Modern Physics Letters A
Vol. 35, No. 36 (2020) 2030020 (21 pages)

DOI: 10.1142/S0217732320300207

The spacetime between Einstein and Kaluza–Klein

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Received 24 July 2020
Accepted 10 August 2020
Published 31 August 2020

In this paper I introduce tensor multinomials, an algebra that is dense in the space of nonlinear smooth differential operators, and use a subalgebra to create an extension of Einstein's theory of general relativity. In a mathematical sense this extension falls between Einstein's original theory of general relativity in four dimensions and the Kaluza–Klein theory in five dimensions. The theory has elements in common with both the original Kaluza–Klein and Brans–Dicke, but emphasizes a new and different underlying mathematical structure. Despite there being only four physical dimensions, the use of tensor multinomials naturally leads to expanded operators that can incorporate other fields. The equivalent Ricci tensor of this geometry is robust and yields vacuum general relativity and electromagnetism, as well as a Klein–Gordon-like quantum scalar field. The formalism permits a time-dependent cosmological function, which is the source for the scalar field. I develop and discuss several candidate Lagrangians. Trial solutions of the most natural field equations include a singularity-free dark energy dust cosmology.

Keywords: Kaluza–Klein; general relativity; dark energy; unified field theory.

PACS Nos.: 04.50.-h, 04.50.Cd, 04.50.Kd, 12.10.-g, 93.36.+x, 98.80.Cq, 98.80.Es, 98.90-k

1. Introduction

The theory I present here grew out of an idea I had in graduate school in 1988, that of creating a probabilistic theory of gravitation. In doing so, I hoped to understand the connections between quantum mechanics and general relativity (GR). The idea was that the paths of particles in spacetime, determined by curves of maximal length in GR, might also be determined or approximated by some variation of a quantum mechanical quantity. Comparing those two approaches might then shed light on the similarities and differences between them, leading to a better understanding of the quantum in spacetime.

That idea raised a question: what quantum mechanical or statistical expression might be suitable? Probability distributions are important in both statistical and quantum mechanics. If $\tilde{g}(x^a, v^a)$ is a distribution or other function defined on spacetime, it should affect the orbits of particles. We can take an arbitrary path between two points A and B and integrate along it,

$$N = \int_A^B \tilde{g}(x^a, v^a) ds \tag{1}$$

obtaining a number N for that path. We can then ask on which of many possible paths the particle will travel. The intuitive answer is the particle will travel along the path that maximizes the integral in Eq. (1). That yields a variational principle for finding particle orbits that is analogous to that used in GR.

On implementing this idea, it became clear that the integrand in Eq. (1) required the use of smooth nonlinear differential operators on an otherwise ordinary spacetime manifold M. Probability distributions on spacetime might require, for example, exponential functions of both position and four velocities. Evidently, such mathematical operators will not be multilinear, and using them will require a mathematical theory that does not yet exist.

It is possible, however, to perform a multivariable expansion of such operators in powers of four velocities. Such an infinite sum is a multinomial in terms of one or more four velocities, so that $\tilde{g}(x^a, v^a, w^b)$, for example, could be written as

$$\tilde{g}(x^a, v^a, w^b) = \psi + A_a v^a + Q_b w^b + g_{ab} v^a w^b + \cdots \tag{2}$$

where ψ, A_a, Q_b, and g_{ab} are functions of the four spacetime coordinates and by the basis theorem, are tensors of appropriate rank. Truncating the infinite series at the second rank, we recognize that such objects are similar to Lagrangians for combining tensor fields of rank zero, one, and two.

The form of Eq. (2) suggests that $\tilde{g}$ maps pairs of vector fields v^a, w^b to $F(M)$, the space of real-valued functions on the spacetime manifold M. Evidently, the "component fields" ψ, A_a, Q_b, g_{ab}, and so forth operate on the arbitrary vectors v^a and w^b. That suggests that the truncated form $\tilde{g} = \psi \oplus A_a \oplus Q_b \oplus g_{ab}$, where $\oplus$ is a direct sum, may be considered a single operator or field. That operator is *not* multilinear in its arguments. For it to be so, substituting $v^a = \alpha p^a + \beta s^a$, with α, β real numbers, would have to result in

$$\tilde{g}(x^a, \alpha p^a + \beta s^a, w^b) = \alpha \tilde{g}(x^a, p^a, w^b) + \beta \tilde{g}(x^a, s^a, w^b) \tag{3}$$

which it clearly does not. Specifically, neither of the components ψ and Q_b in the expression on the left will be multiplied by either α or β during the operation. Therefore Eq. (3) does not hold, and $\tilde{g}$ is not multilinear in its four-velocity arguments. In going forward, we assume that $Q_b = A_b$. The truncated operator $\tilde{g}$ then takes on characteristics similar to the metric of Kaluza–Klein spacetime.

Kaluza and Klein independently proposed their five-dimensional theory so that the metric would have places for the components of the electromagnetic field.[1,2] The theory did not succeed, and that may be due in part to the fact that the electromagnetic field transforms as a vector, not as components of a tensor of rank two. In addition, the Kaluza–Klein theory did not predict correct particle dynamics. Here, I develop an alternate spacetime operator that has the needed extra components but does not require defining a fifth dimension.[3,4] The construction involves taking a direct sum of a scalar field and a vector field in four dimensions and creating all tensor products of that fundamental object.

The use of direct sums to create new mathematical spaces is well known and described in any elementary text on abstract algebra, e.g., Herstein.[5] A simple example is the creation of the Cartesian plane, $\mathbb{R}\oplus\mathbb{R}$, from the real line, $\mathbb{R}$. Direct sums are used routinely in mathematics and physics, usually without notice, because a universal convention is to drop the direct sum sign, $\oplus$, in favor of an arithmetic $+$ sign, which the reader must understand as indicating a direct sum. Direct sums create polynomials and multinomials, such as $f(x,y) = 1 + x + y + xy$, where the $+$ signs are actually direct sums, $\oplus$. When evaluated at a particular point (x_0, y_0) of $\mathbb{R}\oplus\mathbb{R}$, the foregoing direct sum becomes an arithmetic sum, mapping to a single value of $\mathbb{R}$. Vectors and higher order tensors are also the result of applying the concept of direct sums.

An important and unrecognized tensor multinomial occurs during variation of a Lagrangian involving tensors of different rank. For example, variation of the Lagrangian

$$\mathcal{L} = \int \alpha\nabla_a\psi\nabla^a\psi + \beta\nabla_a A_b\nabla^a A^b + g^{ab}R_{ab}\sqrt{-g}\,d^4x \tag{4}$$

results in separate equations for ψ, A_a, and g_{ab} that can be understood as components of a single tensor multinomial. Similarly, Lagrangians for particle dynamics may be considered tensor multinomials that have operated on four velocities, mapping them to $F(M)$, the space of real-valued functions on the manifold M. That correspondence will be exploited in this paper to find the effective Christoffel symbols and Ricci curvatures by working in reverse.

Arbitrary direct sums of tensors of different rank and type create elements of a universal covering algebra for the tensor algebra. These tensor multinomials are elements of the space of nonlinear differential operators on manifolds. Just as arbitrary polynomials are dense in the space of real-valued functions, tensor polynomials are dense in the space of nonlinear differential operators. For any given nonlinear differential operator, there is a tensor multinomial arbitrarily close to it.

The construction presented here naturally involves a scalar field, necessary so that the extended metric of the spacetime is invertible. The inclusion of both a scalar field and vector field means the spacetime bears some resemblance to both Brans–Dicke theory and Kaluza–Klein theory. This new theory provides a way of understanding three fundamental fields as resulting from a single equation. In addition, it provides an alternative or supplement to the inclusion of a cosmological constant, and a number of new ways of constructing Lagrangians that may facilitate creating new mathematical models of physical phenomena.

Plan of the Paper

i. Develop the concept of tensor multinomials.
ii. Define the analog of the covariant derivative.
iii. Derive the analog of the geodesic equation.
iv. From the geodesic equation, deduce the implied Christoffel symbols.
v. Using the Christoffel symbols, calculate the equivalent Ricci curvature.
vi. Propose field equations based on the equivalent Ricci curvature.
vii. Discuss candidate Lagrangian formulations.
viii. Present trial solutions to some of the field equations.

After working through these eight steps, we will find that the spacetime structure described by the implied Christoffel symbols and equivalent Ricci tensor is more complex than that of standard four-dimensional GR. At the same time, that spacetime structure is much simpler than that of the five-dimensional Kaluza–Klein theory. This new spacetime, therefore, lies between the spacetime of Einstein and that of Kaluza and Klein.

2. *X*-Tensor Analysis

2.1. *Tensor multinomials*

An example of a tensor multinomial is

$$\tilde{Q}_A = Q \oplus Q_a \tag{5}$$

where Q is a scalar field and Q_a is a covector field. Naturally, there are similar operators having a contravariant index. The index A does not refer to spinor indices, here, but to a scalar plus a vector index: A takes two values: $A = 0$ corresponds to the scalar index, whereas $A = a$ corresponds to a vector or covector index. Tildes will often serve to emphasize and distinguish the scalar extended field operators from their conventional analogs. Because the scalar Q extends the covector Q_a, $\tilde{Q}_A$ is called an *extended covector*, or x-covector. The expression in Eq. (5) is also a direct sum of tensors of different rank, and is an element of the universal covering algebra for tensors, as originally pointed out by Christodoulu.[6] All tensor products of such operators form a subalgebra of the universal covering algebra for tensors.

A particle with charge e and momentum p^a will be represented as:

$$p^A = e \oplus p^a. \tag{6}$$

Charge e will be taken as associated with electromagnetic fields, although other interpretations are possible. The direct sum of a scalar and a four-vector is similar to but distinct from a five-vector because it has different transformation properties under a change of coordinates. Here the first component, a scalar, is invariant under coordinate transformations. The property that the zeroth index corresponds to a scalar is particularly appropriate for charge, which is known to be an invariant under coordinate transformations. Such vectors that have a scalar extension will be referred to as *scalar-extended vectors*, or as *extended vectors*, or simply as x-vectors.

The coordinate transformation matrix $\tilde{\Lambda}^A_B$ linking x-vectors in coordinates x^a to coordinates $\bar{x}^a$ is

$$\tilde{\Lambda}^A_B = \begin{pmatrix} 1 & 0_b \\ 0^a & \frac{\partial \bar{x}^a}{\partial x^b} \end{pmatrix} \tag{7}$$

where 0^a is the zero column matrix and 0_b a zero row matrix, respectively. A similar transformation matrix can be defined for x-covectors.

2.2. *Generalized Christoffel symbols*

Now consider a second-order x-tensor $\tilde{g}$, which can be represented as an array:

$$\tilde{g}_{AB} = \begin{pmatrix} \psi & A_b \\ A_a & g_{ab} \end{pmatrix}. \tag{8}$$

The resulting geometric object is similar to a second-order tensor of rank $(0, 2)$ in a five-dimensional space but is distinct for two reasons: First, these objects are functions of only four coordinates; Second, as a fiber bundle, the transition functions form a group isomorphic to $GL(4)$, whereas the transition functions in a five-dimensional space form the group $GL(5)$. The operator $\tilde{g}$ can be evaluated using the x-vector $\tilde{p}^A$, obtaining:

$$\tilde{g}(p^A, p^B) = e^2\psi + eA_a p^a + eA_b p^b + g_{ab}p^a p^b. \tag{9}$$

Now $\tilde{g}$ will be taken as the integrand of a functional to be maximized:

$$S = \int_A^B \tilde{g}(p^A, p^B) d\lambda \tag{10}$$

where λ is an arbitrary path parameter. Note that massless, uncharged particles such as photons would naturally be handled differently; they would not react with the vector or scalar fields except through their momentum and energy. Dividing by m^2 and varying with respect to the four-velocity results in the following equation:

$$\frac{dv^f}{d\lambda} + \frac{1}{2}g^{df}\left(\frac{\partial g_{db}}{\partial x^a} + \frac{\partial g_{ad}}{\partial x^b} - \frac{\partial g_{ab}}{\partial x^d}\right)v^a v^b + \frac{1}{2}\frac{e}{m}g^{df}\left(\frac{\partial A_d}{\partial x^a} - \frac{\partial A_a}{\partial x^d}\right)v^a$$
$$+ \frac{1}{2}\frac{e}{m}g^{df}\left(\frac{\partial A_d}{\partial x^b} - \frac{\partial A_b}{\partial x^d}\right)v^b - \frac{1}{2}\frac{e^2}{m^2}g^{df}\frac{\partial \psi}{\partial x^d} = 0. \tag{11}$$

This equation has the form of a geodesic, however, it has vector and scalar potentials in addition to the usual Christoffel symbols derived from the metric. The Christoffel symbols, however, can be redefined to take into account these new terms. Let zero correspond to a scalar index, and a lower case roman letter correspond to the four components of a vector. Further, define the partial derivative operator as:

$$\frac{\partial}{\partial \tilde{x}^A} = \begin{cases} 0, & \text{if } A = 0, \\ \dfrac{\partial}{\partial x^a}, & \text{if } A = a. \end{cases} \tag{12}$$

The reason for the definition of this operator is that there are only four coordinates: the dimension associated with a scalar field does not introduce another coordinate, nor are there paths in spacetime associated with that purely mathematical dimension. It facilitates the mathematical formalism, however, to define a place-holding zero operator. Now consider a general extended vector $v^A = \frac{e}{m} \oplus v^a$. The directional derivative can be written:

$$v^A \tilde{\nabla}_A v^F = v^A \frac{\partial v^F}{\partial \tilde{x}^A} + v^A \tilde{\Gamma}^F_{AB} v^B \tag{13}$$

where $\tilde{\nabla}_A$ is the extended connection. Expanding this expression, and taking it to be zero as in a geodesic equation, results in

$$\frac{dv^f}{d\lambda} + \frac{e^2}{m^2}\tilde{\Gamma}^F_{00} + \frac{e}{m} v^a \tilde{\Gamma}^F_{a0} + \frac{e}{m} v^b \tilde{\Gamma}^F_{0b} + \tilde{\Gamma}^F_{ab} v^a v^b = 0. \tag{14}$$

The nonzero Christoffel symbols can be obtained from a general expression of the geodesic equation.[7] Comparing the expression in Eq. (14) to the expression in Eq. (11), it is possible to read off the equivalent of the Christoffel symbols:

$$\tilde{\Gamma}^0_{AB} = 0, \tag{15}$$

$$\tilde{\Gamma}^f_{00} = -\frac{1}{2} g^{df} \frac{\partial \psi}{\partial x^d}, \tag{16}$$

$$\tilde{\Gamma}^f_{a0} = \tilde{\Gamma}^f_{0a} = \frac{1}{2} g^{df} \left(\frac{\partial A_d}{\partial x^a} - \frac{\partial A_a}{\partial x^d} \right). \tag{17}$$

$$\tilde{\Gamma}^f_{ab} = \Gamma^f_{ab}. \tag{18}$$

By construction, a theory having connection coefficients given by Eqs. (15)–(18) should have the correct particle dynamics, if solutions to the field equations generated from them are sufficiently similar to those of existing well-tested theories. Further, familiar mathematical forms associated with scalar, vector, and tensor forces are in evidence: the negative gradient of a scalar field, the Maxwell tensor, and the Christoffel symbols of GR. Using the extended Christoffel symbols, the next section develops the concept of extended curvature.

2.3. *The Ricci x-tensor*

The equivalent Ricci tensor, found by calculating a commutator of x-covariant derivatives on an x-vector, is given by the expression

$$\tilde{R}_{AB} = \frac{\partial \tilde{\Gamma}^C_{AB}}{\partial \tilde{x}^C} - \frac{\partial \tilde{\Gamma}^C_{AC}}{\partial \tilde{x}^B} + \tilde{\Gamma}^E_{AB}\tilde{\Gamma}^C_{CE} - \tilde{\Gamma}^E_{AC}\tilde{\Gamma}^C_{BE}. \tag{19}$$

Using the extended C-symbols, this equation generates three component equations, each of different tensorial order. The rank two tensor equation is

$$\tilde{R}_{ab} = \frac{\partial \Gamma^c_{ab}}{\partial x^c} - \frac{\partial \Gamma^c_{ac}}{\partial x^b} + \Gamma^e_{ab}\Gamma^c_{ce} - \Gamma^e_{ac}\Gamma^c_{be} \tag{20}$$

which is the usual Ricci tensor of GR. The vector component yields

$$\tilde{R}_{a0} = \frac{1}{2}\left(\frac{\partial F^c{}_a}{\partial x^c} + \Gamma^c_{cd}F^d{}_a - \Gamma^d_{ca}F^c{}_d\right) \tag{21}$$

with

$$F_{ba} = \frac{\partial A_b}{\partial x^a} - \frac{\partial A_a}{\partial x^b} \tag{22}$$

which is the covariant form of the left side of Maxwell's equation in curved spacetime. Finally, the scalar component is given by

$$\tilde{R}_{00} = -\frac{1}{2}\left[\frac{\partial}{\partial x^c}\left(g^{dc}\frac{\partial \psi}{\partial x^d}\right) + \Gamma^c_{ce}\left(g^{de}\frac{\partial \psi}{\partial x^d}\right)\right] - \frac{1}{4}g^{de}g^{fc}F_{dc}F_{fe} \tag{23}$$

which is a second-order scalar wave equation for ψ with the electromagnetic energy density acting as a source.

So in vacuum, the x-Ricci gives the equations of GR and electromagnetism, however they are coupled to a scalar field. That raises the concern that the scalar field could lead to predictions that are not supported by experiment. If, however, a coupling constant associated with the scalar field is sufficiently small, deviations might not be observable in most contexts while creating physical observables near singularities, such as cosmic inflation at the beginning of the universe. The only way to determine whether predictions of the theory are physical is to complete the theory by introducing the sources of the field and writing the equation connecting the Ricci x-tensor to the analog of stress-energy. After solving the system, predictions can be compared to observations.

2.4. *The extended stress-energy*

For systems of particles, the energy-momentum tensor is described as relating to the flux of momentum across a surface. If $N^a = nv^a$ is the flux number, where n is the number of particles and v^a the four-velocity, and $p^b = mv^b$ is the particle momentum, then the stress-energy can be represented as $N^a p^b = nmv^a v^b$. This formulation suggests a similar definition, taking into account the differences in the fundamental mathematical objects. By analogy, $N^A = n(q/m \oplus v^a)$ and $p^B = m(q/m \oplus v^b)$. The tensor product of those two forms yields:

$$\tilde{T}^{AB} = N^A p^B = \begin{pmatrix} n\frac{q^2}{m} & nqv^b \\ nqv^a & nmv^a v^b \end{pmatrix}. \tag{24}$$

So the natural generalization of the energy-momentum tensor yields the expected charge current for the Maxwell fields and the usual expression for the stress-energy of a swarm of particles. In addition, the source of the scalar field appears as the density of q^2/m. In general, the extended stress-energy tensor might be taken as

$$\tilde{T}^{AB} = \begin{pmatrix} \alpha\rho_\psi & \beta J^b \\ \beta J^a & \kappa T^{ab} \end{pmatrix} \tag{25}$$

where α, β, and κ are constants, and ρ_ψ is the density of quantity q^2/m, which may also be presumed proportional to the number density. Using the x-covariant derivative, the divergence of the x-stress-energy can be defined in the natural way:

$$Div(\tilde{T}) = \tilde{\nabla}_A \tilde{T}^{AB}. \tag{26}$$

For $B = 0$, the result is

$$\tilde{\nabla}_A \tilde{T}^{A0} = \beta \nabla_a J^a \tag{27}$$

whereas for $B = b$, the result is

$$\tilde{\nabla}_A \tilde{T}^{Ab} = \kappa \nabla_a T^{ab} + \beta g^{db} J^e F_{de} - \frac{\alpha}{2} g^{db} \frac{\partial \psi}{\partial x^d} \rho_\psi. \tag{28}$$

The ordinary divergence of the current density, $\nabla_a J^a$, is indeed zero, but it is not clear whether Eq. (28) should be zero, or not. This lack of clarity is likely due to the divergence operation being undefined on scalar fields. An alternate operator $\nabla_A^{alt} = 0 \oplus \nabla_a$, on the other hand, does return zero when operating on the extended stress-energy. In the cosmological trial exact solutions, using $\tilde{\nabla}_A$ as defined in Eq. (26) on both sides of the generalization of Einstein's equations results in a tautology. A tautology may be an acceptable generalization of Einstein's requirement because it does not generate a new, independent equation.

2.5. *X-covariant derivative of the x-metric*

The x-covariant derivative, as defined, does not result in zero when applied to the second rank x-tensor $\tilde{g}$. The computation of $\tilde{\nabla}_A \tilde{g}_{BC}$ yields:

$$\tilde{\nabla}_0 \tilde{g}_{BC} = \begin{pmatrix} \frac{\partial \psi}{\partial x^d} A^d & \frac{1}{2}\frac{\partial \psi}{\partial x^c} - \frac{1}{2} A^d F_{dc} \\ \frac{1}{2}\frac{\partial \psi}{\partial x^b} - \frac{1}{2} A^d F_{db} & 0 \end{pmatrix} \tag{29}$$

$$\tilde{\nabla}_a \tilde{g}_{BC} = \begin{pmatrix} \frac{\partial \psi}{\partial x^a} - A^d F_{da} & \frac{1}{2}\left(\frac{\partial A_c}{\partial x^a} + \frac{\partial A_a}{\partial x^c}\right) - \Gamma^\lambda_{ac} A_\lambda \\ \frac{1}{2}\left(\frac{\partial A_b}{\partial x^a} + \frac{\partial A_a}{\partial x^b}\right) - \Gamma^\lambda_{ab} A_\lambda & \frac{\partial g_{bc}}{\partial x^a} - \Gamma^\lambda_{ab} g_{\lambda c} - \Gamma^\lambda_{ac} g_{b\lambda} \end{pmatrix}. \tag{30}$$

For $\tilde{\nabla}_A \tilde{g}_{BC} = 0$ to hold true, the following four constraints must apply:

$$A^d \nabla_d \psi = 0, \tag{31}$$

$$\frac{\partial \psi}{\partial x^b} - A^d F_{db} = 0, \tag{32}$$

$$\frac{1}{2}\left(\frac{\partial A_b}{\partial x^a} + \frac{\partial A_a}{\partial x^b}\right) - \Gamma^\lambda_{ab} A_\lambda = 0, \tag{33}$$

$$\frac{\partial g_{bc}}{\partial x^a} - \Gamma^{\lambda}_{ab} g_{\lambda c} - \Gamma^{\lambda}_{ac} g_{b\lambda} = 0. \tag{34}$$

Taking the inner product of Eq. (32) with A^b shows that Eq. (31) follows from Eq. (32), so there are only three independent constraints: Eqs. (32), (33), and (34), the latter being the usual assumption in a four-dimensional spacetime. Equation (33) is equivalent to

$$\nabla_a A_d + \nabla_d A_a = 0. \tag{35}$$

Garfinkle[8] suggests, given Eq. (35), that A^a is a Killing field associated with a symmetry of the theory, as it satisfies the Killing equation. Further, Eq. (31) implies that ψ is constant in the direction of A^a, whereas Eq. (32) suggests that ψ is related to the magnitude of A^a. Garfinkle's insights may lead to an understanding of this theory when $\nabla_A \tilde{g}_{BC} = 0$.

Otherwise, Eq. (35) amounts to a constraint on the gauge of the electromagnetic field. In this theory, therefore, making the choice that the covariant derivative is zero automatically chooses that gauge. Note, however, that Eq. (35) implies that $\nabla_a A_d$ is an antisymmetric second rank tensor. Although some special solutions may exist, that constraint is likely too stringent. Further, in the absence of electromagnetic fields, these constraints require a constant scalar field.

Applying all these constraints may therefore result in only a trivial extension of GR. Whether that is always so is worthy of investigation. Here, the focus will be on applying only Eq. (34), as in standard GR. That suggests that while $\tilde{g}_{AB}$ may be considered an extension of the usual metric g_{ab}, it does not share all of its properties under the defined x-covariant derivative.

As in four-dimensional GR, a five-dimensional Kaluza–Klein spacetime has Christoffel symbols defined entirely in terms of the metric. Assuming a five-dimensional spacetime metric analogous in form to the x-metric is independent of the new fifth coordinate, and that it yields proper dynamics from a particle Lagrangian, leads to the same set of constraints. That strongly suggests that the original Kaluza–Klein theory may be untenable in principle.

3. Field Lagrangians and Their Variations

An article of faith in theoretical physics is that equations of physics must derive from Lagrangians. As described in this section, there are a number of choices of field Lagrangians that result in different predictions. The usual field Lagrangian giving a minimally-coupled massless scalar field, electromagnetic field, and gravitation can be written

$$\mathcal{L} = \int \alpha g^{ab} \nabla_a \psi \nabla_b \psi + \beta F_{ab} F^{ab} + g^{ab} R_{ab} \sqrt{-g}\, d^4x \tag{36}$$

where α and β are chosen so as to give suitable expressions for stress-energy contributions to the gravity field. Those constants are adjustable, and must be so, because experiments have not yet determined the magnitude of the contribution of those fields to gravitation.

An important observation is that the fields ψ and A_a in Eq. (36) are quadratic in their first derivatives. In contrast, the x-Ricci Lagrangians include second derivatives in ψ and A_a.

In some sense, Eq. (36) leads to an "already unified" field theory because its variation yields a single x-tensor equation. It is somewhat unsatisfactory because it involves cobbling together three completely separate theories. Using the theory of tensor multinomials, on the other hand, delivers three fields through the extension of curvature.

3.1. *Scalar forms and integration*

Here I present some operators useful in constructing Lagrangians. In addition, I address the question of integration.

3.1.1. *Scalar forms*

Given a general x-metric

$$\tilde{g}_{AB} = \begin{pmatrix} \psi & A_b \\ A_a & g_{ab} \end{pmatrix} \tag{37}$$

an inverse x-metric can be easily calculated:

$$\tilde{g}^{BC} = \begin{pmatrix} 1 & -A^c \\ -A^b & [g^{bc}(\psi - A_d A^d) + A^b A^c] \end{pmatrix} [\psi - A_d A^d]^{-1}. \tag{38}$$

As in standard GR, the inverse x-metric and x-metric can be used to raise and lower indices, and to create Lagrangian densities using the x-Ricci. There are other choices. Lowering and raising indices could also be effected with the operator

$$\tilde{H}_{AB} = \begin{pmatrix} 1 & 0_b \\ 0_a & g_{ab} \end{pmatrix} \tag{39}$$

and its inverse:

$$\tilde{H}^{AB} = \begin{pmatrix} 1 & 0^b \\ 0^a & g^{ab} \end{pmatrix}. \tag{40}$$

The $\tilde{H}$ operators would preserve the canonical mapping of standard GR for the components of the tensor multinomials. They can also create new x-tensors. Applying $\tilde{H}^{AC}$ and $\tilde{H}^{BD}$ to $\tilde{g}_{AB}$ results in the following *star operator*, g^{*CD}:

$$\tilde{g}^{*CD} = \tilde{H}^{AC}\tilde{H}^{BD}\tilde{g}_{AB} = \begin{pmatrix} \psi & A^d \\ A^c & g^{cd} \end{pmatrix}. \tag{41}$$

The operators $\tilde{g}^{AB}$, $\tilde{H}^{AB}$, and $\tilde{g}^{*AB}$ can be used in conjunction with $\tilde{R}_{AB}$ and other operators to construct various Lagrangian densities.

The most natural scalar for consideration in a Lagrangian-based theory is $\tilde{g}^{AB}\tilde{R}_{AB}$. However, $\tilde{R}_{AB}$ involves ψ, A_a, g_{ab}, and g^{ab}, rather than any of the inverse x-metric components. Consequently, there may be a mismatch between the inverse x-metric components of $\tilde{g}^{AB}$ and the components of $\tilde{R}_{AB}$. That suggests using the scalar $\tilde{g}^{*AB}\tilde{R}_{AB}$ may be a good alternative.

3.1.2. *Integration*

An important issue to address is that of integration. Because there are only four physical dimensions, a natural choice of volume element is $\sqrt{-g}\,d^4x$, where $g = \det(g_{ab})$. Intuitively, however, if the vector and scalar fields affect particle motion and are considered extensions of the metric, then the form $\sqrt{-\tilde{g}}\,d^4x$ may be more appropriate, where $\tilde{g} = \det(\tilde{g}_{AB})$. The latter choice also makes sense if the theory is considered a five-dimensional spacetime that restricts the component fields to four dimensions.

In effect, there may already be an implicit fifth dimension in this theory: the charge-to-mass ratio. Given $v^A = p^A/m$, it is evident in expressions such as

$$\tilde{g}(v^A, v^B) = \left(\frac{e}{m}\right)^2 \psi + \frac{e}{m} A_a v^a + \frac{e}{m} A_b v^b + g_{ab} v^a v^b \tag{42}$$

that e/m is analogous to a velocity component, but is constant and scalar-valued. It suggests a relationship with any path parameter λ of $q_m = (e/m)\lambda + C$, where C is a constant and q_m is a linear parameterization. These considerations suggest the use of $\sqrt{-\tilde{g}}$ in Lagrangians. Further investigation of this quantity as a possible discrete or approximately continuous fifth dimension would be of interest. Going forward, the form of the volume element will be considered provisional.

3.2. *Candidate Lagrangians*

Here I present several possible field Lagrangians and/or field equations and briefly comment on them. In many cases, the variational calculations are highly complex, with many of the complications resulting from the presence of the electromagnetic field. Any Lagrangian giving the fields ψ, A_a, and g_{ab} is possible, but if the vacuum field equations deviate too greatly from $\tilde{R}_{AB} = 0$, then observations would find measurable differences between the unified theory and the standard theories of Einstein and Maxwell. The breadth of possibilities is substantial, and ideally, some principle or set of physical observations will point to one formulation over all the others.

3.2.1. *"Already-unified" Lagrangian*

Given the stress-energy of Eq. (25), a field equation might take a form such as

$$\tilde{R}_{AB} - \frac{1}{2} R g_{ab} = \tilde{T}_{AB} \tag{43}$$

where R is the ordinary Ricci scalar of four-dimensional GR and the components of the x-stress tensor are lowered with the operator $\tilde{H}_{AB}$. This equation yields Einstein's theory of GR together with Maxwell's equations and a scalar field equation. Further, a divergence operation using $\tilde{H}^{AC}\nabla_C^{alt}$ gives zero on both sides of this equation.

An approximate Lagrangian for Eq. (43) might have the form of Eq. (36) with an additional term:

$$\mathcal{L} = \int \alpha g^{ab}\nabla_a\psi\nabla_b\psi + (\beta + \alpha\psi)F_{ab}F^{ab} + g^{ab}R_{ab}\sqrt{-g}\, d^4x. \tag{44}$$

The constants α and β, after conversion to dimensionless variables, can be considered scale factors as well as constants yielding proper stress-energy contributions. The extra term in the equation for A_a would not affect vacuum solutions, and if $\beta >> \alpha$, then the term would be negligible in the presence of currents. Although it may give a similar x-tensor equation, Eq. (44) lacks an explicit x-Ricci. Nonetheless, Eq. (44) would lead to expressions similar to those in Eq. (43) and have good chances of preserving GR and electromagnetism.

3.2.2. *LMF Lagrangian*

Another option in developing a Lagrangian for Eq. (43) is to use a Lagrange multiplier field (*LMF*), Ω^{AB}. Here, the constraint equation is given by Eq. (43), itself, and constitutes the full Lagrangian density:

$$\mathcal{L} = \int \Omega^{AB}\left(\tilde{R}_{AB} - \frac{1}{2}Rg_{ab} - \tilde{T}_{AB}\right)\sqrt{-g}\, d^4x. \tag{45}$$

The variation with respect to Ω^{AB} immediately yields Eq. (43). Variations with respect to the metric, the electromagnetic, and scalar fields result in equations for the various components of Ω^{AB}. That said, Ω^{AB} is not easy to find explicitly from the derived equations. The study of special cases may lead the way to discovering an explicit, exact Lagrangian using this method.

3.2.3. *Pseudo-five-dimensional GR Lagrangian*

Equation (43) can be modified by replacing Rg_{ab} with $\tilde{R}\tilde{g}_{AB}$:

$$\tilde{R}_{AB} - \frac{1}{2}\tilde{R}\tilde{g}_{AB} = \tilde{T}_{AB}. \tag{46}$$

The field equations of Eq. (46) can be derived from the usual field Lagrangian for five-dimensional GR. To yield the Christoffel symbols corresponding to the x-Ricci of Eqs. (20)–(23), however, the system would have to be independent of the fifth coordinate and the constraints of Eqs. (32)–(34) would have to hold, probably resulting in a trivial extension of four-dimensional GR, where the new fields are constant. It would therefore be inappropriate to invoke the GR Lagrangian as leading to Eq. (46). If the equation, nonetheless, is predictive and otherwise sound, it could be regarded as a Lagrangian-free theory, or in some cases as a zeroth-order equation in perturbation.

Lagrangian-derived field equations represent extrema in a function space. A unification of fields may require field equations falling between those extrema. As in the previous case, however, it is possible to write a Lagrangian in terms of Lagrange multiplier fields:

$$\mathcal{L} = \int \Omega^{AB}\left(\tilde{R}_{AB} - \frac{1}{2}\tilde{R}\tilde{g}_{AB} - \tilde{T}_{AB}\right)\sqrt{-\tilde{g}}\, d^4x. \tag{47}$$

In principle, therefore, Eq. (47) provides an implicit Lagrangian for the field equations given by Eq. (46). Variation will yield the desired field equations and equations determining the necessary form of Ω^{AB}, although those equations are not easy to solve. I provisionally prefer the Lagrangian of Eq. (47) and that of Eq. (45) because they leave four-dimensional GR and the electromagnetic field largely intact. Equation (47) appears to offer new pathways to modeling certain unexplained physical phenomena, with either $\sqrt{-g}$ or $\sqrt{-\tilde{g}}$ in the volume element, and with either $\tilde{g}_{AB}$ or g_{ab} in the factor multiplied by $\tilde{R}$.

3.2.4. *X-metric Lagrangian*

The most direct try for an explicit Lagrangian-based theory is

$$\mathcal{L} = \int \tilde{g}^{AB}\tilde{R}_{AB}\sqrt{-g}\, d^4x \tag{48}$$

where $\tilde{g}^{AB}$ is the inverse x-metric. The idea that the metric of spacetime is somehow extended to include vector and scalar parts is attractive and geometrically fascinating. Formally, this Lagrangian is the same as for five-dimensional GR, but the integration is over a four-manifold and the constraints not invoked. As discussed previously, $\sqrt{-g}$ might be replaced by $\sqrt{-\tilde{g}}$. Either way, the resulting field equations will include an x-Ricci, but will also involve numerous other new terms.

That a single quantity in Eq. (48), the x-Ricci tensor, contains all three fields can create unexpected difficulties. Variation of such Lagrangians shows how the electromagnetic or scalar field interacts with the gravitational field. In standard GR, those separate fields create stress-energies that by assumption have an effect on gravitation, although coupling constants often weaken that effect. The resulting theory can often be studied through perturbation theory. When those individual fields are already coupled into a single, indivisible field, it's less clear how to proceed.

One method of proceeding would be to place coupling constants in front of the various terms in the expanded contraction, as is done in Eq. (36). That would weaken the concept of three fields given by a single mathematical operator, but allow more control over the mathematical models. Because an x-tensor has a scalar portion, it is not difficult to create second-rank x-tensors that introduce scale factors through contraction. For example, first decompose $\tilde{R}_{AB}$ into two x-tensors:

$$\tilde{R}_{AC} = \tilde{D}^1_{AC} + \tilde{D}^2_{AC} = \begin{pmatrix} \tilde{R}_{00} & 0_c \\ 0_a & \tilde{R}_{ac} \end{pmatrix} + \begin{pmatrix} 0 & \tilde{R}_{0c} \\ \tilde{R}_{a0} & 0_{ac} \end{pmatrix}. \tag{49}$$

Now create a new x-tensor:

$$\tilde{R}^{\text{mod}}_{AB} = \tilde{D}^1_{AC}\tilde{S}^C{}_B + \tilde{D}^2_{AC}\tilde{P}^C{}_B \tag{50}$$

where

$$\tilde{S}^C{}_B = \begin{pmatrix} \alpha & 0_b \\ 0^c & \delta^c_b \end{pmatrix} \tag{51}$$

$$\tilde{P}^C{}_B = \beta \begin{pmatrix} 1 & 0_b \\ 0^c & \delta^c_b \end{pmatrix}. \tag{52}$$

Another way to insert user-adjustable constants involves redefining all fields as scale factors multiplied by the fields, and then defining dimensionless quantities. For example, put $\psi \to a\phi$ and $A_a \to bU_a$, where a and b are constants carrying the units. Then make all quantities and coordinates dimensionless and adjust the x-stress-tensor constants as necessary.

A further difficulty of this Lagrangian is that the field equations it generates are extremely complex, raising concerns that they may make predictions at odds with well-tested theories. Such problems can be mitigated to an extent with the introduction of scale factors.

The variation of $\tilde{R}_{AB}$ can be taken assuming geodesic coordinates, so that the ordinary Christoffel symbols vanish, $\Gamma^a_{bc} = 0$. Variation of $\tilde{R}_{a0}$ and $\tilde{R}_{00}$ are both nonzero. The expanded variation is

$$\begin{aligned} \delta\mathcal{L} = \int & \delta\tilde{g}^{AB}\tilde{R}_{AB} + \tilde{g}^{00}\delta\tilde{R}_{00} + \tilde{g}^{a0}\delta\tilde{R}_{a0} + \tilde{g}^{0b}\delta\tilde{R}_{0b} \\ & - \frac{1}{2} g_{ab}\tilde{R}\delta g^{ab}\sqrt{-g}\, d^4x = 0 \end{aligned} \tag{53}$$

where $\delta R_{ab} = 0$ has already been dropped. Note that the volume element results in a term containing δg^{ab}, which must be written as a combination of $\delta\tilde{g}^{AB}$. For that reason, using $\sqrt{-\tilde{g}}$ in the volume element actually leads to simpler equations. One method of calculating δg^{ab} involves using the fact that $\delta(g_{ab}g^{bc}) = 0$ and $\delta(\tilde{g}_{AB}\tilde{g}^{BC}) = 0$ both hold, resulting in the two formulas:

$$\delta g^{ab} = -g^{ac}g^{db}\delta g_{cd} \tag{54}$$

and

$$\delta\tilde{g}_{CD} = -\tilde{g}_{CK}\tilde{g}_{LD}\delta\tilde{g}^{KL}. \tag{55}$$

Simply write the quantity δg^{ab} in terms of lowered indices with Eq. (54), then identify g_{cd} with $\tilde{g}_{cd}$, which is identical,and use Eq. (55) to raise the x-tensor indices.

The other, more intuitive method is to write g^{ab} in terms of $\tilde{g}^{AB}$, as follows:

$$g^{ab} = \tilde{g}^{ab} - \frac{A^a A^b}{\psi - A_c A^c} = \tilde{g}^{ab} - \tilde{g}^{a0}\tilde{g}^{0b}(\tilde{g}^{00})^{-1} \tag{56}$$

so that

$$\delta g^{ab} = \delta\tilde{g}^{ab} + A^b\delta\tilde{g}^{a0} + A^a\delta\tilde{g}^{0b} + A^a A^b\delta\tilde{g}^{00}. \tag{57}$$

Both methods yield the same answer. In other variational calculations, it may also be necessary to write δA_a as a linear combination of $\delta\tilde{g}_{AB}$. It must be symmetrized, because A_a could be either $\tilde{g}_{a0}$ or $\tilde{g}_{0a}$. The result, using Eq. (55), is

$$\begin{aligned}\delta A_a = \frac{1}{2}\delta(\tilde{g}_{a0} + \tilde{g}_{0a}) &= -\psi A_a\delta\tilde{g}^{00} - \frac{1}{2}(A_b A_a + \psi g_{ba})\delta\tilde{g}^{b0} \\ &\quad - \frac{1}{2}(A_c A_a + \psi g_{ca})\delta\tilde{g}^{0c} - A_b g_{ca}\delta\tilde{g}^{bc}.\end{aligned} \tag{58}$$

With these and similar results, it is now possible to write the full set of field equations for the x-metric Lagrangian, either using the given volume element or $\sqrt{-\tilde{g}}\,d^4x$. Those field equations are of interest, but are reasonably tractable only in the absence of an electromagnetic field and will not be pursued here.

The field equations may be simplified by noting that $-\frac{1}{4}\tilde{g}^{00}F_{cd}F^{dc}$ is a scalar, and that an arbitrary multiple of it may be added to the Lagrangian. That multiple could either remove the term entirely from the Lagrangian or reduce it sufficiently for perturbation theory. That procedure, called "bleaching" the Lagrangian, simplifies the resulting field equations considerably, both here and in the Star Lagrangian. Given that the term naturally arises from a consideration of particle dynamics, however, strongly suggests it should be retained.

3.2.5. *Star Lagrangian*

The advantage of using the Star Lagrangian is that the variation process and the infinitesimals are very similar to those of well-known theories. The Lagrangian in vacuum can be taken as

$$\mathcal{L} = \int \tilde{R}^*\sqrt{-g}\,d^4x \tag{59}$$

where $R^* = \tilde{g}^{*AB}\tilde{R}_{AB}$. Expanding the contraction, obtain

$$\begin{aligned}\mathcal{L} = \int \psi\left(-\frac{1}{2}\nabla^d\nabla_d\psi - \frac{1}{4}F_{cd}F^{dc}\right) + A^a\left(\frac{1}{2}\nabla_d F^d{}_a\right) \\ + A^b\left(\frac{1}{2}\nabla_d F^d{}_b\right) + R\sqrt{-g}\,d^4x.\end{aligned} \tag{60}$$

Equation (60) involves second derivatives of the fields ψ and A_a, each contracted with their contravariant counterparts, whereas Eq. (36) involves squares of first derivatives for ψ and A_a. Unlike the x-metric Lagrangian, there is no need to convert to combinations of $\delta\tilde{g}^{AB}$. Choosing $\sqrt{-\tilde{g}}$ in the Lagrangian, however, would result in having to write $\delta\tilde{g}^{AB}$ in terms of $\delta\psi$, δA^a, and δg^{ab}.

As with the x-metric Lagrangian, the electromagnetic energy density in $\tilde{R}_{00}$ can be bleached out by adding a multiple of the scalar $\frac{1}{4}\psi F_{cd}F^{dc}$ to the Lagrangian, somewhat simplifying the resulting field equations.

4. Trial Solutions of the Field Equations

To give the reader a feel for the modeling possibilities of this formalism, I present some illustrative solutions to Eq. (46). The eventual choice of Lagrangian or other system of equations will depend on what form of the theory can best be put in agreement with observations and answer unsolved questions.

Physically, the scalar source ρ_ψ acts in concert with Einstein's equations to produce the equivalent of a cosmological constant, although it need not be constant to do that. By the derivation of the stress-energy, it is related to the distribution of the electric charge squared, but other interpretations are possible. Here, I will take $\rho_\psi \to \mu\psi$ for the type (0, 2) x-tensor equations, with μ considered constant, although in general μ could depend on the coordinates.

4.1. *Static vacuum solution*

With the given form of the x-Ricci, it's natural to first determine a possible vacuum solution,

$$\tilde{R}_{AB} = 0 \tag{61}$$

for the case of static spherical symmetry, assuming a trial x-metric of

$$d\tilde{s}^2 = -e^\nu\, dt^2 + e^\lambda\, dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + 2A_4\, dt + \psi \tag{62}$$

where direct sums are understood, as appropriate. It is clear that the Schwarzschild solution and $A_4 = a/r$, where a is a constant, immediately follow. That leaves the static scalar field equation, which reads

$$-\frac{1}{2}e^{-\lambda}\left(\psi'' + \left(\frac{2}{r} + \frac{\nu'}{2} - \frac{\lambda'}{2}\right)\psi'\right) - \frac{1}{2}e^{-\nu-\lambda}A_4'^2 = 0. \tag{63}$$

With $\nu = -\lambda$ and $e^\nu = 1 - 2m/r$, this equation can be readily solved, giving

$$\psi(r) = \frac{a^2/2m}{r} + \left(\frac{b}{2m} + \frac{a^2}{4m^2}\right)\ln\left(1 - \frac{2m}{r}\right) + \psi_\infty \tag{64}$$

where a is a constant associated with the electrostatic field, and b is an arbitrary constant. Eq. (64) yields an additional attractive inverse square force if $r \gg 2m$ and $b > 0$. In principle, it should be possible to measure b experimentally, although if it couples to e^2 the effect might be unobservable except under extreme circumstances.

4.2. *Plane-symmetric cosmologies*

For this calculation, the field equation is again Eq. (46) and raising and lowering is performed using the x-metric, $\tilde{g}_{AB}$. The stress-energy is taken to be a plane-symmetric perfect fluid. Note here that the Lagrangian involves $\sqrt{-\tilde{g}}$, as it would in Kaluza–Klein theory, instead of $\sqrt{-g}$. The distinction is that $\sqrt{-\tilde{g}}$ depends only on four coordinates. Using $\sqrt{-g}$ yields the usual cosmologies when $\mu = 0$. Regardless, a factor of $\sqrt{\tilde{g}_{00}}$ could always be appended to the Lagrangian, if desired. The x-metric is

$$d\tilde{s}^2 = -d\tau^2 + a^2(\tau)(dx^2 + dy^2 + dz^2) + \psi(\tau). \tag{65}$$

The field equations then become:

$$\tilde{G}_{00} = \frac{1}{2}\left(\ddot{\psi} + 3\frac{\dot{a}}{a}\dot{\psi}\right) - \frac{1}{2}\psi(\psi^{-1}\tilde{R}_{00} + R) = \mu\psi, \tag{66}$$

$$\tilde{G}_{\tau\tau} = 3\frac{\dot{a}^2}{a^2} + \frac{1}{2}\psi^{-1}\tilde{R}_{00} = \kappa\rho, \tag{67}$$

$$\tilde{G}_{xx} = -2a\ddot{a} - \dot{a}^2 - \frac{1}{2}a^2\psi^{-1}\tilde{R}_{00} = \kappa a^2 P \tag{68}$$

with $\tilde{G}_{zz} = \tilde{G}_{yy} = \tilde{G}_{xx}$. Note that in Eqs. (67) and (68), $\tilde{R} = \psi^{-1}\tilde{R}_{00} + R$, with $R = g^{ab}R_{ab}$, was split into two terms, with the R term in each case combined with similar terms involving $a(\tau)$. For reference, it also holds that

$$R = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) \tag{69}$$

where R is the ordinary scalar curvature. Further, an additional equation can be generated by requiring that the divergence of the left side equals the divergence of the other side. Raising with the inverse x-metric, this results in:

$$-\frac{1}{2}\psi^{-1}\dot{\psi}\left(3\frac{\dot{a}^2}{a^2} + F\right) - \frac{dF}{d\tau} = -\frac{\kappa}{2}\psi^{-1}\dot{\psi}\rho - \kappa\left(\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P)\right) \tag{70}$$

where $F = \frac{1}{2}\psi^{-1}\tilde{R}_{00}$. Now, using Eq. (67), its derivative, and Eq. (68) divided by a^2, it can be shown that Eq. (70) is a tautology, so going forward, it can be disregarded.

4.2.1. *Vacuum cosmology with scalar source*

Under these assumptions, the field equations become:

$$\tilde{G}_{00} = \frac{1}{2}\left(\ddot{\psi} + 3\frac{\dot{a}}{a}\dot{\psi}\right) - \frac{1}{2}\psi(\psi^{-1}\tilde{R}_{00} + R) = \mu\psi, \tag{71}$$

$$\tilde{G}_{\tau\tau} = 3\frac{\dot{a}^2}{a^2} + \frac{1}{2}\psi^{-1}\tilde{R}_{00} = 0, \tag{72}$$

$$\tilde{G}_{xx} = -2a\ddot{a} - \dot{a}^2 - \frac{1}{2}a^2\psi^{-1}\tilde{R}_{00} = 0 \tag{73}$$

with $\tilde{G}_{zz} = \tilde{G}_{yy} = \tilde{G}_{xx}$.

After dividing Eq. (73) by a^2, Eqs. (72) and (73) can be summed, obtaining

$$2\frac{\dot{a}^2}{a^2} - 2\frac{\ddot{a}}{a} = 0 \tag{74}$$

which gives the de Sitter solution:

$$a(\tau) = c_0 e^{k\tau}. \tag{75}$$

With essentially two different equations for ψ, consistency of this solution requires $\mu = -9k^2$. Equation (71) now gives a solution for ψ:

$$\ddot{\psi} + 3k\dot{\psi} + 12k^2\psi = 0 \tag{76}$$

resulting in two complex roots, which can be converted to a negative exponential times a real cosine:

$$\psi = \psi_0 e^{-\frac{3}{2}k\tau}\cos\left(\frac{\sqrt{39}}{2}k\tau\right). \tag{77}$$

Exactly the same solution, with $\mu = -6k^2$, results if Eq. (46) features a term $\frac{1}{2}\tilde{R}g_{ab}$ instead of $\frac{1}{2}\tilde{R}\tilde{g}_{AB}$.

4.2.2. *Dust cosmologies*

In this case $P = 0$, so the field equations become:

$$\tilde{G}_{00} = \frac{1}{2}\left(\ddot{\psi} + 3\frac{\dot{a}}{a}\dot{\psi}\right) - \frac{1}{2}\psi(\psi^{-1}\tilde{R}_{00} + R) = \mu\psi, \tag{78}$$

$$\tilde{G}_{\tau\tau} = 3\frac{\dot{a}^2}{a^2} + \frac{1}{2}\psi^{-1}\tilde{R}_{00} = \kappa\rho, \tag{79}$$

$$\tilde{G}_{xx} = -2a\ddot{a} - \dot{a}^2 - \frac{1}{2}a^2\psi^{-1}\tilde{R}_{00} = 0 \tag{80}$$

with $\tilde{G}_{zz} = \tilde{G}_{yy} = \tilde{G}_{xx}$. From Eq. (80), it follows that

$$\frac{1}{2}\psi^{-1}\tilde{R}_{00} = -2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}. \tag{81}$$

Substituting the expression for $\frac{1}{2}\psi^{-1}\tilde{R}_{00}$ in Eq. (78), obtain

$$\ddot{\psi} + 3\frac{\dot{a}}{a}\dot{\psi} + \left(2\frac{\ddot{a}}{a} + 4\frac{\dot{a}^2}{a^2} + 2\mu\right)\psi = 0. \tag{82}$$

Equation (81) can be written as:

$$\ddot{\psi} + 3\frac{\dot{a}}{a}\dot{\psi} + \left(-8\frac{\ddot{a}}{a} - 4\frac{\dot{a}^2}{a^2}\right)\psi = 0. \tag{83}$$

Subtracting Eq. (83) from Eq. (82) yields:

$$\left(5\frac{\ddot{a}}{a} + 4\frac{\dot{a}^2}{a^2} + \mu\right)\psi = 0. \tag{84}$$

If μ is taken to be constant, there are several solutions depending on whether μ is positive, negative, or zero, and depending on the sign of an integration constant.

Case 1: $\mu = 0$

It is straightforward to integrate Eq. (84) and find that

$$a(\tau) = (k\tau + c_0)^{5/9}. \tag{85}$$

After setting $u = k\tau + c_0$, the equation for ψ turns out to be an Euler equation:

$$\psi'' + \frac{5}{3}u^{-1}\psi' + \frac{20}{27}u^{-2}\psi = 0. \tag{86}$$

This equation can be converted into a linear equation by putting $z = \ln(u)$, resulting in

$$\frac{d^2\psi}{dz^2} + \frac{2}{3}\frac{d\psi}{dz} + \frac{20}{27}\psi = 0. \tag{87}$$

The general solution, after choosing the origin with $c_0 = 0$, is elementary:

$$\psi = \psi_0 e^{-\frac{1}{3}k\tau}\cos\left(\sqrt{17/27}\ln(k\tau)\right). \tag{88}$$

Case 2: $\mu \neq 0$

If $\mu \neq 0$ and is constant, it is tempting to guess a simple exponential form, but that would imply $\rho = 0$, returning the vacuum case. Instead, the Eq. (84) must be solved in more generality. Dividing the equation by five and multiplying through by $a^{9/5}$ results in:

$$\frac{d}{d\tau}(a^{4/5}\dot{a}) = -\frac{1}{5}\mu a^{9/5}. \tag{89}$$

Now, multiplying both sides by $a^{4/5}\dot{a}$, it is possible to arrive at a simple antiderivative for τ, which can then be inverted to obtain the expansion factor, $a(\tau)$:

$$\int \frac{a^{4/5}da}{\left(-\frac{\mu}{9}a^{18/5} + C\right)^{1/2}} = \pm(\tau - \tau_0). \tag{90}$$

The positive sign on $(\tau - \tau_0)$ will be taken going forward. Some choices of the constants will result in contradictions, such as $\rho = 0$ or an oscillating solution admitting negative values of $a(\tau)$. However, two such solutions for the scale factor $a(\tau)$ are of interest. The choices $C > 0$ and $\mu < 0$ yield

$$a(\tau) = \alpha \sinh^{5/9}(\beta\tau) \tag{91}$$

where $\alpha = (9|C|/|\mu|)^{5/18}$ and $\beta = 3|\mu|^{1/2}/5$ are both constants. The solution of Eq. (91) is singular at the origin, has positive energy density, and inflates exponentially. The choices $C < 0$, $\mu < 0$ yield

$$a(\tau) = \alpha \cosh^{5/9}(\beta\tau) \tag{92}$$

which is non-singular at the origin, has a positive energy density and again, inflates exponentially thereafter. The equations for the scalar field are linear with hyperbolic function coefficients, and will not be presented, here.

So the source $\mu\psi$ of the scalar field equation, as shown here, assumes the same role as a cosmological constant. That is also true in the static, spherically symmetric case. There are two important differences: First, μ need not be constant; Second, the source of a field contributes to the dynamics of the system. Intuitively, $\mu\psi$ is related to the density of charge squared, although some other interpretation may be required for cosmological solutions.

5. Concluding Remarks

In this paper, I have demonstrated that there is a relationship between the idea of a nonlinear differential operator on spacetime and the classical fields of physics. By using a simple variational principle, it's possible to develop GR, electromagnetism, and a quantum mechanical scalar field as parts of a single entity, a tensor multinomial associated with a subalgebra of the universal covering algebra of tensors in four spacetime dimensions. Further, the formalism yields a new way of introducing the equivalent of a cosmological constant, including the possibility of a "time-dependent cosmological constant", i.e. a cosmological function, as called for recently by James Peebles.[9]

The mathematical ideas here could readily be carried out with spinors, as well, which might yield some new insights. Three fields could be represented as a single mathematical object by the direct sum of a spin-one-half spinor, a spin-one spinor, and a two-spinor.

This theory could be of interest to those seeking explanations or models for cosmic inflation, dark matter, or dark energy, the "cosmic trifecta". The theory is not complete as it stands, because it is unclear which of the many variations connects best to current theory and also makes correct predictions concerning those three unexplained phenomena. The field equations can be derived and tested, however, and through that process it may be possible to determine the best way forward.

Acknowledgments

I thank Jim Ipser of the University of Florida, my doctoral thesis advisor, for three decades of periodic discussions on the subject of this article, and for reading and extensively commenting on this paper over the last year. Kip Thorne, Demetrios Christodoulu, David Garfinkle, and especially Fred Elston (deceased) provided many valuable comments on earlier versions of the theory. Garfinkle and Omer Farooq read and critiqued the current manuscript. I offer special thanks to Drs. K. Phua and Han Sun for requesting and encouraging the paper's completion. I dedicate this work to my late parents, Janet and James "Gentleman Jim" Vuille.

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