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Scrutinizing $\pi\pi$ scattering in light of recent lattice phase shifts

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In this paper, the $IJ = 00, 11, 20$ partial wave $\pi\pi$ scattering phase shifts determined by the lattice QCD approach are analyzed by using a novel dispersive solution of the S-matrix, i.e. the PKU representation, in which the unitarity and analyticity of scattering amplitudes are automatically satisfied and the phase shifts are conveniently decomposed into the contributions of the cuts and various poles, including bound states, virtual states and resonances. The contribution of the left-hand cut is estimated by the $SU(2)$ chiral perturbation theory to $O(p^4)$. The Balanchandran-Nuyts-Roskies relations are considered as constraints to meet the requirements of the crossing symmetry. It is found that the $IJ = 00$ $\pi\pi$ scattering phase shifts obtained at $m_\pi = 391$ MeV by Hadron Spectrum Collaboration (HSC) reveal the presence of both a bound state pole and a virtual state pole below the $\pi\pi$ threshold rather than only one bound state pole for the σ. To reproduce the lattice phase shifts at $m_\pi = 391$ MeV, a virtual-state pole in the $IJ = 20$ channel is found to be necessary in order to balance the left-hand cut effects from the chiral amplitudes. Similar discussions are also carried out for the lattice results with $m_\pi = 236$ MeV from HSC. The observed behaviors of the pole positions with respect to the variation of the pion masses can provide deep insights into our understanding of the dynamical origin of σ resonance.

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PoS(HADRON2025)095

1. Introduction

The $\pi\pi$ scattering is a basic testing ground for non-perturbative QCD. Near threshold it is accurately described by chiral perturbation theory (χPT) [1]. In the scalar–isoscalar channel ($IJ = 00$) the phase shift rises smoothly and reaches $\pi/2$ around 1 GeV without a typical Breit–Wigner shape [2–4]. The long-standing controversy about the existence of the σ resonance has been settled by dispersive analyses (Roy equations and related methods) that respect unitarity, analyticity and crossing, which consistently locate the σ pole [5–9]; see the PDG mini review [10].

Recent lattice-QCD developments now allow precise determinations of isoscalar $\pi\pi$ amplitudes including annihilation diagrams, albeit at heavier-than-physical pion masses. Results at $m_\pi = 236$ and 391 MeV for $IJ = 00, 11, 20$ provide complementary information that cannot be accessed experimentally [11–13].

In this work we analyze those lattice phase shifts using the PKU representation of the S-matrix [14], with the left-hand cut estimated from χPT up to $O(p^4)$, and impose the Balachandran–Nuyts–Roskies (BNR) relations to enforce crossing among partial waves [15, 16]. We extract pole information in $IJ = 00, 11, 20$ and find, in particular, that the $IJ = 00$ data at $m_\pi = 391$ MeV favor a *bound-state plus nearby virtual-state* scenario over a single bound state, and that a *virtual state* in $IJ = 20$ is required to describe the data. These findings offer new constraints on the dynamics and nature of the light scalar sector.

2. Theoretical background

The PKU representation provides a model–independent parametrization of two-body partial-wave scattering amplitudes, obtained from generalized unitarity and analyticity [14]. The S-matrix is factorized into contributions from poles (bound, virtual, resonant) and cuts (left- and right-hand):

$$S(s) = S_{cut}(s) \prod_v S_v(s) \prod_b S_b(s) \prod_r S_r(s) \,, \tag{1}$$

where S_{cut} encodes the smooth background from left-hand cuts and inelastic thresholds, and $S_{v,b,r}$ represent virtual states, bound states and resonances, respectively. This construction automatically satisfies unitarity and analyticity, and separates pole effects from smooth backgrounds, which is especially convenient for extracting resonance parameters [6, 17].

2.1 Estimation of the left-hand cut

The left-hand cut (l.h.c.) integral in the PKU representation is dominated by contributions far from the physical region. We estimate it using chiral perturbation theory (χPT) up to one-loop order ($O(p^4)$), with amplitudes derived from the $SU(2)$ chiral Lagrangian [18]. The imaginary part of the l.h.c. can be written as

$$\mathrm{Im}_L f(s) = -\frac{1}{2\rho(s)} \ln\big|1 + 2i\rho(s)T_{\chi PT}(s)\big| \,, \tag{2}$$

and the dispersion integral is truncated by a cutoff Λ_L to suppress the unreliable high-energy behavior:

$$f(s) = -\frac{s}{\pi} \int_{-\Lambda_L}^{0} \frac{\ln|1 + 2i\rho(s')T_{\chi PT}(s')|}{2\rho(s')s'(s'-s)} ds' \,. \tag{3}$$

The resulting background phase is simply $\delta_{BG}(s) = \rho(s) f(s)$. This treatment has been widely applied in $\pi\pi$ and πN scatterings, providing a controlled estimate of non-pole contributions [8, 19].

2.2 Crossing symmetry and BNR relations

Crossing symmetry requires the s-, t- and u-channel $\pi\pi$ amplitudes to be related by analytic continuation. Since the PKU representation does not impose crossing automatically, we enforce the Balachandran–Nuyts–Roskies (BNR) relations [15, 16], which connect partial-wave amplitudes of different isospin and angular momenta. For instance, one of the five BNR constraints reads

$$\int_0^{4m_\pi^2} (s - 4m_\pi^2)(3s - 4m_\pi^2)\,[t_0^0(s) + 2t_0^2(s)]\, ds = 0, \tag{4}$$

with analogous expressions involving $t_1^1(s)$. In practice, these relations are imposed as soft constraints in the fits to lattice data at different pion masses, ensuring approximate crossing symmetry.

In summary, the PKU representation combined with χPT input for the left-hand cut and BNR constraints provides a consistent framework to analyze $\pi\pi$ scattering amplitudes and extract pole structures from both experimental and lattice-QCD data.

3. Numerical analyses and discussions

Our analysis is based on the lattice simulation data from the Hadron Spectrum Collaboration with m_π = 391 MeV (Data391) [11, 12, 20, 21] and m_π = 236 MeV (Data236) [11, 13]. The considered channels include IJ = 00, 11, and 20 $\pi\pi$ scattering. Phase shifts with heavier pion mass exhibit qualitative differences compared with the physical case, which provides useful information about the underlying pole structure.

Fitting strategy. We first performed single-channel fits using the PKU representation, and then a combined fit to all channels. Crossing symmetry was imposed through the Balachandran–Nuyts–Roskies (BNR) relations introduced as penalty functions in the global χ^2. Uncertainties from both phase shifts and lattice momentum were included consistently.

Key findings.

- $IJ = 11$ *(ρ channel).* The lattice phase shifts at both pion masses clearly show a resonance structure. Our extracted ρ pole positions agree with the lattice determinations, e.g. $\sqrt{s_\rho} \approx 863 - i\,5$ MeV at m_π = 391 MeV and $\sqrt{s_\rho} \approx 782 - i\,48$ MeV at m_π = 236 MeV.

- $IJ = 20$ *channel.* At m_π = 391 MeV, the background alone fails to reproduce the data once crossing constraints are applied. A virtual-state pole near threshold is required, consistent with expectations from phenomenological analyses [22].

- $IJ = 00$ *(σ channel).* For m_π = 236 MeV, the lattice phase shifts feature a broad resonance resembling the physical σ. For m_π = 391 MeV, however, the data cannot be described by a single bound-state pole. A much better fit is obtained when both a bound-state pole and a nearby virtual-state pole are included, indicating a "bound+virtual" pair scenario. This mechanism provides a natural explanation of the rapid phase-shift decrease near threshold.

PoS(HADRON2025)095

Illustration of key findings. For clarity, we display in Fig. 1 a representative example of the lattice phase shifts together with our fits in the $IJ = 00$ channel at $m_\pi = 391$ MeV. The figure highlights the necessity of including both a bound state and a nearby virtual state pole in order to reproduce the lattice data.

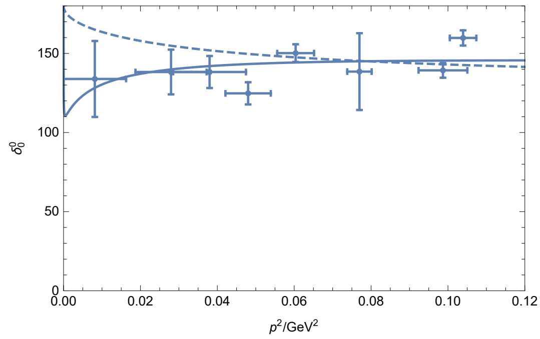

Figure 1: Phase shifts in the $IJ = 00$ channel at $m_\pi = 391$ MeV: lattice data from HSC (points) compared with our fits. Dashed line: fit with only a bound state; solid line: fit with both a bound state and a virtual state.

Pole trajectory. Combining lattice results at different pion masses with the physical case, we obtain a qualitative picture of the σ pole trajectory: at the physical pion mass it corresponds to a conjugate pole pair deep in the second Riemann sheet; as m_π increases, the poles move toward the real axis, eventually becoming a bound state and a virtual state. This behavior is consistent with unitarized χPT and Friedrichs–Lee model expectations [9, 23].

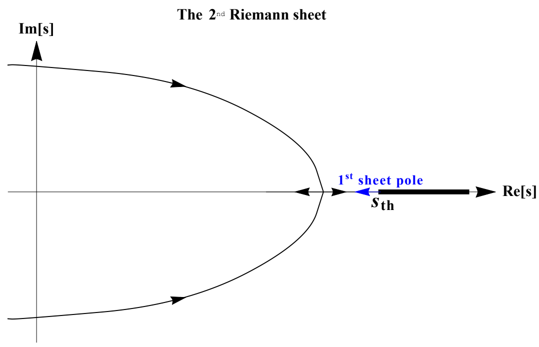

Figure 2: Qualitative picture of the σ pole trajectory as the pion mass increases.

In summary, our fits suggest that both the $IJ = 20$ virtual pole and the $IJ = 00$ bound+virtual scenario are strongly favored by lattice data when crossing symmetry is enforced. These findings provide new insight into the nature of the σ state in QCD.

4. Summary

We have analyzed lattice QCD data of $\pi\pi$ scattering using the PKU representation, which respects unitarity and analyticity, with crossing symmetry imposed via the BNR relations.

Our main findings are:

PoS(HADRON2025)095

- For $m_\pi = 391$ MeV in the $IJ = 00$ channel, the data strongly favor a "bound + virtual" pole pair for the σ, rather than a single bound state.
- In the $IJ = 20$ channel, a near-threshold virtual pole is required once crossing symmetry is enforced.
- The $IJ = 11$ (ρ) channel is well reproduced, and the pole positions agree with lattice determinations.

These results suggest a consistent picture of the σ pole trajectory: as the pion mass increases, the conjugate poles of the physical resonance move toward the real axis, eventually becoming a bound state and a virtual state. Such a behavior agrees with expectations from unitarized χPT and Friedrichs–Lee models.

In conclusion, lattice simulations at unphysical pion masses provide valuable insights into the nature of light scalar mesons. Future lattice data with higher precision near threshold will be crucial to determine pole positions more accurately and to further test crossing symmetry in two-meson scattering.

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PoS(HADRON2025)095

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