeplitz section T(f) is $\varXi$-expandable.*

Proof. Use the Toeplitz section $A = T(f)$ in (27) — setting the dequantization $\varUpsilon = \varXi$ — to obtain a condition for the Berezin transform of f,

$$\lim_{\hbar\to 0} \frac{1}{\hbar^k} \left\| (\varXi_\hbar \circ T_\hbar)(f) - \sum_{j=0}^{k} f_j \hbar^j \right\| = 0. \tag{108}$$

This condition is fulfilled by the functions

$$f_j(z) = \frac{1}{j!}\left(\delta^{i\bar{\imath}}\frac{\partial}{\partial z^i}\frac{\partial}{\partial \bar{z}^i}\right)^j f(z) \tag{109}$$

for all orders $j,k \in \mathbb{N}$. To show that the functions f_j indeed fulfill the condition, first consider a Schwartz function $f \in \mathrm{C}^\infty_{\mathrm{S}}(\mathcal{S},\mathbb{C})$, use the Fourier transforms $\hat{f}$ and $\hat{b}_\hbar(\phi) = (2\pi)^{2N}\exp(-\hbar|\phi|^2)$ with the convolution theorem. The derivatives in (109) become $\mathrm{i}\phi_i$ and $\mathrm{i}\overline{\phi}_{\bar{\imath}}$, respectively, so

$$\begin{aligned}\left\|(\Xi_\hbar \circ T_\hbar)(f) - \sum_{j=0}^{k} f_j \hbar^j\right\| &= \left\|\int_{\mathcal{S}^*} \mathrm{er}_k\left(-\hbar|\phi|^2\right)\hat{f}(\phi)\,\mathrm{e}^{\mathrm{i}\phi}\,\mathrm{dvol}^*(\phi)\right\| \\ &\leqslant \int_{\mathcal{S}^*}\left|\mathrm{er}_k\left(-\hbar|\phi|^2\right)\right|\left|\hat{f}(\phi)\right|\mathrm{dvol}^*(\phi). \end{aligned} \tag{110}$$

Now apply Lemma 4 and note that here the argument ζ is non-positive, such that

$$\left\|(\Xi_\hbar \circ T_\hbar)(f) - \sum_{j=0}^{k} f_j \hbar^j\right\| \leqslant 2C_k\hbar^{k+1}\int_{\mathcal{S}^*}|\phi|^{2(k+1)}\left|\hat{f}(\phi)\right|\mathrm{dvol}^*(\phi). \tag{111}$$

When f is Schwartz, then $\hat{f}$ is Schwartz, the integral is finite, and we obtain an upper bound given by some finite constant times $\hbar^{k+1}$. So the limit expression (108) vanishes for all $k \in \mathbb{N}$. □

The Toeplitz quantization star product $\star_T$ for Schwartz functions $f_1, f_2 \in \mathrm{C}^\infty_{\mathrm{S}}(\mathcal{S},\mathbb{C})$ is determined by the conditions (24). It has an exponential expression with directed derivatives that act only on the function to the left or right as indicated with an arrow,

$$f_1 \star_T f_2 = f_1 \exp\left(-\hbar\frac{\overleftarrow{\partial}}{\partial z^i}\delta^{i\bar{\imath}}\frac{\overrightarrow{\partial}}{\partial \bar{z}^{\bar{\imath}}}\right) f_2. \tag{112}$$

Similarly, the dequantization star product $\star_\Xi$ determined by the conditions (29) is a star product for functions $f_1, f_2 \in \mathrm{C}^\infty_{\mathrm{S}}(\mathcal{S},\mathbb{C})$ with the exponential expression

$$f_1 \star_\Xi f_2 = f_1 \exp\left(\hbar\frac{\overleftarrow{\partial}}{\partial \bar{z}^{\bar{\imath}}}\delta^{\bar{\imath}i}\frac{\overrightarrow{\partial}}{\partial z^i}\right) f_2. \tag{113}$$

Note that the holomorphic and anti-holomorphic derivatives act in different directions and the exponentials have opposite sign.

Proposition 6. *Toeplitz quantization with the star product (112) is an infinite order strict deformation quantization over the algebra of Schwartz functions $\mathcal{A}_0 = \mathrm{C}^\infty_{\mathrm{S}}(\mathcal{S},\mathbb{C})$.*

Proof. In [13, chapter II, section 2.6], it was shown that there exists a continuous field of C^*-algebras $\left(I, (\mathfrak{A}_\hbar)_{\hbar\in I}, \Gamma\right)$ including the Toeplitz sections $T(f)$ for $f \in \mathfrak{A}_0 = \mathrm{C}_0(\mathcal{S},\mathbb{C})$ as sections, $T(f) \in \Gamma$. It remains to show that the star product (112) fulfills the conditions (24) in all orders $k \in \mathbb{N}$.

Recall the Fourier decomposition (97) of the Toeplitz operators $T_\hbar(f)$ and $T_\hbar(f')$ for any functions $f,f' \in \mathrm{C}^\infty_{\mathrm{S}}(\mathcal{S},\mathbb{C})$ into Toeplitz operators $T_\hbar(\mathrm{e}^{\mathrm{i}\phi})$ and $T_\hbar(\mathrm{e}^{\mathrm{i}\phi'})$ (or Weyl generators). The kth remainder (23) is then bounded from above by the double integral

$$R^k_T(f,f',\hbar) \leqslant \iint_{\mathcal{S}^*} \left|\hat{f}(\phi)\right| \left|\hat{f'}(\phi')\right| R^k_T\left(\mathrm{e}^{\mathrm{i}\phi}, \mathrm{e}^{\mathrm{i}\phi'}, \hbar\right) \mathrm{dvol}^*(\phi)\, \mathrm{dvol}^*(\phi'). \tag{114}$$

The norm inside the integral is given by

$$R^k_T\left(\mathrm{e}^{\mathrm{i}\phi}, \mathrm{e}^{\mathrm{i}\phi'}, \hbar\right) = \frac{1}{\hbar^k}\left\| T_\hbar\left(\mathrm{e}^{\mathrm{i}\phi}\right) T_\hbar\left(\mathrm{e}^{\mathrm{i}\phi'}\right) - \sum_{j=0}^{k} \frac{\hbar^j}{j!}\left(\delta^{i\bar{\imath}}\phi_i\overline{\phi'}_{\bar{\imath}}\right)^j T_\hbar\left(\mathrm{e}^{\mathrm{i}(\phi+\phi')}\right)\right\|, \tag{115}$$

where the Weyl relations imply

$$T_\hbar\left(\mathrm{e}^{\mathrm{i}\phi}\right) T_\hbar\left(\mathrm{e}^{\mathrm{i}\phi'}\right) = \exp\left(\hbar\delta^{i\bar{\imath}}\phi_i\overline{\phi'}_{\bar{\imath}}\right) T_\hbar\left(\mathrm{e}^{\mathrm{i}(\phi+\phi')}\right). \tag{116}$$

To apply lemma 4, set $\zeta \in \mathbb{C}$ as

$$\zeta = \hbar\delta^{i\bar{\imath}}\phi_i\overline{\phi'}_{\bar{\imath}}, \qquad \mathrm{Re}\,\zeta = \frac{\hbar}{2}\delta^{i\bar{\imath}}\left(\phi_i\overline{\phi'}_{\bar{\imath}} + \overline{\phi}_{\bar{\imath}}\phi'_i\right). \tag{117}$$

Thus, we have

$$\begin{aligned} R^k_T\left(\mathrm{e}^{\mathrm{i}\phi}, \mathrm{e}^{\mathrm{i}\phi'}, \hbar\right) &= \frac{1}{\hbar^k}\left|\mathrm{er}_k(\zeta)\right| \left\| T_\hbar\left(\mathrm{e}^{\mathrm{i}(\phi+\phi')}\right)\right\| \\ &\leqslant C_k\hbar\left|\delta^{i\bar{\imath}}\phi_i\overline{\phi'}_{\bar{\imath}}\right|^{k+1}\left(\left\|T_\hbar\left(\mathrm{e}^{\mathrm{i}(\phi+\phi')}\right)\right\| + \mathrm{e}^{\mathrm{Re}\,\zeta}\left\|T_\hbar\left(\mathrm{e}^{\mathrm{i}(\phi+\phi')}\right)\right\|\right). \end{aligned} \tag{118}$$

The two terms with the operator norm follow from (91) and $\|W_\hbar(\phi)\| = 1$ (for any $\phi \in \mathcal{S}^*$), implying that the Toeplitz map is norm contracting,

$$\begin{aligned} \left\|T_\hbar\left(\mathrm{e}^{\mathrm{i}(\phi+\phi')}\right)\right\| &= \exp\left(-\frac{\hbar}{2}|\phi+\phi'|^2\right), \\ \mathrm{e}^{\mathrm{Re}\,\zeta}\left\|T_\hbar\left(\mathrm{e}^{\mathrm{i}(\phi+\phi')}\right)\right\| &= \exp\left(-\frac{\hbar}{2}|\phi|^2 - \frac{\hbar}{2}|\phi'|^2\right). \end{aligned} \tag{119}$$

Both of these exponentials are bounded by 1. So (118) is bounded by

$$R^k_T\left(\mathrm{e}^{\mathrm{i}\phi}, \mathrm{e}^{\mathrm{i}\phi'}, \hbar\right) \leqslant 2C_k\hbar\left|\delta^{i\bar{\imath}}\phi_i\overline{\phi'}_{\bar{\imath}}\right|^{k+1}. \tag{120}$$

The modulus in the $(k+1)$ order polynomial is bounded from above by the sum of $|\phi_i|$ and $|\phi'_i|$ all to the power of $k+1$. Inserted back into the integration (114) yields

$$R^k_T(f,f',\hbar) \leqslant 2C_k\hbar\iint_{\mathcal{S}^*}\left(\sum_{i=1}^{N}|\phi_i|\,|\phi'_i|\right)^{k+1}\left|\hat{f}(\phi)\right|\left|\hat{f'}(\phi')\right| \mathrm{dvol}^*(\phi)\,\mathrm{dvol}^*(\phi'). \tag{121}$$

The factor with the sum is a polynomial in ϕ and ϕ' and the integration with the Schwartz functions $\hat{f}$ and $\hat{f'}$ is finite. Therefore, the remainder $R^k_T(f,f',\hbar)$ is bounded by a constant (independent of $\hbar$) times $\hbar$, which vanishes in the limit $\hbar \to 0$ for all $k \in \mathbb{N}$.

Poisson compatibility of this star product follows from the first order terms

$$f\star_T f' - f'\star_T f = -\hbar\sum_{i=1}^{N}\left(\frac{\partial f}{\partial z^i}\frac{\partial f'}{\partial \bar{z}^i} - \frac{\partial f'}{\partial z^i}\frac{\partial f}{\partial \bar{z}^i}\right) + \mathcal{O}\left(\hbar^2\right) = \mathrm{i}\hbar\{f,f'\} + \mathcal{O}\left(\hbar^2\right). \quad (122)$$

Notice that the star product is also self-adjoint and differential, which follows immediately from the differential form (112).

□

Proposition 7. *Berezin–Toeplitz dequantization with the star product (113) is a strict deformation dequantization of the algebra of Schwartz functions* $\mathcal{A}_0 = \mathrm{C}^\infty_{\mathrm{S}}(\mathcal{S},\mathbb{C})$.

Proof. According to [13, chapter II, theorem 2.6.5], the continuous fields of the Weyl quantization and Berezin–Töplitz quantization coincide.

Even though the Weyl operators $W_\hbar(\phi)$ are not elements of $\mathfrak{A}_\hbar = \mathcal{K}(\mathcal{H}_\hbar)$, they are Ξ-expandable following a similar argument as in proposition 5 with the coefficients

$$w_j^\phi = \frac{1}{j!}\left(-\frac{1}{2}|\phi|^2\right)^j \mathrm{e}^{\mathrm{i}\phi}. \quad (123)$$

So, we write a Ξ-expandable section $A_f \in \Gamma$ as

$$A_f(\hbar) = \int_{\mathcal{S}^*} \hat{f}(\phi,\hbar)\, W_\hbar(\phi)\, \mathrm{dvol}^*(\phi) \quad (124)$$

with a continuous amplitude function $\hat{f} \in \mathrm{C}(\mathcal{S}^* \times I,\mathbb{C})$ such that $f(\cdot,\hbar) \in \mathcal{A}_0$ for all $\hbar \in I$. The conditions of the continuous field of C*-algebras $\left(I,(\mathfrak{A}_\hbar)_{\hbar\in I},\Gamma\right)$ in definition 5 imply that all sections of the form A_f span a total subspace of Γ. This means that for any section $A \in \Gamma$ there exists such a Ξ-expandable section $A_f \in \Gamma$ such that for all $\delta > 0$ there exists a neighborhood $N_0 \subset \mathbb{R}_+$ around $\hbar = 0$ such that for all $\hbar' \in N_0 : \|A(\hbar') - A_f(\hbar')\| \leqslant \delta$.

Taking the dequantization yields the smooth function

$$\Xi_\hbar(A_f(\hbar)) = \int_{\mathcal{S}^*} \hat{f}(\phi,\hbar) \exp\left(-\frac{\hbar}{2}|\phi|^2\right) \mathrm{e}^{\mathrm{i}\phi}\, \mathrm{dvol}(\phi), \quad (125)$$

which is the convolution of the pointwise Fourier transformed function $f(\cdot,\hbar)$ with 'half' the Berezin kernel.

Now consider the dequantization of a product $A_f A_{f'}$ of two such sections. With the same identification of ζ as in (117), rewrite the Weyl relations in terms of the complex conjugated value $\bar{\zeta} = \hbar\delta^{i\bar{\imath}}\overline{\phi}_{\bar{\imath}}\phi'_i$,

$$W_\hbar(\phi)\, W_\hbar(\phi') = \mathrm{e}^{-\mathrm{i}\,\mathrm{Im}\,\bar{\zeta}}\, W_\hbar(\phi + \phi'). \quad (126)$$

Similar to the previous proof, notice that

$$\begin{aligned}
\Xi_\hbar(W_\hbar(\phi+\phi')) &= \exp\left(-\frac{\hbar}{2}|\phi+\phi'|^2\right)\mathrm{e}^{\mathrm{i}(\phi+\phi')}, \\
\mathrm{e}^{\mathrm{Re}\,\bar{\zeta}}\,\Xi_\hbar(W_\hbar(\phi+\phi')) &= \exp\left(-\frac{\hbar}{2}|\phi|^2 - \frac{\hbar}{2}|\phi'|^2\right)\mathrm{e}^{\mathrm{i}(\phi+\phi')} \\
&= \Xi_\hbar(W_\hbar(\phi))\,\Xi_\hbar(W_\hbar(\phi')). \quad (127)
\end{aligned}$$

We combine the exponentials as $\overline{\zeta} = \mathrm{Re}\,\overline{\zeta} + \mathrm{i}\,\mathrm{Im}\,\overline{\zeta}$, so that the dequantization of the product (126) reads

$$\Xi_\hbar\left(W_\hbar(\phi)\, W_\hbar(\phi')\right) = \Xi_\hbar\left(W_\hbar(\phi)\right) \mathrm{e}^{-\overline{\zeta}}\, \Xi_\hbar\left(W_\hbar(\phi')\right). \tag{128}$$

Thus the dequantization of the product of sections becomes

$$\begin{aligned}(A_f A_{f'})(\hbar) &= \iint_{\mathcal{S}^*} \hat{f}(\phi,\hbar)\hat{f'}(\phi',\hbar)\, W_\hbar(\phi)\, W_\hbar(\phi')\, \mathrm{d}^2\mathrm{vol}^*,\\ \Xi_\hbar\left((A_f A_{f'})(\hbar)\right) &= \iint_{\mathcal{S}^*} \hat{f}(\phi,\hbar)\hat{f'}(\phi',\hbar)\, \Xi_\hbar\left(W_\hbar(\phi)\right) \mathrm{e}^{-\overline{\zeta}}\, \Xi_\hbar\left(W_\hbar(\phi')\right) \mathrm{d}^2\mathrm{vol}^*. \end{aligned}\tag{129}$$

The exponential $\mathrm{e}^{-\overline{\zeta}}$ is the Fourier transform of the derivatives that act on the $\mathrm{e}^{\mathrm{i}\phi}$ and $\mathrm{e}^{\mathrm{i}\phi'}$ functions of the Weyl generator dequantizations,

$$\mathrm{e}^{\mathrm{i}\phi}\mathrm{e}^{-\overline{\zeta}}\mathrm{e}^{\mathrm{i}\phi'} = \mathrm{e}^{\mathrm{i}\phi}\left(\sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{\overleftarrow{\partial}}{\partial \overline{z}^{\overline{\imath}}}\delta^{\overline{\imath} i}\frac{\overrightarrow{\partial}}{\partial z^i}\right)^j \hbar^j\right)\mathrm{e}^{\mathrm{i}\phi'}. \tag{130}$$

Hence, the integration in (129) separates into the integrals for the sections A_f and A_g. From the assumptions, we know that these sections are Ξ-expandable such that for any $k \in \mathbb{N}$:

$$\Sigma^k_{\Xi}(A_f,\hbar) = \sum_{j=0}^{k} f_j \hbar^j \tag{131}$$

and similarly for $A_{f'}$. We express the dequantization (125) for both A_f and $A_{f'}$ by the respective expansions (131) leading to

$$\Sigma^\infty_{\Xi}(A_f,\hbar)\exp\left(\hbar\frac{\overleftarrow{\partial}}{\partial \overline{z}^{\overline{\imath}}}\delta^{\overline{\imath} i}\frac{\overrightarrow{\partial}}{\partial z^i}\right)\Sigma^\infty_{\Xi}(A_{f'},\hbar) = \Sigma^\infty_{\Xi}(A_f,\hbar) \star_{\Xi} \Sigma^\infty_{\Xi}(A_{f'},\hbar). \tag{132}$$

The dequantization star product is again Poisson compatible, self-adjoint and differential, which is analogously shown as in the previous proposition for the quantization star product. □

The 'gauge transformation' that relates the quantization star product $\star_T$ to the dequantization star product $\star_\Xi$, $\forall f_1, f_2 \in \mathrm{C}^\infty_{\mathrm{S}}(\mathcal{S},\mathbb{C})$:

$$(\Xi_\hbar \circ T_\hbar)(f_1 \star_T f_2) = \left((\Xi_\hbar \circ T_\hbar)(f_1)\right) \star_\Xi \left((\Xi_\hbar \circ T_\hbar)(f_2)\right), \tag{133}$$

is the series expansion with the coefficients (109), determined by the expansion terms of the Berezin transform.

6. Conclusion

We considered the method of geometric quantization for a symplectic manifold with Riemannian metric [27]. When this method is applied to a symplectic vector space with an inner product, as is the case in QFT on causal sets, it naturally yields the Sorkin–Johnston state.

For our case of a real, finite-dimensional vector space $\mathcal{S}$, we analyzed the spectrum of the Bochner Laplacian of the quantization bundle to find the eigenspace of the lowest spectral value in section 4. This eigenspace is spanned by the holomorphic sections (66) with respect to some complex structure J. We choose this subspace of square-integrable sections as the physical Hilbert space $\mathcal{H}_\hbar$. We showed that Toeplitz quantization gives a strict deformation quantization, which induces a star product (112). The adjoint operation to Toeplitz quantization, referred to as dequantization, induces another star product (113). Dequantization maps quantum observables to classical observables, and by evaluation at 0, this defines a state; we showed that this is precisely the Sorkin–Johnston state.

The above construction was done for a finite-dimensional symplectic vector space. Such a finite dimensional system appears as the space of on-shell fields for the Klein–Gordon equation over a (subset) of a causal set (locally finite, partial ordered set) in causal set theory [32]. We hope that our results will find applications in quantum field theory on causal sets, as well as in generalizations to symplectic manifolds, at least in those cases where the construction of quantum observables via geometric quantization is suitable.

Our construction also suggests a generalization to interacting theories, for which the phase space is no longer naturally described as a symplectic vector space. If a Riemannian metric is available on the phase space, then geometric quantization may be applied. Berezin–Toeplitz dequantization and evaluation at 0, would then give a state that generalizes the Sorkin–Johnston state.

A further challenge is to extend this construction to fermionic systems and compare it with the construction of fermionic projector states [33, 34].

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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Appendix. Normal and anti-normal ordering, and the Berezin transform

In this appendix, we want to demonstrate how the Berezin–Toeplitz quantization and dequantization relate to anti-normal and normal ordering, respectively. Heuristically, we may extend the Toeplitz quantization map (39) from continuous, unbounded functions to unbounded operators. In particular for the coordinate functions (with any $i \in [1,N]$), we may write

$$T_\hbar\left(\delta_{i\bar{\imath}} z^i\right) = \sqrt{\hbar} a_{\bar{\imath}}^+ \qquad T_\hbar\left(\delta_{i\bar{\imath}} \bar{z}^{\bar{\imath}}\right) = \sqrt{\hbar} a_i^- . \tag{A.1}$$

The dequantization map Ξ may be extended in a similar way, giving

$$\Xi_\hbar\left(\sqrt{\hbar} a_{\bar{\imath}}^+\right) = \delta_{i\bar{\imath}} z^i \qquad \Xi_\hbar\left(\sqrt{\hbar} a_i^-\right) = \delta_{i\bar{\imath}} \bar{z}^{\bar{\imath}}. \tag{A.2}$$

For a monomial, we have

$$T_\hbar\left(\left(\delta_{i\bar{\imath}} z^i\right)^m \left(\delta_{j\bar{\jmath}} \bar{z}^{\bar{\jmath}}\right)^n\right) = \hbar^{\frac{m+n}{2}} \left(a_j^-\right)^n \left(a_{\bar{\imath}}^+\right)^m \tag{A.3}$$

and

$$\Xi_\hbar\left(\hbar^{\frac{m+n}{2}} \left(a_{\bar{\imath}}^+\right)^m \left(a_j^-\right)^n\right) = \left(\delta_{i\bar{\imath}} z^i\right)^m \left(\delta_{j\bar{\jmath}} \bar{z}^{\bar{\jmath}}\right)^n, \tag{A.4}$$

both extending to any polynomials by linearity. Notice the correspondence of quantization with anti-normal ordering and dequantization with normal ordering, respectively.

As a less heuristic example, consider a Schwartz function $f \in \mathrm{C}^\infty_{\mathrm{S}}(\mathcal{S},\mathbb{C})$ that is an N-fold product of Gaussian functions with variances $\beta_i > \hbar$ for all $i \in [1,N]$, given by

$$f(z) := \prod_{i=1}^N \frac{1}{2\pi\beta_i} \exp\left(-\frac{\left(x^i\right)^2 + \left(y^i\right)^2}{\beta_i}\right) = \prod_{i=1}^N \frac{1}{2\pi\beta_i} \exp\left(-\frac{|z^i|^2}{\beta_i}\right). \tag{A.5}$$

It expands as a product of Taylor series

$$f(z) = \prod_{i=1}^N \frac{1}{2\pi\beta_i} \sum_{k=0}^\infty \frac{1}{k!} \left(-\frac{1}{\beta_i}\right)^k \left(z^i\right)^k \left(\bar{z}^i\right)^k \tag{A.6}$$

and its Toeplitz operator may be expanded similarly in terms of the unbounded ladder operators,

$$T_\hbar(f) = \prod_{i=1}^N \frac{1}{2\pi\beta_i} \sum_{k=0}^\infty \frac{1}{k!} \left(-\frac{\hbar}{\beta_i}\right)^k \left(a_i^-\right)^k \left(a_i^+\right)^k. \tag{A.7}$$

Note that the kth power of the lowering operator appears to the left of the kth power of the raising operator. Thus the function f is the anti-normal ordering corresponding to the Toeplitz operator $T_\hbar(f)$.

Now consider an observable A with a similar expansion as (A.7), but with the opposite ordering of the ladder operators,

$$A := \prod_{i=1}^N \frac{1}{2\pi\beta_i} \sum_{k=0}^\infty \frac{1}{k!} \left(-\frac{\hbar}{\beta_i}\right)^k \left(a_i^-\right)^k \left(a_i^+\right)^k. \tag{A.8}$$

In contrast to (A.7), here the kth power of the creation operator $(a_i^+)^k$ appear to the left of the kth power of the annihilation operator $(a_i^-)^k$. The Ξ-dequantization of the operator A is

$$\Xi_\hbar(A) = f. \tag{A.9}$$

Thus the operator A corresponds to the function f by normal-ordering. This example shows the correspondence of Toeplitz quantization to anti-normal, and Toeplitz dequantization to normal ordered expressions.

Taking the dequantization of the Toeplitz operator $T_\hbar(f)$, we obtain the Berezin transform, here written with the convolution $\circledast$ as defined in (79),

$$\begin{aligned}(\Xi_\hbar \circ T_\hbar)(f)(z) &= \frac{1}{(2\pi\hbar)^N}\exp\left(-\frac{1}{\hbar}|z|^2\right)\circledast f(z)\\ &= \frac{1}{(2\pi\hbar)^N}\int_S \exp\left(-\frac{1}{\hbar}|\zeta|^2\right)\prod_{i=1}^N \frac{1}{2\pi\beta_i}\exp\left(-\frac{1}{\beta_i}\left|z^i-\zeta^i\right|^2\right)\mathrm{dvol}(\zeta).\end{aligned} \tag{A.10}$$

Because our Gaussian function f is a product of independent Gaussian functions, the $2N$-fold integration splits into N double-integrals which are solved by completing the square in the exponents,

$$\begin{aligned}&= \prod_{i=1}^N \frac{1}{(2\pi\beta_i)(2\pi\hbar)}\iint \exp\left(-\frac{1}{\hbar}\left|\zeta^i\right|^2 - \frac{1}{\beta_i}\left|z^i-\zeta^i\right|^2\right)\mathrm{i}\,\mathrm{d}\zeta^i\mathrm{d}\overline{\zeta}^i\\ &= \prod_{i=1}^N\left(\frac{2\pi\frac{\beta_i\hbar}{\beta_i+\hbar}}{(2\pi\beta_i)(2\pi\hbar)}\exp\left(-\frac{|z^i|^2}{\beta_i+\hbar}\right)\right.\\ &\qquad\times\left.\underbrace{\iint\frac{1}{2\pi\frac{\beta_i\hbar}{\beta_i+\hbar}}\exp\left(-\frac{\beta_i+\hbar}{\beta_i\hbar}\left|\zeta^i-\frac{\hbar}{\beta_i+\hbar}z^i\right|^2\right)\mathrm{i}\,\mathrm{d}\zeta^i\mathrm{d}\overline{\zeta}^i}_{=1}\right)\\ &= \prod_{i=1}^N\frac{1}{2\pi(\beta_i+\hbar)}\exp\left(-\frac{|z^i|^2}{\beta_i+\hbar}\right).\end{aligned} \tag{A.11}$$

The Berezin transform of the Gaussian function f is again a Gaussian function with variances increased by $\hbar$.

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References

[1] Afshordi N, Aslanbeigi S and Sorkin R D 2012 A distinguished vacuum state for a quantum field in a curved spacetime: formalism, features and cosmology *J. High Energy Phys.* JHEP08(2012)137
[2] Johnston S 2009 Feynman propagator for a free scalar field on a causal set *Phys. Rev. Lett.* **103** 180401
[3] Sorkin R D 2011 Scalar field theory on a causal set in histories form *J. Phys.: Conf. Ser.* **306** 012017
[4] Fewster C J and Verch R 2012 On a recent construction of 'vacuum-like' quantum field states in curved spacetime *Class. Quantum Grav.* **29** 205017
[5] Fewster C J and Verch R 2013 The necessity of the Hadamard condition *Class. Quantum Grav.* **30** 235027
[6] Brum M and Fredenhagen K 2014 'Vacuum-like' Hadamard states for quantum fields on curved spacetimes *Class. Quantum Grav.* **31** 025024
[7] Wingham F L 2018 Generalised Sorkin-Johnston and Brum-Fredenhagen states for quantum fields on curved spacetimes *PhD Dissertation* University of York, United Kingdom (available at: https://etheses.whiterose.ac.uk/23631)
[8] Sorkin R D 2017 From Green function to quantum field *Int. J. Geom. Methods Mod. Phys.* **14** 1740007
[9] Minz C 2021 Algebraic field theory on causal sets: local structures and quantization methods *PhD Dissertation* University of York, United Kindgom (available at: https://etheses.whiterose.ac.uk/29866)
[10] Dable-Heath E, Fewster C J, Rejzner K and Woods N 2020 Algebraic classical and quantum field theory on causal sets *Phys. Rev.* D **101** 065013
[11] Schlichenmaier M 2010 Berezin-Toeplitz quantization for compact Kähler manifolds. A review of results *Adv. Math. Phys.* **2010** 927280
[12] Moretti V 2013 *Spectral Theory and Quantum Mechanics: With an Introduction to the Algebraic Formulation* vol 64 (Springer)
[13] Landsman N P 1998 *Mathematical Topics Between Classical and Quantum Mechanics* (Springer)
[14] Dixmier J 1964 *Les C^*-algèbres et Leurs Représentations* (Cahiers Scientifiques, Fasc. vol 29) (Gauthier-Villars)
[15] Dixmier J 1977 *C^*-Algebras* vol 15 (North-Holland, Elsevier) translation of Les C^*-algèbres et Leurs Repésentations by Francis Jellett
[16] Kirchberg E and Wassermann S 1995 Operations on continuous bundles of C^*-algebras *Math. Ann.* **303** 677–97
[17] Johnston S 2010 Quantum fields on causal sets *PhD Dissertation* Imperial College London, United Kingdom (arXiv:1010.5514)
[18] Dereziński J and Gérard C 2013 *Mathematics of Quantization and Quantum Fields* (Cambridge University Press)
[19] Ali S T and Engliš M 2005 Quantization methods: a guide for physicists and analysts *Rev. Math. Phys.* **17** 391–490
[20] Bordemann M, Meinrenken E and Schlichenmaier M 1994 Toeplitz quantization of Kähler manifolds and $gl(N)$, $N \to \infty$ limits *Commun. Math. Phys.* **165** 281–96
[21] Karabegov A V and Schlichenmaier M 2001 Identification of Berezin-Toeplitz deformation quantization *J. Reine Angew. Math.* **2001** 49–76
[22] De Monvel L B and Guillemin V 1981 *The Spectral Theory of Toeplitz Operators (Annals of Mathematics Studies* vol 99) (Princeton University Press)
[23] Ioos L, Kaminker V, Polterovich L and Shmoish D 2020 Spectral aspects of the Berezin transform *Ann. Henri Lebesgue* **2020** 1343–87
[24] Berezin F A 1972 Covariant and contravariant symbols of operators *Mathematics of the USSR-Izvestiya* **6** 1117–51 (translated by L W Longdon)
[25] Guillemin V and Uribe A 1988 The Laplace operator on the N-th tensor power of a line bundle: eigenvalues which are uniformly bounded in N *Asymptot. Anal.* **1** 105–13
[26] Borthwick D and Uribe A 1996 Almost complex structures and geometric quantization *Math. Res. Lett.* **3** 845–61
[27] Ma X and Marinescu G 2008 Generalized Bergman kernels on symplectic manifolds *Adv. Math.* **217** 1756–815

[28] Ma X and Marinescu G 2008 Toeplitz operators on symplectic manifolds *J. Geom. Anal.* **18** 565–611
[29] Kordyukov Y A, Ma X and Marinescu G 2019 Generalized Bergman kernels on symplectic manifolds of bounded geometry *Commun. PDE* **44** 1037–71
[30] Ma X and Marinescu G 2002 The spin^c Dirac operator on high tensor powers of a line bundle *Math. Z.* **240** 651–64
[31] Ma X and Marinescu G 2007 *Holomorphic Morse Inequalities and Bergman Kernels* (*Progress in Mathematics* vol 254) (Birkhäuser Verlag)
[32] Bombelli L, Lee J, Meyer D and Sorkin R D 1987 Space-time as a causal set *Phys. Rev. Lett.* **59** 521–24
[33] Araki H 1970 On quasifree states of CAR and Bogoliubov automorphisms *Publ. Res. Inst. Math. Sci.* **6** 385–442
[34] Finster F, Murro S and Röken C 2016 The fermionic projector in a time-dependent external potential: mass oscillation property and Hadamard states *J. Math. Phys.* **57** 072303