RELATIVISTIC COULOMB SCATTERING*

By E. H. de Groot** and Th. W. Ruijgrok

Institute for Theoretical Physics, Princetonplein 5, P.O. Box 80.006, 3508 TA Utrecht, The Netherlands

(Received August 7, 1985)

Despite the extensive literature about Coulomb scattering a few problems remain to be solved. Especially the divergence of the phase of the Coulomb wave function is a source of difficulties. Although they have been overcome, not only in the Schrödinger equation, but also in the Dirac equation, a correct treatment in the framework of a relativistic two-body theory was still lacking. Recoil effects in particular were never taken into account, although for high values of the nuclear charge they might be important. Z-values close to one hundred are beginning to play a role in high-energy scattering of heavy ions, so that a relativistic two-body theory is called for. Of the many existing quasi-potential theories which could be used for this purpose, we decided to choose the one proposed by us ten years ago. Since the theory can be cast into a form which is almost identical to the nonrelativistic theory, it is possible to take advantage of the known exact solution for this case. In the last section we present our results and compare them with those of the Dirac equation. Also the difference in cross sections for positively and negatively charged projectiles is mentioned there.

PACS numbers: 11.10.Qr

1. Introduction

When trying to give a correct theory for Rutherford scattering it is not easy to avoid all pitfalls. It is known that the exact scattering amplitude differs from the Born amplitude by a mere phase factor, so that lowest order perturbation theory already gives the exact expression for the differential cross section. However, the calculation of this phase factor, using an expansion in partial waves, leads to a divergent series. The history of the efforts to overcome or to ignore this difficulty was written by Marquez [1].

Another way to obtain the correct phase factor is to treat the Coulomb potential as the limit of a Yukawa potential with infinite range. This method was first used by Dalitz [2], who showed for the Dirac equation that divergences in the perturbation series should be considered as arising from the expansion of a phase factor, with a phase which

* A preliminary version of this paper was presented by one of us (Th.W.R.) at the XXV Cracow School of Theoretical Physics in Zakopane, Poland, June 2–14, 1985.

** On leave of absence from the University of Bielefeld, GFR.

approaches infinity with the range of the potential. Therefore this infinity is harmless. For the Klein-Gordon equation the same was shown by Kang and Brown [3].

Still another method to obtain the scattering amplitude was given by Schwinger [4], who derived an exact formula for the Green's function, which is finite as long as the energy is off-shell. Later in this paper we will use Schwinger's method for the scattering amplitude.

In the approaches mentioned so far the reduced mass was used to account for the two-body character of the problem. For particles described by the Schrödinger equation this is certainly correct. For the Dirac and Klein-Gordon equation, however, it is doubtful whether this procedure gives a good approximation, especially for high-energy scattering of heavy ions, when recoil effects may become important.

Therefore a relativistic two-body theory is called for. Although many of such so called quasi-potential theories exist, we prefer the formulation we introduced many years ago [5]. A detailed discussion of this theory was presented recently [6] and we will therefore restrict ourselves to a short exposition of the main features. This will be the content of the next section.

So far the theory was applied to the effect of unitarity on multiparticle production [5], to the fine structure of the hydrogen spectrum [7] and to the binding of electrons to heavy ions [8].

In the present paper we will use the same theory to describe relativistic Coulomb scattering of two charged particles. Special attention will be paid to the problem of the infinite phase.

2. *Resumé of RQM*

In this section we will give a brief exposition of our theory [5, 6] for relativistic quantum mechanical systems, which we call RQM.

Let $M_{\alpha\beta}$ be the scattering amplitude for a transition from a state β to a state α. In this paper we will restrict ourselves to the case where not only β but also α is a two-particle state, given for instance by the four-momenta $\alpha = (k_1, k_2)$. A summation over states will be abbreviated as

$$\int_\alpha \dots = \int dk_1 dk_2 \delta(k_1^2 - M^2)\theta(k_1^0)\delta(k_2^2 - m^2)\theta(k_2^0) \dots, \tag{1}$$

so that the particles always remain on their mass shell, even for intermediate states.

The scattering amplitudes are normalised in such a way that the total cross section is given by

$$\sigma(\beta) = \frac{(2\pi)^4}{2\sqrt{\lambda(s, M^2, m^2)}} \int_\alpha |M_{\alpha\beta}|^2 \delta_4(P_\alpha - P_\beta), \tag{2}$$

where P_α and P_β are the total four-momentum of α and β, where $\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2xz$ and where $s = (P_\alpha + P_\beta)^2$.

Unitarity of the S-matrix is expressed by

$$M^*_{\alpha\beta} - M_{\beta\alpha} = 2\pi i \int_\gamma M^*_{\gamma\beta} M_{\gamma\alpha} \delta_4(P_\gamma - P_\alpha) \quad \text{for} \quad P_\alpha = P_\beta. \tag{3}$$

As the Lorentz invariant generalisation of the Lippmann-Schwinger equation we now choose (P is the four-momentum of the initial state):

$$M_{\beta\alpha}(P) = V_{\beta\alpha} - \int_\gamma V_{\beta\gamma} L_\gamma(P) M_{\gamma\alpha}(P), \tag{4}$$

where the potential $V_{\beta\gamma}$ and the propagator $L_\gamma(P)$ are Lorentz invariant and must be chosen in such a way that the nonrelativistic limit can be recovered and the potential is hermitian $V^*_{\beta\gamma} = V_{\gamma\beta}$. Also the unitarity in the form of Eq. (3) must be a consequence of Eq. (4). This can be guaranteed by choosing $L_\gamma(P)$ properly. In order to show this we write Eq. (4) as $M = V - VLM$, with $V^+ = V$. From this follows

$$M^+ - M = -M^+L^+V + V^+LM = -M^+L^+(M+VLM) + (M^+ + M^+L^+V)LM$$
$$= M^+(L - L^+)M.$$

This is just the expression of Eq. (3) for unitarity, provided the imaginary part of the propagator is given by

$$\operatorname{Im} L_\gamma(P_\alpha) = \pi\delta_4(P_\gamma - P_\alpha). \tag{5}$$

In order to satisfy this relation and at the same time to fix the real part, we choose the following manifestly Lorentz invariant dispersion relation for the propagator

$$L_\gamma(P) = \int_0^\infty \frac{2x\,dx}{x^2 - 1 - i\varepsilon}\, \delta_4(P_\gamma - xP). \tag{6}$$

Since the total velocity of the state is equal to the three-momentum divided by the energy, we see from the δ_4-function in Eq. (6) that

$$\vec{V}_\gamma = \vec{P}_\gamma / P^0_\gamma = x\vec{P}/xP^0 = \vec{P}/P^0 = \vec{V}.$$

Therefore the intermediate states γ in Eq. (4) all have the same velocity as the initial state. This conservation of velocity replaces the conservation of three-momentum in other quasi--potential equations. In figure 1 we show in which respect the sets of allowed intermediate states for this relativistic theory and for the nonrelativistic case are different from each other. It also demonstrates that the nonrelativistic theory is recovered when the velocity of light goes to infinity.

In order to define the potential as a function of relativistic invariant variables we consider Fig. 2. Usually the square of the momentum transfer in the upper and in the lower vertex are defined by $t_1 = (q_1 - k_1)^2$ and $t_2 = (q_2 - k_2)^2$. If in the interaction the four--momentum is conserved, then t_1 and t_2 are equal. In our approach, however, we do not

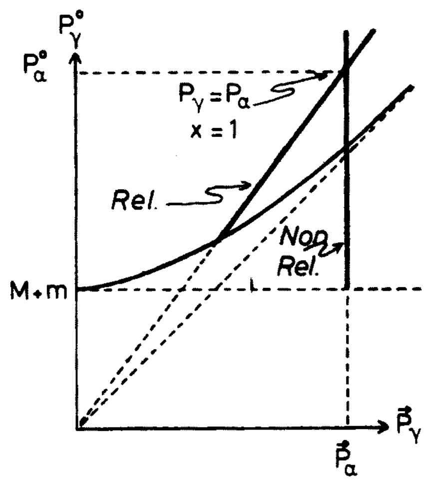

Fig. 1. The heavy lines show the allowed momenta in the relativistic and in the nonrelativistic case. $(M+m)^2 = \operatorname{Min}_{\gamma} P_{\gamma}^2$

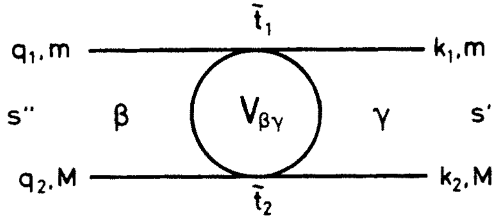

Fig. 2. Interaction as occurring in Eq. (4). The intermediate state γ is off four-momentum shell

have energy-momentum conservation in the intermediate states and consequently $t_1 \neq t_2$. Instead we keep the particles on the mass shell and go off the four-momentum shell in a way which treats energy and momentum on the same footing. According to the new conservation law the velocity of the intermediate state is the same as of the initial state. This is expressed by $k_1 + k_2 = x(q_1 + q_2)$. From this follows

$$\sqrt{s'} \equiv \sqrt{(k_1+k_2)^2} = x\sqrt{(q_1+q_2)^2} \equiv x\sqrt{s''},$$

so that

$$\frac{k_1+k_2}{\sqrt{s'}} = \frac{q_1+q_2}{\sqrt{s''}}. \tag{7}$$

Defining

$$\tilde{t}_1 = \sqrt{s''s'}\left(\frac{q_1}{\sqrt{s''}} - \frac{k_1}{\sqrt{s'}}\right)^2 \quad \text{and} \quad \tilde{t}_2 = \sqrt{s''s'}\left(\frac{q_2}{\sqrt{s''}} - \frac{k_2}{\sqrt{s'}}\right)^2, \tag{8}$$

it is seen from Eq. (7) that $\tilde{t}_1 = \tilde{t}_2 \equiv t$. For $s' = s''$ this definition coincides with the usual t-variable. An invariant form of the potential is therefore obtained by assuming it to be a function only of the variables s', s'' and $\tilde{t}$.

In particular it was shown [6–8] that for the attractive Coulomb interaction the potential is given by

$$V(\tilde{t}) = \frac{2\alpha mM}{\pi^2 \tilde{t}} \quad \text{with} \quad \alpha = \tfrac{1}{137}. \tag{9}$$

With this potential the fine structure of the hydrogen spectrum [7] and the binding of electrons to heavy ions [8] was calculated before.

In the present paper we want to apply this theory to Coulomb scattering. Since, however, the nonrelativistic theory will be at the basis of our considerations, we will first discuss the connection between the two theories.

3. The relation with the nonrelativistic theory

We begin by reminding the reader of the Lippmann-Schwinger equation for the scattering of two particles with masses M and m and reduced mass $\mu = Mm/(M+m)$:

$$M_{\mathrm{NR}}(\vec{k}', \vec{k}|k_0) = V(\vec{k}'-\vec{k}) - 2\mu \int \frac{V(\vec{k}'-\vec{k}'')M_{\mathrm{NR}}(\vec{k}'', \vec{k}|k_0)}{k''^2 - k_0^2 - i\varepsilon}\, d\vec{k}''. \tag{10}$$

From the solution the differential cross section is calculated as

$$\sigma_{\mathrm{NR}} \equiv \frac{d\sigma}{d\cos\theta} = 32\pi^5\mu^2 |M_{\mathrm{NR}}(\vec{k}', \vec{k})|^2 \quad \text{with} \quad |\vec{k}'| = |\vec{k}|, \tag{11}$$

where $M_{\mathrm{NR}}(\vec{k}', \vec{k}) = \lim\limits_{k_0 \to |\vec{k}|} M_{\mathrm{NR}}(\vec{k}', \vec{k}|k_0)$. If we try to solve Eq. (10) by way of an expansion in spherical waves:

$$M_{\mathrm{NR}}(\vec{k}', \vec{k}|k_0) = \sum_{l=0}^{\infty} (2l+1) P_l(\cos\theta) M_l^{\mathrm{NR}}(k', k|k_0), \tag{12}$$

the partial wave amplitudes have to satisfy

$$M_l^{\mathrm{NR}}(k', k|k_0) = V_l(k', k) - 8\pi\mu \int_0^{\infty} \frac{k''^2 dk''}{k''^2 - k_0^2 - i\varepsilon} V_l(k', k'') M_l^{\mathrm{NR}}(k'', k|k_0), \tag{13}$$

where the relation between $V_l(k', k)$ and $V(\vec{k}'-\vec{k})$ is the same as between M_l^{NR} and M_{NR}. Phase shifts are defined by

$$M_l^{\mathrm{NR}}(k, k|k) = -\frac{e^{i\delta_l}}{4\pi^2\mu k} \sin\delta_l. \tag{14}$$

Now in order to find the relation with our amplitude $M_{\beta\alpha}$ of Eq. (4), we again expand in spherical waves:

$$M_{\beta\alpha} = \frac{4mM}{\tau(s', s)} \sum_l (2l+1) M_l(s', s) P_l(\cos\theta), \tag{15}$$

where θ is the scattering angle in the rest system and

$$\tau(s', s) = \tfrac{1}{2}\left[\frac{\lambda(s', M^2, m^2)\lambda(s, M^2, m^2)}{s's}\right]^{1/2}, \tag{16}$$

The potential $V_{\beta\alpha}$ is expanded in the same way as in Eq. (15) with coefficients $V_l(s', s)$. For these partial wave amplitudes the following equation holds [6]

$$M_l(s', s|s_0) = V_l(s', s) - 4\pi mM \int_{(M+m)^2}^{\infty} \frac{V_l(s', s'') M_l(s'', s|s_0) ds''}{\sqrt{\lambda(s'', M^2, m^2)}\,(s''-s_0-i\varepsilon)}. \tag{17}$$

The phase shifts are defined by

$$M_l(s, s|s) = -\frac{\sqrt{\lambda(s, M^2, m^2)}}{4\pi^2 mM} e^{i\delta_l} \sin\delta_l, \tag{18}$$

while the differential cross section is

$$\sigma \equiv \frac{d\sigma}{d\cos\theta} = \frac{2\pi^5}{s} |M_{\beta\alpha}|^2, \tag{19}$$

with $M_{\beta\alpha}$ given by Eq. (15) for $s' = s$.

The resemblance between Eqs. (13) and (17) is striking and can almost be turned into an identity by defining $\overline{M}_l$ and $\overline{V}_l$ as follows:

$$M_l(s', s|s_0) = F\overline{M}_l(k', k|k_0) \quad \text{and} \quad V_l(s', s) = F\overline{V}_l(k', k), \tag{20}$$

with

$$F = \frac{\sqrt{k'k}}{M+m} [\lambda(s', M^2, m^2)\lambda(s, M^2, m^2)]^{1/4}. \tag{21}$$

The relation between s and k and between s' and k' is

$$s = (M+m)^2 + k^2/v; \quad s' = (M+m)^2 + k'^2/v; \quad v = \frac{Mm}{(M+m)^2} = \frac{m/M}{(1+m/M)^2} \tag{22}$$

and similarly for s_0 and k_0.

Eq. (17) then turns into

$$\overline{M}_l(k', k|k_0) = \overline{V}_l(k', k) - 8\pi\mu \int_0^{\infty} \frac{k''^2 dk''}{k''^2 - k_0^2 - i\varepsilon} \overline{V}_l(k', k'')\overline{M}_l(k'', k|k_0). \tag{23}$$

Since this equation is of exactly the same form as Eq. (13) we deduce from Eq. (12) that

$$\overline{M}(\vec{k}', \vec{k}|k_0) \equiv \sum_l (2l+1)P_l(\cos\theta)\overline{M}_l(k', k|k_0) \tag{24}$$

can be found as the solution of the equation

$$\overline{M}(\vec{k}', \vec{k}|k_0) = \overline{V}(\vec{k}', \vec{k}) - 2\mu \int d\vec{k}'' \frac{\overline{V}(\vec{k}', \vec{k}'')\overline{M}(\vec{k}'', \vec{k}|k_0)}{k''^2 - k_0^2 - i\varepsilon} \tag{25}$$

which differs from the nonrelativistic Eq. (10) only in that the potential is not a function of the difference $\vec{k}' - \vec{k}$, so that it is nonlocal.

Using Eqs. (15), (20) and (24) the relation with the original amplitude $M_{\beta\alpha}$ becomes

$$M_{\beta\alpha} = \frac{8\mu\sqrt{k'ks's}}{[\lambda(s', M^2, m^2)\lambda(s, M^2, m^2)]^{1/4}} \overline{M}(\vec{k}', \vec{k}|k_0). \tag{26}$$

This gives for the differential cross section of Eq. (19):

$$\sigma = 32\pi^5\mu^2 A|\overline{M}(\vec{k}', \vec{k}|k)|^2, \quad \text{with} \quad |\vec{k}'| = |\vec{k}| = k_0 \tag{27}$$

and

$$A = \frac{4k^2 s}{\lambda(s, M^2, m^2)} = \frac{4vs}{s-(M-m)^2} = \frac{1+k^2/Mm}{1+k^2/4\mu^2}. \tag{28}$$

The relation between $\overline{V}(\vec{k}', \vec{k})$ and $V(\tilde{t})$ is the same as between $\overline{M}(\vec{k}', \vec{k}|k_0)$ and $M_{\beta\alpha}$. Inserting Eq. (9) this gives

$$\overline{V}(\vec{k}', \vec{k}) = \frac{\alpha(M+m)}{4\pi^2\sqrt{k'ks's}} \cdot \frac{[\lambda(s', M^2, m^2)\lambda(s, M^2, m^2)]^{1/4}}{\tilde{t}} \tag{29}$$

in which $\tilde{t}$ must be read as

$$\tilde{t} = t_0(s', s) + \tau(s', s)\cos\theta, \tag{30}$$

with τ given by Eq. (16) and where

$$t_0(s', s) = \frac{-1}{2\sqrt{s's}}[s's - (M^2+m^2)(s'+s) + (M^2-m^2)^2]. \tag{31}$$

This potential seems to be completely different from the nonrelativistic potential $V(\vec{k}' - \vec{k})$ occurring in Eq. (10), which for the Coulomb interaction $V(r) = -\dfrac{\alpha}{r}$ is given by

$$V(\vec{k}' - \vec{k}) = \frac{1}{(2\pi)^3}\int e^{-i(\vec{k}'-\vec{k})\cdot\vec{r}}\, V(r)d\vec{r} = \frac{-\alpha}{2\pi^2|\vec{k}' - \vec{k}|^2}. \tag{32}$$

However, it is possible to show that $\overline{V}(\vec{k}', \vec{k})$ and $V(\vec{k}'-\vec{k})$ are closely related and that for low energy they become identical. For that purpose we make a change of variables from $\vec{k}$ to

$$\vec{z} = \frac{\vec{k}}{\sqrt{k^2+4\mu^2}}, \tag{33}$$

which can be interpreted as the velocity of a particle with mass 2μ and momentum $\vec{k}$. In the same way we define $z_0 = \dfrac{k_0}{\sqrt{k_0^2+4\mu^2}}$. The quantities s, $\lambda(s, M^2, m^2)$, $t_0(s', s)$ and $\tau(s', s)$ can all be expressed in terms of these new variables. After substitution into Eq. (29) this gives the following form of the potential:

$$\overline{V}(\vec{z}', \vec{z}) = -\frac{\alpha}{2\pi^2(2\mu)^2} \cdot \frac{[(1-z'^2)(1-z^2)]^{3/4}}{|\vec{z}'-\vec{z}|^2}. \tag{34}$$

If in the nonrelativistic potential of Eq. (32) we substitute $\vec{k} = 2\mu\vec{z}$, we obtain

$$V(\vec{z}'-\vec{z}) = -\frac{\alpha}{2\pi^2(2\mu)^2} \cdot \frac{1}{|\vec{z}'-\vec{z}|^2}, \tag{35}$$

which for low energies is the same as Eq. (34).

In order to bring Eqs. (10) and (25) in the simplest possible form, we once more define new scattering amplitudes as a function of the variables of Eq. (33) for the full relativistic case and as a function of $\vec{z} = \vec{k}/2\mu$ for the nonrelativistic equation. By writing

$$\overline{M}(\vec{k}', \vec{k}|k_0) = \frac{\alpha}{4\mu^2}[(1-z'^2)(1-z^2)]^{3/4} G(\vec{z}', \vec{z}|z_0) \tag{36}$$

and

$$M_{NR}(\vec{k}', \vec{k}|k_0) = \frac{\alpha}{4\mu^2} G_{NR}(\vec{z}', \vec{z}|z_0), \tag{37}$$

the equations (25) and (10) become

$$G(\vec{z}, \vec{z}'|z_0) = \frac{-1}{2\pi^2|\vec{z}-\vec{z}'|^2} + \frac{\bar{\alpha}}{2\pi^2} \int_* \frac{G(\vec{z}'', \vec{z}'|z_0)d\vec{z}''}{|\vec{z}-\vec{z}''|^2 \cdot (z''^2-z_0^2-i\varepsilon)} \tag{38}$$

and

$$G_{NR}(\vec{z}, \vec{z}'|z_0) = \frac{-1}{2\pi^2|\vec{z}-\vec{z}'|^2} + \frac{\alpha}{2\pi^2} \int \frac{G_{NR}(\vec{z}'', \vec{z}'|z_0)d\vec{z}''}{|\vec{z}-\vec{z}''|^2 \cdot (z''^2-z_0^2-i\varepsilon)}. \tag{39}$$

The difference between the two equations is twofold:

a) The coupling constant is reduced from α to

$$\bar{\alpha} = \alpha(1-z_0^2) = \frac{4\mu^2}{k_0^2+4\mu^2}\alpha. \tag{40}$$

b) In Eq. (38) the integration is restricted to $|\vec{z}''| \leqslant 1$ (which is indicated by an asterisk), while in the nonrelativistic case of Eq. (39) there is no such restriction.
Finally the differential cross sections calculated with these amplitudes are

$$\sigma = \frac{2\pi^5\bar{\alpha}^2}{\mu^2} \cdot \frac{1+k^2/Mm}{(1+k^2/4\mu^2)^2} |G(\vec{z}', \vec{z}|z)|^2 \tag{41}$$

and

$$\sigma_{\rm NR} = \frac{2\pi\alpha^5}{\mu^2} \cdot |G_{\rm NR}(\vec{z}', \vec{z}|z)|^2. \tag{42}$$

Before applying these formulae, however, a common correction factor must be applied to the scattering amplitudes, which takes into account that the asymptotic states are not really plane waves. This factor will be given in the next section.

4. The nonrelativistic equation

Since the real problem of solving Eq. (38) is so close to the nonrelativistic scattering problem of Eq. (39), we will first address ourselves to the latter. In doing so we will use the method of Schwinger [4] for the calculation of the Green's function.

We first consider the function $\bar{G}(\vec{z}, \vec{z}'|z_1)$, which, for positive real values of z_1 is defined by the equation

$$\bar{G}(z, \vec{z}'|\vec{z}_1) = \frac{-1}{2\pi^2|\vec{z}-\vec{z}'|^2} + \frac{\alpha}{2\pi^2} \int \frac{\bar{G}(\vec{z}'', \vec{z}'|z_1)d\vec{z}''}{|\vec{z}-\vec{z}''|^2 \cdot (z''^2+z_1^2)} \cdot \tag{43}$$

Later the function $G_{\rm NR}(\vec{z}, \vec{z}'|z_0)$ is then obtained by analytic continuation of $\bar{G}(\vec{z}, \vec{z}'|z_1)$ to the complex value $z_1 = -iz_0+\varepsilon$.

We apply a stereographic projection onto the surface of a four-dimensional sphere with unit radius, by introducing the variables

$$\vec{\xi} = \frac{2z_1\vec{z}}{z_1^2+z^2}; \quad \xi_0 = \frac{z_1^2-z^2}{z_1^2+z^2}; \quad \xi^2 \equiv \xi_0^2+\vec{\xi}^2 = 1 \tag{44}$$

and similarly for the primed variables. Defining $H(\xi, \xi'|z_1)$ by

$$\bar{G}(\vec{z}, \vec{z}'|z_1) = z_1^{-2}(1+\xi_0)(1+\xi_0')H(\xi, \xi'|z_1), \tag{45}$$

Eq. (43) reduces to

$$H(\xi, \xi'|z_1) = -\frac{1}{2\pi^2(\xi-\xi')^2} + \frac{\beta}{2\pi^2}\int \frac{H(\xi'', \xi'|z_1)}{(\xi-\xi'')^2}\, d\Omega'', \tag{46}$$

where $\beta = \dfrac{\alpha}{2z_1}$ and $d\Omega'' = \sin^2 u \sin\theta\, du d\theta d\phi$ is the surface element of this four-dimensional sphere with $\xi_0'' = \cos u$ and θ and ϕ the polar and azimuthal angles of $\vec{z}''$. Since this equation is manifestly O(4)-symmetric, a solution can be obtained by expanding in spherical harmonics $Y_{nlm}(\xi)$, which are properly normalised, i.e.,

$$\int Y^*_{nlm}(\xi)Y_{n'l'm'}(\xi)d\Omega = \delta_{nn'}\delta_{ll'}\delta_{mm'}. \tag{47}$$

Some of the series which are going to occur do not converge, however. For that reason we consider $H(\xi, \xi'|z_1)$ as the limit of another function,

$$H(\xi, \xi'|z_1) = \lim_{\varrho \uparrow 1} H_\varrho(\xi, \xi'|z_1), \tag{48}$$

which is defined by the equation

$$H_\varrho(\xi, \xi'|z_1) = -\frac{1}{2\pi^2}\cdot\frac{1}{(1-\varrho)^2+\varrho(\xi-\xi')^2}+\frac{\beta}{2\pi^2}\int\frac{H_\varrho(\xi'', \xi'|z_1)d\Omega''}{(1-\varrho)^2+\varrho(\xi-\xi'')^2}. \tag{49}$$

With the help of

$$\frac{1}{2\pi^2}\cdot\frac{1}{(1-\varrho)^2+\varrho(\xi-\xi')^2} = \sum_{nlm}\frac{\varrho^{n-1}}{n}\,Y_{nlm}(\xi)Y^*_{nlm}(\xi') \tag{50}$$

this function is found to be given by

$$H_\varrho(\xi, \xi'|z_1) = -\sum_{nlm}\frac{\varrho^{n-1}}{n-\beta\varrho^{n-1}}\,Y_{nlm}(\xi)Y^*_{nlm}(\xi'). \tag{51}$$

Using $\dfrac{1}{n-\beta\varrho^{n-1}} = \dfrac{1}{n}-\dfrac{\beta\varrho^{n-1}}{n(n-\beta\varrho^{n-1})} \rightarrow \dfrac{1}{n}-\dfrac{\beta\varrho^{n-1}}{n(n-\beta)}$ and $\dfrac{1}{n-\beta} = \displaystyle\int_0^1 x^{n-\beta-1}dx$, we obtain

$$H(\xi, \xi'|z_1) = \frac{-1}{2\pi^2(\xi-\xi')^2}-\frac{\beta}{2\pi^2}\int_0^1\frac{x^{-\beta}dx}{(1-x)^2+x(\xi-\xi')^2}. \tag{52}$$

This is also equal to

$$H(\xi, \xi'|z_1) = \frac{-1}{2\pi^2}\int_0^1 dx\cdot x^{-\beta}\frac{d}{dx}\left[\left(\frac{1}{\sqrt{x}}-\sqrt{x}\right)^2+(\xi-\xi')^2\right]^{-1}$$

$$= -\frac{1}{2\pi^2}\int_0^1 dx\cdot x^{-\beta}\cdot\frac{d}{dx}\left(\frac{\xi}{\sqrt{x}}-\sqrt{x}\xi'\right)^{-2}. \tag{53}$$

In the analytic continuation to the point $z_1 = -iz_0+\varepsilon$ the parameter β is to be replaced by $\beta = i\eta$ with $\eta = \dfrac{\alpha}{2z_0}$. The quantity $(\xi-\xi')^2$ occurring in the integral (53) can be written as $(\xi-\xi')^2 = \delta^{-1}$ with

$$\delta = -\frac{(z^2-z_0^2-i\varepsilon)(z'^2-z_0^2-i\varepsilon)}{4z_0^2|\vec{z}-\vec{z}'|^2}. \tag{54}$$

In the next section it will be seen that for the calculation of the differential cross section in the relativistic case at least one of the vectors $\vec{z}$ and $\vec{z}'$ will have a length equal to z_0. This implies that we need the integral of Eq. (53) only for $\delta \to 0$. Therefore $\left(\dfrac{1}{\sqrt{x}} - \sqrt{x}\right)^2$ can always be neglected with respect to δ^{-1}, except for very small x, when it can be replaced by x^{-1}. The integral can then be evaluated and we find

$$H(\vec{z}, \vec{z}'|z_0) = -\frac{\delta^{1-i\eta}}{2\pi^2}\cdot\frac{\pi\eta}{\sinh \pi\eta} \quad \text{for} \quad \delta \to 0. \tag{55}$$

Using Eq. (45) the nonrelativistic amplitude can now be calculated either fully on-shell, i.e., $|\vec{z}| = |\vec{z}'| = z_0$, gving $G^{\mathrm{f}}_{\mathrm{NR}}$, or half-on-shell, i.e., $|\vec{z}| = z_0$ and $|z'| > z_0$ or $|\vec{z}'| = z_0$ and $|\vec{z}| > z_0$, giving $G^{\mathrm{h}}_{\mathrm{NR}}$. Writing $\varepsilon = 4z_0^2\bar{\varepsilon}$, where $\bar{\varepsilon}$ now is a dimensionless positive infinitesimal number, the functions $G^{\mathrm{f}}_{\mathrm{NR}}$ and $G^{\mathrm{h}}_{\mathrm{NR}}$ are found to be

$$G^{\mathrm{f}}_{\mathrm{NR}}(\vec{z}, \vec{z}'|z_0) = \frac{-1}{2\pi^2|z-z'|^2}\, e^{-i\eta \log \frac{4z_0^2}{|z-z'|^2}} \cdot \left(\frac{\pi\eta}{\sinh \pi\eta}\right)\cdot(\bar{\varepsilon})^{-2i\eta} \tag{56}$$

for $|\vec{z}| = |\vec{z}'| = z_0$ and

$$G^{\mathrm{h}}_{\mathrm{NR}}(\vec{z}, \vec{z}'|z_0) = \frac{1}{2\pi^2|z-z'|^2}\, e^{-i\eta \log \frac{z'^2-z_0^2}{|\vec{z}-\vec{z}'|^2}} \cdot \left(\frac{\pi\eta}{\sinh \pi\eta}\right)\cdot(\bar{\varepsilon})^{-i\eta} \tag{57}$$

for $|\vec{z}| = z_0$, but $z'^2 > z_0^2$.

The differential cross section for nonrelativistic scattering is proportional to $|G^{\mathrm{f}}_{\mathrm{NR}}|^2$. It was argued, however, by Schwinger [4] that the factor $\dfrac{\pi\eta}{\sinh \pi\eta}$ belongs to the asymptotic in- and outgoing states, which are not plane waves, and should therefore be omitted. Although the phase of $(\bar{\varepsilon})^{-2i\eta} = e^{-2i\eta \log \bar{\varepsilon}}$ becomes infinite for $\bar{\varepsilon} \to 0$, it does not contribute to the cross section and neither does the other exponential in Eq. (56). In this way we obtain the well known result that the remaining part of $G^{\mathrm{f}}_{\mathrm{NR}}$ is equal to the square of the Born-amplitude.

The conclusion is that also for the relativistic case, to be treated in the next section, a factor $\dfrac{\pi\eta}{\sinh \pi\eta}$ should be omitted from the fully on-shell amplitude.

5. *The relativistic equation*

We begin by writing Eq. (38) in operator form

$$G_c = K - KP_cG_c = K - G_cP_cK, \tag{58}$$

in which the matrix elements of K are given by

$$\langle \vec{z}|K|\vec{z}'\rangle = \frac{-1}{2\pi^2|\vec{z}-\vec{z}'|^2} \tag{59}$$

and P_c is a diagonal operator in the $\vec{z}$-representation, which vanishes outside the sphere $|\vec{z}| = c$:

$$\langle \vec{z}|P_c|\vec{z}'\rangle = \bar{\alpha}\,\frac{\delta(\vec{z}-\vec{z}')\theta(c-|\vec{z}|)}{z'^2-z_0^2-i\varepsilon}. \tag{60}$$

For Eq. (38) the value of the cut-off is $c = 1$, while for $c \to c' = \infty$ Eq. (58) reduces to the nonrelativistic Eq. (39), provided we there replace α by $\bar{\alpha}$ (Eq. (40)) and take z_0 as

$$z_0 = \frac{k_0}{\sqrt{k_0^2+4\mu^2}} = \sqrt{\frac{s_0-(M+m)^2}{s_0-(M-m)^2}}. \tag{61}$$

The same replacements should be made in Eqs. (56) and (57), so that η becomes

$$\eta = \frac{\bar{\alpha}}{2z_0}. \tag{62}$$

We now want to derive an equation for G_c in which K is replaced by $G_{c'} = G_{\text{NR}}$. The motivation is that in this way the difficulties connected with the diverging phase factor (the infrared catastrophe) are located and can easily be taken care of through Eqs. (56) and (57). Using Eq. (58) we derive

$$(1-G_cP_c)G_{c'} = (1-G_cP_c)K(1-P_{c'}K_{c'}) = G_c(1-P_{c'}G_{c'}),$$

from which follows immediately

$$G_c = G_{c'} - G_c(P_c - P_{c'})G_{c'},$$

or with $c \to 1$ and $c' \to \infty$:

$$G(\vec{z}, \vec{z}'|z_0) = G_{\text{NR}}(\vec{z}, \vec{z}'|z_0) + \bar{\alpha}\int\limits_{|\vec{z}''|>1} \frac{G(\vec{z}, \vec{z}''|z_0)G_{\text{NR}}(\vec{z}'', \vec{z}'|z_0)}{z''^2-z_0^2}\,dz''. \tag{63}$$

The term $i\varepsilon$ in the denominator of the integrand could be omitted because $z_0^2 < 1$ and $z''^2 > 1$ in the whole integration region.

We have not been able to solve this equation. We did, however, evaluate the first two terms in the Born series, at least for $|\vec{z}| = |\vec{z}'| \equiv z_0$, which is required for the calculation of the differential cross section in this approximation. For this case we find

$$G(\vec{z}, \vec{z}'|z_0) = G^{\mathrm{f}}_{\mathrm{NR}}(\vec{z}, \vec{z}'|z_0) + \bar{\alpha} \int\limits_{|v|>1} \frac{G^{\mathrm{h}}_{\mathrm{NR}}(\vec{z}, \vec{v}|z_0) G^{\mathrm{h}}_{\mathrm{NR}}(\vec{v}, \vec{z}'|z_0)}{v^2 - z_0^2} d\vec{v}, \tag{64}$$

in which $G^{\mathrm{f}}_{\mathrm{NR}}$ and $G^{\mathrm{h}}_{\mathrm{NR}}$ are the fully on shell and half on shell nonrelativistic amplitudes as given in Eqs. (56) and (57), but with the new definitions of z_0 and η. Substituting these amplitudes we find for G as a function of z_0 and of the angle θ between $\vec{z}$ and $\vec{z}'$:

$$G(\theta; z_0) = \frac{-\pi\eta(\bar{\varepsilon})^{-2i\eta}}{\sinh \pi\eta} \left[\frac{e^{2i\eta \log S}}{8\pi^2 z_0^2 S^2} - \frac{\bar{\alpha}}{4\pi^4} \cdot \frac{\pi\eta e^{\pi\eta}}{\sinh \pi\eta} H(\theta, z_0) \right], \tag{65}$$

in which $S = \sin \frac{1}{2}\theta$ and

$$H(\theta, z_0) = \int\limits_{|\vec{v}|>1} \left\{ \frac{|\vec{z}-\vec{v}|^2 \cdot |\vec{z}'-\vec{v}|^2}{(v^2 - z_0^2)^2} \right\}^{i\eta} \cdot \frac{d\vec{v}}{(v^2 - z_0^2) \cdot |\vec{z}-\vec{v}|^2 \cdot |\vec{z}'-\vec{v}|^2} \cdot \tag{66}$$

As mentioned before the factor in front of the brackets in Eq. (65) should be omitted when the differential cross section as given in Eq. (41) is calculated. Taking only the first term inside the brackets, and using the relations between z_0, k and s as given in Eqs. (22) and (23), we obtain in Born approximation

$$\sigma^{\mathrm{B}} = \frac{8\pi\alpha^2 M^2 m^2 s}{\sin^4 \frac{1}{2}\theta \cdot \lambda^2(s, M^2, m^2)} \cdot \tag{67}$$

In the nonrelativistic limit this reduces to

$$\sigma^{\mathrm{B}}_{\mathrm{NR}} = \frac{\pi\alpha^2}{2\mu^2 (P/m)^4 \sin^4 \frac{1}{2}\theta}, \tag{68}$$

which is the Rutherford formula. In this expression P is the momentum of the particle with mass m in the system in which the other particle with mass M is at rest. As a function of the invariant variable s it is given by $P = \sqrt{\lambda(s, M^2, m^2)}/2M$.

The correction to the Born cross section of Eq. (67) is given by $R = (\sigma - \sigma^{\mathrm{B}})/\sigma^{\mathrm{B}}$ and upon substitution of Eq. (65) is found to be

$$R = -\frac{8 z_0^3 \eta^2}{\pi} \sin^2 \tfrac{1}{2}\theta \cdot \frac{e^{\pi\eta}}{\sinh \pi\eta} \{\mathrm{Re}\, H \cdot \cos(\eta \log \sin^2 \tfrac{1}{2}\theta) + \mathrm{Im}\, H \cdot \sin(\eta \log \sin^2 \tfrac{1}{2}\theta)\} \tag{69}$$

with

$$\eta = \frac{\alpha(1 - z_0^2)}{2 z_0}. \tag{70}$$

The calculation of this R depends on our ability to evaluate the integral of Eq. (66) for $H(\theta, z_0)$. For zero energy, i.e. for $|\vec{z}|^2 = |\vec{z}'|^2 = z_0^2 = 0$ it is easily found to be equal to

$$H(\theta, z_0 = 0) = \frac{4\pi}{3}. \tag{71}$$

For low energies we find in this way

$$R = -\frac{4P\alpha^2}{3m}\sin^2\tfrac{1}{2}\theta\cdot\frac{e^{\pi\eta}}{\sinh\pi\eta}\cdot\cos(\eta\log\sin^2\tfrac{1}{2}\theta), \tag{72}$$

with $\eta = \alpha m/P$ and P as before. For low velocities P/m of the projectile particle η therefore will be large and R will have many oscillations as a function of θ. The amplitude, however, will be very small if $\alpha = 1/137$. It will become larger if the target particle carries a charge $Z \gg 1$. Then α must be replaced by $Z\alpha$ and P need not be too small in order to get a larger value of η.

For higher energies we see from Eq. (70) that η becomes smaller, so that the oscillations will disappear. In this case it may be a good approximation to replace the phase factor in the integral of Eq. (66) by unity. The function $H(\theta, z_0)$ can then again be calculated in closed form. The result is

$$H(\theta, z_0) = \frac{\pi}{4z_0^3S^2}\left\{\log C\cdot\log\left(\frac{q+1}{q-1}\right) + SF(\theta, z_0)\right\}, \tag{73}$$

with $S = \sin\frac{1}{2}\theta$, $C = \cos\frac{1}{2}\theta$, $q = \frac{1}{2}\left(z_0 + \frac{1}{z_0}\right)$ and

$$F(\theta, z_0) = \int_C^1 \frac{\log\dfrac{(q-x)(q+1)}{(q+x)(q-1)}}{(1-x^2)\sqrt{x^2-C^2}}\,dx. \tag{74}$$

This function can be expressed in terms of dilogarithms, but a numerical calculation is more reliable. For high energies z_0 goes to 1, η goes to zero and the correction factor behaves as

$$R \simeq -\frac{32}{3\pi}\eta\sin^2\tfrac{1}{2}\theta. \tag{75}$$

It is therefore in the intermediate energy range that the effect is largest.

For a repulsive Coulomb interaction the sign of η is changed. Because of the factor $e^{\pi\eta}$ the correction R is reduced drastically in this case.

The results of our calculations, based on Eqs. (69) and (73), are shown in the next section.

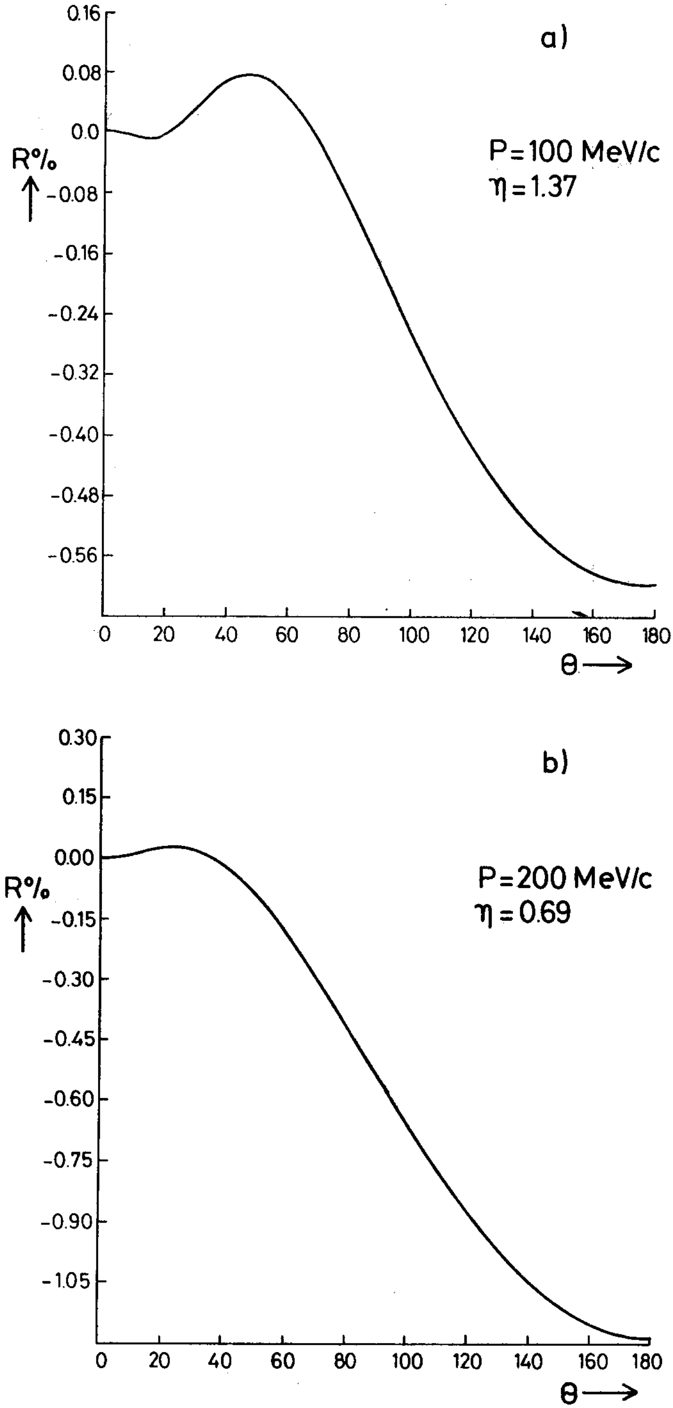
a)
P=100 MeV/c
η=1.37
R%
0.16
0.08
0.0
-0.08
-0.16
-0.24
-0.32
-0.40
-0.48
-0.56
0
20
40
60
80
100
120
140
160
180
θ
b)
P=200 MeV/c
η=0.69
R%
0.30
0.15
0.00
-0.15
-0.30
-0.45
-0.60
-0.75
-0.90
-1.05
θ

Fig. 3

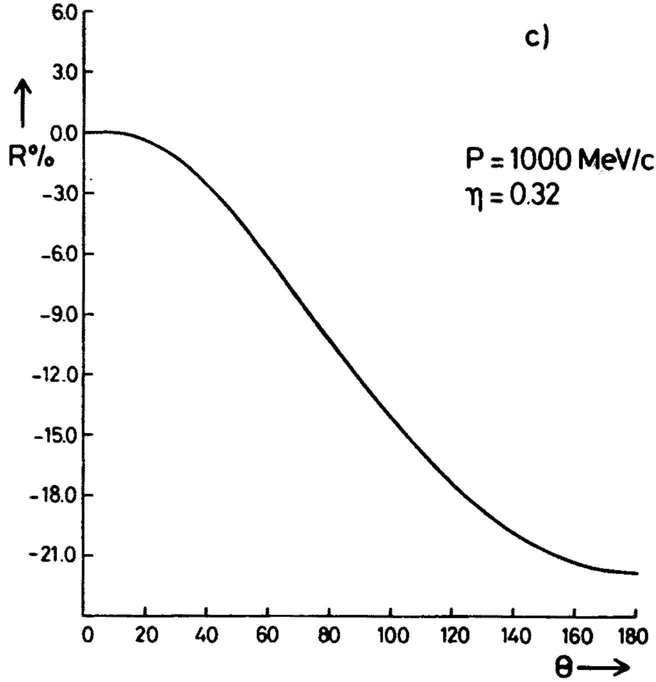

Fig. 3. Relative correction in % to differential cross section in Born approximation for anti-protons scattered by Ca^{40}, for three values of the incident momentum in the rest system of the target. a) $P = 100$ MeV/c, b) $P = 200$ MeV/c, c) $P = 300$ MeV/c

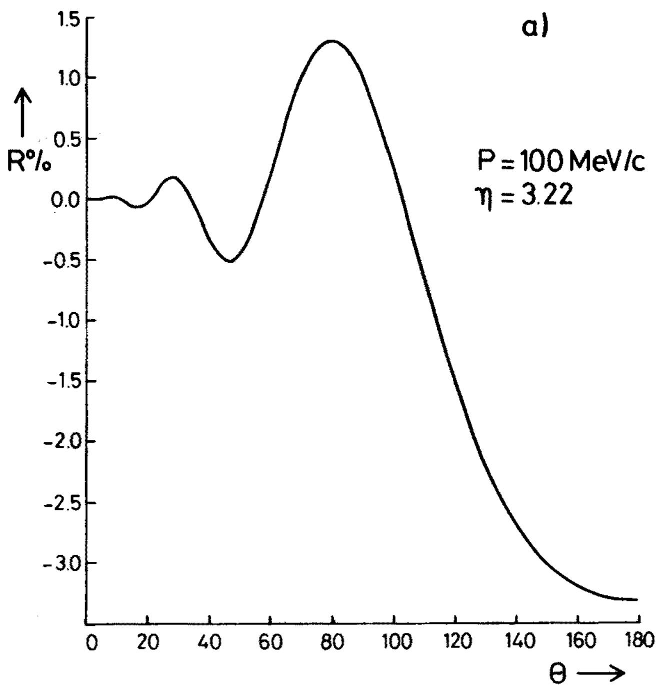

Fig. 4

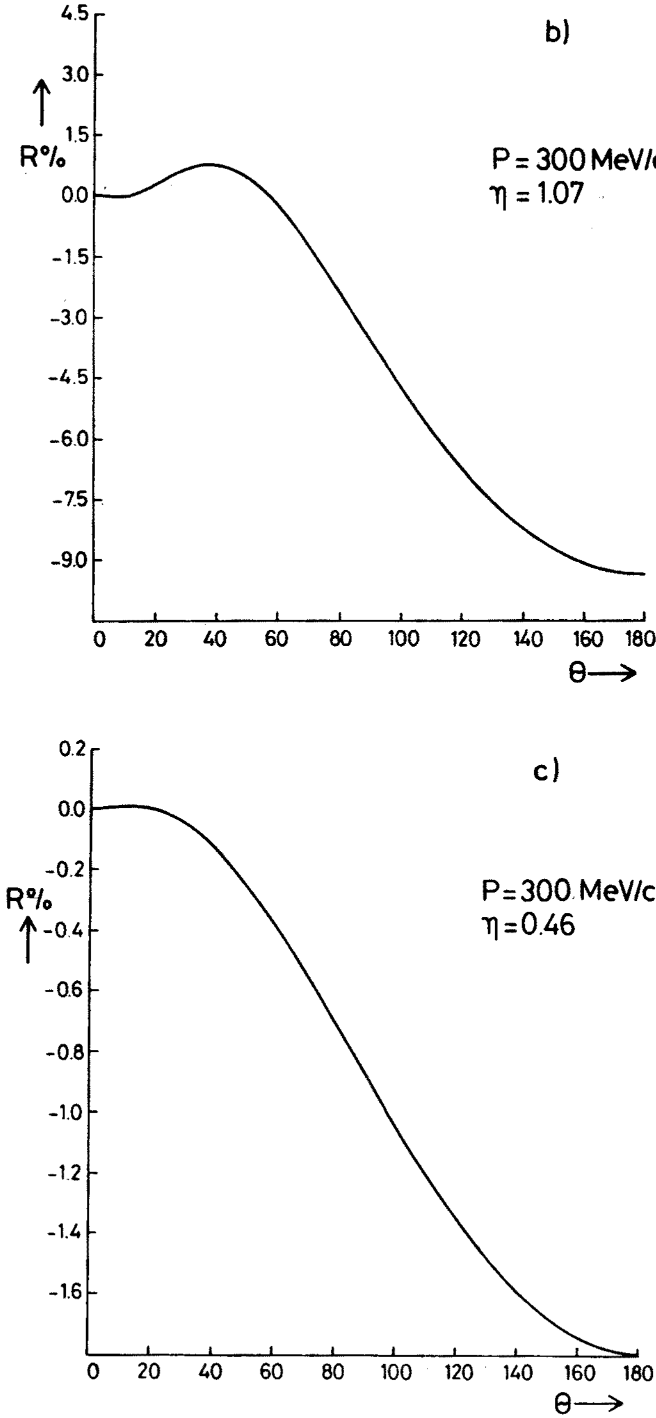

Fig. 4. As in the Fig. 3 for Ag^{108}. a) P = 100 MeV/c, b) P = 300 MeV/c, c) P = 1000 MeV/c

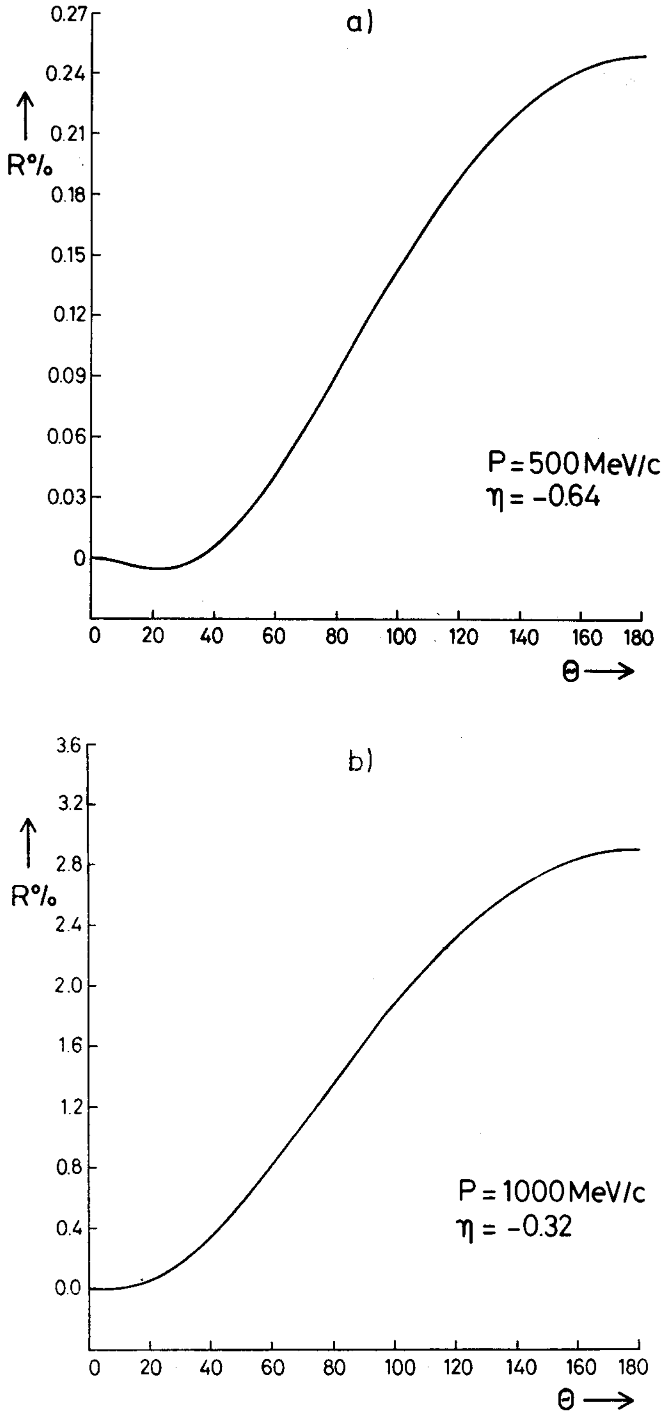

Fig. 5. As in Fig. 4 for protons on Ag^{108}. a) P = 500 MeV/c, b) P = 1000 MeV/c

6. *Summary and conclusions*

For high energies and large scattering angles the elastic collision of protons or anti--protons with heavy nuclei is dominated by nuclear forces. Only for small angles and not too high energies the Coulomb interaction plays an important rôle. Relativistic effects may nevertheless be important and are therefore taken into account in this paper, neglecting hadronic interactions altogether.

Electron scattering by a point charge was calculated by Fradkin, Weber and Hammer [9], who used the single particle Dirac theory to incorporate relativity. Virtually no two--particle theory of the quasi-potential type has been applied to the problem of Coulomb scattering so far. Especially the difficulties connected with the infinite phase shift and the distortion of the asymptotic plane waves have not been discussed in this framework.

Returning to a relativistic two-particle theory which we formulated many years ago [5], we now calculated the corrections to the Rutherford formula for the differential cross section. As the authors of reference [9], we also find corrections which are oscillatory in the scattering angle. Figures 3 and 4 show this effect for the scattering of anti-protons on a Ca- or Ag nucleus. Rather pronounced oscillations occur at low energy, but the magnitude of the corrections is largest at intermediate energies. In comparing figures 3 and 4 it is seen that a higher nuclear charge gives a larger correction factor. Extensive calculations, however, are needed for still higher charges, when third order effects become important.

New effects are expected for the scattering of positively charged particles. Due to the repulsion the correction factor should, and indeed is much smaller than in the attractive case, as can be seen in figure 5 for proton-Ag scattering.

The authors are grateful for the many times they attended the Cracow summerschool in Zakopane. One of us (E.H.de G.) thanks the Institute of Theoretical Physics in Utrecht for the hospitality extended to him.

REFERENCES

[1] L. Marquez, *Am. J. Phys.* **40**, 1420 (1972).
[2] R. H. Dalitz, *Proc. Roy. Soc.* (London) **A206**, 509 (1951).
[3] Ik-Ju Kang, L. M. Brown, *Phys. Rev.* **128**, 2828 (1962).
[4] J. Schwinger, *J. Math. Phys.* **5**, 1606 (1964).
[5] E. H. de Groot, Th. W. Ruijgrok, *Nucl. Phys.* **B101**, 95 (1975).
[6] Th. W. Ruijgrok, in B. Jancewicz and J. Lukierski (eds.), *Quantum Theory of Particles and Fields*, World Scientific, 1983, p. 117.
[7] Th. W. Ruijgrok, *Acta Phys. Pol.* **B7**, 43 (1976).
[8] E. van Meijgaard, Th. W. Ruïjgrok, *Physica* **131A**, 570 (1985).
[9] D. M. Fradkin, T. A. Weber, C. L. Hammer, *Ann. Phys.* (NY) **27**, 338 (1964).