

Nonlinear Higher Spin Theories in Various Dimensions

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ABSTRACT. In this article, an introduction to the nonlinear equations for completely symmetric bosonic higher spin gauge fields in anti de Sitter space of any dimension is provided. To make the presentation self-contained we explain in detail some related issues such as the MacDowell-Mansouri-Stelle-West formulation of gravity, unfolded formulation of dynamical systems in terms of free differential algebras and Young tableaux symmetry properties in terms of Howe dual algebras.

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1 Introduction

The Coleman-Mandula theorem [1] and its generalization [2] strongly restrict the S-matrix symmetries of a nontrivial (*i.e.* interacting) relativistic field theory in flat space-time. More precisely, the extension of the space-time symmetry algebra is at most the (semi)direct sum of a (super)conformal algebra and an internal symmetry algebra spanned by elements that commute with the generators of the Poincaré algebra. Ruling out higher symmetries via these theorems, one rules out higher spin (HS) gauge fields associated with them, allowing in practice only gauge fields of low spins (*i.e.* $s \leq 2$). However, as will be reviewed here, going beyond some assumptions of these no-go theorems allows to overcome both restrictions, on higher spins (*i.e.* $s > 2$) and on space-time symmetry extensions.

By now, HS gauge fields are pretty well understood at the free field level. Therefore, the main open problem in this topic is to find proper nonAbelian HS gauge symmetries which extend the space-time symmetries. These symmetries can possibly mix fields of different spins, as supersymmetry does. Even though one may never find HS particles in accelerators, nonAbelian HS symmetries might lead us to a better understanding of the true symmetries of unification models. From the supergravity perspective, the theories with HS fields may have more than 32 supercharges and may live in dimensions higher than 11. From the superstring perspective, several arguments support the conjecture [3] that the Stueckelberg symmetries of massive HS string excitations result from a spontaneous breaking of some HS gauge symmetries. In this picture, tensile string theory appears as a spontaneously broken phase of a more symmetric phase with only massless fields. In that case, superstrings should exhibit higher symmetries in the high-energy limit as was argued long ago by Gross [4]. A more recent argument came from the *AdS/CFT* side after it was realized [5, 6, 7, 8] that HS symmetries should be unbroken in the Sundborg–Witten limit

$$\lambda = g_{YM}^2 N = \left(\frac{R_{AdS}^2}{\alpha'} \right)^2 \longrightarrow 0,$$

because the boundary conformal theory becomes free. A dual string theory in the highly curved *AdS* space-time is therefore expected to be a HS theory (see also [9, 10] and refs therein for recent developments).

One way to provide an explicit solution of the nonAbelian HS gauge symmetry problem is by constructing a consistent nonlinear theory of massless HS fields. For several decades, a lot of efforts has been put into this direction although, from the very beginning, this line of research faced several difficulties. The first explicit attempts to introduce interactions between HS gauge fields and gravity encountered severe problems [11]. However, some positive results [12] were later obtained in flat space-time on the existence of consistent vertices for HS gauge fields interacting with each other, but not with gravity.

Seventeen years ago, the problem of consistent HS gravitational interactions was partially solved in four dimensions [13]. In order to achieve this result, the following conditions of the no-go theorems [1, 2] were relaxed:

- (1) the theory is formulated around a flat background.
- (2) the spectrum contains a finite number of HS fields.

The nonlinear HS theory in four dimensions was shown to be consistent up to cubic order at the action level [13] and, later, at all orders at the level of equations of motion [14, 15]. The second part of these results was recently extended to arbitrary space-time dimensions [16]. The nonlinear HS theory exhibits some rather unusual properties of HS gauge fields:

- (1') the theory is perturbed around an $(A)dS$ background and does not admit a flat limit as long as HS symmetries are unbroken.
- (2') the allowed spectra contain infinite towers of HS fields and do not admit a consistent finite truncation with $s > 2$ fields.
- (3') the vertices have higher-derivative terms (that is to say, the higher derivatives appear in HS interactions - not at the free field level).

The properties (2') and (3') were also observed by the authors of [12] for HS gauge fields. Though unusual, these properties are familiar to high-energy theorists. The property (1') is verified by gauged

supergravities with charged gravitinos [17, 18]. The property (2') plays an important role in the consistency of string theory. The property (3') is also shared by Witten's string field theory [19].

An argument in favor of an *AdS* background is that the S-matrix theorems [1, 2] do not apply since there is no well-defined S-matrix in *AdS* space-time [20]. The *AdS* geometry plays a key role in the nonlinear theory because cubic higher derivative terms are added to the free Lagrangian, requiring a nonvanishing cosmological constant Λ . These cubic vertices are schematically of the form

$$\mathcal{L}^{int} = \sum_{n,p} \Lambda^{-\frac{1}{2}(n+p)} D^n(\varphi \dots) D^p(\varphi \dots) \mathcal{R} \dots,$$

where $\varphi \dots$ denotes some spin- s gauge field, and \mathcal{R} stands for the fluctuation of the Riemann curvature tensor around the *AdS* background. Such vertices do not admit a $\Lambda \rightarrow 0$ limit. The highest order of derivatives which appear in the cubic vertex increases linearly with the spin [12, 13]: $n + p \sim s$. Since all spins $s > 2$ must be included in the nonAbelian HS algebra, the number of derivatives is not bounded from above. In other words, the HS gauge theory is *nonlocal*¹.

The purpose of these lecture notes is to present, in a self-contained² way, the nonlinear equations for completely symmetric bosonic HS gauge fields in *AdS* space of any dimension. The structure of the present lecture notes is as follows.

1.1 Plan of the paper

In Section 2, the MacDowell-Mansouri-Stelle-West formulation of gravity is recalled. In Section 3, some basics about Young tableaux and irreducible tensor representations are introduced. In Section 4, the approach of Section 2 is generalized to HS fields, *i.e.* the free HS gauge theory is formulated as a theory of one-form connections. In Section 5, a nonAbelian HS algebra is constructed. The general definition of a free differential algebra is given in Section 6 and a strategy is explained on how to formulate nonlinear HS field equations in these terms. Section 7 presents the unfolded form of the free massless scalar field and linearized gravity field equations, which are generalized to free HS field equations in Section 8. In Section 9 is explained how the cohomologies of some operator σ_- describe the dynamical content of a theory. The relevant cohomologies are calculated in the HS case in Section 10. Sections 11 and 12 introduce some tools (the star product and the twisted adjoint representation) useful for writing the nonlinear equations, which is done in Section 13. The nonlinear equations are analyzed perturbatively in Section 14 and they are further discussed in Section 15. A brief conclusion inviting to further readings completes these lecture notes. In Appendix 1 some elementary material of lower spin gauge theories is reviewed, while in Appendix 2 some technical points of the nonlinear HS equations are addressed in more details.

For an easier reading of the lecture notes, here is a guide to the regions related to the main topics addressed in these lecture notes:

- Abelian HS gauge theory: In section 4 is reviewed the quadratic actions of free (constrained) HS gauge fields in the metric-like and frame-like approaches.
- Non-Abelian HS algebra: The definition and some properties of the two simplest HS algebras are given in Section 5.
- Unfolded formulation of free HS fields: The unfolding of the free HS equations is very important as a starting point towards nonlinear HS equations at all orders. The general unfolding procedure and its application to the HS gauge theory is explained in many details in Sections 7, 8, 9, 10 and 12. (Sections 9 and 10 can be skipped in a first quick reading since the corresponding material is not necessary for understanding Sections from 11 till 15.)

¹Nonlocal theories do not automatically suffer from the higher-derivative problem. Indeed, in some cases like string field theory, the problem is somehow cured [21, 19, 22] if the free theory is well-behaved and if nonlocality is treated perturbatively (see [23] for a comprehensive review on this point).

²For these lecture notes, the reader is only assumed to have basic knowledge of Yang-Mills theory, general relativity and group theory. The reader is also supposed to be familiar with the notions of differential forms and cohomology groups.

- Non-linear HS equations: Consistent nonlinear equations, that are invariant under the non-Abelian HS gauge transformations and diffeomorphisms, and correctly reproduce the free HS dynamics at the linearized level, are presented and discussed in Sections 6.2, 10.3, 13, 14 and 15.
- Material of wider interest: Sections 2, 3, 6.1, 9 and 11 introduce tools which prove to be very useful in HS gauge theories, but which may also appear in a variety of different contexts.

1.2 Conventions

Our conventions are as follows:

A generic space-time is denoted by \mathcal{M}^d and is a (pseudo)-Riemannian smooth manifold of dimension d , where the metric is taken to be “mostly minus”.

Greek letters $\mu, \nu, \rho, \sigma, \dots$ denote curved (*i.e.*, base) indices, while Latin letters a, b, c, d, \dots denote fiber indices often referred to as tangent space indices. Both types of indices run from 0 to $d - 1$. The tensor η_{ab} is the mostly minus Minkowski metric. Capital Latin letters A, B, C, D, \dots denote ambient space indices and their range of values is $0, 1, \dots, d - 1, \hat{d}$, where the (timelike) $(d + 1)$ -th direction is denoted by \hat{d} (in order to distinguish the tangent space index d from the value \hat{d} that it can take). The tensor η_{AB} is diagonal with entries $(+, -, \dots, -, +)$.

The bracket $[\dots]$ denotes complete antisymmetrization of indices, with strength one (*e.g.* $A_{[a}B_{b]} = \frac{1}{2}(A_aB_b - A_bB_a)$), while the bracket $\{\dots\}$ denotes complete symmetrization of the indices, with strength one (*e.g.* $A_{\{a}B_{b\}} = \frac{1}{2}(A_aB_b + A_bB_a)$). Analogously, the commutator and anticommutator are respectively denoted as $[\ , \]$ and $\{ \ , \ \}$.

The de Rham complex $\Omega^*(\mathcal{M}^d)$ is the graded commutative algebra of differential forms that is endowed with the wedge product (the wedge symbol will always be omitted in this paper) and the exterior differential d . $\Omega^p(\mathcal{M}^d)$ is the space of differential p -forms on the manifold \mathcal{M}^d , which are sections of the p -th exterior power of the cotangent bundle $T^*\mathcal{M}^d$. In the topologically trivial situation discussed in this paper $\Omega^p(\mathcal{M}^d) = C^\infty(\mathcal{M}^d) \otimes \Lambda^p\mathbb{R}^{d*}$, where the space $C^\infty(\mathcal{M}^d)$ is the space of smooth functions from \mathcal{M}^d to \mathbb{R} . The generators dx^μ of the exterior algebra $\Lambda\mathbb{R}^{d*}$ are Grassmann odd (*i.e.* anticommuting). The exterior differential is defined as $d = dx^\mu \partial_\mu$.

2 Gravity à la MacDowell - Mansouri - Stelle - West

Einstein’s theory of gravity is a nonAbelian gauge theory of a spin-2 particle, in a similar way as Yang-Mills theories³ are nonAbelian gauge theories of spin-1 particles. Local symmetries of Yang-Mills theories originate from internal global symmetries. Similarly, the gauge symmetries of Einstein gravity in the vielbein formulation⁴ originate from global space-time symmetries of its most symmetric vacua. The latter symmetries are manifest in the formulation of MacDowell, Mansouri, Stelle and West [25, 26].

This section is devoted to the presentation of the latter formulation. In the first subsection 2.1, the Einstein-Cartan formulation of gravity is reviewed and the link with the Einstein-Hilbert action without cosmological constant is explained. A cosmological constant can be introduced into the formalism, which is done in Subsection 2.2. This subsection also contains an elegant action for gravity, written by MacDowell and Mansouri. In Subsection 2.3, the improved version of this action introduced by Stelle and West is presented, the covariance under all symmetries being made manifest.

2.1 Gravity as a Poincaré gauge theory

In this subsection, the frame formulation of gravity with zero cosmological constant is reviewed. We first introduce the dynamical fields and sketch the link to the metric formulation. Then the action is written.

³See Appendix 1.2 for a brief review of Yang-Mills theories.

⁴See *e.g.* [24] for a pedagogical review on the gauge theory formulation of gravity and some of its extensions, like supergravity.

The basic idea is as follows: instead of considering the metric $g_{\mu\nu}$ as the dynamical field, two new dynamical fields are introduced: the vielbein or frame field e_μ^a and the Lorentz connection $\omega_\mu^{L\ ab}$.

The relevant fields appear through the one-forms $e^a = e_\mu^a dx^\mu$ and $\omega^{L\ ab} = -\omega^{L\ ba} = \omega_\mu^{L\ ab} dx^\mu$. The number of one-forms is equal to $d + \frac{d(d-1)}{2} = \frac{(d+1)d}{2}$, which is the dimension of the Poincaré group $ISO(d-1, 1)$. So they can be collected into a single one-form taking values in the Poincaré algebra as $\omega = e^a P_a + \frac{1}{2} \omega^{L\ ab} M_{ab}$, where P_a and M_{ab} generate $iso(d-1, 1)$ (see Appendix 1.1). The corresponding curvature is the two-form (see Appendix 1.2):

$$R = d\omega + \omega^2 \equiv T^a P_a + \frac{1}{2} R^{L\ ab} M_{ab},$$

where T^a is the torsion, given by

$$T^a = D^L e^a = de^a + \omega^{L\ a\ b} e^b,$$

and $R^{L\ ab}$ is the Lorentz curvature

$$R^{L\ ab} = D^L \omega^{L\ ab} = d\omega^{L\ ab} + \omega^{L\ a\ c} \omega^{L\ cb},$$

as follows from the Poincaré algebra (1.1)-(1.3).

To make contact with the metric formulation of gravity, one must assume that the frame e_μ^a has maximal rank d so that it gives rise to the nondegenerate metric tensor $g_{\mu\nu} = \eta_{ab} e_\mu^a e_\nu^b$. As will be shown further in this section, one is allowed to require the absence of torsion, $T_a = 0$. Then one solves this constraint and expresses the Lorentz connection in terms of the frame field, $\omega^L = \omega^L(e, \partial e)$. It can be checked that the tensor $R_{\rho\sigma, \mu\nu} = e_\mu^a e_\nu^b R_{\rho\sigma\ ab}^L$ is then expressed solely in terms of the metric, and is the Riemann tensor.

The first order action of the frame formulation of gravity is due to Weyl [27]. In any dimension $d > 1$ it can be written in the form

$$S[e_\mu^a, \omega_\mu^{L\ ab}] = \frac{1}{2\kappa^2} \int_{\mathcal{M}^d} R^{L\ bc} e^{a_1} \dots e^{a_{d-2}} \epsilon_{a_1 \dots a_{d-2} bc}, \quad (2.1)$$

where $\epsilon_{a_1 \dots a_d}$ is the invariant tensor of the special linear group $SL(d)$ and κ^2 is the gravitational constant, so that κ has dimension $(length)^{\frac{d}{2}-1}$. The Euler-Lagrange equations of the Lorentz connection

$$\frac{\delta S}{\delta \omega^{L\ bc}} \propto \epsilon_{a_1 \dots a_{d-2} bc} e^{a_1} \dots e^{a_{d-3}} T^{a_{d-2}} = 0$$

imply that the torsion vanishes. The Lorentz connection is then an auxiliary field, which can be removed from the action by solving its own (algebraic) equations of motion. The action $S = S[e, \omega^L(e, \partial e)]$ is now expressed only in terms of the vielbein⁵. Actually, only combinations of vielbeins corresponding to the metric appear and the action $S = S[g_{\mu\nu}]$ is indeed the second order Einstein-Hilbert action.

The Minkowski space-time solves $R^{L\ ab} = 0$ and $T^a = 0$. It is the most symmetrical solution of the Euler-Lagrange equations, whose global symmetries form the Poincaré group. The gauge symmetries of the action (2.1) are the diffeomorphisms and the local Lorentz transformations. Together, these gauge symmetries correspond to the gauging of the Poincaré group (see Appendix 1.3 for more comments).

2.2 Gravity as a theory of $o(d-1, 2)$ gauge fields

In the previous section, the Einstein-Cartan formulation of gravity with vanishing cosmological constant has been presented. We will now show how a nonvanishing cosmological constant can be added to this formalism. In these lectures, we will restrict ourselves to the AdS case but, for the bosonic case we focus on, everything can be rephrased for dS . One is mostly interested in the AdS case for the reason that it

⁵In the context of supergravity, this action principle [18, 28] is sometimes called the “1.5 order formalism” [29] because it combines in some sense the virtues of first and second order formalism.

is more suitable for supersymmetric extensions. Furthermore, dS and AdS have rather different unitary representations (for dS there are unitary irreducible representations the energy of which is not bounded from below).

It is rather natural to reinterpret P_a and M_{ab} as the generators of the AdS_d isometry algebra $o(d-1, 2)$. The curvature $R = d\omega + \omega^2$ then decomposes as $R = T^a P_a + \frac{1}{2} R^{ab} M_{ab}$, where the Lorentz curvature $R^{L\ ab}$ is deformed to

$$R^{ab} \equiv R^{L\ ab} + R^{cosm\ ab} \equiv R^{L\ ab} + \Lambda e^a e^b, \quad (2.2)$$

since (1.3) is deformed to (1.4).

MacDowell and Mansouri proposed an action [25], the Lagrangian of which is the (wedge) product of two curvatures (2.2) in $d = 4$

$$S^{MM}[e, \omega] = \frac{1}{4\kappa^2 \Lambda} \int_{\mathcal{M}^4} R^{a_1 a_2} R^{a_3 a_4} \epsilon_{a_1 a_2 a_3 a_4}. \quad (2.3)$$

Expressing R^{ab} in terms of $R^{L\ ab}$ and $R^{cosm\ ab}$ by (2.2), the Lagrangian is the sum of three terms: a term $R^{L\ ab} R^{cosm\ ab}$, which is the previous Lagrangian (2.1) without cosmological constant, a cosmological term $R^{cosm\ ab} R^{cosm\ ab}$ and a Gauss-Bonnet term $R^{L\ ab} R^{L\ ab}$. The latter term contains higher-derivatives but it does not contribute to the equations of motion because it is a total derivative.

In any dimension, the AdS_d space-time is defined as the most symmetrical solution of the Euler-Lagrange equations of pure gravity with the cosmological term. As explained in more detail in Section 2.3, it is a solution of the system $R^{ab} = 0$, $T^a = 0$ such that $\text{rank}(e_\mu^a) = d$.

The MacDowell-Mansouri action admits a higher dimensional generalization [30]

$$S^{MM}[e, \omega] = \frac{1}{4\kappa^2 \Lambda} \int_{\mathcal{M}^d} R^{a_1 a_2} R^{a_3 a_4} e^{a_5} \dots e^{a_d} \epsilon_{a_1 \dots a_d}. \quad (2.4)$$

Because the Gauss-Bonnet term

$$S^{GB}[e, \omega] = \frac{1}{4\kappa^2 \Lambda} \int_{\mathcal{M}^d} R^{L\ a_1 a_2} R^{L\ a_3 a_4} e^{a_5} \dots e^{a_d} \epsilon_{a_1 \dots a_d}$$

is not topological beyond $d = 4$, the field equations resulting from the action (2.4) are different from the Einstein equations in d dimensions. However the difference is by nonlinear terms that do not contribute to the free spin-2 equations [30] apart from replacing the cosmological constant Λ by $\frac{2(d-2)}{d} \Lambda$ (in such a way that no correction appears in $d = 4$, as expected). One way to see this is by considering the action

$$S^{nonlin}[e, \omega] \equiv S^{GB}[e, \omega] + \frac{d-4}{4\kappa^2} \int_{\mathcal{M}^d} \left(\frac{2}{d-2} R^{L\ a_1 a_2} e^{a_3} \dots e^{a_d} + \frac{\Lambda}{d} e^{a_1} \dots e^{a_d} \right) \epsilon_{a_1 \dots a_d}, \quad (2.5)$$

which is the sum of the Gauss-Bonnet term plus terms of the same type as the Einstein-Hilbert and cosmological terms (note that the latter are absent when $d = 4$). The variation of (2.5) is equal to

$$\delta S^{nonlin}[e, \omega] = \frac{1}{4\kappa^2 \Lambda} \int_{\mathcal{M}^d} R^{a_1 a_2} R^{a_3 a_4} \delta(e^{a_5} \dots e^{a_d}) \epsilon_{a_1 \dots a_d}, \quad (2.6)$$

when the torsion is required to be zero (*i.e.* applying the 1.5 order formalism to see that the variation over the Lorentz connection does not contribute). Indeed, the variation of the action (2.5) vanishes when $d = 4$, but when $d > 4$ the variation (2.6) is bilinear in the AdS_d field strength R^{ab} . Since the AdS_d field strength is zero in the vacuum AdS_d solution, variation of the action S^{nonlin} is nonlinear in the fluctuations near the AdS_d background. As a consequence, at the linearized level the Gauss-Bonnet term does not affect the form of the free spin-2 equations of motion, it merely redefines an overall factor in front of the action and the cosmological constant via $\kappa^2 \rightarrow (\frac{d}{2} - 1)\kappa^2$ and $\Lambda \rightarrow \frac{2(d-2)}{d}\Lambda$, respectively (as can be seen by substituting S^{GB} in (2.4) with its expression in terms of S^{nonlin} from (2.5)). Also let us note that from (2.6) it is obvious that the variation of the Gauss-Bonnet term contains second order

derivatives of the metric, *i.e.* it is of the Lanczos-Lovelock type (see [24] for a review).

Beyond the free field approximation the corrections to Einstein's field equations resulting from the action (2.4) are nontrivial for $d > 4$ and nonanalytic in Λ (as can be seen from (2.6)), having no meaningful flat limit. As will be shown later, this is analogous to the structure of HS interactions which also contain terms with higher derivatives and negative powers of Λ . The important difference is that in the case of gravity one can subtract the term (2.5) without destroying the symmetries of the model, while this is not possible in the HS gauge theories. In both cases, the flat limit $\Lambda \rightarrow 0$ is perfectly smooth at the level of the algebra (*e.g.* $o(d-1, 2) \rightarrow iso(d-1, 1)$ for gravity, see Appendix 1.1) and at the level of the free equations of motion, but it may be singular at the level of the action and nonlinear field equations.

2.3 MacDowell-Mansouri-Stelle-West gravity

The action (2.4) is not manifestly $o(d-1, 2)$ gauge invariant. Its gauge symmetries are the diffeomorphisms and the local Lorentz transformations. It is possible to make the $o(d-1, 2)$ gauge symmetry manifest by combining the vielbein and the Lorentz connection into a single field $\omega = dx^\mu \omega_\mu^{AB} M_{AB}$ and by introducing a vector V^A called *compensator*⁶. The fiber indices A, B now run from 0 to d . They are raised and lowered by the invariant mostly minus metric η_{AB} of $o(d-1, 2)$ (see Appendix 1.1).

In this subsection, the MacDowell-Mansouri-Stelle-West (MMSW) action [26] is written and it is shown how to recover the action presented in the previous subsection. The particular vacuum solution which corresponds to AdS space-time is also introduced. Finally the symmetries of the MMSW action and of the vacuum solution are analyzed.

To have a formulation with manifest $o(d-1, 2)$ gauge symmetries, a time-like vector compensator V^A has to be introduced which is constrained to have a constant norm ρ ,

$$V^A V^B \eta_{AB} = \rho^2. \quad (2.7)$$

As one will see, the constant ρ is related to the cosmological constant by

$$\rho^2 = -\Lambda^{-1}. \quad (2.8)$$

The MMSW action is ([26] for $d = 4$ and [30] for arbitrary d)

$$S^{MMSW}[\omega^{AB}, V^A] = -\frac{\rho}{4\kappa^2} \int_{\mathcal{M}^d} \epsilon_{A_1 \dots A_{d+1}} R^{A_1 A_2} R^{A_3 A_4} E^{A_5} \dots E^{A_d} V^{A_{d+1}}, \quad (2.9)$$

where the curvature or field strength R^{AB} is defined by

$$R^{AB} \equiv d\omega^{AB} + \omega^{AC} \omega_C^B$$

and the frame field E^A by

$$E^A \equiv DV^A = dV^A + \omega_B^A V^B.$$

Furthermore, in order to make link with Einstein gravity, two constraints are imposed: (i) the norm of V^A is fixed, and (ii) the frame field E_μ^A is assumed to have maximal rank equal to d . As the norm of V^A is constant, the frame field satisfies

$$E^A V_A = 0. \quad (2.10)$$

If the condition (2.7) is relaxed, then the norm of V^A corresponds to an additional dilaton-like field [26].

Let us now analyze the symmetries of the MMSW action. The action is manifestly invariant under

⁶This compensator field compensates additional symmetries serving for them as a Higgs field. The terminology is borrowed from application of conformal supersymmetry for the analysis of Poincaré supermultiplets (see *e.g.* [31]). It should *not* be confused with the homonymous - but unrelated - gauge field introduced in another approach to free HS fields [32, 33].

- Local $o(d-1, 2)$ transformations:

$$\delta\omega_\nu^{AB}(x) = D_\nu\epsilon^{AB}(x), \quad \delta V^A(x) = -\epsilon^{AB}(x)V_B(x); \quad (2.11)$$

- Diffeomorphisms:

$$\delta\omega_\nu^{AB}(x) = \partial_\nu\xi^\mu(x)\omega_\mu^{AB}(x) + \xi^\mu(x)\partial_\mu\omega_\nu^{AB}(x), \quad \delta V^A(x) = \xi^\nu(x)\partial_\nu V^A(x). \quad (2.12)$$

Let us define the covariantized diffeomorphism as the sum of a diffeomorphism with parameter ξ^μ and a $o(d-1, 2)$ local transformation with parameter $\epsilon^{AB}(\xi^\mu) = -\xi^\mu\omega_\mu^{AB}$. The action of this transformation is thus

$$\delta^{cov}\omega_\mu^{AB} = \xi^\nu R_{\nu\mu}^{AB}, \quad \delta^{cov}V^A = \xi^\nu E_\nu^A \quad (2.13)$$

by (2.11)-(2.12).

The compensator vector is pure gauge. Indeed, by local $O(d-1, 2)$ rotations one can gauge fix $V^A(x)$ to any value with $V^A(x)V_A(x) = \rho^2$. In particular, one can reach the standard gauge

$$V^A = \rho\delta_d^A. \quad (2.14)$$

Taking into account (2.10), one observes that the covariantized diffeomorphism also makes it possible to gauge fix fluctuations of the compensator $V^A(x)$ near any fixed value. Because the full list of symmetries can be represented as a combination of covariantized diffeomorphisms, local Lorentz transformations and diffeomorphisms, in the standard gauge (2.14) the gauge symmetries are spontaneously broken to the $o(d-1, 1)$ local Lorentz symmetry and diffeomorphisms. In the standard gauge, one therefore recovers the field content and the gauge symmetries of the MacDowell-Mansouri action. Let us note that covariantized diffeomorphisms (2.13) do not affect the connection ω_μ^{AB} if it is flat (*i.e.* has zero curvature $R_{\nu\mu}^{AB}$). In particular covariantized diffeomorphisms do not affect the background AdS geometry.

To show the equivalence of the action (2.9) with the action (2.4), it is useful to define a Lorentz connection by

$$\omega^{L\ AB} \equiv \omega^{AB} - \rho^{-2}(E^A V^B - E^B V^A). \quad (2.15)$$

In the standard gauge, the curvature can be expressed in terms of the vielbein $e^a \equiv E^a = \rho\omega^{a\hat{d}}$ and the nonvanishing components of the Lorentz connection $\omega^{L\ ab} = \omega^{ab}$ as

$$\begin{aligned} R^{ab} &= d\omega^{ab} + \omega^{aC}\omega_C^b = d\omega^{L\ ab} + \omega^{L\ a}{}_c\omega^{L\ cb} - \rho^{-2}e^a e^b = R^{L\ ab} + R^{cosm\ ab}, \\ R^{a\hat{d}} &= d\omega^{a\hat{d}} + \omega^{ac}\omega_c^{\hat{d}} = \rho^{-1}T^a. \end{aligned}$$

Inserting these gauge fixed expressions into the MMSW action yields the action (2.4), where $\Lambda = -\rho^{-2}$. The MMSW action is thus equivalent to (2.4) by partially fixing the gauge invariance. Let us note that a version of the covariant compensator formalism applicable to the case with zero cosmological constant was developed in [34].

Let us now consider the vacuum equations $R^{AB}(\omega_0) = 0$. They are equivalent to $T^a(\omega_0) = 0$ and $R^{ab}(\omega_0) = 0$ and, under the condition that $\text{rank}(E_\nu^A) = d$, they uniquely define the local geometry of AdS_d with parameter ρ , in a coordinate independent way. The solution ω_0 also obviously satisfies the equations of motion of the MMSW action. To find the symmetries of the vacuum solution ω_0 , one first notes that vacuum solutions are sent onto vacuum solutions by diffeomorphisms and local AdS transformations, because they transform the curvature homogeneously. Since covariantized diffeomorphisms do not affect ω_0 , to find symmetries of the chosen solution ω_0 it is enough to check its transformation law under local $o(d-1, 2)$ transformation. Indeed, by adjusting an appropriate covariantized diffeomorphism it is always possible to keep the compensator invariant.

The solution ω_0 is invariant under those $o(d-1, 2)$ gauge transformations for which the parameter $\epsilon^{AB}(x)$ satisfies

$$0 = D_0\epsilon^{AB}(x) = d\epsilon^{AB}(x) + \omega_0^A{}_C(x)\epsilon^{CB}(x) - \omega_0^B{}_C(x)\epsilon^{CA}(x). \quad (2.16)$$

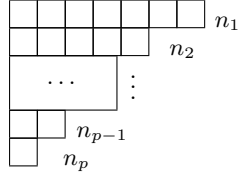
This equation fixes the derivatives $\partial_\mu \epsilon^{AB}(x)$ in terms of $\epsilon^{AB}(x)$ itself. In other words, once $\epsilon^{AB}(x_0)$ is chosen for some x_0 , $\epsilon^{AB}(x)$ can be reconstructed for all x in a neighborhood of x_0 , since by consistency⁷ all derivatives of the parameter can be expressed as functions of the parameter itself. The parameters $\epsilon^{AB}(x_0)$ remain arbitrary, being parameters of the global symmetry $o(d-1, 2)$. This means that, as expected for AdS_d space-time, the symmetry of the vacuum solution ω_0 is the global $o(d-1, 2)$.

The lesson is that, to describe a gauge model that has a global symmetry h , it is useful to reformulate it in terms of the gauge connections ω and curvatures R of h in such a way that the zero curvature condition $R = 0$ solves the field equations and provides a solution with h as its global symmetry. If a symmetry h is not known, this observation can be used the other way around: by reformulating the dynamics à la MacDowell-Mansouri one might guess the structure of an appropriate curvature R and thereby the nonAbelian algebra h .

3 Young tableaux and Howe duality

In this section, the Young tableaux are introduced. They characterize the irreducible representations of $gl(M)$ and $o(M)$. A representation of these algebras that will be useful in the sequel is built.

A *Young tableau* $\{n_i\}$ ($i = 1, \dots, p$) is a diagram which consists of a finite number $p > 0$ of rows of identical squares. The lengths of the rows are finite and do not increase: $n_1 \geq n_2 \geq \dots \geq n_p \geq 0$. The Young tableau $\{n_i\}$ is represented as follows:



Let us consider covariant tensors of $gl(M)$: $A_{a,b,c,\dots}$ where $a, b, c, \dots = 1, 2, \dots, M$. Simple examples of these are the symmetric tensor $A_{a,b}^S$ such that $A_{a,b}^S - A_{b,a}^S = 0$, or the antisymmetric tensor $A_{a,b}^A$ such that $A_{a,b}^A + A_{b,a}^A = 0$.

A complete set of covariant tensors irreducible under $gl(M)$ is given by the tensors $A_{a_1^1 \dots a_{n_1}^1, \dots, a_1^p \dots a_{n_p}^p}$ ($n_i \geq n_{i+1}$) that are symmetric in each set of indices $\{a_1^i \dots a_{n_i}^i\}$ with fixed i and that vanish when one symmetrizes the indices of a set $\{a_1^i \dots a_{n_i}^i\}$ with any index a_l^j with $j > i$. The properties of these irreducible tensors can be conveniently encoded into Young tableaux. The Young tableau $\{n_i\}$ ($i = 1, \dots, p$) is associated with the tensor $A_{a_1^1 \dots a_{n_1}^1, \dots, a_1^p \dots a_{n_p}^p}$. Each box of the Young tableau is related to an index of the tensor, boxes of the same row corresponding to symmetric indices. Finally, the symmetrization of all the indices of a row with an index from any row below vanishes. In this way, the irreducible tensors A_{ab}^S and $A_{a,b}^A$ are associated with the Young tableaux $\square\square$ and $\begin{smallmatrix} \square \\ \square \end{smallmatrix}$, respectively.

Let us introduce the polynomial algebra $\mathbb{R}[Y_i^a]$ generated by commuting generators Y_i^a where $i = 1, \dots, p$; $a = 1, \dots, M$. Elements of the algebra $\mathbb{R}[Y_i^a]$ are of the form

$$A(Y) = A_{a_1^1 \dots a_{n_1}^1, \dots, a_1^p \dots a_{n_p}^p} Y_1^{a_1^1} \dots Y_1^{a_{n_1}^1} \dots Y_p^{a_1^p} \dots Y_p^{a_{n_p}^p}.$$

⁷The identity $D_0^2 = R_0 = 0$ ensures consistency of the system (2.16), which is overdetermined because it contains $\frac{d^2(d+1)}{2}$ equations for $\frac{d(d+1)}{2}$ unknowns. Consistency in turn implies that higher space-time derivatives $\partial_{\nu_1} \dots \partial_{\nu_n} \epsilon^{AB}(x)$ obtained by hitting (2.16) $n-1$ times with $D_{0\nu_k}$ are guaranteed to be symmetric in the indices $\nu_1 \dots \nu_n$.

The condition that A is irreducible under $gl(M)$, *i.e.* is a Young tableau $\{n_i\}$, can be expressed as

$$\begin{aligned} Y_i^a \frac{\partial}{\partial Y_i^a} A(Y) &= n_i A(Y), \\ Y_i^b \frac{\partial}{\partial Y_j^b} A(Y) &= 0, \quad i < j, \end{aligned} \quad (3.1)$$

where no sum on i is to be understood in the first equation. Let us first note that, as the generators Y_i^a commute, the tensor A is automatically symmetric in each set of indices $\{a_1^i \dots a_{n_i}^i\}$. The operator on the l.h.s. of the first equation of (3.1) then counts the number of Y_i^a 's, the index i being fixed and the index a arbitrary. The first equation thus ensures that there are n_i indices contracted with Y_i , forming the set $a_1^i \dots a_{n_i}^i$. The second equation of (3.1) is equivalent to the vanishing of the symmetrization of a set of indices $\{a_1^i \dots a_{n_i}^i\}$ with an index b to the right. Indeed, the operator on the l.h.s. replaces a generator Y_j^b by a generator Y_i^b , $j > i$, thus projecting $A_{\dots, a_1^i \dots a_{n_i}^i, \dots, b, \dots}$ on its component symmetric in $\{a_1^i \dots a_{n_i}^i b\}$.

Two types of generators appear in the above equations: the generators

$$t_b^a = Y_i^a \frac{\partial}{\partial Y_i^b} \quad (3.2)$$

of $gl(M)$ and the generators

$$l_i^j = Y_i^a \frac{\partial}{\partial Y_j^a} \quad (3.3)$$

of $gl(p)$. These generators commute

$$[l_i^j, t_b^a] = 0 \quad (3.4)$$

and the algebras $gl(p)$ and $gl(M)$ are said to be *Howe dual* [35]. The important fact is that the irreducibility conditions (3.1) of $gl(M)$ are the highest weight conditions with respect to $gl(p)$.

When all the lengths n_i have the same value n , $A(Y)$ is invariant under $sl(p) \subset gl(p)$. Moreover, exchange of any two rows of this rectangular Young tableau only brings a sign factor $(-1)^n$, as is easy to prove combinatorically. The conditions (3.1) are then equivalent to:

$$\left(Y_i^a \frac{\partial}{\partial Y_j^a} - \frac{1}{p} \delta_i^j Y_k^a \frac{\partial}{\partial Y_k^a} \right) A(Y) = 0. \quad (3.5)$$

Indeed, let us first consider the equation for $i = j$. The operator $Y_k^a \frac{\partial}{\partial Y_k^a}$ (where there is a sum over k and a) counts the total number m of Y 's in $A(Y)$, while $Y_i^a \frac{\partial}{\partial Y_i^a}$ counts the number of Y_i 's for some fixed i . The condition can only be satisfied if $\frac{m}{p}$ is an integer, *i.e.* $m = np$ for some integer n . As the condition is true for all i 's, there are thus n Y_i 's for every i . In other words, the tensorial coefficient of $A(Y)$ has p sets of n indices, *i.e.* is rectangular. The fact that it is a Young tableau is insured by the condition (3.5) for $i < j$, which is simply the second condition of (3.1). That the condition (3.5) is true both for $i < j$ and for $i > j$ is a consequence of the simple fact that any finite-dimensional $sl(p)$ module with zero $sl(p)$ weights (which are differences of lengths of the rows of the Young tableau) is $sl(p)$ invariant. Alternatively, this follows from the property that exchange of rows leaves a rectangular Young tableau invariant.

If there are p_1 rows of length n_1 , p_2 rows of length n_2 , etc., then $A(Y)$ is invariant under $sl(p_1) \oplus sl(p_2) \oplus sl(p_3) \oplus \dots$, as well as under permutations within each set of p_i rows of length n_i .

To construct irreducible representations of $o(M - N, N)$, one needs to add the condition that A is

traceless⁸, which can be expressed as:

$$\frac{\partial^2}{\partial Y_i^a \partial Y_j^b} \eta^{ab} A(Y) = 0, \quad \forall i, j, \quad (3.6)$$

where η_{ab} is the invariant metric of $o(M - N, N)$. The generators of $o(M - N, N)$ are given by

$$t_{ab} = \frac{1}{2}(\eta_{ac} t_b^c - \eta_{bc} t_a^c). \quad (3.7)$$

They commute with the generators

$$k_{ij} = \eta_{ab} Y_i^a Y_j^b, \quad l_i^j = Y_i^a \frac{\partial}{\partial Y_j^a}, \quad m^{ij} = \eta^{ab} \frac{\partial^2}{\partial Y_i^a \partial Y_j^b} \quad (3.8)$$

of $sp(2p)$. The conditions (3.1) and (3.6) are highest weight conditions for the algebra $sp(2p)$ which is Howe dual to $o(M - N, N)$.

In the notation developed here, the irreducible tensors are manifestly symmetric in groups of indices. This is a convention: one could as well choose to have manifestly antisymmetric groups of indices corresponding to columns of the Young tableau. An equivalent implementation of the conditions for a tensor to be a Young tableau can be performed in the antisymmetric convention, by taking fermionic generators Y .

To end up with this introduction to Young diagrams, we give two “multiplication rules” of one box with an arbitrary Young tableau. More precisely, the tensor product of a vector (characterized by one box) with an irreducible tensor under $gl(M)$ characterized by a given Young tableau decomposes as the direct sum of irreducible tensors under $gl(M)$ corresponding to all possible Young tableaux obtained by adding one box to the initial Young tableau, e.g.

$$\begin{array}{|c|c|} \hline & \\ \hline \end{array} \otimes \begin{array}{|c|} \hline * \\ \hline \end{array} \simeq \begin{array}{|c|c|c|} \hline & & * \\ \hline & & \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline & \\ \hline * & \\ \hline \end{array} \oplus \begin{array}{|c|} \hline & \\ \hline * & \\ \hline \end{array}.$$

For the (pseudo)orthogonal algebras $o(M - N, N)$, the tensor product of a vector (characterized by one box) with a traceless tensor characterized by a given Young tableau decomposes as the direct sum of traceless tensors under $o(M - N, N)$ corresponding to all possible Young tableaux obtained by adding or removing one box from the initial Young tableau (a box can be removed as a result of contraction of indices), e.g.

$$\begin{array}{|c|c|} \hline & \\ \hline \end{array} \otimes \begin{array}{|c|} \hline \\ \hline \end{array} \simeq \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \oplus \begin{array}{|c|} \hline & \\ \hline & \\ \hline * & \\ \hline \end{array} \oplus \begin{array}{|c|} \hline \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}.$$

4 Free symmetric higher spin gauge fields as one-forms

Properties of HS gauge theories are to a large extent determined by the HS global symmetries of their most symmetric vacua. The HS symmetry restricts interactions and fixes spectra of spins of massless fields in HS theories as ordinary supersymmetry does in supergravity. To elucidate the structure of a global HS algebra h it is useful to follow the approach in which fields, action and transformation laws are formulated in terms of the connection of h .

⁸For $M = 2N$ modulo 4, the irreducibility conditions also include the (anti)selfduality conditions on the tensors described by Young tableaux with $M/2$ rows. However, these conditions are not used in the analysis of HS dynamics in this paper.

4.1 Metric-like formulation of higher spins

The free HS gauge theories were originally formulated in terms of completely symmetric and double-traceless HS-fields $\varphi_{\nu_1 \dots \nu_s}$ [36, 37], in a way analogous to the metric formulation of gravity (see [38] for recent reviews on the metric-like formulation of HS theories). In Minkowski space-time $\mathbb{R}^{d-1,1}$, the spin- s Fronsdal action is

$$\begin{aligned} S_2^{(s)}[\varphi] = & \frac{1}{2} \int d^d x \left(\partial_\nu \varphi_{\mu_1 \dots \mu_s} \partial^\nu \varphi^{\mu_1 \dots \mu_s} \right. \\ & - \frac{s(s-1)}{2} \partial_\nu \varphi^\lambda{}_{\lambda \mu_3 \dots \mu_s} \partial^\nu \varphi^\rho{}_{\rho \mu_3 \dots \mu_s} + s(s-1) \partial_\nu \varphi^\lambda{}_{\lambda \mu_3 \dots \mu_s} \partial_\rho \varphi^{\nu \rho \mu_3 \dots \mu_s} \\ & \left. - s \partial_\nu \varphi^\nu{}_{\mu_2 \dots \mu_s} \partial_\rho \varphi^{\rho \mu_2 \dots \mu_s} - \frac{s(s-1)(s-2)}{4} \partial_\nu \varphi^{\nu \rho}{}_{\rho \mu_2 \dots \mu_s} \partial_\lambda \varphi^\sigma{}_{\sigma \mu_2 \dots \mu_s} \right), \end{aligned} \quad (4.1)$$

where the metric-like field is double-traceless ($\eta^{\mu_1 \mu_2} \eta^{\mu_3 \mu_4} \varphi_{\mu_1 \dots \mu_s} = 0$) and has the dimension of $(length)^{1-d/2}$. This action is invariant under Abelian HS gauge transformations

$$\delta \varphi_{\mu_1 \dots \mu_s} = \partial_{\{\mu_1} \epsilon_{\mu_2 \dots \mu_s\}}, \quad (4.2)$$

where the gauge parameter is a completely symmetric and traceless rank- $(s-1)$ tensor,

$$\eta^{\mu_1 \mu_2} \epsilon_{\mu_1 \dots \mu_{s-1}} = 0.$$

For spin $s = 2$, (4.1) is the Pauli-Fierz action that is obtained from the linearization of the Einstein-Hilbert action via $g_{\mu\nu} = \eta_{\mu\nu} + \kappa \varphi_{\mu\nu}$ and the gauge transformations (4.2) correspond to linearized diffeomorphisms. The approach followed in this section is to generalize the MMSW construction of gravity to the case of free HS gauge fields in AdS backgrounds. The free HS dynamics will then be expressed in terms of one-form connections taking values in certain representations of the AdS_d isometry algebra $o(d-1, 2)$.

The procedure is similar to that of Section 2 for gravity. The HS metric-like field is replaced by a frame-like field and a set of connections. These new fields are then united in a single connection. The action is given in terms of the latter connection and a compensator vector. It is constructed in such a way that it reproduces the dynamics of the Fronsdal formulation, once auxiliary fields are removed and part of the gauge invariance is fixed.

4.2 Frame-like formulation of higher spins

The double-traceless metric-like HS gauge field $\varphi_{\mu_1 \dots \mu_s}$ is replaced by a frame-like field $e_\mu^{a_1 \dots a_{s-1}}$, a Lorentz-like connection $\omega_\mu^{a_1 \dots a_{s-1}, b}$ [39] and a set of connections $\omega_\mu^{a_1 \dots a_{s-1}, b_1 \dots b_t}$ called *extra fields*, where $t = 2, \dots, s-1$, $s > 2$ [40, 41]. All fields e, ω are traceless in the fiber indices a, b , which have the symmetry of the Young tableaux $\boxed{}_t^{s-1}$, where $t = 0$ for the frame-like field and $t = 1$ for the Lorentz-like connection. The metric-like field arises as the completely symmetric part of the frame field [39],

$$\varphi_{\mu_1 \dots \mu_s} = \rho^{3-s-\frac{d}{2}} e_{\{\mu_1, \mu_2 \dots \mu_s\}},$$

where the dimensionful factor of $\rho^{3-s-\frac{d}{2}}$ is introduced for the future convenience (ρ is a length scale) and all fiber indices have been lowered using the AdS or flat frame field $e_{0\mu}^a$ defined in Section 2. From the fiber index tracelessness of the frame field follows automatically that the field $\varphi_{\mu_1 \dots \mu_s}$ is double traceless.

The frame-like field and other connections are then combined [30] into a connection one-form $\omega^{A_1 \dots A_{s-1}, B_1 \dots B_{s-1}}$ (where $A, B = 0, \dots, d-1, \hat{d}$) taking values in the irreducible $o(d-1, 2)$ -module

characterized by the two-row traceless rectangular Young tableau $\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}$ of length $s-1$, that is

$$\begin{aligned} \omega_\mu^{A_1 \dots A_{s-1}, B_1 \dots B_{s-1}} &= \omega_\mu^{\{A_1 \dots A_{s-1}\}, B_1 \dots B_{s-1}} = \omega_\mu^{A_1 \dots A_{s-1}, \{B_1 \dots B_{s-1}\}}, \\ \omega_\mu^{\{A_1 \dots A_{s-1}, A_s\} B_2 \dots B_{s-1}} &= 0, \quad \omega_\mu^{A_1 \dots A_{s-3} C}_{C, B_1 \dots B_{s-1}} = 0. \end{aligned} \quad (4.3)$$

One also introduces a time-like vector V^A of constant norm ρ . The component of the connection $\omega^{A_1 \dots A_{s-1}, B_1 \dots B_{s-1}}$ that is most parallel to V^A is the frame-like field

$$E^{A_1 \dots A_{s-1}} = \omega^{A_1 \dots A_{s-1}, B_1 \dots B_{s-1}} V_{B_1} \dots V_{B_{s-1}},$$

while the less V -longitudinal components are the other connections. Note that the contraction of the connection with more than $s-1$ compensators V^A is zero by virtue of (4.3). Let us be more explicit in a specific gauge. As in the MMSW gravity reformulation, one can show that V^A is a pure gauge field and that one can reach the standard gauge $V^A = \delta_d^A \rho$ (the argument will not be repeated here). In the standard gauge, the frame field and the connections are given by

$$\begin{aligned} e^{a_1 \dots a_{s-1}} &= \rho^{s-1} \omega^{a_1 \dots a_{s-1}, \hat{d} \dots \hat{d}}, \\ \omega^{a_1 \dots a_{s-1}, b_1 \dots b_t} &= \rho^{s-1-t} \Pi(\omega^{a_1 \dots a_{s-1}, b_1 \dots b_t \hat{d} \dots \hat{d}}), \end{aligned}$$

where the powers of ρ originate from a corresponding number of contractions with the compensator vector V^A and Π is a projector to the Lorentz-traceless part of a Lorentz tensor, which is needed for $t \geq 2$. These normalization factors are consistent with the fact that the auxiliary fields $\omega_\mu^{a_1 \dots a_{s-1}, b_1 \dots b_t}$ will be found to be expressed via t partial derivatives of the frame field $e_\mu^{a_1 \dots a_{s-1}}$ at the linearized level.

The linearized field strength or curvature is defined as the $o(d-1, 2)$ covariant derivative of the connection $\omega^{A_1 \dots A_{s-1}, B_1 \dots B_{s-1}}$, *i.e.* by

$$\begin{aligned} R_1^{A_1 \dots A_{s-1}, B_1 \dots B_{s-1}} &= D_0 \omega^{A_1 \dots A_{s-1}, B_1 \dots B_{s-1}} \\ &= d\omega^{A_1 \dots A_{s-1}, B_1 \dots B_{s-1}} + \omega_0^{A_1}{}_C \omega^{CA_2 \dots A_{s-1}, B_1 \dots B_{s-1}} + \dots \\ &\quad + \omega_0^{B_1}{}_C \omega^{A_1 \dots A_{s-1}, CB_2 \dots B_{s-1}} + \dots, \end{aligned} \quad (4.4)$$

where the dots stand for the terms needed to get an expression symmetric in $A_1 \dots A_{s-1}$ and $B_1 \dots B_{s-1}$, and $\omega_0^A{}_B$ is the $o(d-1, 2)$ connection associated to the AdS space-time solution, as defined in Section 2. The connection $\omega_\mu^{A_1 \dots A_{s-1}, B_1 \dots B_{s-1}}$ has dimension $(length)^{-1}$ in such a way that the field strength $R_{\mu\nu}^{A_1 \dots A_{s-1}, B_1 \dots B_{s-1}}$ has proper dimension $(length)^{-2}$.

As $(D_0)^2 = R_0 = 0$, the linearized curvature R_1 is invariant under Abelian gauge transformations of the form

$$\delta_0 \omega^{A_1 \dots A_{s-1}, B_1 \dots B_{s-1}} = D_0 \epsilon^{A_1 \dots A_{s-1}, B_1 \dots B_{s-1}}. \quad (4.5)$$

The gauge parameter $\epsilon^{A_1 \dots A_{s-1}, B_1 \dots B_{s-1}}$ has the symmetry $\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}$ and is traceless.

Before writing the action, let us analyze the frame field and its gauge transformations, in the standard gauge. According to the multiplication rule formulated in the end of Section 3, the frame field $e_\mu^{a_1 \dots a_{s-1}}$ contains three irreducible (traceless) Lorentz components characterized by the symmetry of their indices:

$\begin{array}{|c|} \hline \\ \hline \end{array}_s$, $\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}_1^{s-1}$ and $\begin{array}{|c|} \hline \\ \hline \end{array}_{s-2}$, where the last tableau describes the trace component of the frame field $e_\mu^{a_1 \dots a_{s-1}}$. Its gauge transformations are given by (4.5) and read

$$\delta_0 e^{a_1 \dots a_{s-1}} = D_0^L \epsilon^{a_1 \dots a_{s-1}} - e_0^c \epsilon^{a_1 \dots a_{s-1}, c}.$$

The parameter $\epsilon^{a_1 \dots a_{s-1}, c}$ is a generalized local Lorentz parameter. It allows us to gauge away the traceless component $\begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}$ of the frame field. The other two components of the latter just correspond

to a completely symmetric double traceless Fronsdal field $\varphi_{\mu_1 \dots \mu_s}$. The remaining invariance is then the Fronsdal gauge invariance (4.2) with a traceless completely symmetric parameter $\epsilon^{a_1 \dots a_{s-1}}$.

4.3 Action of higher spin gauge fields

For a given spin s , the most general $o(d-1,2)$ -invariant action that is quadratic in the linearized curvatures (4.4) and, for the rest, built only from the compensator V^C and the background frame field $E_0^B = D_0 V^B$ is

$$S_2^{(s)}[\omega_\mu^{A_1 \dots A_{s-1}, B_1 \dots B_{s-1}}, \omega_0^{AB}, V^C] = \frac{1}{2} \sum_{p=0}^{s-2} a(s, p) S^{(s,p)}[\omega_\mu^{A_1 \dots A_{s-1}, B_1 \dots B_{s-1}}, \omega_0^{AB}, V^C], \quad (4.6)$$

where $a(s, p)$ is the *a priori* arbitrary coefficient of the term

$$S^{(s,p)}[\omega, \omega_0, V] = \epsilon_{A_1 \dots A_{d+1}} \int_{M^d} E_0^{A_5} \dots E_0^{A_d} V^{A_{d+1}} V_{C_1} \dots V_{C_{2(s-2-p)}} \times \\ \times R_1^{A_1 B_1 \dots B_{s-2}, A_2 C_1 \dots C_{s-2-p} D_1 \dots D_p} R_1^{A_3 B_1 \dots B_{s-2}, A_4 C_{s-1-p} \dots C_{2(s-2-p)} D_1 \dots D_p}.$$

This action is manifestly invariant under diffeomorphisms, local $o(d-1,2)$ transformations (2.11) and Abelian HS gauge transformations (4.5), which leave invariant the linearized HS curvatures (4.4). Having fixed the AdS_d background gravitational field ω_0^{AB} and compensator V^A , the diffeomorphisms and the local $o(d-1,2)$ transformations break down to the AdS_d global symmetry $o(d-1,2)$.

As will be explained in Sections 6 and 8, the connections $\omega_\mu^{a_1 \dots a_{s-1}, b_1 \dots b_t}$ can be expressed as t derivatives of the frame-like field, via analogues of the torsion constraint. Therefore, to make sure that higher-derivative terms are absent from the free theory, the coefficients $a(s, p)$ are chosen in such a way that the Euler-Lagrange derivatives are nonvanishing only for the frame field and the first connection ($t = 1$). All extra fields, *i.e.* the connections $\omega_\mu^{a_1 \dots a_{s-1}, b_1 \dots b_t}$ with $t > 1$, should appear only through total derivatives⁹. This requirement fixes uniquely the spin- s free action up to a coefficient $b(s)$ in front of the action. More precisely, the coefficient $a(s, p)$ is essentially a relative coefficient given by [30]

$$a(s, p) = b(s)(-\Lambda)^{-(s-p-1)} \frac{(d-5+2(s-p-2))!!(s-p-1)}{(s-p-2)!},$$

where $b(s)$ is the arbitrary spin-dependent factor.

The equations of motion for $\omega_\mu^{a_1 \dots a_{s-1}, b}$ are equivalent to the “zero-torsion condition”

$$R_{1 A_1 \dots A_{s-1}, B_1 \dots B_{s-1}} V^{B_1} \dots V^{B_{s-1}} = 0.$$

They imply that $\omega_\mu^{a_1 \dots a_{s-1}, b}$ is an auxiliary field that can be expressed in terms of the first derivative of the frame field modulo a pure gauge part associated with the symmetry parameter $\epsilon_\mu^{a_1 \dots a_{s-1}, b_1 b_2}$. Substituting the found expression for $\omega_\mu^{a_1 \dots a_{s-1}, b}$ into the HS action yields an action only expressed in terms of the frame field and its first derivative, modulo total derivatives. As gauge symmetries told us, the action actually depends only on the completely symmetric part of the frame field, *i.e.* the Fronsdal field. Moreover, the action (4.6) has the same gauge invariance as Fronsdal’s one, thus it must be proportional to the Fronsdal action (4.1) because the latter is fixed up to a front factor by the requirements of being gauge invariant and of being of second order in the derivatives of the field [42].

⁹The extra fields show up in the nonlinear theory and are responsible for the higher-derivatives as well as for the terms with negative powers of Λ in the interaction vertices.

5 Simplest higher spin algebras

In the previous section, the dynamics of free HS gauge fields has been expressed as a theory of one-forms, the $o(d-1, 2)$ fiber indices of which have symmetries characterized by two-row rectangular Young tableaux. This suggests that there exists a nonAbelian HS algebra $h \supset o(d-1, 2)$ that admits a basis formed by a set of elements $T_{A_1 \dots A_{s-1}, B_1 \dots B_{s-1}}$ in irreducible representations of $o(d-1, 2)$ characterized by such Young tableaux. More precisely, the basis elements $T_{A_1 \dots A_{s-1}, B_1 \dots B_{s-1}}$ satisfy the following properties $T_{\{A_1 \dots A_{s-1}, A_s\} B_2 \dots B_{s-1}} = 0$, $T_{A_1 \dots A_{s-3} C C, B_1 \dots B_{s-1}} = 0$, and the basis contains the $o(d-1, 2)$ basis elements $T_{A,B} = -T_{B,A}$ such that all generators transform as $o(d-1, 2)$ tensors

$$[T_{C,D}, T_{A_1 \dots A_{s-1}, B_1 \dots B_{s-1}}] = \eta_{DA_1} T_{C A_2 \dots A_{s-1}, B_1 \dots B_{s-1}} + \dots \quad (5.1)$$

The question is whether a nonAbelian algebra h with these properties really exists. If yes, the Abelian curvatures R_1 can be understood as resulting from the linearization of the nonAbelian field curvatures $R = dW + W^2$ of h with the h gauge connection $W = \omega_0 + \omega$, where ω_0 is some fixed flat (*i.e.* vanishing curvature) zero-order connection of the subalgebra $o(d-1, 2) \subset h$ and ω is the first-order dynamical part which describes massless fields of various spins.

The HS algebras with these properties were originally found for the case of AdS_4 [43, 44, 45, 46] in terms of spinor algebras. Then this construction was extended to HS algebras in AdS_3 [47, 48, 49] and to $d = 4$ conformal HS algebras [50, 51] equivalent to the AdS_5 algebras of [7]. The $d = 7$ HS algebras [52] were also built in spinorial terms. Conformal HS conserved currents in any dimension, generating HS symmetries with the parameters carrying representations of the conformal algebra $o(d, 2)$ described by various rectangular two-row Young tableaux, were found in [53]. The realization of the conformal HS algebra h in any dimension in terms of a quotient of the universal enveloping algebra¹⁰ $\mathcal{U}(o(d, 2))$ was given by Eastwood in [54]. Here we use the construction of the same algebra as given in [16], which is based on vector oscillator algebra (*i.e.* Weyl algebra).

5.1 Weyl algebras

The Weyl algebra A_{d+1} is the associative algebra generated by the oscillators \hat{Y}_i^A , where $i = 1, 2$ and $A = 0, 1, \dots, d$, satisfying the commutation relations

$$[\hat{Y}_i^A, \hat{Y}_j^B] = \epsilon_{ij} \eta^{AB}, \quad (5.2)$$

where $\epsilon_{ij} = -\epsilon_{ji}$ and $\epsilon_{12} = \epsilon^{12} = 1$. The invariant metrics $\eta_{AB} = \eta_{BA}$ and symplectic form ϵ^{ij} of $o(d-1, 2)$ and $sp(2)$, respectively, are used to raise and lower indices in the usual manner $A^A = \eta^{AB} A_B$, $a^i = \epsilon^{ij} a_j$, $a_i = a^j \epsilon_{ji}$. The Weyl algebra A_{d+1} can be realized by taking as generators

$$\hat{Y}_1^A = \eta^{AB} \frac{\partial}{\partial X^B}, \quad \hat{Y}_2^A = X^A,$$

i.e. the Weyl algebra is realized as the algebra of differential operators acting on formal power series $\Phi(X)$ in the variable X^A . One can consider both complex ($A_{d+1}(\mathbb{C})$) and real Weyl ($A_{d+1}(\mathbb{R})$) algebras. One can also construct the (say, real) Weyl algebra $A_{d+1}(\mathbb{R})$ starting from the associative algebra $\mathbb{R} \langle \hat{Y}_i^A \rangle$ freely generated by the variables \hat{Y}_i^A , *i.e.* spanned by all (real) linear combinations of all possible products of the variables \hat{Y}_i^A . The real Weyl algebra A_{d+1} is realized as the quotient of $\mathbb{R} \langle \hat{Y}_i^A \rangle$ by the ideal made of all elements proportional to

$$\hat{Y}_i^A \hat{Y}_j^B - \hat{Y}_j^B \hat{Y}_i^A - \epsilon_{ij} \eta^{AB}.$$

In order to pick one representative of each equivalence class, we work with Weyl ordered operators. These are the operators completely symmetric under the exchange of \hat{Y}_i^A 's. The generic element of

¹⁰A universal enveloping algebra is defined as follows. Let \mathcal{S} be the associative algebra that is freely generated by the elements of a Lie algebra s . Let \mathcal{I} be the ideal of \mathcal{S} generated by elements of the form $xy - yx - [x, y]$ ($x, y \in s$). The quotient $\mathcal{U}(s) = \mathcal{S}/\mathcal{I}$ is called the universal enveloping algebra of s .

A_{d+1} is then of the form

$$f(\hat{Y}) = \sum_{p=0}^{\infty} \phi_{A_1 \dots A_p}^{i_1 \dots i_p} \hat{Y}_{i_1}^{A_1} \dots \hat{Y}_{i_p}^{A_p}, \quad (5.3)$$

where $\phi_{A_1 \dots A_p}^{i_1 \dots i_p}$ is symmetric under the exchange $(i_k, A_k) \leftrightarrow (i_l, A_l)$. Equivalently, one can define basis elements $S^{A_1 \dots A_m, B_1 \dots B_n}$ that are completely symmetrized products of m \hat{Y}_1^A 's and n \hat{Y}_2^B 's (e.g. $S^{A,B} = \{\hat{Y}_1^A, \hat{Y}_2^B\}$), and write the generic element as

$$f(\hat{Y}) = \sum_{m,n} f_{A_1 \dots A_m, B_1 \dots B_n} S^{A_1 \dots A_m, B_1 \dots B_n}, \quad (5.4)$$

where the coefficients $f_{A_1 \dots A_m, B_1 \dots B_n}$ are symmetric in the indices A_i and B_j .

The elements

$$T^{AB} = -T^{BA} = \frac{1}{4} \{\hat{Y}^{Ai}, \hat{Y}_i^B\} \quad (5.5)$$

satisfy the $o(d-1, 2)$ algebra

$$[T^{AB}, T^{CD}] = \frac{1}{2} (\eta^{BC} T^{AD} - \eta^{AC} T^{BD} - \eta^{BD} T^{AC} + \eta^{AD} T^{BC})$$

because of (5.2). When the Weyl algebra is realized as the algebra of differential operators, then $T^{AB} = X^{[A} \partial^{B]}$ generates rotations of $\mathbb{R}^{d-1,2}$ acting on a scalar $\Phi(X^A)$.

The operators

$$t_{ij} = t_{ji} = \frac{1}{2} \{\hat{Y}_i^A, \hat{Y}_j^B\} \eta_{AB} \quad (5.6)$$

generate $sp(2)$. The various bilinears T^{AB} and t_{ij} commute

$$[T^{AB}, t_{ij}] = 0, \quad (5.7)$$

thus forming a Howe dual pair $o(d-1, 2) \oplus sp(2)$.

5.2 Definition of the higher spin algebras

Let us consider the subalgebra \mathcal{S} of elements $f(\hat{Y})$ of the complex Weyl algebra $A_{d+1}(\mathbb{C})$ that are invariant under $sp(2)$, i.e. $[f(\hat{Y}), t_{ij}] = 0$. Replacing $f(\hat{Y})$ by its Weyl symbol $f(Y)$, which is the ordinary function of commuting variables Y that has the same power series expansion as $f(\hat{Y})$ in the Weyl ordering, the $sp(2)$ invariance condition takes the form (a simple proof will be given in Section 11)

$$\left(\epsilon_{kj} Y_i^A \frac{\partial}{\partial Y_k^A} + \epsilon_{ki} Y_j^A \frac{\partial}{\partial Y_k^A} \right) f(Y_i^A) = 0, \quad (5.8)$$

which is equivalent to (3.5) for $p = 2$. This condition implies that the coefficients $f_{A_1 \dots A_n, B_1 \dots B_m}$ vanish except when $n = m$, and the nonvanishing coefficients carry irreducible representations of $gl(d+1)$ corresponding to two-row rectangular Young tableaux. The $sp(2)$ invariance condition means in particular that (the symbol of) any element of \mathcal{S} is an even function of Y_i^A . Let us note that the rôle of $sp(2)$ in our construction is reminiscent of that of $sp(2)$ in the conformal framework description of dynamical models (two-time physics) [55, 56].

However, the associative algebra \mathcal{S} is not simple. It contains the ideal \mathcal{I} spanned by the elements of \mathcal{S} of the form $g = t_{ij} g^{ij} = g^{ij} t_{ij}$. Due to the definition of t_{ij} (5.6), all traces of two-row Young tableaux are contained in \mathcal{I} . As a result, the associative algebra $\mathcal{A} = \mathcal{S}/\mathcal{I}$ contains only all traceless two-row rectangular tableaux. Let us choose a basis $\{T_s\}$ of \mathcal{A} where the elements T_s carry an irreducible representation of $o(d-1, 2)$ characterized by a two-row Young tableau with $s-1$ columns:

$$T_s \sim \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} \begin{array}{l} s-1 \\ s-1 \end{array}.$$

Now consider the complex Lie algebra $h_{\mathbb{C}}$ obtained from the associative algebra \mathcal{A} by taking the commutator as Lie bracket, the associativity property of \mathcal{A} thereby translating into the Jacobi identity of $h_{\mathbb{C}}$. It admits several inequivalent real forms $h_{\mathbb{R}}$ such that $h_{\mathbb{C}} = h_{\mathbb{R}} \oplus i h_{\mathbb{R}}$. The particular real form that corresponds to a unitary HS theory is denoted by $hu(1|2:[d-1, 2])$. This notation¹¹ refers to the Howe dual pair $sp(2) \oplus o(d-1, 2)$ and to the fact that the related spin-1 Yang-Mills subalgebra is $u(1)$. The algebra $hu(1|2:[d-1, 2])$ is spanned by the elements satisfying the following reality condition

$$(f(\hat{Y}))^\dagger = -f(\hat{Y}) , \quad (5.9)$$

where \dagger is an involution¹² of the complex Weyl algebra defined by the relation

$$(\hat{Y}_i^A)^\dagger = i\hat{Y}_i^A . \quad (5.10)$$

Thanks to the use of the Weyl ordering prescription, reversing the order of the oscillators has no effect so that $(f(\hat{Y}))^\dagger = \bar{f}(i\hat{Y})$ where the bar means complex conjugation of the coefficients in the expansion (5.3)

$$\bar{f}(\hat{Y}) = \sum_{p=0}^{\infty} \bar{\phi}_{A_1 \dots A_p}^{i_1 \dots i_p} \hat{Y}_{i_1}^{A_1} \dots \hat{Y}_{i_p}^{A_p} . \quad (5.11)$$

As a result, the reality condition (5.9) implies that the coefficients in front of the generators T_s (*i.e.* the basis elements $S^{A_1 \dots A_{s-1}, B_1 \dots B_{s-1}}$) with even and odd s are, respectively, real and pure imaginary. In particular, the spin-2 generator T^{AB} enters with a real coefficient.

What singles out $hu(1|2:[d-1, 2])$ as the physically relevant real form is that it allows lowest weight unitary representations to be identified with the spaces of single particle states in the free HS theory [57]. For these unitary representations (5.9) becomes the antihermiticity property of the generators with \dagger defined via a positive definite Hermitian form. As was argued in [44, 45, 46] for the similar problem in the case of $d=4$ HS algebras, the real HS algebras that share this property are obtained by imposing the reality conditions based on an involution of the underlying complex associative algebra (*i.e.*, Weyl algebra).

Let us note that, as pointed out in [75] and will be demonstrated below in Section 10.1, the Lie algebra with the ideal \mathcal{I} included, *i.e.*, resulting from the algebra \mathcal{S} with the commutator as a Lie product and the reality condition (5.9), (5.10), underlies the off-mass-shell formulation of the HS gauge theory. We call this off-mass-shell HS algebra $hu_{\infty}(1|2:[d-1, 2])$.¹³

5.3 Properties of the higher spin algebras

The real Lie algebra $hu(1|2:[d-1, 2])$ is infinite-dimensional. It contains the space-time isometry algebra $o(d-1, 2)$ as the subalgebra generated by T^{AB} . The basis elements T_s ($s \geq 1$) will be associated with a spin- s gauge field. In Section 14 we will show that $hu(1|2:[d-1, 2])$ is indeed a global symmetry algebra of the AdS_d vacuum solution in the nonlinear HS gauge theory.

Taking two HS generators T_{s_1} and T_{s_2} , being homogeneous polynomials of degrees $2(s_1-1)$ and $2(s_2-1)$ in \hat{Y} , respectively, one obtains (modulo some coefficients)

$$[T_{s_1}, T_{s_2}] = \sum_{m=1}^{\min(s_1, s_2)-1} T_{s_1+s_2-2m} = T_{s_1+s_2-2} + T_{s_1+s_2-4} + \dots + T_{|s_1-s_2|+2} . \quad (5.12)$$

¹¹This notation was introduced in [57] instead of the more complicated one $hu(1/sp(2)[d-1, 2])$ of [16].

¹²This means that \dagger conjugates complex numbers, reverses the order of operators and squares to unity: $(\mu f)^\dagger = \bar{\mu} f^\dagger$, $(fg)^\dagger = g^\dagger f^\dagger$, $((f)^\dagger)^\dagger = f$. To be an involution of the Weyl algebra, \dagger is required to leave invariant its defining relation (5.2). Any condition of the form (5.9) singles out a real linear space of elements that form a closed Lie algebra with respect to commutators.

¹³There exist also intermediate factor algebras $hu_P(1|2:[d-1, 2])$ with the smaller ideals \mathcal{I}_P factored out, where \mathcal{I}_P is spanned by the elements of \mathcal{S} of the form $t_{ij}P(c_2)g^{ij}$, where $c_2 = \frac{1}{2}t_{ij}t^{ij}$ is the quadratic Casimir operator of $sp(2)$ and $P(c_2)$ is some polynomial. The latter HS algebras are presumably of less importance because they should correspond to HS models with higher derivatives in the field equations even at the free field level.

Let us notice that the formula (5.12) is indeed consistent with the requirement (5.1). Furthermore, once a gauge field of spin $s > 2$ appears, the HS symmetry algebra requires an infinite tower of HS gauge fields to be present, together with gravity. Indeed, the commutator $[T_s, T_s]$ of two spin- s generators gives rise to generators T_{2s-2} , corresponding to a gauge field of spin $s' = 2s - 2 > s$, and also gives rise to generators T_2 of $o(d-1, 2)$, corresponding to gravity fields. The spin-2 barrier separates theories with usual finite-dimensional lower-spin symmetries from those with infinite-dimensional HS symmetries. More precisely, the maximal finite-dimensional subalgebra of $hu(1|2:[d-1, 2])$ is the direct sum: $u(1) \oplus o(d-1, 2)$, where $u(1)$ is the center associated with the elements proportional to the unit. Another consequence of the commutation relations (5.12) is that even spin generators T_{2p} ($p \geq 1$) span a proper subalgebra of the HS algebra, denoted as $ho(1|2:[d-1, 2])$ in [57].

The general structure of the commutation relations (5.12) follows from the simple fact that the associative Weyl algebra possesses an antiautomorphism such that $\rho(f(\hat{Y})) = f(i\hat{Y})$ in the Weyl ordering. (The difference between ρ and \dagger is that the former does not conjugate complex numbers.) It induces an automorphism τ of the Lie algebra $hu(1|2:[d-1, 2])$ with

$$\tau(f(\hat{Y})) = -f(i\hat{Y}). \quad (5.13)$$

This automorphism is involutive in the HS algebra, *i.e.* $\tau^2 = \text{identity}$, because $f(-\hat{Y}) = f(\hat{Y})$. Therefore the algebra decomposes into subspaces of τ -odd and τ -even elements. Clearly, these are the subspaces of odd and even spins, respectively. This determines the general structure of the commutation relation (5.12), implying in particular that the even spin subspace forms the proper subalgebra $ho(1|2:[d-1, 2]) \subset hu(1|2:[d-1, 2])$.

Alternatively, the commutation relation (5.12) can be obtained from the following reasoning. The oscillator commutation relation (5.2) contracts two \hat{Y} variables and produces a tensor ϵ_{ij} . Thus, since the commutator of two polynomials is antisymmetric, only odd numbers of contractions can survive. A HS generator T_s is a polynomial of degree $2(s-1)$ in \hat{Y} with the symmetries associated with the two-row Young tableau of length $s-1$. Computing the commutator $[T_{s_1}, T_{s_2}]$, only odd numbers $2m-1$ of contractions survive ($m \geq 1$) leading to polynomials of degree $2(s_1 + s_2 - 2m - 1)$ in \hat{Y} . They correspond to two-row rectangular Young tableaux¹⁴ of length $s_1 + s_2 - 2m - 1$ that are associated to basis elements $T_{s_1+s_2-2m}$. The maximal number, say $2n-1$, of possible contractions is at most equal to the lowest polynomial degree in \hat{Y} of the two generators. Actually, it must be one unit smaller since the numbers of surviving contractions are odd numbers while the polynomial degrees of the generators are even numbers. The lowest polynomial degree in Y of the two generators T_{s_1} and T_{s_2} is equal to $2(\min(s_1, s_2) - 1)$. Hence, $n = \min(s_1, s_2) - 1$. Consequently, the lowest possible polynomial degree of a basis element appearing on the right-hand-side of (5.12) is equal to $2(s_1 + s_2 - 2n - 1) = 2(|s_1 - s_2| + 1)$. The corresponding generators are $T_{|s_1 - s_2| + 2}$.

The gauge fields of $hu(1|2:[d-1, 2])$ are the components of the connection one-form

$$\omega(\hat{Y}, x) = \sum_{s=1}^{\infty} dx^\mu i^{s-2} \omega_{\mu \ A_1 \dots A_{s-1}, B_1 \dots B_{s-1}}(x) \hat{Y}_1^{A_1} \dots \hat{Y}_1^{A_{s-1}} \hat{Y}_2^{B_1} \dots \hat{Y}_2^{B_{s-1}}. \quad (5.14)$$

They take values in the traceless two-row rectangular Young tableaux of $o(d-1, 2)$. It is obvious from this formula why the basis elements T_s are associated to spin- s fields. The curvature and gauge transformations have the standard Yang-Mills form

$$R = d\omega + \omega^2 \Big|_{\mathcal{I} \sim 0}, \quad \delta\omega = D\epsilon \equiv d\epsilon + [\omega, \epsilon] \Big|_{\mathcal{I} \sim 0} \quad (5.15)$$

except that the product of two elements (5.14) with traceless coefficients is not necessarily traceless so that the ideal \mathcal{I} has to be factored out in the end. More precisely, the products ω^2 and $[\omega, \epsilon]$ have to be represented as sums of elements of the algebra with traceless coefficients and others of the form $g_{ij} t^{ij}$

¹⁴Note that the formal tensor product of two two-row rectangular Young tableaux contains various Young tableaux having up to four rows. The property that only two-row Young tableaux appear in the commutator of HS generators is a consequence of the $sp(2)$ invariance condition.

(equivalently, taking into account the $sp(2)$ invariance condition, $t^{ij}\tilde{g}_{ij}$). The latter terms have then to be dropped out, and the resulting factorization is denoted by the symbol $\Big|_{\mathcal{I}\sim 0}$ (see also the discussion in Subsections 12.2 and 14.3).

The formalism here presented is equivalent [57] to the spinor formalism developed previously for lower dimensions, where the HS algebra was realized in terms of commuting spinor oscillators $\hat{y}^\alpha, \hat{y}^{\dot{\alpha}}$ (see, for example, [3, 58] for reviews) as

$$[\hat{y}^\alpha, \hat{y}^\beta] = i\epsilon^{\alpha\beta}, \quad [\hat{y}^{\dot{\alpha}}, \hat{y}^{\dot{\beta}}] = i\epsilon^{\dot{\alpha}\dot{\beta}}, \quad [\hat{y}^\alpha, \hat{y}^{\dot{\beta}}] = 0.$$

Though limited to $d = 3, 4$, the definition of the HS algebra with spinorial oscillators is simpler than that with vectorial oscillators Y_i^A , since the generators are automatically traceless (because $\hat{y}^\alpha \hat{y}_\alpha = \hat{y}^{\dot{\alpha}} \hat{y}_{\dot{\alpha}} = \text{const}$), and there is no ideal to be factored out. However, spinorial realizations of $d = 4$ conformal HS algebras [50, 51] (equivalent to the AdS_5 algebras [7]) and AdS_7 HS algebras [52] require the factorization of an ideal.

6 Free differential algebras and unfolded dynamics

Subsection 6.1 reviews some general definitions of the unfolded formulation of dynamical systems, a particular case of which are the HS field equations [59, 60]. The strategy of the unfolded formalism is presented in Subsection 6.2. It makes use of free differential algebras [61] in order to write consistent nonlinear dynamics. In more modern terms the fundamental underlying concept is L_∞ algebra [62].

6.1 Definition and examples of free differential algebras

Let us consider an arbitrary set of differential forms $W^\alpha \in \Omega^{p_\alpha}(\mathcal{M}^d)$ with degree $p_\alpha \geq 0$ (zero-forms are included), where α is an index enumerating various forms, which, generically, may range in the infinite set $1 \leq \alpha < \infty$.

Let $R^\alpha \in \Omega^{p_\alpha+1}(\mathcal{M}^d)$ be the generalized curvatures defined by the relations

$$R^\alpha = dW^\alpha + G^\alpha(W), \quad (6.1)$$

where

$$G^\alpha(W^\beta) = \sum_{n=1}^{\infty} f_{\beta_1 \dots \beta_n}^\alpha W^{\beta_1} \dots W^{\beta_n} \quad (6.2)$$

are some power series in W^β built with the aid of the exterior product of differential forms. The (anti)symmetry properties of the structure constants $f_{\beta_1 \dots \beta_n}^\alpha$ are such that $f_{\beta_1 \dots \beta_n}^\alpha \neq 0$ for $p_\alpha + 1 = \sum_{i=1}^n p_{\beta_i}$ and the permutation of any two indices β_i and β_j brings a factor of $(-1)^{p_{\beta_i} p_{\beta_j}}$ (in the case of bosonic fields, *i.e.* with no extra Grassmann grading in addition to that of the exterior algebra).

The choice of a function $G^\alpha(W^\beta)$ satisfying the generalized Jacobi identity

$$G^\beta \frac{\delta^L G^\alpha}{\delta W^\beta} \equiv 0 \quad (6.3)$$

(the derivative with respect to W^β is left) defines a free differential algebra¹⁵ [61] introduced originally in the field-theoretical context in [64]. We emphasize that the property (6.3) is a condition on the function

¹⁵We remind the reader that a differential d is a Grassmann odd nilpotent derivation of degree one, *i.e.* it satisfies the (graded) Leibnitz rule and $d^2 = 0$. A differential algebra is a graded algebra endowed with a differential d . Actually, the “free differential algebras” (in physicist terminology) are more precisely christened “graded commutative free differential algebra” by mathematicians (this means that the algebra does not obey algebraic relations apart from graded commutativity). In the absence of zero-forms (which however play a key role in the unfolded dynamics construction) the structure of these algebras is classified by Sullivan [63].

$G^\alpha(W)$ to be satisfied identically for all W^β . It is equivalent to the following generalized Jacobi identity on the structure coefficients

$$\sum_{n=0}^m (n+1) f_{[\beta_1 \dots \beta_{m-n}}^\gamma f_{\gamma \beta_{m-n+1} \dots \beta_m]}^\alpha = 0, \quad (6.4)$$

where the brackets $[\]$ denote an appropriate (anti)symmetrization of all indices β_i . Strictly speaking, the generalized Jacobi identities (6.4) have to be satisfied only at $p_\alpha < d$ for the case of a d -dimensional manifold \mathcal{M}^d where any $d+1$ -form is zero. We will call a free differential algebra *universal* if the generalized Jacobi identity is true for all values of indices, *i.e.*, independently of a particular value of space-time dimension. The HS free differential algebras discussed in this paper belong to the universal class. Note that every universal free differential algebra defines some L_∞ algebra¹⁶.

The property (6.3) guarantees the generalized Bianchi identity

$$dR^\alpha = R^\beta \frac{\delta^L G^\alpha}{\delta W^\beta},$$

which tells us that the differential equations on W^β

$$R^\alpha(W) = 0 \quad (6.5)$$

are consistent with $d^2 = 0$ and supercommutativity. Conversely, the property (6.3) is necessary for the consistency of the equation (6.5).

For universal free differential algebras one defines the gauge transformations as

$$\delta W^\alpha = d\varepsilon^\alpha - \varepsilon^\beta \frac{\delta^L G^\alpha}{\delta W^\beta}, \quad (6.6)$$

where $\varepsilon^\alpha(x)$ has form degree equal to $p_\alpha - 1$ (so that zero-forms W^α do not give rise to any gauge parameter). With respect to these gauge transformations the generalized curvatures transform as

$$\delta R^\alpha = -R^\gamma \frac{\delta^L}{\delta W^\gamma} \left(\varepsilon^\beta \frac{\delta^L G^\alpha}{\delta W^\beta} \right)$$

due to the property (6.3). This implies the gauge invariance of the equations (6.5). Also, since the equations (6.5) are formulated entirely in terms of differential forms, they are explicitly general coordinate invariant.

Unfolding means reformulating the dynamics of a system into an equivalent system of the form (6.5), which, as is explained below, is always possible by virtue of introducing enough auxiliary fields. Note that, according to (6.1), in this approach the exterior differential of all fields is expressed in terms of the fields themselves. A nice property of the universal free differential algebras is that they allow an equivalent description of unfolded systems in larger (super)spaces simply by adding additional coordinates corresponding to a larger (super)space as was demonstrated for some particular examples in [51, 89, 17, 71].

Let h be a Lie (super)algebra, a basis of which is the set $\{T_\alpha\}$. Let $\omega = \omega^\alpha T_\alpha$ be a one-form taking values in h . If one chooses $G(\omega) = \omega^2 \equiv \frac{1}{2} \omega^\alpha \omega^\beta [T_\alpha, T_\beta]$, then the equation (6.5) with $W = \omega$ is the zero-curvature equation $d\omega + \omega^2 = 0$. The relation (6.3) amounts to the usual Jacobi identity for the Lie (super)algebra h as is most obvious from (6.4) (or its super version). In the same way, (6.6) is the usual gauge transformation of the connection ω .

If the set W^α also contains some p -forms denoted by \mathcal{C}^i (*e.g.* zero-forms) and if the functions G^i are linear in ω and \mathcal{C} ,

$$G^i = \omega^\alpha (T_\alpha)^i_j \mathcal{C}^j, \quad (6.7)$$

¹⁶The minor difference is that a form degree p_α of W^α is fixed in a universal free differential algebra while W^α in L_∞ are treated as coordinates of a graded manifold. A universal free differential algebra can therefore be obtained from L_∞ algebra by an appropriate projection to specific form degrees.

then the relation (6.3) implies that the coefficients $(T_\alpha)^i_j$ define some matrices T_α forming a representation T of h , acting in a module V where the C^i take their values. The corresponding equation (6.5) is a covariant constancy condition $D_\omega \mathcal{C} = 0$, where $D_\omega \equiv d + \omega$ is the covariant derivative in the h -module V .

6.2 Unfolding strategy

From the previous considerations, one knows that the system of equations

$$d\omega_0 + \omega_0^2 = 0, \quad (6.8)$$

$$D_{\omega_0} \mathcal{C} = 0 \quad (6.9)$$

forms a free differential algebra. The first equation usually describes a background (for example Minkowski or AdS) along with some pure gauge modes. The connection one-form ω_0 takes value in some Lie algebra h . The second equation may describe nontrivial dynamics if \mathcal{C} is a zero-form C that forms an infinite-dimensional h -module T appropriate to describe the space of all moduli of solutions (*i.e.*, the initial data). One can wonder how the set of equations (6.8) and

$$D_{\omega_0} C = 0 \quad (6.10)$$

could describe any dynamics, since it implies that (locally) the connection ω_0 is pure gauge and C is covariantly constant, so that

$$\omega_0(x) = g^{-1}(x) dg(x), \quad (6.11)$$

$$C(x) = g^{-1}(x) \cdot C, \quad (6.12)$$

where $g(x)$ is some function of the position x taking values in the Lie group H associated with h (by exponentiation), C is a constant vector of the h -module T and the dot stands for the corresponding action of H on T . Since the gauge parameter $g(x)$ does not carry any physical degree of freedom, all physical information is contained in the value $C(x_0) = g^{-1}(x_0) \cdot C$ of the zero-form $C(x)$ at a fixed point x_0 of space-time. But as one will see in Section 7, if the zero-form $C(x)$ somehow parametrizes all the derivatives of the original dynamical fields, then, supplemented with some algebraic constraints (that, in turn, single out an appropriate h -module), it can actually describe nontrivial dynamics. More precisely, the restrictions imposed on values of some zero-forms at a fixed point x_0 of space-time can lead to nontrivial dynamics if the set of zero-forms is rich enough to describe all space-time derivatives of the dynamical fields at a fixed point of space-time, provided that the constraints just single out those values of the derivatives that are compatible with the original dynamical equations. By knowing a solution (6.12) one knows all the derivatives of the dynamical fields compatible with the field equations and can therefore reconstruct these fields by analyticity in some neighborhood of x_0 .

The p -forms with $p > 0$ contained in \mathcal{C} (if any) are still pure gauge in these equations. As will be clear from the examples below, the meaning of the zero-forms C contained in \mathcal{C} is that they describe all gauge invariant degrees of freedom (*e.g.* the spin-0 scalar field, the spin-1 Maxwell field strength, the spin-2 Weyl tensor, etc., and all their on-mass-shell nontrivial derivatives). When the gauge invariant zero-forms are identified with derivatives of the gauge fields which are $p > 0$ forms, this is expressed by a deformation of the equation (6.9)

$$D_{\omega_0} \mathcal{C} = P(\omega_0) \mathcal{C}, \quad (6.13)$$

where $P(\omega_0)$ is a linear operator (depending on ω_0 at least quadratically) acting on \mathcal{C} . The equations (6.8) and (6.13) are of course required to be consistent, *i.e.* to describe some free differential algebra, which is a deformation of (6.8) and (6.9). If the deformation is trivial, one can get rid of the terms on the right-hand-side of (6.13) by a field redefinition. The interesting case therefore is when the deformation is nontrivial. A useful criterium of whether the deformation (6.13) is trivial or not is given in terms of the σ_- cohomology in Section 9.

The next step is to interpret the equations (6.8) and (6.13) as resulting from the linearization of some nonlinear system of equations with

$$W = \omega_0 + \mathcal{C} \quad (6.14)$$

in which ω_0 is some fixed zero-order background field chosen to satisfy (6.8) while \mathcal{C} describes first order fluctuations. Consistency of this identification however requires nonlinear corrections to the original linearized equations because the full covariant derivatives built of $W = \omega_0 + \mathcal{C}$ develop nonzero curvature due to the right hand side of (6.13). Finding these nonlinear corrections is equivalent to finding interactions.

This suggests the following strategy for the analysis of HS gauge theories:

1. One starts from a space-time with some symmetry algebra s (*e.g.* Poincaré or anti-de Sitter algebra) and a vacuum gravitational gauge field ω_0 , which is a one-form taking values in s and satisfying the zero curvature equations (6.8).
2. One reformulates the field equations of a given free dynamical system in the “unfolded form” (6.13). This can always be done in principle (the general procedure is explained in Section 7). The only questions are: “how simple is the explicit formulation?” and “what are the modules T of s for which the unfolded equation (6.10) can be interpreted as a covariant constancy condition?”.
3. One looks for a nonlinear free differential algebra such that (6.5) with (6.14) correctly reproduces the free field equations (6.8) and (6.13) at the linearized level. More precisely, one looks for some function $G(W)$ verifying (6.3) and the Taylor expansion of which around ω_0 is given by

$$G(W) = \omega_0^2 + (\omega_0 - P(\omega_0))\mathcal{C} + O(\mathcal{C}^2),$$

where $\omega_0\mathcal{C}$ denotes the action of ω_0 in the h -module \mathcal{C} and the terms denoted by $O(\mathcal{C}^2)$ are at least quadratic in the fluctuation.

It is not *a priori* guaranteed that some nonlinear deformation exists at all. If not, this would mean that no consistent nonlinear equations exist. But if the deformation is found, then the problem is solved because the resulting equations are formally consistent, gauge invariant and generally coordinate invariant¹⁷ as a consequence of the general properties of free differential algebras, and, by construction, they describe the correct dynamics at the free field level.

To find some nonlinear deformation, one has to address two related questions. The first one is “what is a relevant s -module T in which zero-forms that describe physical degrees of freedom in the model can take values?” and the second is “which infinite-dimensional (HS) extension h of s , in which one-form connections take their values, can act on the s -module T ?”. A natural candidate is a Lie algebra h constructed via commutators from the associative algebra \mathcal{A}

$$\mathcal{A} = \mathcal{U}(s)/\text{Ann}(T),$$

where $\mathcal{U}(s)$ is the universal enveloping algebra of s while $\text{Ann}(T)$ is the annihilator, *i.e.* the ideal of $\mathcal{U}(s)$ spanned by the elements which trivialize on the module T . Of course, this strategy may be too naive in general because not all algebras can be symmetries of a consistent field-theoretical model and only some subalgebras of h resulting from this construction may allow a consistent nonlinear deformation. A useful criterium is the *admissibility condition* [66] which requires that there should be a unitary h -module which describes a list of quantum single-particle states corresponding to all HS gauge fields described in terms of the connections of h . If no such representation exists, there is no chance to find a nontrivial consistent (in particular, free of ghosts) theory that admits h as a symmetry of its most symmetric vacuum. In any case, $\mathcal{U}(s)$ is the reasonable starting point to look for a HS algebra¹⁸. It

¹⁷Note that any fixed choice of ω_0 breaks down the diffeomorphism invariance to a global symmetry of the vacuum solution. This is why the unfolding formulation works equally well both in the theories with fixed background field ω_0 and no manifest diffeomorphism invariance and those including gravity where ω_0 is a zero order part of the dynamical gravitational field.

¹⁸Based on somewhat different arguments, this idea was put forward by Fradkin and Linetsky in [65].

seems to be most appropriate, however, to search for conformal HS algebras. Indeed, the associative algebra \mathcal{A} introduced in Section 5 is a quotient $\mathcal{A} = \mathcal{U}(s)/\text{Ann}(T)$, where T represents its conformal realization in $d - 1$ dimensions [54, 57]. The related real Lie algebra h is $hu(1|2:[d - 1, 2])$. The space of single-particle quantum states of free massless HS fields of Section 8 provides a unitary module of $hu(1|2:[d - 1, 2])$ in which all massless completely symmetric representations of $o(d - 1, 2)$ appear just once [57].

7 Unfolding lower spins

The dynamics of any consistent system can in principle be rewritten in the unfolded form (6.5) by adding enough auxiliary variables [67]. This technique is explained in Subsection 7.1. Two particular examples of the general procedure are presented: the unfolding of the Klein-Gordon equation and the unfolding of gravity, in Subsections 7.2 and 7.3, respectively.

7.1 Unfolded dynamics

Let $\omega_0 = e_0^a P_a + \frac{1}{2}\omega_0^{ab} M_{ab}$ be a vacuum gravitational gauge field taking values in some space-time symmetry algebra s . Let $C^{(0)}(x)$ be a given space-time field satisfying some dynamical equations to be unfolded. Consider for simplicity the case where $C^{(0)}(x)$ is a zero-form. The general procedure of unfolding free field equations goes schematically as follows:

For a start, one writes the equation

$$D_0^L C^{(0)} = e_0^a C_a^{(1)}, \quad (7.1)$$

where D_0^L is the covariant Lorentz derivative and the field $C_a^{(1)}$ is auxiliary. Next, one checks whether the original field equations for $C^{(0)}$ impose any restrictions on the first derivatives of $C^{(0)}$. More precisely, some part of $D_{0,\mu}^L C^{(0)}$ might vanish on-mass-shell (*e.g.* for Dirac spinors). These restrictions in turn impose some restrictions on the auxiliary fields $C_a^{(1)}$. If these constraints are satisfied by $C_a^{(1)}$, then these fields parametrize all on-mass-shell nontrivial components of first derivatives.

Then, one writes for these first level auxiliary fields an equation similar to (7.1)

$$D_0^L C_a^{(1)} = e_0^b C_{a,b}^{(2)}, \quad (7.2)$$

where the new fields $C_{a,b}^{(2)}$ parametrize the second derivatives of $C^{(0)}$. Once again one checks (taking into account the Bianchi identities) which components of the second level fields $C_{a,b}^{(2)}$ are nonvanishing provided that the original equations of motion are satisfied.

This process continues indefinitely, leading to a chain of equations having the form of some covariant constancy condition for the chain of fields $C_{a_1, a_2, \dots, a_m}^{(m)}$ ($m \in \mathbb{N}$) parametrizing all on-mass-shell nontrivial derivatives of the original dynamical field. By construction, this leads to a particular unfolded equation (6.5) with G^i in (6.1) given by (6.7). As explained in Section 6.1, this means that the set of fields realizes some module T of the space-time symmetry algebra s . In other words, the fields $C_{a_1, a_2, \dots, a_m}^{(m)}$ are the components of a single field C living in the infinite-dimensional s -module T . Then the infinite chain of equations can be rewritten as a single covariant constancy condition $D_0 C = 0$, where D_0 is the s -covariant derivative in T .

7.2 The example of the scalar field

As a preliminary to the gravity example considered in the next subsection, the simplest field-theoretical case of unfolding is reviewed, *i.e.* the unfolding of a massless scalar field, which was first described in [67].

For simplicity, for the remaining of Section 7, we will consider the flat space-time background. The Minkowski solution can be written as

$$\omega_0 = dx^\mu \delta_\mu^a P_a \quad (7.3)$$

i.e. the flat frame is $(e_0)_\mu^a = \delta_\mu^a$ and the Lorentz connection vanishes. The equation (7.3) corresponds to the “pure gauge” solution (6.11) with

$$g(x) = \exp(x^\mu \delta_\mu^a P_a), \quad (7.4)$$

where the space-time Lie algebra s is identified with the Poincaré algebra $iso(d-1, 1)$. Though the vacuum ω_0 solution has a pure gauge form (7.4), this solution cannot be gauged away because of the constraint $\text{rank}(e_0) = d$ (see Section 2.1).

The “unfolding” of the massless Klein-Gordon equation

$$\square C(x) = 0 \quad (7.5)$$

is relatively easy to work out, so we give directly the final result and we comment about how it is obtained afterwards.

To describe the dynamics of the spin-0 massless field $C(x)$, let us introduce the infinite collection of zero-forms $C_{a_1 \dots a_n}(x)$ ($n = 0, 1, 2, \dots$) that are completely symmetric traceless tensors

$$C_{a_1 \dots a_n} = C_{\{a_1 \dots a_n\}}, \quad \eta^{bc} C_{bca_3 \dots a_n} = 0. \quad (7.6)$$

The “unfolded” version of the Klein-Gordon equation (7.5) has the form of the following infinite chain of equations

$$dC_{a_1 \dots a_n} = e_0^b C_{a_1 \dots a_n b} \quad (n = 0, 1, \dots), \quad (7.7)$$

where we have used the opportunity to replace the Lorentz covariant derivative D_0^L by the ordinary exterior derivative d . It is easy to see that this system is formally consistent because applying d on both sides of (7.7) does not lead to any new condition,

$$d^2 C_{a_1 \dots a_n} = -e_0^b dC_{a_1 \dots a_n b} = -e_0^b e_0^c C_{a_1 \dots a_n bc} = 0 \quad (n = 0, 1, \dots),$$

since $e_0^b e_0^c = -e_0^c e_0^b$ because e_0^b is a one-form. As we know from Section 6.1, this property implies that the space T of zero-forms $C_{a_1 \dots a_n}(x)$ spans some representation of the Poincaré algebra $iso(d-1, 1)$. In other words, T is an infinite-dimensional $iso(d-1, 1)$ -module¹⁹.

To show that this system of equations is indeed equivalent to the free massless field equation (7.5), let us identify the scalar field $C(x)$ with the member of the family of zero-forms $C_{a_1 \dots a_n}(x)$ at $n = 0$. Then the first two equations of the system (7.7) read

$$\partial_\nu C = C_\nu,$$

$$\partial_\nu C_\mu = C_{\mu\nu},$$

where we have identified the world and tangent indices via $(e_0)_\mu^a = \delta_\mu^a$. The first of these equations just tells us that C_ν is the first derivative of C . The second one tells us that $C_{\nu\mu}$ is the second derivative of C . However, because of the tracelessness condition (7.6) it imposes the Klein-Gordon equation (7.5). It is easy to see that all other equations in (7.7) express highest tensors in terms of the higher-order derivatives

$$C_{\nu_1 \dots \nu_n} = \partial_{\nu_1} \dots \partial_{\nu_n} C \quad (7.8)$$

and impose no new conditions on C . The tracelessness conditions (7.6) are all satisfied once the Klein-Gordon equation is true. From this formula it is clear that the meaning of the zero-forms $C_{\nu_1 \dots \nu_n}$ is that they form a basis in the space of all on-mass-shell nontrivial derivatives of the dynamical field $C(x)$ (including the derivative of order zero which is the field $C(x)$ itself).

¹⁹Strictly speaking, to apply the general argument of Section 6.1 one has to check that the equation remains consistent for any flat connection in $iso(d-1, 1)$. It is not hard to see that this is true indeed.

Let us note that the system (7.7) without the constraints (7.6), which was originally considered in [68], remains formally consistent but is dynamically empty just expressing all highest tensors in terms of derivatives of C according to (7.8). This simple example illustrates how algebraic constraints like tracelessness of a tensor can be equivalent to dynamical equations.

The above consideration can be simplified further by means of introducing the auxiliary coordinate u^a and the generating function

$$C(x, u) = \sum_{n=0}^{\infty} \frac{1}{n!} C_{a_1 \dots a_n}(x) u^{a_1} \dots u^{a_n}$$

with the convention that

$$C(x, 0) = C(x).$$

This generating function accounts for all tensors $C_{a_1 \dots a_n}$ provided that the tracelessness condition is imposed, which in these terms implies that

$$\square_u C(x, u) \equiv \frac{\partial}{\partial u^a} \frac{\partial}{\partial u_a} C = 0. \quad (7.9)$$

In other words, the $iso(d-1, 1)$ -module T is realized as the space of formal harmonic power series in u^a . The equations (7.7) then acquire the simple form

$$\frac{\partial}{\partial x^\mu} C(x, u) = \delta_\mu^a \frac{\partial}{\partial u^a} C(x, u). \quad (7.10)$$

From this realization one concludes that the translation generators in the infinite-dimensional module T of the Poincaré algebra are realized as translations in the u -space, *i.e.*

$$P_a = -\frac{\partial}{\partial u^a},$$

for which the equation (7.10) reads as a covariant constancy condition (6.9)

$$dC(x, u) + e_0^a P_a C(x, u) = 0. \quad (7.11)$$

One can find a general solution of the equation (7.11) in the form

$$C(x, u) = C(x + u, 0) = C(0, x + u) \quad (7.12)$$

from which it follows in particular that

$$C(x) \equiv C(x, 0) = C(0, x) = \sum_{n=0}^{\infty} \frac{1}{n!} C_{\nu_1 \dots \nu_n}(0) x^{\nu_1} \dots x^{\nu_n}. \quad (7.13)$$

From (7.6) and (7.8) one can see that this is indeed the Taylor expansion for any solution of the Klein-Gordon equation which is analytic at $x_0 = 0$. Moreover the general solution (7.12) expresses the covariant constancy of the vector $C(x, u)$ of the module T ,

$$C(x, u) = C(0, x + u) = \exp(-x^\mu \delta_\mu^a P_a) C(0, u).$$

This is a particular realization of the pure gauge solution (6.12) with the gauge function $g(x)$ of the form (7.4) and $C = C(0, u)$.

The example of a free scalar field is so simple that one might think that the unfolding procedure is always like a trivial mapping of the original equation (7.5) to the equivalent one (7.9) in terms of additional variables. This is not true, however, for the less trivial cases of dynamical systems in nontrivial backgrounds and, especially, nonlinear systems. The situation here is analogous to that in the Fedosov quantization prescription [69] which reduces the nontrivial problem of quantization in a curved

background to the standard problem of quantization of the flat phase space, that, of course, becomes an identity when the ambient space is flat itself. It is worth to mention that this parallelism is not accidental because, as one can easily see, the Fedosov quantization prescription provides a particular case of the general unfolding approach [59] in the dynamically empty situation (*i.e.*, with no dynamical equations imposed).

7.3 The example of gravity

The set of fields in Einstein-Cartan's formulation of gravity is composed of the frame field e_μ^a and the Lorentz connection ω_μ^{ab} . One supposes that the torsion constraint $T_a = 0$ is satisfied, in order to express the Lorentz connection in terms of the frame field. The Lorentz curvature can be expressed as $R^{ab} = e_c e_d R^{[cd]; [ab]}$, where $R^{ab; cd}$ is a rank four tensor with indices in the tangent space and which

is antisymmetric both in ab and in cd , having the symmetries of the tensor product $\begin{smallmatrix} a \\ b \end{smallmatrix} \otimes \begin{smallmatrix} c \\ d \end{smallmatrix}$. The algebraic Bianchi identity $e_b R^{ab} = 0$, which follows from the zero torsion constraint, imposes that the tensor $R^{ab; cd}$ possesses the symmetries of the Riemann tensor, *i.e.* $R^{[ab; c]d} = 0$. More precisely, it carries an irreducible representation of $GL(d)$ characterized by the Young tableau $\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}$ in the antisymmetric basis. The vacuum Einstein equations state that this tensor is traceless, so that it is actually irreducible under the pseudo-orthogonal group $O(d-1, 1)$ on-mass-shell. In other words, the Riemann tensor is equal on-mass-shell to the Weyl tensor.

For HS generalization, it is more convenient to use the symmetric basis. In this convention, the Einstein equations can be written as

$$T^a = 0, \quad R^{ab} = e_c e_d C^{ac, bd}, \quad (7.14)$$

where the zero-form $C^{ac, bd}$ is the Weyl tensor in the symmetric basis. More precisely, the tensor $C^{ac, bd}$ is symmetric in the pairs ac and bd and it satisfies the algebraic identities

$$C^{\{ac, b\}d} = 0, \quad \eta_{ac} C^{ac, bd} = 0.$$

Let us now start the unfolding of linearized gravity around the Minkowski background described by a frame one-form e_0^a and Lorentz covariant derivative D_0^L . The linearization of the second equation of (7.14) is

$$R_1^{ab} = e_0^c e_0^d C^{ac, bd}, \quad (7.15)$$

where R_1^{ab} is the linearized Riemann tensor. This equation is a particular case of the equation (6.13). What is lacking at this stage is the equations containing the differential of the Weyl zero-form $C^{ac, bd}$. Since we do not want to impose any additional dynamical restrictions on the system, the only restrictions on the derivatives of the Weyl zero-form $C^{ac, bd}$ may result from the Bianchi identities applied to (7.15).

A priori, the first Lorentz covariant derivative of the Weyl tensor is a rank five tensor in the following representation

$$\begin{smallmatrix} \square \end{smallmatrix} \otimes \begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix} = \begin{smallmatrix} \square & \square \\ \square & \square \\ \square \end{smallmatrix} \oplus \begin{smallmatrix} \square & \square & \square \\ \square & \square \end{smallmatrix} \quad (7.16)$$

decomposed according to irreducible representations of $gl(d)$. Since the Weyl tensor is traceless, the right hand side of (7.16) contains only one nontrivial trace, that is for traceless tensors we have the $o(d-1, 1)$ Young decomposition by adding a three cell hook tableau, *i.e.*

$$\begin{smallmatrix} \square \end{smallmatrix} \otimes \begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix} = \begin{smallmatrix} \square & \square \\ \square & \square \\ \square \end{smallmatrix} \oplus \begin{smallmatrix} \square & \square & \square \\ \square & \square \end{smallmatrix} \oplus \begin{smallmatrix} \square & \square \\ \square \end{smallmatrix} \quad (7.17)$$

The linearized Bianchi identity $D_0^L R_1^{ab} = 0$ leads to

$$e_0{}_c e_0{}_d D_0^L C^{ac, bd} = 0. \quad (7.18)$$

The components of the left-hand-side written in the basis $dx^\mu dx^\nu dx^\rho$ have the symmetry property corresponding to the tableau

$$\begin{array}{|c|c|} \hline \mu & a \\ \hline \nu & b \\ \hline \rho & \\ \hline \end{array} \sim D_0^L {}_{[\rho} C^a{}_{\mu}{}^{b}{}_{\nu]},$$

which also contains the single trace part with the symmetry properties of the three-cell hook tableau.

Therefore the consistency condition (7.18) says that in the decomposition (7.17) of the Lorentz covariant derivative of the Weyl tensor, the first and third terms vanish and the second term is traceless and otherwise arbitrary. Let $C^{abf, cd}$ be the traceless tensor corresponding to the second term in the decomposition (7.16) of the Lorentz covariant derivative of the Weyl tensor. This is equivalent to say that

$$D_0^L C^{ac, bd} = e_0{}_f (2C^{acf, bd} + C^{acb, df} + C^{acd, bf}), \quad (7.19)$$

where the right hand side is fixed by the Young symmetry properties of the left hand side modulo an overall normalization coefficient. This equation looks like the first step (7.1) of the unfolding procedure. $C^{acf, bd}$ is irreducible under $o(d-1, 1)$.

One should now perform the second step of the general unfolding scheme and write the analogue of (7.2). This process goes on indefinitely. To summarize the procedure, one can analyze the decomposition of the k -th Lorentz covariant derivatives (with respect to the Minkowski vacuum background, so they commute) of the Weyl tensor $C^{ac, bd}$. Taking into account the Bianchi identity, the decomposition goes as follows

$$\boxed{}^k \otimes \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \rightarrow \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \boxed{}^{k+2} \quad (7.20)$$

As a result, one obtains

$$D_0^L C^{a_1 \dots a_{k+2}, b_1 b_2} = e_0{}_c \left((k+2) C^{a_1 \dots a_{k+2} c, b_1 b_2} + C^{a_1 \dots a_{k+2} b_1, b_2 c} + C^{a_1 \dots a_{k+2} b_2, b_1 c} \right), \quad (0 \leq k \leq \infty), \quad (7.21)$$

where the fields $C^{a_1 \dots a_{k+2}, b_1 b_2}$ are in the irreducible representation of $o(d-1, 1)$ characterized by the traceless two-row Young tableau on the right hand side of (7.20), *i.e.*

$$C^{\{a_1 \dots a_{k+2}, b_1\} b_2} = 0, \quad \eta_{a_1 a_2} C^{a_1 a_2 \dots a_{k+2}, b_1 b_2} = 0.$$

Note that, as expected, the system (7.21) is consistent with $(D_0^L)^2 C^{a_1 \dots a_{k+2}, b_1 b_2} = 0$.

Analogously to the spin-0 case, the meaning of the zero-forms $C^{a_1 \dots a_{k+2}, b_1 b_2}$ is that they form a basis in the space of all on-mass-shell nontrivial gauge invariant combinations of the derivatives of the spin-2 gauge field.

In order to extend this analysis to nonlinear gravity, one replaces the background derivative D_0^L and frame field e_0^a by the full Lorentz covariant derivative D^L and dynamical frame e^a satisfying the zero torsion condition $D^L e^a = 0$ and

$$D^L D^L = R, \quad (7.22)$$

where R is the Riemann tensor taking values in the adjoint representation of the Lorentz algebra. The unfolding procedure goes the same way up to the equation (7.19) but needs nonlinear corrections starting from the next step. The reason is that Bianchi identities for (7.19) and analogous higher equations give rise to terms nonlinear in C via (7.22) and (7.14). All terms of second order in C in the nonlinear deformation of (7.21) were obtained in [60] for the case of four dimensions. The problem of unfolding nonlinear gravity in all orders remains unsolved.

8 Free massless equations for any spin

In order to follow the strategy exposed in Subsection 6.2 and generalize the example of gravity treated along these lines in Subsection 7.3, we shall start by writing unfolded HS field equations in terms of the linearized HS curvatures (4.4). This result is christened the “central on-mass-shell theorem”. It was originally obtained in [40, 59] for the case of $d = 4$ and then extended to any d in [41, 30]. That these HS equations of motion indeed reproduce the correct physical degrees of freedom will be shown later in Section 10, via a cohomological approach explained in Section 9.

8.1 Connection one-form sector

The linearized curvatures $R_1^{A_1 \dots A_{s-1}, B_1 \dots B_{s-1}}$ were defined by (4.4). They decompose into the linearized curvatures with Lorentz (*i.e.* V^A transverse) fibre indices which have the symmetry properties associated with the two-row traceless Young tableau $\begin{array}{|c|} \hline \square \\ \hline \end{array}_t^{s-1}$. It is convenient to use the standard gauge $V^A = \delta_d^A$ (from now on we normalize V to unity). In the Lorentz basis, the linearized HS curvatures have the form

$$R_1^{a_1 \dots a_{s-1}, b_1 \dots b_t} = D_0^L \omega^{a_1 \dots a_{s-1}, b_1 \dots b_t} + e_0{}_c \omega^{a_1 \dots a_{s-1}, b_1 \dots b_t c} + O(\Lambda). \quad (8.1)$$

For simplicity, in this section we discard the complicated Λ -dependent terms which do not affect the general analysis, *i.e.* we present explicitly the flat-space-time part of the linearized HS curvatures. It is important to note however that the Λ -dependent terms in (8.1) contain only the field $\omega^{a_1 \dots a_{s-1}, b_1 \dots b_{t-1}}$ which carries one index less than the linearized HS curvatures. The explicit form of the Λ -dependent terms is given in [41].

For $t = 0$, these curvatures generalize the torsion of gravity, while for $t > 0$ the curvature corresponds to the Riemann tensor. In particular, as we will demonstrate in Section 10, the analogues of the Ricci tensor and scalar curvature are contained in the curvatures with $t = 1$ while the HS analog of the Weyl tensor is contained in the curvatures with $t = s - 1$. (For the case of $s = 2$ they combine into the level $t = 1$ traceful Riemann tensor.)

The first on-mass-shell theorem states that the following free field equations in Minkowski or $(A)dS$ space-time

$$R_1^{a_1 \dots a_{s-1}, b_1 \dots b_t} = \delta_{t, s-1} e_0{}_c e_0{}_d C^{a_1 \dots a_{s-1} c, b_1 \dots b_{s-1} d}, \quad (0 \leq t \leq s-1) \quad (8.2)$$

properly describe completely symmetric gauge fields of generic spin $s \geq 2$. This means that they are equivalent to the proper generalization of the $d = 4$ Fronsdal equations of motion to any dimension, supplemented with certain algebraic constraints on the auxiliary HS connections which express the latter via derivatives of the dynamical HS fields. The zero-form $C^{a_1 \dots a_s, b_1 \dots b_s}$ is the spin- s Weyl-like tensor. It is irreducible under $o(d-1, 1)$ and is characterized by a rectangular two-row Young tableau $\begin{array}{|c|} \hline \square \\ \hline \end{array}_s^s$. The field equations generalize (7.15) of linearized gravity. The equations of motion put to zero all curvatures with $t \neq s-1$ and require $C^{a_1 \dots a_s, b_1 \dots b_s}$ to be traceless.

8.2 Weyl zero-form sector

Note that the equations (8.2) result from the first step in the unfolding of the Fronsdal equations.²⁰ The analysis of the Bianchi identities of (8.2) works for any spin $s \geq 2$ in a way analogous to gravity. The

²⁰Actually, the action and equations of motion for totally symmetric massless HS fields in AdS_d with $d > 4$ were originally obtained in [41] in the frame-like formalism.

final result is the following equation [30], which presents itself like a covariant constancy condition

$$0 = \tilde{D}_0 C^{a_1 \dots a_{s+k}, b_1 \dots b_s} \equiv D_0^L C^{a_1 \dots a_{s+k}, b_1 \dots b_s} - e_{0\ c} \left((k+2) C^{a_1 \dots a_{s+k} c, b_1 \dots b_s} + s C^{a_1 \dots a_{s+k} \{b_1, b_2 \dots b_s\} c} \right) + O(\Lambda), \quad (0 \leq k \leq \infty), \quad (8.3)$$

where $C^{a_1 \dots a_{s+k}, b_1 \dots b_s}$ are $o(d-1, 1)$ irreducible (*i.e.*, traceless) tensors characterized by the Young tableaux $\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}_s s+k$. They describe on-mass-shell nontrivial k -th derivatives of the spin- s Weyl-like tensor, thus forming a basis in the space of gauge invariant combinations of $(s+k)$ -th derivatives of a spin- s HS gauge field. The system (8.3) is the generalization of the spin-0 system (7.7) and the spin-2 system (7.21) to arbitrary spin and to AdS background (the explicit form of the Λ -dependent terms is given in [30]). Let us stress that for $s \geq 2$ the infinite system of equations (8.3) is a consequence of (8.2) by the Bianchi identity. For $s = 0$ and $s = 1$, the system (8.3) contains the dynamical Klein-Gordon and Maxwell equations, respectively. Note that (8.2) makes no sense for $s = 0$ because there is no spin-0 gauge potential while (8.3) with $s = 0$ reproduces the unfolded spin-0 equation (7.7) and its AdS generalization. For the spin-1 case, (8.2) only gives a definition of the spin-1 Maxwell field strength $C^{a,b} = -C^{b,a}$ in terms of the potential ω_μ . The dynamical equations for spin-1, *i.e.* Maxwell equations, are contained in (8.3). The fields $C^{a_1 \dots a_{k+1}, b}$, characterized by the Lorentz irreducible (*i.e.* traceless) two-row Young tableaux with one cell in the second row, form a basis in the space of on-mass-shell nontrivial derivatives of the Maxwell tensor $C^{a,b}$.

It is clear that the complete set of zero-forms $C^{a_1 \dots a_{s+k}, b_1 \dots b_s} \sim \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}_s s+k$ covers the set of all two-row Young tableaux. This suggests that the Weyl-like zero-forms take values in the linear space of $hu(1|2;[d-1, 2])$, which obviously forms an $o(d-1, 1)$ -(*i.e.* Lorentz) module. Following Sections 6.1 and 6.2, one expects that the zero-forms belong to an $o(d-1, 2)$ -module T . But the idea to use the adjoint representation of $hu(1|2;[d-1, 2])$ does not work because, according to the commutation relation (5.1), the commutator of the background gravity connection $\omega_0 = \omega_0^{AB} T_{AB}$ with a generator of $hu(1|2;[d-1, 2])$ preserves the rank of the generator, while the covariant derivative \tilde{D}_0 in (8.3) acts on the infinite set of Lorentz tensors of infinitely increasing ranks. Fortunately, the appropriate representation only requires a slight modification compared to the adjoint representation. As will be explained in Section 12, the zero-forms C belong to the so-called “twisted adjoint representation”.

Since k goes from zero to infinity for any fixed s in (8.3), in agreement with the general arguments of Section 6, each irreducible spin- s submodule of the twisted adjoint representation is infinite-dimensional. This means that, in the unfolded formulation, the dynamics of any fixed spin- s field is described in terms of an infinite set of fields related by the first-order unfolded equations. Of course, to make it possible to describe a field-theoretical dynamical system with an infinite number of degrees of freedom, the set of auxiliary zero-forms associated with all gauge invariant combinations of derivatives of dynamical fields should be infinite. Let us note that the right-hand-side of the equation (8.2) is a particular realization of the deformation terms (6.13) in free differential algebras.

The system of equations (8.2)-(8.3) provides the unfolded form of the free equations of motion for completely symmetric massless fields of all spins in any dimension. This fact is referred to as “central on-mass-shell theorem” because it plays a distinguished role in various respects. The idea of the proof will be explained in Section 10. The proof is based on a very general cohomological reformulation of the problem, which is reviewed in Section 9.

9 Dynamical content via σ_- cohomology

In this section, we perform a very general analysis of equations of motion of the form

$$\hat{D}_0 \mathcal{C} = 0, \quad (9.1)$$

via a cohomological reformulation [68, 51, 17, 71] of the problem²¹. It will be applied to the HS context in the next section.

9.1 General properties of σ_-

Let us introduce the number of Lorentz indices as a grading G of the space of tensors with fiber (tangent) Lorentz indices. In the unfolded HS equations the background covariant derivative \widehat{D}_0 decomposes as the sum

$$\widehat{D}_0 = \sigma_- + D_0^L + \sigma_+, \quad (9.2)$$

where the operator σ_{\pm} modifies the rank of a Lorentz tensor by ± 1 and the background Lorentz covariant derivative D_0^L does not change it.

In the present consideration we do not fix the module V on which \widehat{D} acts and, in (9.1), \mathcal{C} denotes a set of differential forms taking values in V . In the context of the HS theory, the interesting cases are when \widehat{D} acts either on the adjoint representation or on the twisted adjoint representation of the HS algebra. For example, the covariant constancy condition (8.3) takes the form

$$\widetilde{D}_0 C = (D_0^L + \sigma_- + \sigma_+) C = 0,$$

where σ_+ denotes the Λ -dependent terms. The cohomological classification exposed here only assumes the following abstract properties:

- (i) The grading operator G is diagonalizable in the vector space V and it possesses a spectrum bounded from below.
- (ii) The grading properties of the Grassmann odd operators D_0^L and σ_- are summarized in the commutation relations

$$[G, D_0^L] = 0, \quad [G, \sigma_-] = -\sigma_-.$$

The operator σ_+ is a sum of operators of strictly positive grade. (In HS applications σ_+ has grade one, *i.e.* $[G, \sigma_+] = \sigma_+$, but this is not essential for the general analysis.)

- (iii) The operator σ_- acts vertically in the fibre V , *i.e.* it does not act on space-time coordinates. (In HS models, only the operator D_0^L acts nontrivially on the space-time coordinates (differentiates).)
- (iv) The background covariant derivative \widehat{D}_0 defined by (9.2) is nilpotent. The graded decomposition of the nilpotency equation $(\widehat{D}_0)^2 = 0$ gives the following identities

$$(\sigma_-)^2 = 0, \quad D_0^L \sigma_- + \sigma_- D_0^L = 0, \quad (D_0^L)^2 + \sigma_+ \sigma_- + \sigma_- \sigma_+ + D_0^L \sigma_+ + \sigma_+ D_0^L + (\sigma_+)^2 = 0. \quad (9.3)$$

If σ_+ has definite grade $+1$ (as is the case in the HS theories under consideration) the last relation is equivalent to the three conditions $(\sigma_+)^2 = 0$, $D_0^L \sigma_+ + \sigma_+ D_0^L = 0$, $(D_0^L)^2 + \sigma_+ \sigma_- + \sigma_- \sigma_+ = 0$.

An important property is the nilpotency of σ_- . The point is that the analysis of Bianchi identities (as was done in details in Section 7.3 for gravity) is, in fact, equivalent to the analysis of the cohomology of σ_- , that is

$$H(\sigma_-) \equiv \frac{\text{Ker}(\sigma_-)}{\text{Im}(\sigma_-)}.$$

9.2 Cohomological classification of the dynamical content

Let \mathcal{C} denote a differential form of degree p taking values in V , that is an element of the complex $V \otimes \Omega^p(\mathcal{M}^d)$. The field equation (9.1) is invariant under the gauge transformations

$$\delta \mathcal{C} = \widehat{D}_0 \varepsilon, \quad (9.4)$$

²¹Note that a similar approach was applied in a recent paper [72] in the context of *BRST* formalism.

since \widehat{D}_0 is nilpotent by the hypothesis (iv). The gauge parameter ε is a $(p-1)$ -form. These gauge transformations contain both differential gauge transformations (like linearized diffeomorphisms) and Stueckelberg gauge symmetries (like linearized local Lorentz transformations²²).

The following terminology will be used. By *dynamical field*, we mean a field that is not expressed as derivatives of something else by field equations (*e.g.* the frame field in gravity or a frame-like HS one-form field $\omega_\mu^{a_1 \dots a_{s-1}}$). The fields that are expressed by virtue of the field equations as derivatives of the dynamical fields modulo Stueckelberg gauge symmetries are referred to as *auxiliary fields* (*e.g.* the Lorentz connection in gravity or its HS analogues $\omega_\mu^{a_1 \dots a_{s-1}, b_1 \dots b_t}$ with $t > 0$). A field that is neither auxiliary nor pure gauge by Stueckelberg gauge symmetries is said to be a *nontrivial dynamical field* (*e.g.* the metric tensor or the metric-like gauge fields of Fronsdal's approach).

Let $\mathcal{C}(x)$ be an element of the complex $V \otimes \Omega^p(\mathcal{M}^d)$ that satisfies the dynamical equation (9.1). Under the hypotheses (i)-(iv) one can prove the following propositions [68, 51] (see also [17, 71]):

- A. Nontrivial dynamical fields \mathcal{C} are nonvanishing elements of $H^p(\sigma_-)$.
- B. Differential gauge symmetry parameters ε are classified by $H^{p-1}(\sigma_-)$.
- C. Inequivalent differential field equations on the nontrivial dynamical fields contained in $\widehat{D}_0 \mathcal{C} = 0$ are in one-to-one correspondence with representatives of $H^{p+1}(\sigma_-)$.

Proof of A: The first claim is almost obvious. Indeed, let us decompose the field \mathcal{C} according to the grade G :

$$\mathcal{C} = \sum_{n=0} \mathcal{C}_n, \quad G\mathcal{C}_n = n\mathcal{C}_n, \quad (n = 0, 1, 2, \dots).$$

The field equation (9.1) thus decomposes as

$$\widehat{D}_0 \mathcal{C}|_{n-1} = \sigma_- \mathcal{C}_n + D_0^L \mathcal{C}_{n-1} + \left(\sigma_+ \sum_{m \leq n-2} \mathcal{C}_m \right) \Big|_{n-1} = 0. \quad (9.5)$$

By a straightforward induction on $n = 1, 2, \dots$, one can convince oneself that all fields \mathcal{C}_n that contribute to the first term of the right hand side of the equation (9.5) are thereby expressed in terms of derivatives of lower grade (*i.e.* $< n$) fields, hence they are auxiliary²³. As a result only fields annihilated by σ_- are not auxiliary. Taking into account the gauge transformation (9.4)

$$\delta \mathcal{C}_n = \widehat{D}_0 \varepsilon|_n = \sigma_- \varepsilon_{n+1} + D_0^L \varepsilon_n + \left(\sigma_+ \sum_{m \leq n-1} \varepsilon_m \right) \Big|_n \quad (9.6)$$

one observes that, due to the first term in this transformation law, all components \mathcal{C}_n which are σ_- exact, *i.e.* which belong to the image of σ_- , are Stueckelberg and they can be gauged away. Therefore, a nontrivial dynamical p -form field in \mathcal{C} should belong to the quotient $\text{Ker}(\sigma_-)/\text{Im}(\sigma_-)$. \square

For Einstein-Cartan's gravity, the Stueckelberg gauge symmetry is the local Lorentz symmetry and indeed what distinguishes the frame field from the metric tensor is that the latter actually belongs to the cohomology $H^1(\sigma_-)$ while the former contains a σ_- exact part.

Proof of B: The proof follows the same lines as the proof of A. The first step has already been performed in the sense that (9.6) already told us that the parameters such that $\sigma_- \varepsilon \neq 0$ are Stueckelberg and can be used to completely gauge away trivial parts of the field \mathcal{C} . Thus differential parameters must be σ_- closed. The only subtlety is that one should make use of the fact that the gauge transformation $\delta_\varepsilon \mathcal{C} = \widehat{D}_0 \varepsilon$ are reducible. More precisely, gauge parameters obeying the reducibility identity

$$\varepsilon = \widehat{D}_0 \zeta \quad (9.7)$$

²²Recall that, at the linearized level, the metric tensor corresponds to the symmetric part $e_{\{\mu a\}}$ of the frame field. The antisymmetric part of the frame field $e_{[\mu a]}$ can be gauged away by fixing locally the Lorentz symmetry, because it contains as many independent components as the Lorentz gauge parameter ε^{cd} .

²³Here we use the fact that the operator σ_- acts vertically (that is, it does not differentiate space-time coordinates) thus giving rise to algebraic conditions which express auxiliary fields via derivatives of the other fields.

are trivial in the sense that they do not perform any gauge transformation, $\delta_{\varepsilon=\widehat{D}_0\zeta}\mathcal{C}=0$. The second step of the proof is a mere decomposition of the reducibility identity (9.7) in order to see that σ_- exact parameters correspond to reducible gauge transformations²⁴. \square

Proof of C: Given a nonnegative integer number n_0 , let us suppose that one has already obtained and analyzed (9.1) in grades ranging from $n=0$ up to $n=n_0-1$. Let us analyze (9.1) in grade G equal to n_0 by looking at the constraints imposed by the Bianchi identities. Applying the operator \widehat{D}_0 on the covariant derivative $\widehat{D}_0\mathcal{C}$ gives identically zero, which is the Bianchi identity $(\widehat{D}_0)^2\mathcal{C}=0$. Decomposing the latter Bianchi identity gives, in grade equal to n_0-1 ,

$$(\widehat{D}_0)^2\mathcal{C}|_{n_0-1} = \sigma_- \left(\widehat{D}_0\mathcal{C}|_{n_0} \right) + D_0^L \left(\widehat{D}_0\mathcal{C}|_{n_0-1} \right) + \left(\sigma_+ \sum_{m \leq n_0-2} \widehat{D}_0\mathcal{C}|_m \right) \Big|_{n_0-1} = 0. \quad (9.8)$$

By the induction hypothesis, the equations $\widehat{D}_0\mathcal{C}|_m = 0$ with $m \leq n_0-1$ have already been imposed and analyzed. Therefore (9.8) leads to

$$\sigma_- \left(\widehat{D}_0\mathcal{C}|_{n_0} \right) = 0.$$

In other words, $\widehat{D}_0\mathcal{C}|_{n_0}$ belongs to $\text{Ker}(\sigma_-)$. Thus it can contain a σ_- exact part and a nontrivial cohomology part:

$$\widehat{D}_0\mathcal{C}|_{n_0} = \sigma_-(E_{n_0+1}) + F_{n_0}, \quad F_{n_0} \in H^{p+1}(\sigma_-).$$

The exact part can be compensated by a field redefinition of the component \mathcal{C}_{n_0+1} which was not treated before (by the induction hypothesis). More precisely, if one performs

$$\mathcal{C}_{n_0+1} \rightarrow \mathcal{C}'_{n_0+1} := \mathcal{C}_{n_0+1} - E_{n_0+1},$$

then one is left with $\widehat{D}_0\mathcal{C}'|_{n_0} = F_{n_0}$. The field equation (9.1) in grade n_0 is $\widehat{D}_0\mathcal{C}'|_{n_0} = 0$. This not only expresses the auxiliary p -forms \mathcal{C}'_{n_0+1} (that are not annihilated by σ_-) in terms of derivatives of lower grade p -forms \mathcal{C}_k ($k \leq n_0$), but also sets F_{n_0} to zero. This imposes some \mathcal{C}_{n_0+1} -independent conditions on the derivatives of the fields \mathcal{C}_k with $k \leq n_0$, thus leading to differential restrictions on the nontrivial dynamical fields. Therefore, to each representative of $H^{p+1}(\sigma_-)$ corresponds a differential field equation. \square

Note that if $H^{p+1}(\sigma_-) = 0$, the equation (9.1) contains only constraints which express auxiliary fields via derivatives of the dynamical fields, imposing no restrictions on the latter. If D_0^L is a first order differential operator and if σ_+ is at most a second order differential operator (which is true in HS applications) then, if $H^{p+1}(\sigma_-)$ is nonzero in the grade k sector, the associated differential equations on a grade ℓ dynamical field are of order $k+1-\ell$. In the next section, two concrete examples of operator σ_- will be considered in many details, together with the physical interpretation of their cohomologies.

10 σ_- cohomology in higher spin gauge theories

As was shown in the previous section, the analysis of generic unfolded dynamical equations amounts to the computation of the cohomology of σ_- . In this section we apply this technique to the analysis of the specific case we focus on: the free unfolded HS gauge field equations of Section 8. The computation of the σ_- cohomology groups relevant for the zero and one-form sectors of the theory is sketched in the subsection 10.1. The physical interpretation of these cohomological results is discussed in the subsection 10.2.

²⁴Note that factoring out the σ_- exact parameters accounts for algebraic reducibility of gauge symmetries. The gauge parameters in $H^{p-1}(\sigma_-)$ may still have differential reducibility analogous to differential gauge symmetries for nontrivial dynamical fields. For the examples of HS systems considered below the issue of reducibility of gauge symmetries is irrelevant however because there are no p -form gauge parameters with $p > 0$.

10.1 Computation of some σ_- cohomology groups

As explained in Section 8, the fields entering in the unfolded formulation of the HS dynamics are either zero- or one-forms both taking values in various two-row traceless (*i.e.* Lorentz irreducible) Young tableaux. These two sectors of the theory have distinct σ_- operators and thus require separate investigations.

Following Section 3, two-row Young tableaux in the symmetric basis can be conveniently described as a subspace of the polynomial algebra $\mathbb{R}[Y^a, Z^b]$ generated by the $2d$ commuting generators Y^a and Z^b . (One makes contact with the HS algebra convention via the identification of variables (Y, Z) with (Y_1, Y_2) . Also let us note that the variable Z^a in this section has no relation with the variables Z_i^A of sections 13 and 14.) Vectors of the space $\Omega^p(\mathcal{M}^d) \otimes \mathbb{R}[Y, Z]$ are p -forms taking values in $\mathbb{R}[Y, Z]$. A generic element reads

$$\alpha = \alpha_{a_1 \dots a_{s-1}, b_1 \dots b_t}(x, dx) Y^{a_1} \dots Y^{a_{s-1}} Z^{b_1} \dots Z^{b_t},$$

where $\alpha_{a_1 \dots a_{s-1}, b_1 \dots b_t}(x, dx)$ are differential forms. The Lorentz irreducibility conditions of the HS fields are two-fold. Firstly, there is the Young tableau condition (3.1), *i.e.* in this case

$$Y^a \frac{\partial}{\partial Z^a} \alpha = 0. \quad (10.1)$$

The condition singles out an irreducible $gl(d)$ -module $W \subset \Omega^p(\mathcal{M}^d) \otimes \mathbb{R}[Y, Z]$. Secondly, the HS fields are furthermore irreducible under $o(d-1, 1)$, which is equivalent to the tracelessness condition (3.6), *i.e.* in our case it is sufficient to impose

$$\eta^{ab} \frac{\partial^2}{\partial Y^a \partial Y^b} \alpha = 0. \quad (10.2)$$

This condition further restricts to the irreducible $o(d-1, 1)$ -module $\hat{W} \subset W$. Note that from (10.1) and (10.2) follows that all traces are zero

$$\eta^{ab} \frac{\partial^2}{\partial Z^a \partial Y^b} \alpha = 0, \quad \eta^{ab} \frac{\partial^2}{\partial Z^a \partial Z^b} \alpha = 0.$$

Connection one-form sector

By looking at the definition (8.1) of the linearized curvatures, and taking into account that the Λ -dependent terms in this formula denote some operator σ_+ that increases the number of Lorentz indices, it should be clear that the σ_- operator of the unfolded field equation (8.2) acts as

$$\sigma_- \omega(Y, Z) \propto e_0^a \frac{\partial}{\partial Z^a} \omega(Y, Z). \quad (10.3)$$

In other words, σ_- is the “de Rham differential” of the “manifold” parametrized by the Z -variables where the generators dZ^a of the exterior algebra are identified with the background vielbein one-forms e_0^a . This remark is very helpful because it already tells us that the cohomology of σ_- is zero in the space $\Omega^p(\mathcal{M}^d) \otimes \mathbb{R}[Y, Z]$ with $p > 0$ because its topology is trivial in the Z -variable sector. The actual physical situation is less trivial because one has to take into account the Lorentz irreducibility properties of the HS fields. Both conditions, (10.1) and (10.2), do commute with σ_- , so one can restrict the cohomology to the corresponding subspaces. Since the topology in Z space is not trivial any more, the same is true for the cohomology groups.

As explained in Section 9, for HS equations formulated in terms of the connection one-forms ($p = 1$), the cohomology groups of dynamical relevance are $H^q(\sigma_-)$ with $q = 0, 1$ and 2 . The computation of the cohomology groups obviously increases in complexity as the form degree increases. In our analysis we consider simultaneously the cohomology $H^p(\sigma_-, W)$ of traceful two-row Young tableaux (*i.e.* relaxing the tracelessness condition (10.2)) and the cohomology $H^p(\sigma_-, \hat{W})$ of traceless two-row Young tableaux.

Form degree zero: This case corresponds to the gauge parameters ε . The cocycle condition $\sigma_- \varepsilon = 0$ states that the gauge parameters do not depend on Z . In addition, they cannot be σ_- -exact since they are at the bottom of the form degree ladder. Therefore, the elements of $H^0(\sigma_-, W)$ are the completely symmetric tensors which correspond to the unconstrained zero-form gauge parameters $\varepsilon(Y)$ in the traceful case (like in [73]), while they are furthermore traceless in $H^0(\sigma_-, \hat{W})$ and correspond to Fronsdal's gauge parameters [36] $\eta^{ab} \frac{\partial^2}{\partial Y^a \partial Y^b} \varepsilon(Y) = 0$ in the traceless case.

Form degree one: Because of the Poincaré Lemma, any σ_- closed one-form $\alpha(Y, Z)$ admits a representation

$$\alpha(Y, Z) = e_0^a \frac{\partial}{\partial Z^a} \phi(Y, Z). \quad (10.4)$$

The right hand side of this relation should satisfy the Young condition (10.1), *i.e.*, taking into account that it commutes with σ_- ,

$$e_0^a \frac{\partial}{\partial Z^a} \left(Y^b \frac{\partial}{\partial Z^b} \phi(Y, Z) \right) = 0.$$

From here it follows that $\phi(Y, Z)$ is either linear in Z^a (a Z -independent ϕ does not contribute to (10.4)) or satisfies the Young property itself. In the latter case the $\alpha(Y, Z)$ given by (10.4) is σ_- exact. Therefore, nontrivial cohomology can only appear in the sector of elements of the form $\alpha(Y, Z) = e_0^a \beta_a(Y)$, where $\beta_a(Y)$ are arbitrary in the traceful case of W and harmonic in Y in the traceless case of \hat{W} . Decomposing $\beta_a(Y)$ into irreps of $gl(d)$

$$\square \otimes \square^{s-1} \cong \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}^{s-1} \oplus \square^s$$

one observes that the hook (*i.e.*, the two-row tableau) is the σ_- exact part, while the one-row part describes $H^1(\sigma_-, W)$. These are the rank s totally symmetric traceful dynamical fields which appear in the unconstrained approach [37, 73].

In the traceless case, decomposing $\beta_a(Y)$ into irreps of $o(d-1, 1)$ one obtains

$$\square \otimes \square^{s-1} \cong \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}^{s-1} \oplus \square^s \oplus \square^{s-2} \quad (10.5)$$

where all tensors associated with the various Young tableaux are traceless. Again, the hook (*i.e.* two-row tableau) is the σ_- exact part, while the one-row traceless tensors in (10.5) describe $H^1(\sigma_-, \hat{W})$ which just matches the Fronsdal fields [36] because a rank s double traceless symmetric tensor is equivalent to a pair of rank s and rank $s-2$ traceless symmetric tensors.

Form degree two: The analysis of $H^2(\sigma_-, W)$ and $H^2(\sigma_-, \hat{W})$ is still elementary, but a little bit more complicated than that of $H^0(\sigma_-)$ and $H^1(\sigma_-)$. Skipping technical details we therefore give the final results.

By following a reasoning similar to the one in the previous proof, one can show that, in the traceful case, the cohomology group $H^2(\sigma_-, W)$ is spanned by two-forms of the form

$$F = e_0^a e_0^b \frac{\partial^2}{\partial Y^a \partial Y^b} C(Y, Z), \quad (10.6)$$

where the zero-form $C(Y, Z)$ satisfies the Howe dual $sp(2)$ invariance conditions

$$Z^b \frac{\partial}{\partial Y^b} C(Y, Z) = 0, \quad Y^b \frac{\partial}{\partial Z^b} C(Y, Z) = 0, \quad (10.7)$$

and, therefore,

$$(Z^b \frac{\partial}{\partial Z^b} - Y^b \frac{\partial}{\partial Y^b})C(Y, Z) = 0. \quad (10.8)$$

In accordance with the analysis of Section 3, this means that

$$C(Y, Z) = C_{a_1 \dots a_s, b_1 \dots b_s} Y^{a_1} \dots Y^{a_s} Z^{b_1} \dots Z^{b_s}, \quad (10.9)$$

where the zero-form components $C_{a_1 \dots a_s, b_1 \dots b_s}$ have the symmetry properties corresponding to the rectangular two-row Young tableau of length s

$$C_{a_1 \dots a_s, b_1 \dots b_s} \sim \begin{array}{|c|} \hline \\ \hline \\ \hline \end{array} \begin{array}{c} s \\ s \end{array}. \quad (10.10)$$

From (10.6) it is clear that F is σ_- closed and $F \in W$. It is also clear that it is not σ_- exact in the space W . Indeed, suppose that $F = \sigma_- G$, $G \in W$. For any polynomial $G \in W$ its power in Z cannot be higher than the power in Y (because of the Young property, the second row of a Young tableau is not longer than the first row). Since σ_- decreases the power in Z , the degree in Z of exact elements $\sigma_- G$ is strictly less than the degree in Y . This is not true for the elements (10.6) because of the condition (10.8). The tensors (10.9) correspond to the linearized curvature tensors introduced by de Wit and Freedman [37].

Let us now consider the traceless case of $H^2(\sigma_-, \hat{W})$. The formula (10.6) still gives cohomology but now $C(Y, Z)$ must be traceless,

$$\eta^{ab} \frac{\partial^2}{\partial Y^a \partial Y^b} C(Y, Z) = 0, \quad \eta^{ab} \frac{\partial^2}{\partial Z^a \partial Z^b} C(Y, Z) = 0, \quad \eta^{ab} \frac{\partial^2}{\partial Y^a \partial Z^b} C(Y, Z) = 0.$$

In this case the tensors (10.9) correspond to Weyl-like tensors, *i.e.* on-shell curvatures. They form the so-called “Weyl cohomology”. But this is not the end of the story because there are other elements in $H^2(\sigma_-, \hat{W})$. They span the “Einstein cohomology” and contain two different types of elements:

$$r_1 = e_0^a e_0^b \left((Z_b Y^c - Y_b Z^c) \frac{\partial^2}{\partial Y^a \partial Y^c} \rho_1(Y) \right), \quad (10.11)$$

$$\begin{aligned} r_2 &= e_0^a e_0^b \left((d-1)(d+Y^c \frac{\partial}{\partial Y^c} - 2) Y_a Z_b + d Y_c Y^c Z_a \frac{\partial}{\partial Y^b} \right. \\ &\quad \left. - (d+Y^c \frac{\partial}{\partial Y^c} - 2) Y_e Z^e Y_a \frac{\partial}{\partial Y^b} + Y_e Y^e Y_a Z^c \frac{\partial}{\partial Y^c} \frac{\partial}{\partial Y^b} \right) \rho_2(Y), \end{aligned} \quad (10.12)$$

where $\rho_{1,2}(Y)$ are arbitrary harmonic polynomials

$$\eta^{ab} \frac{\partial^2}{\partial Y^a \partial Y^b} \rho_{1,2}(Y) = 0,$$

thus describing completely symmetric traceless tensors.

One can directly see that r_1 and r_2 belong to \hat{W} (*i.e.*, satisfy (10.1) and (10.2)) and are σ_- closed, $\sigma_- r_{1,2} = 0$ (the check is particularly simple for r_1). It is also easy to see that $r_{1,2}$ are in the nontrivial cohomology class. Indeed, the appropriate trivial class is described in tensor notations by the two-form $e_0^c \omega_{a_1 \dots a_{s-1}, bc}$, where

$$\omega_{a_1 \dots a_{s-1}, bc} = e_0^f \omega_{f; a_1 \dots a_{s-1}, bc} \quad (10.13)$$

is a one-form that has the properties of traceless two-row Young tableau with $s-1$ cells in the first row and two cells in the second row. The trivial cohomology class neither contains a rank $s-2$ tensor like ρ_2 , that needs a double contraction in $\omega_{f; a_1 \dots a_{s-1}, bc}$ in (10.13), nor a rank s symmetric tensor like ρ_1 , because symmetrization of a contraction of the tensor $\omega_{f; a_1 \dots a_{s-1}, bc}$ over any s indices gives zero. It can be shown that the Einstein cohomology (10.11) and (10.12) together with the Weyl cohomology (10.6) span $H^2(\sigma_-, \hat{W})$.

Weyl zero-form sector

The Λ -dependent terms in the formula (8.3) denote some operator σ_+ that increases the number of Lorentz indices, therefore the operator σ_- of the unfolded field equation (8.3) is given by

$$(\sigma_- C)^{a_1 \dots a_{s+k}, b_1 \dots b_s} = -e_0{}_c \left((2+k) C^{a_1 \dots a_{s+k}c, b_1 \dots b_s} + s C^{a_1 \dots a_{s+k} \{b_1, b_2 \dots b_s\}c} \right) \quad (10.14)$$

both in the traceless and in the traceful twisted adjoint representations. One can analogously compute the cohomology groups $H^q(\sigma_-, W)$ and $H^q(\sigma_-, \hat{W})$ with $q = 0$ and 1 , which are dynamically relevant in the zero-form sector ($p = 0$).

Form degree zero: It is not hard to see that $H^0(\sigma_-, W)$ and $H^0(\sigma_-, \hat{W})$ consist, respectively, of the generalized Riemann (*i.e.* traceful) and Weyl (*i.e.* traceless) tensors $C^{a_1 \dots a_s, b_1 \dots b_s}$.

Form degree one: The cohomology groups $H^1(\sigma_-, W)$ and $H^1(\sigma_-, \hat{W})$ consist of one-forms of the form

$$w^{a_1 \dots a_s, b_1, \dots, b_s} = e_0{}_c C^{a_1 \dots a_s, b_1 \dots b_s, c}, \quad (10.15)$$

where $C^{a_1 \dots a_s, b_1 \dots b_s, c}$ is, respectively, a traceful and traceless zero-form described by the three row Young tableaux with s cells in the first and second rows and one cell in the third row. Indeed, a one-form $w^{a_1 \dots a_s, b_1, \dots, b_s}$ given by (10.15) is obviously σ_- closed (by definition, σ_- gives zero when applied to a rectangular Young tableau) and not σ_- exact, thus belonging to nontrivial cohomology.

In addition, in the traceless case, $H^1(\sigma_-, \hat{W})$ includes the “Klein-Gordon cohomology”

$$w^a = e_0^a k \quad (10.16)$$

and the “Maxwell cohomology”

$$w^{a,b} = e_0^a m^b - e_0^b m^a, \quad (10.17)$$

where k and m^a are arbitrary scalar and vector, respectively. It is obvious that the one-forms (10.16) and (10.17) are σ_- closed for σ_- (10.14). They are in fact exact in the traceful case with $w = \sigma_-(\varphi)$, where

$$\varphi^{a_1 a_2} \propto k \eta^{a_1 a_2}, \quad \varphi^{a_1 a_2, b} \propto 2m^b \eta^{a_1 a_2} - m^{a_2} \eta^{a_1 b} - m^{a_1} \eta^{a_2 b} \quad (10.18)$$

but are not exact in the traceless case since the tensors φ in (10.18) are not traceless although the resulting $w = \sigma_-(\varphi)$ belong to the space of traceless tensors.

Unfolded system

One may combine the connection one-forms ω and Weyl zero-forms C into the set $\mathcal{C} = (\omega, C)$ and redefine $\sigma_- \rightarrow \hat{\sigma}_-$ in such a way that $\hat{\sigma}_- = \sigma_- + \Delta\sigma_-$ where σ_- acts on ω and C following (10.3) and (10.14), respectively, and $\Delta\sigma_-$ maps the rectangular zero-form Weyl tensors to the two-form sector via

$$\Delta\sigma_- C(Y, Z) \propto e_0^a e_0^b \frac{\partial^2}{\partial Y^a \partial Z^b} C(Y, Z), \quad (10.19)$$

thus adding the term with the Weyl zero-form to the linearized curvature of one-forms.

The dynamical content of the unfolded system of equations (8.2)-(8.3) is encoded in the cohomology groups of $\hat{\sigma}_-$. The gauge parameters are those of $H^0(\hat{\sigma}_-)$ in the adjoint module (*i.e.* the connection one-form sector). The dynamical fields are symmetric tensors of $H^1(\hat{\sigma}_-)$ in the adjoint module, along with the scalar field in the zero-form sector. There are no nontrivial field equations in the traceful case. As explained in more detail in Subsection 10.2, in the traceless case, the field equations are associated with the Einstein cohomology (10.11)-(10.12), Maxwell cohomology (10.17) and Klein-Gordon cohomology (10.16). Note that the cohomology (10.15) disappears as a result of gluing the one-form adjoint and zero-form twisted adjoint modules by (10.19).

10.2 Physical interpretation of some σ_- cohomology groups

These cohomological results tell us that there are several possible choices for gauge invariant differential equations on HS fields.

The form of $r_{1,2}$ (10.11) and (10.12) indicates that the Einstein cohomology is responsible for the Lagrangian field equations of completely symmetric double traceless fields. Indeed, carrying one power of Z^a , they are parts of the HS curvatures $R_{a_1 \dots a_{s-1}, b}$ with one cell in the second row of the corresponding Young tableau. For spin s , $\rho_1(Y)$ is a harmonic polynomial of homogeneity degree s , while $\rho_2(Y)$ is a harmonic polynomial of homogeneity degree $s - 2$. As a result, the field equations which follow from $r_1 = 0$ and $r_2 = 0$ are of second order²⁵ in derivatives of the dynamical Fronsdal fields taking values in $H^1(\sigma_-, \hat{W})$ and, as expected for Lagrangian equations in general, there are as many equations as dynamical fields.

For example, spin-2 equations on the trace and traceless parts of the metric tensor associated with the elements of $H^1(\sigma_-, \hat{W})$ result from the conditions $r_1 = 0$ and $r_2 = 0$ with $\rho_{1,2}$ in (10.11) and (10.12) of the form $\rho_1(Y) = r_{ab} Y^a Y^b$, $\rho_2(Y) = r$, with arbitrary r and traceless r_{ab} . These are, respectively, the traceless and trace parts of the linearized Einstein equations. Analogously, the equations $r_1 = 0$ and $r_2 = 0$ of higher orders in Y correspond to the traceless and first trace parts of the Fronsdal spin $s > 1$ HS equations (which are, of course, double traceless).

Thus, in the traceless case, the proper choice to reproduce dynamical field equations equivalent to the equations resulting from the Fronsdal Lagrangian is to keep only the Weyl cohomology nonzero. Setting elements of the Einstein cohomology to zero, which imposes the second-order field equations on the dynamical fields, leads to

$$R_1 = e_0^a e_0^b \epsilon_{ij} \frac{\partial^2}{\partial Y_i^a \partial Y_j^b} C(Y_k^c), \quad (10.20)$$

where one makes contact with the HS algebra convention via the identification of variables (Y, Z) with (Y_1, Y_2) . Here $C(Y_k^c)$ satisfies the $sp(2)$ invariance condition

$$Y_i^a \frac{\partial}{\partial Y_j^a} C(Y) = \frac{1}{2} \delta_j^i Y_k^a \frac{\partial}{\partial Y_k^a} C(Y) \quad (10.21)$$

along with the tracelessness condition

$$\eta^{ab} \frac{\partial^2}{\partial Y_j^a \partial Y_i^b} C(Y) = 0 \quad (10.22)$$

to parametrize the Weyl cohomology. The equation (10.20) is exactly (8.2). Thus, the generalized Weyl tensors on the right hand side of (8.2) parametrize the Weyl cohomology in the HS curvatures exactly so as to make the equations (8.2) for $s > 1$ equivalent to the HS field equations that follow from Fronsdal's action.²⁶

Alternatively, one can set the Weyl cohomology to zero, keeping the Einstein cohomology arbitrary. It is well known that in the spin-2 case of gravity the generic solution of the condition that the Weyl tensor is zero leads to conformally flat metrics. Analogous analysis for free spin 3 was performed in [74]. It is tempting to conjecture that this property is true for any spin $s > 2$, *i.e.*, the condition that Weyl cohomology is zero singles out the “conformally flat” single trace HS fields of the form

$$\varphi_{\nu_1 \dots \nu_s}(x) = g_{\{\nu_1 \nu_2}(x) \psi_{\nu_3 \dots \nu_s\}}(x) \quad (10.23)$$

with traceless symmetric $\psi_{\nu_3 \dots \nu_s}(x)$. Indeed, the conformally flat free HS fields (10.23) have zero generalized Weyl tensor simply because it is impossible to build a traceless tensor (10.10) from derivatives of

²⁵The cohomological analysis outlined here can be extended to the space \hat{W}_n of tensors required to have their n -th trace equal zero, *i.e.* with (10.2) replaced by $(\eta^{ab} \frac{\partial^2}{\partial Y_i^a \partial Y_j^b})^n \alpha(Y, Z) = 0$. It is tempting to conjecture that the resulting gauge invariant field equations will contain $2n$ derivatives.

²⁶The subtle relationship between Fronsdal's field equations and the tracelessness of the HS curvatures was discussed in details for metric-like spin-3 fields in [74].

$$\psi_{\nu_1 \dots \nu_{s-2}}(x).$$

The spin-0 and spin-1 equations are not described by the cohomology in the one-form adjoint sector. The Klein-Gordon and Maxwell equations result from the cohomology of the zero-form twisted adjoint representation, which encodes equations that can be imposed in terms of the generalized Weyl tensors which contain the spin-1 Maxwell tensor and the spin-0 scalar as the lower spin particular cases. More precisely, the spin-0 and spin-1 field equations are contained in the parts of equations (8.3) associated with the cohomology groups (10.16) and (10.17). For spin $s \geq 1$, the cohomology (10.15) encodes the Bianchi identities for the definition of the Weyl (Riemann) tensors by (10.20). This is equivalent to the fact that, in the traceless case, the equations (10.20) and (8.3) describe properly free massless equations of all spins supplemented with an infinite set of constraints for auxiliary fields, which is the content of the central on-mass-shell theorem.

In the traceful case, the equation (10.20) with traceful $C(Y_i^a)$ does not impose any differential restrictions on the fields in $H^1(\sigma_-, W)$ because $e_0^a e_0^b \epsilon_{ij} \frac{\partial^2}{\partial Y_i^a \partial Y_j^b} C(Y_k^c)$ span the full $H^2(\sigma_-, W)$ of the one-form adjoint sector. This means that the equation (10.20) describes a set of constraints which express all fields in terms of derivatives of the dynamical fields in $H^1(\sigma_-, W)$. In this sense, the equation (10.20) for a traceful field describes off-mass-shell constraints identifying the components of $C(Y)$ with the deWit-Freedman curvature tensors²⁷. Supplementing (10.20) by the covariant constancy equation (8.3) on the traceful zero-forms $C(Y)$ one obtains an infinite set of constraints for any spin which express infinite sets of auxiliary fields in terms of derivatives of the dynamical fields, imposing no differential restrictions on the latter. These constraints provide unfolded off-mass-shell description of massless fields of all spins. We call this fact “central off-mass-shell theorem”.

Let us stress that our analysis works both in the flat space-time and in the $(A)dS_d$ case originally considered in [41]. Indeed, although the nonzero curvature affects the explicit form of the background frame and the Lorentz covariant derivative D_0^L and also requires a nonzero operator σ_+ denoted by $O(\Lambda)$ in (8.1), all this does not affect the analysis of the σ_- cohomology because the operator σ_- remains of the form (10.3) with a nondegenerate frame field e_μ^a .

Let us make the following comment. The analysis of the dynamical content of the covariant constancy equations $\hat{D}_0 C = 0$ may depend on the choice of the grading operator G and related graded decomposition (9.2). This may lead to different definitions of σ_- and, therefore, different interpretations of the same system of equations. For example, one can choose a different definition of σ_- in the space W of traceful tensors simply by decomposing W into a sum of irreducible Lorentz tensors (*i.e.*, traceless tensors) and then defining σ_- within any of these subspaces as in \hat{W} . In this basis, the equations (8.2) will be interpreted as dynamical equations for an infinite set of traceless dynamical fields. This phenomenon is not so surprising, taking into account the well-known analogous fact that, say, an off-mass-shell scalar can be represented as an integral over the parameter of mass of an infinite set of on-mass-shell scalar fields. More generally, to avoid paradoxical conclusions one has to take into account that σ_- may or may not have a meaning in terms of the elements of the Lie algebra $\text{Lie } h$ that gives rise to the covariant derivative (9.2).

10.3 Towards nonlinear equations

Nonlinear equations should replace the linearized covariant derivative \tilde{D}_0 with the full one, \tilde{D} , containing the h -valued connection ω . They should also promote the linearized curvature R_1 to R . Indeed, (8.3) and (10.20) cannot be correct at the nonlinear level because the consistency of $\tilde{D}C = 0$ implies arbitrarily high powers of C in the r.h.s. of the modified equations, since

$$\tilde{D}\tilde{D}C \sim RC \sim O(C^2) + \text{higher order terms} ,$$

²⁷However, as pointed out in [75], if one imposes $C(Y_i^a)$ to be harmonic in Y , then the corresponding field equations (10.20) imposes the deWit-Freedman curvature to be traceless. In this sense, it may be possible to remove the tracelessness requirement in the frame-like formulation (see Section 4.2) without changing the physical content of the free field equations.

the last relation being motivated by (10.20).

Apart from dynamical field equations, the unfolded HS field equations contain constraints on the auxiliary components of the HS connections, expressing the latter via derivatives of the nontrivial dynamical variables (*i.e.* Fronsdal fields), modulo pure gauge ambiguity. Originally, all HS gauge connections $\omega_{\mu}^{A_1 \dots A_{s-1}, B_1 \dots B_{s-1}}$ have dimension $length^{-1}$ so that the HS field strength (5.15) needs no dimensionful parameter to have dimension $length^{-2}$. However, this means that when some of the gauge connections are expressed via derivatives of the others, these expressions must involve space-time derivatives in the dimensionless combination $\rho \frac{\partial}{\partial x^\nu}$, where ρ is some parameter of dimension $length$. The only dimensionful parameter available in the analysis of the free dynamics is the radius ρ of the AdS space-time related to the cosmological constant by (2.8). Recall that it appears through the definition of the frame field (2.10) with $V^A \sim \rho$ adapted to make the frame E_μ^A (and, therefore, the metric tensor) dimensionless. As a result, the HS gauge connections are expressed by the unfolded field equations through the derivatives of the dynamical fields as

$$\omega^{a_1 \dots a_{s-1}, b_1 \dots b_t \hat{d} \dots \hat{d}} = \Pi \left(\rho^t \frac{\partial}{\partial x^{b_1}} \dots \frac{\partial}{\partial x^{b_t}} \omega^{a_1 \dots a_{s-1}, \hat{d} \dots \hat{d}} \right) + \text{lower derivative terms}, \quad (10.24)$$

where Π is some projector that permutes indices (including the indices of the forms) and projects out traces. Plugging these expressions back into the HS field strength (5.15) one finds that HS connections with $t > 1$ (*i.e.* extra fields that appear for $s > 2$) contribute to the terms with higher derivatives which blow up in the flat limit $\rho \rightarrow \infty$. This mechanism brings higher derivatives and negative powers of the cosmological constant into HS interactions (but not into the free field dynamics because the free action is required to be independent of the extra fields). Note that a similar phenomenon takes place in the sector of the generalized Weyl zero-forms $C(Y)$ in the twisted adjoint representation.

11 Star product

We shall formulate consistent nonlinear equations using the star product. In other words we shall deal with ordinary commuting variables Y_i^A instead of operators \hat{Y}_i^A . In order to avoid ordering ambiguities, we use the Weyl prescription. An operator is said to be Weyl ordered if it is completely symmetric under the exchange of operators \hat{Y}_i^A . One establishes a one to one correspondence between each Weyl ordered polynomial $f(\hat{Y})$ (5.4) and its symbol $f(Y)$, defined by substituting each operator \hat{Y}_i^A with the commuting variable Y_i^A . Thus $f(Y)$ admits a formal expansion in power series of Y_i^A identical to that of $f(\hat{Y})$, *i.e.* with the same coefficients,

$$f(Y) = \sum_{m,n} f_{A_1 \dots A_m, B_1 \dots B_n} Y_1^{A_1} \dots Y_1^{A_m} Y_2^{B_1} \dots Y_2^{B_n}. \quad (11.1)$$

To reproduce the algebra Ad_{+1} , one defines the *star product* in such a way that, given any couple of functions f_1, f_2 , which are symbols of operators \hat{f}_1, \hat{f}_2 respectively, $f_1 * f_2$ is the symbol of the operator $\hat{f}_1 \hat{f}_2$. The result is nontrivial because the operator $\hat{f}_1 \hat{f}_2$ should be Weyl ordered. It can be shown that this leads to the well-known Weyl-Moyal formula

$$(f_1 * f_2)(Y) = f_1(Y) e^{\frac{1}{2} \overleftarrow{\partial}_A^j \overrightarrow{\partial}_B^i \eta^{AB} \epsilon_{ji}} f_2(Y), \quad (11.2)$$

where $\partial_A^j \equiv \frac{\partial}{\partial Y_j^A}$ and $\overleftarrow{\partial}$, as usual, means that the partial derivative acts to the left while $\overrightarrow{\partial}$ acts to the right. The rationale behind this definition is simply that higher and higher powers of the differential operator in the exponent produce more and more contractions. One can show that the star product is an associative product law, and that it is regular, which means that the star product of two polynomials in Y is still a polynomial. From (11.2) it follows that the star product reproduces the proper commutation relation of oscillators,

$$[Y_i^A, Y_j^B]_* \equiv Y_i^A * Y_j^B - Y_j^B * Y_i^A = \epsilon_{ij} \eta^{AB}.$$

The star product has also an integral definition, equivalent to the differential one given by (11.2), which is

$$(f_1 * f_2)(Y) = \frac{1}{\pi^{2(d+1)}} \int dS dT f_1(Y + S) f_2(Y + T) \exp(-2S_i^A T_A^i). \quad (11.3)$$

The whole discussion of Section 5 can be repeated here, with the prescription of substituting operators with their symbols and operator products with star products. For example, the $o(d-1, 2)$ generators (5.5) and the $sp(2)$ generators (5.6) are realized as

$$T^{AB} = -T^{BA} = \frac{1}{2} Y^{iA} Y_i^B, \quad t_{ij} = t_{ji} = Y_i^A Y_{jA}, \quad (11.4)$$

respectively. Note that

$$Y_i^A * = Y_i^A + \frac{1}{2} \frac{\overrightarrow{\partial}}{\partial Y_A^i}$$

and

$$*Y_i^A = Y_i^A - \frac{1}{2} \frac{\overleftarrow{\partial}}{\partial Y_A^i}.$$

From here it follows that

$$[Y_i^A, f(Y)]_* = \frac{\partial}{\partial Y_A^i} f(Y) \quad (11.5)$$

and

$$\{Y_i^A, f(Y)\}_* = 2Y_i^A f(Y). \quad (11.6)$$

With the help of these relations it is easy to see that the $sp(2)$ invariance condition $[t_{ij}, f(Y)]_* = 0$ indeed has the form (5.8) and singles out two-row rectangular Young tableaux, *i.e.* it implies that the coefficients $f_{A_1 \dots A_m, B_1 \dots B_n}$ are nonzero only if $n = m$, and symmetrization of any $m+1$ indices of $f_{A_1 \dots A_m, B_1 \dots B_m}$ gives zero. Let us also note that if $[t_{ij}, f(Y)]_* = 0$ then

$$t_{ij} * f = f * t_{ij} = \left(t_{ij} + \frac{1}{4} \frac{\partial^2}{\partial Y_A^i \partial Y^{jA}} \right) f. \quad (11.7)$$

One can then introduce the gauge fields taking values in this star algebra as functions $\omega(Y|x)$ of oscillators,

$$\omega(Y|x) = \sum_{s \geq 1} i^{s-2} \omega^{A_1 \dots A_{s-1}, B_1 \dots B_{s-1}}(x) Y_{1A_1} \dots Y_{1A_{s-1}} Y_{2B_1} \dots Y_{2B_{s-1}}, \quad (11.8)$$

with their field strength defined by

$$R(Y) = d\omega(Y) + (\omega * \omega)(Y) \quad (11.9)$$

and gauge transformations

$$\delta\omega(Y) = d\epsilon(Y) + [\omega, \epsilon]_*(Y) \quad (11.10)$$

(where the dependence on the space-time coordinates x is implicit). For the subalgebra of $sp(2)$ singlets we have

$$D(t_{ij}) = 0, \quad [t_{ij}, \epsilon]_* = 0, \quad [t_{ij}, R]_* = 0. \quad (11.11)$$

Note that $d(t_{ij}) = 0$ and, therefore from the first of these relations it follows that $[t_{ij}, \omega]_* = 0$, which is the $sp(2)$ invariance relation.

Furthermore, one can get rid of traces by factoring out the ideal \mathcal{I} spanned by the elements of the form $t_{ij} * g^{ij}$. For factoring out the ideal \mathcal{I} it is convenient to consider [57] elements of the form $\Delta * g$ where Δ is an element satisfying the conditions $\Delta * t_{ij} = t_{ij} * \Delta = 0$. The explicit form of Δ is [57, 75]

$$\Delta = \int_{-1}^1 ds (1-s^2)^{\frac{1}{2}(d-3)} \exp(s\sqrt{z}) = 2 \int_0^1 ds (1-s^2)^{\frac{1}{2}(d-3)} ch(s\sqrt{z}), \quad (11.12)$$

where

$$z = \frac{1}{4} Y_i^A Y_{Aj} Y^{Bi} Y_B^j . \quad (11.13)$$

Indeed, one can see [57] that $\Delta * f = f * \Delta$ and that all elements of the form $f = u^{ij} * t_{ij}$ or $f = t_{ij} * u^{ij}$ disappear in $\Delta * f$, *i.e.* the factorization of \mathcal{I} is automatic. The operator Δ , which we call quasiprojector, is not a projector because $\Delta * \Delta$ does not exist (diverges) [57, 75]. One therefore cannot define a product of two elements $\Delta * f$ and $\Delta * g$ in the quotient algebra as the usual star product. A consistent definition for the appropriately modified product law \circ is

$$(\Delta * f) \circ (\Delta * g) = \Delta * f * g \quad (11.14)$$

(for more detail see section 3 of [57]). Note that from this consideration it follows that the star product $g_1 * g_2$ of any two elements satisfying the strong $sp(2)$ invariance condition $t_{ij} * g_{1,2} = 0$ of [75] is ill-defined because such elements admit a form $g_{1,2} = \Delta * g'_{1,2}$.

12 Twisted adjoint representation

As announced in Section 8, we give here a precise definition of the module in which the Weyl-like zero-forms take values, in such a way that (8.3) is reproduced at the linearized level. Taking the quotient by the ideal \mathcal{I} is a subtle step the procedure of which is explained in Subsection 12.2.

12.1 Definition of the twisted adjoint module

To warm up, let us start with the adjoint representation. Let \mathcal{A} be an associative algebra endowed with a product denoted by $*$. The $*$ -commutator is defined as $[a, b]_* = a * b - b * a$, $a, b \in \mathcal{A}$. As usual for an associative algebra, one constructs a Lie algebra g from \mathcal{A} , the Lie bracket of which is the $*$ -commutator. Then g has an adjoint representation the module of which coincides with the algebra itself and such that the action of an element $a \in g$ is given by

$$[a, X]_*, \forall X \in \mathcal{A} .$$

Let τ be an automorphism of the algebra \mathcal{A} , that is to say

$$\tau(a * b) = \tau(a) * \tau(b), \quad \tau(\lambda a + \mu b) = \lambda \tau(a) + \mu \tau(b), \quad \forall a, b \in \mathcal{A},$$

where λ and μ are any elements of the ground field \mathbb{R} or \mathbb{C} . The τ -twisted adjoint representation of g has the same definition as the adjoint representation, except that the action of g on its elements is modified by the automorphism τ :

$$a(X) \rightarrow a * X - X * \tau(a) .$$

It is easy to see that this gives a representation of g .

The appropriate choice of τ , giving rise to the infinite bunch of fields contained in the zero-form C (matter fields, generalized Weyl tensors and their derivatives), is the following:

$$\tau(f(Y)) = \tilde{f}(Y) , \quad (12.1)$$

where

$$\tilde{f}(Y) = f(\tilde{Y}), \quad \tilde{Y}_i^A = Y_i^A - 2V^A V^B Y_{Bi}, \quad (12.2)$$

i.e., \tilde{Y}_i^A is the oscillator Y_i^A reflected with respect to the compensator (recall that we use the normalization $V_A V^A = 1$). So one can say that the automorphism τ is some sort of parity transformation in the V -direction, leaving unaltered the Lorentz components of the oscillators. More explicitly, in terms

of the transverse and longitudinal components

$${}^\perp Y_i^A = Y_i^A - V^A V_B Y_i^B, \quad \|Y_i^A = V^A V^B Y_{Bi},$$

the automorphism τ is the transformation

$${}^\perp Y_i^A \rightarrow {}^\perp Y_i^A, \quad \|Y_i^A \rightarrow -\|Y_i^A,$$

or, in the standard gauge, $Y_i^a \rightarrow Y_i^a$, $Y_i^{\hat{d}} \rightarrow -Y_i^{\hat{d}}$. From (11.2) it is obvious that τ is indeed an automorphism of the star product algebra.

The automorphism τ (12.1) leaves invariant the $sp(2)$ generators

$$\tau(t_{ij}) = t_{ij}.$$

This allows us to require the zero-form $C(Y|x)$ in the twisted adjoint module to satisfy the $sp(2)$ invariance condition

$$t_{ij} * C = C * t_{ij} \quad (12.3)$$

and to define the covariant derivative in the twisted adjoint module of $hu(1|2:[d-1,2])$ as

$$\tilde{D}C = dC + \omega * C - C * \tilde{\omega}. \quad (12.4)$$

At the linearized level one obtains

$$\tilde{D}_0 C = dC + \omega_0 * C - C * \tilde{\omega}_0. \quad (12.5)$$

One decomposes ω_0 into its Lorentz and translational part, $\omega_0 = \omega_0^L + \omega_0^{transl}$, via (2.15), which gives

$$\begin{aligned} \omega_0^L &\equiv \frac{1}{2} \omega_0^{AB} {}^\perp Y_A^i {}^\perp Y_{Bi} = \frac{1}{2} \omega_0^{ab} Y_a^i Y_{bi}, \\ \omega_0^{transl} &\equiv \omega_0^{AB} {}^\perp Y_A^i \|Y_{Bi} = e_a^A Y_a^i Y_{Bi} V^B. \end{aligned}$$

Taking into account the definition (12.1), it is clear that τ changes the sign of ω_0^{transl} while leaving ω_0^L untouched. This is tantamount to say that \tilde{D}_0 contains an anticommutator with the translational part of the connection instead of a commutator,

$$\tilde{D}_0 C = D_0^L C + \{\omega_0^{transl}, C\}_*,$$

where D_0^L is the usual Lorentz covariant derivative, acting on Lorentz indices. Expanding the star products, we have

$$\tilde{D}_0 = D_0^L + 2E_0^A V^B ({}^\perp Y_A^i \|Y_{Bi} - \frac{1}{4} \epsilon^{ij} \frac{\partial^2}{\partial {}^\perp Y^{Ai} \partial \|Y^{Bj}}) , \quad (12.6)$$

the last term being due to the noncommutative structure of the star algebra.

The equation (12.6) suggests that there exists a grading operator

$$N^{tw} = N_\perp - N_\parallel = {}^\perp Y_i^A \frac{\partial}{\partial {}^\perp Y_i^A} - \|Y_i^A \frac{\partial}{\partial \|Y_i^A} \quad (12.7)$$

commuting with \tilde{D}_0 , and whose eigenvalues $N_\perp - N_\parallel = 2s$, where s is the spin, classify the various irreducible submodules into which the twisted adjoint module decomposes as $o(d-1,2)$ -module. In other words, the system of equations $\tilde{D}_0 C = 0$ decomposes into an infinite number of independent subsystems, the fields of each subset satisfying $N^{tw} C = 2sC$, for some nonnegative integer s . Let us give some more detail about this fact. Recall that requiring $sp(2)$ invariance restricts us to the rectangular two row AdS_d

Young tableaux $\begin{array}{|c|c|} \hline & \\ \hline \end{array}$. By means of the compensator V^A we then distinguish between transverse (Lorentz) and longitudinal indices. Clearly $N^{tw} \geq 0$, since having more than half of vector indices in the

extra direction V would imply symmetrization over more than half of all indices, thus giving zero because of the symmetry properties of Young tableaux. Then, each independent sector $N^{tw} = 2s$ of the twisted adjoint module starts from the rectangular Lorentz-Young tableau corresponding to the (generalized) Weyl tensor $\begin{array}{|c|} \hline \square \\ \hline \end{array}_s^s$, and admits as further components all its “descendants” $\begin{array}{|c|} \hline \square \\ \hline \end{array}_s^{s+k}$, which the equations themselves set equal to k Lorentz covariant derivatives of $\begin{array}{|c|} \hline \square \\ \hline \end{array}_s^s$. From the AdS_d -Young tableaux point of view, the set of fields forming an irreducible submodule of the twisted adjoint module with some fixed s is nothing but the components of the fields $C^{A_1 \dots A_u, B_1 \dots B_u}$ ($u = s, \dots, \infty$) that have $k = u - s$ indices parallel to V^A : $\begin{array}{|c|} \hline \square \\ \hline \end{array}_s^{s+k=u} \sim \begin{array}{|c|} \hline \square \\ \hline \end{array}_u^u$.

12.2 Factorization procedure

The twisted adjoint module as defined in the previous section is off-mass-shell because the zero-form $C(Y|x)$ is traceful in the oscillators Y_i^A . To put it on-mass-shell one has to factor out those elements of the twisted adjoint module T that also belong to the ideal \mathcal{I} of the associative algebra \mathcal{S} of $sp(2)$ singlets (see Subsection 5.2) and form a submodule $T \cap \mathcal{I}$ of the HS algebra. By a slight abuse of terminology, we will also refer to this submodule as “ideal \mathcal{I} ” in the sequel. Therefore, “quotienting by the ideal \mathcal{I} ” means dropping terms $g \in T \cap \mathcal{I}$, that is

$$g \in \mathcal{I} \iff [t_{ij}, g]_* = 0, \quad g = t_{ij} * g^{ij} = g^{ij} * t_{ij}. \quad (12.8)$$

The factorization of \mathcal{I} admits an infinite number of possible ways of choosing representatives of the equivalence classes (recall that f_1 and f_2 belong to the same equivalence class iff $f_1 - f_2 \in \mathcal{I}$). The tracelessness condition (3.6), which in the case of HS gauge one-forms amounts to

$$\frac{\partial^2}{\partial Y_i^A \partial Y_{Aj}} \omega(Y) = 0,$$

is not convenient for the twisted adjoint representation because it does not preserve the grading (12.7), *i.e.* it does not commute with N^{tw} . A version of the factorization condition that commutes with N^{tw} and is more appropriate for the twisted adjoint case is

$$\left(\frac{\partial^2}{\partial^\perp Y_i^A \partial^\perp Y_{Aj}} - 4 \|Y^{Ai}\| Y_A^j \right) C(Y) = 0.$$

For the computations, both in the adjoint and the twisted adjoint representation, it may be convenient to require Lorentz tracelessness, *i.e.* the tracelessness with respect to transversal indices. Recall that the final result is insensitive to a particular choice of the factorization condition, that is, choosing one or another condition is a matter of convenience. In practice the factorization procedure is implemented as follows. Let A^{tr} denote either a one-form HS connection or a HS Weyl zero-form satisfying a chosen tracelessness (*i.e.* factorization) condition. The left hand side of the field equations contains the covariant derivative D in the adjoint, $D = D_0$, or twisted adjoint, $D = \tilde{D}_0$, representation. $D(A^{tr})$ does not necessarily satisfy the chosen tracelessness condition, but it can be represented in the form

$$D(A^{tr}) = (D(A^{tr}))^{tr} + t_{ij} X_1^{ij}, \quad (12.9)$$

where the ordinary product of t_{ij} with some X_1^{ij} parametrizes the traceful part, while $(D(A^{tr}))^{tr}$ satisfies the chosen tracelessness condition. Note that X_1^{ij} contains less traces than $D(A^{tr})$. One rewrites (12.9) in the form

$$D(A^{tr}) = (D(A^{tr}))^{tr} + t_{ij} * X_1^{ij} + B, \quad (12.10)$$

where $B = t_{ij} X_1^{ij} - t_{ij} * X_1^{ij}$. Taking into account that

$$B = t_{ij} X_1^{ij} - t_{ij} * X_1^{ij} = -\frac{1}{4} \frac{\partial^2}{\partial Y_A^i \partial Y_{jA}} X_1^{ij} \quad (12.11)$$

by virtue of (11.7), one observes that B contains less traces than $D(A^{tr})$. The factorization is performed by dropping out the term $t_{ij} * X_1^{ij}$. The resulting expression $(D(A^{tr}))^{tr} + B$ contains less traces. If necessary, the procedure has to be repeated to get rid of lower traces. At the linearized level it terminates in a finite number of steps (two steps is enough for the Lorentz tracelessness condition). The resulting expression gives the covariant derivative in the quotient representation which is the on-mass-shell twisted adjoint representation.

Equivalently, one can use the factorization procedure with the quasiprojector Δ as explained in the end of Section 11. Upon application of one or another procedure for factorizing out the traces, the spin- s submodule of the twisted adjoint module forms an irreducible $o(d-1, 2)$ -module. The subset of the fields C in the twisted adjoint module with some fixed s matches the set of spin- s generalized Weyl zero-forms of Section 8. Not surprisingly, they form equivalent $o(d-1, 2)$ -modules. In particular, it can be checked that, upon an appropriate rescaling of the fields, the covariant constancy condition in the twisted adjoint representation

$$\tilde{D}_0 C = 0 \quad (12.12)$$

reproduces (8.3) in the standard gauge. The precise form of the Λ -dependent terms in (8.3) follows from this construction. It is straightforward to compute the value of the Casimir operator in this irreducible $o(d-1, 2)$ -module (see also [75])

$$2T^{AB}T_{AB} = (s-1)(s+d-3),$$

where $2s$ is the eigenvalue of N^{tw} . This value coincides with that for the unitary massless representations of any spin in AdS_d [76]. This fact is in agreement with the general observation [77, 51] that the representations carried by the zero-form sector in the unfolded dynamics are dual by a nonunitary Bogolyubov transform to the Hilbert space of single-particle states in the quantized theory.

13 Nonlinear field equations

We are now ready to search for nonlinear corrections to the free field dynamics. We will see that it is indeed possible to find a unique form for interactions, modulo field redefinitions, if one demands that the $sp(2)$ invariance of Section 5.2 is maintained at the nonlinear level. This condition is of crucial importance because, if the $sp(2)$ invariance was broken, then the resulting nonlinear equations might involve new tensor fields, different from the two-row rectangular Young tableaux one started with, and this might have no sense (for example, the new fields may contain ghosts). Thus, to have only usual HS fields as independent degrees of freedom, one has to require that $sp(2)$ invariance survives at the nonlinear level, or, in other words, that there should be a modified $sp(2)$ generator,

$$t_{ij}^{int} = t_{ij} + O(C),$$

that still satisfies $D(t_{ij}^{int}) = 0$, which is a deformation of the free field condition (11.11).

The construction of nonlinear corrections to the free field dynamics and the check of consistency order by order is quite cumbersome. These have been performed explicitly up to second order in the Weyl zero-forms [59, 60, 78] in terms of the spinorial formulation of the $d = 4$ HS theory. More refined methods have been developed to formulate the full dynamics of HS gauge fields in a closed form first in four dimensions [14, 15] and more recently in any dimension [16]. The latter is presented now.

13.1 Doubling of oscillators

A trick that simplifies the formulation is to introduce additional noncommutative variables Z . This allows one to describe complicate nonlinear corrections as solutions of some simple differential equations with respect to such variables. The form of these equations is fixed by formal consistency and by the existence of nonlinear $sp(2)$ generators that guarantee the correct spectrum of fields and the gauge invariance of all nonlinear terms they encode.

More precisely, this step amounts to the doubling of the oscillators $Y_i^A \rightarrow (Z_i^A, Y_i^A)$, and correspondingly one needs to enlarge the star product law. It turns out that a sensible definition is the following,

$$(f * g)(Z, Y) = \frac{1}{\pi^{2(d+1)}} \int dS dT e^{-2S^A T_A^i} f(Z + S, Y + S) g(Z - T, Y + T) , \quad (13.1)$$

which is an associative and regular product law in the space of polynomial functions $f(Z, Y)$, and gives rise to the commutation relations

$$[Z_i^A, Z_j^B]_* = -\epsilon_{ij} \eta^{AB} , \quad [Y_i^A, Y_j^B]_* = \epsilon_{ij} \eta^{AB} , \quad [Y_i^A, Z_j^B]_* = 0 .$$

The definition (13.1) has the meaning of a normal ordering with respect to the “creation” and “annihilation” operators $Z - Y$ and $Z + Y$, respectively. Actually, from (13.1) follows that the left star multiplication by $Z - Y$ and the right star multiplication by $Z + Y$ are equivalent to usual multiplications by $Z - Y$ and $Z + Y$, respectively. Note that Z independent functions $f(Y)$ form a proper subalgebra of the star product algebra (13.1) with the Moyal star product (11.3).

One can also check that the following formulae are true:

$$Y_i^A * = Y_i^A + \frac{1}{2} \left(\frac{\vec{\partial}}{\partial Y_A^i} - \frac{\vec{\partial}}{\partial Z_A^i} \right) , \quad * Y_i^A = Y_i^A - \frac{1}{2} \left(\frac{\overleftarrow{\partial}}{\partial Y_A^i} + \frac{\overleftarrow{\partial}}{\partial Z_A^i} \right) , \quad (13.2)$$

$$Z_i^A * = Z_i^A + \frac{1}{2} \left(\frac{\vec{\partial}}{\partial Y_A^i} - \frac{\vec{\partial}}{\partial Z_A^i} \right) , \quad * Z_i^A = Z_i^A + \frac{1}{2} \left(\frac{\overleftarrow{\partial}}{\partial Y_A^i} + \frac{\overleftarrow{\partial}}{\partial Z_A^i} \right) . \quad (13.3)$$

Furthermore, the appropriate reality conditions for the Lie algebra built from this associative star product algebra via commutators are

$$\bar{f}(Z, Y) = -f(-iZ, iY) , \quad (13.4)$$

where the bar denotes complex conjugation of the coefficients of the expansion of $f(Z, Y)$ in powers of Z and Y . This condition results from (5.9) with the involution \dagger defined by the relations

$$(Y_i^A)^\dagger = iY_i^A , \quad (Z_i^A)^\dagger = -iZ_i^A . \quad (13.5)$$

13.2 Klein operator

The distinguishing property of the extended definition (13.1) of the star product is that it admits the inner Klein operator

$$\mathcal{K} = \exp(-2z_i y^i) , \quad (13.6)$$

where

$$y_i \equiv V_A Y_i^A , \quad z_i \equiv V_A Z_i^A$$

are the projections of the oscillators along V^A . Using the definitions (13.1)-(13.6), one can show that \mathcal{K} (i) generates the automorphism τ as an inner automorphism of the extended star algebra,

$$\mathcal{K} * f(Z, Y) = f(\tilde{Z}, \tilde{Y}) * \mathcal{K} , \quad (13.7)$$

and (ii) is involutive,

$$\mathcal{K} * \mathcal{K} = 1 \quad (13.8)$$

(see Appendix 2.1 for a proof of these properties).

Let us note that *a priori* the star product (13.1) is well-defined for the algebra of polynomials (which means that the star product of two polynomials is still a polynomial). Thus the star product admits an ordinary interpretation in terms of oscillators as long as we deal with polynomial functions. But \mathcal{K} is not a polynomial because it contains an infinite number of terms with higher and higher powers of $z_i y^i$. So, *a priori* the star product with \mathcal{K} might give rise to divergencies arising from the contraction of an

infinite number of terms (for example, an infinite contribution might appear in the zeroth order like a sort of vacuum energy). What singles out the particular star product (13.1) is that this does not happen for the class of functions which extends the space of polynomials to include \mathcal{K} and similar functions.

Indeed, the evaluation of the star product of two exponentials like $\mathcal{A} = \exp(A_{AB}^{ij} W_{1i}^A W_{2j}^B)$, where A_{AB}^{ij} are constant coefficients and the W 's are some linear combinations of Y_i^A and Z_i^A , amounts to evaluating the Gaussian integral resulting from (13.1). The potential problem is that the bilinear form B of the integration variables in the Gaussian integral in $\mathcal{A}_1 * \mathcal{A}_2$ may be degenerate for some exponentials \mathcal{A}_1 and \mathcal{A}_2 , which leads to an infinite result because the Gaussian evaluates $\det^{-1/2}|B|$. As was shown originally in [15] (see also [79]) for the analogous spinorial star product in four dimensions, the star product (13.1) is well-defined for the class of functions, which we call *regular*, that can be expanded into a finite sum of functions f of the form

$$f(Z, Y) = P(Z, Y) \int_{M^n} d^n \tau \rho(\tau) \exp\left(\phi(\tau) z_i y^i\right), \quad (13.9)$$

where the integration is over some compact domain $M^n \subset \mathbb{R}^n$ parametrized by the coordinates τ_i ($i = 1, \dots, n$), the functions $P(Z, Y)$ and $\phi(\tau)$ are arbitrary polynomials of (Z, Y) and τ_i , respectively, while $\rho(\tau)$ is integrable in M^n . The key point of the proof is that the star product (13.1) is such that the exponential in the Ansatz (13.9) never contributes to the quadratic form in the integration variables simply because $s_i s^i = t_i t^i \equiv 0$ (where $s_i = S_i^A V_A$ and $t_i = T_i^A V_A$). As a result, the star product of two elements (13.9) never develops an infinity and the class (13.9) turns out to be closed under star multiplication pretty much as usual polynomials. The complete proof is given in Appendix 2.2.

The Klein operator \mathcal{K} obviously belongs to the regular class, as can be seen by putting $n = 1$, $\rho(\tau) = \delta(\tau + 2)$, $\phi(\tau) = \tau$ and $P(Z, Y) = 1$ in (13.9). Hence our manipulations with \mathcal{K} are safe. This property can be lost however if one either goes beyond the class of regular functions (in particular, this happens when the quasiprojector Δ (11.12) is involved) or uses a different star product realization of the same oscillator algebra. For example, usual Weyl ordering prescription is not helpful in that respect.

13.3 Field content

The nonlinear equations are formulated in terms of the fields $W(Z, Y|x)$, $S(Z, Y|x)$ and $B(Z, Y|x)$, where B is a zero-form, while

$$W(Z, Y|x) = dx^\mu W_\mu(Z, Y|x), \quad S(Z, Y|x) = dZ_i^A S_A^i(Z, Y|x)$$

are connection one-forms, in space-time and auxiliary Z_i^A directions, respectively. They satisfy the reality conditions analogous to (13.4)

$$\bar{W}(Z, Y|x) = -W(-iZ, iY|x), \quad \bar{S}(Z, Y|x) = -S(-iZ, iY|x), \\ \bar{B}(Z, Y|x) = -\tilde{B}(-iZ, iY|x).$$

The fields ω and C are identified with the “initial data” for the evolution in Z variables as follows:

$$\omega(Y|x) = W(0, Y|x), \quad C(Y|x) = B(0, Y|x).$$

The differentials satisfy the standard anticommutation relations

$$dx^\mu dx^\nu = -dx^\nu dx^\mu, \quad dZ_i^A dZ_j^B = -dZ_j^B dZ_i^A, \quad dx^\mu dZ_i^A = -dZ_i^A dx^\mu,$$

and commute with all other variables. The dependence on Z variables will be reconstructed by the imposed equations (modulo pure gauge ambiguities).

We require that all $sp(2)$ indices are contracted covariantly. This is achieved by imposing the conditions

$$[t_{ij}^{tot}, W]_* = 0, \quad [t_{ij}^{tot}, B]_* = 0, \quad [t_{ij}^{tot}, S_k^A]_* = \epsilon_{jk} S_i^A + \epsilon_{ik} S_j^A, \quad (13.10)$$

where the diagonal $sp(2)$ generator

$$t_{ij}^{tot} \equiv Y_i^A Y_{Aj} - Z_i^A Z_{Aj} \quad (13.11)$$

generates inner $sp(2)$ rotations of the star product algebra

$$[t_{ij}^{tot}, Y_k^A]_* = \epsilon_{jk} Y_i^A + \epsilon_{ik} Y_j^A, \quad [t_{ij}^{tot}, Z_k^A]_* = \epsilon_{jk} Z_i^A + \epsilon_{ik} Z_j^A. \quad (13.12)$$

Note that the first of the relations (13.10) can be written covariantly as $D(t_{ij}^{tot}) = 0$, by taking into account that $d(t_{ij}^{tot}) = 0$.

13.4 Nonlinear system of equations

The full nonlinear system of equations for completely symmetric HS fields is

$$dW + W * W = 0, \quad (13.13)$$

$$dB + W * B - B * \widetilde{W} = 0, \quad (13.14)$$

$$dS + W * S + S * W = 0, \quad (13.15)$$

$$S * B - B * \widetilde{S} = 0, \quad (13.16)$$

$$S * S = -\frac{1}{2}(dZ_A^i dZ_i^A + 4\Lambda^{-1} dZ_{Ai} dZ_B^i V^A V^B B * \mathcal{K}), \quad (13.17)$$

where we define

$$\widetilde{S}(dZ, Z, Y) = S(\widetilde{dZ}, \widetilde{Z}, \widetilde{Y}). \quad (13.18)$$

One should stress that the twisting of the basis elements dZ_i^A of the exterior algebra in the auxiliary directions is *not* implemented via the Klein operator as in (13.7). Solutions of the system (13.13)-(13.17) admit factorization over the ideal generated by the nonlinear $sp(2)$ generators (13.22) defined in Section 13.6 as nonlinear deformations of the generators (11.4) used in the free field analysis. The system resulting from this factorization gives the nonlinear HS interactions to all orders.

The first three equations are the only ones containing space-time derivatives, via the de Rham differential $d = dx^\mu \frac{\partial}{\partial x^\mu}$. They have the form of zero-curvature equations for the space-time connection W (13.13) and the Z -space connection S (13.15) together with a covariant constancy condition for the zero-form B (13.14). These equations alone do not allow any nontrivial dynamics, so the contribution coming from (13.16) and (13.17) is essential. Note that the last two equations are constraints from the space-time point of view, not containing derivatives with respect to the x -variables, and that the nontrivial part only appears with the “source” term $B * \mathcal{K}$ in the V^A longitudinal sector of (13.17) (the first term on the right hand side of (13.17) is a constant). The inverse power of the cosmological constant Λ is present in (13.17) to obtain a Weyl tensor with Λ independent coefficients in the linearized equations in such a way that their flat limit also makes sense. In the following, however, we will again keep V normalized to 1, which means $\Lambda = -1$ (see Section 2.3).

13.5 Formal consistency

The system is formally consistent, *i.e.* compatible with $d^2 = 0$ and with associativity. A detailed proof of this statement can be found in the appendix 2.3. Let us however point out here the only tricky step. To prove the consistency, one has to show that the associativity relation $S * (S * S) = (S * S) * S$ is compatible with the equations. This is in fact the form of the Bianchi identity with respect to the Z variables, because S actually acts as a sort of exterior derivative in the noncommutative space (as will be shown in the next section). Associativity seems then to be broken by the source term $B * \mathcal{K}$ which

anticommutes with dz_i as a consequence of (13.16) and the definition (13.18), and brings in a term proportional to $dz_i dz^i dz_j$ (where $dz_i = V_A dZ_i^A$). This is not a problem however: since i is an $sp(2)$ index, it can take only two values and, as a result, the antisymmetrized product of three indices vanishes identically: $dz_i dz^i dz_j = 0$.

In a more compact way, one can prove consistency by introducing the noncommutative extended covariant derivative $\mathcal{W} = d + W + S$ and assembling eqs. (13.13)-(13.17) into

$$\mathcal{W} * \mathcal{W} = -\frac{1}{2} dZ_A^i dZ_i^A + 2 dZ_{Ai} dZ_B^i V^A V^B B * \mathcal{K} ,$$

$$\mathcal{W} * B = B * \widetilde{\mathcal{W}} .$$

In other words, $S * S$ is nothing but the ZZ component of an (x, Z) -space curvature, and it is actually the only component of the curvature allowed to be nonvanishing, xx and xZ being trivial according to (13.13) and (13.15), respectively. Consistency then amounts to the fact that the associativity relations $\mathcal{W} * (\mathcal{W} * \mathcal{W}) = (\mathcal{W} * \mathcal{W}) * \mathcal{W}$ and $(\mathcal{W} * \mathcal{W}) * B = \mathcal{W} * (\mathcal{W} * B)$ are respected by the nonlinear equations. Recall, however, that it was crucial for the consistency that the symplectic indices take only two values.

According to the general scheme of free differential algebras, the consistency of the nonlinear equations implies gauge symmetry under the local transformations

$$\delta \mathcal{W} = [\epsilon, \mathcal{W}]_* , \quad \delta B = \epsilon * B - B * \tilde{\epsilon} , \quad (13.19)$$

where $\epsilon = \epsilon(Z, Y|x)$ is $sp(2)$ invariant, *i.e.* $[t_{ij}^{tot}, \epsilon]_* = 0$, and otherwise arbitrary.

The consistency of the system, which means compatibility of the equations with the Bianchi identities, both in the x and in the Z sector (the latter case being verified by associativity of S), guarantees that the perturbative analysis works systematically at all orders.

13.6 $Sp(2)$ invariance

The $sp(2)$ symmetry is crucial for the consistency of the free system, to avoid unwanted ghost degrees of freedom. But as said before, survival of the $sp(2)$ invariance at the full nonlinear level is also very important, in the sense that it fixes the form of the nonlinear equations and prevents a mixture of unwanted degrees of freedom at the nonlinear level.

The rationale behind this is as follows. The conditions (13.10) guarantee the $sp(2)$ covariance of the whole framework. But this is not enough because one has to remove traces by factoring out terms which are themselves proportional to the $sp(2)$ generators. The third commutation relation in (13.10) makes this difficult. Indeed, it means that the operators S_i^A transform elements of the algebra proportional to t_{ij}^{tot} into t_{ij}^{tot} independent elements, *i.e.* the equations (13.13)-(13.17) do not allow a factorization with respect to the ideal generated by t_{ij}^{tot} .

To avoid this problem at the full nonlinear level one has to build proper generators

$$t_{ij}^{int} = t_{ij} + t_{ij}^1 + \dots ,$$

where t_{ij}^1 and higher terms denote the field-dependent corrections to the original $sp(2)$ generators (5.6), such that they satisfy the $sp(2)$ commutation relations

$$[t_{ij}^{int}, t_{kl}^{int}] = \epsilon_{ik} t_{jl}^{int} + \epsilon_{jk} t_{il}^{int} + \epsilon_{il} t_{jk}^{int} + \epsilon_{jl} t_{ik}^{int}$$

and

$$Dt_{ij}^{int} = 0 , \quad [S, t_{ij}^{int}]_* = 0 , \quad [B * \mathcal{K}, t_{ij}^{int}]_* = 0 . \quad (13.20)$$

What fixes the form of the nontrivial equations (13.16) and (13.17) is just the requirement that such nonlinearly deformed $sp(2)$ generators t_{ij}^{int} do exist. Actually, getting rid of the dZ 's in (13.16) and (13.17) in the longitudinal sector, these equations read as

$$[s^i, s^j]_* = -\epsilon^{ij} (1 - 4B * \mathcal{K}) , \quad \{s^i, B * \mathcal{K}\}_* = 0$$

(where $s^i \equiv V^A S_A^i$). This is just a realization [49] of the so called deformed oscillator algebra found originally by Wigner [80] and discussed by many authors [81]

$$[\hat{y}^i, \hat{y}^j]_* = \epsilon^{ij}(1 + \nu \hat{k}) , \quad \{\hat{y}^i, \hat{k}\}_* = 0 , \quad (13.21)$$

ν being a central element. The main property of this algebra is that, for any ν , the elements $\tau_{ij} = -\frac{1}{2}\{s_i, s_j\}_*$ form the $sp(2)$ algebra that rotates properly s_i

$$[\tau_{ij}, s_k]_* = \epsilon_{ik}s_j + \epsilon_{jk}s_i .$$

As a consequence, there exists an $sp(2)$ generator

$$\mathcal{T}_{ij} = -\frac{1}{2}\{S_i^A, S_{Aj}\}_* ,$$

which acts on S_i^A as

$$[\mathcal{T}_{ij}, S_k^A]_* = \epsilon_{ik}S_j^A + \epsilon_{jk}S_i^A .$$

As a result, the difference

$$t_{ij}^{int} \equiv t_{ij}^{tot} - \mathcal{T}_{ij} \quad (13.22)$$

satisfies the $sp(2)$ commutation relation and the conditions (13.20), taking into account (13.10) and (13.13) - (13.17). Moreover, at the linearized level, where $S_i^A = Z_i^A$ as will be shown in the next section, t_{ij}^{int} reduces to t_{ij} . This means that, if nonlinear equations have the form (13.13) - (13.17), interaction terms coming from the evolution along noncommutative directions do not spoil the $sp(2)$ invariance and allow the factorization of the elements proportional to t_{ij}^{int} . This, in turn, implies that the nonlinear equations admit an interpretation in terms of the tensor fields we started with in the free field analysis. Let us also note that by virtue of (13.22) and (13.13)-(13.17) the conditions (13.20) are equivalent to (13.10).

Finally, let us mention that an interesting interpretation of the deformed oscillator algebra (13.21) is [49] that it describes a two-dimensional fuzzy sphere of a ν -dependent radius. Comparing this with the equations (13.16) and (13.17) we conclude that the nontrivial HS equations describe a two-dimensional fuzzy sphere embedded into a noncommutative space of variables Z^A and Y^A . Its radius varies from point to point of the usual (commutative) space-time with coordinates x , depending on the value of the HS curvatures collectively described by the Weyl zero-form $B(Z, Y|x)$.

13.7 Factoring out the ideal

The factorization procedure is performed analogously to the linearized analysis of Subsection 12.2 by choosing one or another tracelessness condition for representatives of the equivalence classes and then dropping the terms of the form $f = t_{ij}^{int} * g^{ij}$, $[f, t_{ij}^{int}] = 0$ as explained in more detail in Subsection (14.3).

Equivalently, one can use the quasiprojector approach of [57] exposed in Section 11. To this end one defines a nonlinear quasiprojector $\Delta^{int} = \Delta + \dots$ as a nonlinear extension of the operator Δ (11.12), satisfying

$$[S, \Delta^{int}]_* = 0 , \quad D(\Delta^{int}) = 0 \quad (13.23)$$

and

$$\Delta^{int} * t_{ij}^{int} = t_{ij}^{int} * \Delta^{int} = 0 . \quad (13.24)$$

The equations with factored out traces then take the form²⁸

$$\Delta^{int} * (dW + W * W) = 0 , \quad (13.25)$$

²⁸This form of the HS field equations was also considered by E. Sezgin and P. Sundell (unpublished) as one of us (MV) learned from a private discussion during the Solvay workshop.

$$\Delta^{int} * (dB + W * B - B * \widetilde{W}) = 0 , \quad (13.26)$$

$$\Delta^{int} * (dS + W * S + S * W) = 0 , \quad (13.27)$$

$$\Delta^{int} * (S * B - B * \widetilde{S}) = 0 , \quad (13.28)$$

$$\Delta^{int} * \left(S * S + \frac{1}{2} (dZ_A^i dZ_i^A + 4\Lambda^{-1} dZ_{Ai} dZ_B^i V^A V^B B * \mathcal{K}) \right) = 0 . \quad (13.29)$$

The factors of Δ^{int} here insure that all terms proportional to t_{ij}^{int} drop out. Note that Δ and, therefore, Δ^{int} do not belong to the regular class, and their star products with themselves and similar operators are ill-defined. However, as pointed out in Appendix E, since Δ and Δ^{int} admit expansions in power series in Z_i^A and Y_j^A , their products with regular functions are well-defined (free of infinities), so that the equations (13.25)-(13.29) make sense at all orders. Note that, in practice, to derive manifest component equations on the physical HS modes within this approach it is anyway necessary to choose a representative of the quotient algebra in one or another form of the tracelessness conditions as discussed in Sections 4.2 and 12.

Let us note that the idea to use the strong $sp(2)$ condition suggested in [75]

$$t_{ij}^{int} * B = B * t_{ij}^{int} = 0 \quad (13.30)$$

is likely to lead to a problem beyond the linearized approximation. The reason is that the elements satisfying (13.30) are themselves of the form $\Delta^{int} * B'$ [4, 57], which is beyond the class of regular functions even in the linearized approximation with Δ in place of Δ^{int} . From (13.17) it follows that the corresponding field S is also beyond the regular class. As a result, star products of the corresponding functions that appear in the perturbative analysis of the field equations may be ill-defined (infinite), *i.e.* imposing this condition may cause infinities in the analysis of HS equations [75]. Let us note that this unlucky situation is not accidental. A closely related point is that the elements satisfying (13.30) would form a subalgebra of the off-mass-shell HS algebra if their product existed. This is not true, however: the on-mass-shell HS algebra is a quotient algebra over the ideal \mathcal{I} but not a subalgebra. This fact manifests itself in the nonexistence of products of elements satisfying (13.30) (see also [4, 57, 75]), having nothing to do with any inconsistency of the HS field equations. The factorization of the ideal \mathcal{I} in (13.25)-(13.29) is both sufficient and free of infinities.

14 Perturbative analysis

Let us now expand the equations around a vacuum solution, checking that the full system of HS equations reproduces the free field dynamics at the linearized level.

14.1 Vacuum solution

The vacuum solution (W_0, S_0, B_0) around which we will expand is defined by $B_0 = 0$, which is clearly a trivial solution of (13.14) and (13.16). Furthermore, it cancels the source term in (13.17), which is then solved by

$$S_0 = dZ_A^i Z_A^i . \quad (14.1)$$

Let us point out that $dS_0 = 0$ and $\widetilde{S}_0 = S_0$ by the definition (13.18). From (14.1) and

$$[Z_i^A, f]_* (Z, Y) = - \frac{\partial}{\partial Z_A^i} f(Z, Y) \quad (14.2)$$

(see (13.3)) follows that the (twisted or not) adjoint action of S_0 is equivalent to the action of the exterior differential $d_Z = dZ_i^A \frac{\partial}{\partial Z_i^A}$ in the auxiliary space. The equation (13.15) at the zeroth order then becomes $\{W_0, S_0\}_* = d_Z W_0 = 0$ and one concludes that W_0 can only depend on Y and not on Z . One solution of (13.13) is the AdS connection bilinear in Y

$$W_0 = \omega_0^{AB}(x) T_{AB}(Y) , \quad (14.3)$$

which thus appears as a natural vacuum solution of HS nonlinear equations. The vacuum solution (14.1), (14.3) satisfies also the $sp(2)$ invariance condition (13.10).

The symmetry of the chosen vacuum solution is $hu(1|2:[d-1, 2])$. Indeed, the vacuum symmetry parameters $\epsilon^{gl}(Z, Y|x)$ must satisfy

$$[S_0, \epsilon^{gl}]_* = 0, \quad D_0(\epsilon^{gl}) = 0. \quad (14.4)$$

The first of these conditions implies that $\epsilon^{gl}(Z, Y|x)$ is Z -independent, *i.e.*, $\epsilon^{gl}(Z, Y|x) = \epsilon^{gl}(Y|x)$ while the second condition reconstructs the dependence of $\epsilon^{gl}(Y|x)$ on space-time coordinates x in terms of values of $\epsilon^{gl}(Y|x_0)$ at any fixed point x_0 of space-time. Since the parameters are required to be $sp(2)$ invariant, one concludes that, upon factorization of the ideal \mathcal{I} , the global symmetry algebra is $hu(1|2:[d-1, 2])$.

Our goal is now to see whether free HS equations emerge from the full system as first order correction to the vacuum solution. We thus set

$$W = W_0 + W_1, \quad S = S_0 + S_1, \quad B = B_0 + B_1,$$

and keep terms up to the first order in W_1, S_1, B_1 in the nonlinear equations.

14.2 First order correction

As a result of the fact that the adjoint action of S_0 is equivalent to the action of d_Z , if treated perturbatively, the space-time constraints (13.16) and (13.17) actually correspond to differential equations with respect to the noncommutative Z variables.

We begin by looking at (13.16). The zero-form $B = B_1$ is already first order, so we can substitute S by S_0 , to obtain that B_1 is Z -independent

$$B_1(Z, Y) = C(Y|x). \quad (14.5)$$

Inserting this solution into (13.14) just gives the twisted adjoint equation (12.12) (with \tilde{D}_0 defined by (12.5)), one of the two field equations we are looking for.

Next we attempt to find S_1 substituting (14.5) into (13.17), taking into account that

$$f(Z, Y) * \mathcal{K} = \exp(-2z_i y^i) f(Z_i^A - V^A(z_i + y_i), Y_i^A - V^A(z_i + y_i))$$

(see Appendix 2.1), which one can write as

$$f(Z, Y) * \mathcal{K} = \exp(-2z_i y^i) f({}^\perp Z - {}^\parallel Y, {}^\perp Y - {}^\parallel Z).$$

This means that \mathcal{K} acts on functions of Z and Y by interchanging their respective longitudinal parts (taken with a minus sign) and multiplying them by a factor of $\exp(-2z_i y^i)$.

Looking at the ZZ part of the curvature, one can see that the V^A transversal sector is trivial at first order and that the essential Z -dependence is concentrated only in the longitudinal components. One can then analyze the content of (13.17) with respect to the longitudinal direction only, getting

$$\partial^i s_1^j - \partial^j s_1^i = 4\epsilon^{ij} C({}^\perp Y - {}^\parallel Z) \exp(-2z_i y^i) \quad (14.6)$$

with $\partial^i = \frac{\partial}{\partial z_i}$ and $s_1^i = S_{1A}^i V^A$. The general solution of the equation $\partial_i f^i(z) = g(z)$ is $f_i(z) = \partial_i \epsilon + \int_0^1 dt t z_i g(z)$. Applying this to (14.6) one has

$$s_1^i = \partial^i \epsilon_1 - 2z^i \int_0^1 dt t C(\perp Y - t \parallel Z) \exp(-2tz_k y^k) .$$

Analogously, in the V transverse sector one obtains that $\perp S_i^A$ is pure gauge so that

$$S_{1A}^i = \frac{\partial}{\partial Z_i^A} \epsilon_1 - 2V_A z^i \int_0^1 dt t C(\perp Y - t \parallel Z) \exp(-2tz_k y^k) ,$$

where the first term on the r.h.s. is the Z -exact part. This term is the pure gauge part with the gauge parameter $\epsilon_1 = \epsilon_1(Z; Y|x)$ belonging to the Z -extended HS algebra. One can conveniently set $\frac{\partial}{\partial Z_i^A} \epsilon_1 = 0$ by using part of the gauge symmetry (13.19). This choice fixes the Z -dependence of the gauge parameters to be trivial and leaves exactly the gauge freedom one had at the free field level, $\epsilon_1 = \epsilon_1(Y|x)$. Moreover, let us stress that with this choice one has reconstructed S_1 entirely in terms of B_1 . Note that s_1^i belongs to the regular class of functions (13.9) compatible with the star product.

We now turn our attention to the equation (13.15), which determines the dependence of W on z . In the first order, it gives

$$\partial^i W_1 = ds_1^i + W_0 * s_1^i - s_1^i * W_0 .$$

The general solution of the equation $\frac{\partial}{\partial z_i} \varphi(z) = \chi^i(z)$ is given by the line integral

$$\varphi(z) = \varphi(0) + \int_0^1 dt z_i \chi^i(tz) ,$$

provided that $\frac{\partial}{\partial z_i} \chi^i(z) = 0$ ($i = 1, 2$). Consequently,

$$W_1 = \omega(Y|x) + z^j \int_0^1 dt (1-t) e^{-2tz_i y^i} E_0^B \frac{\partial}{\partial \perp Y^j B} C(\perp Y - t \parallel Z) , \quad (14.7)$$

taking into account that the term $z_i ds_1^i$ vanishes because $z_i z^i = 0$. Again, W_1 is in the regular class. Note also that in (14.7) only the frame field appears, while the dependence on the Lorentz connection cancels out. This is the manifestation of the local Lorentz symmetry which forbids the presence of ω^{ab} if not inside Lorentz covariant derivatives.

One still has to analyze (13.13), which at first order reads

$$dW_1 + \{W_0, W_1\}_* = 0 .$$

Plugging in (14.7), one gets

$$R_1 = O(C) , \quad R_1 \equiv d\omega + \{\omega, W_0\}_* , \quad (14.8)$$

where corrections on the r.h.s. of the first equation in (14.8) come from the second term in (14.7), and prevent (14.8) from being trivial, that would imply ω to be a pure gauge solution²⁹. The formal consistency of the system with respect to Z variables allows one to restrict the study of (13.13) to the physical space $Z = 0$ only (with the proviso that the star products are to be evaluated before sending Z to zero). This is due to the fact that the dependence on Z is reconstructed by the equations in such a way that if (13.13) is satisfied for $Z = 0$, it is true for all Z . By elementary algebraic manipulations one obtains the final result

$$R_1 = \frac{1}{2} E_0^A E_0^B \frac{\partial^2}{\partial \perp Y^A \partial \perp Y^B} C(\perp Y) ,$$

²⁹In retrospective, one sees that this is the reason why it was necessary for the Z coordinates to be noncommutative, allowing nontrivial contractions, since corrections are obtained from perturbative analysis as coefficients of an expansion in powers of z obtained by solving for the z -dependence of the fields from the full system.

which, together with the equation for the twisted adjoint representation previously obtained, reproduces the free field dynamics for all spins (10.20) and (12.12).

14.3 Higher order corrections and factorization of the ideal

Following the same lines one can now reconstruct order-by-order all nonlinear corrections to the free HS equations of motion. Note that all expressions that appear in this analysis belong to the regular class (13.9), and therefore the computation as a whole is free from divergencies, being well defined. At the same time, the substitution of expressions like (10.24) for auxiliary fields will give rise to nonlinear corrections with higher derivatives, which are nonanalytic in the cosmological constant.

Strictly speaking, the analysis explained so far is off-mass-shell. To put the theory on-mass-shell one has to factor out the ideal \mathcal{I} . To this end, analogously to the linearized analysis of Section 12, one has to fix representatives of $\omega(Y|x)$ and $C(Y|x)$ in one or another way (for example, demanding $\omega(Y|x)$ to be *AdS* traceless and $C(Y|x)$ to be Lorentz traceless). The derived component HS equations may or may not share these tracelessness properties. Let us consider a resulting expression containing some terms of the form

$$A = A_0^{tr} + t_{ij} A_1^{ij} ,$$

where A_0^{tr} satisfies the chosen tracelessness condition while the second term, with t_{ij} defined by (11.4), describes extra traces. One has to rewrite such terms as

$$A = A_0^{tr} + t_{ij}^{int} * A_1^{ij} + A' + A'' , \quad (14.9)$$

where the third term

$$A' \equiv t_{ij} A_1^{ij} - t_{ij} * A_1^{ij} = -\frac{1}{4} \frac{\partial^2}{\partial Y_A^i \partial Y^{jA}} A^{ij}$$

contains less traces (we took into account (11.7)), and the fourth term

$$A'' \equiv (t_{ij}^{int} - t_{ij}) * A_1^{ij} = O(C) * A_1^{ij}$$

contains higher-order corrections due to the definitions (13.11) and (13.22). The factorization is performed by dropping out the terms of the form $t_{ij}^{int} * A_1^{ij}$ in (14.9). The resulting expression $A = A_0^{tr} + A' + A''$ contains either less traces or higher order nonlinearities. If necessary, the procedure has to be repeated to get rid of lower traces at the same order or new traces at the nonlinear order. Although this procedure is complicated, it can be done in a finite number of well-defined steps for any given type of HS tensor field and given order of nonlinearity. The process is much nicer of course within the spinorial realization available in the lower dimension cases [14, 78, 79] where the factorization over the ideal \mathcal{I} is automatic.

15 Discussion

A surprising issue related to the structure of the HS equations of motion is that, within this formulation of the dynamics, one can get rid of the space-time variables. Indeed, (13.13)-(13.15) are zero curvature and covariant constancy conditions, admitting pure gauge solutions. This means that, locally,

$$\begin{aligned} W(x) &= g^{-1}(x) * dg(x) , \\ B(x) &= g^{-1}(x) * b * \tilde{g}(x) , \\ S(x) &= g^{-1}(x) * s * g(x) , \end{aligned} \quad (15.1)$$

the whole dependence on space-time points being absorbed into a gauge function $g(x)$, which is an arbitrary invertible element of the star product algebra, while $b = b(Z, Y)$ and $s = s(Z, Y)$ are arbitrary x -independent elements of the star product algebra. Since the system is gauge invariant, the gauge functions disappear from the remaining two equations (13.16)-(13.17), which then encode the whole

dynamics though being independent of x . This turns out to be possible, in the unfolded formulation, just because of the presence of an infinite bunch of fields, supplemented by an infinite number of appropriate constraints, determined by consistency. As previously seen in the lower spin examples (see Section 7), the zero-form B turns out to be the generating function of all on-mass-shell nontrivial derivatives of the dynamical fields. Thus it locally reconstructs their x -dependence through their Taylor expansion which in turn is just given by the formulas (15.1). So, within the unfolded formulation, the dynamical problem is well posed once all zero-forms assembled in b are given at one space-time point x_0 , because this is sufficient to obtain the whole evolution of fields in some neighborhood of x_0 (note that s is reconstructed in terms of b up to a pure gauge part). This way of solving the nonlinear system, getting rid of x variables, is completely equivalent to the one used in the previous section, or, in other words, the unfolded formulation involves a trade between space-time variables and auxiliary noncommutative variables (Z, Y) . Nevertheless, the way we see and perceive the world seems to require the definition of local events, and it is this need for locality that makes the reduction to the “physical” subspace $Z = 0$ (keeping the x -dependence instead of gauging it away) more appealing. On the other hand, as mentioned in Section 13.6, the HS equations in the auxiliary noncommutative space have the clear geometrical meaning of describing embeddings of a two-dimensional noncommutative sphere into the Weyl algebra.

The system of gauge invariant nonlinear equations for all spins in AdS_d here presented can be generalized [16] to matrix-valued fields, $W \rightarrow W_\alpha^\beta$, $S \rightarrow S_\alpha^\beta$ and $B \rightarrow B_\alpha^\beta$, $\alpha, \beta = 1, \dots, n$, giving rise to Yang-Mills groups $U(n)$ in the $s = 1$ sector while remaining consistent. It is also possible to truncate to smaller inner symmetry groups $USp(n)$ and $O(n)$ by imposing further conditions based on certain antiautomorphisms of the star product algebra [46, 16]. Apart from the possibility of extending the symmetry group with matrix-valued fields, and modulo field redefinitions, it seems that there is no ambiguity in the form of nonlinear equations. As previously noted, this is due to the fact that the $sp(2)$ invariance requires (13.16) and (13.17) to have the form of a deformed oscillator algebra.

HS models have just one dimensionless coupling constant

$$g^2 = |\Lambda|^{\frac{d}{2}-1} \kappa^2.$$

To introduce the coupling constant, one has to rescale the fluctuations ω_1 of the gauge fields (*i.e.* additions to the vacuum field) as well as the Weyl zero-forms by a coupling constant g so that it cancels out in the free field equations. In particular, g is identified with the Yang-Mills coupling constant in the spin-1 sector. Its particular value is artificial however because it can be rescaled away in the classical theory (although it is supposed to be a true coupling constant in the quantum theory where it is a constant in front of the whole action in the exponential inside the path integral). Moreover, there is no dimensionful constant allowing us to discuss a low-energy expansion, *i.e.* an expansion in powers of a dimensionless combination of this constant and the covariant derivative. The only dimensionful constant present here is Λ . The dimensionless combinations $\bar{D}_\mu \equiv \Lambda^{-1/2} D_\mu$ are not good expansion entities, since the commutator of two of them is of order 1, as a consequence of the fact that the AdS curvature is roughly $R_0 \sim D^2 \sim \Lambda g$. For this reason also it would be important to find solutions of HS field equations different from AdS_d , thus introducing in the theory a massive parameter different from the cosmological constant.

Finally, let us note that a variational principle giving the nonlinear equations (13.13)-(13.17) is still unknown in all orders. Indeed, at the action level, gauge invariant interactions have been constructed only up to the cubic order [13, 4, 82].

16 Conclusion

The main message of these lectures is that nonlinear dynamics of HS gauge fields can be consistently formulated in all orders in interactions in anti de Sitter space-time of any dimension $d \geq 4$. The level of generality of the analyzes covered has been restricted in the following points: only completely symmetric bosonic HS gauge fields have been considered, and only at the level of the equations of motion.

Since it was impossible to cover all the interesting and important directions of research in the modern

HS gauge theory, an invitation to further readings is provided as a conclusion. For general topics in HS gauge theories, the reader is referred to the review papers [3, 9, 10, 38, 58, 83, 84, 85]. Among the specific topics that have not been addressed here, one can mention:

- (i) Spinor realizations of HS superalgebras in $d = 3, 4$ [43, 44, 46, 79], $d = 5$ [50, 7], $d = 7$ [52] and the recent developments in any dimension [57],
- (ii) Cubic action interactions [13, 4, 82],
- (iii) Spinor form of $d = 4$ nonlinear HS field equations [14, 3, 86],
- (iv) HS dynamics in larger (super)spaces, *e.g.* free HS theories in tensorial superspaces [87, 51, 71] and HS theories in usual superspace [88, 89],
- (v) Group-theoretical classification of invariant equations via unfolded formalism [90],
- (vi) HS gauge fields different from the completely symmetric Fronsdal fields: *e.g.* mixed symmetry fields [91], infinite component massless representations [92] and partially massless fields [93],
- (vii) Light cone formulation for massless fields in AdS_d [94],
- (viii) Tensionless limit of quantized (super)string theory [95, 33].

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1 Basic material on lower spin gauge theories

1.1 Isometry algebras

By “space-time” symmetries one means symmetries of the corresponding space-time manifold \mathcal{M}^d of dimension d , which may be isometries or conformal symmetries. The most symmetrical solutions of vacuum Einstein equations, with or without cosmological constant Λ , are (locally) Minkowski space-time ($\Lambda = 0$), de Sitter ($\Lambda > 0$) and Anti de Sitter ($\Lambda < 0$) spaces. In this paper, we only consider $\mathbb{R}^{d-1,1}$ and AdS_d spaces though the results generalize easily to dS_d space.

The Poincaré group $ISO(d-1, 1) = \mathbb{R}^{d-1,1} \rtimes SO(d-1, 1)$ has translation generators P_a and Lorentz generators M_{ab} ($a, b = 0, 1, \dots, d-1$) satisfying the algebra

$$[M_{ab}, M_{cd}] = \eta_{ac} M_{db} - \eta_{bc} M_{da} - \eta_{ad} M_{cb} + \eta_{bd} M_{ca}, \quad (1.1)$$

$$[P_a, M_{bc}] = \eta_{ab} P_c - \eta_{ac} P_b, \quad (1.2)$$

$$[P_a, P_b] = 0. \quad (1.3)$$

“Internal” symmetries are defined as transformations that commute with the translations generated by P_a and the Lorentz transformations generated by M_{ab} [1]. The relation (1.1) defines the Lorentz algebra $o(d-1, 1)$ while the relations (1.2)-(1.3) state that the Poincaré algebra is a semi-direct product $iso(d-1, 1) = \mathbb{R}^d \rtimes o(d-1, 1)$.

The algebra of isometries of the AdS_d space-time is given by the commutation relations (1.1)-(1.2) and

$$[P_a, P_b] = -\frac{1}{\rho^2} M_{ab}, \quad (1.4)$$

where ρ is proportional to the radius of curvature of AdS_d and is related to the cosmological constant via $\Lambda = -\rho^{-2}$. The (noncommuting) transformations generated by P_a are called transvections in AdS_d to distinguish them from the (commuting) translations. By defining $M_{\hat{d}a} = \rho P_a$, it is possible to collect all generators into the generators M_{AB} where $A = 0, 1, \dots, \hat{d}$. These generators M_{AB} span $o(d-1, 2)$ algebra since they satisfy the commutation relations

$$[M_{AB}, M_{CD}] = \eta_{AC} M_{DB} - \eta_{BC} M_{DA} - \eta_{AD} M_{CB} + \eta_{BD} M_{CA},$$

where η_{AB} is the mostly minus invariant metric of $o(d-1, 2)$. This is easily understood from the geometrical construction of AdS_d as the hyperboloid defined by $X^A X_A = \frac{(d-1)(d-2)}{2} \rho^2$ which is obviously invariant under the isometry group $O(d-1, 2)$. Since transvections are actually rotations in ambient space, it is normal that they do not commute. It is possible to derive the Poincaré algebra from the AdS_d isometry algebra by taking the infinite-radius limit $\rho \rightarrow \infty$. This limiting procedure is called Inönü-Wigner contraction [96].

1.2 Gauging internal symmetries

In this subsection, a series of tools used in any gauge theory is briefly introduced. One considers the most illustrative example of Yang-Mills theory, which corresponds to gauging an internal symmetry group.

Let g be a (finite-dimensional) Lie algebra of basis $\{T_\alpha\}$ and Lie bracket $[\cdot, \cdot]$. The structure constants are defined by $[T_\alpha, T_\beta] = T_\gamma f^\gamma_{\alpha\beta}$. The Yang-Mills theory is conveniently formulated by using differential forms taking values in the Lie algebra g .

- The connection $A = dx^\mu A_\mu^\alpha T_\alpha$ is defined in terms the vector gauge field A_μ^α .
- The curvature $F = dA + A^2 = \frac{1}{2} dx^\mu dx^\nu F_{\mu\nu}^\alpha T_\alpha$ is associated with the field strength tensor $F_{\mu\nu}^\alpha = \partial_{[\mu} A_{\nu]}^\alpha + f^\alpha_{\beta\gamma} A_\mu^\beta A_\nu^\gamma$.
- The Bianchi identity $dF + AF - FA = 0$ is a consequence of $d^2 = 0$ and the Jacobi identity in the Lie algebra.
- The gauge parameter $\epsilon = \epsilon^\alpha T_\alpha$ is associated with the infinitesimal gauge transformation $\delta_\epsilon A = d\epsilon + A\epsilon - \epsilon A$ which transforms the curvature as $\delta_\epsilon F = F\epsilon - \epsilon F$. In components, this reads as $\delta_{\epsilon^\alpha} A_\mu^\alpha = \partial_\mu \epsilon^\alpha + f_{\beta\gamma}^\alpha A_\mu^\beta \epsilon^\gamma$ and $\delta F_{\mu\nu}^\alpha = f_{\beta\gamma}^\alpha F_{\mu\nu}^\beta \epsilon^\gamma$.

The algebra $\Omega(\mathcal{M}^d) \otimes g$ is a Lie superalgebra, the product of which is the graded Lie bracket denoted by $[\cdot, \cdot]$ ³⁰. The elements of $\Omega^p(\mathcal{M}^d) \otimes g$ are p -forms taking values in g . The interest of the algebra $\Omega(\mathcal{M}^d) \otimes g$ is that it contains the gauge parameter $\epsilon \in \Omega^0(\mathcal{M}^d) \otimes g$, the connection $A \in \Omega^1(\mathcal{M}^d) \otimes g$, the curvature $F \in \Omega^2(\mathcal{M}^d) \otimes g$, and the Bianchi identity takes place in $\Omega^3(\mathcal{M}^d) \otimes g$.

To summarize, the Yang-Mills theory is a fibre bundle construction where the Lie algebra g is the fiber, A the connection and F the curvature. The Yang-Mills action takes the form

$$S^{YM}[A_\mu^a] \propto \int_{\mathcal{M}^d} Tr[F^* F],$$

³⁰Here, the grading is identified with that in the exterior algebra so that the graded commutator is evaluated in terms of the original Lie bracket $[\cdot, \cdot]$.

in which case g is taken to be compact and semisimple so that the Killing form is negative definite (in order to ensure that the Hamiltonian is bounded from below). Here $*$ is the Hodge star producing a dual form. Note that, via Hodge star, the Yang-Mills action contains the metric tensor which is needed to achieve invariance under diffeomorphisms. Furthermore, because of the cyclicity of the trace, the Yang-Mills Lagrangian $\text{Tr}[F^*F]$ is also manifestly invariant under the gauge transformations.

An operatorial formulation is also useful for its compactness. Let us now consider some matter fields Φ living in some space V on which acts the Lie algebra g , via a representation T_α . In other words, the elements T_α are reinterpreted as operators acting on some representation space (also called module) V . The connection A becomes thereby an operator. For instance, if T is the adjoint representation then the module V is identified with the Lie algebra g and A acts as $A \cdot \Phi = [A, \Phi]$. The connection A defines the covariant derivative $D \equiv d + A$. For any representation of g , the transformation law of the matter field is $\delta\Phi = -\epsilon \cdot \Phi$, where ϵ is a constant or a function of x according to whether g is a global or a local symmetry. The gauge transformation law of the connection can also be written as $\delta A = \delta D = [D, \epsilon]$ because of the identity $[d, A] = dA$, and is such that $\delta(D\Phi) = -\epsilon \cdot (D\Phi)$. The curvature is economically defined as an operator $F = \frac{1}{2}[D, D] = D^2$. In space-time components, the latter equation reads as usual $[D_\mu, D_\nu] = F_{\mu\nu}$. The Bianchi identity is a direct consequence of the associativity of the differential algebra and Jacobi identities of the Lie algebra and reads in space-time components as $[D_\mu, F_{\nu\rho}] + [D_\nu, F_{\rho\mu}] + [D_\rho, F_{\mu\nu}] = 0$. The graded Jacobi identity leads to $\delta F = \frac{1}{2} \left([[D, \epsilon], D] + [D, [D, \epsilon]] \right) = [D^2, \epsilon] = [F, \epsilon]$.

The present notes make an extension of the previous compact notations and synthetic identities. Indeed, they generalize it straightforwardly to other gauge theories formulated via a nonAbelian connection, *e.g.* HS gauge theories³¹.

1.3 Gauging space-time symmetries

The usual Einstein-Hilbert action $S[g]$ is invariant under diffeomorphisms. The same is true for $S[e, \omega]$, defined by (2.1), since everything is written in terms of differential forms. The action (2.1) is also manifestly invariant under local Lorentz transformations $\delta\omega = d\epsilon + [\omega, \epsilon]$ with gauge parameter $\epsilon = \epsilon^{ab}M_{ab}$, because $\epsilon_{a_1 \dots a_d}$ is an invariant tensor of $SO(d-1, 1)$. The gauge formulation of gravity shares many features with a Yang-Mills theory formulated in terms of a connection ω taking values in the Poincaré algebra.

However, gravity is actually *not* a Yang-Mills theory with Poincaré as (internal) gauge group. The aim of this section is to express precisely the distinction between gauge symmetries which are either internal or space-time.

To warm up, let us mention several obvious differences between Einstein-Cartan's gravity and Yang-Mills theory. First of all, the Poincaré algebra $iso(d-1, 1)$ is not semisimple (since it is not a *direct* sum of simple Lie algebras, containing a nontrivial Abelian ideal spanned by translations). Secondly, the action (2.1) cannot be written in a Yang-Mills form $\int \text{Tr}[F^*F]$. Thirdly, the action (2.1) is not invariant under the gauge transformations $\delta\omega = d\epsilon + [\omega, \epsilon]$ generated by *all* Poincaré algebra generators, *i.e.* with gauge parameter $\epsilon(x) = \epsilon^a(x)P_a + \epsilon^{ab}(x)M_{ab}$. For $d > 3$, the action (2.1) is invariant only when $\epsilon^a = 0$. (For $d = 3$, the action (2.1) describes a genuine Chern-Simons theory with local $ISO(2, 1)$ symmetries.)

This latter fact is not in contradiction with the fact that one actually gauges the Poincaré group in gravity. Indeed, the torsion constraint allows one to relate the local translation parameter ϵ^a to the infinitesimal change of coordinates parameter ξ^μ . Indeed, the infinitesimal diffeomorphism $x^\mu \rightarrow x^\mu + \xi^\mu$ acts as the Lie derivative

$$\delta_\xi = \mathcal{L}_\xi \equiv i_\xi d + di_\xi,$$

³¹More precisely, one can take g as an infinite-dimensional Lie algebra that arises from an associative algebra with product law $*$ and is endowed with the (sometimes twisted) commutator as bracket. Up to these subtleties and some changes of notation, all previous relations hold for HS gauge theories considered here, and they might simplify some explicit checks by the reader.

where the inner product i is defined by

$$i_\xi \equiv \xi^\mu \frac{\partial}{\partial(dx^\mu)},$$

where the derivative is understood to act from the left. Any coordinate transformation of the frame field can be written as

$$\delta_\xi e^a = i_\xi(de^a) + d(i_\xi e^a) = i_\xi T^a + \underbrace{\epsilon^a_b e^b + D^L \epsilon^a}_{=\delta_\epsilon e^a},$$

where the Poincaré gauge parameter is given by $\epsilon = i_\xi \omega$. Therefore, when T^a vanishes any coordinate transformation of the frame field can be interpreted as a local Poincaré transformation of the frame field, and reciprocally.

To summarize, the Einstein-Cartan formulation of gravity is indeed a fibre bundle construction where the Poincaré algebra $iso(d-1, 1)$ is the fiber, ω the connection and R the curvature, but, unlike for Yang-Mills theories, the equations of motion imposes some constraints on the curvature ($T^a = 0$), and some fields are auxiliary (ω^{ab}). A fully covariant formulation is achieved in the AdS_d case with the aid of compensator formalism as explained in Section 2.3.

2 Technical issues on nonlinear higher spin equations

2.1 Two properties of the inner Klein operator

In this appendix, we shall give a proof of the properties (13.7) and (13.8). One can check the second property with the help of (13.1), which in this case amounts to

$$\mathcal{K} * \mathcal{K} = \frac{1}{\pi^{2(d+1)}} \int dS dT e^{-2S^A T^A} e^{-2(s_i + z_i)(y^i + s^i)} e^{-2(z_i - t_i)(y^i + t^i)},$$

with

$$s_i \equiv V_A S_i^A, \quad t_i \equiv V_A T_i^A. \quad (2.1)$$

Using the fact that $s_i s^i = t_i t^i = 0$ and rearranging the terms yields

$$\begin{aligned} \mathcal{K} * \mathcal{K} &= \frac{1}{\pi^{2(d+1)}} e^{-4z_i y^i} \int dS dT e^{-2S^A T^A} e^{-2s^i(z_i - y_i)} e^{-2t^i(z_i + y_i)} \\ &= \frac{1}{\pi^{2(d+1)}} e^{-4z_i y^i} \int dS dT e^{-2S^A [V^A(z_i - y_i) - T_i^A]} e^{-2t^i(z_i + y_i)}. \end{aligned}$$

Since

$$\frac{1}{\pi^{2(d+1)}} \int dS e^{-2S^A (Z_i^A - Y_i^A)} = \delta(Z_i^A - Y_i^A), \quad (2.2)$$

one gets

$$\mathcal{K} * \mathcal{K} = e^{-4z_i y^i} \int dT \delta(V^A(z_i - y_i) - T_i^A) e^{-2T_B^i V^B(z_i + y_i)}$$

which, using $V_B V^B = 1$, gives $\mathcal{K} * \mathcal{K} = e^{-4z_i y^i} e^{-2(-y^i z_i + z^i y_i)} = 1$. The formula (2.2) might seem unusual because of the absence of an i in the exponent. It is consistent however, since one can assume that all oscillator variables, including integration variables, are genuine real variables times (some fixed) square root of i , *i.e.* that the integration is along appropriate lines in the complex plane. One is allowed to do so without coming into conflict with the definition of the star algebra, because its elements are analytic functions of the oscillators, and can then always be continued to real values of the variables.

The proof of (13.7) is quite similar. Let us note that, with the help of (13.8), it amounts to check that $\mathcal{K} * f(Z, Y) * \mathcal{K} = f(\tilde{Z}, \tilde{Y})$. One can prove, by going through almost the same steps shown above,

that

$$\mathcal{K} * f(Z, Y) = \mathcal{K} f(Z_A^i + V_A(y^i - z^i), Y_A^i - V_A(y^i - z^i)) .$$

Explicitly, one has

$$\begin{aligned} \mathcal{K} * f(Z, Y) &= \frac{\mathcal{K}}{\pi^{2(d+1)}} \int dS dT e^{-2S_A^i(T_A^i + V_A(y^i - z^i))} f(Z - T, Y + T) \\ &= \mathcal{K} \int dT \delta(T_A^i + V_A(y^i - z^i)) f(Z - T, Y + T) \\ &= \mathcal{K} f(Z_A^i + V_A(y^i - z^i), Y_A^i - V_A(y^i - z^i)) , \end{aligned}$$

where we have made use of (2.2). Another star product with \mathcal{K} leads to

$$\mathcal{K} * f(Z, Y) * \mathcal{K} =$$

$$\frac{1}{\pi^{2(d+1)}} e^{-4z_i y^i} \int dS dT e^{-2T_A^i[V^A(z_i + y_i) + S_A^i]} e^{-2s_i(y^i - z^i)} f(Z_A^i + V_A(y^i - z^i) + S_A^i, Y_A^i - V_A(y^i - z^i) + S_A^i) ,$$

which, performing the integral over T and using (2.2), gives in the end

$$e^{-4z_i y^i} e^{2(z_i + y_i)(y^i - z^i)} f(Z_A^i + V_A(y^i - z^i) - V_A(z^i + y^i), Y_A^i - V_A(y^i - z^i) - V_A(z^i + y^i)) = f(\tilde{Z}, \tilde{Y}) .$$

2.2 Regularity

We will prove that the star product (13.1) is well-defined for the regular class of functions (13.9). This extends the analogous result for the spinorial star product in three and four dimension obtained in [15, 79].

Theorem 2.1. *Given two regular functions $f_1(Z, Y)$ and $f_2(Z, Y)$, their star product (13.1) ($f_1 * f_2$)(Z, Y) is a regular function.*

Proof:

$$\begin{aligned} f_1 * f_2 &= P_1(Z, Y) \int_{M_1} d\tau_1 \rho_1(\tau_1) \exp[2\phi_1(\tau_1) z_i y^i] * P_2(Z, Y) \int_{M_2} d\tau_2 \rho_2(\tau_2) \exp[2\phi_2(\tau_2) z_i y^i] \\ &= \int_{M_1} d\tau_1 \rho_1(\tau_1) \exp[2\phi_1(\tau_1) z_i y^i] \int_{M_2} d\tau_2 \rho_2(\tau_2) \exp[2\phi_2(\tau_2) z_i y^i] \frac{1}{\pi^{2(d+1)}} \int dS dT \times \\ &\quad \times \exp\{-2S_A^i T_A^i + 2\phi_1(\tau_1)[s^i(z - y)_i] + 2\phi_2(\tau_2)[t^i(z + y)_i]\} P_1(Z + S, Y + S) P_2(Z - T, Y + T) , \end{aligned}$$

with s_i and t_i defined in (2.1). Inserting

$$P(Z + U, Y + U) = \exp\left[U_i^A \left(\frac{\partial}{\partial Z_{1i}^A} + \frac{\partial}{\partial Y_{1i}^A}\right)\right] P(Z_1, Y_1) \Big|_{\substack{Z_1=Z \\ Y_1=Y}} , \quad (2.3)$$

one gets

$$\begin{aligned} f_1 * f_2 &= \int_{M_1 \times M_2} d\tau_1 d\tau_2 \rho_1(\tau_1) \rho_2(\tau_2) \exp\{2[\phi_1(\tau_1) + \phi_2(\tau_2)] z_i y^i\} \times \\ &\quad \times \int dS dT \exp\left\{-2S_A^i \left[-\phi_1(\tau_1) V^A(z - y)_i + \frac{1}{2} \left(\frac{\partial}{\partial Z_{1i}^A} + \frac{\partial}{\partial Y_{1i}^A}\right)\right]\right\} P_1(Z_1, Y_1) \Big|_{\substack{Z_1=Z \\ Y_1=Y}} \times \\ &\quad \times \exp\left\{-2T_A^i \left[S_A^i - \phi_2(\tau_2) V^A(z + y)_i + \frac{1}{2} \left(-\frac{\partial}{\partial Z_{2i}^A} + \frac{\partial}{\partial Y_{2i}^A}\right)\right]\right\} P_2(Z_2, Y_2) \Big|_{\substack{Z_2=Z \\ Y_2=Y}} \end{aligned}$$

$$\begin{aligned}
&= \int_{M_1 \times M_2} d\tau_1 d\tau_2 \rho_1(\tau_1) \rho_2(\tau_2) \exp\{2[\phi_1(\tau_1) + \phi_2(\tau_2) + 2\phi_1(\tau_1)\phi_2(\tau_2)]z_i y^i\} \times \\
&\times \exp\left\{-\frac{1}{2}\left(\frac{\partial}{\partial Z_{1A}^i} + \frac{\partial}{\partial Y_{1A}^i}\right)\left(\frac{\partial}{\partial Z_{2A}^i} - \frac{\partial}{\partial Y_{2A}^i}\right)\right\} P_1[Z_1 - \phi_2(\tau_2) \parallel (Z + Y), Z_2 - \phi_2(\tau_2) \parallel (Z + Y)] \times \\
&\quad P_2(Z_2 + \phi_1(\tau_1) \parallel (Z - Y), Y_2 - \phi_1(\tau_1) \parallel (Z - Y)) \Big|_{\substack{Z_1=Z_2=Z \\ Y_1=Y_2=Y}},
\end{aligned}$$

as one can check by using (2.2) (or, equivalently, Gaussian integration, the rationale for the equivalence being given in Appendix 2.1). The product of two compact domains $M_1 \subset R^n$ and $M_2 \subset R^m$ is a compact domain in R^{n+m} and P_1, P_2 are polynomials, so one concludes that the latter expression is a finite sum of regular functions of the form (13.9). \square

As explained in Section 13.1, it is important for this proof that the exponential in the Ansatz (13.9) never contributes to the quadratic form in the integration variables, which is thus independent of the particular choice of the functions f_1 and f_2 . This property makes sure that the class of functions (13.9) is closed under star multiplication, and it is a crucial consequence of the definition (13.1).

Analogously one can easily prove that the star products $f * g$ and $g * f$ of $f(Z, Y)$ being a power series in Z_i^A and Y_i^A with a regular function $g(Z, Y)$ are again some power series. In other words, the following theorem is true:

Theorem 2.2. *The space of power series $f(Z, Y)$ forms a bimodule of the star product algebra of regular functions.*

2.3 Consistency of the nonlinear equations

We want to show explicitly that the system of equations (13.13)-(13.17) is consistent with respect to both x and Z variables.

We can start by acting on (13.13) with the x -differential d . Imposing $d^2 = 0$, one has

$$dW * W - W * dW = 0 ,$$

which is indeed satisfied by associativity, as can be checked by using (13.13) itself. So (13.13) represents its own consistency condition.

Differentiating (13.14), one gets

$$dW * B - W * dB - dB * \widetilde{W} - B * d\widetilde{W} = 0 .$$

Using (13.14), one can substitute each dB with $-W * B + B * \widetilde{W}$, obtaining

$$dW * B + W * W * B - B * \widetilde{W} * \widetilde{W} - B * d\widetilde{W} = 0 .$$

This is identically satisfied by virtue of (13.13), which is thus the consistency condition of (13.14).

The same procedure works in the case of (13.15), taking into account that, although S is a space-time zero-form, $dx^\mu dZ_i^A = -dZ_i^A dx^\mu$. Again one gets a condition which amounts to an identity because of (13.13).

Hitting (13.16) with d and using (13.14) and (13.15), one obtains

$$-W * S * B - S * B * \widetilde{W} + W * B * \widetilde{S} + B * \widetilde{S} * \widetilde{W} = 0 ,$$

which is identically solved taking into account (13.16).

Finally, (13.15) turns the differentiated l.h.s. of (13.17) into $-W * S * S + S * S * W$, while using (13.14) the differentiated r.h.s. becomes $-2\Lambda^{-1} dz_i dz^i (-W * B * \mathcal{K} + B * \widetilde{W} * \mathcal{K})$. Using (13.7), one is then able to show that the two sides of the equation obtained are indeed equal if (13.17) holds.

S being an exterior derivative in the noncommutative directions, in the Z sector the consistency check is more easily carried on by making sure that a covariant derivative of each equation does not lead to any new condition, *i.e.* leads to identities. This amounts to implementing $d_Z^2 = 0$.

So commuting S with (13.13) gives identically 0 by virtue of (13.13) itself. The same is true for (13.15), (13.14) and (13.16), with the proviso that in these latter two cases one has to take an anticommutator of the equations with S because one is dealing with odd-degree-forms (one-forms). The only nontrivial case then turns out to be (13.17), which is treated in Section 13.3.

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