



Model-independent constraints on Lorentz invariance violation via the cosmographic approach

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ABSTRACT

Since Lorentz invariance plays an important role in modern physics, it is of interest to test the possible Lorentz invariance violation (LIV). The time-lag (the arrival time delay between light curves in different energy bands) of Gamma-ray bursts (GRBs) has been extensively used to this end. However, to our best knowledge, one or more particular cosmological models were assumed *a priori* in (almost) all of the relevant works in the literature. So, this makes the results on LIV in those works model-dependent and hence not so robust in fact. In the present work, we try to avoid this problem by using a model-independent approach. We calculate the time delay induced by LIV with the cosmic expansion history given in terms of cosmography, without assuming any particular cosmological model. Then, we constrain the possible LIV with the observational data, and find weak hints for LIV.

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1. Introduction

As is well known, Lorentz invariance plays an important role in modern physics. Actually, it is one of the foundation stones of special/general relativity and particle physics, which have been well tested in solar system and colliders. If Lorentz invariance is violated, the pillars of modern physics will be shocked and new physics is needed. So, it is of interest to test the possible Lorentz invariance violation (LIV) with various terrestrial experiments and astrophysical/cosmological observations [1,2].

In the literature, there exist many theories inducing LIV. Here we are interested in the possible violation of Lorentz invariance induced by quantum gravity (QG). Commonly, most theories of QG (e.g. string theory, loop quantum gravity, doubly special relativity) predict that LIV might happen on high energy scales [3–17,49]. The propagation of high energy photons through the spacetime foam might exhibit a non-trivial dispersion relation in vacuum (which should be regarded as a non-trivial medium in QG). The deformed dispersion relation for photons usually takes the form $p^2c^2 = E^2[1 + f(E/E_{QG})]$, where E_{QG} is the effective QG energy scale, f is a dimensionless function depending on the particular QG model, c is the limiting speed of light on low energy scales, p and E are the momentum and energy of photons, respectively. On low energy scales $E \ll E_{QG}$, one can consider a series expansion

of this dispersion relation, namely $p^2c^2 = E^2[1 + \xi E/E_{QG} + \mathcal{O}(E^2/E_{QG}^2)]$, where $\xi = \pm 1$ is a sign ambiguity [3]. Such a series expansion corresponds to an energy-dependent speed of light, $v = \partial E/\partial p \sim c(1 - \xi E/E_{QG})$ [3]. So, the high and low energy photons will not reach us at the same time. A signal of energy E that travels a distance L acquires a time delay (measured with respect to the ordinary case of an energy-independent speed c), namely $\Delta t \sim \xi(E/E_{QG})(L/c)$. Although the QG effect is expected to be very weak (since E_{QG} is typically close to the Planck energy scale $E_P \sim 10^{19}$ GeV), a very long distance L can still make it testable. In the pioneer work [3], Amelino-Camelia et al. proposed that Gamma-ray bursts (GRBs) at a cosmological distance can be used to test the possible LIV, while time-lag (the arrival time delay between light curves in different energy bands) is a common feature in GRBs [18].

Following the pioneer work of Amelino-Camelia et al. [3], many constraints on LIV have been obtained from the time-lag of GRBs in the literature (e.g. [5–16]). It is worth noting that the cosmic expansion history (characterized by the Hubble parameter $H(z)$ as a function of redshift z) should be taken into account when one calculates the time delay Δt (see Sec. 2 for details), since photons propagate through the expanding space. Thus, the cosmic expansion history $H(z)$ should be given in advance. Actually, a particular cosmological model was assumed *a priori* in (almost) all of the relevant works. For example, the well-known spatially flat Λ CDM model $H(z) = H_0\sqrt{\Omega_M(1+z)^3 + \Omega_\Lambda}$ is usually assumed in the literature (e.g. [5–15]), while all the values of the model parameters Ω_M , $\Omega_\Lambda = 1 - \Omega_M$, and the Hubble constant H_0 are fixed (typ-

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ically taken from the WMAP/Planck results). In a few of works, some other cosmological models instead of Λ CDM model are also considered. For instance, in e.g. [9], the dark energy models with $w = \text{const.}$ and $w = w_0 + w_1 z$, the generalized Chaplygin gas (GCG) model, and the Dvali–Gabadadze–Porrati (DGP) braneworld model are assumed, while all the values of the corresponding model parameters have also been fixed. Different from the above works, in [16], the corresponding model parameters are not fixed *a priori*, and they are constrained together with the LIV parameters by using the observational data. However, some particular cosmological models still should be assumed in [16], i.e. the flat Λ CDM model, the w CDM model, and the Chevallier–Polarski–Linder (CPL) model, although their model parameters are free. In summary, to our best knowledge, one or more particular cosmological models have been assumed *a priori* in (almost) all of the relevant works in the literature. So, this makes the results on LIV in those works model-dependent and hence not so robust in fact.

In the present work, we try to avoid this problem by using a model-independent approach. As is well known, one of the powerful model-independent approaches is the so-called cosmography [19–29,50]. In fact, the only necessary assumption of cosmography is the cosmological principle. With cosmography, one can analyze the evolution of the universe without assuming any underlying theoretical model. Essentially, cosmography is the Taylor series expansion of the quantities related to the cosmic expansion history, such as the scale factor $a(t)$, the Hubble parameter $H(z)$ and the luminosity distance $d_L(z)$. Therefore, this makes cosmography model-independent indeed. We refer to e.g. [19–29,50] and references therein for more details of cosmography. So, in the present work, we can calculate the time delay Δt induced by LIV with the cosmic expansion history given in terms of cosmography, without assuming any particular cosmological model, unlike the relevant works on LIV mentioned above. The results on LIV obtained via the cosmographic approach will be model-independent and robust.

The rest of this paper is organized as follow. In Sec. 2, we briefly review the formalism of the time delay Δt induced by LIV. In Sec. 3, we derive the cosmic expansion history in terms of cosmography with respect to redshift z at first. Then, we constrain the LIV parameters together with the cosmographic parameters by using the time delay data from GRBs, the observational data from type Ia supernovae (SNIa) and the baryon acoustic oscillation (BAO). Note that we adopt the Markov Chain Monte Carlo (MCMC) technique in doing this. As is well known, there exists a divergence problem in cosmography with respect to redshift z when $z > 1$. Thus, another cosmography with respect to the so-called y -shift $y \equiv z/(1+z)$ has been proposed in the literature, which alleviates the divergence problem since $y < 1$ in the range of $0 \leq z < \infty$. In Sec. 4, we obtain the observational constraints on LIV via the cosmographic approach with respect to y -shift. The brief concluding remarks are given in Sec. 5.

2. Time delay of GRB photons induced by LIV

As mentioned in Sec. 1, the deformed dispersion relation for photons [3] usually takes the form $p^2 c^2 = E^2 [1 + f(E/E_{QG})]$, where E_{QG} is the effective QG energy scale, f is a dimensionless function depending on the particular QG model, c is the limiting speed of light on low energy scales, p and E are the momentum and energy of photons, respectively. On low energy scales $E \ll E_{QG}$, one can always expand this deformed dispersion relation as a Taylor series (see e.g. [14]),

$$E^2 = p^2 c^2 \left[1 - \sum_{n=1}^{\infty} s_{\pm} \left(\frac{E}{\xi_n E_{QG}} \right)^n \right], \quad (1)$$

where $s_{\pm} = \pm 1$ is the “sign of LIV”, a theory-dependent factor equal to $+1$ (-1) for a decrease (increase) in photon speed with an increasing photon energy [14]. ξ_n is a dimensionless parameter, and $E_{QG,n} = \xi_n E_{QG}$ is actually the effective energy scale where LIV happens for the order n term [14]. For $E \ll E_{QG}$, the lowest order term in the series not suppressed by theory (usually the $n = 1$ term) is expected to dominate the sum. If the $n = 1$ term is suppressed (say, by a symmetry law), the next term $n = 2$ will dominate, and so forth. If the dominated LIV correction is of order n , Eq. (1) can be approximated by

$$E^2 = p^2 c^2 \left[1 - s_{\pm} \left(\frac{E}{\xi_n E_{QG}} \right)^n \right], \quad (2)$$

which is the well-known form adopted in most of the relevant works in the literature (e.g. [5–16]). Note that E in the right hand side of Eq. (2) can be freely substituted with pc since we are only interested in the leading order correction [6]. Keeping this in mind and using Eq. (2), the energy-dependent speed of photons is given by (see e.g. [11,12,14,15])

$$v = \frac{\partial E}{\partial p} = c \left[1 - s_{\pm} \frac{n+1}{2} \left(\frac{E}{\xi_n E_{QG}} \right)^n \right]. \quad (3)$$

Following the standard procedures given in e.g. [6] (calculating the comoving path in the expanding universe is the key), one can finally get the LIV-induced time delay between photons with energies E_{high} and E_{low} as (see e.g. [11,12,14])

$$\Delta t_{LIV} = s_{\pm} \frac{1+n}{2H_0} \frac{E_{\text{high}}^n - E_{\text{low}}^n}{E_{QG,n}^n} \int_0^z \frac{(1+\tilde{z})^n d\tilde{z}}{h(\tilde{z})}, \quad (4)$$

where $h(z) \equiv H(z)/H_0$ is the dimensionless Hubble parameter, and z is the redshift of GRB. Following most of the relevant works in the literature (e.g. [5–14,16]), we only consider the case of $n = 1$, $s_{\pm} = +1$ and $\xi_1 = 1$ in the present work. In this case, Eq. (4) becomes

$$\Delta t_{LIV} = \frac{\Delta E}{H_0 E_{QG}} \int_0^z \frac{(1+\tilde{z}) d\tilde{z}}{h(\tilde{z})}. \quad (5)$$

Note that Δt_{LIV} in the original published version of [5] lacked the factor $(1+z)$ in the integration, and it has been corrected in the Erratum while the conclusion was modified accordingly.

For a cosmic transient source (e.g. GRB), the observed time delay between two different energy bands should include five terms [30,31],

$$\Delta t_{obs} = \Delta t_{LIV} + \Delta t_{int} + \Delta t_{spe} + \Delta t_{DM} + \Delta t_{gra}, \quad (6)$$

where Δt_{LIV} is the LIV-induced time delay as mentioned above. Δt_{int} is the intrinsic (astrophysical) time delay, which means that photons with high and low energies do not leave the source simultaneously. This term is difficult to predict because we have no good understanding on the physics of source evolution. As in the literature (e.g. [5,9,16]), one can write it as $\Delta t_{int} = b(1+z)$ while the cosmic expansion has been taken into account, and the constant parameter b characterizes our ignorance. Δt_{spe} represents the potential time delay due to special relativistic effects if photons have a non-zero rest mass. Since modern experiments have provided the upper limits for the photon rest mass as $m_{ph} < 10^{-18}$ eV/ c^2 [32], this term is negligible in fact [31]. Δt_{DM} is the time delay contribution from the dispersion by the line-of-sight free electron content,

which is also negligible for GRB photons [30]. Δt_{gra} represents the effect of gravitational potential along the propagation path of photons if the Einstein's equivalence principle (EEP) is violated. This term can be dropped since EEP is preserved in our case. Substituting Eq. (5) and $\Delta t_{\text{int}} = b(1+z)$ into Eq. (6), and neglecting other terms, we obtain [5,7,9,11,12,16]

$$\frac{\Delta t_{\text{obs}}}{1+z} = a_{\text{LIV}} K + b, \quad (7)$$

where $a_{\text{LIV}} \equiv \Delta E/(H_0 E_{\text{QG}})$, and

$$K \equiv \frac{1}{1+z} \int_0^z \frac{(1+\bar{z}) d\bar{z}}{h(\bar{z})}. \quad (8)$$

Obviously, if $a_{\text{LIV}} = 0$, there is no LIV. On the other hand, if the evidence of $a_{\text{LIV}} \neq 0$ is found, LIV happens on the energy scales above E_{QG} .

3. Constraints on LIV via the cosmographic approach with respect to redshift z

3.1. Cosmographic approach with respect to redshift z

For convenience, we recast Eq. (7) as

$$\Delta t_{\text{obs}} = a_{\text{LIV}} \mathcal{K} + b(1+z), \quad (9)$$

where

$$\mathcal{K} \equiv (1+z) K = \int_0^z \frac{(1+\bar{z}) d\bar{z}}{h(\bar{z})}. \quad (10)$$

In order to constrain the possible LIV, one should calculate the theoretical time delay induced by LIV, $\Delta t_{\text{th}} = a_{\text{LIV}} \mathcal{K} + b(1+z)$, and then confront it with the observed one, Δt_{obs} . From Eq. (10), it is easy to see that the cosmic expansion history (characterized by $h(z)$) should be given in advance. As mentioned in Sec. 1, we will present it via the cosmographic approach which is model-independent, rather than assuming a particular cosmological model (e.g. Λ CDM) as in the literature.

The only necessary assumption of cosmography is the cosmological principle, so that the spacetime metric is the one of the Friedmann–Robertson–Walker (FRW) universe,

$$ds^2 = -c^2 dt^2 + a^2(t) \left[\frac{dr^2}{1-kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right], \quad (11)$$

in terms of the comoving coordinates (t, r, θ, ϕ) , where a is the scale factor. Motivated by the inflation paradigm and the observational results from e.g. Planck 2015 data [33], in this work we only consider a spatially flat FRW universe with $k=0$. Introducing the so-called cosmographic parameters, namely the Hubble constant H_0 , the deceleration q_0 , the jerk j_0 , the snap s_0 ,

$$\begin{aligned} H_0 &\equiv \left. \frac{1}{a} \frac{da}{dt} \right|_{t=t_0}, & q_0 &\equiv - \left. \frac{1}{aH^2} \frac{d^2 a}{dt^2} \right|_{t=t_0}, \\ j_0 &\equiv \left. \frac{1}{aH^3} \frac{d^3 a}{dt^3} \right|_{t=t_0}, & s_0 &\equiv \left. \frac{1}{aH^4} \frac{d^4 a}{dt^4} \right|_{t=t_0}, \end{aligned} \quad (12)$$

one can expand the scale factor a in terms of a Taylor series with respect to cosmic time t [19,20],

$$\begin{aligned} a(t) = a(t_0) &\left[1 + H_0(t-t_0) - \frac{q_0}{2} H_0^2 (t-t_0)^2 + \frac{j_0}{3!} H_0^3 (t-t_0)^3 \right. \\ &\left. + \frac{s_0}{4!} H_0^4 (t-t_0)^4 + \mathcal{O}((t-t_0)^5) \right]. \end{aligned} \quad (13)$$

One of the most important quantities in cosmology is the luminosity distance $d_L = (c/H_0) D_L$, where the dimensionless luminosity distance D_L is defined by [19]

$$D_L \equiv (1+z) \int_0^z \frac{d\bar{z}}{h(\bar{z})}. \quad (14)$$

We can also expand the dimensionless luminosity distance D_L (equivalent to d_L) in terms of a Taylor series with respect to redshift z (see e.g. [19–21,29] for details),

$$\begin{aligned} D_L(z) = z + \frac{1}{2} (1-q_0) z^2 - \frac{1}{6} (1-q_0-3q_0^2+j_0) z^3 \\ + \frac{1}{24} (2-2q_0-15q_0^2-15q_0^3+5j_0+10q_0j_0+s_0) z^4 \\ + \mathcal{O}(z^5). \end{aligned} \quad (15)$$

Differentiating Eq. (14), we get

$$\frac{1+z}{h(z)} = \frac{dD_L}{dz} - \frac{D_L}{1+z}. \quad (16)$$

Substituting Eqs. (16) and (15) into Eq. (10), we obtain

$$\begin{aligned} \mathcal{K}(z) = z - \frac{q_0}{2} z^2 + \left(\frac{q_0}{3} + \frac{q_0^2}{2} - \frac{j_0}{6} \right) z^3 \\ + \left(-\frac{q_0}{4} - \frac{3}{4} q_0^2 - \frac{5}{8} q_0^3 + \frac{j_0}{4} + \frac{5}{12} q_0 j_0 + \frac{s_0}{24} \right) z^4 \\ + \mathcal{O}(z^5). \end{aligned} \quad (17)$$

In this approach, one can calculate the theoretical time delay $\Delta t_{\text{th}} = a_{\text{LIV}} \mathcal{K} + b(1+z)$ by using Eq. (17), without assuming any particular cosmological model. The LIV parameters a_{LIV} and b , together with the cosmographic parameters, will be determined by using the observational data. Note that one can also obtain $h(z)$ or $1/h(z)$ by using Eq. (16) with D_L given in Eq. (15). This is useful when one calculates other quantities (e.g. $D_V(z)$ used below).

3.2. Observational data

Ellis et al. [4] have developed the systematic analysis of statistical samples of GRBs at a range of different redshifts, and they have introduced techniques from signal processing such as wavelet analysis to identify and correlate genuine features in the intensities observed in different energy bands. Later, using these techniques, Ellis et al. [5] compiled a time delay dataset from 35 GRBs with known redshifts from $z=0.168$ to $z=6.29$. The numerical dataset can be found in Table 1 of [5]. To constrain the possible LIV, we perform the χ^2 statistics. The χ^2 from the time-lags of GRBs is given by

$$\chi_{\text{GRB}}^2 = \sum_{i=1}^{35} \left[\frac{\Delta t_{\text{th}}(z_i) - \Delta t_{\text{obs}}(z_i)}{\sigma_i} \right]^2, \quad (18)$$

where Δt_{obs} and σ_i are the observed time delay and the corresponding uncertainty given in Table 1 of [5], and $\Delta t_{\text{th}} = a_{\text{LIV}} \mathcal{K} + b(1+z)$ is the theoretical time delay, while \mathcal{K} is given in Eq. (17).

Clearly, the cosmographic parameters (characterizing the cosmic expansion history) cannot be well constrained by using only

the time delay data from GRBs. So, the other observational data are needed. Here, we consider the JLA (joint light-curve analysis) dataset [34] consisting of 740 SNIa obtained by the SDSS-II and SNLS collaborations. The theoretical distance modulus is defined by [19,34]

$$\mu_{th} = 5 \log_{10} \frac{d_L}{\text{Mpc}} + 25, \quad (19)$$

where the luminosity distance $d_L = (c/H_0) D_L$, and D_L is given in Eq. (15). On the other hand, in the JLA dataset, the observed distance modulus is given by [34]

$$\mu_{obs} = m_B - (\mathcal{M} - \alpha X_1 + \beta C), \quad (20)$$

where m_B corresponds to the observed peak magnitude in rest frame B band. α and β are both nuisance parameters. X_1 and C are the stretch measure and the color measure of SNIa, respectively. \mathcal{M} is a nuisance parameter representing some combination of the absolute magnitude of a fiducial SNIa and the Hubble constant H_0 . The χ^2 from JLA SNIa reads [34]

$$\chi_{JLA}^2 = \Delta \vec{\mu}^T \mathbf{C}^{-1} \Delta \vec{\mu}, \quad (21)$$

where $\Delta \mu = \mu_{obs} - \mu_{th}$, and \mathbf{C} is the covariance matrix of $\vec{\mu}$. It is equivalent to [35] (see also e.g. [36])

$$\chi_{JLA}^2 = \Delta \vec{m}^T \mathbf{C}^{-1} \Delta \vec{m}, \quad (22)$$

where $\Delta m = m_B - m_{mod}$, and

$$m_{mod} = 5 \log_{10} D_L - \alpha X_1 + \beta C + \mathcal{M}, \quad (23)$$

while H_0 in d_L can be absorbed into \mathcal{M} . The numerical data of m_B , X_1 , C , and the covariance matrix \mathbf{C} can be found from the JLA dataset [34,37]. The nuisance parameters \mathcal{M} , α and β could be marginalized (note that H_0 is also a nuisance parameter and it can be absorbed into \mathcal{M} , and hence the JLA SNIa dataset is Hubble-free in fact). It is worth noting that since March 2014, the JLA likelihood plugin is included in the official release of the MCMC code CosmoMC [38,39].

We can further consider the observational data from BAO. Here we use the data $D_V(0.35)/D_V(0.2) = 1.736 \pm 0.065$ from SDSS Collaboration [40], where $D_V(z)$ is related to the angular diameter distance d_A and the luminosity distance d_L or D_L according to [40, 41]

$$\begin{aligned} D_V(z) &\equiv \left[(1+z)^2 d_A^2 \frac{cz}{H(z)} \right]^{1/3} = \left[\frac{d_L^2}{(1+z)^2} \frac{cz}{H(z)} \right]^{1/3} \\ &= \frac{c}{H_0} \left[\frac{D_L^2}{(1+z)^2} \frac{z}{h(z)} \right]^{1/3}, \end{aligned} \quad (24)$$

in which we have used the well-known relation $d_A = d_L/(1+z)^2$ (see e.g. [19,22,24]) and $d_L = (c/H_0) D_L$. Note that the factor c/H_0 will be cancelled in $D_V(0.35)/D_V(0.2)$ and hence it is also Hubble-free. One can calculate $D_V(z)$ by using D_L given in Eq. (15), and $z/h(z) = [z/(1+z)] \cdot [dD_L/dz - D_L/(1+z)]$ from Eq. (16). The χ^2 from BAO is given by

$$\chi_{BAO}^2 = \left[\frac{D_V(0.35)/D_V(0.2) - 1.736}{0.065} \right]^2. \quad (25)$$

Note that actually there exist other BAO data in the literature, such as the observational data of $A \equiv \Omega_{m0}^{1/2} H_0 D_V(z)/z$ [41–43], and $d_z \equiv r_s(z_d)/D_V(z)$ [40,44,45]. However, they will introduce one or

more free model parameters such as Ω_{m0} , Ω_b , while the Hubble constant H_0 is no longer a nuisance parameter. This is a drawback, and makes the constraints loose. So, we do not consider such types of BAO data in this work. Similarly, we also do not consider the observational data from cosmic microwave background (CMB), since one or more free model parameters, e.g. Ω_{m0} , and/or Ω_b , H_0 , should be introduced.

3.3. Constraints on LIV

Here, we constrain the LIV parameters a_{LIV} and b , together with the cosmographic parameters, by using the observational data mentioned in Sec. 3.2. Note that we use the MCMC code CosmoMC [38,39] in doing this. The cosmographic formulae for D_L and \mathcal{K} are given in Eqs. (15) and (17) respectively, while one can calculate $D_V(z)$ by using D_L given in Eq. (15), and $z/h(z) = [z/(1+z)] \cdot [dD_L/dz - D_L/(1+z)]$ from Eq. (16).

As mentioned above, the Hubble constant H_0 (which is also the first cosmographic parameter) has not been involved in the time delay data from GRBs and the $D_V(0.35)/D_V(0.2)$ data from BAO, while it is a nuisance parameter in the JLA SNIa dataset and can be marginalized. On the other hand, if one consider the cosmographic formulae only up to 2nd order, the corresponding Taylor series cannot be enough general to include many cosmological models as its special cases, and hence the conclusions become not so robust. However, if one consider the cosmographic formulae up to very high order, too many cosmographic parameters will be involved and hence the constraints become very loose. On balance, here we consider the cosmographic formulae up to 3rd and 4th orders one by one.

In the case of 3rd order cosmography with respect to redshift z (labeled as “ $z-j$ ”), namely we only consider the Taylor series for $D_L(z)$, $\mathcal{K}(z)$, $D_V(z)$... up to 3rd order and ignore all the higher order terms $\mathcal{O}(z^4)$, there are only two free cosmographic parameters q_0 and j_0 besides the nuisance parameter H_0 . So, the free parameters under consideration are $\{q_0, j_0, a_{LIV}, b\}$. The total χ^2 is given by $\chi_{tot}^2 = \chi_{GRB}^2 + \chi_{JLA}^2 + \chi_{BAO}^2$. By fitting the cosmographic formulae to the combined GRB + JLA + BAO observational data, we obtain the 1σ and 2σ constraints on the LIV parameters (a_{LIV}, b) and the cosmographic parameters (q_0, j_0), which are presented in the 2nd column of Table 1. We also present the 2D marginalized 1σ and 2σ contours and the 1D distributions of the LIV parameters and the cosmographic parameters in Fig. 1. From Fig. 1 and the 2nd column of Table 1, we find a fairly weak hint for LIV with a non-zero a_{LIV} (slightly beyond 1σ confidence region), while $a_{LIV} = 0$ is still consistent with the observational data within 2σ confidence region.

In the case of 4th order cosmography with respect to redshift z (labeled as “ $z-s$ ”), namely we only consider the Taylor series for $D_L(z)$, $\mathcal{K}(z)$, $D_V(z)$... up to 4th order and ignore all the higher order terms $\mathcal{O}(z^5)$, there are three free cosmographic parameters q_0 , j_0 and s_0 besides the nuisance parameter H_0 . So, the free parameters under consideration are $\{q_0, j_0, s_0, a_{LIV}, b\}$. By fitting the cosmographic formulae to the combined GRB + JLA + BAO observational data, we obtain the 1σ and 2σ constraints on the LIV parameters (a_{LIV}, b) and the cosmographic parameters (q_0, j_0, s_0), which are presented in the 3rd column of Table 1. We also present the 2D marginalized 1σ and 2σ contours and the 1D distributions of the LIV parameters and the cosmographic parameters in Fig. 2. From Fig. 2 and the 3rd column of Table 1, we find again a weak hint for LIV with a non-zero a_{LIV} (beyond 1σ confidence region), while $a_{LIV} = 0$ is still consistent with the observational data within 2σ confidence region.

So, in both cases of cosmography with respect to redshift z , LIV with a non-zero a_{LIV} is slightly favored by the observational data.

Table 1

The mean with 1σ and 2σ uncertainties of the LIV parameters and the cosmographic parameters for the cases of 3rd order (labeled as “ $z-j$ ”, the 2nd column) and 4th order (labeled as “ $z-s$ ”, the 3rd column) cosmography with respect to redshift z , respectively. Note that a_{LIV} and b are given in units of seconds, while q_0 , j_0 , s_0 are all dimensionless. See the text for details.

Parameters	Case $z-j$	Case $z-s$
a_{LIV}	$-0.0056134^{+0.0032712}_{-0.0033430} (1\sigma) ^{+0.0073291}_{-0.0074206} (2\sigma)$	$-0.0358845^{+0.0236884}_{-0.0180408} (1\sigma) ^{+0.0367396}_{-0.0385662} (2\sigma)$
b	$-0.0009220^{+0.0018053}_{-0.0019603} (1\sigma) ^{+0.0041510}_{-0.0038621} (2\sigma)$	$0.0153007^{+0.0098136}_{-0.0135162} (1\sigma) ^{+0.0214681}_{-0.0202124} (2\sigma)$
q_0	$-0.4516519^{+0.0802934}_{-0.0799675} (1\sigma) ^{+0.1530627}_{-0.1641987} (2\sigma)$	$-0.5464667^{+0.0501635}_{-0.0431929} (1\sigma) ^{+0.0958649}_{-0.0981299} (2\sigma)$
j_0	$0.5062922^{+0.1333122}_{-0.1669237} (1\sigma) ^{+0.3155159}_{-0.3016949} (2\sigma)$	$1.1111259^{+0.1301341}_{-0.2501652} (1\sigma) ^{+0.5109921}_{-0.3991027} (2\sigma)$
s_0		$-0.2812477^{+0.3564938}_{-0.5722327} (1\sigma) ^{+1.0999906}_{-0.9656683} (2\sigma)$

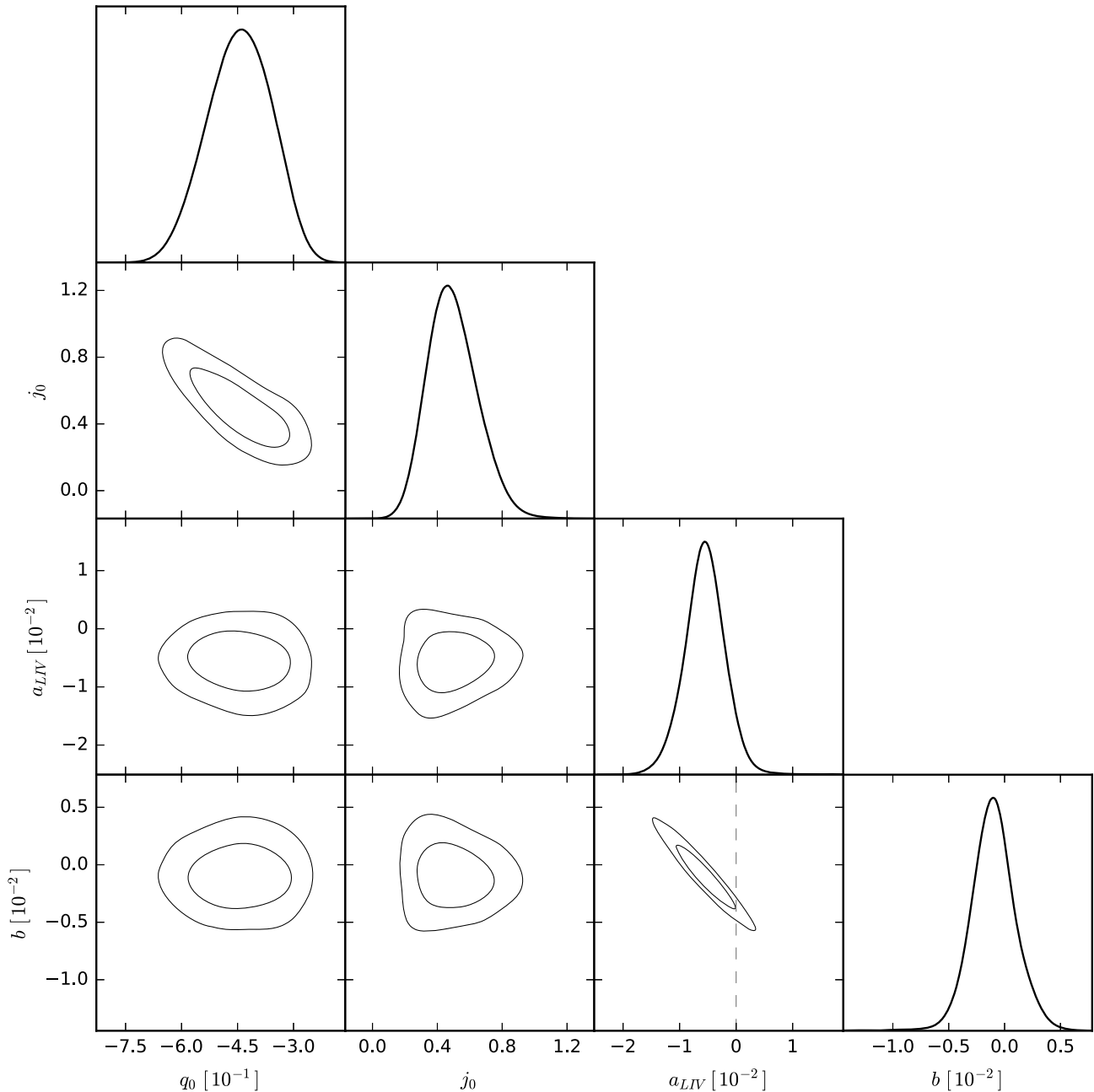


Fig. 1. The 2D marginalized 1σ and 2σ contours and the 1D distributions of the LIV parameters and the cosmographic parameters for the case of 3rd order cosmography with respect to redshift z (labeled as “ $z-j$ ”). Note that $a_{LIV} = 0$ is also indicated by a dashed line in the $a_{LIV} - b$ panel, and q_0 , a_{LIV} , b are given in units of 10^{-1} , 10^{-2} , 10^{-2} , respectively. See the text and Table 1 for details.

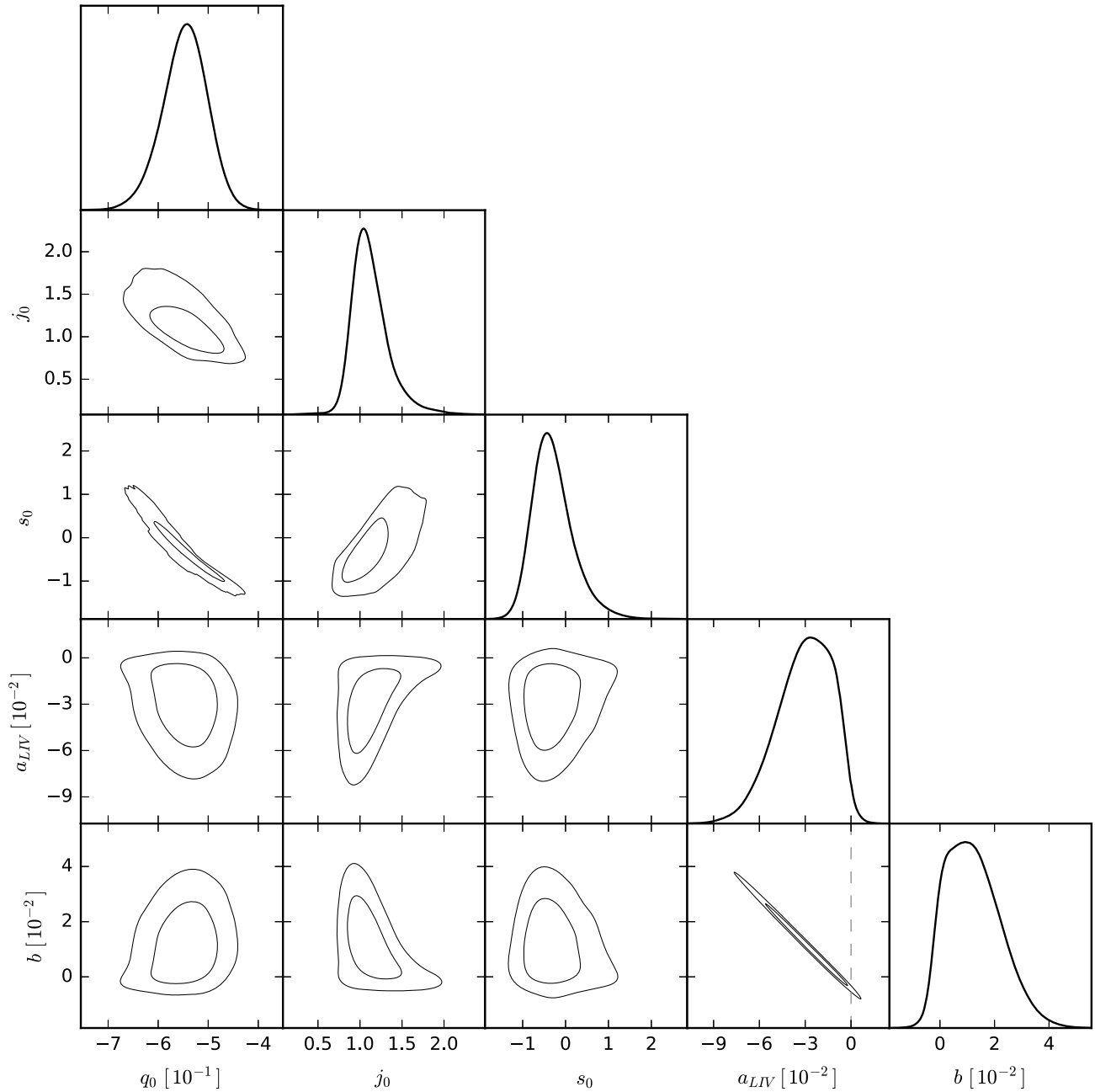


Fig. 2. The 2D marginalized 1σ and 2σ contours and the 1D distributions of the LIV parameters and the cosmographic parameters for the case of 4th order cosmography with respect to redshift z (labeled as “ $z - s$ ”). Note that $a_{LIV} = 0$ is also indicated by a dashed line in the $a_{LIV} - b$ panel, and q_0, a_{LIV}, b are given in units of $10^{-1}, 10^{-2}, 10^{-2}$, respectively. See the text and Table 1 for details.

On the other hand, from Figs. 1, 2, and Table 1, it is easy to see that in both cases, the deceleration parameter q_0 is negative beyond 2σ confidence region, and the jerk j_0 is positive also beyond 2σ confidence region. Thus, from the definitions in Eq. (12), this means that an accelerating universe ($q_0 < 0$) is strongly favored, while the acceleration is still increasing ($j_0 > 0$).

4. Constraints on LIV via the cosmographic approach with respect to y -shift

4.1. Cosmographic approach with respect to y -shift

It is easy to see that the key of cosmography is to expand the quantities under consideration as a Taylor series. In the original

version of cosmography, the relevant quantities are expanded with respect to redshift z . However, it is well known that such a Taylor series converges only for small z around 0, and it might diverge at high redshift $z > 1$. A possible remedy is to replace z with the so-called y -shift, $y \equiv z/(1+z)$ (see e.g. [22–24,26,29]). In this case, $y < 1$ holds in the whole cosmic past $0 \leq z < \infty$, and hence the Taylor series with respect to y -shift converges. So, here we also consider the cosmographic approach with respect to y -shift, $y \equiv z/(1+z)$.

We can expand the dimensionless luminosity distance D_L (equivalent to d_L) in terms of a Taylor series with respect to y (see e.g. [21,23,29] for details),

$$D_L(y) = y + \frac{1}{2} (3 - q_0) y^2 + \frac{1}{6} (11 - 5q_0 + 3q_0^2 - j_0) y^3$$

Table 2

The mean with 1σ and 2σ uncertainties of the LIV parameters and the cosmographic parameters for the cases of 3rd order (labeled as “ $y - j$ ”, the 2nd column) and 4th order (labeled as “ $y - s$ ”, the 3rd column) cosmography with respect to y -shift $y = z/(1 + z)$, respectively. Note that a_{LIV} and b are given in units of seconds, while q_0 , j_0 , s_0 are all dimensionless. See the text for details.

Parameters	Case $y - j$	Case $y - s$
a_{LIV}	$-0.0839179^{+0.0386001}_{-0.0347786} (1\sigma)^{+0.0692573}_{-0.0695886} (2\sigma)$	$0.0045945^{+0.0150052}_{-0.0126229} (1\sigma)^{+0.0290441}_{-0.0296695} (2\sigma)$
b	$0.0384035^{+0.0176107}_{-0.0190885} (1\sigma)^{+0.0349654}_{-0.0346907} (2\sigma)$	$-0.0063665^{+0.0070200}_{-0.0082512} (1\sigma)^{+0.0163490}_{-0.0158987} (2\sigma)$
q_0	$-0.6736552^{+0.2455526}_{-0.3089059} (1\sigma)^{+0.5268171}_{-0.5047697} (2\sigma)$	$-0.8314270^{+0.2482399}_{-0.4659630} (1\sigma)^{+0.7825488}_{-0.6406471} (2\sigma)$
j_0	$2.5771079^{+3.4483891}_{-3.3925562} (1\sigma)^{+6.1780295}_{-6.2831097} (2\sigma)$	$8.7896032^{+10.6351976}_{-3.7547503} (1\sigma)^{+11.2103968}_{-13.7152157} (2\sigma)$
s_0		$136.7073059^{+118.7269897}_{-132.2172699} (1\sigma)^{+195.5156860}_{-189.9219513} (2\sigma)$

$$+ \frac{1}{24}(50 - 26q_0 + 21q_0^2 - 15q_0^3 - 7j_0 + 10q_0j_0 + s_0)y^4 + \mathcal{O}(y^5). \quad (26)$$

Alternatively, one can derive Eq. (26) by substituting $z = y/(1 - y) = y + y^2 + y^3 + y^4 + \mathcal{O}(y^5)$ into Eq. (15) and then rearranging it as a series in terms of y . Similarly, we can also substitute $z = y/(1 - y) = y + y^2 + y^3 + y^4 + \mathcal{O}(y^5)$ into Eq. (17) and then rearrange it as a series in terms of y ,

$$\begin{aligned} \mathcal{K}(y) = & y + \left(1 - \frac{q_0}{2}\right)y^2 + \left(1 - \frac{2}{3}q_0 + \frac{q_0^2}{2} - \frac{j_0}{6}\right)y^3 \\ & + \left(1 - \frac{3}{4}q_0 + \frac{3}{4}q_0^2 - \frac{5}{8}q_0^3 - \frac{j_0}{4} + \frac{5}{12}q_0j_0 + \frac{s_0}{24}\right)y^4 \\ & + \mathcal{O}(y^5). \end{aligned} \quad (27)$$

Of course, one can also expand D_V as a Taylor series in terms of y by substituting $z = y/(1 - y) = y + y^2 + y^3 + y^4 + \mathcal{O}(y^5)$ into Eq. (24) and then rearranging it as a series in terms of y . Note that Eq. (16) is still useful in doing this.

4.2. Constraints on LIV

Again, we constrain the LIV parameters a_{LIV} and b , together with the cosmographic parameters, by using the observational data mentioned in Sec. 3.2. This is similar to Sec. 3.3 actually, but the cosmographic formulae for D_L , \mathcal{K} , D_V ... should instead use the ones with respect to y -shift $y = z/(1 + z)$ given in Sec. 4.1.

In the case of 3rd order cosmography with respect to y -shift $y = z/(1 + z)$ (labeled as “ $y - j$ ”), namely we only consider the Taylor series for $D_L(y)$, $\mathcal{K}(y)$, $D_V(y)$... up to 3rd order and ignore all the higher order terms $\mathcal{O}(y^4)$, there are only two free cosmographic parameters q_0 and j_0 besides the nuisance parameter H_0 . So, the free parameters under consideration are $\{q_0, j_0, a_{LIV}, b\}$. The total χ^2 is given by $\chi_{tot}^2 = \chi_{GRB}^2 + \chi_{JLA}^2 + \chi_{BAO}^2$. By fitting the cosmographic formulae to the combined GRB + JLA + BAO observational data, we obtain the 1σ and 2σ constraints on the LIV parameters (a_{LIV} , b) and the cosmographic parameters (q_0 , j_0), which are presented in the 2nd column of Table 2. We also present the 2D marginalized 1σ and 2σ contours and the 1D distributions of the LIV parameters and the cosmographic parameters in Fig. 3. From Fig. 3 and the 2nd column of Table 2, we find a notable hint for LIV with a non-zero a_{LIV} (around 2σ confidence region).

In the case of 4th order cosmography with respect to y -shift $y = z/(1 + z)$ (labeled as “ $y - s$ ”), namely we only consider the Taylor series for $D_L(y)$, $\mathcal{K}(y)$, $D_V(y)$... up to 4th order and ignore all the higher order terms $\mathcal{O}(y^5)$, there are three free cosmographic parameters q_0 , j_0 and s_0 besides the nuisance parameter H_0 . So, the free parameters under consideration are

$\{q_0, j_0, s_0, a_{LIV}, b\}$. By fitting the cosmographic formulae to the combined GRB + JLA + BAO observational data, we obtain the 1σ and 2σ constraints on the LIV parameters (a_{LIV} , b) and the cosmographic parameters (q_0 , j_0 , s_0), which are presented in the 3rd column of Table 2. We also present the 2D marginalized 1σ and 2σ contours and the 1D distributions of the LIV parameters and the cosmographic parameters in Fig. 4. From Fig. 4 and the 3rd column of Table 2, we find that $a_{LIV} = 0$ is fully consistent with the observational data, namely there is no evidence for LIV.

Although the results about LIV are quite different in both the cases of cosmography with respect to y -shift $y = z/(1 + z)$, from Figs. 3, 4, and Table 2, it is easy to see that in both cases, the deceleration parameter q_0 is negative beyond 2σ confidence region. Thus, from the definition in Eq. (12), this means that an accelerating universe ($q_0 < 0$) is strongly favored.

5. Concluding remarks

Since Lorentz invariance plays an important role in modern physics, it is of interest to test the possible LIV. The time-lag (the arrival time delay between light curves in different energy bands) of GRBs has been extensively used to this end. However, to our best knowledge, one or more particular cosmological models were assumed *a priori* in (almost) all of the relevant works in the literature. So, this makes the results on LIV in those works model-dependent and hence not so robust in fact. In the present work, we try to avoid this problem by using a model-independent approach. We calculate the time delay induced by LIV with the cosmic expansion history given in terms of cosmography, without assuming any particular cosmological model. Then, we constrain the possible LIV with the observational data from GRBs, SNIa and BAO, and find weak hints for LIV with non-zero a_{LIV} in 3 of 4 cases of cosmography considered in the present work.

It is of interest to compare the 4 cases of cosmography considered here. As mentioned above, they are labeled as “ $z - j$ ”, “ $z - s$ ”, “ $y - j$ ” and “ $y - s$ ”, respectively. Since they have different free parameters and the correlations between model parameters are fairly different, it is not suitable to directly compare their confidence level contours. Instead, it is more appropriate to compare them from the viewpoint of goodness-of-fit. A conventional criterion for model comparison in the literature is χ_{min}^2/dof , in which the degree of freedom $dof = \mathcal{N} - \kappa$, while \mathcal{N} and κ are the number of data points and the number of free model parameters, respectively. On the other hand, there are other criteria for model comparison in the literature. The most sophisticated criterion is the Bayesian evidence (see e.g. [46] and references therein). However, the computation of Bayesian evidence usually consumes a large amount of time and power. As an alternative, one can consider some approximations of Bayesian evidence, such as the so-called Bayesian

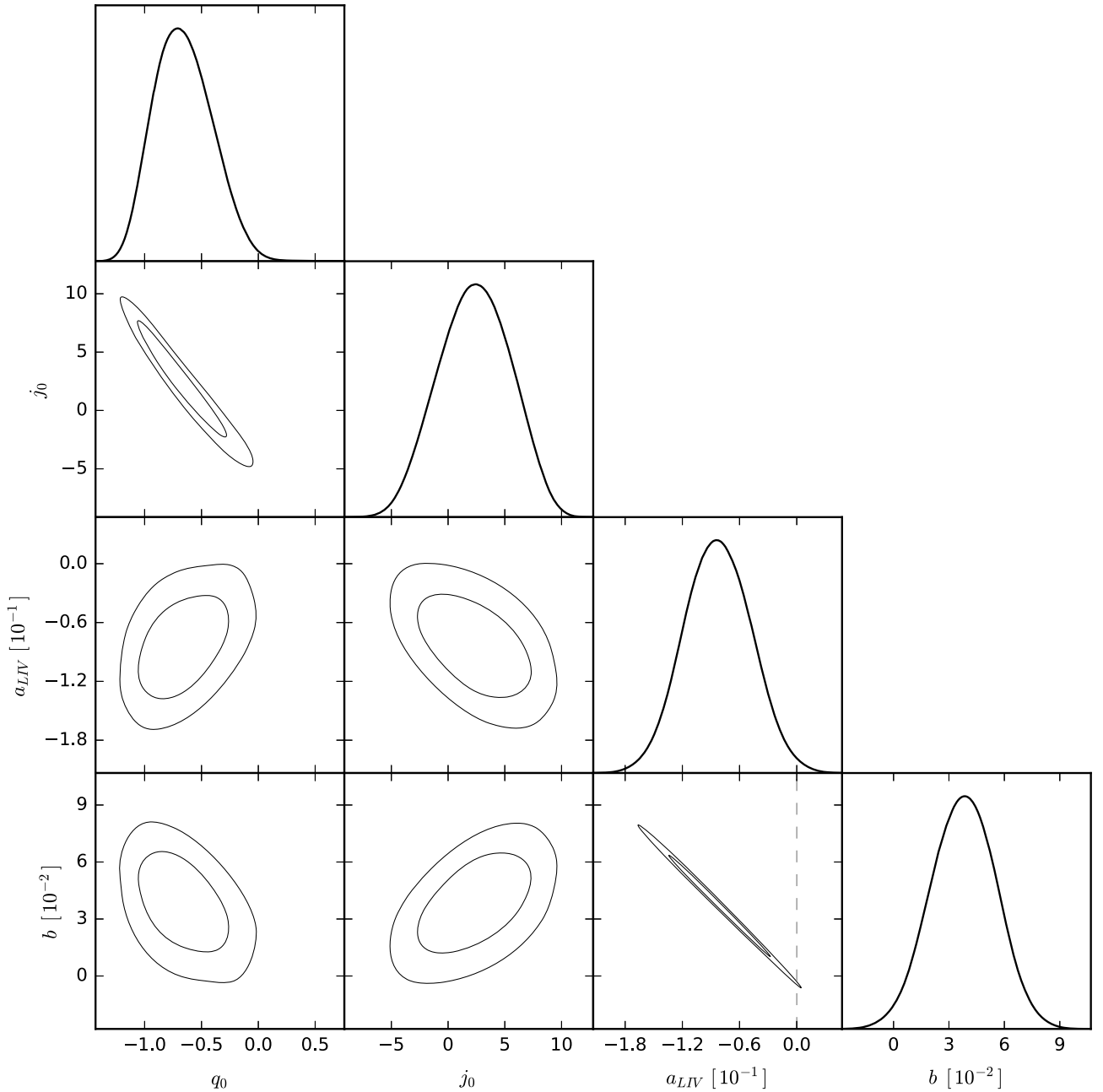


Fig. 3. The 2D marginalized 1σ and 2σ contours and the 1D distributions of the LIV parameters and the cosmographic parameters for the case of 3rd order cosmography with respect to y -shift (labeled as “ $y - j$ ”). Note that $a_{LIV} = 0$ is also indicated by a dashed line in the $a_{LIV} - b$ panel, and a_{LIV} , b are given in units of 10^{-1} , 10^{-2} , respectively. See the text and Table 2 for details.

Information Criterion (BIC) and Akaike Information Criterion (AIC). The BIC is defined by [47]

$$\text{BIC} = -2 \ln \mathcal{L}_{\max} + \kappa \ln \mathcal{N}, \quad (28)$$

where \mathcal{L}_{\max} is the maximum likelihood. In the Gaussian cases, $\chi_{\min}^2 = -2 \ln \mathcal{L}_{\max}$. So, the difference in BIC between two models is given by $\Delta \text{BIC} = \Delta \chi_{\min}^2 + \Delta \kappa \ln \mathcal{N}$. The AIC is defined by [48]

$$\text{AIC} = -2 \ln \mathcal{L}_{\max} + 2\kappa. \quad (29)$$

The difference in AIC between two models is given by $\Delta \text{AIC} = \Delta \chi_{\min}^2 + 2\Delta \kappa$. In Table 3, we present χ_{\min}^2/dof , ΔBIC and ΔAIC for the 4 cases of cosmography considered in this work. Note that “ $z - s$ ” has been chosen to be the fiducial model when we calculate ΔBIC and ΔAIC . Clearly, from the viewpoint of all the three

criteria χ_{\min}^2/dof , BIC and AIC, “ $z - s$ ” is the best, and “ $y - s$ ” is the worst. Together with the fact that we find no evidence for LIV only in the case “ $y - s$ ” while there are weak hints for LIV in the other three cases “ $z - s$ ”, “ $y - j$ ” and “ $z - j$ ”, we conclude that LIV with non-zero a_{LIV} is slightly favored by the observational data.

In the literature, there exist many relevant works on the possible LIV. In addition to the weak evidence for LIV found in [5] (its Erratum should be seriously considered), some further works (e.g. [9,11,12,15]) supported this conclusion of [5]. In particular, a strong evidence for LIV was claimed in [11,12]. Our results obtained in the present work could be regarded as a new support. On the other hand, one should also be aware of the contrary claim that there is no evidence for LIV (see e.g. [16]). Besides, most of the relevant works in the literature kept silence and just put a

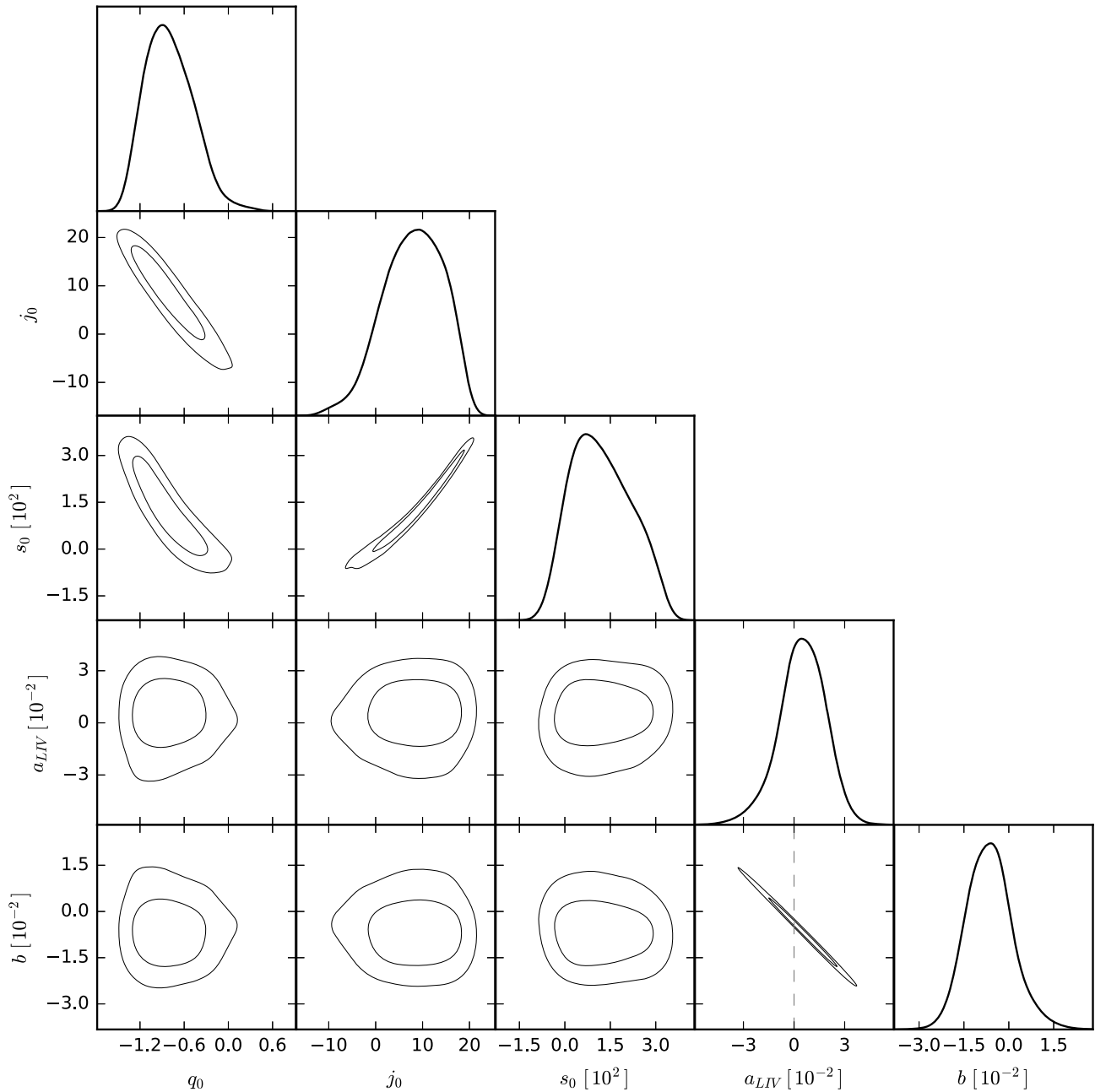


Fig. 4. The 2D marginalized 1σ and 2σ contours and the 1D distributions of the LIV parameters and the cosmographic parameters for the case of 4th order cosmography with respect to y -shift (labeled as “ $y-s$ ”). Note that $a_{LIV} = 0$ is also indicated by a dashed line in the $a_{LIV} - b$ panel, and s_0, a_{LIV}, b are given in units of $10^2, 10^{-2}, 10^{-2}$, respectively. See the text and Table 2 for details.

Table 3

Comparing the four cases of cosmography considered in the present work, namely 3rd order (labeled as “ $z-j$ ”) and 4th order (labeled as “ $z-s$ ”) cosmography with respect to redshift z , as well as 3rd order (labeled as “ $y-j$ ”) and 4th order (labeled as “ $y-s$ ”) cosmography with respect to y -shift $y = z/(1+z)$. See the text for details.

Cosmography	$z-j$	$z-s$	$y-j$	$y-s$
χ^2_{min}	871.3198	860.0310	868.3670	871.9442
κ	4	5	4	5
χ^2_{min}/dof	1.1287	1.1155	1.1248	1.1309
ΔBIC	4.6346	0	1.6818	11.9132
ΔAIC	9.2888	0	6.3360	11.9132
Rank	3	1	2	4

lower bound on the possible LIV energy scale E_{QG} . Therefore, the debate on LIV has not been settled so far. More and better observational data from e.g. GRBs are needed. New ideas to test LIV are also desirable.

In the present work, we consider the cosmographic approaches with respect to redshift z and y -shift $y = z/(1+z)$. The first one might diverge at high redshift $z > 1$, while the second one can alleviate this problem since $y < 1$ in the whole cosmic past $0 \leq z < \infty$, and hence the Taylor series with respect to y converges. However, there still exist several serious problems in the case of $y = z/(1+z)$. The first is that the error of a Taylor approximation throwing away the higher order terms will become unacceptably large when y is close to 1 (say, when $z > 9$). The second is that the cosmography in terms of $y = z/(1+z)$ cannot

work well in the cosmic future $-1 < z < 0$. The Taylor series with respect to $y = z/(1+z)$ does not converge when $y < -1$ (namely $z < -1/2$), and it drastically diverges when $z \rightarrow -1$ (it is easy to see that $y \rightarrow -\infty$ in this case). So, the y -shift cosmography fails to predict the future evolution of the universe. In [29], two new generalizations of cosmography inspired by the Padé approximant have been proposed, which can avoid or at least alleviate the problems of ordinary cosmography mentioned above. The model-independent constraints on LIV via these two new cosmographic approaches proposed in [29] deserve further investigation.

In most of the relevant works on LIV (including the present work), the intrinsic time-lag of GRBs is actually oversimplified by assuming $\Delta t_{\text{int}} = b(1+z)$. Noting that the factor $(1+z)$ comes from the cosmic expansion, this assumption means that all GRBs have the same intrinsic time delay (characterized by the constant parameter b) in the source frame. In e.g. [15], an energy-dependent time-lag was proposed. We consider that a more realistic assumption for the intrinsic time-lag of GRBs is important to robustly constrain the possible LIV in the future work. Deeper understanding on the observed time-lag of GRBs is also desirable (nb. Eq. (6)), especially the ones other than LIV-induced time delay.

As mentioned in Sec. 1, the possible LIV is commonly accompanied with a non-trivial dispersion relation. A signal of energy E that travels a distance L acquires a time delay (measured with respect to the ordinary case of an energy-independent speed c), namely $\Delta t \sim \xi(E/E_{QG})(L/c)$. Although the QG effect is expected to be very weak, a very long distance L can still make it testable. This point can be clearly seen from Eq. (5), namely a large time delay Δt_{LIV} follows a high redshift z . GRBs are among the most powerful sources in the universe. Their high energy photons in the gamma-ray band are almost immune to dust extinction, and hence they have been observed up to redshift $z \sim 8-9$ [51,52], while the maximum redshift of GRBs is expected to be 10 or even larger [53,54]. Therefore, GRBs at high redshift can be used to test the possible LIV which induces a large time delay. In the present work, we find weak hints for LIV by using the time delay dataset from 35 GRBs with redshifts up to $z = 6.29$ [5] via the cosmographic approach. Our results suggest that the possible LIV should be taken seriously. The physical mechanism for LIV might be the spacetime foam predicted in most of the quantum gravity theories (e.g. string theory, loop quantum gravity, and doubly special relativity). In addition, we would like to mention the so-called Standard-Model Extension (SME) [49], which also provides a field theory framework for LIV. The deep physics behind LIV deserves serious consideration.

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