A group theoretical approach to quantum gravity in (A)dS

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Submitted in partial fulfillment of the
requirements for the degree
of Doctor of Philosophy
in the Graduate School of Arts and Sciences

COLUMBIA UNIVERSITY

2021

ABSTRACT

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This thesis is devoted to developing a group-theoretical approach towards quantum gravity in (Anti)-de Sitter spacetime. We start with a comprehensive review of the representation theory of de Sitter (dS) isometry group, focusing on the construction of unitary irreducible representations and the computation of characters. The three chapters that follow present the results of novel research conducted as a graduate student.

Chapter 4 is based on [1]. We provide a general algebraic construction of higher spin quasinormal modes of de Sitter horizon and identify the boundary operator insertions that source the quasinormal modes from a local QFT point of view. Quasinormal modes of a single higher spin field in dS$_D$ furnish two nonunitary lowest-weight representations of the dS isometry group SO$(1, D)$. We also show that quasinormal mode spectrums of higher spin fields are precisely encoded in the Harish-Chandra characters of the corresponding SO$(1, D)$ unitary irreducible representations.

Chapter 5 is based on work with D. Anninos, F. Denef and A. Law [2]. With potential application to constraining UV-complete microscopic models of de Sitter quantum gravity, we compute de Sitter entropy as the logarithm of the sphere path integral, for any possible low energy effective field theory containing a massless graviton, in arbitrary dimensions. The path integral is performed exactly at the one-loop level. The one-loop correction to the dS entropy is found to take a universal "bulk − edge" form, with the bulk part being an integral transformation of a Harish-Chandra character encoding quasinormal modes spectrum in a static patch of dS and the edge part being the same integral transformation of an edge character encoding degrees of freedom frozen on the dS horizon. In 3D de Sitter spacetime, the one-loop exact entropy is promoted to an all-loop exact result for truncated higher spin gravity, the latter admitting an SL$(n, \mathbb{C})$ Chern-Simons formulation with n being the spin cut-off.

Chapter 6 is based on [3]. Inspired by [2], we revisit the one-loop partition function of any higher spin field in $(d+1)$-dimensional Anti-de Sitter spacetime and show that it can be universally expressed as an integral transform of an SO$(2,d)$ bulk character and an SO$(2,d-2)$ edge character. We apply this character integral formula to various higher-spin Vasiliev gravities and find miraculous (almost) cancellations between bulk and edge characters, leading to striking agreement with the predictions of higher spin holography. We also comment on the relation between our character integral formula and Rindler-AdS [4] thermal partition functions.

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Acknowledgments

I am forever indebted to my advisor Frederik Denef, for teaching me physics, for sharing brilliant insights, for countless marathon discussions and for a lot of other things that are certainly beyond the responsibilities of an advisor. You show me how a great physicist thinks and does his job.

I am also thankful to my collaborators: Dionysios Anninos who is a constant source of juicy physics ideas, Ruben Monten who gave me the very first lesson about higher spin gravity, Albert Law who taught me how to do sphere path integrals, Shamit Kachru and Arnav Tripathy who provided me the opportunity to venture out into directions orthogonal to my main PhD research area. My deepest gratitude to the theorists of Pupin Hall for intriguing discussions: Austin Joyce, Robert Penna, Rachel Rosen, Lam Hui, Alberto Nicolis, Klaas Parmentier, Manvir Grewal, Michael Zlotnikov and Luca Santoni.

I am furthermore incredibly lucky to have numerous amazing friends while at Columbia: Wenkai Zou, Bill Yin, Lin Xiong, Guanqun Ge ("the lunch crew"), Yihang Zeng, Albert Law, Dhruv Desai, Zhiyuan Sun, Kiran Salman, Manvir Grewal ("the ballers"), Xinyu Li, Jackson Tu, Qiang Han, Minghao Cheng, Chih-Hsi Lee("the card gang"), Giuliano Chiriaco, Jessica Zhang, Rui Hu, Tianhao Ren and Agnes Gui.

I finally want to thank my parents for supporting me along the path to my Ph.D. I am grateful for all the experiences we have shared and cannot thank you enough for being a part of my life.

Chapter 1: Introduction

Symmetry is ubiquitous in modern physics. It permeates physics at all energy scales, from sub-eV scales (e.g. condensed matter) up to the Planck scale (e.g. string theory). In these theories, symmetry has been used as a guiding line for model building, e.g. low energy effective theories [7, 8], and as a powerful tool to magically simplify computations, e.g. supersymmetric localization [9]. The mathematical tool to describe symmetry precisely and rigorously is group theory. Discrete symmetries are described by finite groups (e.g. the $\mathbb{Z}_2$ symmetry in Ising model without external magnetic field) and continuous symmetries are described by Lie groups/algebras (e.g. the $U(1)$ symmetry of QED). Apart from the discrete vs. continuous classification, symmetries in a physical theory can also be divided into spacetime and internal symmetries. While internal symmetry depends on the dynamical details, spacetime symmetry is determined by the background spacetime geometry. For example, the spacetime symmetry in a flat spacetime $\mathbb{R}^{1,d}$ is the Poincaré group $\mathbb{R}^d \rtimes \mathrm{SO}(1,d)$. As noted in the seminal work of Bargmann and Wigner [10], the Poincaré group plays a fundamental role in particle physics, classifying elementary particles by its unitary irreducible representations (UIRs). More precisely, the single-particle Hilbert space of a free elementary particle in flat spacetime furnishes a UIR of the Poincaré group. These UIRs can be divided into two classes: massive and massless, which are fundamentally different.

Wigner's classification can be directly generalized to de Sitter spacetime dS_{d+1} or Anti-de Sitter spacetime AdS_{d+1} which are also maximally symmetric spacetimes like $\mathbb{R}^{1,d}$, under the condition that the Poincaré group gets replaced by the corresponding isometry groups: $\mathrm{SO}(1,d+1)$ for dS_{d+1} and $\mathrm{SO}(2,d)$ for AdS_{d+1}. The physically relevant UIRs of $\mathrm{SO}(2,d)$ are familiar to physicists since $\mathrm{SO}(2,d)$ is also the symmetry group of a conformal field theory (CFT) on the flat spacetime $\mathbb{R}^{1,d-1}$. For this reason, studying the UIRs of $\mathrm{SO}(2,d)$ is a crucial step in understanding the structure of CFT Hilbert spaces. A complete classification of these UIRs can be found in [11, 12]. On the other hand, the UIRs of $\mathrm{SO}(1,d+1)$ are less familiar to physicists since they are not contained in the Hilbert space of any unitary Lorentzian CFT. The mathematical classification problem was solved in the early 60s [13]. More recently, as a technical tool, the representation theory of SO$(1, d+1)$ has led to many profound analytical results in CFTs, e.g. conformal partial wave expansion [14], conformal Regge theory [15] and CFT inversion formula [16].

This thesis is devoted to developing a group-theoretical approach, based on the representation theory of SO$(1, d+1)$ (SO$(2, d)$), towards quantum gravity with a positive (negative) cosmological constant. In particular, we will see how quantum corrections to the horizon entropy are universally calculable from purely group-theoretic data, for arbitrary effective field theories of quantum gravity. Before that, we want to give the motivation for this work and review some useful background.

1.1 Motivation

Mathematically, dS (AdS) spacetime is the maximally symmetric vacuum solution of Einstein's equations with a positive (negative) cosmological constant Λ. They can be represented as hypersurfaces in higher dimensional flat space

$$-X_0^2 + X_1^2 + \cdots + X_d^2 \pm X_{d+1}^2 = \pm \ell^2, \quad \ell \propto \frac{1}{\sqrt{\pm \Lambda}} \tag{1.1.1}$$

where the "$+$" sign corresponds to dS and the "$-$" sign corresponds to AdS. More details about the geometry for (A)dS are shown in their Penrose diagrams (1.1.1). The diamond denoted by "S" or "N" in the dS Penrose diagram (1.1.1a) is called the southern or northern static patch, the maximal region accessible to a local observer. The static patch metric is

$$ds^2 = -\left(1 - \frac{r^2}{\ell^2}\right) dt^2 + \frac{dr^2}{1 - \frac{r^2}{\ell^2}} + r^2 d\Omega_{d-1}^2, \quad 0 \leq r \leq \ell \tag{1.1.2}$$

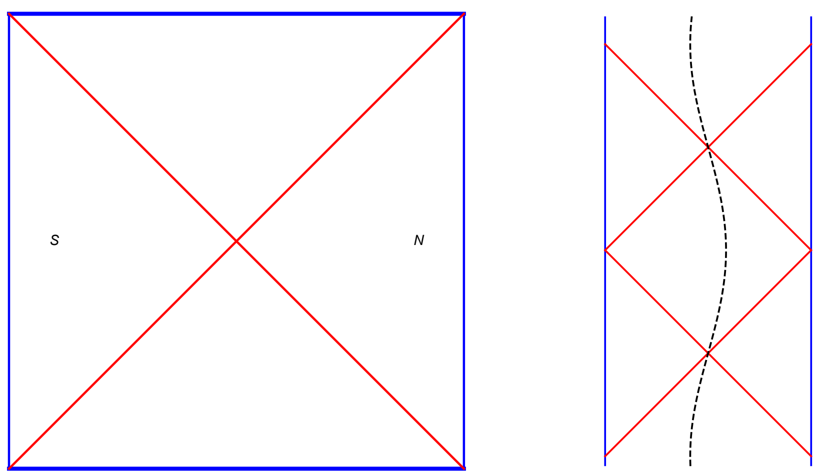
where $d\Omega_{d-1}^2$ denotes the standard metric of a $(d-1)$-dimensional unit sphere. The de Sitter horizon sits at $r = \ell$ and the classical de Sitter horizon entropy is given by the area law [17]

$$\mathcal{S}_0 = \frac{A}{4\, G_N} \tag{1.1.3}$$

where A is the area of dS horizon. In global coordinates, which cover the whole strip of (1.1.1b), the AdS metric takes a similar form, with $\ell \to i\ell$:

$$ds^2 = -\left(1 + \frac{r^2}{\ell^2}\right) dt^2 + \frac{dr^2}{1 + \frac{r^2}{\ell^2}} + r^2 d\Omega_{d-1}^2, \quad 0 \leq r < \infty \tag{1.1.4}$$

The AdS boundary is located at $r \to \infty$. Other coordinates and the corresponding metrics of dS (AdS) can be found in the appendix C.3.2 (D.5).



(a) dS Penrose diagram ("a square") (b) AdS Penrose diagram ("a strip")

Figure 1.1.1: In (a): the two horizontal lines (thick blue) denote future and past boundaries, the two vertical lines (blue) are actually in the interior of spacetime manifold and the two diagonal lines (red) correspond to cosmological horizon. In (b): the two vertical lines (blue) represent the cylinder-shaped boundary, the diagonal lines (red) are trajectories of light and the dashed line correspond to trajectory of a massive particle.

dS is an important realistic model in cosmology while AdS is more like an ideal lab for string theory/quantum gravity that only exists in our heads. For concreteness, we will focus on dS in what follows. But one should keep in mind that our strategy towards dS quantum gravity, which will be discussed below, can also be directly applied to AdS.

According to the recent cosmological experiments, including supernovae observations [18–20], CMB measurements [21] from Planck and the baryonic acoustic oscillations [22], our universe is accelerating. In the framework of ΛCDM, this accelerating expansion will persist and eventually with matter and light diluting away fast and with vacuum energy dominating, the observable universe will end up being a de Sitter static patch, surrounded by a sphere-shaped cosmological horizon. So a local observer is causally disconnected from everything outside his/her static patch. In view of this and other reasons, it is of evident importance to try to understand quantum gravity in de Sitter spacetime. However, unlike in the AdS case where, thanks to the AdS-CFT correspondence [23], quantum gravity (at least in certain examples) can be microscopically formulated in terms of CFTs, we still do not have a UV-complete microscopic model of quantum gravity in dS. We even do not know what are the basic ingredients in such a theory. There have been many attempts towards this problem, including string theoretical construction of metastable dS vacua [24–27] and holographic considerations [17, 28–40]. But none of them can answer one of the central problems in dS quantum gravity:

What is the microscopic origin of de Sitter entropy?

In contrast, string theory has been very successful in the microscopic accounting of Bekenstein-Hawking entropy for certain BPS black holes, starting from the seminal work [41] by Strominger and Vafa. Roughly speaking, the Bekenstein-Hawking entropy $S_{\text{BH}}(Q)$ of a given set of charges Q can be obtained by counting degeneracy $d(Q)$ of BPS states in large charge limit $Q \to \infty$

$$S_{\text{BH}}(Q) = \log d(Q) \tag{1.1.5}$$

Being completely clueless about the fundamental microscopic theory of quantum gravity in dS, we would like to take a systematic approach to this problem, using only machinery that is fully understood, namely quantum field theory. In other words, even though the UV-completed theory is far from being known, the low-energy theory of quantum gravity is assumed to be perturbatively described by low-energy effective fields, including the massless graviton in particular. In the regime of low-energy effective field theory, we can ask the following two questions:

(a) What effective field theories are allowed?

(b) What can we learn about the underlying microscopic theory of quantum gravity?

The dS isometry group $\text{SO}(1, d+1)$ and its representation theory appear quite naturally when we try to answer these questions.

To answer question (a), we should specify both field content and interactions of the fields. A fundamental constraint on the field content is unitarity. According to our previous discussion about Wigner's classification, the unitarity requirement is the same as the unitarity of the corresponding SO$(1, d+1)$ representations. For example, looking at the list of SO$(1, d+1)$ UIRs given in section 3.3.6, more specifically the principal and complementary series, we see that there are massive spin-s UIRs with a lower bound on the mass for $s \geq 2$, the unitarity bound for this family of SO$(1, d+1)$ representations. From a field theory point of view, the same lower bound arises as the condition for the theory to be ghost free, famously known as the Higuchi bound [42]. In addition to these massive spin-s particles/UIRs, there exists a gigantic zoo of more exotic particle species, including the exceptional and discrete series of UIRs in the list of section 3.3.6. The corresponding zoo of fields includes p-form gauge fields, massless higher spin fields, partially massless higher spin fields, etc, many of which do not have an analog in flat space, but all of which could appear in low-energy effective field theories of dS quantum gravity.

The existence of nontrivial interactions is a very strong constraint, beyond the scope of representation theory. For example, in flat spacetime, there exist various no-go theorems [43–47] forbidding interacting massless higher-spin field theories. However, these no-go theorems can be bypassed by turning on a non-zero cosmological constant and allowing an infinite tower of higher spin fields. In a series paper by Fradkin and Vasiliev[48–52], a consistent interacting action up to the cubic order was found on a fixed (A)dS background, and in [53–55] the fully nonlinear equations of motion for interacting massless higher spin fields (known as the Vasiliev equations) was written down in the so-called unfolding formalism. The Vasiliev theories of gravity provide important examples in AdS/CFT, going beyond string theory. For instance, the type-A Vasiliev gravity in AdS$_4$ whose field content consists of a conformally coupled real scalar and massless fields of spin $s = 0, 1, 2, 3, ..., \infty$, is conjectured to be dual to free $U(N)$ vector model on the boundary [56].

Altogether, this illustrates higher-spin fields and more exotic variants thereof can be equally important as low-spin fields in low-energy theories of quantum gravity in (A)dS, providing further motivation for a general group-theoretic approach.

For question (b), our toolkit for dS is much more limited compared to flat spacetime and AdS. There is no S-matrix, there are no boundary correlation functions, and no asymptotic physical charges available to define physically meaningful quantities that could be used as sharp quantitative tests of microscopic models. However, there is at least one physically meaningful quantity that could serve this purpose: the de Sitter entropy $\mathcal{S}$ itself. The entropy, including quantum corrections, is computed macroscopically as the Euclidean path integral of the effective field theory [57]

$$\mathcal{S} = \log \mathcal{Z} \qquad \mathcal{Z} = \int \mathcal{D}g \, \mathcal{D}\Phi \, e^{-S_E[g,\Phi]} \tag{1.1.6}$$

where g denotes the metric fluctuation, Φ is a collective symbol for other fields, and the path integral is expanded around its round sphere saddle point which is the Wick rotation of dS static patch in eq. (1.1.2). Concretely, we wish to extract unambiguously calculable, UV-insensitive terms in the semiclassical loop expansion of $\mathcal{S}$, for the purpose of comparison with microscopic models attempting to match this in the form of a large-N expansion or otherwise. At tree level, the above path integral yields the classical horizon entropy $\mathcal{S}_0$ of eq. (1.1.3). But this cannot be used to constrain microscopic models, because unlike for black holes, it is just a number, a renormalized dimensionless coupling constant, instead of a function of physical charges like $S_{\mathsf{BH}}(Q)$ in eq. (1.1.5). So we have to go beyond tree level. As we will see in chapter 5, certain (nonlocal) quantum correction to the dS horizon entropy, including in particular nonlocal 1-loop corrections, are UV-insensitive, and moreover exactly calculable. Therefore, our goal boils down to computing the one-loop sphere path integrals, for arbitrary effective field theories.

However, the one-loop sphere path integral is not as innocent as it looks. For a massless spin-s field, we need to fix a gauge and introduce a spin-$(s-1)$ ghost field. For a massive spin-s field, defined using the Stueckelberg trick, we need to introduce a tower of massless auxiliary fields [58] and each auxiliary field requires the same steps as in the massless case. It gets extremely involved to keep track of all the off-shell degrees of freedom as the spin increases. As we will see in section 5.4 and 5.5, by using SO$(1, d+1)$ representation theory, we find a way to compute the one-loop path integral by using only on-shell data. The latter is precisely and elegantly encoded in SO$(1, d+1)$ Harish-Chandra characters, which we will review extensively in chapter 3.

Altogether, the answer to question (b) is that we propose a nontrivial, quantitatively precise test constraining possible microscopic models of de Sitter quantum gravity. Let's illustrate this test more explicitly with 3D Einstein gravity as an example. Assume that 3D Einstein gravity is the low energy effective theory of some UV-complete microscopic of quantum gravity in dS. The path integral of 3D pure gravity as in (1.1.6) yields

$$\mathcal{S} = \mathcal{S}_0 - 3\log\mathcal{S}_0 + 5\log(2\pi) + \sum_{n\geq 1} c_n \mathcal{S}_0^{-2n} \tag{1.1.7}$$

where the leading term is dS horizon entropy and the rest corresponds to exactly calculable, nonlocal quantum corrections (i.e. cannot be absorbed into local counterterms). $\mathcal{S}$ in (1.1.7) has to be matched by the microscopic entropy spit out by the microscopic model, say $\mathcal{S}_{\text{mic}} = \log d(N)$, for some degeneracy $d(N)$. This imposes a very strong constraint on the function $d(N)$, and in turn, the microscopic model. As a simple illustration, consider for instance a hypothetic microscopic model spitting out $d(N) = \binom{2N}{N}$. Then

$$\mathcal{S}_{\text{mic}} = \mathcal{S}_0 - \frac{1}{2}\log\mathcal{S}_0 + \log\left(\frac{\pi}{2\log 2}\right) + \mathcal{O}\left(\mathcal{S}_0^{-2}\right) \tag{1.1.8}$$

where we have identified $\mathcal{S}_0 = N\log 4 + \mathcal{O}(N^{-1})$. The evident discrepancy between (1.1.7) and (1.1.8) rules out the model.

1.2 Structure of this thesis

This thesis consists of four main parts. The chapter 2 and chapter 3 form an extensive and quasi self-contained review of the representation theory of the de Sitter isometry group SO$(1, d+1)$. The unitary irreducible representations are constructed and classified. The list of UIRs and their corresponding de Sitter quantum fields are given in the 3.3.6. The Harish-Chandra characters of these unitary irreducible representations (and some non-unitary representations) are defined and computed. More explicitly, given a UIR ρ, the Harish-Chandra character associated to ρ is defined as $\Theta_\rho(t) \equiv \operatorname{tr}_\rho e^{-itL}$, where $t \in \mathbb{R}$ and L is a *boost* in $\mathfrak{so}(1, d+1)$ that generates time translations in the southern static patch, c.f. (1.1.2) [1]. For instance, the Harish-Chandra

[1] In contrast, the element H in $\mathfrak{so}(2, d)$ generating AdS global time translation (c.f. (1.1.4)), is positive definite. So given a UIR R of SO$(2, d)$, the most natural definition (which is also the definitions of character used in CFT [12]) of an AdS character is $\operatorname{tr}_R e^{-tH}$, where $t > 0$. With this definition, the character of linearized gravity in

character corresponding to linearized gravity in dS_5 is

$$\Theta(t) = \frac{10\,q^2}{(1-q)^4}, \quad q = e^{-|t|} \tag{1.2.1}$$

Such characters will be a key ingredient in all of the work presented in this thesis.

Chapter 4 studies higher spin quasinormal modes in the static patch of dS_{d+1}. The traditional separation of variables method of solving quasinormal modes becomes almost impossible in practice when the spin is large. We develop an algebraic approach to this problem, based on the ambient space formalism of higher spin fields and the observation that quasinormal modes of a single field field in dS furnish two (non-unitary) lowest-weight representation of the dS isometry group $\mathrm{SO}(1, d+1)$ [59, 60]. In particular, we obtain the explicit expression of the two lowest-weight quasinormal modes for scalar fields, massive spinning fields and massless spinning fields. The whole quasinormal spectrum is then built from these lowest-weight quasinormal modes by using conformal algebra. More importantly, we define a generating function that encodes the corresponding quasinormal spectrum of any unitary field and we find the generating function is exactly the Harish-Chandra character associated to this field. This result provides a physical interpretation for Harish-Chandra characters, namely that the characters count quasinormal modes in dS. For instance, let's take the Harish-Chandra character in eq. (1.2.1), expanded around $q = 0$

$$\Theta = 10\,q^2 + 40\,q^3 + 100\,q^4 + 200\,q^5 + \mathcal{O}\left(q^6\right) \tag{1.2.2}$$

from which we can immediately read off the quasinormal mode spectrum of linearized gravity in dS_5, i.e. there are 10 quasinormal modes with quasinormal frequency $\omega_0 = -2i$, 40 quasinormal modes with quasinormal frequency $\omega_1 = -3i$, 100 quasinormal modes with quasinormal frequency $\omega_2 = -4i$, etc.

The long chapter 5 is devoted to computing de Sitter entropy from sphere path integral. We first develop an intuitive thermal picture, i.e. the one-loop path integral on S^{d+1} should be *formally* equal to the thermal partition function in static patch of dS_{d+1} up to corrections

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AdS_5 is $\frac{9\,q^4 - 4\,q^5}{(1-q)^4}$, $q = e^{-t}$. But one should be aware that a character defined in this way is *not* a group character because e^{-tH} does not belong to the isometry group $\mathrm{SO}(2, d)$.

associated to the horizon. Harich-Chandra characters naturally appear in the thermal partition function. Given the physical interpretation of Harish-Chandra characters obtained in the previous chapter, we get to learn how quasinormal modes contribute to the one-loop partition function. Then we perform the one-loop sphere partition function rigorously using heat kernel regularization for scalars, Dirac spinors and massive/massless higher spin fields, from which we obtain the one-loop correction to de Sitter entropy for all effective field theories by using eq. (5.1.1). The result, which is summarized in the eq. (5.1.4), has a universal "bulk-edge" structure for all field content supplemented by a group volume factor for gauge fields:

$$\mathcal{S}^{(1)} = \log \prod_{a=0}^{K} \frac{(2\pi\gamma_a)^{\dim G_a}}{\operatorname{vol} G_a} + \int_0^\infty \frac{dt}{2t}\left(\frac{1+q}{1-q}\,\Theta_{\rm tot}^{\rm bos} - \frac{2\sqrt{q}}{1-q}\,\Theta_{\rm tot}^{\rm fer}\right) + \mathcal{S}_{\rm ct} \tag{1.2.3}$$

where G_a denote gauge groups and $\Theta_{\rm tot} = \Theta_{\rm bulk} - \Theta_{\rm edge}$. The bulk character $\Theta_{\rm bulk}$ contains a bunch of SO$(1,d+1)$ Harish-Chandra characters and hence the bulk contribution to $\mathcal{S}^{(1)}$ admits the interpretation as (logarithm of) thermal partition function. In three dimensional de Sitter, higher spin gravity can be formulated in terms of SL$(n,\mathbb{C})$ Chern Simons theory with n being the spin cut-off. Classically, we find a landscape of vacua for higher spin gravity, corresponding to different embeddings of $\mathfrak{sl}(2)$ into $\mathfrak{sl}(n)$. At quantum level, as a Chern Simons theory is exactly solvable, we can compute all-loop quantum entropy and show that is given by the absolute value squared of a topological string partition function.

The chapter 6 is of independent interest. Inspired by the result of chapter 5 and the higher spin holography, we revisit the one-loop path integral of higher spin fields in $(d+1)$ dimensional Euclidean AdS (EAdS) and find a character integral representation for the one-loop free energy. Compared to the de Sitter case, the "bulk-edge" structure persists but the group volume factor disappears as the zero modes are non-normalizable in EAdS. The bulk part is captured by SO$(2,d)$ characters and the edge part is argued to be associated to a EAdS$_{d-1}$-shaped horizon in Rindler-AdS coordinate. For example, the Euclidean one-loop path integral of a Maxwell field in AdS$_4$ is given by (dropping UV regularization)

$$\log Z = \int_0^\infty \frac{dt}{2t}\,\frac{1+q}{1-q}\left(\Theta_b - \Theta_e\right), \quad q = e^{-t} \tag{1.2.4}$$

where the bulk character $\Theta_b = \frac{3q^2-q^3}{(1-q)^3}$ and the edge character $\Theta_e = \frac{q}{1-q}$. We apply the character integral representation to the one-loop free energy of various Vasiliev theories of gravity. The vanishing of one-loop free energy of Vasiliev theories, predicted by higher spin holography, is strikingly realized by the algebraic relation:

$$\text{total bulk character} - \text{total edge character} = \text{boundary character}$$

where the boundary character vanishes for non-minimal type-A Vasiliev gravity and is the SO$(1,d)$ Harish-Chandra character corresponding to a conformally coupled scalar for minimal type-A Vasiliev gravity.

It should be mentioned that some part of the authors's work has been left out of this thesis. It includes the following two papers: [61] with F.Denef, S.Kachru and A. Tripathy, and [40] with D. Anninos, F. Denef and R. Monten. In [61], we conjecture that the three-center BPS bound states in type II string theory compactified on $K3\times T^2$ is counted by the degree three Siegel modular form. We test the conjecture by checking wall-crossing properties (including the degenerating limits where one-center and two-center objects appear) and holographic bounds. In [40], we propose a microscopic model for minimal higher spin de Sitter quantum gravity. In particular, this model provides a precise definition of the Hilbert space of higher spin dS quantum gravity, its operator algebra and its Hartle-Hawking vacuum state. This model can be used to compute late time cosmological correlation functions, probabilities of arbitrary field configurations (beyond the perturbative regime) and reconstruct bulk Heisenberg algebra (up to errors exponentially suppressed by de Sitter Bekenstein-Hawking entropy).

Chapter 2: Representation theory of SO(K)

In this short chapter, we review some basic facts about the irreducible finite-dimensional representations of special orthogonal groups. We will also derive a new expression for the Weyl characters and a relation of Weyl dimension formulae. They are important technical tools when we study the de Sitter isometry group and perform sphere path integrals of higher spin fields.

2.1 General facts

The special orthogonal group SO(K) $(K \geq 2)$ consists of $K \times K$ matrices satisfying $O^T O = 1$ and $\det O = 1$ and its Lie algebra $\mathfrak{so}(K)$ consists of antisymmetric matrices. We choose a basis $L_{AB} = -L_{BA}, 1 \leq A, B \leq K$ of $\mathfrak{so}(K)$ that satisfy the following commutation relation

$$[L_{AB}, L_{CD}] = \delta_{BC} L_{AD} + \text{permutations} \tag{2.1.1}$$

The Cartan subalegbra is generated by $J_1 \equiv L_{12}, J_2 \equiv L_{34}, \cdots, J_r \equiv L_{2r-1,2r}$ for $K = 2r$ or $K = 2r + 1$.

Every irreducible finite-dimensional representation of SO(K) with $K = 2r$ or $K = 2r + 1$ is labelled by a highest weight vector $\mathbf{s} = (s_1, \ldots, s_r)$ ordered from large to small, satisfying

- **Positive condition**: all s_i are nonnegative except s_r which can be negative when $K = 2r$. Different signs of s_r distinguish the chirality of a representation.
- **Integral condition**: s_i are either all integer (bosons) or all half-integer (fermions).

For example, the spin-s representation corresponds to $\mathbf{s} = (s, 0, \cdots, 0)$. In the bosonic case, we can associate a Young diagram $\mathbb{Y}_{\mathbf{s}}$ to the highest weight vector $\mathbf{s}$ such that there are s_i boxes in the i-th row of $\mathbb{Y}_{\mathbf{s}}$ (ignoring the subtlety when s_r becomes negative). We shall use the shorthand notations $\mathbb{Y}_s$ and $\mathbb{Y}_{n,s}$ when $\mathbf{s} = (s, 0, \cdots, 0)$ and $\mathbf{s} = (n, s, 0, \cdots, 0)$ respectively.

2.2 Weyl dimension formula

For various applications in this thesis we need the dimensions $D^K_{\mathbf{s}}$ of these SO(K) representations $\mathbf{s}$. The Weyl dimension formula gives a general expression for the dimensions of irreducible representations of simple Lie groups. For the SO(K) this is

• $K = 2r$:

$$D^K_{\mathbf{s}} = \mathcal{N}_K^{-1} \prod_{1\le i<j\le r} (\ell_i + \ell_j)(\ell_i - \ell_j)\,, \qquad \ell_i \equiv s_i + \tfrac{K}{2} - i \tag{2.2.1}$$

with $\mathcal{N}_K$ independent of $\mathbf{s}$, hence fixed by $D^K_0 = 1$, i.e. $\mathcal{N}_K = \prod_{1\le i<j\le r}(K - i - j)(j - i)$.

• $K = 2r + 1$:

$$D^K_{\mathbf{s}} = \mathcal{N}_K^{-1} \prod_{1\le i\le r} (2\ell_i) \prod_{1\le i<j\le r} (\ell_i + \ell_j)(\ell_i - \ell_j)\,, \qquad \ell_i \equiv s_i + \tfrac{K}{2} - i\,, \tag{2.2.2}$$

where $\mathcal{N}_K$ is fixed as above: $\mathcal{N}_K = \prod_{1\le i\le r}(K - 2i)\prod_{1\le i<j\le r}(K - i - j)(j - i)$.

When $\mathbf{s}$ has only one or two nonvanishing entries, $D^K_{\mathbf{s}}$ takes a simpler form that works for both odd and even K

$$D^K_s = \binom{s + K - 1}{K - 1} - \binom{s + K - 3}{K - 1} = (2s + K - 2)\frac{\Gamma(s + K - 2)}{\Gamma(s + 1)\Gamma(K - 1)} \tag{2.2.3}$$

$$D^K_{n,s} = \frac{(K + 2n - 2)(K + 2s - 4)(n - s + 1)(K + n + s - 3)\Gamma(K + n - 3)\Gamma(K + s - 4)}{\Gamma(K - 3)\Gamma(K - 1)\Gamma(n + 2)\Gamma(s + 1)} \tag{2.2.4}$$

The expression $D^K_{n,s}$ in eq. (2.2.4) as a function of n and s can be analytically continued to $\mathbb{C} \times \mathbb{C}$ apart from some discrete poles. In this sense, it satisfies an interesting relation

$$\boxed{D^K_{s,n} = -D^K_{n-1,s+1}} \tag{2.2.5}$$

For convenience we list here some low-dimensional explicit expressions:

K	D^K_s	$D^K_{n,s}$	$D^K_{k+\frac12,\mathbf{\frac12}}$
2	1		1
3	$2s+1$		$2\binom{k+1}{1}$
4	$(s+1)^2$	$(n-s+1)\,(n+s+1)$	$2\binom{k+2}{2}$
5	$\frac{(s+1)(s+2)(2s+3)}{6}$	$\frac{(2n+3)(n-s+1)(n+s+2)(2s+1)}{6}$	$4\binom{k+3}{3}$
6	$\frac{(s+1)(s+2)^2(s+3)}{12}$	$\frac{(n+2)^2(n-s+1)(n+s+3)(s+1)^2}{12}$	$4\binom{k+4}{4}$
7	$\frac{(s+1)(s+2)(s+3)(s+4)(2s+5)}{120}$	$\frac{(n+2)(n+3)(2n+5)(n-s+1)(n+s+4)(s+1)(s+2)(2s+3)}{720}$	$8\binom{k+5}{5}$
8	$\frac{(s+1)(s+2)(s+3)^2(s+4)(s+5)}{360}$	$\frac{(n+2)(n+3)^2(n+4)(n-s+1)(n+s+5)(s+1)(s+2)^2(s+3)}{4320}$	$8\binom{k+6}{6}$

(2.2.6)

Here $(k+\frac12,\mathbf{\frac12})$ means $(s_1,\ldots,s_r)=(k+\frac12,\frac12,\ldots,\frac12)$, i.e. the spin $s=k+\frac12$ representation.

2.3 A new expression for SO$(d+2)$ characters

Besides the dimensions $D^K_{\mathbf{s}}$, we also need the characters $\mathrm{tr}_{\mathbf{s}} x_1^{J_1}\cdots x_r^{J_r}$ of SO(K) representations. Instead of using the famous Weyl character formula directly, we will show a new expression that works exclusively for the single-row and two-row representations. This new expression is based on the following observation:

Claim. *The character $\Theta^{SO(d+2)}_{\mathbb{Y}_{ns}}(x)\equiv \mathrm{tr}_{\mathbb{Y}_{ns}} x^{L_{AB}}$ has the same polar part (i.e. terms of negative powers in x) and same constant term as the following function*

$$\boxed{P^d_{n,s}(x)\equiv\frac{D^d_s\,x^{-n}-D^d_{n+1}\,x^{1-s}}{(1-x)^d}} \tag{2.3.1}$$

Since $\Theta^{SO(d+2)}_{\mathbb{Y}_{ns}}(x)$ is symmetric under $x\leftrightarrow x^{-1}$ by construction, it is then completely encoded in the function $P^d_{n,s}(x)$. In particular, when $s=0$, i.e. spin-n representation, the character $\Theta^{SO(d+2)}_{\mathbb{Y}_n}(x)$ is encoded in

$$\boxed{Q^d_n(x)\equiv\frac{x^{-n}}{(1-x)^d}} \tag{2.3.2}$$

More explicitly, let $f(x)$ be a function with well-defined Laurent expansion around $x=0$.

Denote the polar part of $f(x)$ by $[f(x)]_-$ and denote the polar part together with the constant term of $f(x)$ by $[f(x)]_0$. With this notations, the claim implies that

$$\boxed{\Theta^{\mathsf{SO}(d+2)}_{\mathbb{Y}_{ns}}(x) = \Big[P^d_{n,s}(x)\Big]_0 + \left(\Big[P^d_{n,s}(x)\Big]_-\Big|_{x\to x^{-1}}\right)} \tag{2.3.3}$$

A simple way to prove this claim is by induction. It only involves using branching rules of special orthogonal groups. Here we want to present a more illuminating approach that requires information about the detailed structure of the representation $\mathbb{Y}_{ns}$, which will be reviewed in a CFT-style language as follows. Mimicking the conformal algebra, we define the following basis

$$M_{ij} = L_{ij}, \quad H = iL_{0,d+1}, \quad P_i = L_{d+1,i} + iL_{0i}, \quad K_i = -L_{d+1,i} + iL_{0i}, \tag{2.3.4}$$

where $1 \leq i \leq d$. The new basis leads to some interesting commutators

$$[M_{ij}, H] = 0, \quad [H, P_i] = P_i, \quad [H, K_i] = -K_i \quad [K_i, P_j] = 2M_{ij} - 2\delta_{ij}H \tag{2.3.5}$$

In particular, P_i raises the eigenvalue of H by 1 while K_i lowers the eigenvalue of H by 1. First, consider spin-n representation of SO$(d+2)$ generated by the lowest weight state $|lw\rangle$ that satisfies

$$H|lw\rangle = -n|lw\rangle, \quad K_i|lw\rangle = 0, \quad M_{ij}|lw\rangle = 0 \tag{2.3.6}$$

A generic state in the Verma module generated by $|lw\rangle$ is a linear combination of the descendants $P_{i_1}\cdots P_{i_k}|lw\rangle$. However, there are very strict constraints on the descendants imposed by the integralness of n. To see this more explicitly, let's switch to the wavefunction picture. In this picture, the representations space consists of degree-n polynomials $\varphi(X)$ of $X^A \in \mathbb{R}^{d+2}$, where $A = 0, 1, \cdots, d+1$, satisfying $\partial_X^2\varphi = 0$ and the generators L_{AB} act as differential operators $X_A\partial_B - X_B\partial_A$. Define complex (lightcone) coordinate $z = X_{d+1} + iX_0, \bar{z} = X_{d+1} - iX_0$ and then the differential operator realization of H, P_i, K_i can be expressed as

$$H = z\partial_z - \bar{z}\partial_{\bar{z}}, \quad P_i = z\partial_i - 2\,X_i\partial_{\bar{z}}, \quad K_i = -\bar{z}\partial_i + 2\,X_i\partial_z \tag{2.3.7}$$

It is easy to check that the lowest weight state $|lw\rangle$ corresponds to a $\psi_{lw}(z,\bar{z}) = \bar{z}^n$. One crucial observation is that all of the states $\psi_{i_1\cdots i_k} \equiv P_{i_1}\cdots P_{i_k}\psi_{lw}$ with $k \le n$ are linearly independent. To show this, it suffices to notice that the top component of $\psi_{i_1\cdots i_k}$ in X_i is $X_{i_1}\cdots X_{i_k}\bar{z}^{n-k}$ for $k \le n$ which arises from the $-2X_i\partial_{\bar{z}}$ part of each P_i. It's apparently that $X_{i_1}\cdots X_{i_k}\bar{z}^{n-k}$ are nonvanishing and linearly independent and hence all $\psi_{i_1\cdots i_k}$ with $k \le n$ are linearly independent. Then it's almost trivial to construct the nonpositive powers in the character $\Theta^{\mathsf{SO}(d+2)}_{\mathbb{Y}_n}(x) = \operatorname{Tr} x^H$

$$\left[\Theta^{\mathsf{SO}(d+2)}_{\mathbb{Y}_n}(x)\right]_0 = \sum_{k=0}^{n}\binom{d+k-1}{d-1}x^{k-n} = \left[\sum_{k=0}^{\infty}\binom{d+k-1}{d-1}x^{k-n}\right]_0 = \left[Q^d_n(x)\right]_0 \qquad (2.3.8)$$

This is exactly the single-row case of the claim. Then the full character is reconstructed using the $x \leftrightarrow x^{-1}$. For example, due to this symmetry the coefficient of x in $\Theta^{\mathsf{SO}(d+2)}_{\mathbb{Y}_n}(x)$ should be $\binom{d+n-2}{d-1}$ rather than the native counting $\binom{d+n}{d-1}$. Even more explicitly, we can find what are these $\binom{d+n-2}{d-1}$ states. The subspace $\mathcal{H}^{(n)}_0$ where $H=0$ is spanned by all $x_{i_1}\cdots x_{i_n}$. Acting P_j on them yields basis for the subspace $\mathcal{H}^{(n)}_1$ where $H=1$

$$P_j x_{i_1}\cdots x_{i_n} = z\delta_{j(i_1}x_{i_2}\cdots x_{i_n)} \qquad (2.3.9)$$

Therefore $\mathcal{H}^{(n)}_1$ is spanned by $z\,x_{i_1}\cdots x_{i_{n-1}}$ and has dimension $\binom{d+n-2}{d-1}$.

Next, we consider more complicated representations like $\mathbb{Y}_{ns}$, for which we have a lowest weight state $|lw\rangle_{i_1\cdots i_s}$ that carries a spin-s representation of SO(d) (rigorously speaking "lowest weight state" is not the correct terminology here but we'll stick to it for convenience). As before, $|lw\rangle_{i_1\cdots i_s}$ has quantum number $-n$ under H and is annihilated by K_i. To represent the lowest weight state as a wavefunction, we need to introduce another copy of $\mathbb{R}^{d+2}$ with coordinate Y^A such that L_{AB} is realized as

$$L_{AB} = X_A\partial_{X_B} - X_B\partial_{X_A} + Y_A\partial_{Y_B} - Y_B\partial_{Y_A} \qquad (2.3.10)$$

and the representation space consists of homogeneous polynomials $\varphi(X,Y)$ satisfying

$$(X\cdot\partial_X - n)\varphi(X,Y) = (Y\cdot\partial_Y - s) = \partial_X^2\varphi(X,Y) = X\cdot\partial_Y\varphi(X,Y) = 0 \qquad (2.3.11)$$

Define complex (lightcone) coordinate for the Y-space $w = Y_{d+1} + iY_0, \bar{w} = Y_{d+1} - iY_0$ and then H, P_i, K_i can be expressed as

$$H = z\partial_z - \bar{z}\partial_{\bar{z}} + w\partial_w - \bar{w}\partial_{\bar{w}}, \quad P_i = z\partial_{X_i} - 2\,X_i\partial_{\bar{z}} + w\partial_{Y_i} - 2\,Y_i\partial_{\bar{w}}$$
$$K_i = -\bar{z}\partial_{X_i} + 2\,X_i\partial_z - \bar{w}\partial_{Y_i} + 2\,Y_i\partial_w \tag{2.3.12}$$

It's easy to check that the following wavefunction actually corresponds to the lowest weight state

$$\psi^{lw}_{i_1\cdots i_s} = \bar{z}^{n-s}(\bar{z}Y - \bar{w}X)_{i_1}\cdots(\bar{z}Y - \bar{w}X)_{i_s} - \text{trace} \tag{2.3.13}$$

Introduce a *null* d-vector u_i and then $\psi^{lw}_{i_1\cdots i_s}$ can be more efficiently written as

$$\psi^{lw}_{ns}(X_A, Y_B; u_i) = \bar{z}^{n-s}(\bar{z}\,Y\cdot u - \bar{w}\,X\cdot u)^s \tag{2.3.14}$$

The claim $[\Theta^{\mathsf{SO}(d+2)}_{\mathbb{Y}_{ns}}(x)]_0 = [\frac{D^d_s x^{-n} - D^d_{n+1}x^{1-s}}{(1-x)^d}]_0$, in particular the x^{1-s} part on the R.H.S, signals that there should be a constraint on the descendants at level $(n+1-s)$. To show this, let's focus on the component of $\psi^{lw}_{i_1\cdots i_s}$with the highest degree in X_i, which is roughly $\bar{z}^{n-s}\bar{w}^s X_{i_1}\cdots X_{i_s}$ up to pure trace component. Similarly, the top X_i components in the descendants of ψ^{lw}_{ns} come from $-2X_i\partial_{\bar{z}} \in P_i$ with $\partial_{\bar{z}}$ acting on $\bar{z}^{n-s}$, that is,

$$P_{i_1}\cdots P_{i_k}\psi^{lw}_{j_1\cdots j_s} \ni (-2)^k\left(\partial^k_{\bar{z}}\bar{z}^{n-s}\right)\,\bar{w}^s X_{i_1}\cdots X_{i_k}X_{j_1}\cdots X_{j_s} \tag{2.3.15}$$

Therefore for level $0 \le k \le n-s$, all the naive descendants $P_{i_1}\cdots P_{i_k}\psi^{lw}_{j_1\cdots j_s}$ are nonvanishing and linearly independent. However at level $k = n+1-s$, one can easily check that $(u\cdot P)^k\psi^{lw}_{ns}$ vanishes identically which means the following spin-$(n+1)$ wavefunction is actually zero

$$P_{(i_1}\cdots P_{i_{n+1-s}}\psi^{lw}_{j_1\cdots j_s)} - \text{trace} = 0 \tag{2.3.16}$$

Without this constraint, the polar part of $\Theta^{\mathsf{SO}(d+2)}_{\mathbb{Y}_{ns}}(x)$ would simply be encoded in $D^d_s\frac{x^{-n}}{(1-x)^d}$, i.e. the result of $s=0$ multiplied by a spin degeneracy. With this constraint, we should subtract the contribution of itself together with its descendants, which is $D^d_{n+1}\frac{x^{1-s}}{(1-x)^d}$ from a simple counting. Altogether, we get

$$\left[\Theta^{\mathsf{SO}(d+2)}_{\mathbb{Y}_{ns}}(x)\right]_0 = \left[\frac{D^d_s x^{-n} - D^d_{n+1} x^{1-s}}{(1-x)^d}\right]_0 = \left[P^d_{n,s}(x)\right]_0 \tag{2.3.17}$$

Before moving to the implication of the claim, let's illustrate it again with the explicit example $\mathbb{Y}_{11}$. In this case, the lowest weight (vector valued) wavefunction is

$$\psi^{lw}_i(X,Y) = \bar{z}\, Y_i - \bar{w}\, X_i \tag{2.3.18}$$

and the states at the next level

$$P_j \psi^{lw}_i(X,Y) = 2(X_i\, Y_j - X_j\, Y_i) + (w\,\bar{z} - z\,\bar{w})\delta_{ij} \tag{2.3.19}$$

where we only have antisymmetric and pure trace components and the symmetric traceless part is missing. Therefore the character is

$$\Theta^{\mathsf{SO}(d+2)}_{\mathbb{Y}_{11}}(x) = d\left(x + x^{-1}\right) + \frac{d(d-1)}{2} + 1 \tag{2.3.20}$$

The claim (2.3) has some important corollaries. We first rewrite $P^d_{n,s}$ as

$$P^d_{n,s}(x) = D^d_s Q^d_n(x) - D^d_{n+1} Q^d_{s-1}(x) \tag{2.3.21}$$

Since $P^d_{n,s}(x)$ encodes the character of $\mathbb{Y}_{n,s}$ and $Q^d_n(x)$ encodes the character of $\mathbb{Y}_n$, the elementary equation (2.3.21) yields a highly nontrivial relation of characters

$$\boxed{\Theta^{\mathsf{SO}(d+2)}_{\mathbb{Y}_{n,s}}(x) = D^d_s \Theta^{\mathsf{SO}(d+2)}_{\mathbb{Y}_n} - D^d_{n+1} \Theta^{\mathsf{SO}(d+2)}_{\mathbb{Y}_{s-1}}} \tag{2.3.22}$$

It allows us to construct the Weyl characters of two-row representations by only using the Weyl characters of single-row representations. Taking the limit $x \to 1$ on both sides of (2.3.22), we obtain a nontrivial relation of dimensions

$$\boxed{D^{d+2}_{n,s} = D^d_s D^{d+2}_n - D^d_{n+1} D^{d+2}_{s-1}} \tag{2.3.23}$$

which can be checked by using the eqs. (2.2.3) and (2.2.4). We have also checked in mathematica that the eq. (2.3.22) (and hence also eq. (2.3.23)) admits a generalization to a special type of hook diagrams:

$$\boxed{\Theta^{\mathsf{SO}(d+2)}_{\mathbb{Y}_{n,s,1^m}}(x) = D^d_{s,1^m}\Theta^{\mathsf{SO}(d+2)}_{\mathbb{Y}_n} - D^d_{n+1,1^m}\Theta^{\mathsf{SO}(d+2)}_{\mathbb{Y}_{s-1}}} \tag{2.3.24}$$

where $\mathbb{Y}_{n,s,1^m}$ corresponds to the highest weight vector $\mathbf{s} = (n, s, 1, 1, \cdots, 1, 0, \cdots, 0)$ with 1 repeated m times. At the level of dimension, (2.3.24) yields

$$D^{d+2}_{n,s,1^m} = D^d_{s,1^m} D^{d+2}_n - D^d_{n+1,1^m} D^{d+2}_{s-1} \tag{2.3.25}$$

Chapter 3: Representation theory of SO$(1, D)$

In this chapter we review the representation theory of de Sitter isometry group focusing on the construction of unitary irreducible representations (UIRs) and computing the corresponding characters. Instead of being rigorous mathematically, we aim to present an elementary and physicist-friendly description about the constructions. Therefore for certain technical proofs, we will simply illustrate the underlying idea with examples and refer the interested readers to literature. With the conventions specified in the section 3.1, we start the construction of UIRs of SO$(1,2)$ in section 3.2 by borrowing ideas from conformal field theory (CFT). Then we generalize this method to the higher dimensional groups SO$(1, d+1)$ in section 3.3. In the last section, we compute the Harish-Chandra characters [62] of these UIRs. The presentation of this chapter is mainly inspired by the unpublished note [63] and the book [64].

3.1 Introduction and conventions

The isometry group of $(d+1)$-dimensional de Sitter spacetime is SO$(1, d+1)$, which is isomorphic to the d-dimensional Euclidean conformal conformal group. A standard basis for the Lie algebra $\mathfrak{so}(1, d+1)$ is $L_{AB} = -L_{BA}, A = 0, 1, \cdots d+1$, with commutation relations

$$[L_{AB}, L_{CD}] = \eta_{BC}L_{AD} - \eta_{AC}L_{BD} + \eta_{AD}L_{BC} - \eta_{BD}L_{AC} \tag{3.1.1}$$

where

$$\eta_{AB} = \text{diag}(-1, 1, \cdots, 1) \tag{3.1.2}$$

The isomorphism between $\mathfrak{so}(1, d+1)$ and the d-dimensional Euclidean conformal algebra is given by the following linear combinations

$$L_{ij} = M_{ij}, \quad L_{0,d+1} = D, \quad L_{d+1,i} = \frac{1}{2}(P_i + K_i), \quad L_{0,i} = \frac{1}{2}(P_i - K_i) \tag{3.1.3}$$

where D is dilatation, P_i $(i = 1, 2, \cdots d)$ are translations, K_i are special conformal transformations and $M_{ij} = -M_{ji}$ are rotations. The commutation relations for the the conformal algebra following from (3.1.1) and (3.1.3) are

$$\begin{aligned}
&[D, P_i] = P_i, \quad [D, K_i] = -K_i, \quad [K_i, P_j] = 2\delta_{ij} D - 2M_{ij} \\
&[M_{ij}, P_k] = \delta_{jk} P_i - \delta_{ik} P_j, \quad [M_{ij}, K_k] = \delta_{jk} K_i - \delta_{ik} K_j \\
&[M_{ij}, M_{k\ell}] = \delta_{jk} M_{i\ell} - \delta_{ik} M_{j\ell} + \delta_{i\ell} M_{jk} - \delta_{j\ell} M_{ik}
\end{aligned} \tag{3.1.4}$$

The generators L_{AB} exponentiate to group elements in $\mathrm{SO}(1, d+1)$

$$U(\theta) = \exp(\theta^{AB} L_{AB}) \tag{3.1.5}$$

where θ^{AB} are real parameters. Some important subgroups of $\mathrm{SO}(1, d+1)$ that will be used later are

$$\begin{aligned}
\mathsf{K} = \mathrm{SO}(d+1), \quad \mathsf{M} = \left\{ e^{\omega^{ij} M_{ij}} = \mathrm{SO}(d), \omega^{ij} \in \mathbb{R} \right\}, \quad \mathsf{A} = \left\{ e^{\lambda D}, \lambda \in \mathbb{R} \right\} \\
\mathsf{N} = \left\{ e^{b \cdot K}, b^i \in \mathbb{R} \right\}, \quad \tilde{\mathsf{N}} = \left\{ e^{x \cdot P}, x^i \in \mathbb{R} \right\}
\end{aligned} \tag{3.1.6}$$

where K is the *maximal compact* subgroup of $\mathrm{SO}(1, d+1)$.

To get a unitary representation of $\mathrm{SO}(1, d+1)$, the Lie algebra generators must be realized as anti-hermitian operators on the Hilbert space, i.e.

$$L_{AB}^\dagger = -L_{AB} \tag{3.1.7}$$

This is the reality condition relevant to $(d+1)$-dimensional unitary quantum field theories on a fixed dS background. Notice it is different from the reality conditions relevant to d-dimensional unitary (Euclidean) CFTs or $(d+1)$-dimensional unitary quantum field theories on a fixed (E)AdS background. The latter corresponds to the reality condition of the $\mathfrak{so}(2,d)$ algebra obtained by Wick-rotating the X^{d+1} direction. This gives for example $P_i^\dagger = K_i$ whereas for $\mathfrak{so}(1,d+1)$ we have $P_i^\dagger = -P_i$.

Given a representation[1], the quadratic Casimir which commutes with all generators, is chosen to be

$$\begin{aligned}\mathcal{C}_2 &= \frac{1}{2}L_{AB}L^{AB} = -D^2 + \frac{1}{4}(P_i + K_i)^2 - \frac{1}{4}(P_i - K_i)^2 + \frac{1}{2}M_{ij}M^{ij} \\ &= -D^2 + \frac{1}{2}(P_iK_i + K_iP_i) + \frac{1}{2}M_{ij}^2 \\ &= D(d-D) + P_iK_i + \frac{1}{2}M_{ij}^2 \end{aligned} \tag{3.1.8}$$

Here $\frac{1}{2}M_{ij}^2 \equiv \frac{1}{2}M_{ij}M^{ij}$ is the quadratic Casimir of SO(d) and it is negative-definite for a unitary representation since M_{ij} are anti-hermitian. For example, for a spin-s representation of SO(d), it takes the value of $-s(s+d-2)$.

3.2 UIRs of SO$(1,2)$

When $d=1$, the Lie algebra $\mathfrak{so}(1,2)$ is generated by $\{P, D, K\}$ satisfying the following commutation conditions

$$[D,P] = P, \quad [D,K] = -K, \quad [K,P] = 2D \tag{3.2.1}$$

The quadratic Casimir is $\mathcal{C}_2 = D(1-D) + PK$.

3.2.1 A direct constrction

Define $L_0 = iL_{12} = -\frac{i}{2}(P+K)$ to be the *Hermitian* operator that generates rotation in dS$_2$. It takes value in integers if we require a group representation since $e^{2\pi i L_0}$ is the identity operator

[1] Rigorously speaking, we do not need a representation to define a Casimir. Instead, it can be defined abstractly as an element of the universal enveloping algebra.

in SO$(1,2)$. [2] As in the $SU(2)$ case, we define ladder operators

$$L_{\pm} = -\frac{i}{2}(P-K) \mp D \tag{3.2.2}$$

and they satisfy

$$[L_0, L_{\pm}] = \pm L_{\pm}, \quad [L_-, L_+] = 2L_0, \quad L_-^{\dagger} = L_+ \tag{3.2.3}$$

Thus L_+ raises the eigenvalue of L_0 by 1 while L_- lowers it by 1. In terms of the basis $\{L_0, L_{\pm}\}$, the quadratic Casimir operator is expressed as

$$\mathcal{C}_2 = L_+L_- + L_0(1-L_0) \tag{3.2.4}$$

To construct the whole representation space of certain unitary irreducible representation R, we start from an eigenstate of L_0, i.e. a state $|N\rangle$ satisfying $L_0|N\rangle = N|N\rangle$ where $N \in \mathbb{Z}$. By acting $L_{\pm}$ on $|N\rangle$ repeatedly, we obtain a whole tower of states: $\{L_+^{p_+}|N\rangle, L_-^{p_-}|N\rangle\}_{p_{\pm}\in\mathbb{N}}$. It is possible that the L_- action gets truncated. That is, there exist some P such that $L_-^k|N\rangle = 0$ for $k \geq P+1$. On the other hand, due to the noncompactness of SO$(1,2)$, the unitary irreducible representation R has to be infinite dimensional. Thus, in this case, the L_+ action cannot be truncated. In addition, due to the irreducibility, other states with L_0-eigenvalue larger than $N-P$ cannot be annihilated by L_-. Altogether, by proper normalization, the action of L_- can be chosen to be

$$L_-|n\rangle = (n-\Delta)|n-1\rangle \tag{3.2.5}$$

where $|n\rangle$ is the eigenstate (not necessarily normalized) of L_0 with eigenvalue n. The truncations discussed above happen when Δ hits integers. Acting the quadratic Casimir on $|\Delta\rangle$ yields the expected value $\Delta(1-\Delta)$ and it can be used to fix the action of L_+:

$$L_+|n\rangle = (n+\Delta)|n+1\rangle \tag{3.2.6}$$

[2] If we consider representations of SL$(2,\mathbb{R})$, the double covering of SO$(1,2)$, then L_0 can be either integers or half-integers and $e^{2\pi i L_0} \in \{\pm 1\}$ is a constant in a unitary irreducible representation.

With the action known, we proceed to analyze the constraint of unitarity which consists of two parts: (a) the inner product $\langle n|n\rangle$ is positive definite and (b) L_+ is the Hermitian conjugate of L_- with respect to this inner product, i.e.

$$\langle n|L_-|n+1\rangle = \langle n+1|L_+|n\rangle^* \tag{3.2.7}$$

The latter implies

$$\frac{\langle n+1|n+1\rangle}{\langle n|n\rangle} = \frac{n+\bar{\Delta}}{n+\Delta^*} \equiv \lambda_n \tag{3.2.8}$$

where $\bar{\Delta} = 1-\Delta$ and Δ^* is the complex conjugate of Δ. The reality of λ_n yields two possible families of solutions:

$$(i)\,\Delta = \frac{1}{2} + i\mu,\ \mu \in \mathbb{R}, \quad (ii)\,\Delta \in \mathbb{R} \tag{3.2.9}$$

The positivity of λ_n requires separate discussions for different values of Δ:

- λ_n is identically equal to 1 when $\Delta = \frac{1}{2} + i\mu$. We can consistently choose $\langle n|n\rangle = 1$ for any $n \in \mathbb{Z}$ and hence the resulting representation, denoted by $\mathcal{P}_\Delta$, is unitary. We call $\mathcal{P}_\Delta$ a (unitary) *principal series* representation.

- When $\Delta \in \mathbb{R}$ but $\Delta \notin \mathbb{Z}_+$, n takes value in all integers and we need

$$\lambda_n = \frac{n+\Delta}{n+1-\Delta} = \frac{(n+\frac{1}{2})^2 - (\Delta+\frac{1}{2})^2}{(n+1-\Delta)^2} > 0 \tag{3.2.10}$$

  to hold for all $n \in \mathbb{Z}$. The most stringent constraint clearly comes from $n=0$, which implies $0 < \Delta < 1$. For Δ in this range, we choose the following normalization

$$\langle n|n\rangle = \frac{\Gamma(n+\bar{\Delta})}{\Gamma(n+\Delta)} \tag{3.2.11}$$

  Altogether, we obtain a new continuous family of UIRs, denoted by $\mathcal{C}_\Delta$ for each fixed $\Delta \in (0,1)$. They are called *complementary series*. At $\Delta = \frac{1}{2}$, the intersection point of principal series and complementary series, the norm (3.2.11) is reduced to $\langle n|n\rangle = 1$, i.e.

the norm we've chosen for principal series.

- When $\Delta \in \mathbb{Z}_+$, the vector space spanned by all $|n\rangle$ contains two $\mathfrak{so}(1,2)$-invariant subspaces:

$$\begin{aligned}&\{|n\rangle\}_{n\geq\Delta} : \text{carries a lowest weight representation } \mathcal{D}^+_\Delta\\ &\{|n\rangle\}_{n\leq-\Delta} : \text{carries a highest weight representation } \mathcal{D}^-_\Delta\end{aligned} \tag{3.2.12}$$

  It's clear that all λ_n are positive when restricted to these two subspaces. Thus $\mathcal{D}^\pm_\Delta$ are also UIRs, called *discrete series*. In each $\mathcal{D}^\pm_\Delta$, we choose the normalization to be

$$\langle n|n\rangle_{\mathcal{D}^\pm_\Delta} = \frac{\Gamma(\pm n+\bar{\Delta})}{\Gamma(\pm n+\Delta)} \tag{3.2.13}$$

- When Δ takes value in negative integers, the vector space spanned by all $|n\rangle$ contains only one finite-dimensional $\mathfrak{so}(1,2)$-invariant subspace which furnishes a nonunitary representation, When $\Delta = 0$, $|0\rangle$ carries the *trivial* representation.

In conclusion, the conformal group SO$(1,2)$ admits four types of unitary irreducible representations (the irreducibility is automatically guaranteed by the construction given above): principal series $\mathcal{P}_\Delta$ for $\Delta \in \frac{1}{2}+i\mathbb{R}$, complementary series $\mathcal{C}_\Delta$ for $0<\Delta<1$, discrete series $\mathcal{D}^\pm_\Delta$ for $\Delta\in\mathbb{Z}_+$ and the trivial representation. In addition, for principal and complementary series, there exist an isomorphism between $\mathcal{P}_\Delta$ ($\mathcal{C}_\Delta$) and $\mathcal{P}_{\bar{\Delta}}$ ($\mathcal{C}_{\bar{\Delta}}$) which can be realized by the following *invertible intertwining map* that preserves the inner product:

$$\mathcal{S}_\Delta : |n\rangle_\Delta \longmapsto \frac{\Gamma(n+\bar{\Delta})}{\Gamma(n+\Delta)}|n\rangle_{\bar{\Delta}} \tag{3.2.14}$$

where $|n\rangle_\Delta$ denotes the $|n\rangle$ basis in $\mathcal{P}_\Delta$ ($\mathcal{C}_\Delta$) and similarly for $|n\rangle_{\bar{\Delta}}$. With this normalization, $\mathcal{S}_{\bar{\Delta}}\circ\mathcal{S}_\Delta = 1$, where $\circ$ means the composition of maps. In the CFT terminologies, this intertwining map $\mathcal{S}_\Delta$ is called a *shadow transformation*.

3.2.2 A CFT-type construction

In this section, we will show that all the UIRs of SO$(1,2)$ constructed above can be realized in certain function spaces by borrowing ideas from conformal field theory. This method can be easily generalized to the higher dimension cases.

3.2.2.1 Representation space

We start with a *primary state of scaling dimension* $\bar{\Delta} = 1 - \Delta$, i.e. a state $|\bar{\Delta}, 0\rangle$ satisfying:

$$K|\bar{\Delta}, 0\rangle = 0, \quad D|\bar{\Delta}, 0\rangle = \bar{\Delta}|\bar{\Delta}, 0\rangle \tag{3.2.15}$$

However, unlike in usual unitary CFT, we do not require this state to be normalizable (indeed, as we shall see, it is not and hence does not belong to the Hilbert spaces we will construct). Acting by translations on this state produces a family of states

$$|\bar{\Delta}, x\rangle = e^{xP}|\bar{\Delta}, 0\rangle \tag{3.2.16}$$

From this definition and the conformal algebra, we then get (dropping the label $\bar{\Delta}$ for $|\bar{\Delta}, x\rangle$)

$$P|x\rangle = \partial_x|x\rangle, \quad D|x\rangle = (x\partial_x + \bar{\Delta})|x\rangle, \quad K|x\rangle = (x^2\partial_x + 2\bar{\Delta}x)|x\rangle \tag{3.2.17}$$

Then a general state $|\psi\rangle$ can be expressed as a linear combination

$$|\psi\rangle \equiv \int_{\mathbb{R}} dx\, \psi(x)|x\rangle \tag{3.2.18}$$

At this point, $\psi(x)$ can be any smooth function [3]. The action of the conformal generators on the wavefunctions $\psi(x)$ is obtained from (3.2.17) and integrating by part. For example, $P|\psi\rangle = \int dx\, \psi(x)\partial_x|x\rangle = \int dx\, (-\partial_x\psi(x))|x\rangle$. This gives

$$P\psi(x) = -\partial_x\psi(x), \quad D\psi(x) = -(x\partial_x + \Delta)\psi(x), \quad K\psi(x) = -(x^2\partial_x + 2\Delta x)\psi(x) \tag{3.2.19}$$

[3] We can also consider more general functions, for example $\psi(x) = \frac{1}{|x|^{2\Delta}}$ which gives the shadow of $|\bar{\Delta}, 0\rangle$, i.e. a primary state of scaling dimension Δ. But these states are *not* normalizable.

The requirement of lifting the Lie algebra action (3.2.19) to a group action imposes constraints on the wavefunctions. In particular, exponentiating the special conformal transformation yields

$$\left(e^{bK}\psi\right)(x) = \left(\frac{1}{1+bx}\right)^{2\Delta}\psi\left(\frac{x}{1+bx}\right) \tag{3.2.20}$$

By taking the limit $x \to -\frac{1}{b}$, we can read off the asymptotic behavior of $\psi(x)$

$$\psi(x) \overset{|x|\to\infty}{\approx} \frac{1}{|x|^{2\Delta}}\left(c_\psi + \mathcal{O}\left(\frac{1}{|x|}\right)\right) \tag{3.2.21}$$

where c_ψ is some constant. Let $\mathcal{F}_\Delta$ be the complex vector space of infinitely differentiable functions satisfying the asymptotic boundary condition (3.2.21) and it furnishes a representation of SO$(1,2)$ whose infinitesimal version is given by (3.2.19).

3.2.2.2 Consequence of unitarity

Given the set of states/wavefunctions, we should define a positive definite inner product $\langle x|y\rangle$ that respects the dS reality condition, i.e. all generators of SO$(1,2)$ are anti-Hermitian. First notice that $P^\dagger = -P$ leads to the relation $\partial_y\langle x|y\rangle = \langle x|P|y\rangle = -\partial_x\langle x|y\rangle$, so $(\partial_x+\partial_y)\langle x|y\rangle = 0$, i.e. $\langle x|y\rangle = f(x-y)$ for some function $f(x)$. Likewise the reality of the dilatation and special conformal transformation requires

$$(x\partial_x + y\partial_y + \bar{\Delta} + \bar{\Delta}^*)f(x-y) = 0 \tag{3.2.22}$$

$$(x^2\partial_x + y^2\partial_y + 2\bar{\Delta}y + 2\bar{\Delta}^*x)f(x-y) = 0 \tag{3.2.23}$$

where the first equation fixes the scaling property of $f(x)$ as $(x\partial_x + \bar{\Delta} + \bar{\Delta}^*)f(x) = 0$, which together with the second equation implies

$$(\Delta - \Delta^*)xf(x) = 0 \tag{3.2.24}$$

The eq. (3.2.24) holds when either Δ is real or $f(x) \propto \delta(x)$. The former further implies $f(x) \propto \frac{1}{|x|^{2\bar{\Delta}}}$ while the latter implies $\Delta + \bar{\Delta} = 1$. Thus we arrive at the conclusion that the reality condition allows two qualitatively different cases

$$\begin{aligned} &\mathsf{I}: \Delta = \frac{1}{2} + i\mu,\ \mu \in \mathbb{R}, \quad && \langle x|y\rangle = c\,\delta(x-y) \\ &\mathsf{II}: \Delta \in \mathbb{R}, && \langle x|y\rangle = \frac{c}{|x-y|^{2\bar{\Delta}}} \end{aligned} \tag{3.2.25}$$

where c is some Δ-dependent constant. Correspondingly the inner product of two states of the form (3.2.18) equals (dropping c)

$$\langle \psi_1|\psi_2\rangle = \begin{cases} \int_{\mathbb{R}} dx\, \psi_1^*(x)\psi_2(x) & \Delta = \frac{1}{2} + i\mu \\ \int_{\mathbb{R}\times\mathbb{R}} dx\, dy\, \frac{\psi_1^*(x)\psi_2(y)}{|x-y|^{2\bar{\Delta}}} & \Delta \in \mathbb{R} \end{cases} \tag{3.2.26}$$

In case I, the inner product (3.2.26) defines the usual L^2-norm for wavefunctions in the space $\mathcal{F}_{\frac{1}{2}+i\mu}$. This norm is clearly well-defined because for any $\psi \in \mathcal{F}_{\frac{1}{2}+i\mu}$, $|\psi|^2$ decays as $\frac{1}{|x|^2}$ near infinity. Therefore, $\psi \in \mathcal{F}_{\frac{1}{2}+i\mu}$ equipped with the L^2-norm carries a unitary irreducible representation of SO$(1,2)$, which as we shall see is a principal series representation. To analyze the positivity of (3.2.26) in case II, it is more convenient to rewrite the kernel $\mathcal{K}_\Delta(x,y) \equiv \frac{c}{|x-y|^{2\bar{\Delta}}}$ in momentum space

$$\begin{aligned} \mathcal{K}_\Delta(p) &= \int_{-\infty}^{\infty} dx \frac{c}{|x|^{2\bar{\Delta}}} e^{ipx} = \frac{c}{\Gamma(\bar{\Delta})} \int_0^\infty \frac{dt}{t}\, t^{\bar{\Delta}} \int_{-\infty}^{\infty} dx\, e^{-tx^2+ipx} \\ &= \frac{c\sqrt{\pi}}{\Gamma(\bar{\Delta})} \int_0^\infty \frac{dt}{t} t^{\bar{\Delta}-\frac{1}{2}} e^{-\frac{p^2}{4t}} = \frac{c'}{p^{2\Delta-1}}, \quad c' = 2^{2\Delta-1}\sqrt{\pi}\frac{\Gamma(\Delta-\frac{1}{2})}{\Gamma(\bar{\Delta})} c \end{aligned} \tag{3.2.27}$$

As long as we choose c properly, the kernel $\mathcal{K}_\Delta(p)$ is positive. However, the positivity of $\mathcal{K}_\Delta(p)$ does *not* necessarily imply that the inner product on the space $\mathcal{F}_\Delta$ defined by $\mathcal{K}_\Delta(p)$ is positive definite. We have to check that all the wavefunctions in $\mathcal{F}_\Delta$ are normalizable with respect to this inner product, otherwise regularization can destroy the positivity. Since $\psi(x) \in \mathcal{F}_\Delta$ is infinitely differentiable, its Fourier transformation decays rapidly as $|p| \to \infty$. It suffices to check the normalizability for small p. Let $\psi(p)$ be the Fourier transformation of $\psi(x)$. Its small p behavior has two types of leading fall-offs: $c_1\, p^0$ and $c_2\, p^{2\Delta-1}$ where c_1, c_2 are constants. The former comes from the regularity of $\psi(x)$ for finite x [4] and the latter arises from the large x asymptotic behavior of $\psi(x)$. As an explicit example, let's take $\psi(x) = \frac{1}{(1+x^2)^\Delta}$, which clearly lives in the space $\mathcal{F}_\Delta$. Its Fourier transformation can be performed analytically $\psi(p) \propto p^{\Delta-\frac{1}{2}} K_{\frac{1}{2}-\Delta}(p)$. Then the two leading fall-offs of $\psi(p)$ are a result of the following property of Bessel K-function

$$K_\alpha(z) \overset{z\to 0}{\approx} z^\alpha \left(2^{-1-\alpha}\Gamma(-\alpha) + \mathcal{O}(z^2)\right) + z^{-\alpha}\left(2^{-1+\alpha}\Gamma(\alpha) + \mathcal{O}(z^2)\right) \tag{3.2.28}$$

Plugging $\psi(p) \overset{p\to 0}{\approx} c_1 + c_2 p^{2\Delta-1}$ into the inner product $\int \frac{dp}{2\pi} \mathcal{K}_\Delta(p)|\psi(p)|^2$ where $\mathcal{K}_\Delta(p) = \frac{c'}{p^{2\Delta-1}}$, we conclude that $\psi(x) \in \mathcal{F}_\Delta$ is normalizable if and only if $2\Delta - 1 > -1$ and $1 - 2\Delta > -1$, i.e.

$$0 < \Delta < 1 \tag{3.2.29}$$

which is exactly the range for complementary series.

3.2.2.3 A discrete basis of $\mathcal{F}_\Delta$

Thus far we have shown that the function spaces $\mathcal{F}_\Delta$ furnish unitary irreducible representations of SO$(1,2)$ when $\Delta \in \frac{1}{2} + i\mathbb{R}$ or $0 < \Delta < 1$. Indeed they are the same as principal and complementary series constructed in the previous section. To prove this claim, we first find the eigenbasis of $L_0 = i\left(\frac{1+x^2}{2}\partial_x + \Delta x\right)$ in $\mathcal{F}_\Delta$, i.e. wavefunctions $\psi_n^{(\Delta)}$ satisfying $L_0\psi_n^{(\Delta)} = n\psi_n^{(\Delta)}$

$$\psi_n^{(\Delta)}(x) = \left(\frac{1-ix}{1+ix}\right)^n \frac{1}{(1+x^2)^\Delta} \tag{3.2.30}$$

In addition, it is straightforward to check

$$L_+\psi_n^{(\Delta)} = (n+\Delta)\psi_{n+1}^{(\Delta)}, \quad L_-\psi_n^{(\Delta)} = (n-\Delta)\psi_{n-1}^{(\Delta)} \tag{3.2.31}$$

which take the same form as the action of $L_\pm$ on $|n\rangle$, c.f. (3.2.5) and (3.2.6).

[4] For example, if $\psi(x) = \frac{1}{x^{2\Delta}}$ which is singular at the origin, then we don not have the p^0 behavior.

For $\boxed{\Delta \in \frac{1}{2} + i\mathbb{R}}$, the $L^2(\mathbb{R})$ inner product yields

$$(\psi_n^{(\Delta)}, \psi_m^{(\Delta)}) = c \int dx\, \overline{\psi_n^{(\Delta)}(x)} \psi_m^{(\Delta)}(x) = c\, \pi \delta_{nm} \tag{3.2.32}$$

By choosing $c = \frac{1}{\pi}$, we get to identify $\psi_n^{(\Delta)}(x)$ as the state $|n\rangle$ in principal series.

For $\boxed{0 < \Delta < 1}$, the inner product can be computed by expanding the kernel $\mathcal{K}_\Delta(x_1 - x_2)$ in terms of $\psi_n^{(\bar{\Delta})}$. Define $x_i = \tan \frac{\theta_i}{2}$ and we rewrite $\mathcal{K}_\Delta(x_1 - x_2)$ as

$$\begin{aligned}
\mathcal{K}_\Delta(x_{12}) &= \frac{2^{\bar{\Delta}}\, c}{(1+x_1^2)^{\bar{\Delta}}(1+x_2^2)^{\bar{\Delta}}} \frac{1}{(1-\cos\theta_{12})^{\bar{\Delta}}} \\
&= \frac{2^{\bar{\Delta}}\, c}{(1+x_1^2)^{\bar{\Delta}}(1+x_2^2)^{\bar{\Delta}}} \frac{1}{\Gamma(\bar{\Delta})} \int_0^\infty \frac{ds}{s}\, s^{\bar{\Delta}} e^{-s+s\theta_{12}}
\end{aligned} \tag{3.2.33}$$

where $x_{12} = x_1 - x_2$ and $\theta_{12} \equiv \theta_1 - \theta_2$. The factor $e^{s\theta_{12}}$ admits a harmonic expansion

$$e^{s\theta_{12}} = \sum_{n\in\mathbb{Z}} I_n(s)\, e^{-in\theta_{12}} \tag{3.2.34}$$

with the coefficients being modified Bessel functions. Plugging the expansion (3.2.34) into (3.2.33), the s-integral can be evaluated analytically [5]

$$\begin{aligned}
\mathcal{K}_\Delta(x_{12}) &= \frac{c\,\Gamma(\Delta - \frac{1}{2})}{\sqrt{\pi}\Gamma(\bar{\Delta})} \frac{1}{(1+x_1^2)^{\bar{\Delta}}(1+x_2^2)^{\bar{\Delta}}} \sum_{n\in\mathbb{Z}} \frac{\Gamma(\bar{\Delta}+n)}{\Gamma(\Delta+n)}\, e^{-in\theta_{12}} \\
&= \frac{c\,\Gamma(\Delta - \frac{1}{2})}{\sqrt{\pi}\,\Gamma(\bar{\Delta})} \sum_{n\in\mathbb{Z}} \frac{\Gamma(\bar{\Delta}+n)}{\Gamma(\Delta+n)}\, \psi_n^{(\bar{\Delta})}(x_1) \overline{\psi_n^{(\bar{\Delta})}(x_2)}
\end{aligned} \tag{3.2.35}$$

where in the last line we've used $e^{-i\theta_i} = \frac{1-ix_i}{1+ix_i}$. By choosing $c = \frac{\Gamma(\bar{\Delta})}{\pi^{\frac{3}{2}}\Gamma(\Delta - \frac{1}{2})}$, the basis $\psi_n^{(\Delta)}$ is normalized as

$$\left(\psi_n^{(\Delta)}, \psi_m^{(\Delta)}\right) = \delta_{nm}\, \frac{\Gamma(\bar{\Delta}+n)}{\Gamma(\Delta+n)} \tag{3.2.36}$$

and hence we get to identify $\psi_n^{(\Delta)}$ as $|n\rangle$ in complementary series, c.f. (3.2.11).

[5] The integral $\int_0^\infty ds\, s^{-\Delta} I_n(s) e^s$ is convergent when $\frac{1}{2} < \Delta < 1$ and admits an analytic continuation to the whole complex plane except some single poles.

3.2.2.4 Discrete series

When Δ is an positive integer, say $\Delta = N \in \mathbb{Z}_+$, the function space $\mathcal{F}_N$ has two invariant subspaces $\mathcal{F}_N^+ = \text{Span}\{\psi_n^{(N)}\}_{n\geq N}$ and $\mathcal{F}_N^- = \text{Span}\{\psi_n^{(N)}\}_{n\leq -N}$, which as we will show soon correspond to the discrete series $\mathcal{D}_N^\pm$ respectively. Similarly, when $\Delta \equiv 1-N$ is a nonpositive integer, $\mathcal{F}_{1-N}$ contains three invariant subspaces $\mathcal{F}_{1-N}^+ = \text{Span}\{\psi_n^{(N)}\}_{n\geq 1-N}$, $\mathcal{F}_{1-N}^- = \text{Span}\{\psi_n^{(N)}\}_{n\leq N-1}$ and $P_N = \mathcal{F}_{1-N}^+ \cap \mathcal{F}_{1-N}^-$, where P_N is nothing but the space of polynomials in x (with coefficients valued in $\mathbb{C}$) up to degree $2N-2$ and hence is annihilated by the differential operator ∂_x^{2N-1}. Indeed, ∂_x^{2N-1} is an intertwining map between $\mathcal{F}_{1-N}$ and $\mathcal{F}_N$, as shown in the commuting diagram 3.2.1. In particular, ∂_x^{2N-1} maps $\psi_n^{(1-N)}(x)$ to $\psi_n^{(N)}(x)$ up to an overall constant for any $|n| \geq N$.

$$\begin{array}{ccc} \mathcal{F}_{1-N} & \xrightarrow{\partial_x^{2N-1}} & \mathcal{F}_N \\ \Big\downarrow{\scriptstyle L_{AB}} & & \Big\downarrow{\scriptstyle L_{AB}} \\ \mathcal{F}_{1-N} & \xrightarrow[\partial_x^{2N-1}]{} & \mathcal{F}_N \end{array}$$

Figure 3.2.1: Intertwining operator $\partial_x^{2N-1} : \mathcal{F}_{1-N} \to \mathcal{F}_N$. L_{AB} denote the action of $\mathfrak{so}(1,2)$ in $\mathcal{F}_N$ and $\mathcal{F}_{1-N}$.

Group theoretically, this claim follows from $L_0(\partial_x^{2N-1}\psi_n^{(1-N)}) = \partial_x^{2N-1} L_0 \psi_n^{(1-N)} = n\, \partial_x^{2N-1}\psi_n^{(1-N)}$. On the other hand, it can also be proved by a direct computation, for example for $n \geq N$

$$\begin{aligned} \partial_x^{2N-1}\psi_n^{(1-N)}(x) &= (-)^{N-1+n}\partial_x^{2N-1}\frac{(1+ix-2)^{N-1+n}}{(1+ix)^{n+1-N}} \\ &= (-)^{N-1+n}\sum_{k=2N-1}^{N+n-1}(-2)^k\binom{N+n-1}{k}\partial_x^{2N-1}(1+ix)^{2N-2-k} \\ &= i(-)^N 2^{2N-1}\frac{\Gamma(N+n)}{\Gamma(n-N+1)}\psi_n^{(N)}(x) \end{aligned} \tag{3.2.37}$$

Taking complex conjugate on both sides of (3.2.37) yields the $n \leq -N$ cases. Thus $\mathcal{F}_N^\pm$ is isomorphic to the quotient space $\mathcal{F}_{1-N}^\pm / P_N$. Altogether, the relation between $\mathcal{F}_N$ and $\mathcal{F}_{1-N}$ can be summarized in the following diagram:

To define an inner product on $\mathcal{F}_N^+$ (and similarly on $\mathcal{F}_N^-$), one would naively expect to use

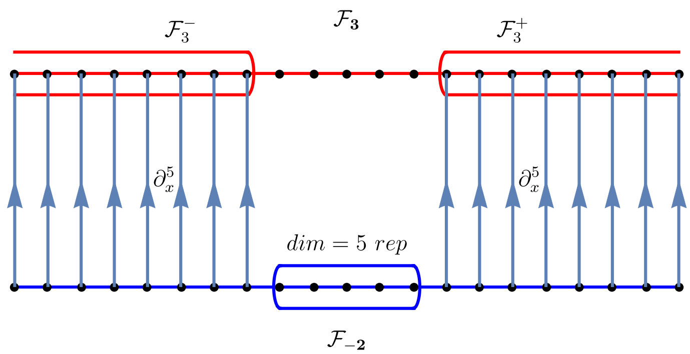

Figure 3.2.2: The relation between $\mathcal{F}_3$ and its shadow $\mathcal{F}_{-2}$. The upper red line represents the space $\mathcal{F}_3$ and each dot denotes the basis $\psi_k^{(3)}$. It contains two invariant subspaces $\mathcal{F}_3^\pm$ corresponding to the discrete series representation $D_3^\pm$. The lower blue line represents the space $\mathcal{F}_{-2}$ and each dot denotes the basis $\psi_k^{(-2)}$. It contains a 5-dimensional invariant space P_3. The upward arrows denote the intertwining operators ∂_x^5. It annihilates the subspace P_3 and maps $\psi_k^{(-2)}$ to $\psi_k^{(3)}$ for $|k| \geq 3$.

the kernel $\mathcal{K}_N(x,y) = (x-y)^{2(N-1)}$ as in the complementary series case. However, this does not work because $\mathcal{K}_N$ is a delta function with some derivatives in momentum space while the wavefunctions $\psi_n^{(N)}(x)$ in $\mathcal{F}_N^+$ are supported on $p > 0$ due to the analyticity of $\psi_n^{(N)}(x)$ in the lower x-plane when $n \geq N$. Therefore, the inner product defined via $\mathcal{K}_N(x,y)$ is identically zero. Thanks to the quotient space realization, i.e. $\mathcal{F}_N^\pm = \mathcal{F}_{1-N}^\pm / P_N$, we can avoid using $\mathcal{K}_N(x,y)$ by working in the space $\mathcal{F}_{1-N}$. The idea is very simple and let's phrase it in a more general setup. Given two representations V_1 and V_2 of some group G and an intertwining map $\varphi : V_1 \to V_2$ which can have a nontrivial kernel, then a positive semi-definite G-invariant inner product $(\ ,\ )_1$ on V_1 induces a positive definite G-invariant inner product $(\ ,\ )_2$ on $\operatorname{Im}\varphi$, if $\ker\varphi$ is orthogonal to the *whole* vector space V_1 with respect to $(\ ,\ )_1$ and $(\ ,\ )_1$ is positive for vectors not in $\ker\varphi$. With these assumptions, the induced inner product $(\ ,\ )_2$ is given by $(v_2, v_2)_2 \equiv (v_1, v_1)_1$ where $v_i \in V_i$ and $\varphi(v_1) = v_2$. The choice of v_1 is clearly not unique when φ has a nontrivial kernel but the inner product is independent of such a choice. We can fix v_1 by choosing a map $\mathfrak{L} : V_2 \to V_1$ such that $\varphi \circ \mathfrak{L}$ is the identity operator on V_2. Then the inner product on V_2

becomes

$$(v_2, v_2)_2 = (\mathfrak{L}v_2, \mathfrak{L}v_2)_1 \tag{3.2.38}$$

Now let's take $V_1 = \mathcal{F}^+_{1-N}, V_2 = \mathcal{F}^+_N$ and $\varphi = \partial_x^{2N-1}$. The relation (3.2.37) yields a positive semi-definite inner product on $V_1 = \mathcal{F}^+_{1-N}$ that satisfies the conditions discussed above

$$(\psi, \psi)_1 \equiv i(-)^N \, c_N \int dx \, \psi(x) \, \partial_x^{2N-1} \overline{\psi(x)} \tag{3.2.39}$$

where c_N is a positive constant. More explicitly, we have

$$(\psi_m^{(1-N)}, \psi_n^{(1-N)})_1 = 2^{2N-1} \pi c_N \frac{\Gamma(N+n)}{\Gamma(n-N+1)} \delta_{mn} \tag{3.2.40}$$

Next we choose a map $\mathfrak{L}_N : \mathcal{F}^+_N \to \mathcal{F}^+_{1-N}$ such that $\partial_x^{2N-1} \circ \mathfrak{L}_N$ is the identity operator on $\mathcal{F}^+_N$. A very natural choice of $\mathfrak{L}_N$ is

$$(\mathfrak{L}_N \psi)(x) = \frac{1}{\Gamma(2N-1)} \int_{\mathbb{R}+i\epsilon} \frac{dy}{2\pi i} \, \mathfrak{L}_N(x-y) \psi(y), \quad \mathfrak{L}_N(z) = z^{2(N-1)} \log(z) \tag{3.2.41}$$

where the limit $\epsilon \to 0^+$ is understood and the branch cut of $\log z$ is chosen to be the $z < 0$ line. When acting ∂_x^{2N-1} on $(\mathfrak{L}_N \psi)(x)$, the kernel $\mathfrak{L}_N(x-y)$ becomes $\frac{\Gamma(2N-1)}{x-y}$ and we can close the contour in the lower half plane picking up the pole at $y = x$ since $\psi(y)$ is holomorphic in the lower half plane and decays fast enough at ∞. More explicitly, the action of $\mathfrak{L}_N$ on the basis $\psi_n^{(N)}$ is computed in the appendix A.1

$$\left(\mathfrak{L}_N \psi_n^{(N)} \right)(x) = \frac{(-)^N}{2^{2N-1} i} \frac{\Gamma(n-N+1)}{\Gamma(N+n)} \psi_n^{(1-N)}(x) + \mathsf{Pol}_{N,n}(x) \tag{3.2.42}$$

where $\mathsf{Pol}_{N,n}(x)$ is a polynomial annihilated by ∂_x^{2N-1}. Altogether, the inner product on the space $\mathcal{F}^+_N$ induced by (3.2.39) and (3.2.41) is

$$(\psi, \psi)_{\mathcal{F}^+_N} = (\mathfrak{L}_N \psi, \mathfrak{L}_N \psi) = \frac{(-)^N c_N}{2\pi \Gamma(2N-1)} \int_{\mathbb{R}} dx \int_{\mathbb{R}+i\epsilon} dy \, \bar{\psi}(x) (x-y)^{2(N-1)} \log(x-y) \psi(y) \tag{3.2.43}$$

In particular, choosing $c_N = \frac{2^{2N-1}}{\pi}$, we get

$$(\psi_m^{(N)}, \psi_n^{(N)})_{\mathcal{F}_N^+} = \frac{\Gamma(n-N+1)}{\Gamma(N+n)}\delta_{mn} \tag{3.2.44}$$

consistent with the inner product (3.2.13) for discrete series $\mathcal{D}_N^+$.

Before moving to the next section, we want to make some remarks about the kernel $\mathfrak{L}_N(x,y)$.

Remark 1. *For complementary series of scaling dimension Δ, the inner product is defined through the kernel $\mathcal{K}_\Delta(x,y) = \frac{1}{|x-y|^{2\bar{\Delta}}}$. As a function of Δ, $\mathcal{K}_\Delta(x,y)$ is well-defined on the whole complex plane as long as $x \neq y$. Taylor expansion of $\mathcal{K}_\Delta(x,y)$ around $\Delta = N$ yields*

$$\mathcal{K}_\Delta(x,y) \overset{\Delta\to N}{\approx} |x-y|^{2(N-1)} + 2(\Delta - N)|x-y|^{2(N-1)}\log(x-y) + \cdots \tag{3.2.45}$$

The first term on the R.H.S is simply $\mathcal{K}_N(x,y)$ and we have argued that it leads to a trivial inner product for $\mathcal{F}_N^+$. The second term, with the numerical factor stripped off, is exactly the kernel $\mathfrak{L}_N(x-y)$ that defines the inner product for discrete series (up to a contour prescription).

Remark 2. *Unlike $\mathcal{K}_\Delta(x,y)$, $\mathfrak{L}_N(x-y)$ is not SO$(1,2)$-invariant. For example, under scaling transformation*

$$e^{\lambda D} : \mathfrak{L}_N(x-y) \to e^{2\lambda(1-N)}\mathfrak{L}_N(e^\lambda x - e^\lambda y) = \mathfrak{L}_N(x-y) + \lambda(x-y)^{2(N-1)} \tag{3.2.46}$$

and under special conformal transformation

$$\begin{aligned} e^{bK} : \mathfrak{L}_N(x-y) &\to \mathfrak{L}_N\left(\frac{x}{1-bx} - \frac{y}{1-by}\right)(1-bx)^{2(N-1)}(1-by)^{2(N-1)} \\ &\to \mathfrak{L}_N(x-y) + (x-y)^{2(N-1)}(\log(1-bx) + \log(1-by)) \end{aligned} \tag{3.2.47}$$

However, these extra terms does not contribute to the inner product (3.2.53).

3.2.2.5 Shadow transformation

In the eq. (3.2.14), we define the so-called shadow transformation as an intertwining map between the two representations with scaling dimension Δ and $\bar{\Delta}$ respectively. In the wavefunction picture, we want to realize $\mathcal{S}_\Delta$ as a linear operator acts on any $\psi(x) \in \mathcal{F}_\Delta$ by using a kernel function $S_\Delta(x,y)$

$$\mathcal{S}_\Delta:\ \psi(x)\longmapsto\int dy S_\Delta(x,y)\psi(y) \tag{3.2.48}$$

The intertwining condition imposes nontrivial constrains on $S_\Delta(x,y)$. These constrains are nothing but conformal Ward identities associated to the conformal group SO$(1,2)$. Thus $S_\Delta(x,y)$ is the two-point function of a scalar primary operator with scaling dimension $\bar{\Delta}$

$$S_\Delta(x,y)=\frac{N_\Delta}{|x-y|^{2\bar{\Delta}}},\quad N_\Delta=\frac{1}{\sqrt{\pi}}\frac{\Gamma(\bar{\Delta})}{\Gamma\left(\frac{1}{2}-\bar{\Delta}\right)} \tag{3.2.49}$$

where the normalization constant N_Δ is chosen such that $\psi_n^{(\Delta)}$ is mapped to $\frac{\Gamma(\bar{\Delta}+n)}{\Gamma(\Delta+n)}\psi_n^{(\bar{\Delta})}$ as in (3.2.14), i.e.

$$\int_{-\infty}^{\infty}dy\,\frac{N_\Delta}{|x-y|^{2\bar{\Delta}}}\left(\frac{1-iy}{1+iy}\right)^n\frac{1}{(1+y^2)^\Delta}=\frac{\Gamma(\bar{\Delta}+n)}{\Gamma(\Delta+n)}\left(\frac{1-ix}{1+ix}\right)^n\frac{1}{(1+x^2)^{\bar{\Delta}}} \tag{3.2.50}$$

This integral is evaluated using the harmonic expansion of $\frac{1}{|x-y|^{2\bar{\Delta}}}$ given by the eq. (3.2.35). For $\Delta\in\frac{1}{2}+i\mu$ or $0<\Delta<1$, the shadow transformation $\mathcal{S}_\Delta$ is an isomorphism between $\mathcal{F}_\Delta$ and $\mathcal{F}_{\bar{\Delta}}$ because it has a well-defined inverse given by $\mathcal{S}_{\bar{\Delta}}$, i.e.

$$\int_{-\infty}^{\infty}dz\,\frac{N_\Delta}{|x-z|^{2\bar{\Delta}}}\frac{N_{\bar{\Delta}}}{|z-y|^{2\Delta}}=\delta(x-y) \tag{3.2.51}$$

Remark 3. *For $\Delta\in\mathbb{Z}$, the shadow transformation is not an isomorphism between $\mathcal{F}_\Delta$ and $\mathcal{F}_{\bar{\Delta}}$. For example, when $\Delta=1-N$ is a nonpositive integer, $\psi_n^{(1-N)}$ with $n\leq N-1$ are annihilated by $\mathcal{S}_{1-N}$. Indeed, in this case, $\mathcal{S}_{1-N}$ is equivalent to the differential operator ∂_x^{2N-1} up to normalization.*

3.2.2.6 Summary

Given a complex constant Δ which we call *scaling dimension*, the SO$(1,2)$ generators act on the wavefunction space $\mathcal{F}_\Delta$ spanned by $\{\psi_n^{(\Delta)}(x) = (\frac{1-ix}{1+ix})^n \frac{1}{(1+x^2)^\Delta}\}_{n\in\mathbb{Z}}$ as follows

$$P\psi(x) = -\partial_x\psi(x), \quad D\psi(x) = -(x\partial_x + \Delta)\psi(x), \quad K\psi(x) = -(x^2\partial_x + 2\Delta x)\psi(x) \tag{3.2.52}$$

The quadratic Casimir takes the value $\Delta(1-\Delta)$ acting on $\mathcal{F}_\Delta$. All unitary irreducible representations of SO$(1,2)$ can be realized as either $\mathcal{F}_\Delta$ or its invariant subspace for certain values of Δ:

- **Case I**: $\Delta = \frac{1}{2} + i\mu,\ \mu \in \mathbb{R}$ with $L^2(\mathbb{R})$ inner product $(\psi, \psi) = \int dx \bar{\psi}(x)\psi(x)$ for any $\psi(x) \in \mathcal{F}_\Delta$. The representations of $\Delta = \frac{1}{2} \pm i\mu$ are equivalent. This is the *principal series*.

- **Case II**: $\Delta = \frac{1}{2} + \nu,\ -\frac{1}{2} < \nu < \frac{1}{2}$ with inner product $(\psi, \psi) = \int dx dy \frac{1}{|x-y|^{2\Delta}} \bar{\psi}(x)\psi(y)$ or equivalently (up to normalization) in momentum space $(\psi, \psi) = \int dp\, p^{1-2\Delta}|\psi(p)|^2$ for any $\psi(x) \in \mathcal{F}_\Delta$. The representations of $\Delta = \frac{1}{2} \pm \nu$ are equivalent. This is the *complementary series*.

- **Case III**: $\Delta = N \in \mathbb{Z}_+$ with inner product

$$(\psi, \psi)_{\mathcal{F}_N^\pm} = \frac{(-4)^N}{(2\pi)^2\Gamma(2N-1)} \int_{\mathbb{R}} dx \int_{\mathbb{R}\pm i\epsilon} dy\, \bar{\psi}(x)(x-y)^{2(N-1)} \log(x-y)\psi(y) \tag{3.2.53}$$

  where the contour $\mathbb{R} + i\epsilon$ works for $\psi(x) \in \mathcal{F}_N^+$ which is spanned by $\{\psi_n^{(N)}(x)\}_{n\geq N}$ and the contour $\mathbb{R} - i\epsilon$ works for $\psi(x) \in \mathcal{F}_N^-$ which is spanned by $\{\psi_n^{(N)}(x)\}_{n\leq -N}$. This is the *discrete series*.

3.3 UIRs of SO$(1, d+1)$

Like in the $d=1$ case, we start from a primary state $|\bar{\Delta}, 0\rangle_\alpha$ of scaling dimension $\bar{\Delta} \equiv d - \Delta$, but now in addition we also need to specify the spin (also known as the highest weight vector) $\mathbf{s} = (s_1, s_2, \cdots, s_r), r \equiv \left\lfloor \frac{d}{2} \right\rfloor$ (carried by the index α) associated to the SO(d) subgroup. In this work, we will focus on the single-row representation of SO(d), i.e. $\mathbf{s} = (s, 0, 0, \cdots), s \in \mathbb{N}$ which in bulk corresponds to scalar fields and (bosonic) spin-s fields. Let $|\bar{\Delta}, 0\rangle_{i_1 \cdots i_s}$ be such a primary state where the indices $i_1 \cdots i_s$ are symmetric and traceless. The action of conformal algebra is

$$
\begin{aligned}
& D|\bar{\Delta}, 0\rangle_{i_1\cdots i_s} = \bar{\Delta}|\bar{\Delta}, 0\rangle_{i_1\cdots i_s}, \quad K_i|\bar{\Delta}, 0\rangle_{i_1\cdots i_s} = 0, \\
& M_{k\ell}|\bar{\Delta}, 0\rangle_{i_1\cdots i_s} = \sum_{j=1}^{s} |\bar{\Delta}, 0\rangle_{i_1\cdots i_{j-1} k\, i_{j+1}\cdots i_s}\delta_{\ell i_j} - |\bar{\Delta}, 0\rangle_{i_1\cdots i_{j-1}\ell\, i_{j+1}\cdots i_s}\delta_{k i_j}
\end{aligned} \tag{3.3.1}
$$

Acting by translations on this state produces a family of position-dependent states (dropping the label $\bar{\Delta}$ in the kets $|\bar{\Delta}, x\rangle_{i_1\cdots i_s}$ again)

$$
|x\rangle_{i_1\cdots i_s} \equiv e^{x\cdot P}|\bar{\Delta}, 0\rangle_{i_1\cdots i_s} \tag{3.3.2}
$$

The action of the $\mathfrak{so}(1, d+1)$ algebra on these states is then easily computed to be:

$$
\begin{aligned}
P_i|x\rangle_{i_1\cdots i_s} &= \partial_i|x\rangle_{i_1\cdots i_s} \\
D|x\rangle_{i_1\cdots i_s} &= (x\cdot\partial_x + \bar{\Delta})|x\rangle_{i_1\cdots i_s} \\
M_{k\ell}|x\rangle_{i_1\cdots i_s} &= \left(x_\ell\partial_k - x_k\partial_\ell + \mathcal{M}^{(s)}_{k\ell}\right)|x\rangle_{i_1\cdots i_s} \\
K_k|x\rangle_{i_1\cdots i_s} &= \left(2x_k(x\cdot\partial_x + \bar{\Delta}) - x^2\partial_k - 2x^\ell\mathcal{M}^{(s)}_{k\ell}\right)|x\rangle_{i_1\cdots i_s}
\end{aligned} \tag{3.3.3}
$$

where $\mathcal{M}^{(s)}_{kl}$ is the spin-s representation of $\mathfrak{so}(d)$

$$
\mathcal{M}^{(s)}_{kl}|x\rangle_{i_1\cdots i_s} = \sum_{j=1}^{s}|x\rangle_{i_1\cdots i_{j-1}k\, i_{j+1}\cdots i_s}\delta_{\ell i_j} - |x\rangle_{i_1\cdots i_{j-1}\ell\, i_{j+1}\cdots i_s}\delta_{k i_j} \tag{3.3.4}
$$

A general state in the vector space spanned by all $|x\rangle$ is of the form

$$|\psi\rangle \equiv \int d^d x \, \psi_{i_1 \cdots i_s}(x) |x\rangle_{i_1 \cdots i_s} \tag{3.3.5}$$

where the wavefunction $\psi_{i_1 \cdots i_s}(x)$ is smooth in x^i and symmetric and traceless with respect to the i_α indices. We package all components of $\psi_{i_1 \cdots i_s}(x)$ into a single x-dependent polynomial by introducing a complex *null* vector $z^i \in \mathbb{C}^d$, i.e.

$$\psi(x,z) \equiv \frac{1}{s!} \psi_{i_1 \cdots i_s}(x) \, z^{i_1} \cdots z^{i_s} \tag{3.3.6}$$

The null vector z^i can be stripped off while respecting the nullness condition by the following *interior derivative*

$$\mathcal{D}_{z^i} \equiv \partial_{z^i} - \frac{1}{d + 2(z \cdot \partial_z - 1)} \, z_i \, \partial_z^2 \tag{3.3.7}$$

The $\mathfrak{so}(1, d+1)$ action on the wavefunctions $\psi(x,z)$ induced by eq. (3.3.3) is then given by

$$\begin{aligned}
P_i \psi(x,z) &= -\partial_i \psi(x,z) \\
D\psi(x,z) &= -(x \cdot \partial_x + \Delta) \psi(x,z) \\
M_{k\ell} \psi(x,z) &= (x_k \partial_\ell - x_\ell \partial_k + z_k \partial_{z^\ell} - z_\ell \partial_{z^k}) \, \psi(x,z) \\
K_k \psi(x,z) &= \Big(x^2 \partial_k - 2x_k (x \cdot \partial_x + \Delta) - 2x^\ell (z_k \partial_{z^\ell} - z_\ell \partial_{z^k}) \Big) \, \psi(x,z)
\end{aligned} \tag{3.3.8}$$

where the shorthand notation ∂_i means *exclusively* the derivative with respect to x^i. To lift the $\mathfrak{so}(1, d+1)$ action (3.3.8) to a group action, it suffices to exponentiate translations, dilatation, rotations and special conformal transformations separately because of the Bruhat decomposition i.e. SO$(1, d+1)$ = ÑNAM up to a lower dimensional submanifold in SO$(1, d+1)$ [64]. The exponentiation of translations, dilatation and rotations is guaranteed by the smoothness condition imposed on the wavefunctions $\psi(x,z)$. However, the same does not hold for special conformation transformations. Indeed, granting the exponentiation of K_i to a group action on certain subspace $\mathcal{C}_K$ of smooth functions, we obtain

$$\left(e^{b\cdot K}\psi\right)(x,z) = \frac{1}{(1+2b\cdot x+b^2x^2)^{\Delta}}\,\psi\left(\frac{x^i+b^ix^2}{1+2b\cdot x+b^2x^2}, R(x_b)^i{}_j R(x)^j{}_k z^k\right) \tag{3.3.9}$$

where

$$x_b^i \equiv \frac{x^i}{x^2}+b^i, \quad R(x)^i{}_j \equiv \frac{2x^ix_j}{x^2}-\delta^{ij} \tag{3.3.10}$$

Replace ψ by $\psi_1 \equiv e^{-b\cdot K}\psi$ which also lives in the same subspace $\mathcal{C}_K$ and rewrite eq. (3.3.9) in terms of $\tilde{x}^i \equiv \frac{x^i}{x^2}$ (except the combination $R(x)^i{}_j z^j$)

$$\psi(x,z) = \frac{1}{(x^2)^{\Delta}}\frac{1}{(\tilde{x}^2+2b\cdot\tilde{x}+b^2)^{\Delta}}\,\psi_1\left(\frac{\tilde{x}^i+b^i}{\tilde{x}^2+2b\cdot\tilde{x}+b^2}, R(b+\tilde{x})^i{}_j R(x)^j{}_k z^k\right) \tag{3.3.11}$$

The R.H.S of (3.3.11), apart from the factor $\frac{1}{(x^2)^{\Delta}}$, admits a Taylor expansion in $\tilde{x}^i$ as $x\to\infty$

$$\psi(x,z) \overset{x\to\infty}{\approx} \frac{1}{(x^2)^{\Delta}}\sum_{n=0}^{\infty} C_{ns}\left(\tilde{x}^i, R(x)^i{}_j z^j\right) \tag{3.3.12}$$

where $C_{ns}(u,v)$ is a homogeneous polynomial of degree in u^i and degree s in v^i. This equation serves as the universal asymptotic boundary condition for all wavefunctions in $\mathcal{C}_K$.

Altogether, let $\mathcal{F}_{\Delta,s}$ be the space of smooth functions on $\mathbb{R}^d$ which are simultaneously polynomials of degree s in the null vector z^i and satisfy the asymptotic condition (3.3.12). It furnishes an SO$(1,d+1)$ representation whose infinitesimal version is given by (3.3.8). We want to mention that it is straightforward to construct the more general representation $\mathcal{F}_{\Delta,\mathbf{s}}$ for an arbitrary SO(d) highest weight vector $\mathbf{s}$. It suffices to choose the spin-$\mathbf{s}$ action of M_{ij} on the primary state and then the wavefunction picture follows accordingly. In the mathematical literature, the representations $\mathcal{F}_{\Delta,\mathbf{s}}$ are constructed using the induced representation method, which is reviewed in the appendix A.2 where we also show explicitly the equivalence of the two constructions for $\mathcal{F}_{\Delta,s}$. These representations are important because [64]

- Almost all $\mathcal{F}_{\Delta,\mathbf{s}}$ are irreducible apart from some discrete values of Δ. We give an elementary proof of this claim for $\mathcal{F}_{\Delta}$ in the appendix A.3. A more general proof for $\mathcal{F}_{\Delta,\mathbf{s}}$ can be found in [65][6] and a full list of irreducible representations (including the SO$(d+1)$ contents of each one) is given in a subsequent paper by the same author [13]. Indeed, there are only four types of reducible representations of this form when $\mathbf{s} = s$ is a single-row representation

$$\mathcal{F}_{1-t,s}, \quad \mathcal{F}_{1-s,t}, \quad \mathcal{F}_{d+t-1,s}, \quad \mathcal{F}_{d+s-1,t} \tag{3.3.13}$$

where $s = 1, 2, 3, \cdots$ and $t = 0, 1, 2, \cdots$. We will see why these representations are reducible in the section 3.3.5. Throughout this paper, for a given spin s, a point in the Δ-plane is called *generic* if $\mathcal{F}_{\Delta,s}$ is irreducible and called *exceptional* if $\mathcal{F}_{\Delta,s}$ is reducible.

- Any irreducible representation of SO$(1, d+1)$ is equivalent to some subrepresentations of $\mathcal{F}_{\Delta,\mathbf{s}}$ (including $\mathcal{F}_{\Delta,\mathbf{s}}$ itself when it is irreducible).

Therefore, in order to understand the underlying representation theory of unitary spin-s fields in dS$_{d+1}$, we should study the structure of $\mathcal{F}_{\Delta,s}$ in detail and look for its unitary irreducible subrepresentations. This will be our main task for the remaining part of this section.

3.3.1 SO$(d+1)$ contents of $\mathcal{F}_{\Delta,s}$

Before imposing the unitarity condition on $\mathcal{F}_{\Delta,s}$, let's first focus on the SO$(d+1)$ subgroup and see how $\mathcal{F}_{\Delta,s}$ decomposes into irreducible representations of SO$(d+1)$. A standard and elegant approach to this problem involves the Iwasawa decomposition, i.e. SO$(d+1)$ = KNA [64] and the induced representation construction which is reviewed in the appendix A.2. Using this approach, it is not hard to see that $\mathcal{F}_{\Delta,s}$ when considered as an SO$(d+1)$ representation is equivalent to the induced representation $\mathrm{ind}_{\mathrm{SO}(d)}^{\mathrm{SO}(d+1)}\mathbb{Y}_s$, whose SO$(d+1)$ contents follow from Frobenius reciprocity theorem[7] [66]. Here we will present a more elementary method that only depends on our CFT-type construction of $\mathcal{F}_{\Delta,s}$. The SO$(d+1)$ generators are $L_{ij} = M_{ij}$ and

[6]The method in this paper is essentially equivalent to what we does in the appendix A.3 except that the author used an abstract basis, instead of the spherical harmonics we use, for each SO$(d+1)$ content of $\mathcal{F}_{\Delta,\mathbf{s}}$ that works universally for any s. In addition, this method heavily relies on the fact, which we will prove for $\mathcal{F}_{\Delta,s}$ in the following section, that each SO$(d+1)$ content contained in $\mathcal{F}_{\Delta,\mathbf{s}}$ has multiplicity 1.

[7]Roughly speaking, the Frobenius reciprocity theorem states that given a unitary irreducible representation σ of H and a unitary irreducible representation ρ of G, then ρ is contained in the induced representation $\mathrm{ind}_H^G\sigma$ as many times as σ contained in the restriction representation $\rho|_H$.

$L_{d+1,i} = \frac{1}{2}(P_i + K_i)$ which act on $\psi(x,z) \in \mathcal{F}_{\Delta,s}$ as

$$L_{ij}\psi(x,z) = (x_i\partial_j - x_j\partial_i + z_i\partial_{z^j} - z_j\partial_{z^i})\psi(x,z)$$
$$L_{d+1,i}\psi(x,z) = \left(-\frac{1-x^2}{2}\partial_i - x_i(x\cdot\partial_x + \Delta) + (x\cdot z\,\partial_{z^i} - z_i\,x\cdot\partial_z)\right)\psi(x,z) \tag{3.3.14}$$

One important observation is that the Δ-dependence in the action of $L_{d+1,i}$ disappears if we perform a rescaling for the wavefunction $\hat{\psi}(x,z) \equiv \left(\frac{1+x^2}{2}\right)^{\Delta-s}\psi(x,z)$ (the purpose of introducing "$-s$" will be clear soon automatically)

$$L_{ij}\hat{\psi}(x,z) = (x_i\partial_j - x_j\partial_i + z_i\partial_{z^j} - z_j\partial_{z^i})\hat{\psi}(x,z)$$
$$L_{d+1,i}\hat{\psi}(x,z) = \left(-\frac{1-x^2}{2}\partial_i - x_i\,(x\cdot\partial_x + s) + (x\cdot z\,\partial_{z^i} - z_i\,x\cdot\partial_z)\right)\hat{\psi}(x,z) \tag{3.3.15}$$

We claim that $\hat{\psi}(x,z)$ defines a spin-s tensor on S^d (in the stereographic coordinate). It suffices to show that $\mathfrak{so}(d+1)$ acts on $\hat{\psi}(x,z)$ as Lie derivatives along the Killing vectors of S^d. In embedding space coordinates, S^d is described as a hypersurface

$$Y_1^2 + Y_2^2 + \cdots + Y_{d+1}^2 = 1 \tag{3.3.16}$$

with the space of Killing vectors spanned by $V_{ab} = Y_a\partial_{Y^b} - Y_b\partial_{Y^a},\ 1 \le a,b \le d+1$, where the vector field V_{ab} is generated by the action of L_{ab}. The stereographic coordinates of S^d correspond to choosing the following embedding

$$Y^a = \left(\frac{2x^i}{x^2+1}, \frac{x^2-1}{x^2+1}\right) \tag{3.3.17}$$

which yields a conformally flat metric $g_{ij} = \left(\frac{2}{1+x^2}\right)^2\delta_{ij}$. Given an arbitrary vector field $V = V^i\partial_i$ and an arbitrary tensor $\varphi_{i_1\cdots i_s}$, the Lie derivative $\mathcal{L}_V$ is defined as

$$\mathcal{L}_V\varphi_{i_1\cdots i_s} = V^k\partial_k\varphi_{i_1\cdots i_s} + \sum_{j=1}^{s}\partial_{i_j}V^k\varphi_{i_1\cdots i_{j-1}ki_{j+1}\cdots i_s} \tag{3.3.18}$$

When $\varphi_{i_1\cdots i_s}$ is an symmetric and traceless tensor, we can use the index-free formalism and replace it by a polynomial $\varphi(x,z) \equiv \frac{1}{s!}\varphi_{i_1\cdots i_s}z^{i_1}\cdots z^{i_s}$, where z^i is null. Then the Lie derivative acting on $\varphi(x,z)$ becomes

$$\mathcal{L}_V\varphi(x,z) = V^k\partial_k\varphi(x,z) + (z\cdot\partial_x V^k)\mathcal{D}_{z^k}\varphi(x,z) \tag{3.3.19}$$

where $\mathcal{D}_{z^i}$ is given by eq. (3.3.7). To compute the Lie derivatives $\mathcal{L}_{V_{ab}}$, we need further to write out the Killing vectors V_{ab} in terms of stereographic coordinates by using

$$(V_{ab})^k = g^{k\ell}(Y_a\partial_\ell Y_b - Y_b\partial_\ell Y_a) \tag{3.3.20}$$

We obtain $(V_{ij})^k = x_i\delta^k_j - x_j\delta^k_i$ and $(V_{d+1,i})^k = -\frac{1-x^2}{2}\delta^k_i - x^k x_i$ respectively. Plugging these explicit forms of V_{ab} into eq. (3.3.19) yields

$$\begin{aligned}
&\mathcal{L}_{V_{ij}}\varphi(x,z) = \left(x_i\partial_j - x_j\partial_i + z_i\partial_{z^j} - z_j\partial_{z^i}\right)\varphi(x,z)\\
&\mathcal{L}_{V_{d+1,i}}\varphi(x,z) = \left(-\frac{1-x^2}{2}\partial_i - x_i\left(x\cdot\partial_x + s\right) + \left(x\cdot z\,\partial_{z^i} - z_i\,x\cdot\partial_z\right)\right)\varphi(x,z)
\end{aligned} \tag{3.3.21}$$

where the number s in the second line comes from $z\cdot\partial_z$ acting on $\varphi(x,z)$. The agreement between eq. (3.3.15) and eq. (3.3.21) implies that the space $\mathcal{F}_{\Delta,s}$ is isomorphic to the space of spin-s tensors [8] on S^d. From the latter, we can easily read off the SO$(d+1)$ contents of $\mathcal{F}_{\Delta,s}$. For example, when $s=1$, we can decompose a spin-1 tensor into a scalar function and a transverse spin-1 tensor, i.e. $\hat{\psi}_i(x) = \partial_i\xi(x) + \eta_i(x),\ \nabla_i\eta^i = 0$, where ξ admits an expansion in terms of scalar spherical harmonics excluding the constant one and η_i admits an expansion in terms of transverse vector spherical harmonics. It is well known that the scalar harmonics on S^d correspond to all single-row representations, i.e. $\mathbb{Y}_n$ of SO$(d+1)$ while the transverse vector harmonics correspond to two-row representations with 1 box in the second row, i.e. $\mathbb{Y}_{n,1}$ [67]. Altogether, $\mathcal{F}_{\Delta,1}$ as an SO$(d+1)$ representation contains the following irreducible components

$$\mathcal{F}_{\Delta,1}\big|_{\mathrm{SO}(d+1)} = \left(\bigoplus_{n\geq1}\mathbb{Y}_n\right)\oplus\left(\bigoplus_{k\geq1}\mathbb{Y}_{k,1}\right) = \bigoplus_{n\geq1}\bigoplus_{0\leq m\leq1}\mathbb{Y}_{n,m} \tag{3.3.22}$$

[8] By spin-s, we mean a symmetric and traceless tensor of rank s.

For higher s, we decompose $\hat{\psi}_{i_1\cdots i_s}$ into a spin-$(s-1)$ tensor $\xi_{i_1\cdots i_{s-1}}$ and a transverse spin-s tensor $\eta_{i_1\cdots i_s}$, i.e.

$$\hat{\psi}_{i_1\cdots i_s} = \nabla_{(i_1}\xi_{i_2\cdots i_s)} - \text{trace} + \eta_{i_1\cdots i_s}, \quad \nabla^{i_1}\eta_{i_1\cdots i_s} = 0 \tag{3.3.23}$$

The latter admits an expansion in terms of transverse spin-s tensor harmonics which group theoretically correspond to all 2-row representations of SO$(d+1)$ with s boxes in the second row. In $\xi_{i_1\cdots i_{s-1}}$, we should exclude the modes such that $\nabla_{(i_1}\xi_{i_2\cdots i_s)}$ is pure trace. These modes are spin-$(s-1)$ conformal Killing tensors on S^d and furnish the representation $\mathbb{Y}_{s-1,s-1}$ of SO$(1,d+1)$ which becomes $\bigoplus_{0\le n\le s-1}\mathbb{Y}_{s-1,n}$ while restricted to the SO$(d+1)$ subgroup. By using induction on s, we can immediately conclude that

$$\boxed{\mathcal{F}_{\Delta,s}\big|_{\text{SO}(d+1)} = \bigoplus_{n\ge s}\bigoplus_{0\le m\le s}\mathbb{Y}_{n,m}} \tag{3.3.24}$$

3.3.2 Shadow transformation

In the SO$(1,2)$ case, we have shown that there exists a linear intertwining operator $\mathcal{S}_\Delta$, called shadow transformation, that maps $\mathcal{F}_\Delta$ to $\mathcal{F}_{\bar{\Delta}}$ and in particular, when $\Delta\notin\mathbb{Z}$ it is an isomorphism. In this section, we will show that a similar operator also exists for higher d. Assume that $\mathcal{S}_{\Delta,s}^{\Delta',s'}$: $\mathcal{F}_{\Delta,s}\to\mathcal{F}_{\Delta',s'}$ is an intertwining operator defined by a kernel function $S_{\Delta,s}^{\Delta',s'}(x_1,x_2)_{i_1\cdots i_{s'},j_1\cdots j_s}$

$$\mathcal{S}_{\Delta,s}^{\Delta',s'} : \psi_{i_1\cdots i_s}(x_1)\in\mathcal{F}_{\Delta,s}\longmapsto\int d^dx\, S_{\Delta,s}^{\Delta',s'}(x_1,x_2)_{i_1\cdots i_{s'},j_1\cdots j_s}\psi_{j_1\cdots j_s}(x_2) \tag{3.3.25}$$

The requirement $\mathcal{S}_{\Delta,s}^{\Delta',s'}[\psi_{i_1\cdots i_s}]\in\mathcal{F}_{\Delta',s'}$ induces a set of differential equations for the kernel function (Suppress the spin indices of $S_{\Delta,s}^{\Delta',s'}(x,y)_{i_1\cdots i_{s'},j_1\cdots j_s}$ for the simplicity of notation. It should be clear that $\mathcal{M}^{(s')}_{k\ell}$ acts on the i indices while $\mathcal{M}^{(s)}_{k\ell}$ acts on the j indices.):

$$(\partial_{x^i} + \partial_{y^i}) S^{\Delta',s'}_{\Delta,s}(x,y) = 0 \tag{3.3.26}$$

$$(x\cdot\partial_x + y\cdot\partial_y + \Delta' + \bar{\Delta}) S^{\Delta',s'}_{\Delta,s}(x,y) = 0 \tag{3.3.27}$$

$$(x_\ell\partial_{x^k} - x_k\partial_{x^\ell} + y_\ell\partial_{y^k} - y_k\partial_{y^\ell} + \mathcal{M}^{(s)}_{k\ell} + \mathcal{M}^{(s')}_{k\ell}) S^{\Delta',s'}_{\Delta,s}(x,y) = 0 \tag{3.3.28}$$

$$(x^2\partial_{x^k} - 2x_k(x\cdot\partial_x + \Delta') + y^2\partial_{y^k} - 2y_k(y\cdot\partial_y + \bar{\Delta}) + 2x^\ell\mathcal{M}^{(s')}_{k\ell} + 2y^\ell\mathcal{M}^{(s)}_{k\ell}) S^{\Delta',s'}_{\Delta,s}(x,y) = 0 \tag{3.3.29}$$

These differential equations are exactly the conformal Ward identities for a two-point function of two primary operators $\mathcal{O}_{i_1\cdots i_s}$ and $\mathcal{O}'_{i_1\cdots i'_s}$ with scaling dimension $\bar{\Delta}$ and Δ' respectively. It is well known that such a two-point function is vanishing unless the two operators have the same spin and scaling dimension. Therefore $\mathcal{S}^{\Delta',s'}_{\Delta,s}$ exists only when $s = s', \Delta' = \bar{\Delta}$ and in this case the corresponding kernel function, denoted by $S_{\Delta,s}(x_1-x_2)_{i_1\cdots i_s, j_1\cdots j_s}$, becomes the conformal two-point function $\langle \mathcal{O}_{i_1\cdots i_s}(x_1)\mathcal{O}_{j_1\cdots j_s}(x_2)\rangle$ which takes the following simple form in the index-free formalism

$$\begin{aligned} S_{\Delta,s}(x_{12}; z, w) &\equiv S_{\Delta,s}(x_{12})_{i_1\cdots i_s, j_1\cdots j_s}\, z^{i_1}\cdots z^{i_s} w^{j_1}\cdots w^{j_s} \\ &= N_{\Delta,s}\frac{(-z\cdot R(x_{12})\cdot w)^s}{(x_{12}^2)^{\bar{\Delta}}}, \quad x^i_{12} \equiv x^i_1 - x^i_2 \end{aligned} \tag{3.3.30}$$

where $N_{\Delta,s}$ is a normalization constant, z^i and w^i are null vectors and $R(x)_{ij} = \frac{2x_i x_j}{x^2} - \delta_{ij}$ is defined in the eq. (3.3.10).

Given shadow transformations $\mathcal{S}_{\Delta,s}$ and $\mathcal{S}_{\bar{\Delta},s}$, the composition $\mathcal{S}_{\bar{\Delta},s}\circ\mathcal{S}_{\Delta,s}$ is an intertwining map that maps $\mathcal{F}_{\Delta,s}$ to itself. When $\mathcal{F}_{\Delta,s}$ is irreducible [9], which is true for almost all Δ, the composition should be proportional to identity map due to Schur's lemma. We want to (partially) fix the normalization by requiring $\mathcal{S}_{\bar{\Delta},s}\circ\mathcal{S}_{\Delta,s} = 1_{\mathcal{F}_{\Delta,s}}$, i.e.

$$\int d^d x_2 S_{\bar{\Delta},s}(x_{12})_{i_1\cdots i_s, j_1\cdots j_s} S_{\Delta,s}(x_{23})_{j_1\cdots j_s, k_1\cdots k_s} = \delta^d(x_{13})\delta_{i_1\cdots i_s, k_1\cdots k_s} \tag{3.3.31}$$

[9] At those exceptional points where $\mathcal{F}_{\Delta,s}$ is reducible, the corresponding shadow transformations are not invertible. We will comment more on these cases later.

or equivalent in momentum space

$$S_{\bar{\Delta},s}(p)_{i_1\cdots i_s,j_1\cdots j_s}S_{\Delta,s}(p)_{j_1\cdots j_s,k_1\cdots k_s}=\delta_{i_1\cdots i_s,k_1\cdots k_s} \tag{3.3.32}$$

The Fourier transformation $S_{\Delta,s}(p,z;w)\equiv\int d^dx\, e^{ip\cdot x}\, S_{\Delta,s}(x,z;w)$ can be performed by using the binomial expansion for the numerator of $S_{\Delta,s}(x,z;w)$ and then applying the following formula to each term

$$\int d^dx\,\frac{1}{|x|^{2\bar{\Delta}}}e^{ip\cdot x}=2^{d-2\bar{\Delta}}\pi^{\frac{d}{2}}\,\frac{\Gamma(\frac{d}{2}-\bar{\Delta})}{\Gamma(\bar{\Delta})}p^{2\bar{\Delta}-d} \tag{3.3.33}$$

The final expression admits a harmonic expansion with respect to SO$(d-1)$, which is the little group of a fixed momentum p. For a detailed derivation of such an expansion, we refer to the book [64]. Here we simply present the result

$$S_{\Delta,s}(p;z,w)=\pi^{\frac{d}{2}}\,N_{\Delta,s}\,\frac{\Gamma(\frac{d}{2}-\bar{\Delta})\,(\frac{1}{2}p)^{2\bar{\Delta}-d}}{(\bar{\Delta}+s-1)\Gamma(\bar{\Delta}-1)}\sum_{\ell=0}^{s}\kappa_{s\ell}(\Delta)\Pi^{s\ell}(\hat{p};z,w),\quad \kappa_{s\ell}(\Delta)=\frac{(\Delta+\ell-1)_{s-\ell}}{(\bar{\Delta}+\ell-1)_{s-\ell}} \tag{3.3.34}$$

where $\Pi^{s\ell}(\hat{p};z,w)\equiv\Pi^{s\ell}(\hat{p})_{i_1\cdots i_s,j_1\cdots j_s}z^{i_1}\cdots z^{i_s}w^{j_1}\cdots w^{j_s}$ are $(s+1)$ *projection operators* that only depend on the unit vector $\hat{p}$ in the direction of p^i and satisfy the following properties

$$\begin{aligned}
&\textbf{Orthogonality}:\Pi^{s\ell}(\hat{p})_{i_1\cdots i_s,j_1\cdots j_s}\Pi^{s\ell'}(\hat{p})_{j_1\cdots j_s,k_1\cdots k_s}=\delta_{\ell\ell'}\Pi^{s\ell}(\hat{p})_{i_1\cdots i_s,k_1\cdots k_s}\\
&\textbf{Completeness}:\sum_{\ell=0}^{s}\Pi^{s\ell}(\hat{p};z,w)=(z\cdot w)^s\\
&\textbf{Helicity}\ \ell:p^{i_\ell}p^{i_{\ell+1}}\cdots p^{i_s}\Pi^{s\ell}(\hat{p})_{i_1\cdots i_s,j_1\cdots j_s}=p^{j_\ell}p^{j_{\ell+1}}\cdots p^{j_s}\Pi^{s\ell}(\hat{p})_{i_1\cdots i_s,j_1\cdots j_s}=0,\ \ell\geq 1
\end{aligned} \tag{3.3.35}$$

As an example, when $s=1$, we have

$$\Pi^{10}(\hat{p})_{i,j}=\hat{p}_i\hat{p}_j,\quad \Pi^{11}(\hat{p})_{i,j}=\delta_{ij}-\hat{p}_i\hat{p}_j \tag{3.3.36}$$

Remark 4. *The set of* $\Pi^{s\ell}$ *can be thought as the manifestation of the branching rule from SO*(d) *(the full rotation group of* p*) to SO*$(d-1)$ *(the little group of* p*), i.e. the spin-*s *representation of SO*(d) *can be decomposed into the direct sum of the spin-*ℓ *representations of SO*$(d-1)$ *with*

ℓ ranging from 0 to s. Each $\Pi^{s\ell}$ projects the spin-s SO(d) representation to its spin-ℓ SO($d-1$) summand. For example, in the $s=1$ case, Π^{10} projects a vector to its component along the direction of p which is clearly invariant under the little group and Π^{11} yields the transverse part which carries the spin-1 representation of the little group.

Using the orthogonality and completeness of $\{\Pi^{s\ell}\}$ and noticing $\kappa_{s\ell}(\Delta)\kappa_{s\ell}(\bar{\Delta})=1$, we find that the normalization condition (3.3.32) is equivalent to

$$\frac{\pi^d\,\Gamma(\frac{d}{2}-\bar{\Delta})\,\Gamma(\frac{d}{2}-\Delta)\,N_\Delta\,N_{\bar{\Delta}}}{(\bar{\Delta}+s-1)(\Delta+s-1)\Gamma(\bar{\Delta}-1)\Gamma(\Delta-1)}=1 \tag{3.3.37}$$

Apparently, this equation cannot fix the normalization constant $N_{\Delta,s}$ completely. Two convenient solutions that we will use are

$$N^+_{\Delta,s}=\frac{(\bar{\Delta}+s-1)\Gamma(\bar{\Delta}-1)}{\pi^{\frac{d}{2}}\,\Gamma(\frac{d}{2}-\bar{\Delta})},\quad N^-_{\Delta,s}=\frac{(\Delta+s-1)\Gamma(\Delta-1)}{\pi^{\frac{d}{2}}\,\Gamma(\frac{d}{2}-\bar{\Delta})} \tag{3.3.38}$$

and the corresponding shadow transformation will be denoted by $\mathcal{S}^\pm_{\Delta,s}$. Let's write out the kernel for $\mathcal{S}^\pm_{\Delta,s}$ in momentum space explicitly

$$\begin{aligned}
S^+_{\Delta,s}(p;z,w)&=\left(\frac{p}{2}\right)^{2\bar{\Delta}-d}\sum_{\ell=0}^{s}\frac{(\Delta+\ell-1)_{s-\ell}}{(\bar{\Delta}+\ell-1)_{s-\ell}}\Pi^{s\ell}(\hat{p};z,w)\\
S^-_{\Delta,s}(p;z,w)&=\left(\frac{p}{2}\right)^{2\bar{\Delta}-d}\frac{\Gamma(\Delta-1)}{\Gamma(\bar{\Delta}-1)}\sum_{\ell=0}^{s}\frac{(\Delta+\ell-1)_{s-\ell+1}}{(\bar{\Delta}+\ell-1)_{s-\ell+1}}\Pi^{s\ell}(\hat{p};z,w)
\end{aligned} \tag{3.3.39}$$

For a generic Δ, the two choices are equivalent since they only differ by a normalization factor. So we will stick to $S^+_{\Delta,s}$ in this case. However, at exceptional points, $S^\pm_{\Delta,s}$ are completely different and the corresponding properties are summarized in the table (3.3.1), where $s=1,2,\cdots$ and $t=0,1,\cdots,s-1$.

	$S^+_{\Delta',s'}$	$S^-_{\Delta',s'}$
$(\Delta',s')=(1-t,s)$	only contains $\Pi^{s\ell}$ with $t+1\le\ell\le s$	ill-defined
$(\bar{\Delta}',s')=(1-t,s)$	ill-defined	only contains $\Pi^{s\ell}$ with $0\le\ell\le t$
$(\Delta',s')=(1-s,t)$	contains all $\Pi^{t\ell},0\le\ell\le t$	ill-defined
$(\bar{\Delta}',s')=(1-s,t)$	contains all $\Pi^{t\ell},0\le\ell\le t$	δ-function in momentum space

Table 3.3.1: Properties of $S^\pm_{\Delta,s}$ at exceptional points.

Notice that $S^-_{d+s-1,t}$ is special because it is a polynomial in x^i of degree $2s-2$

$$S^-_{d+s-1,t}(x;z,w) = \frac{(d+s+t-2)\Gamma(d+s-2)}{\pi^{\frac{d}{2}}\Gamma(\frac{d}{2}+s-1)}\left(x^2\right)^{s-t-1}\left(x^2\, z\cdot w - 2\, x\cdot z\, x\cdot w\right)^t \tag{3.3.40}$$

Therefore, its Fourier transformation is a δ-function with $(2s-2)$ derivatives in the momentum space.

3.3.3 (Nonexceptional) unitary scalar representations

Recall that a wavefunction $\psi \in \mathcal{F}_\Delta$ defines a ket $|\psi\rangle = \int d^dx\, \psi(x)|x\rangle$. So an inner product on $\mathcal{F}_\Delta$ is fixed by defining a bra $\langle x|$:

$$(\psi_1,\psi_2) \equiv \int d^dx \int d^dy\, \psi_1(x)^* K_\Delta(x,y)\psi_2(y), \quad K_\Delta(x,y) = \langle x|y\rangle \tag{3.3.41}$$

Imposing the reality condition $L^\dagger_{AB} = -L_{AB}$ as in section 3.2.2.2, we find that the function $K_\Delta(x,y)$ exists only in the following two cases

$$\begin{aligned} &\text{I}: \Delta = \frac{d}{2}+i\mu,\ \mu\in\mathbb{R}, \quad &&K_\Delta(x,y) = \delta^d(x-y) \\ &\text{II}: \Delta \in \mathbb{R}, &&K_\Delta(x,y) = \frac{c_\Delta}{|x-y|^{2\bar{\Delta}}} \end{aligned} \tag{3.3.42}$$

where c_Δ is a normalization constant. In the case (I), the inner product (3.3.41) is the standard L^2 inner product on $\mathbb{R}^d$. It is positive definite and normalizable since $\psi(x)\in\mathcal{F}_\Delta$ falls off as $\frac{1}{|x|^{2\Delta}}$ for large x. Therefore, the function space $\mathcal{F}_{\frac{d}{2}+i\mu}$ equipped with the standard L^2 inner product furnishes a unitary irreducible representation of SO$(1,d+1)$, which is known as the (unitary) *scalar principal series representation* of scaling dimension $\Delta = \frac{d}{2}+i\mu$. In the case (II), we choose $c_\Delta = N^+_\Delta = \frac{\Gamma(\bar{\Delta})}{\pi^{\frac{d}{2}}\Gamma(\frac{d}{2}-\bar{\Delta})}$, so that $\boxed{K_\Delta(x,y) = S^+_\Delta(x,y)}$, i.e. the kernel of shadow transformation from $\mathcal{F}_\Delta$ to $\mathcal{F}_{\bar{\Delta}}$. Using the Fourier transformation of $S^+_\Delta(x)$, c.f. eq.(3.3.39), we rewrite the inner product (3.3.41) in momentum space as

$$(\psi_1,\psi_2) = \frac{1}{4^{\bar{\Delta}}\pi^d}\int d^dp\, p^{2\bar{\Delta}-d}\psi_1(p)^*\psi_2(p) \tag{3.3.43}$$

where $\psi_i(p) \equiv \int d^dx\, e^{ip\cdot x}\psi_i(x)$ is the Fourier transformation of $\psi(x)$. This inner product is positive definite as long as it is convergent. Due to the smoothness of $\psi(x)$, its Fourier transformation $\psi(p)$ decays exponentially for large p. So the p-integral in (3.3.43) is convergent around $p \to \infty$. For small p, as we have argued in the SO$(1,2)$ case, $\psi(p)$ has two types of leading fall-offs: p^0 and $p^{2\Delta-d}$. Then the requirement of convergence near $p \to 0$ yields $\boxed{0<\Delta<d}$. Therefore, the representations $\mathcal{F}_\Delta$ with $0<\Delta<d$ are unitary, known as the (unitary) *scalar complementary series*.

Before moving to spinning representations, we want to present a different way to expand the kernel K_Δ based on our discussion in the section 3.3.1. It yields the same constraint on the scaling dimension. Using the Weyl transformation $\psi(x) = \left(\frac{2}{1+x^2}\right)^\Delta \hat{\psi}(x)$, where $\hat{\psi}(x)$ is a function on S^d, we rewrite the inner product (3.3.41) as

$$(\psi_1,\psi_2) = N_\Delta^+ \int \frac{d^dx}{\left(\frac{1+x^2}{2}\right)^d}\frac{d^dy}{\left(\frac{1+y^2}{2}\right)^d}\left(\frac{(1+x^2)(1+y^2)}{2(x-y)^2}\right)^{\bar{\Delta}} \hat{\psi}(x)^*\hat{\psi}(y) \tag{3.3.44}$$

Switch to spherical coordinate $\Omega^i = (\sin\theta\,\omega^i, \cos\theta), \omega^i \in S^{d-1}$ which is related the stereographic coordinate x^i by $x^i = \cot\frac{\theta}{2}\omega^i$ and eq. (3.3.44) becomes an integral on $S^d\times S^d$

$$(\psi_1,\psi_2) = \int d^d\Omega_1\, d^d\Omega_2\, \hat{K}(\Omega_1,\Omega_2)\hat{\psi}(\Omega_1)^*\hat{\psi}_2(\Omega_2), \quad \hat{K}_\Delta(\Omega_1,\Omega_2) = \frac{N_\Delta^+}{(1-\Omega_1\cdot\Omega_2)^{\bar{\Delta}}} \tag{3.3.45}$$

Now we need to perform a harmonic expansion for the new kernel $\hat{K}_\Delta$. First, write it as an integral using Schwinger's trick

$$\hat{K}_\Delta(\Omega_1,\Omega_2) = \frac{N_\Delta^+}{\Gamma(\bar{\Delta})}\int_0^\infty \frac{ds}{s}\, s^{\bar{\Delta}} e^{-s+s\Omega_1\cdot\Omega_2} \tag{3.3.46}$$

Then plug it in the expansion of plane waves in Gegenbauer polynomials [68]

$$e^{s\Omega_1\cdot\Omega_2} = \Gamma(\alpha)\left(\frac{s}{2}\right)^{-\alpha}\sum_{\ell\geq 0}(\alpha+\ell)I_{\alpha+\ell}(s)C_\ell^\alpha(\Omega_1\cdot\Omega_2), \quad \alpha = \frac{d-1}{2} \tag{3.3.47}$$

Using the addition theorem, Gegenbauer polynomials can thus be expanded in spherical harmonics on S^d [69]

$$C_\ell^\alpha(\Omega_1 \cdot \Omega_2) = \frac{2\pi^{\frac{d+1}{2}}}{\Gamma(\alpha)(\ell+\alpha)} \sum_{\mathbf{m}} Y_{\ell\mathbf{m}}(\Omega_1) Y_{\ell\mathbf{m}}(\Omega_2)^* \tag{3.3.48}$$

Therefore if we can perform the s-integral, we have the desired expansion of the kernel $\hat{K}_\Delta$ in spherical harmonics

$$\hat{K}_\Delta(\Omega_1, \Omega_2) = 2^\Delta \sum_{\ell \geq 0} \sum_{\mathbf{m}} \frac{\Gamma(\ell+\bar{\Delta})}{\Gamma(\ell+\Delta)} Y_{\ell\mathbf{m}}(\Omega_1) Y_{\ell\mathbf{m}}(\Omega_2)^* \tag{3.3.49}$$

where we have plugged in the explicit form of N_Δ^+. For $\hat{K}_\Delta$ to be positive definite, the matrix element $\frac{\Gamma(\ell+\bar{\Delta})}{\Gamma(\ell+\Delta)}$ should *not* oscillate with ℓ, which requires $\frac{\ell+\bar{\Delta}}{\ell+\Delta} > 0$ for any $\ell \geq 0$. The most stringent constraint comes from $\ell = 0$ and it is $\boxed{0 < \Delta < d}$, in agreement with what we have found in momentum space. With this constraint satisfied, $\frac{\Gamma(\ell+\bar{\Delta})}{\Gamma(\ell+\Delta)}$ stays positive for all ℓ and hence $\hat{K}_\Delta$ is positive definite.

Remark 5. *When Δ is a nonpositive integer, $\frac{\Gamma(\ell+\bar{\Delta})}{\Gamma(\ell+\Delta)}$ vanishes for $0 \leq \ell \leq -\Delta$. Thus the spherical harmonics $Y_{\ell\mathbf{m}}$ with $0 \leq \ell \leq -\Delta$ are null states. Indeed, they furnish an irreducible representation of SO$(1, d+1)$ which is checked explicitly in the appendix A.3. When $\bar{\Delta}$ is a nonpositive integer, we choose $c_\Delta = N_\Delta^-$. It amounts to replacing $\frac{\Gamma(\ell+\bar{\Delta})}{\Gamma(\ell+\Delta)}$ by $\frac{(\Delta)_\ell}{(\bar{\Delta})_\ell}$ in the expansion eq. (3.3.49). In this case, all $Y_{\ell\mathbf{m}}$ with $\ell \geq 1 - \bar{\Delta}$ are null and carry an irreducible representation which is again checked in the appendix A.3.*

3.3.4 (Nonexceptional) unitary spinning representations

For a spinning representation $\mathcal{F}_{\Delta,s}$, the inner product is defined by the kernel

$$K_{\Delta,s}(x_1, x_2)_{i_1\cdots i_s, j_1\cdots j_s} \equiv {}_{i_1\cdots i_s}\langle x_1 | x_2 \rangle_{j_1\cdots j_s} \tag{3.3.50}$$

In the index free formalism, we contract the i-indices with a null vector z and contract the j-indices with a different null vector w

$$K_{\Delta,s}(x_1, z; x_2, w) \equiv K_{\Delta,s}(x_1, x_2)_{i_1\cdots i_s, j_1\cdots j_s} z^{i_1} \cdots z^{i_s} w^{j_1} \cdots w^{j_s} \tag{3.3.51}$$

With the de Sitter reality condition imposed, $K_{\Delta,s}(x_1, z; x_2, w)$ exists only for two cases (I): $\Delta = \frac{d}{2} + i\mu$ and (II): $\Delta \in \mathbb{R}$. In the first case, $K_{\Delta,s}$ is simply a δ-function (in both spacetime coordinate and spin indices), i.e.

$$K_{\Delta,s}(x_1, z; x_2, w) = \delta^d(x_1, x_2)(z \cdot w)^s \tag{3.3.52}$$

and the hence the inner product defined by this $K_{\Delta,s}$ becomes the L^2-inner product for spin-s tensors on $\mathbb{R}^d$

$$\boxed{(\psi, \varphi)_{\mathcal{P}} \equiv \int d^d x\, \psi^*_{i_1 \cdots i_s}(x)\, \varphi_{i_1 \cdots i_s}(x)} \tag{3.3.53}$$

The positivity (and normalizability) of this inner product on $\mathcal{F}_{\frac{d}{2}+i\mu,s}$ is guaranteed by the asymptotic behavior (3.3.12). Therefore, $\mathcal{F}_{\frac{d}{2}+i\mu,s}$ is a unitary irreducible representation, belonging to the so-called (unitary) *spinning principal series*. In the second case, the reality condition is actually equivalent to the conformal Ward identities for two-point functions. Therefore $K_{\Delta,s}$ is the same as $S_{\Delta,s}$ up to normalization, i.e.

$$K_{\Delta,s}(x_1, z; x_2, w) = c_{\Delta,s} \frac{(-z \cdot R(x_{12}) \cdot w)^s}{(x_{12}^2)^{\bar{\Delta}}} \tag{3.3.54}$$

For a generic Δ, we choose $c_{\Delta,s} = N^+_{\Delta,s}$ and hence $K_{\Delta,s} = S^+_{\Delta,s}$, which in momentum space is (c.f. eq. (3.3.39))

$$S^+_{\Delta,s}(p; z, w) = \left(\frac{p}{2}\right)^{2\bar{\Delta}-d} \sum_{\ell=0}^{s} \kappa_{s\ell}(\Delta) \Pi^{s\ell}(\hat{p}; z, w), \quad \kappa_{s\ell}(\Delta) = \frac{(\Delta + \ell - 1)_{s-\ell}}{(\bar{\Delta} + \ell - 1)_{s-\ell}} \tag{3.3.55}$$

As in the scalar case, the inner product defined by $S^+_{\Delta,s}$ is normalizable when $0 < \Delta < d$. However, this condition cannot guarantee the positivity of $S^+_{\Delta,s}$. Additionally, for $S^+_{\Delta,s}$ to be positive, we need all $\kappa_{s\ell}(\Delta)$ to have a fixed sign (insert an overall minus sign if negative) which yields

$$\frac{\kappa_{s,\ell+1}(\Delta)}{\kappa_{s,\ell}(\Delta)} = \frac{\bar{\Delta} + \ell - 1}{\Delta + \ell - 1} > 0 \tag{3.3.56}$$

The eq. (3.3.56) holds for all ℓ if and only if $\boxed{1 < \Delta < d-1}$. It is also straightforward to check that all $\kappa_{s\ell}(\Delta)$ are indeed positive for Δ in this range. Altogether, $\mathcal{F}_{\Delta,s}$ with the following inner product

$$\boxed{(\psi,\varphi)_{\mathcal{C}} = \int d^dx_1\, d^dx_2\, \psi^*(x_1)_{i_1\cdots i_s}\, S^+_{\Delta,s}(x_{12})_{i_1\cdots i_s,j_1\cdots j_s}\varphi(x_2)_{j_1\cdots j_s}} \tag{3.3.57}$$

is a unitary irreducible representation of SO$(1,d+1)$ when $1 < \Delta < d-1$. It belongs to the so-called (unitary) *spinning complementary series*.

3.3.5 Exceptional series $(d \geq 3)$

We have studied the constraint of unitarity on $\mathcal{F}_{\Delta,s}$ for generic Δ and managed to identify the (unitary) principal and complementary series. In this section, we will look into the four types of $\mathcal{F}_{\Delta,s}$ at exceptional points by mainly following the book [64]:

$$\mathcal{F}_{1-t,s}, \quad \mathcal{F}_{d+t-1,s}, \quad \mathcal{F}_{1-s,t}, \quad \mathcal{F}_{d+s-1,t} \tag{3.3.58}$$

The first important observation is that the four representations share the same SO$(1,d+1)$ (quadratic) Casimir

$$\mathcal{C}_2 = -(s-1)(s+d-1) - t(t+d-2) \tag{3.3.59}$$

which is also the Casimir associated to the highest weight representation $\mathbb{Y}_{s-1,t}$ of SO$(1,d+1)$. This observation suggests that these representations are related by a chain of intertwining maps. For representations in the two pairs $(\mathcal{F}_{1-t,s},\mathcal{F}_{d+t-1,s})$ and $(\mathcal{F}_{1-s,t},\mathcal{F}_{d+s-1,t})$, this is certainly true due to shadow transformations, which have been explored in the section 3.3.2. To find intertwining representations relating the two pairs, let's revisit the shadow transformations $\mathcal{S}^+_{1-t,s}$. At the end of the section 3.3.2 (see the table (3.3.1)), we find that $S^+_{1-t,s}(p)$ only contains the projection operators $\Pi^{s\ell}$ with $\ell \geq t+1$. Due to the third property in the eq. (3.3.35), it means

$$p^{i_1}\cdots p^{i_{s-t}} S^+_{1-t,s}(p)_{i_1\cdots i_s,j_1\cdots j_s} = 0 \tag{3.3.60}$$

So the shadow transformation $\mathcal{S}^+_{1-t,s}$ has a nontrivial kernel which consists functions of the form

$$p_{(i_1} \cdots p_{i_{s-t}} f_{i_{s-t+1} \cdots i_s)} - \text{trace} \tag{3.3.61}$$

in momentum space. Switch back to position space and the kernel can be alternatively expressed as $\{(z \cdot \partial_x)^{s-t} f(x,z)\}$ in the index-free formalism. Requiring that $(z \cdot \partial_x)^{s-t} f(x,z)$ transforms under the $\mathcal{F}_{1-t,s}$, we find $f(x,z)$ is actually an element in $\mathcal{F}_{1-s,t}$. Therefore, $(z \cdot \partial_x)^{s-t}$ is an intertwining operator mapping $\mathcal{F}_{1-s,t}$ to $\mathcal{F}_{1-t,s}$. In bulk, it corresponds to the gauge transformation for higher spin gauge fields. Similarly, the eq. (3.3.60) also implies that the image of $\mathcal{S}^+_{1-t,s}$ is annihilated by $(p \cdot \mathcal{D}_z)^{s-t}$ or equivalently in position space $(\partial_x \cdot \mathcal{D}_z)^{s-t}$. Again, it is straightforward to check that $\boxed{\mathcal{D}_{s,t} \equiv (\partial_x \cdot \mathcal{D}_z)^{s-t}}$ is an intertwining operator mapping $\mathcal{F}_{d+t-1,s}$ to $\mathcal{F}_{d+s-1,t}$. Altogether, the diagram 3.3.1 shows the six intertwining maps that relate the four types of exceptional representations

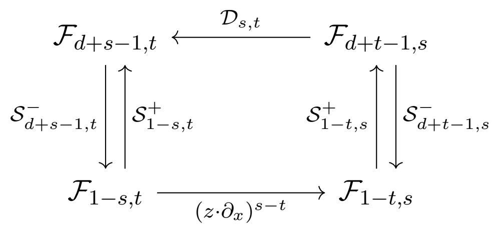

Figure 3.3.1: A sequence of intertwining maps for the exceptional representations

Apart from commuting with group actions, these intertwining maps are important for the following reasons

- Each directed sequence of homomorphisms in the diagram 3.3.1 is *exact*. For example, we have just shown the two sequences $\mathcal{F}_{1-t,s} \xrightarrow{\mathcal{S}^+_{1-t,s}} \mathcal{F}_{d+t-1,s} \xrightarrow{\mathcal{D}_{s,t}} \mathcal{F}_{d+s-1,t}$ and $\mathcal{F}_{1-s,t} \xrightarrow{(z\cdot\partial_x)^{s-t}} \mathcal{F}_{1-t,s} \xrightarrow{\mathcal{S}^+_{1-t,s}} \mathcal{F}_{d+t-1,s}$ are exact. A detailed proof for the rest sequences can be found in [64]).

- The kernel and image of each map are irreducible representations of SO$(1, d+1)$. This claim can be checked by comparing with the full list of irreducible representations given in [13].

The exactness of these intertwining maps implies that there are only three inequivalent irreducible subrepresentations contained in the four exceptional representations

$$\begin{aligned}\ker(z\cdot\partial_x)^{s-t},\quad \mathcal{U}_{s,t}\equiv\ker\mathcal{D}_{s,t}\cong\mathcal{F}_{1-t,s}/\mathrm{Im}\,(z\cdot\partial_x)^{s-t}\\ \mathcal{V}_{s,t}\equiv\mathrm{Im}\,(z\cdot\partial_x)^{s-t}\cong\mathrm{Im}\,\mathcal{S}^{+}_{1-s,t}\cong\mathcal{F}_{1-s,t}/\ker\mathcal{S}^{+}_{1-s,t}\end{aligned}\tag{3.3.62}$$

where $\ker(z\cdot\partial_x)^{s-t}$ carries the finite dimensional representation $\mathbb{Y}_{s-1,t}$ of $\mathsf{SO}(1,d+1)$ which explains the Casimir (3.3.59). For example, when $s=2, t=1$, $\ker(z\cdot\partial_x)^{s-t}$ is spanned by $z_i, x\cdot z, x_iz_j-x_jz_i, 2x_i\,x\cdot z-x^2z_i$. Since $\ker(z\cdot\partial_x)^{s-t}$ is finite dimensional, it cannot be unitary unless it is a trivial representation which corresponds to $s=1,t=0$. For the rest two irreducible representations $\mathcal{U}_{s,t}$ and $\mathcal{V}_{s,t}$, by analyzing their $\mathsf{SO}(d+1)$ contents, i.e.

$$\mathcal{U}_{s,t}|_{\mathsf{SO}(d+1)}=\bigoplus_{n\geq s}\bigoplus_{t+1\leq m\leq s}\mathbb{Y}_{n,m},\quad \mathcal{V}_{s,t}|_{\mathsf{SO}(d+1)}=\bigoplus_{n\geq s}\bigoplus_{0\leq m\leq t}\mathbb{Y}_{n,m}\tag{3.3.63}$$

we manage to identify them as the irreducible representation $\mathsf{D}^{\lceil\frac{d}{2}\rceil-2}_{(\mathbb{Y}_{s,t+1};0)}$ and $\mathsf{D}^{\lceil\frac{d}{2}\rceil-1}_{(\mathbb{Y}_{s};t)}$ on Hirai's list respectively [13].

To define inner product on $\mathcal{U}_{s,t}$ and $\mathcal{V}_{s,t}$, it is more convenient to use their quotient space realization given in the eq. (3.3.62). [10] For example, we can use $\mathcal{S}^{+}_{1-s,t}$ to define a pairing on $\mathcal{F}_{1-s,t}$. Since $\ker\mathcal{S}^{+}_{1-s,t}$ drops out by construction, it naturally induces a pairing on the quotient space $\mathcal{V}_{s,t}$. The explicit form of $\mathcal{S}^{+}_{1-s,t}$ in momentum space is given by the eq. (3.3.39)

$$S^{+}_{1-s,t}(p;z,w)=\left(\frac{p}{2}\right)^{d+2s-2}\sum_{\ell=0}^{t}(-)^{t-\ell}\frac{(s+1-t)_{t-\ell}}{(d+s+\ell-2)_{t-\ell}}\Pi^{t\ell}(\hat{p};z,w)\tag{3.3.64}$$

Due to the alternating sign $(-)^{t-\ell}$, the inner product induced by $\mathcal{S}^{+}_{1-s,t}$ is not positive definite unless $t=0$. Therefore, among the irreducible representations $\mathcal{V}_{s,t}$, only $\mathcal{V}_{s,0}$ is unitary. In this case, the inner product becomes more transparent if we write it in spherical coordinate using

[10] From the bulk point of view, the quotient space realization is quite natural. For example, $(z\cdot\partial_x)^{s-t}$ is the counterpart of the bulk gauge transformation. So factoring out $\mathrm{Im}\,(z\cdot\partial_x)^{s-t}$ amounts to getting rid of the pure gauge degrees of freedom.

the harmonic expansion (3.3.49)

$$\begin{aligned}(\psi_1, \psi_2) &\equiv \int d^d x_1\, d^d x_2\, \psi_1^*(x_1) S^+_{1-s,0}(x_{12}) \psi_2(x_2) \\ &= 2^{1-s} \sum_{\ell \geq s} \sum_{\mathbf{m}} \frac{\Gamma(d+s+\ell-1)}{\Gamma(\ell+1-s)} \left(\int_{S^d} d^d\Omega_1 \hat{\psi}_1(\Omega_1) Y_{\ell \mathbf{m}}(\Omega_1)^* \right)^* \left(\int_{S^d} d^d\Omega_2 \hat{\psi}_2(\Omega_2) Y_{\ell \mathbf{m}}(\Omega_2)^* \right)\end{aligned} \tag{3.3.65}$$

where $\hat{\psi}_i(\Omega) = \left(\frac{1+x^2}{2}\right)^{1-s} \psi_i(x)$. Since the sum starts from $\ell = s$, the spherical harmonics on S^d with $\ell \leq s-1$ get projected out. These spherical harmonics carry the $\mathbb{Y}_{s-1}$ representation of SO$(1, d+1)$ and span the kernel of $\mathcal{S}^+_{1-s,0}$.

Similarly for $\mathcal{U}_{s,t}$, we can define a paring on $\mathcal{F}_{1-t,s}$ by $\mathcal{S}^+_{1-t,s}$ and it naturally induces a paring on $\mathcal{U}_{s,t}$. Since

$$S^+_{1-t,s}(p; z, w) = \left(\frac{p}{2}\right)^{d+2t-2} \sum_{\ell=t+1}^{s} \frac{(\ell - t)_{s-\ell}}{(d+t+\ell-2)_{s-\ell}} \Pi^{s\ell}(\hat{p}; z, w) \tag{3.3.66}$$

is manifestly positive definite with its kernel factored out, the paring on $\mathcal{U}_{s,t}$ defined in this way is positive granting the normalizability. The normalizability is not obvious in this case. For example, when $s = 2, t = 1$, the asymptotic behavior of any wavefunction $\psi(x, z)$ in $\mathcal{F}_{1-t,s}$ is

$$\psi(x, z) \overset{x\to\infty}{\approx} \sum_{k=0}^{\infty} C_k\left(\tilde{x}^i, R(x)^i{}_j z^j\right) \tag{3.3.67}$$

where $C_k(u, v)$ is a homogeneous polynomial of degree k in u^i and degree 2 in v^i. The $k = 0$ term is potentially problematic because its Fourier transform has a p^{-d} behavior around $p = 0$ by a simple power counting and the higher k terms are normalizable. Fortunately any second order tensor in $v^i \equiv R(x)^i{}_j z^j$ is pure gauge, i.e.

$$v^i v^j = z \cdot \partial_x \left(-\frac{x^i}{x^2} \left(2\, x \cdot z\, x^j - x^2 z^j\right) \right) \tag{3.3.68}$$

A rigorous proof about the normalizability for all $\mathcal{U}_{s,t}$ can be found in [64]. Altogether, $\mathcal{U}_{s,t}$ is a unitary irreducible representation for any $s = 1, 2, 3, \cdots$ and $t = 0, 1, \cdots, s-1$. Both $\mathcal{U}_{s,t}$ and $\mathcal{V}_{s,0}$ belong to the (unitary) exceptional series.

Remark 6. *: When $d = 3$, $\mathcal{U}_{s,t}$ is still reducible with respect to SO$(1,4)$. It can be decomposed into two irreducible components $\mathcal{U}^{\pm}_{s,t}$ with the following SO$(d+1)$ contents respectively*

$$\mathcal{U}^{\pm}_{s,t}\Big|_{SO(4)} = \bigoplus_{n\geq s}\bigoplus_{t+1\leq m\leq s} \mathbb{Y}_{n,\pm m} \tag{3.3.69}$$

$\mathcal{U}^{\pm}_{s,t}$ are related by spatial reflection because the SO(4) generator $L_{34} = -\frac{1}{2}(P_3+K_3)$ is mapped to $-L_3$ under spatial reflection while L_{12} stays invariant which means that the SO(4) highest weight vector (n,m) is mapped to $(n,-m)$. Therefore $\mathcal{U}_{s,t}$ is still irreducible with respect to the bigger group O$(1,4)$. In addition, $\mathcal{U}^{\pm}_{s,t}$ belong to the discrete series [64]. For higher d, the exceptional series is completely different from the discrete series [70].

3.3.6 Summary and bulk QFT correspondence

We have identified all unitary irreducible representations contained in $\mathcal{F}_{\Delta,s}$ for the conformal group SO$(1,d+1)$ with $d \geq 3$. In this section, we show a list of these representations and briefly comment on their bulk QFT realizations. For a spin-s field of mass m, its scaling dimension Δ satisfies the following equation (see e.g. [71])

$$\begin{aligned} s = 0: &\quad m^2 = \Delta(d-\Delta) \\ s \geq 1: &\quad m^2 = (d+s-2-\Delta)(\Delta+s-2) \end{aligned} \tag{3.3.70}$$

Then $m = 0$ for the photon, the graviton and their higher-spin generalizations, and for $s = 1$, m is the familiar spin-1 Proca mass.

- **Trivial representation**.
- **Scalar principal series**: $\mathcal{F}_{\frac{d}{2}+i\mu}$ ($\mu \in \mathbb{R}$) equipped with the $L^2(\mathbb{R})$ inner product. It describes a massive scalar field in dS$_{d+1}$ with mass $m \geq \frac{d}{2}$.

- **Scalar complementary series**: $\mathcal{F}_\Delta$ $(0<\Delta<d)$ equipped with the inner product

$$(\psi_1,\psi_2)=\int d^dx_1\,d^dx_2\,\psi_1(x_1)^*S^+_\Delta(x_{12})\,\psi_2(x_2) \tag{3.3.71}$$

It describes a massive scalar field in dS$_{d+1}$ with mass $0<m<\frac{d}{2}$.

- **Spinning principal series**: $\mathcal{F}_{\frac{d}{2}+i\mu,s}$ ($\mu\in\mathbb{R}$ and $s\in\mathbb{Z}_+$) equipped with the L^2 inner product. It describes a massive spin-s in dS$_{d+1}$ with mass $m\geq s+\frac{d}{2}-2$.

- **Spinning complementary series**: $\mathcal{F}_{\Delta,s}$ ($1<\Delta<d-1$ and $s\in\mathbb{Z}_+$) equipped with the inner product

$$(\psi_1,\psi_2)=\int d^dx_1\,d^dx_2\,\psi_1(x_1)^*_{i_1\cdots i_s}S^+_{\Delta,s}(x_{12})_{i_1\cdots i_s,j_1\cdots j_s}\,\psi_2(x_2)_{j_1\cdots j_s} \tag{3.3.72}$$

It describes a massive spin-s field in dS$_{d+1}$ with mass $\sqrt{(s-1)(s+d-3)}<m<s+\frac{d}{2}-2$. The lower bound $\sqrt{(s-1)(s+d-3)}$ is the so-called Higuchi bound [42] which is the equivalent to the requirement of no negative norm states in bulk canonical quantization (a concise summary is given in [72] and [73]).

- **Exceptional series I**: $\mathcal{V}_{s,0}=\mathcal{F}_{1-s}/\ker\mathcal{S}^+_{1-s}$ ($s\in\mathbb{Z}_+$) equipped with the inner product

$$(\psi_1,\psi_2)=\int d^dx_1\,d^dx_2\,\psi_1(x_1)^*S^+_{1-s}(x_{12})\,\psi_2(x_2) \tag{3.3.73}$$

We believe that it should correspond to the shift symmetric scalars with mass square $m^2=(1-s)(s+d-1)$ [74]. In particular, when $s=1$, it is a massless scalar.

- **Exceptional series II**: $\mathcal{U}_{s,t}=\mathcal{F}_{1-t,s}/\ker\mathcal{S}^+_{1-t,s}$ ($s\in\mathbb{Z}_+$ and $t=0,1,\cdots,s-1$) equipped with the inner product

$$(\psi_1,\psi_2)=\int d^dx_1\,d^dx_2\,\psi_1(x_1)^*_{i_1\cdots i_s}S^+_{1-t,s}(x_{12})_{i_1\cdots i_s,j_1\cdots j_s}\,\psi_2(x_2)_{j_1\cdots j_s} \tag{3.3.74}$$

It describes a partially massless spin-s gauge field with mass square $m^2_{s,t}=(s-1-t)(d+s+t-3)$. The parameter t is called *depth*, which is also the spin of the corresponding ghost field. In particular, when $t=s-1$, it describes to a massless spin-s gauge field, i.e. photon, graviton etc and we will call $\mathcal{U}_{s,s-1}$ the massless spin-s representation of SO$(1, d+1)$.

- There also exists the so-called discrete series when d is odd [13, 70], which is not covered in this review. Fields in discrete series are of mixed symmetry.

For principal and complementary series, $\mathcal{F}_{\Delta,s}$ and $\mathcal{F}_{\bar{\Delta},s}$ are isomorphic due to the shadow transformations. So in principal series, it suffices to consider $\mu \geq 0$ and in complementary series, it suffices to consider $\Delta > \frac{d}{2}$.

3.4 Harish-Chandra characters

3.4.1 General theory

With the unitary irreducible representations constructed, the next step is to compute their group characters which collect the information about the representation in a simple function. We are all familiar with the idea of characters of finite dimensional representations. For example, given a finite dimensional representation ρ of a group G, the corresponding group character associated to certain element $g \in G$ is defined as a trace of $\rho(g)$ over the representation space V_ρ, i.e. $\Theta_\rho(g) \equiv \mathrm{Tr}_{V_\rho}\, \rho(g)$. This quantity is clearly well-defined and conjugation invariant since it only involves a finite sum. However, such a trace does not necessarily make sense for an infinite dimensional representation like $\mathcal{F}_{\Delta,s}$. To tell when a "trace" can be defined in the infinite dimensional case, we will need some deep notions and theorems in representation theory [66]:

Definition 1. *Let G be a connected reductive Lie group and let K be a maximal compact subgroup. A representation π of G on a Hilbert space V is called* ***admissible*** *if $\pi|_K$ is unitary and if each unitary irreducible representation τ of K occurs with only finite multiplicity in $\pi|_K$.*

In particular, SO$(1, d+1)$ a is connected reductive Lie group and all $\mathcal{F}_{\Delta,s}$ together all its unitary irreducible subrepresentations are admissible.

Definition 2. *We say an admissible representation π of a linear connected reductive group G has a* ***Harish-Chandra character*** *(or* ***global character****) Θ_π if $\pi(\varphi)$ is of trace class for any compact supported function φ on G and if $\varphi \to \mathrm{Tr}\,\pi(\varphi) \equiv \Theta_\pi(\varphi)$ is a distribution. In this case, the character Θ_π is clearly conjugation invariant.*

Then the following theorem tells us when an admissible representation π has a Harish-Chandra character

Theorem. *Every admissible representation π of a linear connected reductive group G whose decomposition $\pi|_K = \bigoplus_\tau n_\tau \tau$ satisfies $n_\tau \le C \dim\tau$ has a Harish-Chandra character.*

Since the SO$(d+1)$ contents of in $\mathcal{F}_{\Delta,s}$ have multiplicity 1, $\mathcal{F}_{\Delta,s}$ and its irreducible components have a Harish-Chandra character. We shall also use the heuristic notations

$$\Theta_{\mathcal{F}_{\Delta,s}}(g) \equiv \mathrm{Tr}_{\mathcal{F}_{\Delta,s}}\, g, \quad \Theta_{\mathcal{F}_{\Delta,s}}(q) \equiv \Theta_{\mathcal{F}_{\Delta,s}}(q^D), \quad q>0 \tag{3.4.1}$$

which only exist in the distribution sense. In most cases, we shall simply call $\Theta_{\mathcal{F}_{\Delta,s}}(q)$ the Harish-Chandra character of $\mathcal{F}_{\Delta,s}$.

3.4.2 Compute $\Theta_{\mathcal{F}_{\Delta,s}}(q)$

In this section, we show how to compute the character $\Theta_{\mathcal{F}_{\Delta,s}}(q)$. Let's start from the $s=0$ case. By definition, one would find an orthonormal basis of $\mathcal{F}_\Delta$ and compute the matrix element of D with respect to this basis (which is actually done in the appendix A.3 with the basis being spherical harmonics). Then exponentiate the infinite dimensional matrix D and compute the trace. However, this method can be infeasible due to technical difficulty. Instead, we will use the ket basis $|x\rangle$. The action of q^D on $|x\rangle$ is

$$q^D|x\rangle = q^D\,|qx\rangle = \int d^dy\, q^{\bar{\Delta}}\delta^d(qx-y)|y\rangle \tag{3.4.2}$$

from which we can directly read off the matrix elements of q^D with respect to the basis $|x\rangle$. The integral of the diagonal entries of q^D is localized on the fixed points of q^D, i.e. 0 and ∞. But the x^i coordinate system only contains the $x=0$ point and hence we would miss the contribution from ∞ in such a computation. We can bypass this problem by working with spherical basis, i.e. $|\Omega\rangle = \left(\frac{1+x^2}{2}\right)^{\bar{\Delta}}|x\rangle$ as in [2]. In this basis, the fixed points of q^D become southern and northern poles of S^d. Alternatively, we compute $\mathrm{Tr}(e^{-b\cdot K}q^De^{b\cdot K})$. This conjugation maps 0 and ∞ to two finite points in the x-plane while keeping the trace invariant. The action of $e^{-b\cdot K}q^De^{b\cdot K}$

on $|x\rangle$ is given by

$$e^{-b\cdot K}q^{D}e^{b\cdot K}|x\rangle=\Omega(q,x,b)^{\bar{\Delta}}|\tilde{x}(q,x,b)\rangle,\quad \tilde{x}^{i}=q\frac{x^{i}+(q-1)\,x^{2}b^{i}}{1+2(q-1)\,x\cdot b+(q-1)^{2}\,b^{2}x^{2}} \tag{3.4.3}$$

where

$$\Omega(q,x,b)=\frac{q}{1+2(q-1)\,x\cdot b+(q-1)^{2}\,b^{2}x^{2}} \tag{3.4.4}$$

Therefore the character $\Theta_{\mathcal{F}_\Delta}(q)$ should be

$$\Theta_{\mathcal{F}_\Delta}(q)=\int d^{d}x\,\Omega(q,x,b)^{\bar{\Delta}}\delta^{d}(\tilde{x}-x) \tag{3.4.5}$$

This integral is localized to the fixed points of $e^{-b\cdot K}q^{D}e^{b\cdot K}$, which are $x^i=0$ and $x^i=\frac{b^i}{b^2}$. At these points, the scaling factor $\Omega(q,x,b)$ becomes

$$\Omega(q,0,b)=q,\quad \Omega(q,b^{i}/b^{2},b)=q^{-1} \tag{3.4.6}$$

and the Jacobian associated to the map $x^i\to\tilde{x}^i-x^i$ becomes $|q-1|^d$ and $|1-q^{-1}|^d$ respectively. Altogether, the integral in the eq. (3.4.5) yields

$$\Theta_{\mathcal{F}_\Delta}(q)=\frac{q^{\bar{\Delta}}}{|q-1|^{d}}+\frac{q^{-\bar{\Delta}}}{|1-q^{-1}|^{d}}=\frac{q^{\Delta}+q^{\bar{\Delta}}}{|1-q|^{d}} \tag{3.4.7}$$

where the b dependence drops out explicitly as expected from the conjugation invariance.

Remark 7. *The character $\Theta_{\mathcal{F}_\Delta}(q)$ given by the eq. (3.4.7) is symmetric under $q\to q^{-1}$. This property is also a result of the conjugation invariance of Θ since $D=L_{0,d+1}$ is mapped to $-D$ by the conjugation of $e^{i\pi L_{d+1,i}}$.*

Remark 8. *One might be tempted to compute the character $\Theta_{\mathcal{F}_\Delta}(q)$ by diagonalizing D. Its eigenstates are $|\nu,\ell\mathbf{m}\rangle$ where $D=i\nu\in i\mathbb{R}$, $\ell\mathbf{m}$ labels SO(d) angular momentum quantum numbers. However this produces a nonsensical result*

$$\Theta_{\mathcal{F}_\Delta}(q=e^{t})\overset{\text{naive}}{=}\sum_{\mathbf{m}}\int_{\mathbb{R}}d\nu\,e^{i\nu t}\,\delta_{\ell,\ell}\delta_{\mathbf{m},\mathbf{m}}=2\pi\delta^{d}(0)\delta(t) \tag{3.4.8}$$

not even remotely resembling the correct $\Theta_{\mathcal{F}_\Delta}(q)$. Our correct computation actually illuminates why this naive computation fails. Notice that the eigenbasis $|\nu, \ell\mathbf{m}\rangle$ corresponds to wavefunction $\psi_{\nu\ell\mathbf{m}}(x) = x^{-(i\nu+\Delta)}Y_{\ell\mathbf{m}}(\omega)$, which under Weyl transformation is mapped to

$$\hat{\psi}_{\nu\ell\mathbf{m}}(\Omega) = \left(\tan\frac{\Theta}{2}\right)^{\nu}(\sin\Theta)^{-\Delta}\,Y_{\ell\mathbf{m}}(\omega) \tag{3.4.9}$$

It is now clear why the naive computation (3.4.8) of $\Theta_{\mathcal{F}_\Delta}(q)$ in the basis $|\nu, \ell\mathbf{m}\rangle$ fails to produce the correct result: the wave functions $\hat{\psi}_{\nu\ell\mathbf{m}}(\Omega)$ are singular precisely at the fixed points of D. On the other hand, a proper basis of wavefunctions that are well-defined at the the fixed points of D are the SO$(d+1)$ spherical harmonics used in the appendix A.3.

The generalization of our computation to the spinning case is almost straightforward. Let's write the ket basis of $\mathcal{F}_{\Delta,\mathbf{s}}$ as $|x\rangle_\alpha$ where the index α carries the spin-s representation of SO(d). The action of $e^{-b\cdot K}e^{tD}e^{b\cdot K}$ on this basis can be schematically expressed as

$$e^{-b\cdot K}q^D e^{b\cdot K}|x\rangle_\alpha = \mathcal{O}_{\alpha\beta}(q,x,b)\Omega(q,x,b)^{\bar{\Delta}}|\tilde{x}(q,x,b)\rangle_\beta \tag{3.4.10}$$

where $\mathcal{O}_{\alpha\beta}(q,x,b)$ is an SO(d) rotation matrix in the spin-s representation. In general, $\mathcal{O}_{\alpha\beta}(q,x,b)$ takes a very sophisticated form but while evaluated at the two fixed points, i.e. $x^i = 0$ and $x^i = \frac{b^i}{b^2}$, it becomes an identity matrix. Altogether, we have

$$\begin{aligned}\Theta_{\mathcal{F}_{\Delta,\mathbf{s}}}(q) &= \sum_\alpha \int d^d x\, \mathcal{O}_{\alpha\alpha}(q,x,b)\Omega(q,x,b)^{\bar{\Delta}}\delta^d(\tilde{x}-x)\\ &= \Theta_{\mathcal{F}_\Delta}(q)\times\left(\sum_\alpha 1\right) = D^d_{\mathbf{s}}\,\frac{q^\Delta + q^{\bar{\Delta}}}{|1-q|^d}\end{aligned} \tag{3.4.11}$$

In particular, when $\mathbf{s} = (s,0,0,\cdots,0)$, the Harish-Chandra character for both principal and complementary series is

$$\Theta_{\mathcal{F}_{\Delta,s}}(q) = D^d_s\,\frac{q^\Delta + q^{\bar{\Delta}}}{|1-q|^d} \tag{3.4.12}$$

3.4.3 Harish-Chandra character of exceptional series

Now we are able to compute the Harish-Chandra characters of exceptional series $\mathcal{V}_{s,0}$ and $\mathcal{U}_{s,t}$. For $\mathcal{V}_{s,0}$, recall that $\mathcal{V}_{s,t} \equiv \mathrm{Im}\,(z\cdot\partial_x)^{s-t} \cong \mathcal{F}_{1-s,t}/\ker(z\cdot\partial_x)^{s-t}$, which yields

$$\Theta_{\mathcal{V}_{s,0}}(q) = \Theta_{\mathcal{F}_{1-s}}(q) - \Theta_{\ker(z\cdot\partial_x)^s}(q) \tag{3.4.13}$$

Since $\ker(z\cdot\partial_x)^s$ carries the $\mathbb{Y}_{s-1}$ of $\mathsf{SO}(1,d+1)$, $\Theta_{\ker(z\cdot\partial_x)^s}(q)$ is nothing but the usual $\mathsf{SO}(d+2)$ character corresponding to the highest weight representation $\mathbb{Y}_{s-1}$, denoted by $\Theta^{\mathsf{SO}(d+2)}_{\mathbb{Y}_{s-1}}(q)$. Thus we obtain (**assuming** $\boxed{0<q<1}$ to get rid of the absolute value in $|1-q|^d$)

$$\Theta_{\mathcal{V}_{s,0}}(q) = \frac{q^{1-s}+q^{d+s-1}}{(1-q)^d} - \Theta^{\mathsf{SO}(d+2)}_{\mathbb{Y}_{s-1}}(q) \tag{3.4.14}$$

Similarly for $\mathcal{U}_{s,t}$, using the isomorphism $\mathcal{U}_{s,t} \cong \mathcal{F}_{1-t,s}/\mathcal{V}_{s,t}$, we obtain

$$\Theta_{\mathcal{U}_{s,t}}(q) = D^d_s\frac{q^{1-t}+q^{d+t-1}}{(1-q)^d} - D^d_t\frac{q^{1-s}+q^{d+s-1}}{(1-q)^d} + \Theta^{\mathsf{SO}(d+2)}_{\mathbb{Y}_{s-1,t}}(q) \tag{3.4.15}$$

In general, one can write out the characters like $\Theta^{\mathsf{SO}(d+2)}_{\mathbb{Y}_{s-1,t}}(q)$ explicitly by using Weyl character formula. However, in order to compare with the paper [2], we have derived a slightly different expression c.f. eq. (2.3.3) for $\Theta^{\mathsf{SO}(d+2)}_{\mathbb{Y}_{s-1,t}}(q)$ in the section 2.3. Plugging this new expression into eq. (3.4.14) and eq. (3.4.15) yields

$$\begin{aligned}
\Theta_{\mathcal{V}_{s,0}}(q) &= \left[\Theta_{\mathcal{F}_{1-s}}(q)\right]_+ = \left[\frac{q^{1-s}+q^{d+s-1}}{(1-q)^d}\right]_+ \\
\Theta_{\mathcal{U}_{s,t}}(q) &= \Big[\Theta_{\mathcal{F}_{1-t,s}}(q) - \Theta_{\mathcal{F}_{1-s,t}}(q)\Big]_+ = \left[D^d_s\frac{q^{1-t}+q^{d+t-1}}{(1-q)^d} - D^d_t\frac{q^{1-s}+q^{d+s-1}}{(1-q)^d}\right]_+
\end{aligned} \tag{3.4.16}$$

where the “flipping” operator $[\ ]_+$ acts on an arbitrary Laurent series $\sum_k c_k q^k$ as

$$\left[\sum_k c_k q^k\right]_+ = \sum_{k<0}(-c_k)q^{-k} + \sum_{k>0} c_k q^k \tag{3.4.17}$$

We also derived an interesting formula for the "flipping" operator acting on $\frac{q^\Delta}{(1-q)^d}$ by using Mathematica

$$\left[\frac{q^\Delta}{(1-q)^d}\right]_+ = \frac{\mathcal{P}_\Delta(q)}{(1-q)^d}$$
$$\mathcal{P}_\Delta(q) = (-)^{d+1} q^{d-\Delta} + \sum_{m=0}^{\lfloor \frac{d}{2} \rfloor -1} (-)^m \, D^d_{1-\Delta,1^m} \left(q^{1+m} + (-)^d q^{d-1-m}\right) \tag{3.4.18}$$

This equation can be checked by writing out $D^d_{1-\Delta,1^m}$ explicitly using the general Weyl dimension formula (2.2.1) and (2.2.2). But the computation is somewhat tedious and not especially illuminating, so we omit it here. Plugging eq. (3.4.18) into the characters in (3.4.16), we obtain

$$\Theta_{\mathcal{V}_{s,0}}(q) = (1-(-)^d)\frac{q^{d+s-1}}{(1-q)^d} + \sum_{m=0}^{\lfloor \frac{d}{2} \rfloor -1} (-)^m \, D^d_{s,1^m} \left(q^{1+m} + (-)^d q^{d-1-m}\right) \tag{3.4.19}$$

and

$$\Theta_{\mathcal{U}_{s,t}}(q) = (1-(-)^d)\frac{D^d_s q^{d+t-1} - D^d_t q^{d+s-1}}{(1-q)^d} + \sum_{m=0}^{\lfloor \frac{d}{2} \rfloor -2} (-)^m D^d_{s,t+1,1^m} \left(q^{2+m} + (-)^d q^{d-2-m}\right) \tag{3.4.20}$$

where we have used $D^d_t D^d_{s,1^m} - D^d_s D^d_{t,1^m} = D^d_{s,t+1,1^{m-1}}$, which again can be checked by using Weyl dimension formula (2.2.1) and (2.2.2).

Remark 9. *(Literature disagreements). The characters (3.4.19) and (3.4.20) agree with the characters listed in the original work [75], computed by undisclosed methods. They do not agree with those listed in the more recent work [70], computed by Bernstein-Gelfand-Gelfand resolutions. Indeed [70] emphasized they disagreed with [75] for even d. More precisely, in their eq. (2.14) applied to $p = 2, \mathbb{Y}_p = (s, s, 0 \cdots, 0), \mathbf{x} = 0$ (also to $p = 1, \mathbb{Y}_p = (s, 0, \cdots, 0), \mathbf{x} = 0$), they find a factor $2 = (1+(-)^d)$ instead of the factor $(1-(-)^d) = 0$ in (3.4.20). It is stated in [70] that on the other hand their results do agree with [75] for odd d. Actually we find this is not quite true either, as in that case eq. (2.13) in [70] applied to $p = 2, \mathbb{Y}_p = (s, s, 0 \cdots, 0), \mathbf{x} = 0$ (also to $p = 1, \mathbb{Y}_p = (s, 0, \cdots, 0), \mathbf{x} = 0$) has a factor 1 instead of the factor $(1-(-)^d) = 2$. Our Euclidean path integral result of chapter 5 strongly suggests the original results in [75] and*

(3.4.19), (3.4.20) are the correct versions. Further support is provided in the chapter 4 by direct construction of higher-spin quasinormal modes.

Remark 10. *Some lower d examples are*

$$\begin{aligned} d=3: \ \Theta_{\mathcal{U}_{s,t}}(q) &= 2\,\frac{(2s+1)q^{t+2}-(2t+1)q^{s+2}}{(1-q)^3} \\ d=4: \ \Theta_{\mathcal{U}_{s,t}}(q) &= 2(s-t)(s+t+2)\frac{q^2}{(1-q)^4} \end{aligned} \tag{3.4.21}$$

where the overall factor 2 in the first line arises from the reducibility of $\mathcal{U}_{s,t}$ when $d=3$.

Remark 11. *For the discrete series of SO$(1,2)$, we use the quotient space realization $\mathcal{D}_N^+ \bigoplus \mathcal{D}_N^- = \mathcal{F}_{1-N}/P_N$ where P_N carries the spin-$(N-1)$ representation of SO$(1,2)$*

$$\Theta_{\mathcal{D}_N^+ \bigoplus \mathcal{D}_N^-}(q) = \Theta_{\mathcal{F}_{1-N}}(q) - \sum_{k=1-N}^{N-1} q^k = \frac{2\,q^N}{1-q} \tag{3.4.22}$$

The character for each summand $\mathcal{D}_N^\pm$ is $\frac{q^N}{1-q}$. [66]

Chapter 4: Higher spin de Sitter quasinormal modes

In the previous chapter, we have derived the Harish-Chandra characters of the dS isometry group, c.f. eq. 3.4.12, 3.4.19 and 3.4.20 for different UIRs. These expressions admit Taylor expansions for small q with all Taylor coefficients taking values in positive integers. One might immediately wonder about the physical meaning of the coefficients in the sense that if they count any physical objects. In this chapter, we will show that these coefficients count quasinormal modes in de Sitter spacetime by presenting an algebraic construction of de Sitter quasinormal modes.

4.1 Introduction

In the framework of general relativity (GR), quasinormal modes can be defined as the damped modes of some perturbation in a classical gravitational background with a horizon, like black holes and dS. Astrophysically, they are important because the least damped gravitational quasinormal mode of a Schwarzschild black hole is detected and measured by LIGO through gravitational waves emitted during the so-called "ringdown" phase [76]. The measured value of gravitational quasinormal frequency can be used to test GR which predicts that the spin and mass of a black hole completely fix gravitational quasinormal frequencies.

One standard method of finding quasinormal modes in a generic background with a horizon is solving the equation of motion for a perturbation, in most cases numerically, and then imposing in-falling boundary condition at the horizon [77–80]. In the static patch of de Sitter spacetime, c.f. (4.3.2), by separation of variables the radial parts of quasinormal modes are found to satisfy hypergeometric functions and hence can be solved analytically (see [81–83] for a summary and derivation of the analytical results associated to scalars, Dirac spinors, Maxwell fields and linearized gravity in any dimensions). The underlying reason for the existence of these analytical solutions is the large dS isometry group which organizes quasinormal modes according to certain representation structure. Such a representation structure was first discovered in [59, 60] for a massive scalar field with mass $m^2 = \Delta(3-\Delta), 0 < \Delta < 3$ [1] in dS$_4$. In this case, the quasinormal modes of the scalar field comprise two (non-unitary) lowest-weight representations of the isometry algebra $\mathfrak{so}(1,4)$, which is also the conformal algebra of $\mathbb{R}^3$. More explicitly, the authors built two lowest-weight/primary quasinormal modes of quasinormal frequency $i\omega = \Delta$ and $i\omega = \bar{\Delta} = 3-\Delta$ respectively, as the two leading asymptotic behaviors of vacuum-to-vacuum bulk two-point function when one point pushed to the northern pole on the future sphere. Upon each primary quasinormal mode, an infinite tower of quasinormal modes can be generated as $\mathfrak{so}(1,4)$-descendants and the scaling dimension of every descendant can be identified as ($i\times$ quasinormal frequency). The two towers of quasinormal modes together span the whole scalar quasinormal spectrum. These results were later reformulated by [84] in the ambient space formalism and generalized to massive vector fields and Dirac spinors in the same chapter. We'll call the way of constructing quasinormal modes using the dS isometry group as in [59, 60, 84] the *algebraic* approach.

The separation of variables method in [81, 83] is increasingly cumbersome when applied to higher spin fields due to the rapidly increasing number of tensor structures. So in this chapter we will focus on generalizing the algebraic approach to construct quasinormal modes of higher spin fields, which are formulated in the ambient space (see section 4.2 for a review about the ambient space formalism). In section 4.3, we first review the algebraic construction of quasinormal modes of a scalar field φ of mass m in dS$_{d+1}$ using the ambient space formalism. The quasinormal spectrum found in this way can be packaged into a "*quasinormal character*", cf. (4.5.1)

$$\Theta^{\mathsf{QN}} \equiv \sum_{\omega} d_\omega \, q^{i\omega} \tag{4.1.1}$$

where the sum runs over all quasinormal frequencies and d_ω is the degeneracy of quasinormal modes with frequency ω. We show that the quasinormal character of φ is given by

$$\Theta^{\mathsf{QN}}_{\varphi}(q) = \frac{q^{\Delta}+q^{\bar{\Delta}}}{(1-q)^d}, \quad \bar{\Delta} = d-\Delta \tag{4.1.2}$$

[1]The representation carried by such a scalar field is in the (scalar) complementary series.

where $\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} - m^2}$ is the scaling dimension of φ. According to [64, 70, 75], $\Theta_{\varphi}^{\mathrm{QN}}(q)$ is the Harish-Chandra SO$(1, d+1)$ character corresponding to the unitary (scalar) principal series when $m > \frac{d}{2}$ and the unitary (scalar) complementary series when $0 < m \leq \frac{d}{2}$. The $(\Delta \leftrightarrow \bar{\Delta})$ symmetry in (4.1.2) manifests the *two* towers of quasinormal modes. The generalization of the algebraic construction to *massive* higher spin fields is straightforward. The only difference is that the two primary quasinormal modes have spin degeneracy. In this case, the quasinormal character is given by eq. (4.5.4). However, the generalization to the *massless* higher spin case is quite nontrivial because gauge symmetry significantly reshapes quasinormal spectrums compared to the massive case. For example, the naive $(\Delta \leftrightarrow \bar{\Delta})$ symmetry would lead to growing modes instead of damped modes because $\bar{\Delta} = 2 - s < 0$ when $s \geq 3$. Moreover the symmetry disagrees with the result of [83] for $s = 1, 2$. On the representation side, the underlying reason for the difficulty in generalization is that the massless higher spin fields are in the exceptional series for $d \geq 4$ and in the discrete series for $d = 3$ while generic massive fields are in the principal series or the complementary series [2]. In section 4.3.3, we'll discuss the subtleties associated to gauge symmetry in more detail and explain how to take into account gauge symmetry properly while constructing *physical* quasinormal modes. In particular, the two-tower structure still holds and the two primary quasinormal modes are given by eq.(4.3.37) and eq.(4.3.39). In addition, in section 4.4, we argue that the two primary quasinormal modes are produced by insertions of boundary gauge-invariant conserved currents (of scaling dimension $d + s - 2$) and boundary higher-spin Weyl tensors (of scaling dimension 2) at the southern pole of the past sphere (see fig. 4.1.1). Other quasinormal modes are sourced by the descendants of these two operators.

Based on the algebraic construction described in section 4.3.3, we extract the physical quasinormal spectrum of massless higher spin fields in section 4.5 and compute the corresponding quasinormal character. Here, we list some examples at $d = 3, 4, 5$ (see eq. (4.5.12) for a general

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[2]We exclude the partially massless fields while talking about massive fields.

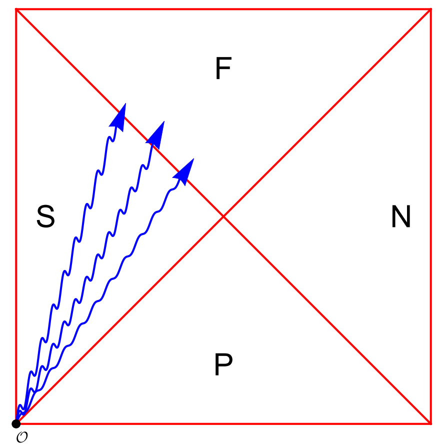


Figure 4.1.1: The Penrose diagram of de Sitter spacetime. Quasinormal modes (in the southern static patch "S") of massless higher spin fields are sourced by certain gauge-invariant operators $\mathcal{O}$ inserted at the southern pole of the past sphere.

expression working in any d):

$$\begin{aligned} d=3: &\quad \Theta_s^{\mathsf{QN}}(q) = 2\,\frac{(2s+1)\,q^{1+s}-(2s-1)\,q^{2+s}}{(1-q)^3} \\ d=4: &\quad \Theta_s^{\mathsf{QN}}(q) = 2\,\frac{(2s+1)\,q^{2}}{(1-q)^4} \\ d=5: &\quad \Theta_s^{\mathsf{QN}}(q) = \frac{1}{3}(s+1)(2s+1)(2s+3)\frac{q^2-q^3}{(1-q)^5}+\frac{s+1}{3}\frac{(s+2)(2s+3)q^{s+3}-s(2s+1)q^{s+4}}{(1-q)^5} \end{aligned} \tag{4.1.3}$$

These quasinormal characters coincide with the original computation of Harish-Chandra SO$(1,d+1)$ characters in [13, 75] and the characters appearing in the one-loop path integral of massless higher spin fields on S^{d+1}, the Wick rotation of dS$_{d+1}$ [2]. Therefore, the quasinormal characters, which are defined in a pure physics setup, connect *nonunitary* lowest-weight representations of $\mathfrak{so}(1,d+1)$ and the *unitary* representations of SO$(1,d+1)$. In addition, in the appendix B.2, we recover the quasinormal spectrum of Maxwell theory and linearized gravity in [83] by using $\Theta_1^{\mathsf{QN}}(q)$ and $\Theta_2^{\mathsf{QN}}(q)$.

When $d\geq 4$, the expansion of quasinormal character $\Theta_s^{\mathsf{QN}}(q)$ always starts from a q^2 term because it corresponds to a boundary higher-spin Weyl tensor insertion. When $d=3$, $\Theta_s^{\mathsf{QN}}(q)$ starts from q^{1+s} since the spin-s Weyl tensor, which vanishes identically on the 3-dimensional boundary, gets replaced by the co-called Cotton tensor [85–88], which involves $(2s+1)$ derivatives on the boundary gauge field of scaling dimension $2-s$.

4.2 Ambient space formalism for fields in de Sitter

In this section, we review ambient space formalism for higher spin fields in $(d+1)$-dimensional de Sitter spacetime, which is realized as a hypersurface

$$\eta_{AB}X^A X^B = 1, \quad \eta_{AB} = (-,+,\cdots,+) \tag{4.2.1}$$

in the ambient space $\mathbb{R}^{1,d+1}$, where $A = 0, 1, \cdots, d+1$. Local intrinsic coordinates y^μ are defined through an embedding $X^A(y)$ that satisfies (4.2.1) and such an embedding induces a local metric $ds^2 = g_{\mu\nu}dy^\mu dy^\nu$ on dS$_{d+1}$. To gain some intuitions about the ambient space description in field theory, let's consider a free scalar field $\varphi(y)$, defined in the local coordinates y^μ, of mass $m^2 = \Delta(d-\Delta)$ and satisfying equation of motion: $\nabla^2\varphi = m^2\varphi$ with ∇^2 being the scalar Laplacian. This scalar field φ admits a unique extension $\phi(X)$ to the ambient space such that

$$\phi(\lambda X) = \lambda^{-\kappa}\phi(X), \quad \phi(X(y)) = \varphi(y) \tag{4.2.2}$$

where κ is an arbitrary constant. Define radial coordinate $R = \sqrt{X^2}$ and hence any point in the ambient space can be parameterized by (R, y^μ) via a dS foliation. In terms of the radial coordinate, the extension condition (4.2.2) can be rephrased as $\phi(X) = \phi(R, y^\mu) = R^{-\kappa}\varphi(y)$ and the ambient space Laplacian can be expressed as $\partial_X^2 = \frac{1}{R^{d+1}}\partial_R(R^{d+1}\partial_R) + \frac{1}{R^2}\nabla^2$, which together with the equation of motion of φ, yields

$$\partial_X^2\phi = \frac{1}{R^2}(\Delta(d-\Delta) - \kappa(d-\kappa))\phi \tag{4.2.3}$$

In particular, if we choose κ to be Δ or $\bar{\Delta}$, $\phi(X)$ becomes a harmonic function in the ambient space. Altogether, the scalar field $\varphi(y)$ in dS$_{d+1}$ of scaling dimension Δ is equivalent to a homogeneous harmonic function $\phi(X)$ in the ambient space $\mathbb{R}^{1,d+1}$, i.e.

$$(X\cdot\partial_X + \kappa)\phi(X) = \phi(X), \quad \partial_X^2\phi(X) = 0 \tag{4.2.4}$$

where $\kappa = \Delta$ or $\kappa = \bar{\Delta}$. The obvious technical advantage of this ambient space description is replacing the cumbersome covariant derivative ∇_μ by the simple ordinary derivative ∂_{X^A}. Such a simplification is more crucial when we deal with higher spin fields. One can find a very good review about the ambient space formalism for spinning fields in AdS in [89, 90]; also see [91, 92] for a more general and systematic discussion using the Howe duality. Here we present an adapted version of the ambient space description in dS.

4.2.1 Higher spin fields in ambient space formalism

The totally symmetric transverse (on-shell) spin-s field $\varphi_{\mu_1...\mu_s}(y)$ of scaling dimension Δ in dS$_{d+1}$ is represented uniquely in ambient space by the symmetric tensor $\phi_{A_1...A_s}(X)$,

$$\varphi_{\mu_1\cdots\mu_s}(y) = \frac{\partial X^{A_1}}{\partial y^{\mu_1}} \cdots \frac{\partial X^{A_s}}{\partial y^{\mu_s}} \phi_{A_1\cdots A_s}(X) \tag{4.2.5}$$

satisfying the following equations:

- Tangentiality to surfaces of constant $R = \sqrt{X^2}$:

$$(X \cdot \partial_U)\, \phi_s(X, U) = 0 \tag{4.2.6}$$

- The homogeneity condition:

$$(X \cdot \partial_X + \kappa)\phi_s(X, U) = 0 \tag{4.2.7}$$

  A convenient choice is $\kappa = \Delta$ or $\kappa = \bar{\Delta}$ because, as we've seen in the scalar case, it yields the simplest equation of motion as follows:

- The Casimir condition, i.e. equation of motion

$$(\partial_X \cdot \partial_X)\, \phi_s(X, U) = 0 \tag{4.2.8}$$

- The transverse condition:

$$(\partial_X \cdot \partial_U)\, \phi_s(X, U) = 0 \tag{4.2.9}$$

- The traceless condition:

$$(\partial_U \cdot \partial_U)\, \phi_s(X,U) = 0 \tag{4.2.10}$$

where we've used the generating function $\phi_s(X,U) \equiv \frac{1}{s!}\phi_{A_1\cdots A_s}(X)U^{A_1}\cdots U^{A_s}$ with U^A being a constant auxiliary vector. The first two conditions ensure that $\phi_s(X,U)$ is the unique uplift of $\varphi_{\mu_1\cdots\mu_s}$ that satisfies (4.2.5) and the last three conditions are equivalent to the Fierz-Pauli system:

$$\text{The Casimir condition :} \quad \nabla^2\varphi_{\mu_1\cdots\mu_s} = (\Delta(d-\Delta)+s)\varphi_{\mu_1\cdots\mu_s} \tag{4.2.11}$$

$$\text{The transverse condition :} \quad \nabla^{\mu_1}\varphi_{\mu_1\cdots\mu_s} = 0 \tag{4.2.12}$$

$$\text{The traceless condition :} \quad g^{\mu_1\mu_2}\varphi_{\mu_1\cdots\mu_s} = 0 \tag{4.2.13}$$

In the remaining part of the chapter, we'll call (4.2.6)-(4.2.7) the *uplift conditions* and (4.2.8)-(4.2.10) the *Fierz-Pauli conditions*.

When $\varphi_{\mu_1\cdots\mu_s}$ is a massless spin-s bulk field, the uplift conditions and Fierz-Pauli conditions have a gauge symmetry, with the gauge transformation takes the following simple form if we choose $\boxed{\kappa = 2 - s = \bar{\Delta}}$ in eq. (4.2.7):

$$\delta_{\xi_{s-1}}\phi_s(X,U) = (U \cdot \partial_X)\, \xi_{s-1}(X,U) \tag{4.2.14}$$

where ξ_{s-1} satisfies

- Tangentiality to surfaces of constant $R = \sqrt{X^2}$:

$$(X \cdot \partial_U)\, \xi_{s-1}(X,U) = 0 \tag{4.2.15}$$

- The homogeneity condition:

$$(X \cdot \partial_X + 1 - s)\xi_{s-1}(X,U) = 0 \tag{4.2.16}$$

- The Casimir condition:

$$(\partial_X \cdot \partial_X)\, \xi_{s-1}(X,U) = 0 \tag{4.2.17}$$

- The transverse condition:

$$(\partial_X \cdot \partial_U)\, \xi_{s-1}(X,U) = 0 \tag{4.2.18}$$

- The traceless condition:

$$(\partial_U \cdot \partial_U)\, \xi_{s-1}(X,U) = 0 \tag{4.2.19}$$

In the intrinsic coordinate language, this set of equations implies that ξ_{s-1} is a transverse traceless symmetric spin-$(s-1)$ field on dS$_{d+1}$ satisfying on-shell equation of motion $(\nabla^2 - (s-1)(s+d-2))\xi_{s-1} = 0$, where spin indices are suppressed.

4.2.2 Isometry group in ambient space formalism

The isometry group SO$(1, d+1)$ of dS$_{d+1}$ acts linearly on fields in the ambient space $\mathbb{R}^{1,d+1}$. In particular, the generators L_{AB} are realized as linear differential operators in both X and U:

$$L_{AB} = (X_A \partial_{X^B} - X_B \partial_{X^A}) + (U_A \partial_{U^B} - U_B \partial_{U^A}) \tag{4.2.20}$$

where the first term corresponds to orbital angular momentum and the second term represents spin angular momentum. The action of M_{ij}, P_i, K_i, D induced by (4.2.20) is

$$M_{ij} = X_i \partial_{X^j} - X_j \partial_{X^i} + U_i \partial_{U^j} - U_j \partial_{U^i} \tag{4.2.21}$$

$$K_i = X^+ \partial_{X^i} + 2X_i \partial_{X^-} + U^+ \partial_{U^i} + 2U_i \partial_{U^-} \tag{4.2.22}$$

$$P_i = -X^- \partial_{X^i} - 2X_i \partial_{X^+} - U^- \partial_{U^i} - 2U_i \partial_{U^+} \tag{4.2.23}$$

$$D = -X^+ \partial_{X^+} + X^- \partial_{X^-} - U^+ \partial_{U^+} + U^- \partial_{U^-} \tag{4.2.24}$$

where we've used lightcone coordinates $X^\pm \equiv X^0 \pm X^{d+1}$ and $U^\pm \equiv U^0 \pm U^{d+1}$.

With a little algebra, one can show that all L_{AB} commute with the following set of differential operators:

$$X \cdot \partial_U, \quad X \cdot \partial_X, \quad \partial_X^2, \quad \partial_X \cdot \partial_U, \quad \partial_U^2, \quad U \cdot \partial_X \tag{4.2.25}$$

The first five operators define the uplift conditions and Fierz-Pauli conditions, cf. (4.2.6)-(4.2.10), and hence the commutation relations imply that the on-shell bulk fields carry representations of $\mathfrak{so}(1, d+1)$. The last operator defines the gauge transformation of a massless higher spin field and therefore one implication of $[L_{AB}, U \cdot \partial_X] = 0$ is that the descendants of a pure gauge mode are also pure gauge. This observation is crucial when we construct quasinormal modes for massless higher spin fields.

4.3 Algebraic construction of quasinormal modes

The southern static patch of de Sitter spacetime, which corresponds to the region denoted by "S" in the fig. (4.1.1), has coordinates

$$X^0 = \sqrt{1-r^2}\sinh t, \quad X^i = r\,\Omega^i, \quad X^{d+1} = \sqrt{1-r^2}\cosh t \tag{4.3.1}$$

and shows a manifest spherical horizon at $r = 1$ or $\rho = \infty$ in its metric

$$\begin{aligned} ds^2 &= -(1-r^2)dt^2 + \frac{dr^2}{1-r^2} + r^2 d\Omega^2 \\ &= \frac{-dt^2 + d\rho^2}{\cosh^2\rho} + \tanh^2\rho\, d\Omega^2 \end{aligned} \tag{4.3.2}$$

where $r = \tanh\rho$ and $d\Omega^2 = h_{ab} d\vartheta^a d\vartheta^b$ is the standard metric on S^{d-1} (ϑ^a are spherical coordinates on S^{d-1}). The traditional analytical approach to quasinormal requires solving the equation of motion in bulk and imposing in-falling boundary condition, i.e. $e^{-i\omega_{\mathsf{QN}}(t-\rho)}$, for the leading asymptotic behavior near the horizon at $\rho = \infty$.

An algebraic method of solving quasinormal modes was first used for scalar fields in dS_4 in [59, 60]. In particular, the authors found that all the quasinormal modes fall into two lowest-weight representations of the conformal algebra $\mathfrak{so}(1, d+1)$. Therefore, it suffices to find the two lowest-weight/primary quasinormal modes, which are solutions to the equation of motion and are annihilated by K_i, and the rest of the quasinormal spectrum can be generated as descendants of them. In this section, we will first reformulate this scalar story using the ambient space formalism and then generalize it to higher spin fields.

4.3.1 Scalar fields

Let $\varphi(X)$ be a free scalar field of mass $m^2 = \Delta(d-\Delta) > 0$ in dS$_{d+1}$. By construction, the equation of motion $(\nabla^2 - m^2)\varphi = 0$ is satisfied by the boundary-to-bulk propagators, which in ambient space take the following form:

$$\alpha_\Delta(X;\xi) = \frac{1}{(X\cdot\xi)^\Delta}, \quad \beta_\Delta(X;\xi) = \frac{1}{(X\cdot\xi)^{\bar\Delta}} \tag{4.3.3}$$

where ξ^A is a constant null vector in $\mathbb{R}^{1,d+1}$ representing a point on the future/past boundary of dS$_{d+1}$. Treating α_Δ and β_Δ as mode functions in X^A, they are primary with respect to the conformal algebra $\mathfrak{so}(1,d+1)$ if $X\cdot\xi = X^+$, because K_i only involves derivatives ∂_{X^i} and ∂_{X^-} while acting on scalar fields (cf. (4.2.21)). By choosing $\xi^A = (-1,0,\cdots,0,1)$ which is the southern pole of the past sphere, we obtain the two primary quasinormal modes

$$\begin{aligned}\alpha_\Delta(X) &= \frac{1}{(X^+)^\Delta} = (\cosh\rho)^\Delta e^{-\Delta t} \overset{\rho\to\infty}{\rightsquigarrow} e^{-\Delta(t-\rho)}\\ \beta_\Delta(X) &= \frac{1}{(X^+)^{\bar\Delta}} = (\cosh\rho)^{\bar\Delta} e^{-\bar\Delta t} \overset{\rho\to\infty}{\rightsquigarrow} e^{-\bar\Delta(t-\rho)}\end{aligned} \tag{4.3.4}$$

with quasinormal frequency $i\omega_\alpha = \Delta$ and $i\omega_\beta = \bar\Delta$ respectively. The rest quasinormal modes can be realized as descendants of $\alpha_\Delta(X)$ and $\beta_\Delta(X)$. Though not explicitly spoken out in [59, 60], this claim actually relies on two facts: (a) P_i preserves the equation of motion and (b) P_i preserves the in-falling boundary condition near horizon. The former is obvious as we've seen at the end of last section that the SO$(1,d+1)$ action preserves uplift conditions and Pauli-Fierz conditions. The latter holds because P_i is dominated by $-2X_i\partial_{X^+}$ near horizon, where $X_i \approx \Omega_i$ and $X^+ \approx 2\,e^{t-\rho}$. With the quasinormal modes known, we need to figure out the corresponding frequency. This is quite straightforward in our formalism. By construction, each quasinormal mode is an eigenfunction of the dilatation operator D, which is just $-\partial_t$ in the static patch.

Using the in-falling boundary condition $e^{-i\omega_{\mathsf{QN}}(t-\rho)}$ near horizon, we can identify the scaling dimension, i.e. eigenvalue with respect to D, as $i\times$(quasinormal frequency ω_{QN}). For example, a descendant of α_Δ at level n is a quasinormal mode of frequency $\omega_{\mathsf{QN}} = -i(\Delta+n)$ and similarly for β_Δ.

To end the discussion about scalar quasinormal modes, let's compare our construction with the known result in literature. For example, in [83], the scalar quasinormal modes in dS$_{d+1}$ are found to be

$$\varphi_\omega^{\mathsf{QN}}(t,r,\Omega) = r^\ell(1-r^2)^{\frac{i\omega}{2}} F\left(\frac{\ell+i\omega+\Delta}{2}, \frac{\ell+i\omega+\bar{\Delta}}{2}, \frac{d}{2}+\ell, r^2\right) Y_{\ell\mathbf{m}}(\Omega) e^{-i\omega t} \tag{4.3.5}$$

where $Y_{\ell\mathbf{m}}(\Omega)$ denote spherical harmonics on S^{d-1} and the quasinormal frequency ω takes the following values

$$\omega_{\ell,n} = -i(\Delta+\ell+2n), \quad \bar{\omega}_{\ell,n} = -i(\bar{\Delta}+\ell+2n), \quad \ell,n\in\mathbb{N} \tag{4.3.6}$$

For fixed ℓ and n, the quasinormal modes of frequency $\omega_{\ell,n}$ or $\bar{\omega}_{\ell,n}$ have degeneracy D^d_ℓ, i.e. the dimension of the spin-ℓ representation of SO(d). In particular, the two quasinormal modes corresponding to $\ell=n=0$ are

$$\begin{aligned}
\varphi^{\mathsf{QN}}_{\omega_{0,0}} &= \frac{1}{\sqrt{A_{d-1}}}\frac{e^{-\Delta t}}{(1-r^2)^{\frac{\Delta}{2}}} = \frac{1}{\sqrt{A_{d-1}}}(\cosh\rho)^\Delta e^{-\Delta t} \\
\varphi^{\mathsf{QN}}_{\bar{\omega}_{0,0}} &= \frac{1}{\sqrt{A_{d-1}}}\frac{e^{-\bar{\Delta} t}}{(1-r^2)^{\frac{\bar{\Delta}}{2}}} = \frac{1}{\sqrt{A_{d-1}}}(\cosh\rho)^{\bar{\Delta}} e^{-\bar{\Delta} t}
\end{aligned} \tag{4.3.7}$$

where A_{d-1} is the area of S^{d-1}. Apart from the normalization constant, these two quasinormal modes are exactly the primary quasonormal modes α_Δ and β_Δ respectively. In addition to the match of the primary quasinormal modes, we can also show that the algebraic construction reproduces the quasinormal spectrum (4.3.6). Define $P=\sqrt{P_iP_i}$ and $\hat{P}_i = P^{-1}P_i$. Then the linear independent descendants of α_Δ are of the form $P^{2n+\ell}Y_{\ell\mathbf{m}}(\hat{P})\alpha_\Delta$ with $\ell,n\in\mathbb{N}$. For fixed ℓ and n, these are quasinormal modes corresponding to $\omega_{\ell,n}$. Similarly $P^{2n+\ell}Y_{\ell\mathbf{m}}(\hat{P})\beta_\Delta$ represents quasinormal modes corresponding to $\bar{\omega}_{\ell,n}$.

4.3.2 Massive higher spin fields

As in the scalar case, we need to start from a solution of the Pauli-Fierz conditions (4.2.8)-(4.2.10), subject to the tangential condition and homogeneous condition. A natural candidate is the higher spin boundary-to-bulk propagator. In AdS, the boundary-to-bulk propagator of a spin-s field with a generic scaling dimension $\Delta(\neq d+s-2)$ is given by [90, 93, 94]

$$K^{\text{AdS}}_{[\Delta,s]}(X,U;\xi,Z) = \frac{[(U\cdot Z)(\xi\cdot X)-(U\cdot\xi)(Z\cdot X)]^s}{(X\cdot\xi)^{\Delta+s}} \tag{4.3.8}$$

where the null vector $\xi\in\mathbb{R}^{1,d+1}$ represents a boundary point and the null vector $Z\in\mathbb{C}^{1,d+1}$, satisfying $\xi\cdot Z=0$, encodes the *boundary* spin. $K^{\text{AdS}}_{[\Delta,s]}$ in (4.3.8) scales like $K^{\text{AdS}}_{[\Delta,s]}(\lambda X) = \lambda^{-\Delta}K^{\text{AdS}}_{[\Delta,s]}(X)$, which is the analogue of $\kappa=\Delta$ in eq. (4.2.7). In AdS, this scaling property corresponds to the choice of ordinary boundary condition. In dS, on the other hand, both near-boundary fall-offs of a bulk field are dynamical and hence there are two boundary-to-bulk propagators

$$K^{\text{dS}}_{[\kappa,s]}(X,U;\xi,Z) = \frac{[(U\cdot Z)(\xi\cdot X)-(U\cdot\xi)(Z\cdot X)]^s}{(X\cdot\xi)^{\kappa+s}} \tag{4.3.9}$$

where $\kappa\in\{\Delta,\bar\Delta\}$. Notice that K_i in (4.2.21) doesn't involve any derivative with respect to X^+ or U^+. So we can obtain primary mode functions from $K^{\text{dS}}_{[\kappa,s]}(X,U;\xi,Z)$ by putting ξ^A at the southern pole of the past sphere, i.e. $\xi^A=(-1,0,\cdots,0,1)$ and choosing $Z^A=(0,z_i,0)$, where z_i itself is a null vector in $\mathbb{C}^d$:

$$\alpha_{[\Delta,s]} = \frac{\Phi^s}{(X^+)^\Delta},\quad \beta_{[\Delta,s]} = \frac{\Phi^s}{(X^+)^{\bar\Delta}} \tag{4.3.10}$$

where $\boxed{\Phi = X^+u\cdot z - U^+x\cdot z}$ (despite the lower case x and u, indeed $x^i\equiv X^i$ and $u^i\equiv U^i$). $\alpha_{[\Delta,s]}$ and $\beta_{[\Delta,s]}$ are clearly primary quasinormal modes since in-falling boundary condition near horizon naturally follows from the lack of dependence on X^- and U^-. Given the two primary quasinormal modes, the whole quasinormal spectrum can be generated by acting P_i on them repeatedly. In this sense, the algebraic construction of quasinormal modes for massive higher spin fields is a straightforward generalization of the scalar case. Before moving to massless higher spin fields, we want to emphasize that rigorously speaking, "$\alpha_{[\Delta,s]}$" or "$\beta_{[\Delta,s]}$" is *not one* primary quasinormal mode because varying z^i would yield different quasinormal modes. Indeed, $\alpha_{[\Delta,s]}$ represents a collection of quasinormal modes with the same frequency and the vector space spanned by these quasinormal modes furnishes a spin-s representation of SO(d). But for convenience, in most part of the chapter, we'll stick to the misnomer by calling, say $\alpha_{[\Delta,s]}$, *the* α-mode.

4.3.3 Massless higher spin fields

When Δ hits $d-2+s$, i.e. the massless limit, $K^{\text{AdS}}_{[\Delta,s]}$ still holds as a boundary-to-bulk propagator in the so-called de Donder gauge [90, 93]. So we can extract de Sitter primary quasinormal modes, in de Donder gauge, from $K^{\text{AdS}}_{[\Delta,s]}$ as in the massive case:

$$\begin{aligned} \alpha\text{-mode} : \alpha^{(s)}(X,U;z) &= \frac{\Phi^s}{(X^+)^2}\left(\frac{R}{X^+}\right)^{d+2(s-2)} \\ \beta\text{-mode} : \beta^{(s)}(X,U;z) &= \frac{\Phi^s}{(X^+)^2} \end{aligned} \tag{4.3.11}$$

where the SO$(1,d+1)$-invariant $R=\sqrt{X^2}$ is inserted in $\alpha^{(s)}$ [3] so that it can have the same scaling property as $\beta^{(s)}$, which corresponds to $\kappa=2-s$ in (4.2.7). With this choice of κ, gauge transformation acts in the same way on both modes, schematically $\delta\alpha^{(s)}=U\cdot\partial_X(\cdots)$ and $\delta\beta^{(s)}=U\cdot\partial_X(\cdots)$.

Naively, one would expect (4.3.11) to be the end of story since we can generate the rest quasinormal modes as descendants of $\alpha^{(s)}$ and $\beta^{(s)}$, just as in the massive case. However, this expectation is only partially correct because, as we'll show in the following, the quasinormal spectrum is significantly affected by gauge symmetry in the massless case compared to its massive counterpart. For example, the $\beta^{(s)}$-mode has quasinormal frequency $i\omega=2-s$, which would lead to an exponentially growing rather than damped behavior at future for $s\geq 3$. Gauge symmetry should be the only cure for this pathological growth and indeed, we do find that the

[3] Including $R^{d+2(s-2)}$ doesn't spoil the Fierz-Pauli conditions (4.2.8)-(4.2.10).

β-mode is pure gauge for any $s \geq 1$:

$$\beta^{(s)} = U \cdot \partial_X \left(\xi_{s-1}\right), \quad \xi_{s-1} = \Phi^{s-1} \frac{x \cdot z}{X^+} \tag{4.3.12}$$

where the gauge parameter ξ_{s-1} can be realized as a descendant of another mode in the following sense:

$$\xi_{s-1} = (z \cdot P)\eta_{s-1}, \quad \eta_{s-1} = -\frac{1}{2}\Phi^{s-1} \log X^+ \tag{4.3.13}$$

As a result, all the quasinormal modes in the β-tower are *unphyical*. The α-mode itself is not pure gauge but it has some pure-gauge descendants:

$$P \cdot \mathcal{D}\, \alpha^{(s)} = \frac{s(s+d-3)}{\frac{d}{2}+s-2}\, U \cdot \partial_X \left[\Phi^{s-1} \left(\frac{R}{X^+}\right)^{d+2s-2}\right] \tag{4.3.14}$$

where $\mathcal{D}_i$ is defined in the eq. (3.3.7). If we write out the indices explicitly, (4.3.14) means $P_{i_1} \alpha^{(s)}_{i_1 \cdots i_s}(X, U) = 0$ up to gauge transformation, which is the reminiscence of a spin-s conserved current.

In the remaining part of this section, we'll show that although the β-tower of quasinormal modes gets killed by gauge transformation, there exist a brand new tower of physical quasinormal modes.

4.3.3.1 Maxwell fields

The primary β-mode of a massless spin-1 field is $\beta^{(1)}_i = \frac{\Phi_i}{(X^+)^2}$, where $\Phi_i = X^+ u_i - U^+ x_i$. It is pure gauge and the corresponding gauge parameter takes a very special form

$$\beta^{(1)}_i = U \cdot \partial_X \left(P_i\, \eta_0\right) \tag{4.3.15}$$

where η_0 is given by eq. (4.3.13) with $s = 1$. Since $\mathfrak{so}(1, d+1)$ action commutes with gauge transformation, we can switch the order of P_i and $U \cdot \partial_X$ in $\beta^{(1)}_i$:

$$\boxed{\beta^{(1)}_i = P_i \left(U \cdot \partial_X\, \eta_0\right)} \tag{4.3.16}$$

This new expression of $\beta_i^{(1)}$ inspires the following crucial observation. Treating $\beta_i^{(1)}$ as a vector field indexed by i and treating P_i as an ordinary derivative like ∂_i, then $\beta_i^{(1)}$ can be thought as a "pure gauge" mode with the gauge parameter being $U \cdot \partial_X \eta_0$. (We want to emphasize that this gauge symmetry structure is completely different from the bulk gauge symmetry, which takes the form $\delta(\cdots) = U \cdot \partial_X(\cdots)$. To distinguish it from the bulk gauge symmetry, we call it a "pseudo" gauge symmetry and its connection with the boundary gauge transformation will be discussed in section 4.4.2). Since $\beta_i^{(1)}$ is pure gauge with respect to the pseudo gauge symmetry, the (pseudo) field strength $\mathcal{F}_{ij} \equiv P_i \beta_j^{(1)} - P_j \beta_i^{(1)}$ vanishes identically. The vanishing of $\mathcal{F}_{ij}$ signals a potential way to obtain the new physical quasinormal modes, which will be implemented step by step as follows:

- First, we define a different β-mode $\hat{\beta}_i^{(1)}$ by deforming the scaling dimension of $\beta_i^{(1)}$ from 1 to $\Delta - 1$:

$$\hat{\beta}_i^{(1)}(X, U) \equiv \frac{\Phi_i}{(X^+)^\Delta} = \left(X^+\right)^{2-\Delta} \beta_i^{(1)}, \quad \Delta \neq 2 \tag{4.3.17}$$

  From the bulk field theory point of view, it amounts to giving a mass term to the Maxwell field to break the *bulk* $U(1)$ gauge symmetry.

- Then the new pseudo field strength $\hat{\mathcal{F}}_{ij} \equiv P_i \hat{\beta}_j^{(1)} - P_j \hat{\beta}_i^{(1)}$ is nonvanishing

$$\hat{\mathcal{F}}_{ij} = 2(\Delta - 2) \frac{x_i\, u_j - u_i\, x_j}{(X^+)^\Delta} \tag{4.3.18}$$

- Stripping off the numerical factor $2(\Delta - 2)$ and taking the limit $\Delta \to 2$ for the remaining part, we obtain a new non-pure-gauge mode function that is antisymmetric in i, j

$$\gamma_{ij}^{(1)}(X, U) \equiv \frac{x_i\, u_j - u_i\, x_j}{(X^+)^2} = \frac{1}{2}\left(P_i \frac{u_j}{X^+} - P_j \frac{u_i}{X^+}\right) \tag{4.3.19}$$

It's straightforward to check that $\gamma_{ij}^{(1)}$ satisfies all the requirements of being a quasinormal mode of frequency $i\omega = 2$. Written in the form of (4.3.19), $\gamma_{ij}^{(1)}$ looks like a descendant of $\frac{u_i}{X^+}$. However, this descendant structure doesn't have any physical meaning because $\frac{u_i}{X^+}$ fails to satisfy the tangentiality condition (4.2.6). Actually, $\gamma_{ij}^{(1)}$ is a primary up to gauge transformation:

$$K_k \gamma_{ij}^{(1)} = U \cdot \partial_X \left(\frac{\delta_{ik} x_j - \delta_{jk} x_i}{X^+} \right) \tag{4.3.20}$$

Therefore, $\gamma_{ij}^{(1)}$ is a physical primary quasinormal mode and we will call the whole Verma module built from $\gamma_{ij}^{(1)}$ the "γ-tower" of quasinormal modes.

In the framework of pseudo gauge symmetry, $\gamma_{ij}^{(1)}$ can be thought as a $U(1)$ field strength with $\frac{u_i}{X^+}$ being the gauge potential. As a field strength, $\gamma_{ij}^{(1)}$ satisfies Bianchi identity $P_{[i}\gamma_{jk]}^{(1)} = 0$ which imposes a nontrivial constraint on the descendants of $\gamma_{ij}^{(1)}$. When $d = 3$, the field strength $\gamma_{ij}^{(1)}$ is dual to a vector $\tilde{\gamma}_i^{(1)}$ and the Bianchi identity is equivalent to a conservation equation $P_i \tilde{\gamma}_i^{(1)} = 0$. In this case, the representation structure of the γ-tower is exactly the same as the α-tower.

We'll leave the comparison with intrinsic coordinate computation of quasinormal modes of Maxwell theory to appendix B.1.

4.3.3.2 Linearized gravity

$\mathbf{d \geq 4}$

The primary β-mode associated to a massless spin-2 field is $\beta^{(2)} = \frac{\Phi^2}{(X^+)^2}$. According to eq. (4.3.12) and (4.3.13), $\beta^{(2)}$ can be alternatively expressed as

$$\beta_{ij}^{(2)} = P_i \left(U \cdot \partial_X \, \eta_1^j \right) + P_j \left(U \cdot \partial_X \, \eta_1^i \right) - \text{trace} \tag{4.3.21}$$

where the null vectors z in $\beta^{(2)}$ are stripped off. Treating P_i as an ordinary derivative, the first two terms in $\beta_{ij}^{(2)}$ have the form of diffeomorphism transformation of (Euclidean) linearized gravity in $\mathbb{R}^d$. Given this pseudo diffeomorphism structure, we can naturally kill these two terms by considering the (pseudo) linearized Riemann tensor $R[\beta^{(2)}]_{ijk\ell}$, which is defined as

$$R[\beta^{(2)}]_{ijk\ell} \equiv \frac{1}{2} \left(P_j P_k \beta_{i\ell}^{(2)} + P_i P_\ell \beta_{jk}^{(2)} - P_i P_k \beta_{j\ell}^{(2)} - P_j P_\ell \beta_{ik}^{(2)} \right) \tag{4.3.22}$$

The remaining pure trace term in $\beta_{ij}^{(2)}$ drops out by projecting $R[\beta^{(2)}]_{ijk\ell}$ to the "linearized Weyl tensor" $\mathcal{C}[\beta^{(2)}]_{ijk\ell}$ which is defined as the traceless part $R[\beta^{(2)}]_{ijk\ell}$ and carries the $\boxplus$

representation of SO(d). By construction, $\mathcal{C}[\beta^{(2)}]_{ijk\ell}$ would vanish just as the $U(1)$ field strength $\mathcal{F}_{ij}$ in the spin-1 case. Due to this similarity, it's quite natural to expect the new spin-2 physical primary quasinormal mode will be produced by the same "deformation+ limiting" procedure, whose steps are listed here again for readers' convenience: (a) deform the $\beta^{(2)}$ mode by sending it to $\hat{\beta}^{(2)} \equiv \frac{\Phi^2}{(X^+)^\Delta} = (X^+)^{2-\Delta}\beta^{(2)}$, (b) compute the Weyl tensor $\mathcal{C}[\hat{\beta}^{(2)}]_{ijk\ell}$ associated to $\hat{\beta}^{(2)}$, (c) strip off the overall factor $(\Delta-2)$ in $\mathcal{C}[\hat{\beta}^{(2)}]_{ijk\ell}$ and take the limit $\Delta \to 2$ for the remaining part. To show the limiting procedure (c) more explicitly, expand, for instance, the first term in $R[\hat{\beta}^{(2)}]$:

$$P_j P_k \hat{\beta}^{(2)}_{i\ell} = 2(\Delta-2)\,\beta^{(2)}_{i\ell}\,P_j \frac{x_k}{(X^+)^{\Delta-1}} + 2(\Delta-2)\frac{x_{(k}\,P_{j)}\beta^{(2)}_{i\ell}}{(X^+)^{\Delta-1}} + \frac{P_j P_k \beta^{(2)}_{i\ell}}{(X^+)^{\Delta-2}} \tag{4.3.23}$$

where the convention for symmetrization is $x_{(k}\,P_{j)} = x_k\,P_j + x_j\,P_k$. The last term in (4.3.23) does not contribute to the Riemann tensor $R[\hat{\beta}^{(2)}]$ and we'll drop it henceforth. The remaining terms are proportional to $\Delta-2$, so the limiting procedure is applicable to them

$$\begin{aligned}\lim_{\Delta\to 2}\frac{P_j P_k \hat{\beta}^{(2)}_{i\ell}}{\Delta-2} &= P_j\left(\frac{2\,x_k}{X^+}\right)\beta^{(2)}_{i\ell} + \frac{2\,x_k}{X^+}P_j\beta^{(2)}_{i\ell} + \frac{2\,x_j}{X^+}P_k\beta^{(2)}_{i\ell} \\ &= P_j P_k(-\log(X^+)\,\beta^{(2)}_{i\ell}) + \log(X^+)P_j P_k\,\beta^{(2)}_{i\ell}\end{aligned} \tag{4.3.24}$$

where the last term drops out from Riemann tensor $R[\hat{\beta}^{(2)}]$. Therefore the new physical quasinormal mode is schematically

$$\gamma^{(2)}_{ijk\ell}(X,U) \equiv \mathcal{C}[h]_{ijk\ell}, \quad h_{ij} = -\log(X^+)\,\beta^{(2)}_{ij} \tag{4.3.25}$$

which has scaling dimension 2 (or quasinormal frequency $i\omega = 2$) and carries the $\begin{array}{|c|c|}\hline & \\ \hline & \\ \hline\end{array}$ representation of SO(d). The primariness of $\gamma^{(2)}_{ijk\ell}$ is proved in appendix B.3.

$\mathbf{d = 3}$

The $d = 3$ case is degenerate and requires a separate discussion because the 3D Weyl tensor $\mathcal{C}[\hat{\beta}^{(2)}]_{ijk\ell}$ vanishes identically for arbitrary choice of Δ. So the deformation and limiting procedure used above fails to yield any quasinormal mode when $d = 3$. The solution to this problem is using Cotton tensor, the 3D analogue of Weyl tensor. On a 3-dimensional Riemann manifold with metric g_{ij}, the Cotton tensor is given by [85, 86]

$$\mathcal{C}_i^j[g] = \nabla_k \left(R_{i\ell} - \frac{1}{4} R\, g_{i\ell} \right) \epsilon^{k\ell j} \tag{4.3.26}$$

and the vanishing of Cotton tensor is the necessary and sufficient condition for the 3-dimensional manifold to be conformally flat. (In spite of the abuse of notation $\mathcal{C}$, it will be clear from the context that $\mathcal{C}$ means Weyl tensor when $d \geq 4$ and means Cotton tensor when $d = 3$). At the linearized level, i.e. $g_{ij} = \delta_{ij} + \phi_{ij}$, the Cotton tensor becomes

$$\mathcal{C}[\phi]_{ij} = \partial_k R_{i\ell}[\phi] \epsilon_{k\ell j} - \frac{1}{4} \epsilon_{kij} \partial_k R[\phi] \tag{4.3.27}$$

where $R[\phi]_{ij}, R[\phi]$ are linearized Ricci tensor and linearized Ricci scalar respectively. In addition, the linearized Cotton tensor is actually symmetric in i, j which can be checked by contracting it with ϵ_{ijm}

$$\mathcal{C}[\phi]_{ij} \epsilon_{ijm} = \frac{1}{2} \partial_m R[\phi] - \partial_i R_{im} = 0 \tag{4.3.28}$$

where the last step is a well-known result of Bianchi identity of Riemann tensor. Since $\mathcal{C}[\phi]_{ij}$ is symmetric and traceless, it will be convenient to restore the null vector z

$$\mathcal{C}[\phi; z] = \partial_k R_{i\ell}[\phi] \epsilon_{k\ell j}\, z_i\, z_j = \frac{1}{2} \epsilon_{k\ell j} (\partial_i \partial_k \partial_m \phi_{m\ell} - \partial^2 \partial_k \phi_{i\ell})\, z_i\, z_j \tag{4.3.29}$$

In the context of quasinormal modes, we can similarly construct a pseudo Cotton tensor with ordinary derivative ∂_i in eq. (4.3.29) replaced by momentum operator P_i

$$\mathcal{C}[\phi; z] = \frac{1}{2} \epsilon_{k\ell j} (P_i P_k P_m \phi_{m\ell} - P^2 P_k \phi_{i\ell})\, z_i\, z_j \tag{4.3.30}$$

Because Cotton tensor is invariant under diffeomorphism and local Weyl transformation by construction, $\mathcal{C}[\phi; z]$ vanishes exactly when $\phi = \beta^{(2)}$. As a result, applying the deformation and limiting procedure to the Cotton tensor $\mathcal{C}[\beta^{(2)}; z]$ would yield a new physical primary quasinormal mode

$$\gamma^{(2)}(X,U;z) = \frac{z\cdot(x\wedge u)}{(X^+)^3}(X^+U^-\,x\cdot z - X^+X^-\,u\cdot z + x^2\,u\cdot z - (u\cdot x)(x\cdot z)) \quad (4.3.31)$$

where $(x\wedge u)_i = \epsilon_{ijk}x_j u_k$. Compared to higher dimensional cases, $\gamma^{(2)}$ in $d=3$ is different mainly in two ways: (a) it has scaling dimension 3 because Cotton tensor involves three derivatives while Weyl tensor only involves 2, (b) it carries a spin-2 representation of SO(3) and the Bianchi identity in higher dimension becomes a "conservation" equation $P_i\gamma^{(2)}_{ij} = 0$. Due to these two properties, the $\gamma^{(2)}$-tower of quasinormal modes in dS$_4$ is isomorphic to the $\alpha^{(2)}$-tower.

4.3.3.3 Massless higher spin fields

With the spin-1 and spin-2 examples worked out explicitly, we'll continue to show that for any massless higher spin field, there exist the primary γ-mode. For a massless spin-s field, the primary β-mode given by eq. (4.3.12) and (4.3.13) can be written as

$$\beta^{(s)}_{i_1\cdots i_s} = P^{(i_1}U\cdot\partial_X\eta^{i_2\cdots i_s)}_{s-1} - \text{trace} \quad (4.3.32)$$

Treating P^i as an ordinary derivative, apart from the pure trace part, $\beta^{(s)}$ has the form of gauge transformation of d-dimensional linearized spin-s gravity. Such gauge transformation can be eliminated by using the higher spin Riemann tensor

$$R[\phi]_{i_1\ell_1,\cdots,i_s\ell_s} \equiv \Pi_{ss}P_{i_1}\cdots P_{i_s}\phi_{\ell_1\cdots\ell_s} \quad (4.3.33)$$

where Π_{ss} is a projection operator ensuring $R[\phi]_{i_1\ell_1,\cdots,i_s\ell_s}$ carries the $\mathbb{Y}_{ss}$ representation of GL$(d,\mathbb{R})$. More explicitly, Π_{ss} can be realized by antisymmetrizing the s pairs of indices: $[i_1,\ell_1],\cdots[i_s,\ell_s]$ [87, 95]. The higher spin Weyl tensor $\mathcal{C}[\phi]_{i_1\ell_1,\cdots,i_s\ell_s}$ is defined as the traceless part of $R[\phi]_{i_1\ell_1,\cdots,i_s\ell_s}$ and thus it carries the $\mathbb{Y}_{ss}$ representation of SO(d) [4]. Since Weyl tensor

[4]For simplicity, we assume $d\geq 4$ so the higher spin Weyl tensor is nonvanishing. When $d=3$, we should use higher spin Cotton tensor $\mathcal{C}_{i_1\cdots i_s}$ [87, 88] that is symmetric and traceless. The Bianchi identity for Cotton tensor is a conservation equation $P_{i_1}\mathcal{C}_{i_1\cdots i_s}=0$. In addition, the definition of Cotton tensor involves $2s-1$ momentum operators and hence the associated primary γ-mode would have scaling dimension $1+s$ instead of 2.

is invariant under diffeomorphism and local Weyl transformation, $\mathcal{C}[\beta^{(s)}]$ vanishes exactly. Thus we can apply the deformation procedure to it and obtain the following quasinormal mode

$$\boxed{\gamma^{(s)}_{i_1\ell_1,\cdots i_s\ell_s}(X,U)\equiv\mathcal{C}[h^{(s)}]_{i_1\ell_1,\cdots,i_s\ell_s},\quad h^{(s)}_{\ell_1\cdots\ell_s}=-\log(X^+)\beta^{(s)}_{\ell_1\cdots\ell_s}} \tag{4.3.34}$$

In appendix B.3, we show that $\gamma^{(s)}_{i_1\ell_1,\cdots,i_s\ell_s}$ represents D^d_{ss} primary quasinormal modes of scaling dimension 2 that carry $\mathbb{Y}_{ss}$ representation of SO(d), where D^d_{ss} is the dimension of the $\mathbb{Y}_{ss}$ representation. In the same appendix, we also show that these primary quasinormal modes can alternatively be expressed in the following form

$$\boxed{\gamma^{(s)}(X,U)=\frac{T_{ss}(x,u)}{(X^+)^2}} \tag{4.3.35}$$

where $T_{ss}(x,u)$ is a homogeneous polynomial in both x and u of degree s and satisfies (B.3.10). The space of such T_{ss} carries the $\mathbb{Y}_{ss}$ representation of SO(d) [96]. One obvious example of T_{ss} is

$$T_{ss}(x,u)=\Big[(x^1+ix^2)(u^3+iu^4)-(x^3+ix^4)(u^1+iu^2)\Big]^s \tag{4.3.36}$$

In the representation language, the example given by eq. (4.3.36) is actually the lowest-weight state in $\mathbb{Y}_{ss}$. Therefore, we are able to generate the whole γ-tower by acting P_i and M_{ij} on the following quansinormal mode:

$$\boxed{\gamma^{(s)}_{lw}(X,U)=\frac{\left[(X^1+iX^2)(U^3+iU^4)-(X^3+iX^4)(U^1+iU^2)\right]^s}{(X^+)^2}} \tag{4.3.37}$$

This is a strikingly universal expression that works for any $s\geq 1$ and $d\geq 4$. In static patch coordinate, the nonvanishing components of (4.3.37) are

$$\gamma^{(s)}_{lw,a_1\cdots a_s}(t,r,\Omega)=\frac{r^{2s}e^{-2\,t}}{(1-r^2)}(\Omega_{12}\partial_{\vartheta^{a_1}}\Omega_{34}-\Omega_{34}\partial_{\vartheta^{a_1}}\Omega_{12})\cdots(\Omega_{12}\partial_{\vartheta^{a_s}}\Omega_{34}-\Omega_{34}\partial_{\vartheta^{a_s}}\Omega_{12}) \tag{4.3.38}$$

where $\Omega_{12}=\Omega_1+i\Omega_2$ and $\Omega_{34}=\Omega_3+i\Omega_4$. One can check that the Ω-dependent part of (4.3.38) is actually a divergence-free spin-s tensor harmonics on S^{d-1}. This is also expected from the representation side because these tensor harmonics also furnish the $\mathbb{Y}_{ss}$ representation of SO(d).

For the completeness of the final result, we also give the unique lowest-weight state in the $\alpha^{(s)}$-tower here

$$\boxed{\alpha_{lw}^{(s)}(X,U) = \frac{[X^+(U^1 + i\,U^2) - U^+(X^1 + i\,X^2)]^s}{(X^+)^2}\left(\frac{R}{X^+}\right)^{d+2(s-2)}} \tag{4.3.39}$$

Then all the quasinormal modes are built from $\alpha_{lw}^{(s)}$ (cf. (4.3.39)) and $\gamma_{lw}^{(s)}$ (cf. (4.3.37)) with the action of P_i and M_{ij}.

4.4 Quasinormal modes from a QFT point of view

In the previous section, we presented a pure algebraic method to construct quasinormal modes of scalars and higher spin fields. In this section, we'll provide a simple physical picture for this method from a bulk QFT point of view. In particular, we'll use scalar fields and Maxwell fields to illustrate this intuitive picture explicitly and then give a brief comment on general massless higher spin fields. Through out this section, the bulk quantum fields are defined in the southern past planar coordinate (η, y^i) of dS$_{d+1}$ (the region "S"+"P" in fig. (4.1.1)):

$$X^0 = \frac{1+y^2-\eta^2}{2\eta}, \quad X^i = -\frac{y^i}{\eta}, \quad X^{d+1} = -\frac{1-y^2+\eta^2}{2\eta} \tag{4.4.1}$$

where $\eta < 0$ and the quasinormal modes are still defined in the southern static patch, which corresponds to $y < -\eta$ in eq. (4.4.1).

4.4.1 Scalar fields

Let φ be a scalar field of scaling dimension Δ. Near the past boundary, it has the following asymptotic behavior

$$\varphi(\eta, y) \approx (-\eta)^{\Delta}\mathcal{O}^{(\alpha)}(y) + (-\eta)^{\bar{\Delta}}\mathcal{O}^{(\beta)}(y) \tag{4.4.2}$$

Define quantum operators $\mathcal{L}_{AB}$ such that $L_{AB}\varphi = -[\mathcal{L}_{AB}, \varphi]$. Then $\mathcal{O}^{(\alpha)}$ and $\mathcal{O}^{(\beta)}$ are primary operators in the sense that

$$[\mathcal{D}, \mathcal{O}^{(\alpha)}(0)] = \Delta\mathcal{O}^{(\alpha)}(0), \quad [\mathcal{K}_i, \mathcal{O}^{(\alpha)}(0)] = 0$$
$$[\mathcal{D}, \mathcal{O}^{(\beta)}(0)] = \bar{\Delta}\mathcal{O}^{(\beta)}(0), \quad [\mathcal{K}_i, \mathcal{O}^{(\beta)}(0)] = 0 \tag{4.4.3}$$

The bulk two-point function of φ defined with respect to the Euclidean vacuum $|E\rangle$ is given by [97]

$$\langle E|\varphi(X)\varphi(X')|E\rangle = \frac{\Gamma(\Delta)\Gamma(\bar{\Delta})}{(4\pi)^{\frac{d+1}{2}}\Gamma(\frac{d+1}{2})} F\left(\Delta, \bar{\Delta}, \frac{d+1}{2}, \frac{1+P}{2}\right) \tag{4.4.4}$$

where $P = X \cdot X'$. We push X' to the past southern pole, i.e. $y'^i = 0$ and $\eta' \to 0^-$, then P is approximately $-\frac{X^+}{2\eta'} \to \infty$. For $P \to \infty$, the hypergeometric function in (4.4.4) has two leading asymptotic behaviors: $P^{-\Delta}$ and $P^{-\bar{\Delta}}$. Schematically, it means

$$\langle E|\varphi(X)\varphi(\eta' \to 0, y'^i = 0)|E\rangle \approx c_\Delta \frac{(-\eta')^\Delta}{(X^+)^\Delta} + c_{\bar{\Delta}} \frac{(-\eta')^{\bar{\Delta}}}{(X^+)^{\bar{\Delta}}} \tag{4.4.5}$$

where c_Δ and $c_{\bar{\Delta}}$ are two constants. Comparing the eq. (4.4.2) and eq. (4.4.5), we find that $\langle E|\varphi(X)\mathcal{O}^{(\alpha)}(0)|E\rangle$ produces the primary quasinormal mode $\alpha_\Delta(X)$ and $\langle E|\varphi(X)\mathcal{O}^{(\beta)}(0)|E\rangle$ produces the primary quasinormal mode $\beta_\Delta(X)$. Altogether, the scalar primary quasinormal modes in southern static patch can be produced by inserting primary operator $\mathcal{O}^{(\alpha)}$ or $\mathcal{O}^{(\beta)}$ at the southern pole of the past sphere and other quasinormal modes can be produced by inserting descendants of $\mathcal{O}^{(\alpha)}$ or $\mathcal{O}^{(\beta)}$.

4.4.2 Maxwell fields

We want to derive the two primary quasinormal modes of Maxwell field in dS_4 using local operators. First, let's pull back $\alpha_i^{(1)}$ and $\gamma_{ij}^{(1)}$ to planar patch coordinates.

The α-mode:

$$\alpha_{i,\eta}^{(1)} = \frac{2\,y_i\,\eta^2}{(\eta^2 - y^2)^3}, \quad \alpha_{i,j}^{(1)} = -\frac{\eta(2\,y_i\,y_j + \delta_{ij}\,(\eta^2 - y^2))}{(\eta^2 - y^2)^3} \tag{4.4.6}$$

For later convenience, we perform a gauge transformation to kill the timelike component, which can be done by choosing the following gauge parameter [5]

$$\xi = \frac{y_i}{4\,y^3}\left(\frac{y\,\eta(\eta^2+y^2)}{(\eta^2-y^2)^2} - \frac{1}{2}\log\frac{\eta+y}{\eta-y}\right) \tag{4.4.7}$$

The resulting spatial part of $\alpha_i^{(1)}$ becomes

$$\boxed{\tilde{\alpha}^{(1)}_{i,j} = \left(\partial_{y^i}\partial_{y^j} - \delta_{ij}\partial_y^2\right)\frac{\log\frac{\eta+y}{\eta-y}}{8\,y}} \tag{4.4.8}$$

The γ-mode:

$$\boxed{\gamma^{(1)}_{ij,\eta} = 0, \quad \gamma^{(1)}_{ij,k} = \frac{y_i\,\delta_{jk} - y_j\,\delta_{ik}}{(\eta^2-y^2)^2}} \tag{4.4.9}$$

The timelike component is automatically vanishing.

Next, we do a mode expansion for a Maxwell field A_μ in the Coulomb gauge [40]:

$$A_i(\eta, y) = -\int \frac{d^3k}{(2\pi)^3}\left(\mathcal{O}_i^{(\alpha)}(k)\,\frac{\sin(k\eta)}{k} - \mathcal{O}_i^{(\beta)}(k)\cos(k\eta)\right)e^{ik\cdot y} \tag{4.4.10}$$

where the two primary operators $\mathcal{O}_i^{(\alpha)}, \mathcal{O}_i^{(\beta)}$ capture the leading asymptotic behavior of A_i near the past boundary

$$A_i(\eta\to 0^-, y) \approx (-\eta)\,\mathcal{O}_i^{(\alpha)}(y) + \mathcal{O}_i^{(\beta)}(y) \tag{4.4.11}$$

[5]Since quasinormal modes are still defined in the static patch, which corresponds to $\eta + y < 0$, we are away from the branch cut of logarithm and the gauge parameter is real.

They also satisfy the following vacuum two-point functions in momentum space:

$$\begin{aligned}
\langle E|\mathcal{O}_i^{(\beta)}(k)\mathcal{O}_j^{(\beta)}(k')|E\rangle &= \frac{1}{2k}\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)(2\pi)^3\delta^3(k+k') \\
\langle E|\mathcal{O}_i^{(\alpha)}(k)\mathcal{O}_j^{(\beta)}(k')|E\rangle &= \frac{i}{2}\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)(2\pi)^3\delta^3(k+k') \\
\langle E|\mathcal{O}_i^{(\alpha)}(k)\mathcal{O}_j^{(\alpha)}(k')|E\rangle &= \frac{k}{2}\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)(2\pi)^3\delta^3(k+k') \\
\langle E|\mathcal{O}_i^{(\beta)}(k)\mathcal{O}_j^{(\alpha)}(k')|E\rangle &= -\frac{i}{2}\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)(2\pi)^3\delta^3(k+k')
\end{aligned} \tag{4.4.12}$$

Like in the scalar case, we insert the primary operator $\mathcal{O}_i^{(\alpha)}$ at the southern pole of the past sphere and it produces a mode in the bulk:

$$\begin{aligned}
\langle E|A_i(\eta,y)\mathcal{O}_j^{(\alpha)}(0)|E\rangle &= -\frac{i}{2}\int\frac{d^3k}{(2\pi)^3}\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right)e^{ik\cdot y-ik\eta} \\
&= -\frac{i}{4\pi^2}(\partial_{y^i}\partial_{y^j}-\delta_{ij}\partial_y^2)\int_0^\infty\frac{dk}{k\,y}\sin(k)e^{-i\,\frac{\eta}{y}k}
\end{aligned} \tag{4.4.13}$$

where the integral over k depends on the relative size of y and $-\eta$ because [6]

$$\int_0^\infty\frac{dk}{k}\sin(k)e^{i\,a\,k} = \begin{cases}\frac{i}{2}\log\frac{a+1}{a-1}, & |a|>1 \\ \frac{\pi}{2}+\frac{i}{2}\log\frac{1+a}{1-a}, & |a|<1\end{cases} \tag{4.4.14}$$

In a physical picture, the jumping at $|a|=1$ reflects horizon crossing. For quasinormal modes defined in southern static patch, which has $y<-\eta$, the k-integral in (4.4.13) corresponds to the top case of (4.4.14):

$$\langle E|A_i(\eta,y)\mathcal{O}_j^{(\alpha)}(0)|E\rangle = (\partial_{y^i}\partial_{y^j}-\delta_{ij}\partial_y^2)\left(\frac{-1}{8\pi^2y}\log\frac{\eta+y}{\eta-y}\right) \tag{4.4.15}$$

Up to normalization, we precisely reproduce the $\alpha_i^{(1)}$ quasinormal mode given by (4.4.8). Simi-

[6] The two cases can be uniformly treated if we give a a small positive imaginary part, i.e. $a\to a+i\epsilon,\epsilon>0$. With this $i\epsilon$ prescription, $\frac{i}{2}\log\frac{a+i\epsilon+1}{a+i\epsilon-1}$ works for both cases. In bulk, it amounts to Wick rotating the planar coordinate time: $\eta\to e^{i\epsilon}\eta$.

larly, we can insert the $\mathcal{O}_i^{(\beta)}$ at the southern pole of the past sphere and it yields

$$\langle E|A_k(\eta,y)\mathcal{O}_i^{(\beta)}(0)|E\rangle = \int \frac{d^3k}{(2\pi)^3}\left(\delta_{ik}-\frac{k_ik_k}{k^2}\right)\frac{e^{ik\cdot y-ik\eta}}{2k} \tag{4.4.16}$$

Due to the extra $\frac{1}{k}$ in the integrand compared to eq.(4.4.13), this mode suffers from an IR divergence around $k=0$. However, this divergence can be eliminated if we replace $\mathcal{O}_i^{(\beta)}(0)$ by a "curvature" $\mathcal{O}_{ij}^{(\gamma)}(0) \equiv P_i\mathcal{O}_j^{(\beta)}(0) - P_j\mathcal{O}_i^{(\beta)}(0)$ as in the construction of $\gamma_{ij}^{(1)}$. The insertion of $\mathcal{O}_i^{(\gamma)}$ at southern pole yields

$$\langle E|A_k(\eta,y)\mathcal{O}_{ij}^{(\gamma)}(0)|E\rangle = \frac{i}{2}\int \frac{d^3k}{(2\pi)^3}\frac{k_i\delta_{jk}-k_j\delta_{ik}}{k}\,e^{ik\cdot y-ik\eta} = \frac{1}{2\pi^2}\frac{y_j\delta_{ik}-y_i\delta_{jk}}{(\eta^2-y^2)^2} \tag{4.4.17}$$

which is exactly the $\gamma_{ij}^{(1)}$ quasinormal mode, cf. (4.4.9), up to normalization. Note that in the definition of $\mathcal{O}_{ij}^{(\gamma)}$, we implicitly use the *pseudo* gauge symmetry structure. On the other hand, P_i reduces to ordinary derivative ∂_i at boundary and hence $\mathcal{O}_{ij}^{(\gamma)}$ is indeed the curvature corresponding to the boundary gauge symmetry. Therefore, the classically pseudo gauge symmetry can be identified as the boundary gauge symmetry in quantum theory. In this sense, $\mathcal{O}_{ij}^{(\gamma)} \sim \epsilon_{ijk}B_k$ has the interpretation as a boundary magnetic field and $\mathcal{O}_i^{(\alpha)} \sim E_i$ has the interpretation as a boundary electric field, subject to the constraint $\nabla\cdot E = 0$. So the quasinormal modes of Maxwell fields are produced by the electric/magnetic field operator, together with their derivatives, inserted at the past southern pole, cf. fig (4.1.1).

In general, a free massless higher spin field $\varphi_{\mu_1\cdots\mu_s}$ in a suitable gauge has the following asymptotic behavior near the past boundary

$$\varphi_{i_1\cdots i_s}(\eta\to 0^-,y) \approx (-\eta)^{d-2}\mathcal{O}^{(\alpha)}_{i_1\cdots i_s}(y) + (-\eta)^{2-2s}\mathcal{O}^{(\beta)}_{i_1\cdots i_s}(y) \tag{4.4.18}$$

where $\mathcal{O}^{(\alpha)}_{i_1\cdots i_s}(y)$ is a gauge-invariant boundary conserved current and $\mathcal{O}^{(\beta)}_{i_1\cdots i_s}(y)$ is a boundary gauge field. [7] From $\mathcal{O}^{(\beta)}_{i_1\cdots i_s}(y)$, we can build a boundary Weyl tensor $\mathcal{O}^{(\gamma)}_{i_1j_1,\cdots,i_sj_s}$ that is gauge invariant. Then inserting operators in the conformal family of $\mathcal{O}^{(\alpha)}_{i_1\cdots i_s}$ at the past southern pole

[7] For a bulk gauge transformation $\delta\varphi_{\mu_1\cdots\mu_s} = \nabla_{(\mu_1}\xi_{\mu_2\cdots\mu_s)}$, the asymptotic behavior of ξ near the past boundary is $\xi_{i_1\cdots i_s}(\eta,y) \approx (-\eta)^d A_{i_1\cdots i_{s-1}}(y) + (-\eta)^{2-2s}B_{i_1\cdots i_{s-1}}(y)$. $\mathcal{O}^{(\alpha)}_{i_1\cdots i_s}$ is clearly invariant under this transformation as the A-mode falls off too fast to affect it. Meanwhile $\mathcal{O}^{(\beta)}_{i_1\cdots i_s}$ undergoes an induced boundary gauge transformation $\delta\mathcal{O}^{(\beta)}_{i_1\cdots i_s} = \partial_{(i_1}B_{i_2\cdots i_s)}$ because the B-mode has the same fall-off as it.

produces the α-tower of quasinormal modes and inserting operators in the conformal family of $\mathcal{O}^{(\gamma)}_{i_1j_1,\cdots,i_sj_s}$ at the past southern pole produces the γ-tower of quasinormal modes

4.5 Quasinormal modes and SO$(1, d+1)$ characters

In the section 4.3, we describe a procedure to construct quasinormal modes of scalar fields and higher spin fields in dS$_{d+1}$. In this section, we will extract the whole quasinormal spectrum by using this construction and show that it's related to the Harish-Chandra group character of SO$(1, d+1)$. To collect the information of quasinormal modes of certain field ϕ in a compact expression, we define a "quasinormal character" :

$$\Theta^{\mathsf{QN}}_{\phi}(q) = \sum_{\omega} d_\omega\, q^{i\omega}, \quad 0 < q < 1 \tag{4.5.1}$$

where the sum runs over all quasinormal frequencies of ϕ and d_ω is the degeneracy of quasinormal modes with frequency ω. Due to the representation structure of the quasinormal modes, the quasinormal character $\Theta^{\mathsf{QN}}_{\phi}(q)$ naturally splits into two different parts, with each part involves either the α-tower or β/γ-tower of quasinormal modes. For example, let ϕ be a real scalar field of scaling dimension Δ. The quasinormal modes in the α-tower have frequencies $i\omega = \Delta+n, n \geq 0$ and for each n the degeneracy is $\binom{d+n-1}{d-1}$. Thus the α-part of the quasinormal character is

$$\Theta^{\mathsf{QN},\alpha}_{\Delta}(q) = \sum_{n\geq 0} \binom{d+n-1}{d-1} q^{\Delta+n} = \frac{q^{\Delta}}{(1-q)^d} \tag{4.5.2}$$

Similarly, the contribution of β-tower is $\Theta^{\mathsf{QN},\beta}_{\Delta}(q) = \frac{q^{\bar{\Delta}}}{(1-q)^d}$. Altogether, we obtain the full quasinormal character of ϕ

$$\Theta^{\mathsf{QN}}_{\Delta}(q) = \Theta^{\mathsf{QN},\alpha}_{\Delta}(q) + \Theta^{\mathsf{QN},\beta}_{\Delta}(q) = \frac{q^{\Delta} + q^{\bar{\Delta}}}{(1-q)^d} \tag{4.5.3}$$

where $\frac{q^{\Delta}+q^{\bar{\Delta}}}{(1-q)^d}$ is exactly the Harish-Chandra character $\Theta_{\mathcal{F}_\Delta}(q)$ for the scalar principal series, i.e. $\Delta \in \frac{d}{2} + i\,\mathbb{R}$ and the scalar complementary series, i.e. $0 < \Delta < d$. Therefore, the quasinormal character of a unitary scalar field is same as its Harish-Chandra character. For a massive spin-s field, the story is almost the same except the α-modes and β-modes have spin degeneracy D^d_s.

Taking into account this spin degeneracy, we obtain the quasinormal character of a spin-s field of scaling dimension Δ,

$$\Theta^{\mathsf{QN}}_{[\Delta,s]}(q) = D^d_s \, \frac{q^{\Delta} + q^{\bar{\Delta}}}{(1-q)^d} \tag{4.5.4}$$

which is exactly the Harish-Chandra character $\Theta_{\mathcal{F}_{\Delta,s}}(q)$ for the spin-s principal series, i.e. $\Delta \in \frac{d}{2} + i\,\mathbb{R}$ and the spin-s complementary series, i.e. $1 < \Delta < d-1$.

In the remaining part of this section, we'll compute quasinormal characters for massless higher spin fields. In this case, the α-part is easy because $\alpha^{(s)}_{i_1\cdots i_s}$ is a conserved current in the sense of (4.3.14). So for $i\omega^\alpha_n = d+s-2+n$ in the α-tower, the degeneracy is $d^\alpha_n = \binom{n+d-1}{d-1} D^d_s - \binom{n+d-2}{d-1} D^d_{s-1}$, which yields

$$\Theta^{\mathsf{QN},\alpha}_s(q) = \sum_{n\geq 0} d^\alpha_n \, q^{i\omega^\alpha_n} = \frac{D^d_s \, q^{d-2+s} - D^d_{s-1}\, q^{d-1+s}}{(1-q)^d} \tag{4.5.5}$$

$\Theta^{\mathsf{QN},\alpha}_s(q)$ is the same as the SO$(2,d)$ character associated to a massless spin-s field in AdS$_{d+1}$[12, 98]. However, this is quite different from the corresponding SO$(1,d+1)$ character $\Theta_{\mathcal{U}_{s,s-1}}(q)$. To compare the quasinormal character $\Theta^{\mathsf{QN}}_s(q)$ with the Harish-Chandra character $\Theta_{\mathcal{U}_{s,s-1}}(q)$, we still need to figure out quasinormal spectrum of the γ-tower.

Maxwell field

Let's start from a Maxwell field. At level 0, i.e. $i\omega = 2$, the degeneracy is $d^\gamma_0 = \binom{d}{2}$ because $\gamma^{(1)}_{ij}$ carries the 2-form representation of SO(d). At level 1, generic descendants are of the form $P_k \gamma^{(1)}_{ij}$, corresponding to the SO(d) representation $\square \otimes \begin{smallmatrix}\square\\\square\end{smallmatrix}$. The 3-form representation in this tensor product is vanishing due to Bianchi identity $P_{[k}\gamma^{(1)}_{ij]} = 0$. Therefore the degeneracy of quasinormal modes with frequency $i\omega = 3$ is

$$d^\gamma_1 = d\binom{d}{2} - \binom{d}{3} = 2\binom{d+1}{3} \tag{4.5.6}$$

The descendants at level 2 are $P_k P_\ell \gamma^{(1)}_{ij}$, corresponding to the SO(d) representation $(\bullet \oplus \square\square) \otimes \begin{smallmatrix}\square\\\square\end{smallmatrix}$. Due to Bianchi identity, we would exclude terms like $P_\ell P_{[k}\gamma^{(1)}_{ij]}$, that carries the $\square \otimes \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}$ representation. However, this is overcounting because the 4-form representation in this tensor product, carried by $P_{[\ell}P_k\gamma^{(1)}_{ij]}$, vanishes automatically without using Bianchi identity. Therefore the degeneracy of quasinormal modes with frequency $i\omega = 4$ is

$$d_2^\gamma = \binom{d+1}{d-1}\binom{d}{2} - d\binom{d}{3} + \binom{d}{4} = 3\binom{d+2}{4} \tag{4.5.7}$$

At any level n, using the same argument, we obtain the degeneracy of quasinormal modes with frequency $i\omega = 2+n$

$$d_n^\gamma = \sum_{k=0}^{n}(-)^k\binom{d}{k+2}\binom{n+d-1-k}{d-1} = (n+1)\binom{n+d}{d-2} \tag{4.5.8}$$

which leads to the γ-tower quasinormal character

$$\Theta_1^{\mathsf{QN},\gamma}(q) = \sum_{n\geq 0}(n+1)\binom{n+d}{d-2}q^{n+2} = 1 - \frac{1-dq}{(1-q)^d} \tag{4.5.9}$$

Combining eq.(4.5.5) for $s=1$ and eq.(4.5.9), we get the full quasinormal character of a Maxwell field

$$\Theta_1^{\mathsf{QN}}(q) = \Theta_1^{\mathsf{QN},\alpha}(q) + \Theta_1^{\mathsf{QN},\gamma}(q) = 1 - \frac{1-dq}{(1-q)^d} + \frac{dq^{d-1}-q^d}{(1-q)^d} \tag{4.5.10}$$

On the other hand, the Harish-Chandra character associated to $\mathcal{U}_{1,0}$ is

$$\Theta_{\mathcal{U}_{1,0}}(q) = \left[\frac{d(q+q^{d-1})-(1+q^d)}{(1-q)^d}\right]_+ = 1 - \frac{1-dq}{(1-q)^d} + \frac{dq^{d-1}-q^d}{(1-q)^d} \tag{4.5.11}$$

where the flipping operator $[\ ]_+$ simply removes the constant -1 in the small q expansion. Again, quasinormal character=Harish-Chandra character. In appendix B.2, we'll show that $\Theta_1^{\mathsf{QN}}(q)$ is also consistent with the spin-1 quasinormal spectrum in [83].

Higher spin fields

To check $\Theta_s^{\mathsf{QN}} = \Theta_{\mathcal{U}_{s,s-1}}(q)$ for any $s\geq 2$, we rewrite the latter as

$$\Theta_{\mathcal{U}_{s,s-1}}(q) = \frac{D_s^d q^{d+s-2} - D_{s-1}^d q^{s+d-1}}{(1-q)^d} + \left[\frac{D_s^d q^{2-s} - D_{s-1}^d q^{1-s}}{(1-q)^d}\right]_+ \tag{4.5.12}$$

Notice that the first term of $\Theta_{\mathcal{U}_{s,s-1}}(q)$ is the same as $\Theta_s^{\mathsf{QN},\alpha}(q)$, so it suffices to compare the second term with $\Theta_s^{\mathsf{QN},\gamma}(q)$. Expand the second term into a Taylor series around $q=0$:

$$\left[\frac{D_s^d q^{2-s} - D_{s-1}^d q^{1-s}}{(1-q)^d}\right]_+ \equiv \sum_{n\geq 0} b_n q^{2+n} \tag{4.5.13}$$

With some simple algebra, one can show that $b_0 = D_{ss}^d, b_1 = D_{s+1,s}^d + D_{s,s-1}^d$ and furthermore b_n satisfies the following recurrence relation

$$b_n - b_{n-2} = D_s^d(D_{s+n}^d + D_{s-n-2}^d) - D_{s-1}^d(D_{s+n+1}^d + D_{s-n-1}^d) \tag{4.5.14}$$

Using the tensor product decomposition of $\mathbb{Y}_s \otimes \mathbb{Y}_t$ (assuming $t \leq s$)

$$\mathbb{Y}_s \otimes \mathbb{Y}_t = \bigoplus_{\ell=0}^{t} \bigoplus_{m=0}^{t-\ell} \mathbb{Y}_{s+t-2\ell-m,m} \tag{4.5.15}$$

the products of dimensions in eq. (4.5.14) can be rewritten as a summation

$$b_n - b_{n-2} = \sum_{\ell=0}^{s} D_{s-\ell+n,s-\ell}^d - \sum_{\ell=0}^{s-n-1} D_{s-\ell-1,s-n-1-\ell}^d = \left(\sum_{\ell=0}^{s} - \sum_{\ell=n+1}^{s}\right) D_{s-\ell+n,s-\ell}^d \tag{4.5.16}$$

When $n \geq s$ the second sum in the bracket vanishes and when $n \leq s$, the first sum over ℓ gets truncated at $\ell = n$. Altogether,

$$b_n - b_{n-2} = \sum_{\ell=0}^{\mathsf{min}(n,s)} D_{s-\ell+n,s-\ell}^d \tag{4.5.17}$$

On the quasinormal modes side, since the primary $\gamma^{(s)}$ carries $\mathbb{Y}_{ss}$ representation, the degeneracy at level 0 is $d_0^\gamma = D_{ss}^d = b_0$. At level 1, the descendants $P_k \gamma^{(s)}_{i_1 j_1, \cdots, i_s j_s}$ are represented by $\mathbb{Y}_1 \otimes \mathbb{Y}_{ss} = \mathbb{Y}_{s+1,s} \oplus \mathbb{Y}_{s,s-1} \oplus \mathbb{Y}_{s,s,1}$. Due to Bianchi identity, the three-row summand in this tensor product vanishes and the level 1 descendants only carry the $\mathbb{Y}_{s+1,s} \oplus \mathbb{Y}_{s,s-1}$ representation. So the degeneracy of quasinormal frequency $i\omega = 3$ is $d_1^\gamma = D_{s+1,s}^d + D_{s,s-1}^d = b_1$. At higher levels, we aim to derive a recurrence relation for the degeneracy d_n^γ. For example, at level n, the descendants are of the form $P_{\ell_1} \cdots P_{\ell_n} \gamma^{(s)}_{i_1 j_1, \cdots, i_s j_s}$, where $P_{\ell_1} \cdots P_{\ell_n}$ should be understood group theoretically the symmetrized tensor product of n spin-1 representations. Compared to level $(n-2)$, the new representation structure is $\mathbb{Y}_n \otimes \mathbb{Y}_{ss}$ where $\mathbb{Y}_n$ corresponds to the traceless part of $P_{\ell_1} \cdots P_{\ell_n}$ and $\mathbb{Y}_{ss}$ corresponds to $\gamma^{(s)}$. Due to Bianchi identity, only two-row representations in the tensor product decomposition of $\mathbb{Y}_n \otimes \mathbb{Y}_{ss}$ are nonvanishing. These two-row representations are exactly $\oplus_{\ell=0}^{\min(n,s)} \mathbb{Y}_{s-\ell+n,s-\ell}$, which yields $d_n^\gamma - d_{n-2}^\gamma = b_n - b_{n-2}$ and furthermore $b_n = d_n^\gamma$ for $n \geq 0$. Altogether, we can conclude

$$\boxed{\Theta_s^{\rm QN}(q) = \Theta_{\mathcal{U}_{s,s-1}}(q)} \tag{4.5.18}$$

4.6 Conclusion and outlook

In this chapter, we present an algebraic method of constructing quasinormal modes of massless higher spin fields in the southern static patch of dS_{d+1} using ambient space formalism. With the action of isometry group $\mathrm{SO}(1, d+1)$, the whole quasinormal spectrum can be built from two primary quasinormal modes, whose properties are summarized in the table. 4.6.1 (assuming $d \geq 4$)

Primaries	$i\,\omega_{\rm QN}$	$\mathrm{SO}(d)$ representation	constraint
$\alpha^{\mu_1\cdots\mu_s}_{i_1\cdots i_s}$	$d+s-2$	$\mathbb{Y}_s$	conservation law
$\gamma^{\mu_1\cdots\mu_s}_{i_1j_1,\cdots,i_sj_s}$	2	$\mathbb{Y}_{ss}$	Bianchi identity

Table 4.6.1: A brief summary about the *physical* primary quasinormal modes of a massless spin-s gauge field in dS_{d+1} ($d \geq 4$). μ_k are bulk spin indices and i_k, j_k indicate the $\mathrm{SO}(d)$ representation whose dimension gives the degeneracy.

For example, when $s = 2$, the primary α-modes $\alpha^{\mu\nu}_{i_1 i_2}$ have quasinormal frequency $\omega_{QN} = -id$ and degeneracy $D_2^d = \frac{(d+2)(d-1)}{2}$ because the i_1, i_2 indices transform as a spin-2 representation of $\mathrm{SO}(d)$. The conservation law means that $P_{i_1}\alpha^{\mu\nu}_{i_1 i_2}$ is pure gauge and hence should be excluded from the physical spectrum of quasinormal modes. On the other hand, the primary γ-modes $\gamma^{\mu\nu}_{i_1j_1,i_2j_2}$ have quasinormal frequency $\omega_{QN} = -2i$ and degeneracy $D_{22}^d = \frac{1}{12}(d+2)(d+1)d(d-3)$ because the indices $[i_1, j_1], [i_2, j_2]$ transform as Weyl tensor under $\mathrm{SO}(d)$. This also explains the Bianchi identity $P_{[k}\gamma^{\mu\nu}_{i_1j_1],i_2j_2} = 0$.

With the higher spin quasinormal modes known, we define a quasinormal character $\Theta_s^{\rm QN}(q)$, cf. (4.5.1) that encodes precisely the information of quasinormal spectrum. We show that $\Theta_s^{\rm QN}(q)$ is equal to the Harish-Chandra character $\Theta_{\mathcal{U}_{s,s-1}}(q)$ of the unitary massless spin-s $\mathrm{SO}(1, d+1)$ representation. In other words, the pure group theoretical object $\Theta_{\mathcal{U}_{s,s-1}}(q)$ knows everything about the physical quasinormal spectrum.

Our algebraic approach to quasinormal modes has some potential generalizations and applications which will be left to investigate in the future:

- Construct quasinormal modes of fields carrying other unitary representations, for example partially massless fields or discrete series fields. The generalization to partially massless fields should be more or less straightforward. In particular, the construction of the primary α-modes would be the same except the conservation law being replaced by a multiply-conservation equation [99]:

$$P_{i_1} \cdots P_{i_{s-t}} \alpha^{\mu_1 \cdots \mu_s}_{i_1 \cdots i_s} = \text{pure gauge} \tag{4.6.1}$$

  where t is the depth. For the primary γ-modes, the higher spin Weyl tensor used in eq. (4.3.34) is expected to be replaced by its partially massless counterpart which carries $\mathbb{Y}_{s,t+1}$ representation of SO(d) [71]. The discrete series case should be different because it is labelled by a maximal height Young diagram. As a result, neither of the primary quasinormal modes can be a curvature like object. This is also confirmed from the character side [70].

- Generalize the "quasinormal quantization" [59, 60] to massless higher spin gauge fields in any higher dimension. It's well known that quasinormal modes are nonnormalizable with respect to the standard Klein-Gordon inner product. However, it is noticed in [59, 60] that, at least for light scalar fields in dS$_4$, there is the so-called "R-norm" such that the quasinormal modes become normalizable and SO$(1,4)$ is effectively Wick rotated to SO$(2,3)$. Granting the existence of "R-norm" in higher dimensions for massless higher spin fields that maps SO$(1,d+1)$ to SO$(2,d)$, then the γ-tower of quasinormal modes carries the $[\Delta = 2, \mathbb{Y}_{ss}]$ representation of SO$(2,d)$, that is below the unitarity bound for sufficiently large s. This simple argument seems to question the naive generalization of "R-norm".

- In this chapter, we have focused on Harish-Chandra character $\Theta_R(q) = \mathrm{tr}_R\, q^D$ with only the scaling operator D turned on, where R denotes some unitary irreducible representation.

In general, we can also include SO(d) generators in the definition of characters:

$$\Theta_R(q,x) = \operatorname{tr}_R \left(q^D x_1^{J_1}\cdots, x_r^{J_r}\right), \quad r = \left\lfloor \frac{d}{2} \right\rfloor \tag{4.6.2}$$

where $J_i = L_{2i-1,2i}$ span the Cartan algebra of SO(d) and x_i are auxiliary variables. For example, for spin-s principal series or complementary series, the full character (4.6.2) reads [70, 75]

$$\Theta_{\mathcal{F}_{\Delta,s}}(q)(q,x) = (q^\Delta + q^{\bar{\Delta}})\Theta^{\mathsf{SO}(d)}_{\mathbb{Y}_s}(x)P_d(q,x) \tag{4.6.3}$$

where $\Theta^{\mathsf{SO}(d)}_{\mathbb{Y}_s}(x) \equiv \operatorname{tr}_{\mathbb{Y}} x_1^{J_1}\cdots x_r^{J_r}$ denotes the SO(d) character of spin-s representation and

$$P_d(q,x) = \frac{1}{\prod_{i=1}^r (1-x_i q)(1-x_i^{-1}q)} \times \begin{cases} 1, & d=2r \\ \frac{1}{1-q}, & d=2r+1 \end{cases} \tag{4.6.4}$$

For massless spin-s representation, the full character originally computed in [75] is:

$$\begin{aligned} d=2r+1: \quad \Theta_{\mathcal{U}_{s,s-1}}(q,x) &= 2\left(\Theta^{\mathsf{SO}(d)}_{\mathbb{Y}_s}(x)\, q^{s+d-2} - \Theta^{\mathsf{SO}(d)}_{\mathbb{Y}_{s-1}}(x)\, q^{s+d-1}\right) P_d(q,x) \\ &\quad + \sum_{n=2}^{r} (-)^n (q^n - q^{d-n}) \Theta^{\mathsf{SO}(d)}_{\mathbb{Y}_{(ss,1^{n-2})}}(x)\, P_d(q,x) \end{aligned} \tag{4.6.5}$$

and

$$\begin{aligned} d=2r: \quad \Theta_{\mathcal{U}_{s,s-1}}(q,x) &= \sum_{n=2}^{r-1} (-)^n (q^n + q^{d-n}) \Theta^{\mathsf{SO}(d)}_{\mathbb{Y}_{(ss,1^{n-2})}}(x) P_d(q,x) \\ &\quad + (-)^r\, q^r \left(\Theta^{\mathsf{SO}(d)}_{\mathbb{Y}_{(ss1,\cdots,+1)}}(x) + \Theta^{\mathsf{SO}(d)}_{\mathbb{Y}_{(ss1,\cdots,-1)}}(x)\right) P_d(q,x) \end{aligned} \tag{4.6.6}$$

We conjecture that the full Harish-Chandra character encodes spin content of quasinormal modes. More precisely, expand $\Theta_{\mathcal{U}_{s,s-1}}(q,x)$ in terms of q and x_i

$$\Theta_{\mathcal{U}_{s,s-1}}(q,x) = \sum_{\omega,\mathbf{j}} d_{\omega,\mathbf{j}}\, q^{i\omega}\, x_1^{j_1}\cdots x_r^{j_r}, \quad \mathbf{j} = (j_1,\cdots,j_r) \tag{4.6.7}$$

then $d_{\omega,\mathbf{j}}$ is conjectured to be the degeneracy of quasinormal modes with frequency ω and

SO(d) spin content $\mathbf{j}$.

Chapter 5: Quantum de Sitter horizon entropy

5.1 Introduction

As seen by local inhabitants [17, 57, 100–104] of a cosmology accelerated by a cosmological constant, the observable universe is evolving towards a semiclassical equilibrium state asymptotically indistinguishable from a de Sitter static patch, enclosed by a horizon of area $A = \Omega_{d-1}\ell^{d-1}$, $\ell \propto 1/\sqrt{\Lambda}$, with the de Sitter universe globally in its Euclidean vacuum state. A picture is shown in fig. 5.1.1b, and the metric in (5.2.1)/(C.3.5)S. The semiclassical equilibrium state locally maximizes the observable entropy at a value $\mathcal{S}$ semiclassically given by [57]

$$\mathcal{S} = \log \mathcal{Z} \,, \tag{5.1.1}$$

where $\mathcal{Z} = \int e^{-S_E[g,\cdots]}$ is the effective field theory Euclidean path integral, expanded about the round sphere saddle related by Wick-rotation (C.3.7) to the de Sitter universe of interest. At tree level in Einstein gravity, the familiar area law is recovered:

$$\mathcal{S}^{(0)} = \frac{A}{4G_{\rm N}} \,. \tag{5.1.2}$$

The interpretation of $\mathcal{S}$ as a (metastable) equilibrium entropy begs for a microscopic understanding of its origin. By aspirational analogy with the Euclidean AdS partition function for effective field theories with a CFT dual (see [105] for a pertinent discussion), a natural question is: are there effective field theories for which the semiclassical expansion of $\mathcal{S}$ corresponds to a large-N expansion of a microscopic entropy? Given a proposal, how can it be tested?

In contrast to EAdS, without making any assumptions about the UV completion of the effective field theory, there is no evident extrinsic data constraining the problem. The sphere has no boundary, all symmetries are gauged, and physically meaningful quantities must be gauge

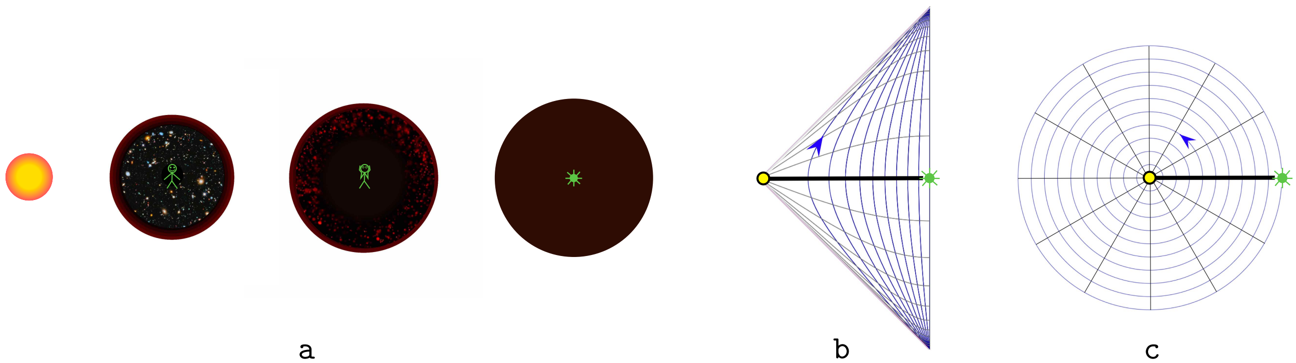


Figure 5.1.1: a: Cartoon of observable universe evolving to its maximal-entropy equilibrium state. The horizon consumes everything once seen, growing until it reaches its de Sitter equilibrium area A. (The spiky dot is a reference point for b, c; it will ultimately be gone, too.) b: Penrose diagram of dS static patch. c: Wick-rotated (b) = sphere. Metric details are given in appendix C.3.2 + fig. C.3.1c,d.

and field-redefinition invariant, leaving little. In particular there is no invariant information contained in the tree-level $\mathcal{S}^{(0)}$ other than its value, which in the low-energy effective field theory merely represents a renormalized coupling constant; an input parameter. However, in the spirit of [105–114], nonlocal *quantum* corrections to $\mathcal{S}$ do offer unambiguous, intrinsic data, directly constraining models. To give a simple example, discussed in more detail under (5.8.17), say someone posits that for pure 3D gravity, the sought-after microscopic entropy is $S_{\rm micro} = \log d(N)$, where $d(N)$ is the number of partitions of N. This is readily ruled out. Both macroscopic and microscopic entropy expansions can uniquely be brought to a form

$$\mathcal{S} = \mathcal{S}_0 - a \log \mathcal{S}_0 + b + \textstyle\sum_n c_n \, \mathcal{S}_0^{-2n} + O(e^{-\mathcal{S}_0/2}) \,, \tag{5.1.3}$$

characterized by absence of odd (=local) powers of $1/\mathcal{S}^{(0)}$. The microscopic theory predicts $(a, b) = (2, \log(\pi^2/6\sqrt{3}))$, refuted by the macroscopic one-loop result $(a, b) = (3, 5\log(2\pi))$. Some of the models in [27, 31, 36, 97, 115–124] are sufficiently detailed to be tested along these lines.

In this work, we focus exclusively on collecting macroscopic data, more specifically the exact one-loop (in some cases all-loop) corrected $\mathcal{S} = \log \mathcal{Z}$. The problem is old, and computations for $s \leq 1$ are relatively straightforward, but for higher spin $s \geq 2$, sphere-specific complications crop up. Even for pure gravity [125–134], virtually no complete, exact results have been obtained at a level brining tests of the above kind to their full potential.

Building on results and ideas from [70, 135–144], we obtain a universal formula solving this problem in general, for all $d \geq 2$ parity-invariant effective field theories, with matter in arbitrary representations, and general gauge symmetries including higher-spin:

$$\boxed{\mathcal{S}^{(1)} = \log \prod_{a=0}^{K} \frac{(2\pi\gamma_a)^{\dim G_a}}{\mathrm{vol}\, G_a} + \int_0^\infty \frac{dt}{2t}\Big(\frac{1+q}{1-q}\,\Theta^{\rm bos}_{\rm tot} - \frac{2\sqrt{q}}{1-q}\,\Theta^{\rm fer}_{\rm tot}\Big) + \mathcal{S}_{\rm ct}} \tag{5.1.4}$$

$q \equiv e^{-t/\ell}$. Below we explain the ingredients in sufficient detail to allow application in practice. A sample of explicit results is listed in (5.1.12). We then summarize the content of the paper by section, with more emphasis on the physics and other results of independent interest.

G_0 is the subgroup of (possibly higher-spin) gravitational gauge transformations acting trivially on the S^{d+1} saddle. This includes rotations of the sphere. $\mathrm{vol}\, G_0$ is the volume for the invariant metric normalized such that the standard rotation generators have unit norm, implying in particular $\mathrm{vol}\,\mathsf{SO}(d+2) =$ (C.3.2). The other G_i, $i = 1, \ldots, K$ are Yang-Mills group factors, with $\mathrm{vol}\, G_i$ the volume in the metric defined by the trace in the action, as in (C.3.3). The γ_a are proportional to the (algebraically defined) gauge couplings:

$$\gamma_0 \equiv \sqrt{\frac{8\pi G_{\rm N}}{A_{d-1}}} = \sqrt{\frac{2\pi}{\mathcal{S}^{(0)}}}\,, \qquad \gamma_i \equiv \sqrt{\frac{g_i^2}{2\pi A_{d-3}}}\,, \tag{5.1.5}$$

with $A_n \equiv \Omega_n \ell^n$, $\Omega_n =$ (C.3.1) for $n \geq 0$, and $A_{-1} \equiv 1/2\pi\ell$ for γ_i in $d = 2$.

The functions $\Theta_{\rm tot}(t)$ are determined by the bosonic/fermionic physical particle spectrum of the theory. They take the form of a "bulk" *minus* an "edge" character:

$$\Theta_{\rm tot} = \Theta_{\rm bulk} - \Theta_{\rm edge}\,. \tag{5.1.6}$$

The bulk character $\Theta_{\rm bulk}(t)$ is defined as follows. Single-particle states on global dS_{d+1} furnish a representation R of the isometry group $\mathsf{SO}(1, d+1)$. The content of R is encoded in its Harish-Chandra character $\Theta(g) \equiv \mathrm{tr}_R\, g$ (see section 3.4). Restricted to $\mathsf{SO}(1,1)$ isometries $g = e^{-itH}$ acting as time translations on the static patch, $\Theta(g)$ becomes $\Theta_{\rm bulk}(t) \equiv \mathrm{tr}_R\, e^{-itH}$.

For example for a massive integer spin-s particle it is given by $\Theta_{\mathcal{F}_{d/2+i\nu,s}}(q)$ (3.4.12):

$$\Theta_{\rm bulk,s} = D_s^d \, \frac{q^{\frac{d}{2}+i\nu} + q^{\frac{d}{2}-i\nu}}{(1-q)^d} \,, \qquad\qquad q \equiv e^{-|t|/\ell} \,, \tag{5.1.7}$$

where ν is related to the mass:

$$s=0: \; \nu^2 = m^2\ell^2 - \left(\tfrac{d}{2}\right)^2 , \qquad\qquad s \geq 1: \; \nu^2 = m^2\ell^2 - \left(\tfrac{d}{2}+s-2\right)^2 . \tag{5.1.8}$$

For arbitrary massive matter $\Theta_{\rm bulk}$ is given by $\Theta_{\mathcal{F}_{\Delta,\mathbf{s}}}(q)$ (3.4.11). Massless spin-s characters are more intricate, but can be obtained by applying a simple "flipping" recipe (3.4.17) to $\Theta_{\mathcal{F}_{2-s,s}}(q) - \Theta_{\mathcal{F}_{1-s,s-1}}(q)$. Some low (d,s) examples are

(d,s)	$(2,1)$	$(2,2)$	$(3,1)$	$(3,2)$	$(4,1)$	$(4,2)$
$\Theta_{\rm bulk,s}$	$\frac{2\,q}{(1-q)^2}$	0	$\frac{6\,q^2-2\,q^3}{(1-q)^3}$	$\frac{10\,q^3-6\,q^4}{(1-q)^3}$	$\frac{6\,q^2}{(1-q)^4}$	$\frac{10\,q^2}{(1-q)^4}$

(5.1.9)

The q-expansion of $\Theta_{\rm bulk}$ gives the static patch *quasinormal* mode degeneracies, its Fourier transform gives the *normal* mode spectral density, and the bulk part of (5.1.4) is the quasi-canonical ideal gas partition function at $\beta = 2\pi\ell$, as we explain below (5.1.15).

The edge character $\Theta_{\rm edge}(t)$ is inferred from path integral considerations in sections 5.3-5.5. It vanishes for spin $s<1$. For integer $s \geq 1$ we get (5.4.8):

$$\Theta_{\rm edge,s} = N_s \cdot \frac{q^{\frac{d-2}{2}+i\nu} + q^{\frac{d-2}{2}-i\nu}}{(1-q)^{d-2}} \,, \qquad N_s = D_{s-1}^{d+2} \,, \tag{5.1.10}$$

e.g. $N_1 = 1$, $N_2 = d+2$. Note this is the bulk character of N_s scalars in *two lower* dimensions. Thus the edge correction effectively *subtracts* the degrees of freedom of N_s scalars living on S^{d-1}, the horizon "edge" of static time slices (yellow dot in fig. 5.1.1). (5.4.14) yields analogous results for more general matter; e.g. $\yng(1,1)$ bulk field $\to$ $\yng(1)$ edge field, $\yng(2,1)$ bulk $\to (d+2) \times \Box$ edge. For massless spin-s, use (C.6.23). The edge companions of (5.1.9) are

(d,s)	$(2,1)$	$(2,2)$	$(3,1)$	$(3,2)$	$(4,1)$	$(4,2)$
$\Theta_{\rm edge,s}$	0	0	$\frac{2\,q}{1-q}$	$\frac{10\,q^2-2\,q^3}{1-q}$	$\frac{2\,q}{(1-q)^2}$	$\frac{10\,q}{(1-q)^2}$

(5.1.11)

The edge correction extends observations of [5, 6, 137, 145–160], reviewed in appendix C.4.5.

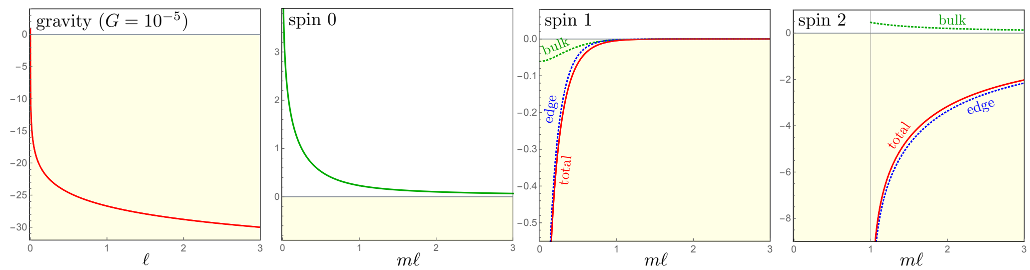


Figure 5.1.2: Contributions to dS_3 one-loop entropy from gravity and massive $s = 0, 1, 2$.

The general closed-form evaluation of the integral in (5.1.4) is given by (C.2.19) in heat kernel regularization. In even d, the finite part is more easily obtained by summing residues.

Finally, $\mathcal{S}_{\rm ct}$ in (5.1.4) is a local counterterm contribution fixed by a renormalization condition specified in section 5.8, which in practice boils down to $\mathcal{S}_{\rm ct}(\ell)$ canceling all divergences and finite terms growing polynomially with ℓ in $\mathcal{S}^{(1)}(\ell)$.

For concreteness here are some examples readily obtained from (5.1.4):

content	$\mathcal{S}^{(1)}$
3D grav	$-3\log\mathcal{S}^{(0)} + 5\log(2\pi)$
3D (s,m)	$\frac{\pi}{3}\big(\nu^3-(m\ell)^3+\frac{3(s-1)^2}{2}m\ell\big)-2\sum_{k=0}^{2}\frac{\nu^k}{k!}\,\frac{\mathrm{Li}_{3-k}(e^{-2\pi\nu})}{(2\pi)^{2-k}}-s^2\big(\pi(m\ell-\nu)-\log(1-e^{-2\pi\nu})\big)$
4D grav	$-5\log\mathcal{S}^{(0)} - \frac{571}{45}\log(\ell/L) - \log\frac{8\pi}{3} + \frac{715}{48} - \frac{47}{3}\zeta'(-1) + \frac{2}{3}\zeta'(-3)$
5D su(4) ym	$-\frac{15}{2}\log(\ell/g^2) - \log\frac{256\pi^9}{3} + \frac{75\,\zeta(3)}{16\,\pi^2} + \frac{45\,\zeta(5)}{16\,\pi^4}$
5D $(\boxplus, m)$	$-15\log(2\pi m\ell) + \frac{5\,\zeta(5)}{8\pi^4} + \frac{65\,\zeta(3)}{24\pi^2} \quad (m\ell\to 0)\,, \qquad \frac{5}{12}(m\ell)^4\,e^{-2\pi m\ell} \quad (m\ell\to\infty)$
11D grav	$-33\log\mathcal{S}^{(0)}+\log\big(\frac{4!\,6!\,8!\,10!}{2^4}(2\pi)^{63}\big)+\frac{1998469\,\zeta(3)}{50400\,\pi^2}+\frac{135619\,\zeta(5)}{60480\,\pi^4}-\frac{34463\,\zeta(7)}{3840\,\pi^6}+\frac{11\,\zeta(9)}{6\pi^8}-\frac{11\,\zeta(11)}{256\,\pi^{10}}$
3D HS_n	$-(n^2-1)\log\mathcal{S}^{(0)} + \log\big[\frac{1}{n}\big(\frac{n(n^2-1)}{6}\big)^{n^2-1}\,\mathtt{G}(n+1)^2\,(2\pi)^{(n-1)(2n+1)}\big]$

(5.1.12)

Comparison to previous results for 3D and 4D gravity is discussed under (5.5.25).[1]

The second line is the contribution of a 3D massive spin-s field, with ν given by (5.1.8). The term $\propto s^2$ is the edge contribution. It is negative for all $m\ell$ and dominates the bulk contribution (fig. 5.1.2). It diverges at the unitarity/Higuchi bound $m\ell = s-1$.

In the 4D gravity example, L is a minimal subtraction scale canceling out of $\mathcal{S}^{(0)} + \mathcal{S}^{(1)}$. In this case, constant terms in $\mathcal{S}^{(1)}$ cannot be distinguished from constants in $\mathcal{S}^{(0)}$ and are as such

[1]In the above and in (5.1.4) we have dropped Polchinski's phase [131] kept in (5.5.25) and generalized in (5.5.17).

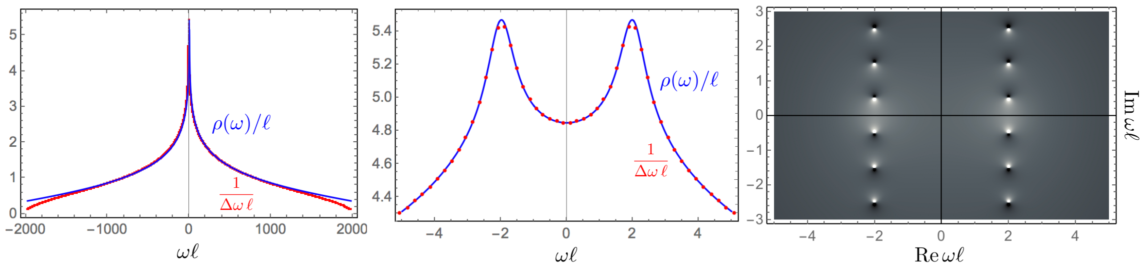


Figure 5.1.3: Regularized dS_2 scalar mode density with $\nu = 2$, $\Lambda_{\rm uv}\ell \approx 4000$. Blue line = Fourier transform of $\Theta_{\rm bulk}$: $\rho(\omega)/\ell = \frac{2}{\pi}\log(\Lambda_{\rm uv}\ell) - \frac{1}{2\pi}\sum\psi(\frac{1}{2} \pm i\nu \pm i\omega\ell)$. Red dots = inverse eigenvalue spacing of numerically diagonalized 4000×4000 matrix H in globally truncated model (appendix C.1.2). Rightmost panel = $|\rho(\omega)|$ on complex ω-plane, with quasinormal mode poles at $\omega\ell = \pm i(\frac{1}{2} \pm i\nu + n)$.

physically ambiguous.[2] The term $\alpha_4 \log(\ell/L)$ with $\alpha_4 = -\frac{571}{45}$ arises from the log-divergent term $\alpha_4 \log(\ell/\epsilon)$ of the regularized character integral.

For any d, in any theory, the coefficient α_{d+1} of the log-divergent term can simply be read off from the $t \to 0$ expansion of the integrand in (5.1.4):

$$\text{integrand} = \cdots + \frac{\alpha_{d+1}}{t} + O(t^0) \tag{5.1.13}$$

For a 4D photon, this gives $\alpha_4 = \alpha_{4,\rm bulk} + \alpha_{4,\rm edge} = -\frac{16}{45} - \frac{1}{3} = -\frac{31}{45}$. The bulk-edge split in this case is the same as the split investigated in [147, 153, 161]. Other illustrations include (partially) massless spin s around (5.5.21), the superstring in (5.9.21), and conformal spin s in (5.9.22).

3D HS_n = higher-spin gravity with $s = 2, 3, \ldots, n$ (section 5.6). $\mathtt{G}$ is the Barnes G-function.

Overview

We summarize the content of sections 5.2-5.9, highlighting other results of interest, beyond (5.1.4).

Quasicanonical bulk thermodynamics of the static patch (section 5.2)

The global dS bulk character $\Theta_{\rm bulk}(t) = \operatorname{tr} e^{-itH}$ locally encodes the quasinormal spectrum and normal mode density of the static patch $ds^2 = -(1 - r^2/\ell^2)dT^2 + (1 - r^2/\ell^2)^{-1}dr^2 + r^2 d\Omega^2$

[2] Comparing different saddles, unambiguous linear combinations can however be extracted, cf. (C.8.70).

on which e^{-itH} acts as a time translation $T \to T + t$. Its expansion in powers of $q = e^{-|t|/\ell}$,

$$\Theta_{\rm bulk} = \sum_r N_r \, q^r \,, \tag{5.1.14}$$

yields the number N_r of *quasinormal* modes decaying as $e^{-rT/\ell}$, in resonance with [1, 59, 60]. The density of *normal* modes $\propto e^{-i\omega T}$ is formally given by its Fourier transform

$$\rho(\omega) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \, \Theta_{\rm bulk}(t) \, e^{i\omega t}. \tag{5.1.15}$$

Because $\Theta_{\rm bulk}$ is singular at $t = 0$, this is ill-defined as it stands. However, a standard Pauli-Villars regularization of the QFT renders it regular (5.2.15), yielding a manifestly covariantly regularized mode density, analytically calculable for arbitrary particle content, including gravitons and higher-spin matter. Some simple examples are shown in figs. 5.1.3, 5.2.2. Quasinormal modes appear as resonance poles at $\omega = \pm ir$, seen by substituting (5.1.14) into (5.1.15).

This effectively solves the problem of making covariant sense of the formally infinite normal mode density universally arising in the presence of a horizon [162]. Motivated by the fact that semiclassical information loss can be traced back to this infinity, [162] introduced a rough model getting rid of it by shielding the horizon by a "brick wall" (reviewed together with variants in C.4.3). Evidently this alters the physics, introduces boundary artifacts, breaks covariance, and is, unsurprisingly, computationally cumbersome. The covariantly regularized density (5.1.15) suffers none of these problems.

In particular it makes sense of the *a priori* ill-defined canonical ideal gas partition function,

$$\log Z_{\rm can}(\beta) = \int_0^\infty d\omega \Big(-\rho_{\rm bos}(\omega) \, \log(e^{\beta\omega/2} - e^{-\beta\omega/2}) + \rho_{\rm fer}(\omega) \, \log(e^{\beta\omega/2} + e^{-\beta\omega/2}) \Big) \,. \tag{5.1.16}$$

Substituting (5.1.15) and integrating out ω, this becomes

$$\boxed{\log Z_{\rm bulk}(\beta) = \int_0^\infty \frac{dt}{2t} \bigg(\frac{1 + e^{-2\pi t/\beta}}{1 - e^{-2\pi t/\beta}} \, \Theta^{\rm bos}_{\rm bulk}(t) \, - \, \frac{2 \, e^{-\pi t/\beta}}{1 - e^{-2\pi t/\beta}} \, \Theta^{\rm fer}_{\rm bulk}(t) \bigg)} \tag{5.1.17}$$

At the static patch equilibrium $\beta = 2\pi\ell$, this is precisely the *bulk* contribution to the one-loop Euclidean partition function $\log \mathcal{Z}^{(1)}$ in (5.1.4). Although $Z_{\rm bulk}$ is not quite a standard canonical partition function, calling it a quasicanonical partition function appears apt.

From (5.1.17), covariantly regularized quasicanonical bulk thermodynamic quantities can be analytically computed for general particle content, as illustrated in section 5.2.3. Substituting the expansion (5.1.14) expresses these quantities as a sum of quasinormal mode contributions, generalizing and refining [163]. In particular the contribution to the entropy and heat capacity from each physical quasinormal mode is finite and positive (fig. 5.2.4).

$S_{\rm bulk}$ can alternatively be viewed as a covariantly regularized entanglement entropy between two hemispheres in the global dS Euclidean vacuum (red and blue lines in figs. 5.2.1, C.3.1). In the spirit of [60], the quasinormal modes can then be viewed as entangled quasiqubits.

Sphere partition functions (sections 5.3,5.4,5.5)

In sections 5.3-5.5 we obtain character integral formulae computing exact heat-kernel regularized one-loop sphere partition functions $Z_{\rm PI}^{(1)}$ for general field content, leading to (5.1.4).

For scalars and spinors (section 5.3), this is easy. For massive spin s (section 5.4), the presence of conformal Killing tensors on the sphere imply naive reduction to a spin-s Laplacian determinant is inconsistent with locality [139]. The correct answer can in principle be obtained by path integrating the full off-shell action [58], but this involves an intricate tower of spin $s' < s$ Stueckelberg fields. Guided by intuition from section 5.2, we combine locality and unitary constraints with path integral considerations to find the terms in $\log Z$ missed by naive reduction. They turn out to be obtained simply by extending the spin-s Laplacian eigenvalue sum to include its "subterranean" levels with formally negative degeneracies, (5.4.6). The extra terms capture contributions from unmatched spin $s' < s$ conformal Killing tensor ghost modes in the gauge-fixed Stueckelberg path integral. The resulting sum yields the bulk−edge character integral formula (5.4.7). Locality and unitarity uniquely determine the generalization to arbitrary parity-symmetric matter representations, (5.4.14).

In the massless case (section 5.5), new subtleties arise: negative modes requiring contour rotations (which translate into the massless character "flipping" recipe mentioned above (5.1.9)), and ghost zeromodes which must be omitted and compensated by a carefully normalized group volume division. Non-universal factors cancel out, yielding (5.5.17) modulo renormalization.

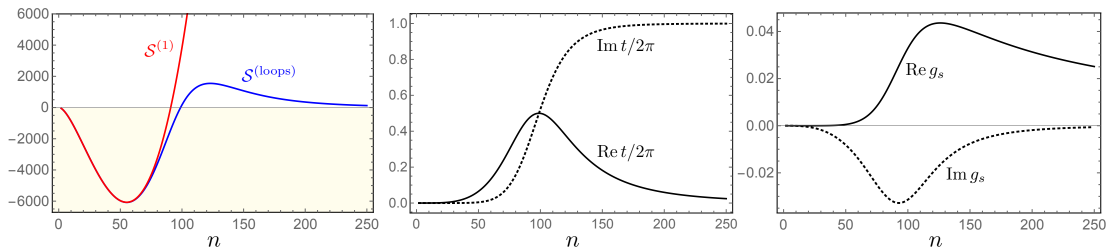


Figure 5.1.4: One- and all-loop entropy corrections, and dual topological string t, g_s, for 3D HS$_n$ theory in its maximal-entropy de Sitter vacuum, for different values of n at fixed $\mathcal{S}^{(0)} = 10^8$, $l = 0$.

3D de Sitter HS$_n$ quantum gravity and the topological string (section 5.6)

The $\mathfrak{sl}(2)$ Chern-Simons formulation of 3D gravity [164, 165] can be extended to an $\mathfrak{sl}(n)$ Chern-Simons formulation of $s \leq n$ higher-spin (HS$_n$) gravity [166–169]. The action for positive cosmological constant is given by (5.6.1). It has a real coupling constant $\kappa \propto 1/G_{\text{N}}$, and an integer coupling constant $l \in \{0, 1, 2, \dots\}$ if a gravitational Chern-Simons term is included.

This theory has a landscape of dS$_3$ vacua, labeled by partitions $\mathbf{m} = \{m_1, m_2, \dots\}$ of n. Different vacua have different values of ℓ/G_{N}, with tree-level entropy

$$\mathcal{S}^{(0)}_{\mathbf{m}} = \frac{2\pi\ell}{4G_{\text{N}}}\bigg|_{\mathbf{m}} = 2\pi\kappa \cdot T_{\mathbf{m}} \,, \qquad T_{\mathbf{m}} = \tfrac{1}{6}\sum_a m_a(m_a^2 - 1)\,. \tag{5.1.18}$$

The number of vacua grows as $\mathcal{N}_{\text{vac}} \sim e^{2\pi\sqrt{n/6}}$. The maximal entropy vacuum is $\mathbf{m} = \{n\}$.

We obtain the all-loop exact quantum entropy $\mathcal{S}_{\mathbf{m}} = \log \mathcal{Z}_{\mathbf{m}}$ by analytic continuation $k_\pm \to l \pm i\kappa$ of the $SU(n)_{k_+} \times SU(n)_{k_-}$ Chern-Simons partition function on S^3, (5.6.7). In the weak-coupling limit $\kappa \to \infty$, this reproduces $\mathcal{S}^{(1)}$ as computed by (5.1.4) in the metric-like formulation of the theory, given in (5.1.12) for the maximal-entropy vacuum $\mathbf{m} = \{n\}$.

When n grows large and reaches a value $n \sim \kappa$, the 3D higher-spin gravity theory becomes strongly coupled. (In the vacuum $\mathbf{m} = \{n\}$ this means $n^4 \sim \ell/G_{\text{N}}$.) In this regime, Gopakumar-Vafa duality [170, 171] can be used to express the quantum de Sitter entropy $\mathcal{S}$ in terms of a weakly-coupled topological string partition function on the resolved conifold, (5.6.8):

$$\boxed{\mathcal{S}_{\mathbf{m}} = \log \left| \tilde{Z}_{\text{top}}(g_s, t)\, e^{-\pi T_{\mathbf{m}} \cdot 2\pi i/g_s} \right|^2} \tag{5.1.19}$$

where $g_s = \frac{2\pi}{n+l+i\kappa}$, and the conifold Kähler modulus $t \equiv \int_{S^2} J + iB = i g_s n = \frac{2\pi i n}{n+l+i\kappa}$.

Euclidean thermodynamics of the static patch (section 5.7)

In section 5.7 we consider the Euclidean thermodynamics of a QFT on a fixed static patch/sphere background. The partition function $Z_{\rm PI}$ is the Euclidean path integral on the sphere of radius ℓ, the Euclidean energy density is $\rho_{\rm PI} = -\partial_V \log Z_{\rm PI}$, where $V = \Omega_{d+1}\ell^{d+1}$ is the volume of the sphere, and the entropy is $S_{\rm PI} = \log Z_{\rm PI} + 2\pi\ell\, U_{\rm PI} = \log Z_{\rm PI} + V\rho_{\rm PI} = (1 - V\partial_V)\log Z_{\rm PI}$, or

$$\boxed{S_{\rm PI} = (1 - \tfrac{1}{d+1}\ell\partial_\ell)\log Z_{\rm PI}} \tag{5.1.20}$$

Using the exact one-loop sphere partition functions obtained in sections 5.3-5.5, this allows general exact computation of the one-loop Euclidean entropy $S_{\rm PI}^{(1)}$, illustrated in section 5.7.2. Euclidean Rindler results are recovered in the limit $m\ell \to \infty$. The sphere computation avoids introducing the usual conical deficit angle, varying the curvature radius ℓ instead.

For minimally coupled scalars, $S_{\rm PI}^{(1)} = S_{\rm bulk}$, but more generally this is false, due to edge (and other) corrections. Our results thus provide a precise and general version of observations made in the work reviewed in appendix C.4.5. Of note, these "corrections" actually *dominate* the one-loop entropy, rendering it negative, increasingly so as s grows large.

Quantum gravitational thermodynamics (section 5.8)

In section 5.8 (with details in appendix C.8), we specialize to theories with dynamical gravity. Denoting $Z_{\rm PI}$, $\rho_{\rm PI}$ and $S_{\rm PI}$ by $\mathcal{Z}$, ϱ and $\mathcal{S}$ in this case, (5.1.20) trivially implies $\varrho = 0$, $\mathcal{S} = \log \mathcal{Z}$, reproducing (5.1.1). All UV-divergences can be absorbed into renormalized coupling constants, rendering the Euclidean thermodynamics well-defined in an effective field theory sense.

Integrating over the geometry is similar in spirit to integrating over the temperature in statistical mechanics, as one does to extract the microcanonical entropy $S(U)$ from the canonical partition function.[3] The analog of this in the case of interest is

$$S(\rho) \equiv \log \int \mathcal{D}g \, \cdots \, e^{-S_E[g,\ldots] + \rho \int \sqrt{g}} \,, \tag{5.1.21}$$

for some suitable metric path integration contour. In particular $S(0) = \mathcal{S}$. The analog of the

[3] Along the lines of $S(U) = \log\left(\frac{1}{2\pi i}\int \frac{d\beta}{\beta}\, {\rm Tr}\, e^{-\beta H + \beta U}\right)$, with contour $\beta = \beta_* + iy$, $y \in \mathbb{R}$, for any $\beta_* > 0$.

microcanonical $\beta \equiv \partial_U S$ is $V \equiv \partial_\rho S$, and the analog of the microcanonical free energy is the Legendre transform $\log Z \equiv S - V\rho$, satisfying $\rho = -\partial_V \log Z$. If we furthermore *define* ℓ by $\Omega_{d+1}\ell^{d+1} \equiv V$, the relation between $\log Z$, ρ and S is by construction identical to (5.1.20).

Equivalently, the free energy $\Gamma \equiv -\log Z$ can be thought of as a quantum effective action for the volume. At tree level, Γ equals the classical action S_E evaluated on the round sphere of radius ℓ. For example for 3D Einstein gravity,

$$\log Z^{(0)} = \frac{2\pi^2}{8\pi G}(-\Lambda\,\ell^3 + 3\,\ell)\,, \qquad S^{(0)} = (1 - \tfrac{1}{3}\ell\partial_\ell)\log Z^{(0)} = \frac{2\pi\ell}{4G}\,. \tag{5.1.22}$$

The tree-level on-shell radius ℓ_0 maximizes $\log Z^{(0)}$, i.e. $\rho^{(0)}(\ell_0) = 0$.

We define renormalized $\Lambda, G, \ldots$ from the $\ell^{d+1}, \ell^{d-1}, \ldots$ coefficients in the $\ell \to \infty$ expansion of the quantum $\log Z$, and fix counterterms by equating tree-level and renormalized couplings for the UV-sensitive subset. For 3D Einstein, the renormalized one-loop correction is

$$\log Z^{(1)} = -3\log\frac{2\pi\ell}{4G} + 5\log(2\pi)\,. \tag{5.1.23}$$

The *quantum* on-shell radius $\bar\ell = \ell_0 + \mathcal{O}(G)$ maximizes $\log Z$, i.e. $\rho(\bar\ell) = 0$. The on-shell entropy can be expressed in two equivalent ways to this order:

$$\boxed{\mathcal{S} = S^{(0)}(\bar\ell) + S^{(1)} = S^{(0)}(\ell_0) + \log Z^{(1)}} \tag{5.1.24}$$

This clarifies why the one-loop correction $\mathcal{S}^{(1)} \equiv \mathcal{S} - \mathcal{S}^{(0)}$ to the dS entropy is given by $\log Z^{(1)}$ rather than $S^{(1)}$: the extra term $-V\rho^{(1)}$ accounts for the change in entropy of the reservoir ($=$ geometry) due to energy transfer to the system ($=$ quantum fluctuations).

The final result is (5.1.4). We work out several examples in detail. We consider higher-order curvature corrections and discuss invariance under local field redefinitions, identifying the invariants $\mathcal{S}_M^{(0)} = -S_E[g_M]$ for different saddles M as and their large-ℓ expanded quantum counterparts $\mathcal{S}_M$ as the $\Lambda > 0$ analogs of tree-level and quantum scattering amplitudes, defining invariant couplings and physical observables of the low-energy effective field theory.

dS, AdS$^{\pm}$, and conformal higher-spin gravity (section 5.9)

Massless $\mathfrak{g} = \mathrm{hs}(\mathfrak{so}(d+2))$ higher-spin gravity theories on dS$_{d+1}$ or S^{d+1} [53, 55, 172] have infinite spin range and infinite $\dim \mathfrak{g}$, obviously posing problems for the one-loop formula (5.1.4):

1. Spin sum divergences untempered by the UV cutoff, for example $\dim G = \frac{1}{3}\sum_s s(4s^2-1)$ for $d=3$ and $\Theta_{\rm tot} = \sum_s (2s+1)\frac{2q^2}{(1-q)^4} - \sum_s \frac{s(s+1)(2s+1)}{6}\frac{2q}{(1-q)^2}$ for $d=4$.
2. Unclear how to make sense of $\mathrm{vol}\, G$.

We compare the situation to analogous one-loop expressions [3, 144, 173] for Euclidean AdS with standard (AdS$^+_{d+1}$) [111–113] and alternate (AdS$^-_{d+1}$) [138] gauge field boundary conditions, and to the associated conformal higher-spin theory on the boundary S^d (CHS$_d$) [139, 174]. For AdS$^+$ the above problems are absent, as $\mathfrak{g}$ is not gauged and $\Delta_s > s$. Like a summed KK tower, the spin-summed bulk character has increased UV dimensionality $d_{\rm eff}^{\rm bulk} = 2d-2$. However, the edge character almost completely cancels this, leading to a *reduced* $d_{\rm eff} = d-1$ in (5.9.10)-(5.9.12). This realizes a version of a stringy picture painted in [5] repainted in fig. C.4.4. A HS "swampland" is identified: lacking a holographic dual, characterized by $d_{\rm eff} > d-1$.

For AdS$^-$ and CHS, the problems listed for dS all reappear. $\mathfrak{g}$ is gauged, and the character spin sum divergences are identical to dS, as implied by the relations (5.9.16):

$$\boxed{\Theta_s(\mathrm{CHS}_d) = \Theta_s(\mathrm{AdS}^-_{d+1}) - \Theta_s(\mathrm{AdS}^+_{d+1}) = \Theta_s(\mathrm{dS}_{d+1}) - 2\,\Theta_s(\mathrm{AdS}^+_{d+1})} \tag{5.1.25}$$

The spin sum divergences are not UV. Their origin lies in low-energy features: an infinite number of quasinormal modes decaying as slowly as $e^{-2T/\ell}$ for $d \geq 4$ (cf. discussion below (C.6.4)). We see no justification for zeta-regularizing such divergences away. However, in certain supersymmetric extensions, the spin sum divergences cancel in a rather nontrivial way, leaving a finite residual as in (5.9.23). This eliminates problem 1, but leaves problem 2. Problem 2 might be analogous to $\mathrm{vol}\, G = \infty$ for the bosonic string or $\mathrm{vol}\, G = 0$ for supergroup Chern-Simons: removed by appropriate insertions. This, and more, is left to future work.

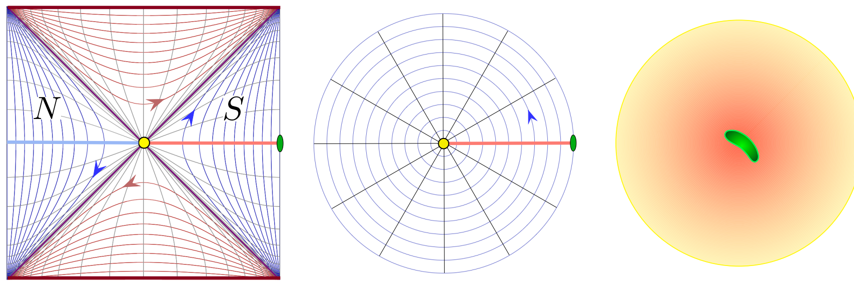


Figure 5.2.1: a: Penrose diagram of global dS, showing flows of SO$(1,1)$ generator $H = M_{0,d+1}$, S = southern static patch. b: Wick-rotated S = sphere; Euclidean time = angle. c: *Pelagibacter ubique* inertial observer in dS with $\ell = 1.2\,\mu\text{m}$ finds itself immersed in gas of photons, gravitons and higher-spin particles at a pleasant $30°\text{C}$. More details are provided in fig. C.3.1 and appendix C.3.2.

5.2 Quasicanonical bulk thermodynamics

5.2.1 Problem and results

From the point of view of an inertial observer, such as *Pelagibacter ubique* in fig. 5.2.1c, the global de Sitter vacuum appears thermal [17, 57, 175]: *P. ubique* perceives its universe, the southern static patch (S in fig. 5.2.1a),

$$ds^2 = -(1 - r^2/\ell^2)dT^2 + (1 - r^2/\ell^2)^{-1}dr^2 + r^2 d\Omega_{d-1}^2, \tag{5.2.1}$$

as a static ball of finite volume, whose boundary $r = \ell$ is a horizon at temperature $T = 1/2\pi\ell$, and whose bulk is populated by field quanta in thermal equilibrium with the horizon. *P. ubique* wishes to understand its universe, and figures the easiest thing to understand should be the thermodynamics of its thermal environment in the ideal gas approximation. The partition function of an ideal gas is

$$\operatorname{Tr} e^{-\beta H} = \exp \int_0^\infty d\omega \Big(-\rho(\omega)_{\rm bos} \log\big(e^{\beta\omega/2} - e^{-\beta\omega/2}\big) + \rho(\omega)_{\rm fer} \log\big(e^{\beta\omega/2} + e^{-\beta\omega/2}\big) \Big), \tag{5.2.2}$$

where $\rho(\omega) = \rho(\omega)_{\rm bos} + \rho(\omega)_{\rm fer}$ is the density of bosonic and fermionic single-particle states at energy ω. However to its dismay, it immediately runs into trouble: the dS static patch mode spectrum is continuous and infinitely degenerate, leading to a pathologically divergent density

$\rho(\omega) = \delta(0)\sum_{\ell m \cdots}$. It soon realizes the unbounded redshift is to blame, so it imagines a brick wall excising the horizon, or some variant thereof (appendix C.4.3). Although this allows some progress, it is aware this alters what it is computing and depends on choices. To check to what extent this matters, it tries to work out nontrivial examples. This turns out to be painful. It feels there should be a better way, but its efforts come to an untimely end.

Here we will make sense of the density of states and the static patch bulk thermal partition function in a different way, manifestly preserving the underlying symmetries, allowing general exact results for arbitrary particle content. The main ingredient is the Harish-Chandra group character (reviewed in the section 3.4) of the SO$(1, d+1)$ representation R furnished by the physical single-particle Hilbert space of the free QFT quantized on *global* dS$_{d+1}$. Letting $H(\equiv iD)$ be the global SO$(1,1)$ generator acting as time translations in the southern static patch and globally as in fig. 5.2.1a, the character restricted to group elements e^{-itH} is

$$\Theta(t) \equiv \mathrm{tr}_G\, e^{-itH} \,. \tag{5.2.3}$$

Here tr_G traces over the *global* dS single-particle Hilbert space furnishing R. (More generally we denote $\mathrm{tr} \equiv$ single-particle trace, $\mathrm{Tr} \equiv$ multi-particle trace, $G \equiv$ global, $S \equiv$ static patch. Our default units set the dS radius $\boxed{\ell \equiv 1}$.)

For example for a scalar field of mass $m^2 = (\frac{d}{2})^2 + \nu^2$, as computed in (3.4.7),

$$\Theta(t) = \frac{e^{-t\Delta_+} + e^{-t\Delta_-}}{|1-e^{-t}|^d} \,, \qquad \Delta_\pm = \tfrac{d}{2} \pm i\nu \,. \tag{5.2.4}$$

For a massive spin-s field this simply gets an additional spin degeneracy factor D^d_s, (3.4.12). Massless spin-s characters take a similar but somewhat more intricate form, (3.4.16)-(3.4.17).

As mentioned in the introduction, (5.1.14), the character has a series expansion

$$\Theta(t) = \sum_r N_r\, e^{-r|t|} \tag{5.2.5}$$

encoding the degeneracy N_r of quasinormal modes $\propto e^{-rT}$ of the dS static patch background. For example expanding the scalar character yields two towers of quasnormal modes with $r_{n\pm} = \frac{d}{2} \pm i\nu + n$ and degeneracy $N_{n\pm} = \binom{n+d-1}{n}$.

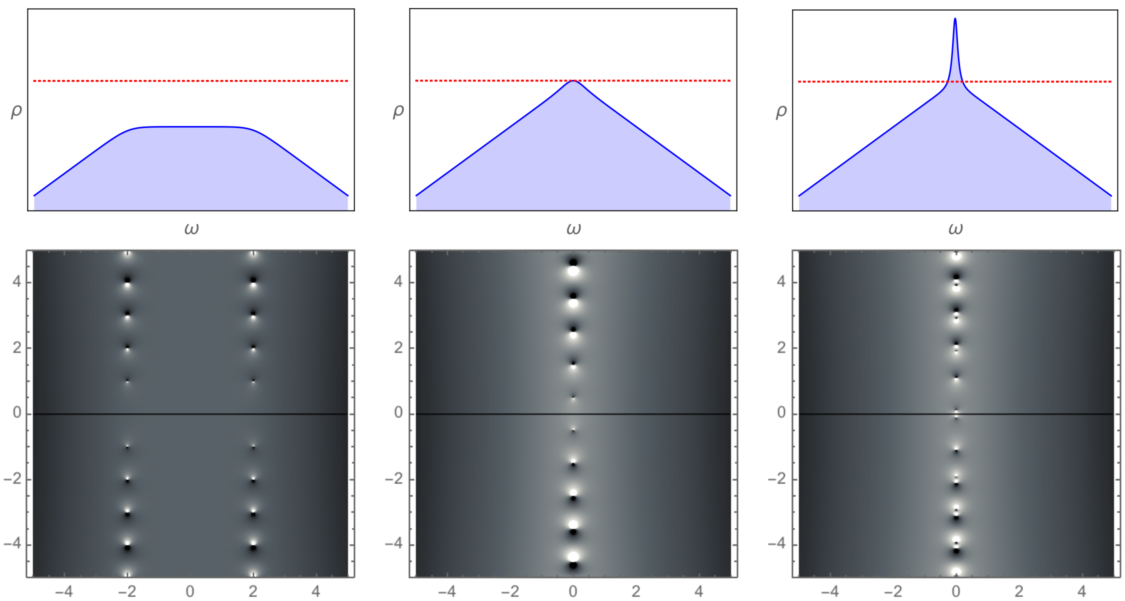


Figure 5.2.2: Regularized scalar $\rho(\omega)$, $d=2$, $\nu=2, i/2, 0.9\,i$; top: $\omega \in \mathbb{R}$; bottom: $\omega \in \mathbb{C}$, showing quasinormal mode poles. See figs. C.1.1, C.1.4 for details.

Our main result, shown in 5.2.2 below, is the observation that

$$\boxed{\log Z_{\rm bulk}(\beta) \equiv \int_0^\infty \frac{dt}{2t}\left(\frac{1+e^{-2\pi t/\beta}}{1-e^{-2\pi t/\beta}}\,\Theta(t)_{\rm bos} - \frac{2\,e^{-\pi t/\beta}}{1-e^{-2\pi t/\beta}}\,\Theta(t)_{\rm fer}\right)}\,, \tag{5.2.6}$$

suitably regularized, provides a physically sensible, manifestly covariant regularization of the static patch bulk thermal partition. The basic idea is that $\rho(\omega)$ can be obtained as a well-defined Fourier transform of the covariantly UV-regularized character $\Theta(t)$, which upon substitution in the ideal gas formula (5.2.2) yields the above character integral formula. Arbitrary thermodynamic quantities at the horizon equilibrium $\beta = 2\pi$ can be extracted from this in the usual way, for example $S_{\rm bulk} = (1-\beta\partial_\beta)\log Z_{\rm bulk}|_{\beta=2\pi}$, which can alternatively be interpreted as the "bulk" entanglement entropy between the northern and southern S^d hemispheres (red and blue lines fig. 5.2.1a).[4] We work out various examples of such thermodynamic quantities in section 5.2.3. General exact solution are easily obtained. The expansion (5.2.5) also allows interpreting the results as a sum over quasinormal modes along the lines of [163].

[4] In part because subregion entanglement entropy does not exist in the continuum, an infinity of different notions of it exist in the literature [176]. Based on [161], $S_{\rm bulk}$ appears perhaps most akin to the "extractable"/"distillable" entropy considered there. Either way, our results are nomenclature-independent.

We conclude this part with some comments on the relation with the Euclidean partition function. As reviewed in appendix C.4, general physics considerations, or formal considerations based on Wick-rotating the static patch to the sphere and slicing the sphere path integral along the lines of fig. 5.2.1b, suggests a relation between the one-loop Euclidean path integral $Z_{\rm PI}^{(1)}$ on S^{d+1} and the bulk ideal gas thermal partition function $Z_{\rm bulk}$ at $\beta = 2\pi$. More refined considerations suggest

$$\log Z_{\rm PI}^{(1)} = \log Z_{\rm bulk} + \text{edge corrections}\,, \tag{5.2.7}$$

where the edge corrections are associated with the S^{d-1} horizon edge of the static patch time slices, i.e. the yellow dot in fig. 5.2.1. The formal slicing argument breaks down here, as does the underlying premise of spatial separability of local field degrees of freedom (for fields of spin $s \geq 1$). Similar considerations apply to other thermodynamic quantities and in other contexts, reviewed in appendix C.4 and more specifically C.4.5.

In sections 5.3-5.5 we will obtain the exact edge corrections by direct computation, logically independent of these considerations, but guided by the physical expectation (5.2.7) and more generally the intuition developed in this section.

5.2.2 Derivation

We first give a formal derivation and then refine this by showing the objects of interest become rigorously well-defined in a manifestly covariant UV regularization of the QFT.

Formal derivation

Our starting point is the observation that the thermal partition function $\operatorname{Tr} e^{-\beta H}$ of a bosonic resp. fermionic oscillator of frequency ω has the integral representation (C.4.14):

$$\begin{aligned} -\log\left(e^{\beta\omega/2} - e^{-\beta\omega/2}\right) &= +\int_0^\infty \frac{dt}{2t}\,\frac{1+e^{-2\pi t/\beta}}{1-e^{-2\pi t/\beta}}\left(e^{-i\omega t} + e^{i\omega t}\right) \\ \log\left(e^{\beta\omega/2} + e^{-\beta\omega/2}\right) &= -\int_0^\infty \frac{dt}{2t}\,\frac{2\,e^{-\pi t/\beta}}{1-e^{-2\pi t/\beta}}\left(e^{-i\omega t} + e^{i\omega t}\right). \end{aligned} \tag{5.2.8}$$

with the pole in the factor $f(t) = c\,t^{-2} + O(t^0)$ multiplying $e^{-i\omega t} + e^{i\omega t}$ resolved by

$$t^{-2} \to \frac{1}{2}\big((t - i\epsilon)^{-2} + (t + i\epsilon)^{-2}\big). \tag{5.2.9}$$

Now consider a free QFT on some space of finite volume, viewed as a system S of bosonic and/or fermionic oscillator modes of frequencies ω with mode (or single-particle) density $\rho_S(\omega) = \rho_S(\omega)_{\rm bos} + \rho_S(\omega)_{\rm fer}$. The system is in thermal equilibrium at inverse temperature β. Using the above integral representation, we can write its thermal partition function (5.2.2) as

$$\log {\rm Tr}_S\, e^{-\beta H_S} = \int_0^\infty \frac{dt}{2t}\bigg(\frac{1+e^{-2\pi t/\beta}}{1-e^{-2\pi t/\beta}}\;\Theta_S(t)_{\rm bos} \;-\; \frac{2\,e^{-\pi t/\beta}}{1-e^{-2\pi t/\beta}}\;\Theta_S(t)_{\rm fer}\bigg)\,, \tag{5.2.10}$$

where we exchanged the order of integration, and we defined

$$\Theta_S(t) \equiv \int_0^\infty d\omega\,\rho_S(\omega)\big(e^{-i\omega t} + e^{i\omega t}\big) \tag{5.2.11}$$

We want to apply (5.2.10) to a free QFT on the southern static patch at inverse temperature β, with the goal of finding a better way to make sense of it than *P. ubique*'s approach. To this end, we note that the global dS$_{d+1}$ Harish-Chandra character $\Theta(t)$ defined in (5.2.3) can formally be written in a similar form by using the general property $\Theta(t) = \Theta(-t)$:

$$\Theta(t) = {\rm tr}_G\, e^{-iHt} = \int_{-\infty}^\infty d\omega\,\rho_G(\omega)\,e^{-i\omega t} = \int_0^\infty d\omega\,\rho_G(\omega)\big(e^{-i\omega t} + e^{i\omega t}\big)\,, \tag{5.2.12}$$

This looks like (5.2.11), except $\rho_G(\omega) = {\rm tr}_G\,\delta(\omega - H)$ is the density of single-particle excitations of the *global* Euclidean vacuum, while $\rho_S(\omega)$ is the density of single-particle excitations of the *southern* vacuum. The global and southern vacua are very different. Nevertheless, there is a simple kinematic relation between their single-particle creation and annihilation operators: the Bogoliubov transformation (C.4.10) (suitably generalized to $d > 0$ [175]). This provides an explicit one-to-one, inner-product-preserving map between southern and global single-particle states with $H = \omega > 0$. Hence, formally,

$$\rho_S(\omega) = \rho_G(\omega) \qquad (\omega > 0)\,, \qquad\qquad \rho_S(\omega) = 0 \qquad (\omega < 0)\,. \tag{5.2.13}$$

While formal in the continuum, this relation becomes precise whenever ρ is rendered effectively finite, e.g. by a brick-wall cutoff or by considering finite resolution projections (say if we restrict to states emitted/absorbed by some apparatus built by *P. ubique*).

At first sight this buys us nothing though, as computing $\rho_G(\omega) = \mathrm{tr}_G\, \delta(\omega - H)$ for say a scalar in dS$_4$ in a basis $|\omega\ell m\rangle_G$ immediately leads to $\rho_G(\omega) = \delta(0) \sum_{\ell m}$, in reassuring but discouraging agreement with *P. ubique*'s result for $\rho_S(\omega)$. On second thought however, substituting this into (5.2.12) leads to a nonsensical $\Theta(t) = 2\pi\delta(t)\delta(0)\sum_{\ell m}$, not remotely resembling the correct expression (5.2.4). How could this happen? As explained in the remark 8, the root cause is the seemingly natural but actually ill-advised idea of computing $\Theta(t) = \mathrm{tr}_G\, e^{-iHt}$ by diagonalizing H: despite its lure of seeming simplicity, $|\omega\ell m\rangle_G$ is in fact the *worst* possible choice of basis to compute the character trace. Its wave functions on the global future boundary S^d of dS$_{d+1}$ are singular at the north and south pole, exactly the fixed points of H at which the correct computation of $\Theta(t)$ in the section 3.4.2 localizes. Although $|\omega\ell m\rangle$ is a perfectly fine basis on the cylinder obtained by a conformal map from sphere, the information needed to compute Θ is irrecoverably lost by this map.

However we can turn things around, and use the properly computed $\Theta(t)$ to extract $\rho_G(\omega)$ as its Fourier transform, inverting (5.2.12). As it stands, this is not really possible, for (5.2.4) implies $\Theta(t) \sim |t|^{-d}$ as $t \to 0$, so its Fourier transform does not exist. Happily, this problem is automatically resolved by standard UV-regularization of the QFT, as we will show explicitly below. For now let us proceed formally, as at this level we have arrived at our desired result: combining (5.2.13) with (5.2.12) and (5.2.11) implies $\Theta_S(t) = \Theta(t)$, which by (5.2.10) yields

$$\mathrm{Tr}_S\, e^{-\beta H_S} = Z_{\mathrm{bulk}}(\beta) \qquad \text{(formal)} \tag{5.2.14}$$

with $Z_{\mathrm{bulk}}(\beta)$ as defined in (5.2.6). The above equation formally gives it its claimed thermal interpretation. In what follows we will make this a bit more precise, and spell out the UV regularization explicitly.

Covariant UV regularization of ρ and $Z_{\rm bulk}$

We begin by showing that $\rho_G(\omega)$ in (5.2.12) becomes well-defined in a suitable standard UV-regularization of the QFT. As in [177], it is convenient to consider Pauli-Villars regularization, which is manifestly covariant and has a conceptually transparent implementation on both the path integral and canonical sides. For e.g. a scalar of mass $m^2 = (\frac{d}{2})^2 + \nu^2$, a possible implementation is adding $\binom{k}{n}$, $n = 1, \ldots, k \geq \frac{d}{2}$ fictitious particles of mass $m^2 = (\frac{d}{2})^2 + \nu^2 + n\Lambda^2$ and positive/negative norm for even/odd n,[5] turning the character $\Theta_{\nu^2}(t)$ of (5.2.4) into

$$\Theta_{\nu^2,\Lambda}(t) = {\rm tr}_{G_\Lambda}\, e^{-itH} = \sum_{n=0}^{k} (-1)^n \binom{k}{n} \Theta_{\nu^2+n\Lambda^2}(t). \tag{5.2.15}$$

This effectively replaces $\Theta(t) \sim |t|^{-d}$ by $\Theta_\Lambda(t) \sim |t|^{2k-d}$ with $2k - d \geq 0$, hence, assuming $\Theta(t)$ falls off exponentially at large t, which is always the case for unitary representations [13, 65, 75], $\Theta_\Lambda(t)$ has a well-defined Fourier transform, analytic in ω:

$$\rho_{G,\Lambda}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, \Theta_\Lambda(t)\, e^{i\omega t} = \frac{1}{2\pi}\int_0^\infty dt\, \Theta_\Lambda(t)\left(e^{i\omega t} + e^{-i\omega t}\right). \tag{5.2.16}$$

The above character regularization can immediately be transported to arbitrary massive SO$(1, d+1)$ representations, as their characters Θ_{s,ν^2} (3.4.11) only differ from the scalar one by an overall spin degeneracy factor.[6]

Although we won't need to in practice for computations of thermodynamic quantities (which are most easily extracted directly as character integrals), $\rho_{G,\Lambda}(\omega)$ can be computed explicitly. For the dS$_{d+1}$ scalar, using (5.2.4) regularized with $k = 1$, we get for $\omega \ll \Lambda$

$$\begin{aligned} d = 1: &\quad \rho_{G,\Lambda}(\omega) = \frac{2}{\pi}\log\Lambda - \frac{1}{2\pi}\sum_{\pm,\pm}\psi(\tfrac{1}{2} \pm i\nu \pm i\omega) + O(\Lambda^{-1}) \\ d = 2: &\quad \rho_{G,\Lambda}(\omega) = \Lambda - \frac{1}{2}\sum_{\pm}(\omega \pm \nu)\coth\big(\pi(\omega \pm \nu)\big) + O(\Lambda^{-1}) \end{aligned} \tag{5.2.17}$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$. Denoting the Λ-independent parts of the above $\omega \ll \Lambda$ expansions

[5] This is equivalent to inserting a heat kernel regulator $f(\tau\Lambda^2) = \left(1 - e^{-\tau\Lambda^2}\right)^k$ in (5.3.2), with $k \geq \frac{d}{2} + 1$.

[6] For massless spin-s, the PV-regulating characters to add to the physical character (e.g. (5.5.7),(C.6.2)) are $\hat{\Theta}_{s,n} = \Theta_{s,\nu_\phi^2+n\Lambda^2} - \Theta_{s-1,\nu_\xi^2+n\Lambda^2}$ where $\nu_\phi^2 = -(s-2+\frac{d}{2})^2$ and $\nu_\xi^2 = -(s-1+\frac{d}{2})^2$, based on (5.5.2) and (5.5.4).

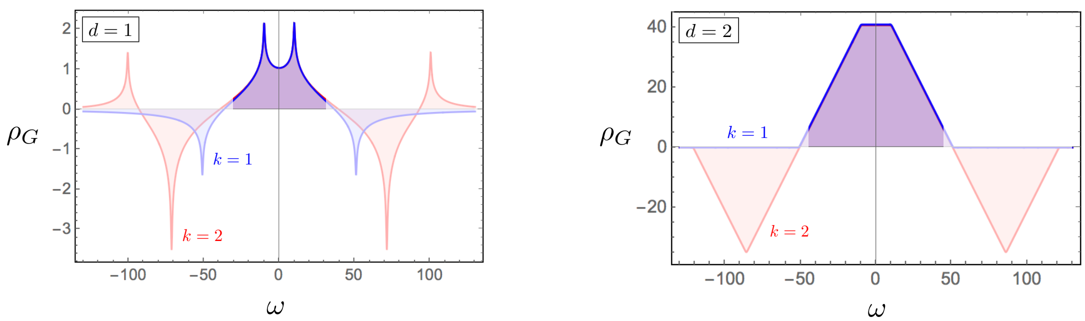


Figure 5.2.3: $\rho_{G,\Lambda}(\omega)$ for dS$_{d+1}$ scalar of mass $m^2 = (\frac{d}{2})^2 + \nu^2$, $\nu = 10$, for $d = 1, 2$ in $k = 1, 2$ Pauli-Villars regularizations (5.2.15). Faint part is unphysical UV regime $\omega \gtrsim \Lambda$. The peaks/kinks appearing at $\omega = \sqrt{\nu^2 + n\Lambda^2}$ are related to quasinormal mode resonances discussed in appendix C.1.3.

by $\tilde{\rho}_{\nu^2}(\omega)$, the exact $\rho_{G,\Lambda}(\omega)$ for general ω and k is $\rho_{G,\Lambda}(\omega) = \sum_{n=0}^{k}(-1)^n \binom{k}{n}\, \tilde{\rho}_{\nu^2+n\Lambda^2}(\omega)$, illustrated in fig. 5.2.3 for $k = 1, 2$. The $\omega \ll \Lambda$ result is independent of k up to rescaling of Λ. The result for massive higher-spin fields is the same up to an overall degeneracy factor D_s^d from (3.4.12).

To make sense of the southern static patch density $\rho_S(\omega)$ directly in the continuum, we *define* its regularized version by mirroring the formal relation (5.2.13), thus ensuring all of the well-defined features and physics this relation encapsulates are preserved:

$$\rho_{S,\Lambda}(\omega) \equiv \rho_{G,\Lambda}(\omega) = (5.2.16) \qquad (\omega > 0)\,. \tag{5.2.18}$$

This definition of the regularized static patch density evidently inherits all of the desirable properties of $\rho_G(\omega)$: manifest general covariance, independence of arbitrary choices such as brick wall boundary conditions, and exact analytic computability. The physical sensibility of this identification is also supported by the fact that the quasinormal mode expansion (5.2.5) of $\Theta(t)$ produces the physically expected static patch quasinormal resonance pole structure $\rho_S(\omega) = \frac{1}{2\pi}\sum_{r,\pm}\frac{N_r}{r \pm i\omega}$, cf. appendix C.1.3.

Putting things together in the way we obtained the formal relation (5.2.14), the correspondingly regularized version of the static patch thermal partition function (5.2.10) is then

$$\boxed{\log Z_{\text{bulk},\Lambda}(\beta) \equiv \int_0^\infty \frac{dt}{2t}\left(\frac{1+e^{-2\pi t/\beta}}{1-e^{-2\pi t/\beta}}\;\Theta_\Lambda(t)_{\text{bos}} - \frac{2\,e^{-\pi t/\beta}}{1-e^{-2\pi t/\beta}}\;\Theta_\Lambda(t)_{\text{fer}}\right)} \tag{5.2.19}$$

Note that if we take $k \geq \frac{d}{2}+1$, then $\Theta_\Lambda(t) \sim t^{2k-d}$ with $2k-d \geq 2$ and we can drop the $i\epsilon$ prescription (5.2.9). $Z_{\rm bulk}$ (or equivalently Θ) can be regularized in other ways, including by cutting off the integral at $t = \Lambda^{-1}$, or by cutting off the angular momentum as in the appendix C.1.2, or by dimensional regularization. For most of the paper we will use yet another variant, defined in section 5.3, equivalent, like Pauli-Villars, to a manifestly covariant heat-kernel regularization of the path integral.

In view of the above observations, $Z_{{\rm bulk},\Lambda}(\beta)$ is naturally interpreted as a well-defined, covariantly regularized and ambiguity-free *definition* of the static patch ideal gas thermal partition in the continuum. However we refrain from denoting $Z_{\rm bulk}(\beta)$ as $\operatorname{Tr}_{S,\Lambda} e^{-\beta H_S}$, because it is not constructed as an actual sum over states of some definite regularized static patch Hilbert space $\mathcal{H}_{S,\Lambda}$. This (together with the role of quasinormal modes) is also why we referred to $Z_{\rm bulk}(\beta)$ as a "quasi"-canonical partition function in the introduction.

5.2.3 Example computations

In this section we illustrate the use and usefulness of the character formalism by computing some examples of bulk thermodynamic quantities at the equilibrium inverse temperature $\beta = 2\pi$ of the static patch. The precise relation of these quantities with their Euclidean counterparts will be determined in 5.3-5.5 and 5.7.

Character formulae for bulk thermodynamic quantities at $\beta = 2\pi$

At $\beta = 2\pi$, the bulk free energy, energy, entropy and heat capacity are obtained by taking the appropriate derivatives of (5.2.6) and putting $\beta = 2\pi$, using the standard thermodynamic relations $F = -\frac{1}{\beta}\log Z$, $U = -\partial_\beta \log Z$, $S = \log Z + \beta U$, $C = -\beta\partial_\beta S$. Denoting $q \equiv e^{-t}$,

$$\log Z_{\rm bulk} = \int_0^\infty \frac{dt}{2t}\biggl(\frac{1+q}{1-q}\,\Theta_{\rm bos} - \frac{2\sqrt{q}}{1-q}\,\Theta_{\rm fer}\biggr), \tag{5.2.20}$$

$$2\pi U_{\rm bulk} = \int_0^\infty \frac{dt}{2}\biggl(-\frac{2\sqrt{q}}{1-q}\,\frac{\sqrt{q}}{1-q}\,\Theta_{\rm bos} + \frac{1+q}{1-q}\,\frac{\sqrt{q}}{1-q}\,\Theta_{\rm fer}\biggr), \tag{5.2.21}$$

and similarly for $S_{\rm bulk}$ and $C_{\rm bulk}$. The characters Θ for general massive representation are given by (3.4.11), and for (partially massless) (s,s') representations by (3.4.20) with $t = s'$. Regularization is implicit here.

Leading divergent term

The leading $t \to 0$ divergence of the scalar character (5.2.4) is $\Theta(t) \sim 2/t^d$. For more general representations this becomes $\Theta(t) \sim 2n/t^d$ with n the number of on-shell internal (spin) degrees of freedom. The generic leading divergent term of the bulk (free) energy is then given by $F_{\rm bulk}, U_{\rm bulk} \sim -\frac{1}{\pi}(n_{\rm bos} - n_{\rm fer}) \int \frac{dt}{t^{d+2}} \sim \pm\Lambda^{d+1}\ell^d$, while for the bulk heat capacity and entropy we get $C_{\rm bulk}, S_{\rm bulk} \sim (\frac{1}{3}n_{\rm bos} + \frac{1}{6}n_{\rm fer}) \int \frac{dt}{t^d} \sim +\Lambda^{d-1}\ell^{d-1}$, where we reinstated the dS radius ℓ. In particular $S_{\rm bulk} \sim +\Lambda^{d-1} \times$ horizon area, consistent with an entanglement entropy area law. The energy diverges more strongly because we included the QFT zero point energy term in its definition, which drops out of S and C.

Coefficient of log-divergent term

The coefficient of the logarithmically divergent part of these thermodynamic quantities is universal. A pleasant feature of the character formalism is that this coefficient can be read off trivially as the coefficient of the $1/t$ term in the small-t expansion of the integrand, easily computed for any representation. In odd $d+1$, the integrand is even in t, so log-divergences are absent. In even $d+1$, the integrand is odd in t, so generically we do get a log-divergence $= a\log\Lambda$. For example the $\log Z$ integrand for a dS$_2$ scalar is expanded as

$$\frac{1}{2t}\frac{1+e^{-t}}{1-e^{-t}}\,\frac{e^{-t(\frac{1}{2}+i\nu)}+e^{-t(\frac{1}{2}-i\nu)}}{1-e^{-t}} = \frac{2}{t^3} + \frac{\frac{1}{12}-\nu^2}{t} + \cdots \quad \Rightarrow \quad a = \tfrac{1}{12} - \nu^2 \,. \tag{5.2.22}$$

For a $\Delta = \frac{d}{2} + i\nu$ spin-s particle in even $d+1$, the $\log\Lambda$ coefficient for $U_{\rm bulk}$ is similarly read off as $a_{U_{\rm bulk}} = -D_s^d \frac{1}{\pi(d+1)!}\prod_{n=0}^d(\Delta - n)$. For a conformally coupled scalar, $\nu = i/2$, so $a_{U_{\rm bulk}} = 0$. Some examples of $a_{S_{\rm bulk}} = a_{\log Z_{\rm bulk}}$ in this case are

$d+1$	2	4	6	8	10	$\cdots$	100	$\cdots$	1000	$\cdots$
$a_{S_{\rm bulk}}$	$\frac{1}{3}$	$-\frac{1}{90}$	$\frac{1}{756}$	$-\frac{23}{113400}$	$\frac{263}{7484400}$	$\cdots$	-8.098×10^{-34}	$\cdots$	-3.001×10^{-306}	$\cdots$

Finite part and exact results

• *Energy:* For future reference (comparison to previously obtained results in section 5.7), we consider dimensional regularization here. The absence of a $1/t$ factor in the integral (5.2.21) for

U_{bulk} then allows straightforward evaluation for general d. For a scalar of mass $m^2 = (\frac{d}{2})^2 + \nu^2$,

$$U_{\text{bulk}}^{\text{fin}} = \frac{m^2 \cosh(\pi\nu)\,\Gamma(\frac{d}{2}+i\nu)\,\Gamma(\frac{d}{2}-i\nu)}{2\pi\,\Gamma(d+2)\cos(\frac{\pi d}{2})} \qquad \text{(dim reg)}\,. \tag{5.2.23}$$

For example for $d = 2$, this becomes

$$U_{\text{bulk}}^{\text{fin}} = -\frac{1}{12}(\nu^2+1)\nu\coth(\pi\nu)\,. \tag{5.2.24}$$

• *Free energy:* The UV-finite part of the $\log Z_{\text{bulk}}$ integral (5.2.20) for a massive field in even d can be computed simply by extending the integration contour to the real line avoiding the pole, closing the contour and summing residues. For example for a $d = 2$ scalar this gives

$$\log Z_{\text{bulk}}^{\text{fin}} = \frac{\pi\nu^3}{6} - \sum_{k=0}^{2} \frac{\nu^k}{k!}\,\frac{\text{Li}_{3-k}(e^{-2\pi\nu})}{(2\pi)^{2-k}}\,, \tag{5.2.25}$$

where Li_n is the polylogarithm, $\text{Li}_n(x) \equiv \sum_{k=1}^{\infty} x^k/k^n$. For future reference, note that

$$\text{Li}_1(e^{-2\pi\nu}) = -\log(1-e^{-2\pi\nu})\,, \qquad \text{Li}_0(e^{-2\pi\nu}) = \frac{1}{e^{2\pi\nu}-1} = \tfrac{1}{2}\coth(\pi\nu) - \tfrac{1}{2}\,. \tag{5.2.26}$$

For odd d, the character does not have an even analytic extension to the real line, so a different method is needed to compute $\log Z_{\text{bulk}}$. The exact evaluation of arbitrary character integrals, for any d and any $\Theta(t)$ is given in (C.2.19) in terms of Hurwitz zeta functions. Simple examples are given in (C.2.21)-(C.2.22). In (C.2.19) we use the covariant regularization scheme introduced in section 5.3. Conversion to PV regularization is obtained from the finite part as explained below.

• *Entropy*: Combined with our earlier result for the bulk energy U_{bulk}, the above also gives the finite part of the bulk entropy $S_{\text{bulk}} = \log Z_{\text{bulk}} + 2\pi U_{\text{bulk}}$. In the Pauli-Villars regularization (5.2.15), the UV-divergent part is obtained from the finite part by mirrorring (5.2.15). For example for $k = 1$, $S_{\text{bulk},\Lambda} = S_{\text{bulk}}^{\text{fin}}|_{\nu^2} - S_{\text{bulk}}^{\text{fin}}|_{\nu^2+\Lambda^2}$. For the $d = 2$ example this gives for $\nu \ll \Lambda$

$$S_{\text{bulk},\Lambda} = \frac{\pi}{6}(\Lambda-\nu) - \frac{\pi}{3}\,\nu\,\text{Li}_0(e^{-2\pi\nu}) - \sum_{k=0}^{3} \frac{\nu^k}{k!}\,\frac{\text{Li}_{3-k}(e^{-2\pi\nu})}{(2\pi)^{2-k}}\,, \tag{5.2.27}$$

where we used (5.2.26). $S_{\rm bulk}$ decreases monotonically with $m^2 = 1+\nu^2$. In the massless limit $m \to 0$, it diverges logarithmically: $S_{\rm bulk} = -\log m + \cdots$. For $\nu \gg 1$, $S_{\rm bulk} = \frac{\pi}{6}(\Lambda - \nu)$ up to exponentially small corrections. Thus $S_{\rm bulk} > 0$ within the regime of validity of the low-energy field theory, consistent with its quasi-canonical/entanglement entropy interpretation. For a conformally coupled scalar $\nu = \frac{i}{2}$, this gives $S_{\rm bulk}^{\rm fin} = \frac{3\zeta(3)}{16\pi^2} - \frac{\log(2)}{8}$.

Quasinormal mode expansion

Substituting the quasinormal mode expansion (5.2.5),

$$\Theta(t) = \sum_r N_r e^{-rt} \tag{5.2.28}$$

in the PV-regularized $\log Z_{\rm bulk}(\beta)$ (5.2.19), rescaling $t \to \frac{\beta}{2\pi} t$, and using (C.2.31) gives

$$\log Z_{\rm bulk}(\beta) = \sum_r N_r^{\rm bos} \log \frac{\Gamma(br+1)}{(b\mu)^{br}\sqrt{2\pi br}} - N_r^{\rm fer} \log \frac{\Gamma(br+\frac{1}{2})}{(b\mu)^{br}\sqrt{2\pi}} \, , \qquad b \equiv \frac{\beta}{2\pi} \, . \tag{5.2.29}$$

Truncating the integral to the IR part (C.2.31) is justified because the Pauli-Villars sum (5.2.15) cancels out the UV part. The dependence on μ likewise cancels out, as do some other terms, but it is useful to keep the above form. At the equilibrium $\beta = 2\pi$, $\log Z_{\rm bulk}$ is given by (5.2.29) with $b = 1$. This provides a PV-regularized version of the quasinormal mode expansion of [163]. Since it is covariantly regularized, it does not require matching to a local heat kernel expansion. Moreover it applies to general particle content, including spin $s \geq 1$.[7]

QNM expansions of other bulk thermodynamic quantities are readily derived from the eq. (5.2.29) by taking derivatives $\beta\partial_\beta = b\partial_b = \mu\partial_\mu + r\partial_r$. For example $S_{\rm bulk} = (1 - \beta\partial_\beta) \log Z_{\rm bulk}|_{\beta=2\pi}$ is

$$S_{\rm bulk} = \sum_r N_r^{\rm bos} s_{\rm bos}(r) + N_r^{\rm fer} s_{\rm fer}(r) \tag{5.2.30}$$

[7] The expansion of [163] pertains to $Z_{\rm PI}$ for $s \leq \frac{1}{2}$. In the following sections we show $Z_{\rm PI} = Z_{\rm bulk}$ for $s \leq \frac{1}{2}$ but not for $s \geq 1$. Hence in general the QNM expansion of [163] computes $Z_{\rm bulk}$, not $Z_{\rm PI}$.

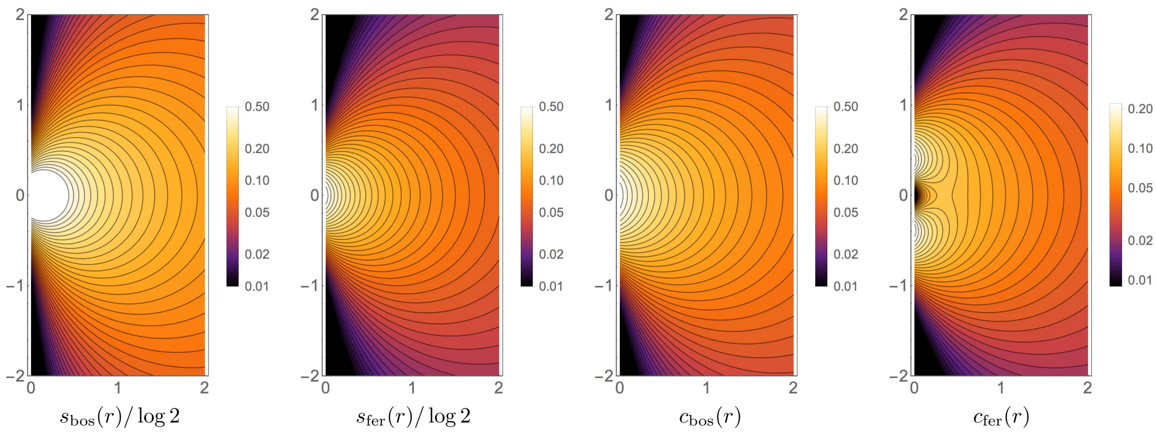


Figure 5.2.4: Contribution to $\beta = 2\pi$ bulk entropy and heat capacity of a quasinormal mode $\propto e^{-rT}$, $r \in \mathbb{C}$, $\mathrm{Re}\, r > 0$. Only the real part is shown here because complex r come in conjugate pairs $r_{n,\pm} = \frac{d}{2} + n \pm i\nu$. The harmonic oscillator case corresponds to the imaginary axis.

where the entropy $s(r)$ carried by a single QNM $\propto e^{-rT}$ at $\beta = 2\pi$ is given by

$$s_{\rm bos}(r) = r + (1 - r\partial_r) \log \frac{\Gamma(r+1)}{\sqrt{2\pi r}} \,, \qquad s_{\rm fer}(r) = -r - (1 - r\partial_r) \log \frac{\Gamma(r+\frac{1}{2})}{\sqrt{2\pi}} \,. \quad (5.2.31)$$

Note the μ-dependence has dropped out, reflecting the fact that the contribution of each *individual* QNM to the entropy is UV-finite, not requiring any regularization. For massive representations, r can be complex, but will always appear in a conjugate pair $r_{n\pm} = \frac{d}{2} + n \pm i\nu$. Taking this into account, all contributions to the entropy are real and positive for the physical part of the PV-extended spectrum. The small and large r asymptotics are

$$r \to 0 : \; s_{\rm bos} \to \frac{1}{2} \log \frac{e}{2\pi r} \,, \quad s_{\rm fer} \to \frac{\log 2}{2} \,, \qquad r \to \infty : \; s_{\rm bos} \to \frac{1}{6\, r} \,, \quad s_{\rm fer} \to \frac{1}{12\, r} \,. \quad (5.2.32)$$

The QNM entropies at general β are obtained simply by replacing

$$r \to \frac{\beta}{2\pi} r \,. \quad (5.2.33)$$

The entropy of a normal bosonic mode of frequency ω, $\tilde{s}(\omega) = -\log(1 - e^{-\beta\omega}) + \frac{\beta\omega}{e^{\beta\omega}-1}$, is recovered for complex conjugate pairs $r_\pm$ in the scaling limit $\beta \to 0$, $\beta\nu = \omega$ fixed, and likewise for fermions. At any finite β, the $n \to \infty$ UV tail of QNM contributions is markedly different however. Instead of falling off exponentially, if falls off as $s \sim 1/n$. PV or any other regularization effectively cuts off the sum at $n \sim \Lambda\ell$, so since $N_n \sim n^{d-1}$, $S_{\rm bulk} \sim \Lambda^{d-1}\ell^{d-1}$.

The bulk heat capacity $C_{\rm bulk} = -\beta\partial_\beta S_{\rm bulk}$, so the heat capacity of a QNM at $\beta = 2\pi$ is

$$c(r) = -r\partial_r s(r)\,. \tag{5.2.34}$$

The real part of $s(r)$ and $c(r)$ on the complex r-plane are shown in fig. 5.2.4.

An application of the quasinormal expansion

The above QNM expansions are less useful for exact computations of thermodynamic quantities than the direct integral evaluations discussed earlier, but can be very useful in computations of certain UV-finite quantities. A simple example is the following. In thermal equilibrium with a 4D dS static patch horizon, which set of particle species has the largest bulk heat capacity: (A) six conformally coupled scalars + graviton, (B) four photons? The answer is not obvious, as both have an equal number of local degrees of freedom: $6+2 = 4\times 2 = 8$. One could compute each in full, but the above QNM expansions offers a much easier way to get the answer. From (5.2.4) and (C.6.2) we read off the scalar and massless spin-s characters:

$$\Theta_0 = \frac{q+q^2}{(1-q)^3}\,, \qquad \Theta_s = \frac{2(2s+1)\,q^{s+1} - 2(2s-1)\,q^{s+2}}{(1-q)^3}\,, \tag{5.2.35}$$

where $q = e^{-|t|}$. We see $\Theta_A - \Theta_B = \Theta_2 + 6\,\Theta_0 - 4\,\Theta_1 = 6\,q$, so Θ_A and Θ_B are *almost* exactly equal: A has just 6 more quasinormal modes than B, all with $r = 1$. Thus, using (5.2.34),

$$C^A_{\rm bulk} - C^B_{\rm bulk} = 6\cdot c_{\rm bos}(1) = \pi^2 - 9\,. \tag{5.2.36}$$

Pretty close, but $\pi > 3$, so A wins. The difference is $\Delta C \approx 0.87$. Along similar lines, $\Delta S = 6\,s_{\rm bos}(1) = 3(2\gamma + 1 - \log(2\pi)) \approx 0.95$.

Another UV-finite example: relative entropies of graviton, photon, neutrino

Less trivial to compute but more real-world in flavor is the following UV-finite linear combination of the 4D graviton, photon, and (assumed massless) neutrino bulk entropies:

$$S_{\mathsf{graviton}} + \tfrac{60}{7} S_{\mathsf{neutrino}} - \tfrac{37}{7} S_{\mathsf{photon}} = \tfrac{48}{7}\zeta'(-1) - \tfrac{60}{7}\zeta'(-3) + 6\gamma + \tfrac{149}{56} - \tfrac{33}{14}\log(2\pi) \approx 0.61 \tag{5.2.37}$$

Finiteness can be checked from the small-t expansion of the total integrand computing this, and the integral can then be evaluated along the lines of (C.2.15)-(C.2.16). We omit the details.

Vasiliev higher-spin example

Non-minimal Vasiliev higher-spin gravity on dS$_4$ has a single conformally coupled scalar and a tower of massless spin-s particles of all spins $s = 1, 2, 3, \ldots$. The prospect of having to compute bulk thermodynamics for this theory by brick wall or other approaches mentioned in appendix C.4.3 would be terrifying. Let us compare this to the character approach. The *total* character obtained by summing the characters of (5.2.35) takes a remarkably simple form:

$$\Theta_{\rm tot} \;=\; \Theta_0 + \sum_{s=1}^{\infty} \Theta_s \;=\; 2 \cdot \left(\frac{q^{1/2} + q^{3/2}}{(1-q)^2} \right)^2 - \frac{q}{(1-q)^2} \;=\; \frac{q + q^3}{(1-q)^4} + 3 \cdot \frac{2q^2}{(1-q)^4} \,. \tag{5.2.38}$$

The first expression is two times the square of the character of a 3D conformally coupled scalar, plus the character of 3D conformal higher-spin gravity (5.9.17).[8] The second expression equals the character of one $\nu = i$ and three $\nu = 0$ scalars on dS$_5$. Treating the character integral as such, we immediately get, in $k = 3$ Pauli-Villars regularization (5.2.15),

$$\begin{aligned} \log Z_{\rm bulk}^{\rm div} &= a_0 \Lambda^5 + a_2 \Lambda^3 - a_4 \Lambda \,, & \log Z_{\rm bulk}^{\rm fin} &= \tfrac{\zeta(5)}{4\pi^4} - \tfrac{\zeta(3)}{24\pi^2} \\ S_{\rm bulk}^{\rm div} &= \tfrac{25}{4} a_2 \Lambda^3 - \tfrac{103}{20} a_4 \Lambda \,, & S_{\rm bulk}^{\rm fin} &= \tfrac{\zeta(5)}{4\pi^4} - \tfrac{\zeta(3)}{24\pi^2} + \tfrac{1}{20} \,. \end{aligned} \tag{5.2.39}$$

[8] For AdS$_4$, the analogous $\Theta_{\rm tot}$ equals *one* copy of the 3D scalar character squared, reflecting the single-trace spectrum of its holographic dual $U(N)$ model (see section 5.9.2). The dS counterpart thus encodes the single-trace spectrum of two copies of this 3D CFT + 3D CHS gravity, reminiscent of [32]. This is generalized by (5.9.16).

where $a_0 = \frac{1-4\sqrt{2}+3\sqrt{3}}{10}\,\pi \approx 0.17$, $a_2 = -\frac{1-2\sqrt{2}+\sqrt{3}}{12}\,\pi \approx 0.025$, and $a_4 = \frac{3-3\sqrt{2}+\sqrt{3}}{48}\,\pi \approx 0.032$. The tower of higher-spin particles alters the *bulk* UV dimensionality much like a tower of KK modes would. (We will later see edge "corrections" rather dramatically alter this.)

5.3 Sphere partition function for scalars and spinors

5.3.1 Problem and result

In this section we consider the one-loop Gaussian Euclidean path integral $Z_{\rm PI}^{(1)}$ of scalar and spinor field fluctuations on the round sphere. For a free scalar of mass m^2 on S^{d+1},

$$Z_{\rm PI} = \int \mathcal{D}\phi \, e^{-\frac{1}{2}\int \phi(-\nabla^2+m^2)\phi} \,, \tag{5.3.1}$$

A convenient UV-regularized version is defined using standard heat kernel methods [178]:

$$\log Z_{{\rm PI},\epsilon} = \int_0^\infty \frac{d\tau}{2\tau} \, e^{-\epsilon^2/4\tau} \; {\rm Tr}\; e^{-\tau(-\nabla^2+m^2)} \,. \tag{5.3.2}$$

The insertion $e^{-\epsilon^2/4\tau}$ implements a UV cutoff at length scale $\sim \epsilon$. We picked this regulator for convenience in the derivation below. We could alternatively insert the PV regulator of footnote 5, which would reproduce the PV regularization (5.2.15). However, being uniformly applicable to all dimensions, the above regulator is more useful for the purpose of deriving general evaluation formulae, as in appendix C.2.

In view of (5.2.7) we wish to compare $Z_{\rm PI}$ to the corresponding Wick-rotated dS static patch bulk thermal partition function $Z_{\rm bulk}(\beta)$ (5.2.6), at the equilibrium inverse temperature $\beta = 2\pi$. Here and henceforth, $Z_{\rm bulk}$ by default means $Z_{\rm bulk}(2\pi)$:

$$\log Z_{\rm bulk} \equiv \int_0^\infty \frac{dt}{2t} \bigg(\frac{1+e^{-t}}{1-e^{-t}} \; \Theta(t)_{\rm bos} \; - \; \frac{2\,e^{-t/2}}{1-e^{-t}} \; \Theta(t)_{\rm fer} \bigg) \tag{5.3.3}$$

Below we show that for free scalars and spinors,

$$\boxed{Z_{\rm PI} = Z_{\rm bulk}} \tag{5.3.4}$$

with the specific regularization (5.3.2) for $Z_{\rm PI}$ mapping to a specific regularization (5.3.9) for $Z_{\rm bulk}$. The relation is exact, for any ϵ. This makes the physical expectation (5.2.7) precise, and shows that for scalars and spinors, there are in fact no edge corrections.

In appendix C.2 we provide a simple recipe for extracting both the UV and IR parts in the $\epsilon \to 0$ limit in the above regularization, directly from the unregularized form of the character formula (5.3.3). This yields the general closed-form solution $\boxed{(C.2.19)}$ for the regularized $Z_{\rm PI}$ in terms of Hurwitz zeta functions. The heat kernel coefficient invariants are likewise read off from the character using (C.2.20). For simple examples see (C.2.21), (C.2.22), (C.2.32)-(C.2.33).

5.3.2 Derivation

The derivation is straightforward:

Scalars:

The eigenvalues of $-\nabla^2$ on a sphere of radius $\ell \equiv 1$ are $\lambda_n = n(n+d)$, $n \in \mathbb{N}$, with degeneracies D_n^{d+2} given by (2.2.3), that is $D_n^{d+2} = \binom{n+d+1}{d+1} - \binom{n+d-1}{d+1}$. Thus (5.3.2) can be written as

$$\log Z_{\rm PI} = \int_0^\infty \frac{d\tau}{2\tau}\, e^{-\epsilon^2/4\tau} e^{-\tau\nu^2} \sum_{n=0}^\infty D_n^{d+2}\, e^{-\tau(n+\frac{d}{2})^2}\,, \qquad \nu \equiv \sqrt{m^2 - \tfrac{d^2}{4}}\,. \tag{5.3.5}$$

To perform the sum over n, we use the Hubbard-Stratonovich trick, i.e. we write

$$\sum_{n=0}^\infty D_n^{d+2}\, e^{-\tau(n+\frac{d}{2})^2} = \int_A du\, \frac{e^{-u^2/4\tau}}{\sqrt{4\pi\tau}}\, f(u)\,, \qquad f(u) \equiv \sum_{n=0}^\infty D_n^{d+2}\, e^{iu(n+\frac{d}{2})}\,. \tag{5.3.6}$$

with integration contour $A = \mathbb{R} + i\delta$, $\delta > 0$, as shown in fig. 5.3.1. The sum evaluates to

$$f(u) = \frac{1+e^{iu}}{1-e^{iu}}\, \frac{e^{i\frac{d}{2}u}}{(1-e^{iu})^d}\,, \tag{5.3.7}$$

We first consider the case $m > \frac{d}{2}$, so ν is real and positive. Then, keeping $\mathrm{Im}\, u = \delta < \epsilon$, we can perform the τ-integral first in (5.3.5) to get

$$\log Z_{\rm PI} = \int_A \frac{du}{2\sqrt{u^2+\epsilon^2}}\, e^{-\nu\sqrt{u^2+\epsilon^2}}\, f(u)\,. \tag{5.3.8}$$

Deforming the contour by folding it up along the two sides of the branch cut to contour B in

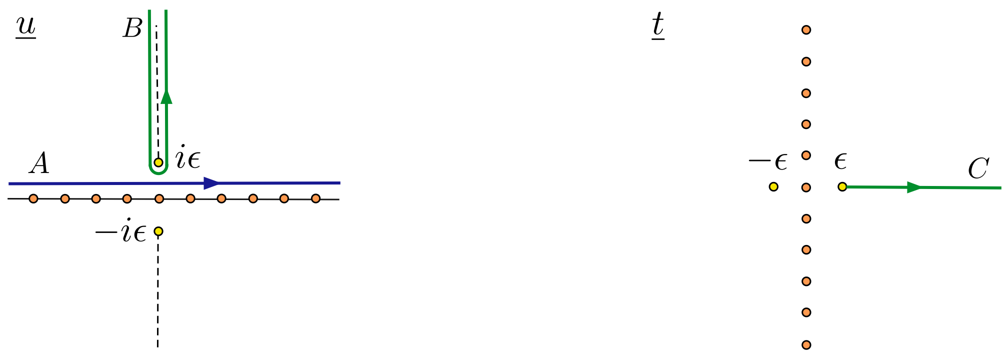


Figure 5.3.1: Integration contours for $Z_{\rm PI}$. Orange dots are poles, yellow dots branch points.

fig. 5.3.1, changing variables $u = it$ and using that the square root takes opposite signs on both sides of the cut, we transform this to an integral over C in fig. 5.3.1:

$$\log Z_{\rm PI} = \int_\epsilon^\infty \frac{dt}{2\sqrt{t^2-\epsilon^2}} \frac{1+e^{-t}}{1-e^{-t}} \frac{e^{-\frac{d}{2}t+i\nu\sqrt{t^2-\epsilon^2}} + e^{-\frac{d}{2}t-i\nu\sqrt{t^2-\epsilon^2}}}{(1-e^{-t})^d}\,, \tag{5.3.9}$$

The result for $0 < m \leq \frac{d}{2}$, i.e. $\nu = i\mu$ with $0 \leq \mu < \frac{d}{2}$ can be obtained from this by analytic continuation. Putting $\epsilon = 0$, this formally becomes

$$\log Z_{\rm PI} = \int_0^\infty \frac{dt}{2t} \frac{1+e^{-t}}{1-e^{-t}}\,\Theta(t)\,, \qquad \Theta(t) = \frac{e^{-(\frac{d}{2}-i\nu)t} + e^{-(\frac{d}{2}+i\nu)t}}{(1-e^{-t})^d}\,, \tag{5.3.10}$$

which we recognize as (5.3.3) with $\Theta(t)$ the scalar character (5.2.4). Thus we conclude that for scalars, $Z_{\rm PI} = Z_{\rm bulk}$, with $Z_{\rm PI}$ regularized as in (5.3.2) and $Z_{\rm bulk}$ as in (5.3.9).

Spinors:

For a Dirac spinor field of mass m we have $Z_{\rm PI} = \int \mathcal{D}\psi\, e^{-\int \bar\psi(\nabla\!\!\!/+m)\psi}$. The relevant formulae for spectrum and degeneracies for general d can be found in the section 2.2. For concreteness we just consider the case $d = 3$ here, but the conclusions are valid for Dirac spinors in general. The spectrum of $\nabla\!\!\!/ + m$ on S^4 is $\lambda_n = m \pm (n+2)i$, $n \in \mathbb{N}$, with degeneracy $D^5_{n+\frac{1}{2},\frac{1}{2}} = 4\binom{n+3}{3}$, so $Z_{\rm PI}$ regularized as in (5.3.2) is given by

$$\log Z_{\rm PI} = -\int_0^\infty \frac{d\tau}{\tau} e^{-\epsilon^2/4\tau} \sum_{n=0}^\infty 4\binom{n+3}{3}\, e^{-\tau((n+2)^2+m^2)} \tag{5.3.11}$$

Following the same steps as for the scalar case, this can be rewritten as

$$\log Z_{\sf PI} = -\int_\epsilon^\infty \frac{dt}{2\sqrt{t^2-\epsilon^2}} \frac{2\,e^{-t/2}}{1-e^{-t}} \cdot 4 \cdot \frac{e^{-\frac{3}{2}t + im\sqrt{t^2-\epsilon^2}} + e^{-\frac{3}{2}t - im\sqrt{t^2-\epsilon^2}}}{(1-e^{-t})^3} \,. \tag{5.3.12}$$

Putting $\epsilon = 0$, this formally becomes

$$\log Z_{\rm PI} = -\int_0^\infty \frac{dt}{2t} \frac{2\,e^{-t/2}}{1-e^{-t}} \,\Theta(t)\,, \qquad \Theta(t) = 4 \cdot \frac{e^{-(\frac{3}{2}+im)t} + e^{-(\frac{3}{2}-im)t}}{(1-e^{-t})^3}\,, \tag{5.3.13}$$

which we recognize as the fermionic (5.3.3) with $\Theta(t)$ the character of the $\Delta = \frac{3}{2} + im$ unitary SO$(1,4)$ representation carried by the single-particle Hilbert space of a Dirac spinor quantized on dS$_4$, given by twice the character (3.4.11) of the irreducible representation $\mathcal{F}_{\Delta,\mathbf{s}}$ with $\mathbf{s} = (\frac{1}{2})$. Thus we conclude $Z_{\rm PI} = Z_{\rm bulk}$. The comment below (5.4.16) generalizes this to all d.

5.4 Massive higher spins

We first formulate the problem, explaining why it is not nearly as simple as one might have hoped, and then state the result, which turns out to be much simpler than one might have feared. The derivation of the result is detailed in appendix C.5.1.

5.4.1 Problem

Consider a massive spin-$s \geq 1$ field, more specifically a totally symmetric tensor field $\phi_{\mu_1\cdots\mu_s}$ on dS$_{d+1}$ satisfying the Fierz-Pauli equations of motion:

$$(-\nabla^2 + \overline{m}_s^2)\phi_{\mu_1\cdots\mu_s} = 0\,, \qquad \nabla^\nu \phi_{\nu\mu_1\cdots\mu_{s-1}} = 0\,, \qquad \phi^\nu{}_{\nu\mu_1\cdots\mu_{s-2}} = 0\,. \tag{5.4.1}$$

Upon quantization, the global single-particle Hilbert space furnishes a massive spin-s representation of SO$(1,d+1)$ with $\Delta = \frac{d}{2} + i\nu$, related to the effective mass $\overline{m}_s$ appearing above (see e.g. [89]), and to the more commonly used definition of mass m (see e.g. [71]) as

$$\overline{m}_s^2 = (\tfrac{d}{2})^2 + \nu^2 + s\,, \qquad m^2 = (\tfrac{d}{2} + s - 2)^2 + \nu^2 = (\Delta + s - 2)(d + s - 2 - \Delta)\,. \tag{5.4.2}$$

The massive spin-s *bulk* thermal partition function is immediately obtained by substituting the massive spin-s character (3.4.12) into the character formula (5.3.3) for $Z_{\rm bulk}$. For $d \geq 3$,[9]

$$\log Z_{\rm bulk} = \int_0^\infty \frac{dt}{2t} \frac{1+q}{1-q} \, D_s^d \cdot \frac{q^{\frac{d}{2}+i\nu} + q^{\frac{d}{2}-i\nu}}{(1-q)^d} \,, \qquad q = e^{-t} \,, \tag{5.4.3}$$

with spin degeneracy factor read off from (2.2.3) or (2.2.6).

The corresponding free massive spin-s Euclidean path integral on S^{d+1} takes the form

$$Z_{\rm PI} = \int \mathcal{D}\Phi \, e^{-S_E[\Phi]}. \tag{5.4.4}$$

where Φ includes at least ϕ. However it turns out that in order to write down a local, manifestly covariant action for massive fields of general spin s, one also needs to include a tower of auxiliary Stueckelberg fields of all spins $s' < s$ [58], generalizing the familiar Stueckelberg action (C.5.19) for massive vector fields. These come with gauge symmetries, which in turn require the introduction of a gauge fixing sector, with ghosts of all spins $s' < s$. The explicit form of the action and gauge symmetries is known, but intricate [58].

Classically, variation of the action with respect the Stueckelberg fields merely enforces the transverse-traceless (TT) constraints in (5.4.1), after which the gauge symmetries can be used to put the Stueckelberg fields equal to zero. One might therefore hope the intimidating off-shell $Z_{\rm PI}$ (5.4.4) likewise collapses to just the path integral $Z_{\rm TT}$ over the TT modes of ϕ with kinetic term given by the equations of motion (5.4.1). This is easy to evaluate. The TT eigenvalue spectrum on the sphere follows from SO$(d+2)$ representation theory. As detailed in eqs. (C.5.1)-(C.5.2), we can then follow the same steps as in section 5.3, ending up with[10]

$$\log Z_{\rm TT} = \int_0^\infty \frac{dt}{2t} \left(q^{i\nu} + q^{-i\nu}\right) \sum_{n \geq s} D_{n,s}^{d+2} \, q^{\frac{d}{2}+n} \,. \tag{5.4.5}$$

Here $D_{n,s}^{d+2}$ is the dimension of the SO$(d+2)$ representation labeled by the two-row Young diagram $\mathbb{Y}_{n,s}$.

Unfortunately, $Z_{\rm TT}$ is *not* equal to $Z_{\rm PI}$ on the sphere. The easiest way to see this is to consider an example in odd spacetime dimensions, such as (C.5.3), and observe the result has a

[9]For $d=2$, the single-particle Hilbert space splits into $(\Delta, \pm s)$ with $D_{\pm s}^2 = 1$, so $D_s^2 \to \sum_\pm D_{\pm s}^2 = 2$.

[10]For $d=2$, $D_{n,s}^4 \to \sum_\pm D_{n,\pm s}^4 = 2D_{n,s}^4$.

logarithmic divergence. A manifestly covariant local QFT path integral on an odd-dimensional sphere cannot possibly have logarithmic divergences. Therefore $Z_{\rm PI} \neq Z_{\rm TT}$. The appearance of such nonlocal divergences in $Z_{\rm TT}$ can be traced to the existence of (normalizable) zeromodes in tensor decompositions on the sphere [128, 139]. For example the decomposition $\phi_\mu = \phi^T_\mu + \nabla_\mu \varphi$ has the constant φ mode as a zeromode, $\phi_{\mu\nu} = \phi^{\rm TT}_{\mu\nu} + \nabla_{(\mu}\varphi_{\nu)} + g_{\mu\nu}\varphi$ has conformal Killing vector zeromodes, and $\phi_{\mu_1\cdots\mu_s} = \phi^{\rm TT}_{\mu_1\cdots\mu_s} + \nabla_{(\mu_1}\varphi_{\mu_2\cdots\mu_s)} + g_{(\mu_1\mu_2}\varphi_{\mu_3\cdots\mu_s)}$ has rank $s-1$ conformal Killing tensor zeromodes. As shown in [128, 139], this implies $\log Z_{\rm TT}$ contains a nonlocal UV-divergent term $c_s \log \Lambda$, where c_s is the number of rank $s-1$ conformal Killing tensors. This divergence cannot be canceled by a local counterterm. Instead it must be canceled by contributions from the non-TT part. Thus, in principle, the full off-shell path integral must be carefully evaluated to obtain the correct result. Computing $Z_{\rm PI}$ for general s on the sphere is not as easy as one might have hoped.

5.4.2 Result

Rather than follow a brute-force approach, we obtain $Z_{\rm PI}$ in appendix C.5.1 by a series of relatively simple observations. In fact, upon evaluating the sum in (5.4.5), writing it in a way that brings out a term $\log Z_{\rm bulk}$ as in (5.4.3), and observing a conspicuous finite sum of terms bears full responsibility for the inconsistency with locality, the answer suggests itself right away: the non-TT part restores locality simply by canceling this finite sum. This turns out to be equivalent to the non-TT part effectively extending the sum $n \geq s$ in (5.4.5) to $n \geq -1$:[11]

$$\log Z_{\rm PI} = \int_0^\infty \frac{dt}{2t} \left(q^{i\nu} + q^{-i\nu}\right) \sum_{n \geq -1} D^{d+2}_{n,s} \, q^{\frac{d}{2}+n} \,, \tag{5.4.6}$$

where $D^{d+2}_{n,s}$ is given by eq. (2.2.4). For $n < s$, this is no longer the dimension of an SO$(d+2)$ representation, but it can be rewritten as *minus* the dimension of such a representation, as $D^{d+2}_{n,s} = -D^{d+2}_{s-1,n+1}$. This extension also turns out to be exactly what is needed for consistency with the unitarity bound (C.5.18) and more refined unitarity considerations. A limited amount of explicit path integral considerations combined with the observation that the coefficients

[11] For $d=2$ use (5.4.14): $D^4_{n,s} \to \sum_\pm D^4_{n,\pm s} = 2D^4_{n,s}$ for $n > -1$, and $D^4_{-1,s} \to \sum_\pm \frac{1}{2} D^4_{-1,\pm s} = D^4_{-1,s} = D^4_{s-1}$.

$D^{d+2}_{s-1,n+1}$ count conformal Killing tensor mode mismatches between ghosts and longitudinal modes then suffice to establish this is indeed the correct answer. We refer to appendix C.5.1 for details.

Using the identity (C.5.8), we can write this in a rather suggestive form:

$$\boxed{\log Z_{\rm PI} = \log Z_{\rm bulk} - \log Z_{\rm edge} = \int_0^\infty \frac{dt}{2t}\,\frac{1+q}{1-q}\left(\Theta_{\rm bulk} - \Theta_{\rm edge}\right)}\,, \tag{5.4.7}$$

where $\Theta_{\rm bulk}$ and $\Theta_{\rm edge}$ are explicitly given by[12]

$$\boxed{\Theta_{\rm bulk} \equiv D^d_s\,\frac{q^{\frac{d}{2}+i\nu} + q^{\frac{d}{2}-i\nu}}{(1-q)^d}\,, \qquad \Theta_{\rm edge} \equiv D^{d+2}_{s-1}\,\frac{q^{\frac{d-2}{2}+i\nu} + q^{\frac{d-2}{2}-i\nu}}{(1-q)^{d-2}}} \tag{5.4.8}$$

The $\log Z_{\rm bulk}$ term is the character integral for the bulk partition function (5.4.3). Strikingly, the correction $\log Z_{\rm edge}$ *also* takes the form a character integral, but with an "edge" character $\Theta_{\rm edge}$ in *two lower* dimensions. By our results of section 5.3 for scalars, $Z_{\rm edge}$ effectively equals the Euclidean path integral of D^{d+2}_{s-1} scalars of mass $\tilde m^2 = \left(\frac{d-2}{2}\right)^2 + \nu^2$ on S^{d-1}:

$$Z_{\rm edge} = \textstyle\int \mathcal{D}\phi\, e^{-\frac{1}{2}\int_{S^{d-1}} \phi^a(-\nabla^2 + \tilde m^2)\phi^a}\,, \qquad a = 1,\ldots,D^{d+2}_{s-1}\,, \tag{5.4.9}$$

In particular this gives 1 scalar for $s=1$ and $d+2$ scalars for $s=2$. The S^{d-1} is naturally identified as the static patch horizon, the edge of the global dS spatial S^d hemisphere at time zero, the yellow dot in fig. 5.2.1. Thus (5.4.7) realizes in a precise way the somewhat vague physical expectation (5.2.7). Notice the relative minus sign here and in (5.4.7): the edge corrections effectively *subtract* degrees of freedom. We do not have a physical interpretation of these putative edge scalars for general s along the lines of the work reviewed in appendix C.4.5.3. Some clues are that their multiplicity equals the number of conformal Killing tensor modes of scalar type appearing in the derivation in appendix C.5.1 (the $\square\!\square\!\square$-modes for $s=4$ in (C.5.14)), and that they become massless at the unitarity bound $\nu = \pm i(\frac{d}{2}-1)$, eq. (C.5.18), where a partially massless field emerges with a scalar gauge parameter.

[12] For $d=2$, $D^2_s \to \sum_\pm D^2_s = 2$ in $\Theta_{\rm bulk}$ as in (5.4.3). $\Theta_{\rm edge}$ remains unchanged.

Independent of any interpretation, we can summarize the result (5.4.7)-(5.4.8) as

$$\log Z_{\rm PI}^{d+1}(s) = \log Z_{\rm bulk}^{d+1}(s) - D_{s-1}^{d+2}\, \log Z_{\rm PI}^{d-1}(0)\,. \tag{5.4.10}$$

Examples

For a $d=2$ spin-$s \geq 1$ field of mass $m^2=(s-1)^2+\nu^2$, $\log Z_{\rm PI} = \int \frac{dt}{2t}\frac{1+q}{1-q}(\Theta_{\rm bulk}-\Theta_{\rm edge})$ with

$$\Theta_{\rm bulk} = 2\,\frac{q^{1+i\nu}+q^{1-i\nu}}{(1-q)^2}\,, \qquad \Theta_{\rm edge} = s^2(q^{i\nu}+q^{-i\nu})\,. \tag{5.4.11}$$

That is, $Z_{\rm PI} = Z_{\rm bulk}/Z_{\rm edge}$, with the finite part of $\log Z_{\rm bulk}$ explicitly given by twice (5.2.25), and with $Z_{\rm edge}$ equal to the Euclidean path integral of $D^4_{s-1}=s^2$ harmonic oscillators of frequency ν on S^1, naturally identified with the S^1 horizon of the dS$_3$ static patch, with finite part

$$Z_{\rm edge}^{\rm fin} = \left(\frac{e^{-\pi\nu}}{1-e^{-2\pi\nu}}\right)^{s^2}\,. \tag{5.4.12}$$

The heat-kernel regularized $Z_{\rm PI}$ is then, restoring ℓ and recalling $\nu=\sqrt{m^2\ell^2-(s-1)^2}$,

$$\log Z_{\rm PI} = 2\Big(\frac{\pi\nu^3}{6} - \sum_{k=0}^{2}\frac{\nu^k}{k!}\,\frac{{\rm Li}_{3-k}(e^{-2\pi\nu})}{(2\pi)^{2-k}} - \frac{\pi\nu^2\ell}{4\epsilon} + \frac{\pi\ell^3}{2\epsilon^3}\Big) - s^2\Big(-\pi\nu - \log(1-e^{-2\pi\nu}) + \frac{\pi\ell}{\epsilon}\Big) \tag{5.4.13}$$

The $d=3$ spin-s case is worked out as another example in (C.2.27).

General massive representations

(5.4.6) has a natural generalization, presented in appendix C.5.2, to arbitrary parity-invariant massive SO$(1,d+1)$ representations $R=\oplus_a(\Delta_a,\mathbf{s}_a)$, $\Delta_a=\frac{d}{2}+i\nu_a$, $\mathbf{s}_a=(s_{a1},\ldots,s_{ar})$:

$$\boxed{\log Z_{\rm PI} = \int_0^\infty \frac{dt}{2t}\sum_a (-1)^{F_a}\big(q^{i\nu_a}+q^{-i\nu_a}\big)\sum_{n\in\frac{F_a}{2}+\mathbb{Z}} \theta\big(\tfrac{d}{2}+n\big)\, D^{d+2}_{n,\mathbf{s}_a}\, q^{\frac{d}{2}+n}} \tag{5.4.14}$$

where $\theta(x)$ is the Heaviside step step function with $\boxed{\theta(0) \equiv \frac{1}{2}}$, and $F_a = 0, 1$ for bosons resp. fermions. This is the unique TT eigenvalue sum extension consistent with locality and unitarity constraints. As in the $\mathbf{s} = (s)$ case, this can be rewritten as a bulk-edge decomposition $\log Z_{\rm PI} = \log Z_{\rm bulk} - \log Z_{\rm edge}$. For example, using (2.3.25), the analog of (5.4.10) for an $\mathbf{s} = (s, 1^m)$ field becomes

$$\log Z_{\rm PI}^{d+1}(s, 1^m) = \log Z_{\rm bulk}^{d+1}(s, 1^m) - D_{s-1}^{d+2} \log Z_{\rm PI}^{d-1}(1^m)\,, \tag{5.4.15}$$

so here $Z_{\rm edge}$ is the path integral of D_{s-1}^{d+2} massive *m-form fields* living on the S^{d-1} edge. In particular this implies the recursion relation $\log Z_{\rm PI}^{d+1}(1^p) = \log Z_{\rm bulk}^{d+1}(1^p) - \log Z_{\rm PI}^{d-1}(1^{p-1})$. Similarly for a spin $s = k + \frac{1}{2}$ Dirac fermion, in the notation explained under table (2.2.6)

$$\log Z_{\rm PI}^{d+1}(s, \tfrac{\mathbf{1}}{\mathbf{2}}) = \log Z_{\rm bulk}^{d+1}(s, \tfrac{\mathbf{1}}{\mathbf{2}}) - \tfrac{1}{2} D_{s-1, \frac{\mathbf{1}}{\mathbf{2}}}^{d+2} \log Z_{\rm PI}^{d-1}(\tfrac{\mathbf{1}}{\mathbf{2}})\,, \tag{5.4.16}$$

where $Z_{\rm bulk}$ now takes the form of the *fermionic* part of (5.3.3), with $\Theta_{\rm bulk}$ as in (3.4.11) with $D_{\mathbf{s}}^d = 2\, D_{s, \frac{\mathbf{1}}{\mathbf{2}}}^d$, the factor 2 due to the field being Dirac. The edge fields are Dirac spinors. Note that because $D_{-\frac{1}{2}, \frac{\mathbf{1}}{\mathbf{2}}}^{d+2} = 0$, the above implies in particular $Z_{\rm PI}(\frac{\mathbf{1}}{\mathbf{2}}) = Z_{\rm bulk}(\frac{\mathbf{1}}{\mathbf{2}})$.

We do not have a systematic group-theoretic or physical way of identifying the edge field content. For evaluation of $Z_{\rm PI}$ using (C.2.19), this identification is not needed however. Actually the original expansions (5.4.6), (5.4.14) are more useful for this, as illustrated in (C.2.23)-(C.2.27).

5.5 Massless higher spins

5.5.1 Problems

Bulk thermal partition function $Z_{\rm bulk}$

Massless spin-s fields on dS$_{d+1}$ are in many ways quite a bit more subtle than their massive spin-s counterparts. This manifests itself already at the level of the characters $\Theta_{{\rm bulk},s}$ needed to compute the bulk ideal gas thermodynamics along the lines of section 5.2. The SO$(1, d+1)$ unitary representations furnished by their single-particle Hilbert space belong to the discrete series for $d = 3$ and to the exceptional series for $d \geq 4$ [70]. The corresponding characters, discussed in the section 3.4.3, are more intricate than their massive (principal and complementary series) counterparts. A brief look at the general formula (3.4.20) with $t = s - 1$ or the table of examples (C.6.2) suffices to make clear they are far from intuitively obvious — as is, for that matter, the identification of the representation itself. Moreover, [70] reported their computation of the exceptional series characters disagrees with the original results in [13, 65, 75].

As noted in section 5.2 and appendix C.1.3, the expansion $\Theta_{\rm bulk}(q) = \sum_q N_k q^k$ can be interpreted as counting the number N_k of static patch quasinormal modes decaying as $e^{-kT/\ell}$. This gives some useful physics intuition for the peculiar form of these characters, explained in the section 4.5. The characters $\Theta_{{\rm bulk},s}(q)$ can in principle be computed by explicitly constructing and counting *physical* quasinormal modes of a massless spin-s field. This is a rather nontrivial problem, however.

Thus we see that for massless fields, complications appear already in the computation of $Z_{\rm bulk}$. Computing $Z_{\rm PI}$ adds even more complications, due to the presence of negative and zero modes in the path integral. Happily, as we will see, the complications of the latter turn out to be the key to resolving the complications of the former. Our final result for $Z_{\rm PI}$ confirms the identification of the representation made in [70] and the original results for the corresponding characters in [13, 65, 75]. This is explicitly verified by counting quasinormal modes in [1].

Euclidean path integral $Z_{\rm PI}$

We consider massless spin-s fields in the metric-like formalism, that is to say totally symmetric double-traceless fields $\phi_{\mu_1 \cdots \mu_s}$, with linearized gauge transformation

$$\delta_\xi^{(0)} \phi_{\mu_1 \cdots \mu_s} = \alpha_s \nabla_{(\mu_1} \xi_{\mu_2 \cdots \mu_s)} \,, \tag{5.5.1}$$

with ξ is traceless symmetric in its $s' = s - 1$ indices, and α_s picked by convention.[13]

We use the notation $s' \equiv s - 1$ as it makes certain formulae more transparent and readily

[13]As explained in appendix C.6.4, for compatibility with certain other conventions we adopt, we will pick $\alpha_s \equiv \sqrt{s}$ with symmetrization conventions such that $\phi_{(\mu_1 \cdots \mu_s)} = \phi_{\mu_1 \cdots \mu_s}$.

generalizable to the partially massless ($0 \leq s' < s$) case. The dimensions of ϕ_s and $\xi_{s'}$ are

$$\tfrac{d}{2} + i\nu_\phi = \Delta_\phi = s' + d - 1\,, \qquad \tfrac{d}{2} + i\nu_\xi = \Delta_\xi = s + d - 1\,. \tag{5.5.2}$$

Note that this value of ν_ϕ assign a mass $m = 0$ to ϕ according to (5.4.2). The Euclidean path integral of a collection of (interacting) gauge fields ϕ on S^{d+1} is formally given by

$$Z_{\rm PI} = \frac{\int \mathcal{D}\phi \, e^{-S[\phi]}}{\mathrm{vol}(\mathcal{G})} \tag{5.5.3}$$

where $\mathcal{G}$ is the group of local gauge transformations. At the one-loop (Gaussian) level $S[\phi]$ is the quadratic Fronsdal action [179]. Several complications arise compared to the massive case:

1. For $s \geq 2$, the Euclidean path integral has negative ("wrong sign" Gaussian) modes, generalizing the well-known issue arising for the conformal factor in Einstein gravity [180]. These can be dealt with by rotating field integration contours. A complication on the sphere is that rotations at the local field level ensuring positivity of short-wavelength modes causes a finite subset of low-lying modes to go negative, requiring these modes to be rotated back [131].

2. The linearized gauge transformations (5.5.1) have zeromodes: symmetric traceless tensors $\bar{\xi}_{\mu_1 \cdots \mu_{s-1}}$ satisfying $\nabla_{(\mu_1} \bar{\xi}_{\mu_2 \cdots \mu_s)} = 0$, the Killing tensors of S^{d+1}. This requires omitting associated modes from the BRST gauge fixing sector of the Gaussian path integral. As a result, locality is lost, and with it the flexibility to freely absorb various normalization constants into local counterterms without having to keep track of nonlocal residuals.

3. At the *nonlinear* level, the Killing tensors generate a subalgebra of the gauge algebra. The structure constants of this algebra are determined by the TT cubic couplings of the interacting theory [140]. At least when it is finite-dimensional, as is the case for Yang-Mills, Einstein gravity and the 3D higher-spin gravity theories of section 5.6, the Killing tensor algebra exponentiates to a group G. For example for Einstein gravity, $G = \mathrm{SO}(d+2)$. To compensate for the zeromode omissions in the path integral, one has to divide by the volume of G. The appropriate measure determining this volume is inherited from the path integral measure, and depends on the UV cutoff and the coupling constants of the theory.

Precisely relating the path integral volume $\mathrm{vol}(G)_{\rm PI}$ to the "canonical" $\mathrm{vol}(G)_{\rm c}$ defined by a theory-independent invariant metric on G requires considerable care in defining and keeping track of normalization factors.

Note that these complications do *not* arise for massless spin-s fields on AdS with standard boundary conditions. In particular the algebra generated by the (non-normalizable) Killing tensors in this case is a global symmetry algebra, acting nontrivially on the Hilbert space.

These problems are not insuperable, but they do require some effort. A brute-force path integral computation correctly dealing with all of them for general higher-spin theories is comparable to pulling a molar with a plastic fork: not impossible, but necessitating the sort of stamina some might see as savage and few would wish to witness. The character formalism simplifies the task, and the transparency of the result will make generalization obvious.

5.5.2 Ingredients and outline of derivation

We derive an exact formula for $Z_{\rm PI}$ in appendix C.6.2-C.6.4. In what follows we merely give a rough outline, just to give an idea what the origin is of various ingredients appearing in the final result. To avoid the $d=2$ footnotes of section 5.4 we assume $\boxed{d \geq 3}$ in what follows.

Naive characters

Naively applying the reasoning of section 5.4 to the massless case, one gets a character formula of the form (5.4.7), with "naive" bulk and edge characters $\hat{\Theta}$ given by

$$\hat{\Theta} \equiv \Theta_\phi - \Theta_\xi \,, \tag{5.5.4}$$

where Θ_ϕ, Θ_ξ are the massive bulk/edge characters for the spin-s, $\Delta = s'+d-1$ field ϕ and the spin-s', $\Delta = s+d-1$ gauge parameter (or ghost) field ξ, recalling $s' \equiv s-1$. The subtraction $-\Theta_\xi$ arises from the BRST ghost path integral. More explicitly, from (5.4.8),

$$\begin{aligned} \hat{\Theta}_{{\rm bulk},s} &= D^d_s \, \frac{q^{s'+d-1}+q^{1-s'}}{(1-q)^d} - D^d_{s'} \, \frac{q^{s+d-1}+q^{1-s}}{(1-q)^d} \\ \hat{\Theta}_{{\rm edge},s} &= D^{d+2}_{s-1} \, \frac{q^{s'+d-2}+q^{-s'}}{(1-q)^{d-2}} - D^{d+2}_{s'-1} \, \frac{q^{s+d-2}+q^{-s}}{(1-q)^{d-2}} \,. \end{aligned} \tag{5.5.5}$$

For example for $s=2$ in $d=3$,

$$\hat{\Theta}_{\rm bulk,2} = \frac{5\,(q^3+1)-3\,(q^4+q^{-1})}{(1-q)^3}\,, \qquad \hat{\Theta}_{\rm edge,2} = \frac{5\,(q^2+q^{-1})-(q^3+q^{-2})}{1-q}\,. \tag{5.5.6}$$

Because of the presence of non-positive powers of q, $\hat{\Theta}_{\rm bulk}$ is manifestly *not* the character of any unitary representation of SO$(1,d+1)$. Indeed, the character integral (5.4.7) using these naive $\hat{\Theta}$ is badly IR-divergent, due to the presence of non-positive powers of q.

Flipped characters

In fact this pathology is nothing but the character integral incarnation of the negative and zeromode mode issues of the path integral mentioned under (5.5.3). The zeromodes must be omitted, and the negative modes are dealt with by contour rotations. These prescriptions turn out to translate to a certain "flipping" operation at the level of the characters. More specifically the flipped character is obtained by acting the flipping operator $[\ ]_+$ (see eq. (3.4.17)) on the native character $\hat{\Theta}$. In particular, $\big[\hat{\Theta}_{{\rm bulk},s}\big]_+$ is the Harish-Chandra character of the exceptional series representation $\mathcal{U}_{s,s-1}$, noticing that $\hat{\Theta}_{{\rm bulk},s} = \Theta_{\mathcal{F}_{2-s,s}} - \Theta_{\mathcal{F}_{1-s,s-1}}$. Explicit expressions for the edge characters are $[\hat{\Theta}_{\rm edge}]_+ = ({\rm C}.6.23)$. Some simple examples are

d	s	$\big[\hat{\Theta}_{{\rm bulk},s}\big]_+ \cdot (1-q)^d$	$\big[\hat{\Theta}_{{\rm edge},s}\big]_+ \cdot (1-q)^{d-2}$
2	≥ 2	0	0
3	≥ 1	$2(2s+1)\,q^{s+1} - 2(2s-1)\,q^{s+2}$	$\frac{1}{3}s(s+1)(2s+1)\,q^s - \frac{1}{3}(s-1)s(2s-1)\,q^{s+1}$
4	≥ 1	$2(2s+1)\,q^2$	$\frac{1}{3}s(s+1)(2s+1)\,q$
≥ 3	1	$d\,(q^{d-1}+q) - q^d + 1 + (1-q)^d$	$q^{d-2} + 1 - (1-q)^{d-2}$

(5.5.7)

Contributions to $Z_{\rm PI}$

To be more precise, after implementing the appropriate contour rotations and zeromode subtractions, we get the following expression for the path integral:

$$Z_{\rm PI} = \frac{1}{{\rm vol}(G)_{\rm PI}} \prod_s \Big(\mathcal{A}_s\, i^{-P_s}\, Z_{{\rm char},s}\Big)^{n_s}\ . \tag{5.5.8}$$

where n_s is the number of massless spin-s fields in the theory, and the different factors appearing here are defined as follows:

1. $Z_{\rm char,s}$ is defined by the character integral

$$\log Z_{\rm char,s} \equiv \int_0^\times \frac{dt}{2t}\,\frac{1+q}{1-q}\left(\big[\hat\Theta_{{\rm bulk},s}\big]_+ - \big[\hat\Theta_{{\rm edge},s}\big]_+ - 2\,D^{d+2}_{s-1,s-1}\right), \tag{5.5.9}$$

   where $\int_0^\times$ means $\int_0^\infty$ with the IR divergence due to the constant term removed:

$$\int_0^\times \frac{dt}{t}\,f(t) \;\equiv\; \lim_{L\to\infty}\int_0^\infty \frac{dt}{t}\,f(t)\,e^{-t/L} \;-\; f(\infty)\,\log L\,. \tag{5.5.10}$$

   The flipped $\big[\hat\Theta_{{\rm bulk},s}\big]_+$ turns out to be precisely the massless spin-s exceptional series character $\Theta_{{\rm bulk},s}$: (3.4.20). Thus the $\Theta_{\rm bulk}$ contribution = ideal gas partition function $Z_{\rm bulk}$, pleasingly consistent with the physics picture. The second term is an edge correction as in the massive case. The third term has no massive counterpart, tied to the presence of gauge zeromodes: $D^{d+2}_{s-1,s-1}$ counts rank $s-1$ Killing tensors on S^{d+1}.

2. $\mathcal{A}_s$ is due to the zeromode omissions. Denoting $M = 2e^{-\gamma}/\epsilon$ as in (C.2.30),

$$\log\mathcal{A}_s \equiv D^{d+2}_{s-1,s-1}\int_0^\times \frac{dt}{2t}\big(2+q^{2s+d-4}+q^{2s+d-2}\big) = \frac{1}{2}D^{d+2}_{s-1,s-1}\log\tfrac{M^4}{(2s+d-4)(2s+d-2)} \tag{5.5.11}$$

   This term looks ugly. Happily, it will drop out of the final result.

3. i^{-P_s} is the spin-s generalization of Polchinski's phase of the one-loop path integral of Einstein gravity on the sphere [131]. It arises because every negative mode contour rotation adds a phase factor $-i$ to the path integral. Explicitly,

$$P_s = \sum_{n=0}^{s-2} D^{d+2}_{s-1,n} + \sum_{n=0}^{s-2} D^{d+2}_{s-2,n} = D^{d+3}_{s-1,s-1} - D^{d+2}_{s-1,s-1} + D^{d+3}_{s-2,s-2}\,. \tag{5.5.12}$$

   In particular $P_1 = 0$, $P_2 = D^{d+2}_1 + D^{d+2}_0 = d+3$ in agreement with [131]. For $d+1=4$, $P_s = \frac{1}{3}s\,(s^2-1)^2$ and $i^{-P_s} = 1,-1,1,1,1,-1,1,1,\ldots$. For $d+1 = 2$ mod 4, $i^{-P_s}=1$.

4. ${\rm vol}(G)_{\rm PI}$ is discussed below.

Volume of G

As mentioned under (5.5.3), G is the subgroup of gauge transformations generated by the Killing tensors $\bar{\xi}_{s-1}$ in the parent *interacting* theory on the sphere. Equivalently it is the subgroup of gauge transformations leaving the background invariant. For Einstein gravity, we have a single massless $s = 2$ field ϕ_2. The Killing vectors $\bar{\xi}_1$ generate diffeomorphisms rigidly rotating the sphere, hence $G = \mathrm{SO}(d+2)$. For $SU(N)$ Yang-Mills, we have $N^2 - 1$ massless $s = 1$ fields ϕ_1^a. The $N^2 - 1$ Killing scalars $\bar{\xi}_0^a$ generate constant $SU(N)$ gauge transformations, hence $G = SU(N)$.[14] For the 3D higher-spin gravity theories introduced in section 5.6, we have massless fields ϕ_s of spin $s = 2, \ldots, n$. The Killing tensors $\bar{\xi}_{s-1}$ turn out to generate $G = SU(n)_+ \times SU(n)_-$.

$\mathrm{vol}(G)_{\mathrm{PI}}$ is the volume of G according to the QFT path integral measure. We wish to relate it to a "canonical" $\mathrm{vol}(G)_{\mathrm{c}}$. We use the word "canonical" in the sense of defined in a theory-independent way. We determine $\mathrm{vol}(G)_{\mathrm{PI}}/\mathrm{vol}(G)_{\mathrm{c}}$ given our normalization conventions in appendix C.6.4. Below we summarize the most pertinent definitions and results.

For Einstein gravity, the Killing vector Lie algebra is $\mathfrak{g} = \mathfrak{so}(d+2)$. Picking a standard basis M_{IJ} satisfying $[M_{IJ}, M_{KL}] = \delta_{IK} M_{JL} + \delta_{JL} M_{IK} - \delta_{IL} M_{JK} - \delta_{JK} M_{IL}$, we define the "canonical" bilinear form $\langle \cdot | \cdot \rangle_{\mathrm{c}}$ on $\mathfrak{g}$ to be the unique invariant bilinear normalized such that

$$\langle M_{IJ} | M_{IJ} \rangle_{\mathrm{c}} \equiv 1 \qquad (I \neq J, \text{no sum})\,. \tag{5.5.13}$$

This invariant bilinear on $\mathfrak{g} = \mathfrak{so}(d+2)$ defines an invariant metric ds_{c}^2 on $G = \mathrm{SO}(d+2)$. Closed orbits generated by M_{IJ} then have length $\oint ds_c = 2\pi$, and $\mathrm{vol}(G)_{\mathrm{c}}$ is given by (C.3.2).

For higher-spin gravity, the Killing tensor Lie algebra $\mathfrak{g}$ contains $\mathfrak{so}(d+2)$ as a subalgebra with generators M_{IJ}. We define $\langle \cdot | \cdot \rangle_c$ on $\mathfrak{g}$ to be the unique $\mathfrak{g}$-invariant bilinear form [140, 141] normalized by (5.5.13). $\mathrm{vol}(G)_{\mathrm{c}}$ is defined using the corresponding metric ds_c^2 on G.

The Killing tensor commutators are determined by the local gauge algebra $[\delta_\xi, \delta_{\xi'}] = \delta_{[\xi,\xi']}$ as in [140]. For Einstein or HS gravity, in our conventions (canonical ϕ + footnote 13), this

[14] or a quotient thereof, such as $SU(N)/\mathbb{Z}_N$, depending on other data such as additional matter content. Here and in other instances, we will not try to be precise about the global structure of G.

gives for the $\mathfrak{so}(d+2)$ Killing vector (sub)algebra of $\mathfrak{g}$

$$[\bar{\xi}_1, \bar{\xi}_1'] = \sqrt{16\pi G_{\rm N}}\, [\bar{\xi}_1', \bar{\xi}_1]_{\sf Lie}\,, \tag{5.5.14}$$

where $[\cdot,\cdot]_{\sf Lie}$ is the standard vector field Lie bracket. In Einstein gravity, $G_{\rm N}$ is the Newton constant. In Einstein + higher-order curvature corrections (section 5.8) or in higher-spin gravity we take it to *define* the Newton constant. It is related to a "central charge" C in (C.6.51).

Building on [140, 143], we find the bilinear $\langle\cdot|\cdot\rangle_{\rm c}$ determining $\mathrm{vol}(G)_{\rm c}$ can then be written as

$$\langle\bar{\xi}|\bar{\xi}\rangle_{\rm c} = \frac{4G_{\rm N}}{A_{d-1}} \sum_s \sum_{\alpha=1}^{n_s} (2s+d-4)(2s+d-2) \int_{S^{d+1}} \bar{\xi}^{(\alpha)}_{s-1} \cdot \bar{\xi}^{(\alpha)}_{s-1}\,, \tag{5.5.15}$$

where $A_{d-1} = \mathrm{vol}(S^{d-1})$ is the dS horizon area. On the other hand, the path integral measure computing $\mathrm{vol}(G)_{\rm PI}$ is derived from the bilinear $\langle\xi|\bar{\xi}\rangle_{\rm PI} = \frac{M^4}{2\pi}\int \bar{\xi}\cdot\bar{\xi}$. From this we can read off the ratio $\mathrm{vol}(G)_{\rm c}/\mathrm{vol}(G)_{\rm PI}$: an awkward product of factors determined by the HS algebra. This turns out to cancel the awkward eigenvalue product of (5.5.11), up to a universal factor:

$$\frac{\mathrm{vol}(G)_{\rm c}}{\mathrm{vol}(G)_{\rm PI}} \prod_s \mathcal{A}_s^{n_s} = \left(\frac{8\pi G_{\rm N}}{A_{d-1}}\right)^{\frac{1}{2}\dim G}, \tag{5.5.16}$$

for all theories covered by [140], i.e. all parity-invariant HS theories consistent at cubic level.

For Yang-Mills, $\mathrm{vol}(G)_{\rm c}$ is computed using the metric ds_c^2 on $\mathfrak{g}$ defined by the canonically normalized YM action $S =: \frac{1}{4}\int\langle F|F\rangle_{\rm c}$. For example for $SU(N)$ YM with $S = -\frac{1}{4}\int \mathrm{Tr}_N\, F^2$, this gives $\mathrm{vol}(G)_{\rm c} = \mathrm{vol}(SU(N))_{\mathrm{Tr}_N} = (C.3.3)$. A similar but simpler computation gives the analog of (5.5.16). See appendix C.6.4 for details on all of the above.

5.5.3 Result and examples

Thus we arrive at the following universal formula for the one-loop Euclidean path integral for parity-symmetric (higher-spin) gravity and Yang-Mills gauge theories on S^{d+1}, $d \geq 3$:

$$\boxed{Z_{\rm PI}^{(1)} = i^{-P} \prod_{a=0}^{K} \frac{\gamma_a^{\dim G_a}}{\mathrm{vol}\, G_a} \cdot \exp \int_0^\times \frac{dt}{2t} \frac{1+q}{1-q} \left(\Theta_{\rm bulk} - \Theta_{\rm edge} - 2\dim G\right)} \tag{5.5.17}$$

• $G = G_0 \times G_1 \times \cdots G_K$ is the subgroup of (higher-spin) gravitational and Yang-Mills gauge transformations acting trivially on the background,

$$\gamma_0 \equiv \sqrt{\frac{8\pi G_{\rm N}}{A_{d-1}}}\,, \qquad \gamma_1 \equiv \sqrt{\frac{g_1^2}{2\pi A_{d-3}}}\,, \qquad \cdots \tag{5.5.18}$$

where $A_n \equiv \Omega_n \ell^n$, $\Omega_n = (C.3.1)$, the gravitational and YM coupling constants $G_{\rm N}$ and $g_1, \ldots, g_K$ are defined by the canonically normalized $\mathfrak{so}(d+2)$ and YM gauge algebras as explained around (5.5.14), and $\mathrm{vol}\, G_a$ is the canonically normalized volume of G_a, defined in the same part.

• For a theory with n_s massless spin-s fields

$$\Theta = \sum_s n_s \Theta_s\,, \qquad \dim G = \sum_s n_s D^{d+2}_{s-1,s-1}\,, \qquad P = \sum_s n_s P_s\,, \tag{5.5.19}$$

where $\Theta_s = [\hat\Theta_s]_+$ are the flipped versions of the naive characters (5.5.5), with examples in (5.5.7) and general formulae in (3.4.20) and (C.6.23), and $P_s = (5.5.12)$ is the spin-s generalization of the $s = 2$ phase $P_2 = d + 3$ found in [131].

• The heat-kernel regularized integral can be evaluated using (C.2.19), as spelled out in appendix C.2.3. For odd $d + 1$, the finite part can alternatively be obtained by summing residues. $\int_0^\times$ means integration with the IR log-divergence from the constant $-2\dim G$ term removed as in (5.5.10). The constant term contribution is then $\dim G \cdot (c\,\epsilon^{-1} + \log(2\pi))$, so when keeping track of linearly divergent terms is not needed, one can replace (5.5.17) by

$$Z_{\rm PI}^{(1)} = i^{-P} \prod_a \frac{(2\pi\gamma_a)^{\dim G_a}}{\mathrm{vol}\, G_a} \cdot \exp \int_0^\infty \frac{dt}{2t}\frac{1+q}{1-q}\left(\Theta_{\rm bulk} - \Theta_{\rm edge}\right) \qquad (\text{mod } \epsilon^{-1}) \tag{5.5.20}$$

• The case $d = 2$ requires some minor amendments, discussed in appendix C.7.1: for $s \geq 2$, nothing changes except P_s, and $\Theta = 0$, resulting in (C.7.5). Yang-Mills gives (C.7.8), or mod ϵ^{-1} (C.7.9), equivalent to putting $A_{-1} \equiv 1/2\pi\ell$ in (5.5.18), and Chern-Simons (C.7.11).

• The above can be extended to more general theories. For examples (s, s') partially massless gauge fields have characters given by (3.4.20) and (C.6.23), and contribute $D^{d+2}_{s-1,s'}$ to $\dim G$. Fermionic counterparts can be derived following the same steps, with $\hat\Theta_{\rm edge}$ given by (5.4.16). Fermionic (s, s') PM fields give negative contributions $-D^{d+2}_{s-1,s',\frac12,\ldots,\frac12}$ to $\dim G$.

Example: coefficient α_{d+1} of log-divergent term

The heat kernel coefficient α_{d+1}, i.e. the coefficient of the log-divergent term of $\log Z$, can be read off simply as the coefficient of the $1/t$ term in the small-t expansion of the integrand. As explained in C.2.3, we can just use the original, naive integrand $\hat{F}(t) = \frac{1}{2t}(\hat{\Theta}_{\rm bulk} - \hat{\Theta}_{\rm edge})$ for this purpose, obtained from (5.5.5). For e.g. a massless spin-s field on S^4 this immediately gives $\alpha_4^{(s)} = -\frac{1}{90}\,(75\,s^4 - 15\,s^2 + 2)$, in agreement with eq. (2.32) of [139]. For $s = 1, 2$,

d	3	5	7	9	11	13	15
$\alpha_{d+1}^{(1)}$	$-\frac{31}{45}$	$-\frac{1271}{1890}$	$-\frac{4021}{6300}$	$-\frac{456569}{748440}$	$-\frac{1199869961}{2043241200}$	$-\frac{893517041}{1571724000}$	$-\frac{17279945447657}{31261590360000}$
$\alpha_{d+1}^{(2)}$	$-\frac{571}{45}$	$-\frac{3181}{140}$	$-\frac{198851}{5670}$	$-\frac{74203873}{1496880}$	$-\frac{75059846731}{1135134000}$	$-\frac{114040703221}{1347192000}$	$-\frac{821333912103503}{7815397590000}$

(5.5.21)

Another case of general interest is a partially massless field with $(s, s') = (42, 26)$ on S^{42}:

$$\begin{aligned} \alpha_{42}^{(42,26)} &= -\frac{5925700837995152105818399547396345088821635783305199815444602762021561970991151947221547}{53488672032485127432027600664556659200000000000} \\ &\sim -10^{42} \end{aligned}$$

Example: $SU(4)$ Yang-Mills on S^5

As a simple illustration and test of (5.5.17), consider $SU(4)$ YM theory on S^5 of radius ℓ with action $S = \frac{1}{4g^2}\int {\rm Tr}_4\, F^2$, so $G = SU(4)$, $n_1 = \dim G = 15$, ${\rm vol}(G)_{\rm c} = \frac{(2\pi)^9}{6}$ as given by (C.3.3), $\gamma = \sqrt{\frac{g^2}{(2\pi)^2 \ell}}$, and $P = 0$. Bulk and edge characters are read off from table (5.5.7). Thus

$$\log Z_{\rm PI}^{(1)} = \log \frac{(g/\sqrt{\ell})^{15}}{(2\pi)^{15} \cdot \frac{(2\pi)^9}{6}} + 15 \cdot \int_0^\times \frac{dt}{2t} \frac{1+q}{1-q} \Big(\frac{6\,q^2}{(q-1)^4} - \frac{2\,q}{(q-1)^2} - 2 \Big) . \tag{5.5.22}$$

The finite part can be evaluated by simply summing residues, similar to (5.2.25):

$$\log Z_{\rm PI}^{\rm fin} = \log \frac{(g/\sqrt{\ell})^{15}}{\frac{1}{6}(2\pi)^9} + 15 \cdot \Big(\frac{5\,\zeta(3)}{16\,\pi^2} + \frac{3\,\zeta(5)}{16\,\pi^4} \Big) . \tag{5.5.23}$$

The $U(1)$ version of this agrees with [181] eq. (2.27). We could alternatively use (C.2.19) as in C.2.3, which includes the UV divergent part: $\log Z_{\rm PI}^{(1)} = \log Z_{\rm PI}^{\rm fin} + 15\big(\frac{9\pi}{8}\epsilon^{-5}\ell^5 - \frac{5\pi}{8}\epsilon^{-3}\ell^3 -$

$\frac{7\pi}{16}\epsilon^{-1}\ell\big)$.

Example: Einstein gravity on S^3, S^4 and S^5

The exact one-loop Euclidean path integral for Einstein gravity on the sphere can be worked out similarly. The S^3 case is obtained in (C.7.7). The S^4 and S^5 cases are detailed in C.2.3, with results including UV-divergent terms given in (C.7.5), (C.2.44), (C.2.47). The finite parts are:

$$Z_{\rm PI}^{\rm fin} = i^{-P} \cdot \frac{1}{\mathrm{vol}(G)_{\rm c}} \left(\frac{8\pi G_{\rm N}}{A_{d-1}}\right)^{\frac{1}{2}\dim G} \cdot Z_{\rm char}^{\rm fin}\,, \tag{5.5.24}$$

S^{d+1}	i^{-P}	$\mathrm{vol}(G)_{\rm c}$	A_{d-1}	$\dim G$	$\log Z_{\rm char}^{\rm fin}$
S^3	$-i$	$(2\pi)^4$	$2\pi\ell$	6	$6\,\log(2\pi)$
S^4	-1	$\frac{2}{3}(2\pi)^6$	$4\pi\ell^2$	10	$-\frac{571}{45}\log(\ell/L)+\frac{715}{48}-\log 2-\frac{47}{3}\zeta'(-1)+\frac{2}{3}\zeta'(-3)$
S^5	i	$\frac{1}{12}(2\pi)^9$	$2\pi^2\ell^3$	15	$15\log(2\pi)+\frac{65\,\zeta(3)}{48\,\pi^2}+\frac{5\,\zeta(5)}{16\,\pi^4}$

(5.5.25)

Checks: We rederive the S^3 result in the Chern-Simons formulation of 3D gravity [164] in appendix C.7.2, and find precise agreement, with the phase matching for odd framing of the Chern-Simons partition function (it vanishes for even framing). The coefficient $-\frac{571}{45}$ of the log-divergent term of the S^4 result agrees with [127]. The phases agree with [131]. The powers of $G_{\rm N}$ agree with zeromode counting arguments of [135, 138]. The full one-loop partition function on S^4 was calculated using zeta-function regularization in [134]. Upon correcting an error in the second number of their equation (A.36) we find agreement. As far as we know, the zeta-function regularized $Z_{\rm PI}^{(1)}$ has not been explicitly computed before for S^{d+1}, $d \geq 4$.

Higher-spin theories

Generic Vasiliev higher-spin gravity theories have infinite spin range and $\dim G = \infty$, evidently posing problems for (5.5.17). We postpone discussion of this case to section 5.9. Below we consider a 3D higher-spin gravity theory with finite spin range $s = 2, \ldots, n$.

5.6 3D HS$_n$ gravity and the topological string

As reviewed in appendix C.7.2, 3D Einstein gravity with positive cosmological constant in Lorentzian or Euclidean signature can be formulated as an $SL(2,\mathbb{C})$ resp. $SU(2) \times SU(2)$ Chern-Simons theory [164].[15] This has a natural extension to an $SL(n,\mathbb{C})$ resp. $SU(n)\times SU(n)$ Chern-Simons theory, discussed in appendix C.7.3, which can be viewed as an $s \leq n$ dS$_3$ higher-spin gravity theory, analogous to the AdS$_3$ theories studied e.g. in [166–169, 182, 183]. The Lorentzian/Euclidean actions S_L/S_E are

$$S_L = iS_E = (l + i\kappa)\, S_{\rm CS}[\mathcal{A}_+] + (l - i\kappa)\, S_{\rm CS}[\mathcal{A}_-]\,, \qquad l \in \mathbb{N}, \quad \kappa \in \mathbb{R}^+\,, \tag{5.6.1}$$

where $S_{\rm CS}[\mathcal{A}] = \frac{1}{4\pi}\int {\rm Tr}_n\big(\mathcal{A}\wedge d\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\big)$ and $\mathcal{A}_\pm$ are ${\rm sl}(n)$-valued connections with reality condition $\mathcal{A}_\pm^* = \mathcal{A}_\mp$ for the Lorentzian theory and $\mathcal{A}_\pm^\dagger = \mathcal{A}_\pm$ for the Euclidean theory.

The Chern-Simons formulation allows all-loop exact results, providing a useful check of our result (5.5.17) for $Z_{\rm PI}^{(1)}$ obtained in the metric-like formulation. Besides this, we observed a number of other interesting features, collected in appendix C.7.3, and summarized below.

Landscape of vacua (C.7.3.1)

The theory has a set of dS$_3$ vacua (or round S^3 solutions in the Euclidean theory), corresponding to different embeddings of $\mathfrak{sl}(2)$ into $\mathfrak{sl}(n)$, labeled by n-dimensional representations

$$R = \oplus_a \mathbf{m_a}\,, \qquad n = \sum_a m_a\,. \tag{5.6.2}$$

of $\mathfrak{su}(2)$, i.e. by partitions of $n = \sum_a m_a$. The radius in Planck units $\ell/G_{\rm N}$ and $Z^{(0)} = e^{-S_E}$ depend on the vacuum R as

$$\log Z^{(0)} = \frac{2\pi\ell}{4G_N} = 2\pi\kappa\, T_R\,, \qquad T_R = \frac{1}{6}\sum_a m_a(m_a^2 - 1)\,. \tag{5.6.3}$$

[15] Or more precisely an SO$(1,3) = SL(2,\mathbb{C})/\mathbb{Z}_2$ or SO$(4) = \big(SU(2)\times SU(2)\big)/\mathbb{Z}_2$ CS theory. For the higher-spin extensions, we could similarly consider quotients. We will use the unquotiented groups here.

Note that $\mathcal{S}^{(0)} = \log Z^{(0)}$ takes the standard Einstein gravity horizon entropy form. The entropy is maximized for the principal embedding, i.e. $R = \mathbf{n}$, for which $T_{\mathbf{n}} = \frac{1}{6}n(n^2-1)$. The number of vacua equals the number of partitions of n:

$$\mathcal{N}_{\rm vac} \sim e^{2\pi\sqrt{n/6}} \,. \tag{5.6.4}$$

For, say, $n \sim 2\times 10^5$, we get $\mathcal{N}_{\rm vac} \sim 10^{500}$, with maximal entropy $\mathcal{S}^{(0)}|_{R=\mathbf{n}} \sim 10^{15}\kappa$.

Higher-spin algebra and metric-like field content (C.7.3.2)

As worked out in detail for the AdS analog in [182], the fluctuations of the Chern-Simons connection for the principal embedding vacuum $R = \mathbf{n}$ correspond in a metric-like description to a set of massless spin-s fields with $s = 2, 3, \ldots, n$. The Euclidean higher-spin algebra is $\mathfrak{su}(n)_+ \oplus \mathfrak{su}(n)_-$, which exponentiates to $G = SU(n)_+ \times SU(n)_-$. The higher-spin field content of the $R = \mathbf{n}$ vacuum can also be inferred from the decomposition of $\mathfrak{su}(n)$ into irreducible representations of $\mathfrak{su}(2)$, with $S \in \mathfrak{su}(2)$ acting on $L \in \mathfrak{su}(n)$ as $\delta L = \epsilon[R(S), L]$, to wit,

$$(\mathbf{n^2} - \mathbf{1})_{{\rm su}(n)} = \sum_{r=1}^{n-1} (\mathbf{2r} + \mathbf{1})_{{\rm su}(2)} \,. \tag{5.6.5}$$

The $(\mathbf{2r}+\mathbf{1}, \mathbf{1})$ and $(\mathbf{1}, \mathbf{2r}+\mathbf{1})$ of $\mathfrak{so}(4) = \mathfrak{su}(2)_+ \oplus \mathfrak{su}(2)_-$ correspond to rank-r self-dual and anti-self-dual Killing tensors on S^3, the zeromodes of (5.5.1) for a massless spin-$(r+1)$ field, confirming $R = \mathbf{n}$ has $n_s = 1$ massless spin-s field for $s = 2, \ldots, n$. For different vacua R, one gets decompositions different from (5.6.5), associated with different field content. For example for $n = 12$ and $R = \mathbf{6} \oplus \mathbf{4} \oplus \mathbf{2}$, we get $n_1 = 2$, $n_2 = 7$, $n_3 = 8$, $n_4 = 6$, $n_5 = 3$, $n_6 = 1$.

One-loop and all-loop partition function (C.7.3.3-C.7.3.4)

In view of the above higher-spin interpretation, we can compute the one-loop Euclidean path integral on S^3 for $l = 0$ from our general formula (5.5.17) for higher-spin gravity theories in the metric-like formalism. The dS_3 version of (5.5.17) is worked out in (C.7.4)-(C.7.6), and applied to the case of interest in (C.7.43), using (5.6.3) to convert from $\ell/G_{\rm N}$ to κ. The result is

$$Z_{\rm PI}^{(1)} = i^{n^2-1} \cdot \frac{\left(2\pi/\sqrt{\kappa}\right)^{\dim G}}{{\rm vol}(G)_{{\rm Tr}_n}}\,, \tag{5.6.6}$$

where ${\rm vol}(G)_{{\rm Tr}_n} = \left(\sqrt{n}\prod_{s=2}^{n}(2\pi)^s/\Gamma(s)\right)^2$ as in (C.3.3).

This can be compared to the weak-coupling limit of the all-loop expression (C.7.45)-(C.7.47), obtained from the known exact partition function of $SU(n)_{k_+} \times SU(n)_{k_-}$ Chern-Simons theory on S^3 by analytic continuation $k_\pm \to l \pm i\kappa$,

$$Z(R)_r = e^{ir\phi} \cdot \left| \frac{1}{\sqrt{n}} \frac{1}{(n+l+i\kappa)^{\frac{n-1}{2}}} \prod_{p=1}^{n-1} \Big(2\sin\frac{\pi p}{n+l+i\kappa}\Big)^{(n-p)} \right|^2 \cdot e^{2\pi\kappa T_R}\,. \tag{5.6.7}$$

Here $\phi = \frac{\pi}{4}\sum_\pm c(l\pm i\kappa)$ with $c(k) \equiv (n^2-1)(1-\frac{n}{n+k})$, and $r \in \mathbb{Z}$ labels the choice of framing needed to define the Chern-Simons theory as a QFT, discussed in more detail below (C.7.25). Canonical framing corresponds to $r=0$. $Z(R)$ is interpreted as the all-loop quantum-corrected Euclidean partition function of the dS$_3$ static patch in the vacuum R.

The weak-coupling limit $\kappa \to \infty$ of (5.6.7) precisely reproduces (5.6.6), with the phase matching for odd framing r. Alternatively this can be seen more directly by a slight variation of the computation leading to (C.7.11). This provides a check of (5.5.17), in particular its normalization in the metric-like formalism, and of the interpretation of (5.6.1) as a higher-spin gravity theory.

Large-n limit and topological string dual (C.7.3.5)

Vasliev-type ${\rm hs}({\rm so}(d+2))$ higher-spin theories (section 5.9) have infinite spin range but finite $\ell^{d-1}/G_{\rm N}$. To mimic this case, consider the $n \to \infty$ limit of the theory at $l=0$. The semiclassical expansion is reliable only if $n \ll \kappa$. Using $\ell/G_{\rm N} \sim \kappa T_R$, this translates to $n\,T_R \ll \ell/G_{\rm N}$, which becomes $n^4 \ll \ell/G_{\rm N}$ for the principal vacuum $R = \mathbf{n}$, and $n \ll \ell/G_{\rm N}$ at the other extreme for $R = \mathbf{2} \oplus \mathbf{1} \oplus \cdots \oplus \mathbf{1}$. Either way, the Vasiliev-like limit $n \to \infty$ at fixed $\mathcal{S}^{(0)} = 2\pi\ell/4G_{\rm N}$ is strongly coupled.

However (5.6.7) continues to make sense in any regime, and in particular *does* have a weak coupling expansion in the $n \to \infty$ 't Hooft limit. Using the large-n duality between $U(n)_k$

Chern-Simons on S^3 and closed topological string theory on the resolved conifold [170, 171], the partition function (5.6.7) of de Sitter higher-spin quantum gravity in the vacuum R can be expressed in terms of the weakly-coupled topological string partition function $\tilde{Z}_{\rm top}$, (C.7.50):

$$\boxed{Z(R)_0 = \left|\tilde{Z}_{\rm top}(g_s,t)\, e^{-\pi T_R \cdot 2\pi i/g_s}\right|^2} \tag{5.6.8}$$

where (in the notation of [171]) the string coupling constant g_s and the resolved conifold Kähler modulus $t \equiv \int_{S^2} J + iB$ are given by

$$g_s = \frac{2\pi}{n+l+i\kappa}\,, \qquad t = i g_s n = \frac{2\pi i n}{n+l+i\kappa}\,. \tag{5.6.9}$$

Note that $\left|e^{-\pi T_R\cdot 2\pi i/g_s}\right|^2 = e^{2\pi\kappa T_R} = e^{\mathcal{S}^{(0)}}$, and that $\kappa > 0$ implies $\int_{S^2} J > 0$ and $\operatorname{Im} g_s \neq 0$. The dependence on n at fixed $\mathcal{S}^{(0)}$ is illustrated in fig. 5.1.4. We leave further exploration of the dS quantum gravity - topological string duality suggested by these observations to future work.

5.7 Euclidean thermodynamics

In section 5.2.3 we defined and computed the bulk partition function, energy and entropy of the static patch ideal gas. In this section we define and compute their Euclidean counterparts, building on the results of the previous sections.

5.7.1 Generalities

Consider a QFT on a dS$_{d+1}$ background with curvature radius ℓ. Wick-rotated to the round sphere metric $g_{\mu\nu}$ of radius ℓ (see appendix C.3.2), we get the Euclidean partition function:

$$Z_{\rm PI}(\ell) \equiv \int \mathcal{D}\Phi\, e^{-S_E[\Phi]} \tag{5.7.1}$$

where Φ collectively denotes all fields. The quantum field theory is to be thought of here as a (weakly) interacting low-energy effective field theory with a UV cutoff ϵ.

Recalling the path integral definition (C.4.18) of the Euclidean vacuum $|O\rangle$ paired with its dual $\langle O|$ as $Z_{\rm PI} = \langle O|O\rangle$, the Euclidean expectation value of the stress tensor is

$$\langle T_{\mu\nu}\rangle \equiv \frac{\langle O|T_{\mu\nu}|O\rangle}{\langle O|O\rangle} = -\frac{2}{\sqrt{g}}\frac{\delta}{\delta g^{\mu\nu}}\log Z_{\rm PI} = -\rho_{\rm PI}\, g_{\mu\nu}\,, \tag{5.7.2}$$

The last equality, in which $\rho_{\rm PI}$ is a constant, follows from SO$(d+2)$ invariance of the round sphere background. Denoting the volume of the sphere by $V = {\rm vol}(S_\ell^{d+1}) = \Omega_{d+1}\ell^{d+1}$,

$$\boxed{-\rho_{\rm PI}V = \tfrac{1}{d+1}\textstyle\int\!\sqrt{g}\,\langle T^\mu_\mu\rangle = \tfrac{1}{d+1}\ell\partial_\ell \log Z_{\rm PI} = V\partial_V \log Z_{\rm PI}} \tag{5.7.3}$$

Reinstating the radius ℓ, the sphere metric in the S coordinates of (C.3.7) takes the form

$$ds^2 = (1-r^2/\ell^2)d\tau^2 + (1-r^2/\ell^2)^{-1}dr^2 + r^2 d\Omega^2\,, \tag{5.7.4}$$

where $\tau \simeq \tau + 2\pi\ell$. Wick rotating $\tau \to iT$ yields the static patch metric. Its horizon at $r=\ell$ has inverse temperature $\beta = 2\pi\ell$. On a constant-T slice, the vacuum expectation value of the Killing energy density corresponding to translations of T equals $\rho_{\rm PI}$ at the location $r=0$ of the inertial observer. Away from $r=0$, it is redshifted by a factor $\sqrt{1-r^2/\ell^2}$. The Euclidean vacuum expectation value $U_{\rm PI}$ of the total static patch energy then equals $\rho_{\rm PI}\sqrt{1-r^2/\ell^2}$ integrated over a constant-T slice:

$$U_{\rm PI} = \rho_{\rm PI}\,\Omega_{d-1}\int_0^\ell dr\, r^{d-1} = \rho_{\rm PI}\, v\,, \qquad v = \frac{\Omega_{d-1}\ell^d}{d} = \frac{V}{2\pi\ell}\,. \tag{5.7.5}$$

Note that v is the volume of a d-dimensional ball of radius ℓ in flat space, so effectively we can think of $U_{\rm PI}$ as the energy of an ordinary ball of volume v with energy density $\rho_{\rm PI}$.

Combining (5.7.3) and (5.7.5), the Euclidean energy on this background is obtained as

$$\boxed{2\pi\ell\, U_{\rm PI} = V\rho_{\rm PI} = -\tfrac{1}{d+1}\,\ell\partial_\ell \log Z_{\rm PI}} \tag{5.7.6}$$

and the corresponding Euclidean entropy $S_{\rm PI} \equiv \log Z_{\rm PI} + \beta\, U_{\rm PI}$ is

$$\boxed{S_{\rm PI} = \big(1 - \tfrac{1}{d+1}\ell\partial_\ell\big)\log Z_{\rm PI} = \big(1 - V\partial_V\big)\log Z_{\rm PI}} \tag{5.7.7}$$

$S_{\rm PI}$ can thus be viewed as the Legendre transform of $\log Z_{\rm PI}$ trading V for $\rho_{\rm PI}$:

$$d\log Z_{\rm PI} = -\rho_{\rm PI}\, dV, \qquad S_{\rm PI} = \log Z_{\rm PI} + V\rho_{\rm PI}\,, \qquad dS_{\rm PI} = V\, d\rho_{\rm PI}\,. \tag{5.7.8}$$

The above differential relations express the first law of (Euclidean) thermodynamics for the system under consideration: using $V = \beta v$ and $\rho_{\rm PI} = U_{\rm PI}/v$, they can be rewritten as

$$d\log Z_{\rm PI} = -U_{\rm PI}\, d\beta - \beta\rho_{\rm PI}\, dv\,, \qquad dS_{\rm PI} = \beta\, dU_{\rm PI} - \beta\rho_{\rm PI}\, dv\,. \tag{5.7.9}$$

Viewing v as the effective thermodynamic volume as under (5.7.5), these take the familiar form of the first law, with pressure $p = -\rho$, the familiar cosmological vacuum equation of state.

The expression (5.7.7) for the Euclidean entropy and (5.7.8) naturally generalize to Euclidean partition functions $Z_{\rm PI}(\ell)$ for *arbitrary* background geometries $g_{\mu\nu}(\ell) \equiv \ell^2\tilde{g}_{\mu\nu}$ with volume $V(\ell) = \ell^{d+1}\tilde{V}$. In contrast, the expression (5.7.6) for the Euclidean energy is specific to the sphere. A generic geometry has no isometries, so there is no notion of Killing energy to begin with. On the other hand, the density $\rho_{\rm PI}$ appearing in (5.7.8) does generalize to arbitrary backgrounds. The last equality in (5.7.2) and the physical interpretation of $\rho_{\rm PI}$ as a Killing energy density no longer apply, but (5.7.3) remains valid.

5.7.2 Examples

Free $d = 0$ scalar

To connect to the familiar and to demystify the ubiquitous ${\rm Li}_n(e^{-2\pi\nu}) = \sum_k e^{-2\pi k\nu}/k^n$ terms encountered later, consider a scalar of mass m on an S^1 of radius ℓ, a.k.a. a harmonic oscillator of frequency m at $\beta = 2\pi\ell = V$. Using (C.2.32) and applying (5.7.6)-(5.7.7) with $\nu(\ell) \equiv m\ell$,

$$\begin{aligned} \log Z_{\rm PI} &= \frac{\pi\ell}{\epsilon} - \pi\nu + {\rm Li}_1(e^{-2\pi\nu}) \\ 2\pi\ell\, U_{\rm PI} = V\rho_{\rm PI} &= -\frac{\pi\ell}{\epsilon} + \pi\nu\coth(\pi\nu) \\ S_{\rm PI} &= {\rm Li}_1(e^{-2\pi\nu}) + 2\pi\nu\, {\rm Li}_0(e^{-2\pi\nu})\,, \end{aligned} \tag{5.7.10}$$

Mod $\Delta E_0 \propto -\epsilon^{-1}$, these are the textbook canonical formulae turned into polylogs by (5.2.26).

Free scalar in general d

The Euclidean action of a free scalar on S^{d+1} is

$$S_E[\phi] = \frac{1}{2}\int \sqrt{g}\,\phi(-\nabla^2 + m^2 + \xi R)\phi\,, \tag{5.7.11}$$

with $R = d(d+1)/\ell^2$ the S^{d+1} Ricci scalar. The total effective mass $m_{\rm eff}^2 = ((\frac{d}{2})^2 + \nu^2)/\ell^2$ is

$$m_{\rm eff}^2 = m^2 + \xi R \qquad \Rightarrow \qquad \nu = \sqrt{(m\ell)^2 - \eta}\,, \quad \eta \equiv \left(\tfrac{d}{2}\right)^2 - d(d+1)\,\xi\,. \tag{5.7.12}$$

Neither $Z_{\rm PI}$ nor the bulk thermodynamic quantities of section 5.2 distinguish between the m^2 and ξR contributions to $m_{\rm eff}^2$, but $U_{\rm PI}$ and $S_{\rm PI}$ do, due to the ∂_ℓ derivatives in (5.7.6)-(5.7.7). This results in an additional explicit dependence on ξ, as

$$\ell\partial_\ell \log Z_{\rm PI} = \big(-\epsilon\partial_\epsilon + J\cdot\nu\partial_\nu\big)\log Z_{\rm PI}\,, \qquad J = \frac{\ell\partial_\ell\nu}{\nu} = \frac{(m\ell)^2}{\nu^2} = \frac{\nu^2+\eta}{\nu^2}\,. \tag{5.7.13}$$

For the minimally coupled case $\xi = 0$, the Euclidean and bulk thermodynamic quantities agree, but in general not if $\xi \neq 0$. To illustrate this we consider the $d = 2$ example. Using (5.2.25) and (C.2.22), restoring ℓ, and putting $\nu \equiv \sqrt{(m\ell)^2 - \eta}$ with $\eta = 1 - 6\xi$,

$$\log Z_{\rm PI} = \frac{\pi\ell^3}{2\epsilon^3} - \frac{\pi\nu^2\ell}{4\epsilon} + \frac{\pi\nu^3}{6} - \sum_{k=0}^{2}\frac{\nu^k}{k!}\,\frac{{\rm Li}_{3-k}(e^{-2\pi\nu})}{(2\pi)^{2-k}}\,. \tag{5.7.14}$$

The corresponding Euclidean energy $U_{\rm PI} = \rho_{\rm PI}\,\pi\ell^2$ (5.7.6) is given by

$$2\pi\ell\,U_{\rm PI} = V\rho_{\rm PI} = -\frac{\pi\ell^3}{2\epsilon^3} + \frac{\pi(\nu^2 + \frac{2}{3}\eta)\ell}{4\epsilon} - \frac{\pi}{6}(\nu^2+\eta)\nu\coth(\pi\nu) \tag{5.7.15}$$

where $V = {\rm vol}(S_\ell^3) = 2\pi^2\ell^3$. For minimal coupling $\xi = 0$ (i.e. $\eta = 1$), $U_{\rm PI}^{\rm fin}$ equals $U_{\rm bulk}^{\rm fin}$ (5.2.24), but not for $\xi \neq 0$. For general d, ξ, $U_{\rm PI}^{\rm fin}$ is given by (5.2.23) with the overall factor m^2 the mass m^2 appearing in the action rather than $m_{\rm eff}^2$, in agreement with [184, 185] or (6.178)-(6.180) of [186]. The entropy $S_{\rm PI} = \log Z_{\rm PI} + 2\pi\ell\,U_{\rm PI}$ (5.7.7) is

$$S_{\rm PI} = \frac{\pi\eta}{6}\Big(\frac{\ell}{\epsilon} - \nu\coth(\pi\nu)\Big) - \sum_{k=0}^{3}\frac{\nu^k}{k!}\,\frac{{\rm Li}_{3-k}(e^{-2\pi\nu})}{(2\pi)^{2-k}}\,, \tag{5.7.16}$$

where we used $\coth(\pi\nu) = 1 + 2\,\mathrm{Li}_0(e^{-2\pi\nu})$ (5.2.26). Since $Z_{\rm PI} = Z_{\rm bulk}$ in general for scalars and $U_{\rm PI} = U_{\rm bulk}$ for minimally coupled scalars, $S_{\rm PI} = S_{\rm bulk}$ for minimally coupled scalars. Indeed, after conversion to Pauli-Villars regularization, (5.7.16) equals (5.2.27) if $\eta = 1$. As a check on the results, the first law $dS_{\rm PI} = V d\rho_{\rm PI}$ (5.7.8) can be verified explicitly.

In the $m\ell \to \infty$ limit, $S_{\rm PI} \to \frac{\pi}{6}\eta(\epsilon^{-1} - m)\ell$, reproducing the well-known scalar one-loop Rindler entropy correction computed by a Euclidean path integral on a conical geometry [5, 6, 145, 146, 150, 187]. Note that $S_{\rm PI} < 0$ when $\eta < 0$. Indeed as reviewed in the Rindler context in appendix C.4.5, $S_{\rm PI}$ does *not* have a statistical mechanical interpretation on its own. Instead it must be interpreted as a correction to the large positive classical gravitational horizon entropy. We discuss this in the de Sitter context in section 5.8.

A pleasant feature of the sphere computation is that it avoids replicated or conical geometries: instead of varying a deficit angle, we vary the sphere radius ℓ, preserving manifest SO$(d+2)$ symmetry, and allowing straightforward exact computation of the Euclidean entropy directly from $Z_{\rm PI}(\ell)$, for arbitrary field content.

Free 3D massive spin s

Recall from (5.4.13) that for a $d = 2$ massive spin-$s \geq 1$ field of mass m, the bulk part of $\log Z_{\rm PI}$ is twice that of a $d = 2$ scalar (5.7.14) with $\nu = \sqrt{(m\ell)^2 - \eta}$, $\eta = (s-1)^2$, while the edge part is $-s^2$ times that of a $d = 0$ scalar, as in (5.7.10), with the important difference however that $\nu = \sqrt{(m\ell)^2 - \eta}$ instead of $\nu = m\ell$. Another important difference with (5.7.10) is that in the case at hand, (5.7.6) stipulates $V\rho_{\rm PI} = 2\pi\ell\, U_{\rm PI} = -\frac{1}{d+1}\ell\partial_\ell \log Z_{\rm PI}$ with $d = 2$ instead of $d = 0$. As a result, for the bulk contribution, we can just copy the scalar formulae (5.7.15) and (5.7.16) for $U_{\rm PI}$ and $S_{\rm PI}$ setting $\eta = (s-1)^2$, while for the edge contribution we get something rather different from the harmonic oscillator energy and entropy (5.7.10):

$$V\rho_{\rm PI} = 2 \times (5.7.15) - s^2\big(-\tfrac{\pi}{3}\tfrac{1}{\epsilon}\ell + \tfrac{\pi}{3}(\nu^2 + \eta)\nu^{-1}\coth(\pi\nu)\big) \tag{5.7.17}$$

$$S_{\rm PI} = 2 \times (5.7.16) - s^2\big(\tfrac{2\pi}{3}(\tfrac{1}{\epsilon}\ell - \nu) + \tfrac{\pi}{3}\eta\nu^{-1}\coth(\pi\nu) + \mathrm{Li}_1(e^{-2\pi\nu}) + \tfrac{2\pi}{3}\nu\,\mathrm{Li}_0(e^{-2\pi\nu})\big) \tag{5.7.18}$$

The edge contribution renders $S_{\rm PI}$ *negative* for all ℓ. In particular, in the $m\ell \to \infty$ limit, $S_{\rm PI} \to \frac{\pi}{3}\big((s-1)^2 - 2s^2\big)\,(\epsilon^{-1} - m)\ell \to -\infty$: although the bulk part gives a large positive contribution for $s \geq 2$, the edge part gives an even larger negative contribution. Going in the opposite direction, to smaller $m\ell$, we hit the $d=2$, $s \geq 1$ unitarity bound at $\nu = 0$, i.e. at $m\ell = \sqrt{\eta} = s-1$. Approaching this bound, the bulk contribution remains finite, while the edge part diverges, again negatively. For $s=1$, $S_{\rm PI} \to \log(m\ell)$, due to the ${\rm Li}_1(e^{-2\pi\nu})$ term, while for $s \geq 2$, more dramatically, we get a pole $S_{\rm PI} \to -\frac{s^2(s-1)}{6}\big(m\ell-(s-1)\big)^{-1}$, due to the $\eta\nu^{-1}\coth(\pi\nu)$ term. Below the unitarity bound, i.e. when $\ell < (s-1)/m$, $S_{\rm PI}$ becomes complex. To be consistent as a perturbative low-energy effective field theory valid down to some length scale l_s, massive spin-$s \geq 2$ particles on dS$_3$ must satisfy $m^2 > (s-1)^2/l_s^2$.

Massless spin 2

From the results and examples in section 5.5.3, $\log Z_{\rm PI}^{(1)} = \log Z_{\rm PI,div}^{(1)} + \log Z_{\rm PI,fin}^{(1)} - \frac{(d+3)\pi}{2}\, i$,

$$\log Z_{\rm PI,fin}^{(1)}(\ell) = -\frac{D_d}{2}\log\frac{A(\ell)}{4G_{\rm N}} + \alpha_{d+1}^{(2)}\log\frac{\ell}{L} + K_{d+1} \qquad (5.7.19)$$

$D_d = \dim\mathfrak{so}(d+2) = \frac{(d+2)(d+1)}{2}$, $A(\ell) = \Omega_{d-1}\ell^{d-1}$, $\alpha_{d+1}^{(2)} = 0$ for even d and given by (5.5.21) for odd d. L is an arbitrary length scale canceling out of the sum of finite and divergent parts, and K_{d+1} an exactly computable numerical constant. Explicitly for $d=2,3,4$, from (5.5.25):

d	$\log Z_{\rm PI,div}^{(1)}$	$\log Z_{\rm PI,fin}^{(1)}$
2	$0 - \frac{9\pi}{2}\frac{1}{\epsilon}\ell$	$-3\log(\frac{\pi}{2G_{\rm N}}\ell) + 5\log(2\pi)$
3	$\frac{8}{3}\frac{1}{\epsilon^4}\ell^4 - \frac{32}{3}\frac{1}{\epsilon^2}\ell^2 - \frac{571}{45}\log(\frac{2e^{-\gamma}}{\epsilon}L)$	$-5\log(\frac{\pi}{G_{\rm N}}\ell^2) - \frac{571}{45}\log(\frac{1}{L}\ell) - \log(\frac{8\pi}{3}) + \frac{715}{48} - \frac{47\,\zeta'(-1)}{3} + \frac{2\,\zeta'(-3)}{3}$
4	$\frac{15\pi}{8}\frac{1}{\epsilon^5}\ell^5 - \frac{65\pi}{24}\frac{1}{\epsilon^3}\ell^3 - \frac{105\pi}{16}\frac{1}{\epsilon}\ell$	$-\frac{15}{2}\log(\frac{\pi^2}{2G_{\rm N}}\ell^3) + \log(12) + \frac{27}{2}\log(2\pi) + \frac{65\,\zeta(3)}{48\,\pi^2} + \frac{5\,\zeta(5)}{16\,\pi^4}$

(5.7.20)

The one-loop energy and entropy (5.7.6)-(5.7.7) are split accordingly. The finite parts are

$$S_{\rm PI,fin}^{(1)} = \log Z_{\rm PI,fin}^{(1)} + V\rho_{\rm fin}^{(1)}\,, \qquad V\rho_{\rm fin}^{(1)} = \tfrac{1}{2}\tfrac{d-1}{d+1}D_d - \tfrac{1}{d+1}\alpha_{d+1}^{(2)}\,, \qquad (5.7.21)$$

where as always $2\pi\ell\, U = V\rho$ with $V = \Omega_{d+1}\ell^{d+1}$. For $d = 2, 3, 4$:

d	$V\rho^{(1)}_{\rm div}$	$V\rho^{(1)}_{\rm fin}$	$S^{(1)}_{\rm PI,div}$
2	$0 + \frac{3\pi}{2}\frac{1}{\epsilon}\ell$	1	$-3\pi\frac{1}{\epsilon}\ell$
3	$-\frac{8}{3}\frac{1}{\epsilon^4}\ell^4 + \frac{16}{3}\frac{1}{\epsilon^2}\ell^2$	$\frac{5}{2} + \frac{571}{180}$	$-\frac{16}{3}\frac{1}{\epsilon^2}\ell^2 - \frac{571}{45}\log(\frac{2e^{-\gamma}}{\epsilon}L)$
4	$-\frac{15\pi}{8}\frac{1}{\epsilon^5}\ell^5 + \frac{13\pi}{8}\frac{1}{\epsilon^3}\ell^3 + \frac{21\pi}{16}\frac{1}{\epsilon}\ell$	$\frac{9}{2}$	$-\frac{13\pi}{12}\frac{1}{\epsilon^3}\ell^3 - \frac{21\pi}{4}\frac{1}{\epsilon}\ell$

(5.7.22)

Like their quasicanonical bulk counterparts, the Euclidean quantities obtained here are UV-divergent, and therefore ill-defined from a low-energy effective field theory point of view. However if the metric itself, i.e. gravity, is dynamical, these the UV-sensitive terms can be absorbed into standard renormalizations of the gravitational coupling constants, rendering the Euclidean thermodynamics finite and physically meaningful. We turn to this next.

5.8 Quantum gravitational thermodynamics

In section 5.7 we considered the Euclidean thermodynamics of effective field theories on a fixed background geometry. In general the Euclidean partition function and entropy depend on the choice of background metric; more specifically on the background sphere radius ℓ. Here we specialize to field theories which include the metric itself as a dynamical field, i.e. we consider gravitational effective field theories. We denote $Z_{\rm PI}$, $\rho_{\rm PI}$ and $S_{\rm PI}$ by $\mathcal{Z}$, ϱ and $\mathcal{S}$ in this case:

$$\mathcal{Z} = \textstyle\int \mathcal{D}g \,\cdots\, e^{-S_E[g,\ldots]}\,, \qquad S_E[g,\ldots] = \frac{1}{8\pi G}\int\!\sqrt{g}\,\left(\Lambda - \tfrac{1}{2}R + \cdots\right). \tag{5.8.1}$$

The geometry itself being dynamical, we have $\partial_\ell \mathcal{Z} = 0$, so (5.7.6)-(5.7.7) reproduce (5.1.1):

$$\varrho = 0\,, \quad \mathcal{S} = \log \mathcal{Z}\,, \tag{5.8.2}$$

We will assume $d \geq 2$, but it is instructive to first consider $d = 0$, i.e. 1D quantum gravity coupled to quantum mechanics on a circle. Then $\mathcal{Z} = \int \frac{d\beta}{2\beta}\,{\rm Tr}\, e^{-\beta H}$, where β is the circle size and H is the Hamiltonian of the quantum mechanical system shifted by the 1D cosmological constant. To implement the conformal factor contour rotation of [180] implicit in (5.8.2), we pick an integration contour $\beta = 2\pi\ell + iy$ with $y \in \mathbb{R}$ and $\ell > 0$ the background circle radius. Then $\mathcal{Z} = \pi i \mathcal{N}(0)$ where $\mathcal{N}(E)$ is the number of states with $H < E$. This being ℓ-independent implies $\varrho = 0$. A general definition of microcanonical entropy is $S_{\rm mic}(E) = \log \mathcal{N}(E)$. Thus, modulo the content-independent πi factor in $\mathcal{Z}$, $\mathcal{S} = \log \mathcal{Z}$ is the microcanonical entropy at zero energy in this case.

Of course $d = 0$ is very different from the general-d case, as there is no classical saddle of the gravitational action, and no horizon. For $d \geq 2$ and $\Lambda \to 0$, the path integral has a semiclassical expansion about a round sphere saddle or radius $\ell_0 \propto 1/\sqrt{\Lambda}$, and $\mathcal{S}$ is dominated by the leading tree-level horizon entropy (5.1.2). As in the AdS-Schwarzschild case reviewed in C.4.5.1, the microscopic degrees of freedom accounting for the horizon entropy, assuming they exist, are invisible in the effective field theory. A natural analog of the dual large-N CFT partition function on $S^1 \times S^{d-1}$ microscopically computing the AdS-Schwarzschild free energy may be some dual large-N quantum mechanics coupled to 1D gravity on S^1 microscopically computing the dS static patch entropy. These considerations suggest interpreting $\mathcal{S} = \log \mathcal{Z}$ as a macroscopic approximation to a microscopic microcanonical entropy, with the semiclassical/low-energy expansion mapping to some large-N expansion.

The one-loop corrected $\mathcal{Z}$ is obtained by expanding the action to quadratic order about its sphere saddle. The Gaussian $Z_{\rm PI}^{(1)}$ was computed in previous sections. Locality and dimensional analysis imply that one-loop divergences are $\propto \int R^n$ with $2n \leq d+1$. Picking counterterms canceling all (divergent and finite) local contributions of this type in the limit $\ell_0 \propto 1/\sqrt{\Lambda} \to \infty$, we get a well-renormalized $\mathcal{S} = \log \mathcal{Z}$ to this order. Proceeding along these lines would be the most straightforward path to the computational objectives of this section. However, when pondering comparisons to microscopic models, one is naturally led to wondering what the actual physics content is of what has been computed. This in turn leads to small puzzles and bigger questions, such as:

1. A natural guess would have been that the one-loop correction to the entropy $\mathcal{S}$ is given by a renormalized version of the Euclidean entropy $S_{\rm PI}^{(1)}$ (5.7.7). However (5.8.2) says it is given by a renormalized version of the free energy $\log Z_{\rm PI}^{(1)}$. In the examples given earlier, these two look rather different. Can these considerations be reconciled?
2. Besides local UV contributions absorbed into renormalized coupling constants determining

the tree-level radius ℓ_0, there will be nonlocal IR vacuum energy contributions (pictorially Hawking radiation in equilibrium with the horizon), shifting the radius from ℓ_0 to $\bar{\ell}$ by gravitational backreaction. The effect would be small, $\bar{\ell} = \ell_0 + O(G)$, but since the leading-order horizon entropy is $S(\ell) \propto \ell^{d-1}/G$, we have $S(\bar{\ell}) = S(\ell_0) + O(1)$, a shift at the one-loop order of interest. The horizon entropy term in (5.8.2) is $\mathcal{S}^{(0)} = S(\ell_0)$, apparently not taking this shift into account. Can these considerations be reconciled?

3. At any order in the large-ℓ_0 perturbative expansion, UV-divergences can be absorbed into a renormalization of a finite number of renormalized coupling constants, but for the result to be physically meaningful, these must be defined in terms of low-energy physical "observables", invariant under diffeomorphisms and local field redefinitions. In asymptotically flat space, one can use scattering amplitudes for this purpose. These are unavailable in the case at hand. What replaces them?

To address these and other questions, we follow a slghtly less direct path, summarized below, and explained in more detail including examples in appendix C.8.

Free energy/quantum effective action for volume

We define an off-shell free energy/quantum effective action $\Gamma(V) = -\log Z(V)$ for the volume, the Legendre transform of the off-shell entropy/moment-generating function $S(\rho)$:[16]

$$S(\rho) \equiv \log \textstyle\int \mathcal{D}g \, e^{-S_E[g] + \rho \int \sqrt{g}} \,, \qquad \log Z(V) \equiv S - V\rho \,, \qquad V = \partial_\rho S = \big\langle \textstyle\int \sqrt{g} \, \big\rangle_\rho \,. \quad (5.8.3)$$

At large V, the geometry semiclassically fluctuates about a round sphere. Parametrizing the mean volume V by a corresponding mean radius ℓ as $V(\ell) \equiv \Omega_{d+1} \ell^{d+1}$, we have

$$Z(\ell) = \textstyle\int_{\text{tree}} d\rho \int \mathcal{D}g \, e^{-S_E[g] + \rho(\int \sqrt{g} - V(\ell))} \,, \qquad (5.8.4)$$

where $\int_{\text{tree}} d\rho$ means saddle point evaluation, i.e. extremization. The Legendre transform (5.8.3)

[16]Non-metric fields in the path integral are left implicit. Note "off-shell" = on-shell for c.c. $\Lambda' = \Lambda - 8\pi G \, \rho$.

is the same as (5.7.8), so we get thermodynamic relations of the same form as (5.7.6)-(5.7.8):

$$dS = V\,d\rho\,, \quad d\log Z = -\rho\,dV\,, \qquad \rho = -\tfrac{1}{d+1}\ell\partial_\ell \log Z\,/\,V\,, \quad S = \big(1 - \tfrac{1}{d+1}\ell\partial_\ell\big)\log Z\,. \tag{5.8.5}$$

On-shell quantities are obtained at $\rho = 0$, i.e. at the minimum $\bar{\ell}$ of the free energy $-\log Z(\ell)$:

$$\varrho = \rho(\bar{\ell}) = 0\,, \qquad \mathcal{S} = S(\bar{\ell}) = \log Z(\bar{\ell})\,, \qquad \langle \textstyle\int \sqrt{g}\,\rangle = \Omega_{d+1}\bar{\ell}^{d+1}\,. \tag{5.8.6}$$

Tree level

At tree level (5.8.4) evaluates to

$$\log Z^{(0)}(\ell) = -S_E[g_\ell]\,, \qquad g_\ell = \text{round } S^{d+1} \text{ metric of radius } \ell\,, \tag{5.8.7}$$

readily evaluated for any action using $R_{\mu\nu\rho\sigma} = (g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})/\ell^2$, taking the general form

$$\log Z^{(0)} = \frac{\Omega_{d+1}\ell^{d+1}}{8\pi G}\big(-\Lambda + \tfrac{d(d+1)}{2}\,\ell^{-2} + z_1\,l_s^2\,\ell^{-4} + z_2\,l_s^4\,\ell^{-6} + \cdots\big)\,. \tag{5.8.8}$$

The z_n are R^{n+1} coupling constants and $l_s \ll \ell$ is the length scale of UV-completing physics. The off-shell entropy and energy density are obtained from $\log Z^{(0)}$ as in (5.8.5).

$$S^{(0)} = \frac{\Omega_{d-1}\ell^{d-1}}{4G}\big(1+s_1\,l_s^2\,\ell^{-2}+\cdots\big), \qquad \rho^{(0)} = \frac{1}{8\pi G}\big(\Lambda-\tfrac{d(d-1)}{2}\,\ell^{-2}+\rho_1\,l_s^2\,\ell^{-4}+\cdots\big) \tag{5.8.9}$$

where $s_n, \rho_n \propto z_n$ and we used $\Omega_{d+1} = \frac{2\pi}{d}\Omega_{d-1}$. The on-shell entropy and radius are given by

$$\mathcal{S}^{(0)} = S^{(0)}(\ell_0)\,, \qquad \rho^{(0)}(\ell_0) = 0\,, \tag{5.8.10}$$

either solved perturbatively for $\ell_0(\Lambda)$ or, more conveniently, viewed as parametrizing $\Lambda(\ell_0)$.

One loop

The one-loop order, (5.8.4) is a by construction tadpole-free Gaussian path integral, (C.8.31):

$$\log Z = \log Z^{(0)} + \log Z^{(1)}\,, \qquad \log Z^{(1)} = \log Z^{(1)}_{\rm PI} + \log Z_{\rm ct}\,, \tag{5.8.11}$$

with $Z_{\rm PI}^{(1)}$ as computed in sections 5.4-5.5 and $\log Z_{\rm ct}(\ell) = -S_{E,{\rm ct}}[g_\ell]$ a polynomial counterterm. We define renormalized coupling constants as the coefficients of the ℓ^{d+1-2n} terms in the $\ell \to \infty$ expansion of $\log Z$, and fix $\log Z_{\rm ct}$ by equating tree-level and renormalized coefficients of the polynomial part, which amounts to the renormalization condition

$$\lim_{\ell\to\infty} \partial_\ell \log Z^{(1)} = 0\,, \tag{5.8.12}$$

in even $d{+}1$ supplemented by $\log Z_{\rm ct}(0) \equiv -\alpha_{d+1} \log(2e^{-\gamma}L/\epsilon)$, implying $L\partial_L \log Z^{(0)} = \alpha_{d+1}$.

Example: 3D Einstein gravity + minimally coupled scalar (C.8.4.1), putting $\nu \equiv \sqrt{m^2\ell^2-1}$,

$$\log Z^{(1)} = -3 \log \frac{2\pi\ell}{4G} + 5\log(2\pi) \;-\; \sum_{k=0}^{2} \frac{\nu^k}{k!} \frac{{\rm Li}_{3-k}(e^{-2\pi\nu})}{(2\pi)^{2-k}} + \frac{\pi\nu^3}{6} \;-\; \frac{\pi m^3\ell^3}{6} + \frac{\pi m\ell}{4}\,. \tag{5.8.13}$$

The last two terms are counterterms. The first two are nonlocal graviton terms. The scalar part is $\mathcal{O}(1/m\ell)$ for $m\ell \gg 1$ but goes nonlocal at $m\ell \sim 1$, approaching $-\log(m\ell)$ for $m\ell \ll 1$.

Defining $\rho^{(1)}$ and $S^{(1)}$ from $\log Z^{(1)}$ as in (5.8.5), and the quantum on-shell $\bar\ell = \ell_0 + O(G)$ as in (5.8.6), the quantum entropy can be expressed in two equivalent ways, (C.8.38)-(C.8.39):

$$A:\; \mathcal{S} = S^{(0)}(\bar\ell) + S^{(1)}(\bar\ell) + \cdots\,, \qquad B:\; \mathcal{S} = S^{(0)}(\ell_0) + \log Z^{(1)}(\ell_0) + \cdots \tag{5.8.14}$$

where the dots denote terms neglected in the one-loop approximation. This simultaneously answers questions 1 and 2 on our list, reconciling intuitive (A) and (5.8.2)-based (B) expectations. To make this physically obvious, consider the quantum static patch as two subsystems, geometry (horizon) + quantum fluctuations (radiation), with total energy $\propto \rho = \rho^{(0)} + \rho^{(1)} = 0$. If $\rho^{(0)} = 0$, the horizon entropy is $S^{(0)}(\ell_0)$. But here we have $\rho = 0$, so the horizon entropy is actually $S^{(0)}(\bar\ell) = S^{(0)}(\ell_0) + \delta S^{(0)}$, where by the first law (5.8.5), $\delta S^{(0)} = V\delta\rho^{(0)} = -V\rho^{(1)}$. Adding the radiation entropy $S^{(1)}$ and recalling $\log Z^{(1)} = S^{(1)} - V\rho^{(1)}$ yields $\mathcal{S} = A = B$. Thus $A = B$ is just the usual small+large = system+reservoir approximation, the horizon being the reservoir, and the Boltzmann factor $e^{-V\rho^{(1)}} = e^{-\beta U^{(1)}}$ in $Z^{(1)}$ accounting for the reservoir's entropy change due to energy transfer to the system.

Viewing the quantum contributions as (Hawking) radiation has its picturesque merits and correctly conveys their nonlocal/thermal character, e.g. $\mathrm{Li}(e^{-2\pi\nu}) \sim e^{-\beta m}$ for $m\ell \gg 1$ in (5.8.13), but might incorrectly convey a presumption of positivity of $\rho^{(1)}$ and $S^{(1)}$. Though positive for minimally coupled scalars (fig. C.8.1), they are in fact *negative* for higher spins (figs. C.8.2, C.8.3), due to edge and group volume contributions. Moreover, although the negative-energy backreaction causes the horizon to grow, partially compensating the negative $S^{(1)}$ by a positive $\delta S^{(0)} = -V\rho^{(1)}$, the former still wins: $\mathcal{S}^{(1)} \equiv \mathcal{S} - \mathcal{S}^{(0)} = S^{(1)} - V\rho^{(1)} = \log Z^{(1)} < 0$.

Computational recipe and examples

For practical purposes, (B) is the more useful expression in (5.8.14). Together with (5.8.10) computing $\mathcal{S}^{(0)}$, the exact results for $Z_{\rm PI}^{(1)}$ obtained in previous sections (with $\gamma_0 = \sqrt{2\pi/\mathcal{S}^{(0)}}$, see (5.8.16) below), and the renormalization prescription outlined above, it immediately gives

$$\mathcal{S} = \mathcal{S}^{(0)} + \mathcal{S}^{(1)} + \cdots\,, \qquad \mathcal{S}^{(0)} = S^{(0)}(\ell_0)\,, \qquad \mathcal{S}^{(1)} = \log Z^{(1)}(\ell_0) \tag{5.8.15}$$

in terms of the renormalized coupling constants, for general effective field theories of gravity coupled to arbitrary matter and gauge fields.

For 3D gravity, this gives $\mathcal{S} = \mathcal{S}^{(0)} - 3\log \mathcal{S}^{(0)} + 5\log(2\pi) + O(1/\mathcal{S}^{(0)})$. We work out and plot several other concrete examples in appendix C.8.4: 3D Einstein gravity + scalar (C.8.4.1, fig. C.8.1), 3D massive spin s (C.8.4.2, fig. C.8.2), 2D scalar (C.8.4.3), 4D massive spin s (C.8.4.4, fig. C.8.3), and 3D,4D,5D gravity (including higher-order curvature corrections) (C.8.4.5). Table 5.1.12 in the introduction lists a few more sample results.

Local field redefinitions, invariant coupling constants and physical observables

Although the higher-order curvature corrections to the tree-level dS entropy $\mathcal{S}^{(0)} = S^{(0)}(\ell_0)$ (5.8.9) seem superficially similar to curvature corrections to the entropy of black holes in asymptotically flat space [188, 189], there are no charges or other asymptotic observables available here to endow them with physical meaning. Indeed, they have no intrinsic low-energy physical meaning at all, as they can be removed order by order in the l_s/ℓ expansion by a metric field redefinition, bringing the entropy to pure Einstein form (5.1.2). In $Z^{(0)}(\ell)$ (5.8.8), this amounts to setting all $z_n \equiv 0$ by a redefinition $\ell \to \ell \sum_n c_n \ell^{-2n}$ (C.8.21). The value of $\mathcal{S}^{(0)} = \max_{\ell \gg l_s} \log Z^{(0)}(\ell)$ remains of course unchanged, providing the unique field-redefinition invariant combination of the coupling constants G, Λ(or ℓ_0)$, z_1, z_2, \ldots$.

Related to this, as discussed in C.8.4.5, caution must be exercised when porting the one-loop graviton contribution in (5.5.17) or (5.7.19): $G_{\rm N}$ appearing in $\gamma_0 = \sqrt{8\pi G_{\rm N}/A}$ is the algebraically defined Newton constant (5.5.14), as opposed to G defined by the Ricci scalar coefficient $\frac{1}{8\pi G}$ in the low-energy effective action. The former is field-redefinition invariant; the latter is not. In Einstein frame $(z_n = 0)$ the two definitions coincide, hence in a general frame

$$\gamma_0 = \sqrt{2\pi/\mathcal{S}^{(0)}}\,. \tag{5.8.16}$$

Since $\log \mathcal{S}^{(0)} = \log \frac{A}{4G} + \log(1 + \mathcal{O}(l_s^2/\ell_0^2))$, this distinction matters only at $\mathcal{O}(l_s^2/\ell_0^2)$, however.

In $d = 2$, $\mathcal{S}^{(0)}$ is in fact the only invariant gravitational coupling: because the Weyl tensor vanishes identically, any 3D parity-invariant effective gravitational action can be brought to Einstein form by a field redefinition. In the Chern-Simons formulation of C.7.2, $\mathcal{S}^{(0)} = 2\pi\kappa$. In $d \geq 3$, the Weyl tensor vanishes on the sphere, but not identically. As a result, there are coupling constants not picked up by the sphere's $\mathcal{S}^{(0)} = -S_E[g_{\ell_0}]$. Analogous $\mathcal{S}_M^{(0)} \equiv -S_E[g_M]$ for different saddle geometries g_M, approaching Einstein metrics in the limit $\Lambda \propto \ell_0^{-2} \to 0$, can be used instead to probe them, and analogous $\mathcal{S}_M \equiv \log \mathcal{Z}_M$ expanded about g_M provide quantum observables. Section C.8.5 provides a few more details, and illustrates extraction of unambiguous linear combinations of the 4D one-loop correction for 3 different M.

This provides the general picture we have in mind as the answer, in principle, to question 3 on our list below (5.8.2): the tree-level $\mathcal{S}_M^{(0)}$ are the analog of tree-level scattering amplitudes, and the analog of quantum scattering amplitudes are the quantum $\mathcal{S}_M$.

Constraints on microscopic models

For pure 3D gravity $\mathcal{S}^{(0)} = \frac{2\pi}{4G}(\ell_0 + s_1\ell_0^{-1} + s_2\,\ell_0^{-3} + \cdots)$, and to one-loop order we have (C.8.62):

$$\mathcal{S} = \mathcal{S}^{(0)} - 3\log \mathcal{S}^{(0)} + 5\log(2\pi) + \cdots\,. \tag{5.8.17}$$

Granting[17] (C.7.24) with $l = 0$ gives the all-loop expansion of pure 3D gravity, taking into

[17] This does not affect the 1-loop based conclusions below, but does affect the c_n. One could leave l general.

account $G \equiv \mathrm{SO}(4)$ here while $G \equiv SU(2) \times SU(2)$ there, to all-loop order,

$$\mathcal{S} = \mathcal{S}_0 + \log\Big|\sqrt{\tfrac{4}{2+i\,\mathcal{S}_0/2\pi}}\ \sin\!\big(\tfrac{\pi}{2+i\,\mathcal{S}_0/2\pi}\big)\Big|^2 = \mathcal{S}_0 - 3\log\mathcal{S}_0 + 5\log(2\pi) + \textstyle\sum_n c_n\,\mathcal{S}_0^{-2n} \quad (5.8.18)$$

where $\mathcal{S}_0 \equiv \mathcal{S}^{(0)}$ to declutter notation. Note all quantum corrections are strictly nonlocal, i.e. no odd powers of ℓ_0 appear, reflected in the absence of odd powers of $1/\mathcal{S}_0$.

Though outside the scope of this paper, let us illustrate how such results may be used to constrain microscopic models identifying large-ℓ_0 and large-N expansions in some way. Say a modeler posits a model consisting of $2N$ spins $\sigma_i = \pm 1$ with $H \equiv \sum_i \sigma_i = 0$. The microscopic entropy is $S_{\rm mic} = \log\binom{2N}{N} = 2\log 2\cdot N - \frac{1}{2}\log(\pi N) + \sum_n c'_n N^{1-2n}$. There is a unique identification of $\mathcal{S}_0$ bringing this in a form with the same analytic/locality structure as (5.8.18), to wit, $\mathcal{S}_0 = \log 4\cdot N + \sum_n c'_n N^{1-2n}$, resulting in

$$S_{\rm mic} \,=\, \mathcal{S}_0 - \tfrac{1}{2}\log\mathcal{S}_0 + \log(\tfrac{\pi}{2\log 2}) + \textstyle\sum_n c''_n\,\mathcal{S}_0^{-2n}\,, \qquad (5.8.19)$$

where $c''_1 = -\frac{1}{8}\log 2, c''_2 = \frac{3}{64}(\log 2)^2 + \frac{1}{48}(\log 2)^3, \ldots$, fully failing to match (5.8.18), starting at one loop. The model is ruled out.

A slightly more sophisticated modeler might posit $S_{\rm mic} = \log d(N)$, where $d(N)$ is the N-th level degeneracy of a chiral boson on S^1. To leading order $S_{\rm mic} \approx 2\pi\sqrt{N/6} \equiv K$. Beyond, $S_{\rm mic} = K - a'\log K + b' + \sum_n c'_n K^{-n} + O(e^{-K/2})$, where $a' = 2$, $b' = \log(\pi^2/6\sqrt{3})$ and c'_n given by [190]. Identifying $\mathcal{S}_0 = K + \sum_n c'_{2n-1}K^{-(2n-1)}$ brings this to the form (5.8.18), yielding $S_{\rm mic} = \mathcal{S}_0 - a'\log\mathcal{S}_0 + b' + \sum_n c''_n\mathcal{S}_0^{-2n} + O(e^{-\mathcal{S}_0/2})$, with $c''_1 = -\frac{5}{2}$, $c''_2 = \frac{37}{12}$, ... — ruled out.

We actually did not need the higher-loop corrections at all to rule out the above models. In higher dimensions, or coupled to more fields, one-loop constraints moreover become increasingly nontrivial, evident in (5.1.12). For pure 5D gravity (C.8.62),

$$\mathcal{S} = \mathcal{S}^{(0)} - \frac{15}{2}\log\mathcal{S}^{(0)} + \log(12) + \frac{27}{2}\log(2\pi) + \frac{65\,\zeta(3)}{48\,\pi^2} + \frac{5\,\zeta(5)}{16\,\pi^4}\,. \qquad (5.8.20)$$

It would be quite a miracle if a microscopic model managed to match this.

5.9 dS, AdS±, and conformal higher-spin gravity

Vasiliev higher-spin gravity theories [53, 55, 172] have infinite spin range and an infinite-dimensional higher-spin algebra, $\mathfrak{g} = \mathrm{hs}(\mathfrak{so}(d+2))$, leading to divergences in the one-loop sphere partition function formula (5.5.17) untempered by the UV cutoff. In this section we take a closer look at these divergences. We contrast the situation to AdS with standard boundary conditions (AdS+), where the issue is entirely absent, and we point out that, on the other hand, for AdS with alternate HS boundary conditions (AdS−) as well as conformal higher-spin (CHS) theories, similar issues arise. We end with a discussion of their significance.

5.9.1 dS higher-spin gravity

Nonminimal type A Vasiliev gravity on dS$_{d+1}$ has a tower of massless spin-s fields for all $s \geq 1$ and a $\Delta = d-2$ scalar. We first consider $d = 3$. The total bulk and edge characters are obtained by summing (5.5.7) and adding the scalar, as we did for the bulk part in (5.2.38):

$$\Theta_{\rm bulk} \,=\, 2\cdot\left(\frac{q^{1/2}+q^{3/2}}{(1-q)^2}\right)^2 - \frac{q}{(1-q)^2}\,, \qquad \Theta_{\rm edge} = 2\cdot\left(\frac{q^{1/2}+q^{3/2}}{(1-q)^2}\right)^2 . \tag{5.9.1}$$

Quite remarkably, the bulk and edge contributions almost exactly cancel:

$$\Theta_{\rm bulk} - \Theta_{\rm edge} = -\frac{q}{(1-q)^2}\,. \tag{5.9.2}$$

For $d = 4$ however, we see from (5.5.7) that due to the absence of overall q^s suppression factors, the total bulk and edge characters each diverge separately by an overall multiplicative factor:

$$\Theta_{\rm bulk} = \sum_s (2s+1)\cdot\frac{2\,q^2}{(1-q)^4}\,, \qquad \Theta_{\rm edge} = \sum_s \tfrac{1}{6}s(s+1)(2s+1)\cdot\frac{2\,q}{(1-q)^2}\,. \tag{5.9.3}$$

This pattern persists for all $d \geq 4$, as can be seen from the explicit form of bulk and edge characters in (3.4.20), (C.6.2) and (C.6.23). For any d, there is moreover an infinite-dimensional group volume factor in (5.5.17) to make sense of, involving a divergent factor $(\ell^{d-1}/G_{\rm N})^{\dim G/2}$ and the volume of an object of unclear mathematical existence [191].

Before we continue the discussion of what, if anything, to make of this, we consider AdS±

and CHS theories within the same formalism. Besides independent interest, this will make clear the issue is neither intrinsic to the formalism, nor to de Sitter.

5.9.2 AdS$\pm$ higher-spin gravity

AdS characters for standard and alternate HS boundary conditions

Standard boundary conditions on massless higher spin fields φ in AdS$_{d+1}$ lead to quantization such that spin-s single-particle states transform in a UIR of $\mathfrak{so}(2,d)$ with primary dimension $\Delta_\varphi = \Delta_+ = s+d-2$. Higher-spin Euclidean AdS one-loop partition functions with these boundary conditions were computed in [111–113]. In [138], the Euclidean one-loop partition function for alternate boundary conditions ($\Delta_\varphi = \Delta_- = 2-s$) was considered. In the EAdS$+$ case, the complications listed under (5.5.3) are absent, but for EAdS$-$ close analogs do appear.

EAdS path integrals can be expressed as character integrals [3, 144, 173], in a form exactly paralleling the formulae and bulk/edge picture of the present work [3].[18] The AdS analog of the dS bulk and edge characters (5.4.8) for a *massive* spin-s field φ with $\Delta_\varphi = \Delta_\pm$ is [3]

$$\Theta^{\mathrm{AdS}\pm}_{\mathrm{bulk},\varphi} \equiv D^d_s \, \frac{q^{\Delta_\pm}}{(1-q)^d} \,, \qquad \Theta^{\mathrm{AdS}\pm}_{\mathrm{edge},\varphi} \equiv D^{d+2}_{s-1} \, \frac{q^{\Delta_\pm - 1}}{(1-q)^{d-2}} \,, \tag{5.9.4}$$

where $\Delta_- = d - \Delta_+$. Thus, as functions of q,

$$\Theta^{\mathrm{dS}}_\varphi = \Theta^{\mathrm{AdS}+}_\varphi + \Theta^{\mathrm{AdS}-}_\varphi \,. \tag{5.9.5}$$

The AdS analog of (5.5.4) for a massless spin-s field ϕ_s with gauge parameter field $\xi_{s'}$ is

$$\hat{\Theta}^{\mathrm{AdS}\pm}_s \equiv \Theta^{\mathrm{AdS}\pm}_\phi - \Theta^{\mathrm{AdS}\pm}_\xi \,, \tag{5.9.6}$$

[18] In this picture, EAdS is viewed as the Wick-rotated AdS-Rindler wedge, with dS$_d$ static patch boundary metric, as in [192, 193]. The bulk character is $\Theta \equiv \mathrm{tr}_G \, q^{iH}$, with H the *Rindler* Hamiltonian, *not* the global AdS Hamiltonian. Its q-expansion counts *quasinormal modes* of the Rindler wedge. The one-loop results are interpreted as corrections to the gravitational thermodynamics of the AdS-Rindler horizon [3, 192, 193].

where $\Delta_{\phi,+} = s' + d - 1$, $\Delta_{\xi,+} = s + d - 1$, $s' \equiv s - 1$. More explicitly, analogous to (5.5.5),

$$\hat{\Theta}^{\text{AdS}+}_{\text{bulk},s} = \frac{D^d_s\, q^{s'+d-1} - D^d_{s'}\, q^{s+d-1}}{(1-q)^d}\,, \qquad \hat{\Theta}^{\text{AdS}+}_{\text{edge},s} = \frac{D^{d+2}_{s-1}\, q^{s'+d-2} - D^{d+2}_{s'-1}\, q^{s+d-2}}{(1-q)^{d-2}} \tag{5.9.7}$$

$$\hat{\Theta}^{\text{AdS}-}_{\text{bulk},s} = \frac{D^d_s\, q^{1-s'} - D^d_{s'}\, q^{1-s}}{(1-q)^d}\,, \qquad \hat{\Theta}^{\text{AdS}-}_{\text{edge},s} = \frac{D^{d+2}_{s-1}\, q^{-s'} - D^{d+2}_{s'-1}\, q^{-s}}{(1-q)^{d-2}}\,. \tag{5.9.8}$$

The presence of non-positive powers of q in $\Theta^{\text{AdS}-}$ has a similar path integral interpretation as in the dS case summarized in section 5.5.2. The necessary negative mode contour rotation and zeromode subtractions are again implemented at the character level by flipping characters. In particular the proper Θ_s to be used in the character formulae for EAdS$\pm$ are

$$\Theta^{\text{AdS}-}_s = \big[\hat{\Theta}^{\text{AdS}-}_s\big]_+\,, \qquad \Theta^{\text{AdS}+}_s = \big[\hat{\Theta}^{\text{AdS}+}_s\big]_+ = \hat{\Theta}^{\text{AdS}+}_s\,, \tag{5.9.9}$$

with $[\hat{\Theta}]_+$ defined as in (3.4.17). The omission of Killing tensor zeromodes for alternate boundary conditions must be compensated by a a division by the volume of the residual *gauge* group G generated by the Killing tensors. Standard boundary conditions on the other hand kill these Killing tensor zeromodes: they are not part of the dynamical, fluctuating degrees of freedom. The group G they generate acts nontrivially on the Hilbert space as a global symmetry group.

AdS$+$

For standard boundary conditions, the character formalism reproduces the original results of [111–113] by two-line computations [3]. We consider some examples:

For nonmimimal type A Vasiliev with $\Delta_0 = d - 2$ scalar boundary conditions, dual to the free $U(N)$ model, using (5.9.7) and the scalar $\Theta_0 = q^{d-2}/(1-q)^d$, the following total bulk and edge characters are readily obtained:

$$\Theta^{\text{AdS}+}_{\text{bulk}} = \sum_{s=0}^{\infty} \Theta^{\text{AdS}+}_{\text{bulk},s} = \left(\frac{q^{\frac{d}{2}-1} + q^{\frac{d}{2}}}{(1-q)^{d-1}}\right)^2, \quad \Theta^{\text{AdS}+}_{\text{edge}} = \sum_{s=0}^{\infty} \Theta^{\text{AdS}+}_{\text{edge},s} = \left(\frac{q^{\frac{d}{2}-1} + q^{\frac{d}{2}}}{(1-q)^{d-1}}\right)^2. \tag{5.9.10}$$

The total bulk character takes the singleton-squared form expected from the Flato-Fronsdal theorem [194]. More interestingly, the edge characters sum up to exactly the same. Thus the generally negative nature of edge "corrections" takes on a rather dramatic form here:

$$\Theta^{\rm AdS+}_{\rm tot} = \Theta^{\rm AdS+}_{\rm bulk} - \Theta^{\rm AdS+}_{\rm edge} = 0 \qquad \Rightarrow \qquad \log Z^{\rm AdS+}_{\rm PI} = 0\,. \tag{5.9.11}$$

As $Z^{\rm AdS+}_{\rm bulk}$ has an Rindler bulk ideal gas interpretation analogous to the static patch ideal gas of section 5.2 [3], the exact bulk-edge cancelation on display here is reminiscent of analogous one-loop bulk-edge cancelations expected in string theory according to the qualitative picture reviewed in appendix C.4.5.2.

For minimal type A, dual to the free O(N) model, the sum yields an expression which after rescaling of integration variables $t \to t/2$ is effectively equivalent to the $\mathfrak{so}(2,d)$ singleton character, which is also the $\mathfrak{so}(1,d)$ character of a conformally coupled ($\nu = i/2$) scalar on S^d. Using (5.3.10), this means $Z^{\rm AdS+}_{\rm PI}$ equals the sphere partition function on S^d, immediately implying the $N \to N-1$ interpretation of [111–113].

For nonminimal type A with $\Delta_0 = 2$ scalar boundary conditions, dual to an interacting U(N) CFT, the cancelation is *almost* exact but not quite:

$$\Theta^{\rm AdS+}_{\rm tot} = \frac{\sum_{k=2}^{d-3} q^k}{(1-q)^{d-1}}\,. \tag{5.9.12}$$

AdS$+$ higher-spin swampland

In the above examples it is apparent that although the spin-summed $\Theta_{\rm bulk}$ has increased effective UV-dimensionality $d^{\rm bulk}_{\rm eff} = 2d-2$, as if we summed KK modes of a compactification manifold of dimension $d-2$, the edge subtraction collapses this back down to a net $d_{\rm eff} = d-1$, *decreasing* the original d. Correspondingly, the UV-divergences of $Z^{(1)}_{\rm PI}$ are not those of a $d+1$ dimensional bulk-local theory, but rather of a d-dimensional *boundary*-local theory. In fact this peculiar property appears necessary for quantum consistency, in view of the non-existence of a nontrivially interacting local *bulk* action [195]. It appears to be true for all AdS$+$ higher spin theories with a known holographic dual [3], but not for all classically consistent higher-spin theories. Thus it appears to be some kind of AdS higher-spin "swampland" criterion:

$$\text{AdS}_{d+1} \text{ HS theory has holographic dual} \qquad \Rightarrow \qquad d_{\rm eff} = d-1\,. \tag{5.9.13}$$

Higher-spin theories violating this criterion do exist. Theories with a tower of massless spins $s \geq 2$ and an a priori undetermined number n of real scalars can be constructed in AdS$_3$ [196, 197]. Assuming all integer spins $s \geq 2$ are present, the total character sums up to

$$\Theta_{\rm tot} = \frac{2\,q^2}{(1-q)^2} - \frac{4\,q}{(1-q)^2} + \sum_{i=1}^{n} \frac{q^{\Delta_i}}{(1-q)^3}\,. \tag{5.9.14}$$

For $t \to 0$ diverges as $\Theta_{\rm HS} \sim (n-2)/t^2 + \mathcal{O}(1/t)$. To satisfy (5.9.13), the number of scalars must be $n = 2$. This is inconsistent with the $n = 4$ AdS$_3$ theory originally conjectured in [197] to be dual to a minimal model CFT$_2$, but consistent with the amended conjecture of [198–200].

AdS$-$

For alternate boundary conditions, one ends up with a massless higher-spin character formula similar to (5.5.17). The factor $\gamma^{\dim G}$ in (5.5.17) is consistent with $\log Z_{\rm PI}^{\rm AdS-} \propto (G_{\rm N})^{\frac{1}{2}\sum_s N_{s-1}^{\rm KT}}$ found in [138]. (5.9.5) implies the massless AdS$\pm$ and dS bulk and edge characters are related as

$$\boxed{\Theta_s^{\rm AdS-} = \Theta_s^{\rm dS} - \Theta_s^{\rm AdS+}} \tag{5.9.15}$$

hence we can read off the appropriate flipped $\Theta_s^{\rm AdS-} = [\hat\Theta_s^{\rm AdS-}]_+$ characters from our earlier explicit results (3.4.20) and (C.6.23) for $\Theta_s^{\rm dS}$. Just like in the dS case, the final result involves divergent spin sums when the spin range is infinite.

5.9.3 Conformal higher-spin gravity

Conformal HS characters

Conformal (higher-spin) gravity theories [201] have (higher-spin extensions of) diffeomorphisms and local Weyl rescalings as gauge symmetries. If one does not insist on a local action, a general way to construct such theories is to view them as induced theories, obtained by integrating out the degrees of freedom of a conformal field theory coupled to a general background metric and other background fields. In particular one can consider a free $U(N)$ CFT$_d$ in a general metric and higher-spin source background. For even d, this results in a local action, which at least at the free level can be rewritten as a theory of towers of partially massless fields with standard kinetic terms [139, 174]. Starting from this formulation of CHS theory on S^d (or equivalently dS$_d$), using our general explicit formulae for partially massless higher-spin field characters (3.4.20) and (C.6.23), and summing up the results, we find

$$\boxed{\Theta_s^{\rm CdS_d} = \Theta_s^{\rm AdS_{d+1}-} - \Theta_s^{\rm AdS_{d+1}+} = \Theta_s^{\rm dS_{d+1}} - 2\,\Theta_s^{\rm AdS_{d+1}+}} \tag{5.9.16}$$

where $\Theta_s^{\rm CdS_d}$ are the CHS bulk and edge characters and the second equality uses (5.9.15). Since we already know the explicit dS and AdS HS bulk and edge characters, this relation also provides the explicit CHS bulk and edge characters. For example

d	s	$\Theta^{\rm CdS_d}_{{\rm bulk},s} \cdot (1-q)^d$	$\Theta^{\rm CdS_d}_{{\rm edge},s} \cdot (1-q)^{d-2}$
2	≥ 2	$-4q^s(1-q)$	$-2\big(s^2q^{s-1} - (s-1)^2 q^s\big)$
3	≥ 1	0	0
3	0	$-q(1-q)$	0
4	≥ 0	$2(2s{+}1)\,q^2 {+} 2s^2 q^{s+3} {-} 2(s{+}1)^2 q^{s+2}$	$\frac{s(s+1)(2s+1)}{3}\,q {+} \frac{(s-1)s^2(s+1)}{6}\,q^{s+2} {-} \frac{s(s+1)^2(s+2)}{6}\,q^{s+1}$
5	≥ 0	$\frac{(s+1)(2s+1)(2s+3)}{3}\,q^2(1-q)$	$\frac{s(s+1)(s+2)(2s+1)(2s+3)}{30}\,q(1-q)$

(5.9.17)

The bulk SO$(1,d)$ q-characters $\Theta^{\rm CdS_d}_{{\rm bulk},s}$ computed from (5.9.16) agree with the $\mathfrak{so}(2,d)$ q-characters obtained in [202]. Edge characters were not derived in [202], as they have no role in the thermal $S^1 \times S^{d-1}$ CHS partition functions studied there.[19]

The one-loop Euclidean path integral of the CHS theory on S^d is given by (5.5.17) using the bulk and edge CHS characters $\Theta_s^{\rm CdS_d}$ and with G the CHS symmetry group generated by the conformal Killing tensors on S^d (counted by $D^{d+3}_{s-1,s-1}$). The coefficient of the log-divergent term, the Weyl anomaly of the CHS theory, is extracted as usual, by reading off the coefficient of the $1/t$ term in the small-t expansion of the integrand in (5.5.17), or more directly from the "naive" integrand $\frac{1}{2t}\frac{1+q}{1-q}\,\hat{\Theta}$. For example for conformal $s=2$ gravity on S^2 coupled to D massless scalars, also known as bosonic string theory in D spacetime dimensions, we have

[19] A priori the interpretation of the bulk characters in (5.9.17) and those in [202] is different. Their mathematical equality is a consequence of the enhanced $\mathfrak{so}(2,d)$ symmetry allowing to map $S^d \to \mathbb{R} \times S^{d-1}$.

$\dim G = \sum_{\pm} D^4_{1,\pm 1} = 6$, generating $G = \mathsf{SO}(1,3)$, and from the above table (5.9.17),

$$\Theta_{\rm tot} = D \cdot \frac{1+q}{1-q} - \frac{4q^2}{1-q} + 2(4q - q^2)\,. \tag{5.9.18}$$

The small-t expansion of the integrand in (5.5.17) for this case is

$$\frac{1}{2t}\frac{1+q}{1-q}(\Theta_{\rm tot} - 12) \to \frac{2(D-2)}{t^3} + \frac{D-26}{3\,t} + \cdots\,, \tag{5.9.19}$$

reassuringly informing us the critical dimension for the bosonic string is $D = 26$. Adding a massless $s = \frac{3}{2}$ field, we get 2D conformal supergravity. For half-integer conformal spin s, $\Theta_{\rm bulk} = -4q^s/(1-q)$ and $\Theta_{\rm edge} = -2\big((s-\frac{1}{2})(s+\frac{1}{2})q^{s-1} - (s-\frac{3}{2})(s-\frac{1}{2})q^s\big)$. Furthermore adding D' massless Dirac spinors, the total fermionic character is

$$\Theta^{\rm fer}_{\rm tot} = D' \cdot \frac{2\,q^{1/2}}{1-q} - \frac{4\,q^{3/2}}{1-q} + 4\,q^{1/2}\,. \tag{5.9.20}$$

The symmetry algebra has $\sum_{\pm} D^4_{\frac{1}{2},\pm\frac{1}{2}} = 4$ fermionic generators, contributing negatively to $\dim G$ in (5.5.17). Putting everything together,

$$\frac{1}{2t}\,\frac{1+q}{1-q}\big(\Theta^{\rm bos}_{\rm tot} - 2(6-4)\big) - \frac{1}{2t}\,\frac{\sqrt{q}}{1-q}\,\Theta^{\rm fer}_{\rm tot} \to \frac{2(D-D')}{t^3} + \frac{2D+D'-30}{6\,t} + \cdots\,, \tag{5.9.21}$$

from which we read off supersymmetry + conformal symmetry requires $D' = D = 10$.

More systematically, the Weyl anomaly $\alpha_{d,s}$ can be read off by expanding $\frac{1}{2t}\frac{1+q}{1-q}\,\hat\Theta^{\rm CS^d}$ with $\hat\Theta^{\rm CS^d} = \hat\Theta^{\rm AdS_{d+1}-} - \hat\Theta^{\rm AdS_{d+1}+}$ given by (5.9.7)-(5.9.8) for integer s. For example,

d	$-\alpha_{d,s}$
2	$\frac{2(6s^2-6s+1)}{3}$
4	$\frac{s^2(s+1)^2(14s^2+14s+3)}{180}$
6	$\frac{(s+1)^2(s+2)^2(22s^6+198s^5+671s^4+1056s^3+733s^2+120s-50)}{151200}$
8	$\frac{(s+1)(s+2)^2(s+3)^2(s+4)(150s^8+3000s^7+24615s^6+106725s^5+261123s^4+351855s^3+225042s^2+31710s-14560)}{2286144000}$

(5.9.22)

This reproduces the $d = 2, 4, 6$ results of [139, 174] and generalizes them to any d.

Physics pictures

Cartoonishly speaking, the character relation (5.9.16) translates to one-loop partition function relations of the form $Z^{\mathrm{CS}^d} \sim Z^{\mathrm{EAdS_{d+1}}-}/Z^{\mathrm{EAdS_{d+1}}+}$ and $Z^{S^{d+1}} \sim Z^{\mathrm{CS}^d}\big(Z^{\mathrm{EAdS_{d+1}}+}\big)^2$. The first relation can then be understood as a consequence of the holographic duality between AdS$_{d+1}$ higher-spin theories and free CFT$_d$ vector models [138, 139, 174], while the second relation can be understood as an expression at the Gaussian/one-loop level of $Z^{S^{d+1}} \sim \int \mathcal{D}\sigma \, |\psi_{\mathrm{HH}}(\sigma)|^2$, where $\psi_{\mathrm{HH}}(\sigma) = \psi_{\mathrm{HH}}(0)\, e^{-\frac{1}{2}\sigma K \sigma + \cdots}$ is the late-time dS Hartle-Hawking wave function, related by analytic continuation to the EAdS partition function with boundary conditions σ [203]. The factor $\big(Z^{\mathrm{EAdS}_{d+1}+}\big)^2$ can then be identified with the bulk one-loop contribution to $|\psi_{\mathrm{HH}}(0)|^2$, and $Z^{\mathrm{cnf}\, S^d}$ with $\int \mathcal{D}\sigma \, e^{-\sigma K \sigma}$, along the lines of [138]. Along the lines of footnote 8, perhaps another interpretation of the spin-summed relation (5.9.16) exists within the picture of [32].

5.9.4 Comments on infinite spin range divergences

Let us return now to the discussion of section 5.9.1. Above we have seen that for EAdS+, summing spin characters leads to clean and effortless computation of the one-loop partition function. The group volume factor is absent because the global higher-spin symmetry algebra $\mathfrak{g}$ generated by the Killing tensors is not gauged. The character spin sum converges, and no additional regularization is required beyond the UV cutoff at $t \sim \epsilon$ we already had in place. The underlying reason for this is that in AdS+, the minimal energy of a particle is bounded below by its spin, hence a UV cutoff is effectively also a spin cutoff. In contrast, for dS, AdS− and CHS theories alike, $\mathfrak{g}$ is gauged, leading to the group volume division factor, and moreover, for $d \geq 4$, the quasinormal mode levels (or energy levels for CHS on $\mathbb{R} \times S^{d-1}$) are infinitely degenerate, not bounded below by spin, leading to character spin sum divergences untempered by the UV cutoff. The geometric origin of quasinormal modes decaying as slowly as $e^{-2T/\ell}$ for every spin s in $d \geq 4$ was explained below (C.6.4).

One might be tempted to use some form of zeta function regularization to deal with divergent sums $\sum_s \Theta_s$ such as (5.9.3), which amounts to inserting a convergence factor $\propto e^{-\delta s}$ and discarding the divergent terms in the limit $\delta \to 0$. This might be justified if the discarded divergences were UV, absorbable into local counterterms, but that is not the case here. The divergence is due to *low*-energy features, the infinite multiplicity of slow-decaying quasinormal modes, analogous to the divergent thermodynamics of an ideal gas in a box with an infinite number of different massless particle species. Zeta function regularization would give a finite result, but the result would be meaningless.

As discussed at the end of section 5.6, the Vasiliev-like[20] limit of the 3D HS_n higher-spin gravity theory, $n \to \infty$ with $l = 0$ and $\mathcal{S}^{(0)}$ fixed, is strongly coupled as a 3D QFT. Unsurprisingly, the one-loop entropy "correction" $\mathcal{S}^{(1)} = \log Z^{(1)}$ diverges in this limit: writing the explicit expression for the maximal-entropy vacuum $R = \mathbf{n}$ in (5.1.12) as a function of $\dim G = 2(n^2 - 1)$, one gets $\mathcal{S}^{(1)} = \dim G \cdot \log\bigl(\dim G/\sqrt{\mathcal{S}^{(0)}}\bigr) + \cdots \to \infty$. The higher-spin decomposition (C.7.41) might inspire an ill-advised zeta function regularization along the lines of $\dim G = 2\sum_{r=1}^{\infty} 2r + 1 = 4\,\zeta(-1) + 2\,\zeta(0) = -\frac{4}{3}$. This gives $\mathcal{S}^{(1)} = \frac{2}{3}\log \mathcal{S}^{(0)} + c$ with c a computable constant — a finite but meaningless answer. In fact, using (5.6.7), the all-loop quantum correction to the entropy can be seen to *vanish* in the limit under consideration, as illustrated in fig. 5.1.4. As discussed around (5.6.8), there are more interesting $n \to \infty$ limits one can consider, taking $\mathcal{S}^{(0)} \to \infty$ together with n. In these cases, the weakly-coupled description is not a 3D QFT, but a topological string theory.

Although these and other considerations suggest massless higher-spin theories with infinite spin range cannot be viewed as weakly-coupled field theories on the sphere, one might wonder whether certain quantities might nonetheless be computable in certain (twisted) supersymmetric versions. We did observe some hints in that direction. One example, with details omitted, is the following. First consider the supersymmetric AdS_5 higher-spin theory dual to the 4D $\mathcal{N} = 2$ supersymmetric free $U(N)$ model, i.e. the $U(N)$ singlet sector of N massless hypermultiplets, each consisting of two complex scalars and a Dirac spinor. The AdS_5 bulk field content is obtained from this following [204]. In their notation, the hypermultiplet corresponds to the $\mathfrak{so}(2,4)$ representation $\mathrm{Di} + 2\,\mathrm{Rac}$. Decomposing $(\mathrm{Di} + 2\,\mathrm{Rac}) \otimes (\mathrm{Di} + 2\,\mathrm{Rac})$ into irreducible $\mathfrak{so}(2,4)$ representations gives the AdS_5 free field content: four $\Delta = 2$ and two $\Delta = 3$ scalars, one $\Delta = 3$, $S = (1, \pm 1)$ 2-form field, six towers of massless spin-s fields for all $s \geq 1$, one tower of massless $S = (s, \pm 1)$ fields for all $s \geq 2$, one $\Delta = \frac{5}{2}$ Dirac spinor, and four towers of massless spin $s = k + \frac{1}{2}$ fermionic gauge fields for all $k \geq 1$. Consider now the same field content on S^5.

[20]"Vasiliev-like" is meant only in a superficial sense here. The higher-spin algebras are rather different [141].

The bulk and edge characters are obtained paralleling the steps summarized in section 5.5.2, generalized to the present field content using (5.4.15) and (5.4.16). Each individual spin tower gives rise to a badly divergent spin sum similar to (5.9.3). However, a remarkable conspiracy of cancelations between various bosonic and fermionic bulk and edge contributions in the end leads to a finite, unambiguous net integrand:[21]

$$\int \frac{dt}{2t}\Big(\frac{1+q}{1-q}\,\Theta_{\rm tot}^{\rm bos} - \frac{2\sqrt{q}}{1-q}\,\Theta_{\rm tot}^{\rm fer}\Big) = -\frac{3}{4}\int \frac{dt}{2t}\,\frac{1+q}{1-q}\,\frac{q}{(1-q)^2}\,. \tag{5.9.23}$$

Note that the effective UV dimensionality is reduced by *two* in this case.

An analogous construction for S^4 starting from the 3D $\mathcal{N}=2\,U(N)$ model, gives two $\Delta_\pm = 1,2$ scalars, a $\Delta=\frac{3}{2}$ Dirac spinor and two massless spin-$1,\frac{3}{2},2,\frac{5}{2},\ldots$ towers, as in [205, 206]. The fermionic bulk and edge characters cancel and the bosonic part is twice (5.9.2). In this case we moreover get a finite and unambiguous $\dim G = \lim_{\delta\to 0}\sum_{s\in\frac{1}{2}\mathbb{N}}^{\infty}(-1)^{2s}\,2\,D^5_{s-1,s-1}\,e^{-\delta s} = \frac{1}{4}$.

The above observations are tantalizing, but leave several problems unresolved, including what to make of the supergroup volume $\mathrm{vol}\,G$. Actually supergroups present an issue of this kind already with a finite number of generators, as their volume is generically zero. In the context of supergroup Chern-Simons theory this leads to indeterminate $0/0$ Wilson loop expectation values [207]. In this case the indeterminacy is resolved by a construction replacing the Wilson loop by an auxiliary worldline quantum mechanics [207]. Perhaps in this spirit, getting a meaningful path integral on the sphere in the present context may require inserting an auxiliary "observer" worldline quantum mechanics, with a natural action of the higher-spin algebra on its phase space, allowing to soak up the residual gauge symmetries.

One could consider other options, such as breaking the background isometries, models with a finite-dimensional higher-spin algebra [208–211], models with an α'-like parameter breaking the higher-spin symmetries, or models of a different nature, perhaps along the lines of [40], or bootstrapped bottom-up. We leave this, and more, to future work.

[21] The spin sums are performed by inserting a convergence factor such as $e^{-\delta s}$, but the end result is finite and unambiguous when taking $\delta\to 0$, along the lines of $\lim_{\delta\to 0}\sum_{s\in\frac{1}{2}\mathbb{N}}(-1)^{2s}(2s+1)\,e^{-\delta s} = \frac{1}{4}$.

Chapter 6: AdS one-loop partition functions from bulk and edge characters

In the previous chapter, we have derived a character integral representation for the Euclidean one-loop path integral on de Sitter spacetime. In this chapter, we first extend the dS character integral representation to AdS spacetime. In particular, we show this extension amounts to replacing the bulk SO$(1, d+1)$ characters by the corresponding SO$(2, d)$ characters and the replacing the edge SO$(1, d-1)$ characters by the corresponding SO$(2, d-2)$ characters. Then we apply the AdS character integral representation to Vasiliev higher spin gravities.

6.1 One-loop partition functions and heat kernels on AdS

The one-loop partition function Z of a (real bosonic) quantum field theory is given by a functional determinant

$$\log Z = -\frac{1}{2} \log \det(\mathcal{D}) \tag{6.1.1}$$

where $\mathcal{D}$ is the "Laplacian" in the quadratic Lagrangian $\mathcal{L}_2 = \frac{1}{2}\phi\mathcal{D}\phi$. (There might be some nontrivial factors that are not captured by the functional determinant due to subtleties like zero modes when the quantum field theory is defined on a compact manifold. We ignore these subtleties in this general discussion as they will not appear in this paper). When the theory has a gauge symmetry, we should also subtract the functional determinant of the corresponding ghost field. In general, the functional determinant is UV-divergent and needs to be regularized. The regularization scheme for $\log Z$ we'll use in this paper is

$$\log Z = \frac{1}{2} \int_0^\infty \frac{dt}{t}\, e^{-\frac{\epsilon^2}{4t}}\, K_{\mathcal{D}}(t) \tag{6.1.2}$$

where $K_{\mathcal{D}}(t) \equiv \mathrm{Tr}\, e^{-t\mathcal{D}}$ is the heat kernel of $\mathcal{D}$. In terms of the spectrum of $\mathcal{D}$, the heat kernel is formally [1] defined as

$$K_{\mathcal{D}}(t) = \sum_{n\geq 0} d_n e^{-t\lambda_n} \tag{6.1.3}$$

where λ_n is the eigenvalue of $\mathcal{D}$ of degeneracy d_n. Performing a Mellin transformation for the heat kernel $K_{\mathcal{D}}(t)$ yields the spectral zeta function

$$\zeta_D(z) \equiv \sum_{n\geq 0} \frac{d_n}{\lambda_n^z} = \frac{1}{\Gamma(z)} \int_0^\infty \frac{dt}{t}\, t^z K_{\mathcal{D}}(t) \tag{6.1.4}$$

which is also a common tool to regularize partition function.

Given a free field in AdS$_{d+1}$ carrying a generic $\mathfrak{so}(2,d)$ [2] UIR of scaling dimension $\Delta = \frac{d}{2}+\nu$ and spin $\mathbf{s}$, denoted by $[\Delta, \mathbf{s}]$, the corresponding heat kernel $K_{\mathbf{s},\nu}(t)$ is constructed explicitly in [212] by a group theoretical method. Alternatively, we can infer the heat kernel from the associated spectral zeta function (see [144, 213] for explicit expressions) by an inverse Mellin transformation:

$$\log Z_{\mathbf{s},\nu} = \frac{\text{Vol}(\text{AdS}_{d+1})}{\text{Vol}(S^d)} \frac{D^d_{\mathbf{s}}}{2^{d-1}\Gamma(\frac{d+1}{2})^2} \int_0^\infty \frac{dt}{2t}\, e^{-\frac{\epsilon^2}{4t}} \int_0^\infty d\lambda\, \mu^{(d)}_{\mathbf{s}}(\lambda) e^{-t(\lambda^2+\nu^2)} \tag{6.1.5}$$

Explanations of the various notations appearing in eq. (6.1.5) are given as follows:

- Vol(S^d): the volume of a d-dimensional sphere.

$$\text{Vol}(S^d) = \frac{2\pi^{\frac{d+1}{2}}}{\Gamma(\frac{d+1}{2})} \tag{6.1.6}$$

- Vol(AdS$_{d+1}$): the regularized volume of a $(d+1)$-dimensional Euclidean AdS [138, 149, 214, 215]. For example, in [214] the authors used unit ball realization of Euclidean AdS

[1] If the spectrum of $\mathcal{D}$ is continuous, the sum over n gets replaced by an integral and the degeneracy d_n gets replaced by the density of eigenmodes.

[2] In CFT language, the single-particle Hilbert space of such a field is equivalent to the Verma module build from a primary state. The Verma module cannot be lifted to an SO$(2,d)$ representation unless the scaling dimension Δ is an integer. Therefore in this chapter, we only consider UIRs of $\mathfrak{so}(2,d)$ rather than SO$(2,d)$.

and computed the volume by dimensional regularization.

$$\text{Vol}(\text{AdS}_{d+1}) = \begin{cases} \frac{2(-\pi)^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)} \log R, & d \text{ even} \\ \pi^{\frac{d}{2}}\Gamma(-\frac{d}{2}), & d \text{ odd} \end{cases} \tag{6.1.7}$$

- $D^d_{\mathbf{s}}$: the dimension of SO(d) representation of highest weight vector $\mathbf{s}$, c.f. eq. (2.2.1) and (2.2.2).
- $\mu^{(d)}_{\mathbf{s}}(\lambda)$: the spin-$\mathbf{s}$ spectral density up to normalization [213] (but we will stick to the name "spectral density" for simplicity)

$$\mu^{(d)}_{\mathbf{s}}(\lambda) = \begin{cases} \prod_{j=1}^r (\lambda^2 + \ell_j^2), & d = 2r \\ \prod_{j=1}^r (\lambda^2 + \ell_j^2)\lambda \tanh^{\epsilon}(\pi\lambda), & d = 2r+1 \end{cases} \tag{6.1.8}$$

where $\ell_j \equiv s_j + \frac{d}{2} - j$, and $\epsilon = \pm 1$ for bosonic/fermionic fields. The $\mathfrak{so}(d)$ spin label $\mathbf{s}$ will be dropped when we deal with a scalar field. We focus on bosonic fields in the main text of this paper and an example about Dirac spinors can be found in the appendix D.1. The spectral density $\mu^{(d)}_{\mathbf{s}}(\lambda)$ with a Wick rotation $\lambda \to i\lambda$, is also the Plancherel measure on the principal series of SO$(1, d+1)$ [64]. The simplest way to derive it [212] is based on an analytical continuation of the SO$(d+2)$ Plancherel measure, which is proportional to the dimension of SO$(d+2)$ representation (we also provide a physical derivation/interpretation of the spectral density in the appendix D.2). As a result of this analytical continuation, the polynomial part $P_{\mathbf{s}}(\lambda) \equiv \lambda \prod_{j=1}^r (\lambda^2 + \ell_j^2)$ of the spectral density also appears in the following formula that relates $D^{d+2}_{(s_0,\mathbf{s})}$ and $D^d_{\mathbf{s}}$ [144]

$$D^{d+2}_{(s_0,\mathbf{s})} = \frac{2D^d_{\mathbf{s}}}{d!}\left(s_0 + \frac{d}{2}\right)\prod_{j=1}^r \left[\left(s_0 + \frac{d}{2}\right)^2 - \ell_j^2\right] \tag{6.1.9}$$

with s_0 replaced by $-\frac{d}{2} + i\lambda$. This equation is extremely useful when we compare the character integral representations for AdS and dS in appendix D.3.

The various Γ-functions in (6.1.5) can be greatly simplified for integer dimension d and we are left with

$$\log Z_{\mathbf{s},\nu} = \frac{D^d_{\mathbf{s}}}{d!}\int_0^\infty \frac{dt}{t}\, e^{-\frac{\epsilon^2}{4t}} \int_0^\infty d\lambda\, \mu^{(d)}_{\mathbf{s}}(\lambda)\, e^{-t(\lambda^2+\nu^2)} \times \begin{cases} \frac{(-)^{\frac{d}{2}}\log R}{\pi}, & d \text{ even} \\ \frac{(-)^{\frac{d+1}{2}}}{2}, & d \text{ odd} \end{cases} \tag{6.1.10}$$

Starting from this equation, we'll show that all partition functions $\log Z_{\mathbf{s},\nu}$ can be expressed as a integral transformations of $\mathfrak{so}(2,d)$ characters up to edge mode corrections.

6.2 Warm-up example: scalar fields in even dimensional AdS

6.2.1 Scalar fields in AdS$_2$

As a warm-up, let's consider a scalar field φ of mass $m^2 = \nu^2 - \frac{1}{4}$ in AdS$_2$ which corresponds to the scaling dimension $\Delta_+ = \frac{1}{2}+\nu$ representation of $\mathfrak{so}(2,1)$. Applying (6.1.10) to this field yields the partition function

$$\log Z_\nu = -\frac{1}{4}\int_0^\infty \frac{dt}{t} e^{-\frac{\epsilon^2}{4t}} \int_{-\infty}^\infty d\lambda\, \lambda \tanh(\pi\lambda) e^{-t(\lambda^2+\nu^2)} \tag{6.2.1}$$

where we've extended the integration domain of λ to the whole real line. To perform the integral over λ, we use the Hubbard-Stratonovich trick:

$$\begin{aligned}\log Z_\nu &= -\frac{1}{4}\int_{-\infty}^\infty du \int_0^\infty \frac{dt}{2\sqrt{\pi}\, t^{3/2}} e^{-\frac{\epsilon^2+u^2}{4t}-t\nu^2} \int_{-\infty}^\infty d\lambda\, \lambda\tanh(\pi\lambda) e^{i\lambda u} \\ &= -\frac{1}{2}\int_0^\infty \frac{du}{\sqrt{\epsilon^2+u^2}} e^{-\nu\sqrt{\epsilon^2+u^2}}\, W(u)\end{aligned} \tag{6.2.2}$$

where

$$W(u) \equiv \int_{-\infty}^\infty d\lambda\, \lambda \tanh(\pi\lambda) e^{i\lambda u} \tag{6.2.3}$$

The naive Fourier transformation (6.2.3) is ill-defined. However for our practical purpose i.e. to get the character integral formula as soon as possible, we pretend that it's well-defined and the contour can be closed at infinity. A more rigorous treatment is postponed until section 6.4 and 6.5 where we give a fully regularized character integral and justify our naive result obtained here is indeed reasonable and sufficient for most applications, in particular the total partition function of Vasiliev theories. From now on, keeping the above comments in mind, we're free to close the λ-contour in $W(u)$ in either upper or lower half-plane depending on the sign of u and we also write the partition function $\log Z_\nu$ in the unregularized form by putting $\epsilon = 0$ formally. When $u > 0$, we close the contour in the upper half-plane picking up simple poles at $\lambda = i(n + \frac{1}{2}), n \in \mathbb{N}$ and when $u < 0$, we close the contour in the lower half-plane picking up simple poles at $\lambda = -i(n + \frac{1}{2}), n \in \mathbb{N}$. By summing over residues in both cases, we find

$$W(u) = -\frac{1+e^{-u}}{1-e^{-u}} \frac{e^{-\frac{1}{2}u}}{1-e^{-u}} \tag{6.2.4}$$

Thus the unregularized partition function is given by

$$\log Z_\nu = \int_0^\infty \frac{du}{2u} \frac{1+e^{-u}}{1-e^{-u}} \, \Theta^{\mathsf{AdS}_2}_{\Delta_+}(u), \quad \Theta^{\mathsf{AdS}_2}_{\Delta_+}(u) = \frac{e^{-\Delta_+ u}}{1-e^{-u}} \tag{6.2.5}$$

$\Theta^{\mathsf{AdS}_2}_{\Delta_+}(u) \equiv \operatorname{tr} e^{-uH}$, with H being the (hermitian and positive) Hamiltonian, is the character of scaling dimension Δ_+ representation of $\mathfrak{so}(2,1)$. Before moving to the higher dimensional examples, let's notice that when evaluating the t-integral in eq. (6.2.2), we implicitly assume $\nu > 0$, so $\Delta_+ = \frac{1}{2} + \nu$ corresponds to the "standard quantization" in bulk. On the other hand, we can safely send ν to $-\nu$ in eq. (6.2.5) by analytic continuation, as long as $0 < \nu < \frac{1}{2}$. Altogether we conclude that, with the standard boundary condition, the partition function of φ is

$$\log Z_\nu = \int_0^\infty \frac{du}{2u} \frac{1+e^{-u}}{1-e^{-u}} \, \Theta^{\mathsf{AdS}_2}_{\Delta_+}(u) \tag{6.2.6}$$

and with the alternate boundary condition, the partition function is

$$\log Z_{-\nu} = \int_0^\infty \frac{du}{2u} \frac{1+e^{-u}}{1-e^{-u}} \, \Theta^{\mathsf{AdS}_2}_{\Delta_-}(u) \tag{6.2.7}$$

where $\Delta_- = \frac{1}{2} - \nu$.

6.2.2 Scalar fields in AdS$_{2r+2}$

The computation above in AdS$_2$ can be generalized straightforwardly to scalar field in any AdS$_{d+1}$ with $d = 2r+1$ odd. Assuming that the scalar field has scaling dimension $\Delta_+ = \frac{d}{2} + \nu$, its partition function is given by

$$\log Z_\nu = \frac{(-)^{r+1}}{2\,d!} \int_0^\infty \frac{dt}{t} e^{-\frac{\epsilon^2}{4t}} \int_0^\infty d\lambda\, \mu^{(d)}(\lambda)\, e^{-t(\lambda^2+\nu^2)} \tag{6.2.8}$$

where the scalar spectral density is

$$\mu^{(d)}(\lambda) = \prod_{j=0}^{r-1} \left[\lambda^2 + \left(j + \frac{1}{2}\right)^2\right] \lambda \tanh(\pi\lambda) \tag{6.2.9}$$

Following the same steps as in section 6.2.1, we obtain

$$\log Z_\nu = \frac{(-)^{r+1}}{d!} \int_0^\infty \frac{du}{2u}\, W^{(d)}(u)\, e^{-\nu\, u} \tag{6.2.10}$$

where $W^{(d)}(u) \equiv \int_{-\infty}^{\infty} d\lambda\, \mu^{(d)}(\lambda) e^{i\lambda u}$. The "Fourier transform" $W^{(d)}(u)$ can be evaluated according to the comments below eq. (6.2.3)

$$W^{(d)}(u) = (-)^{r+1} d! \frac{e^{-\frac{d}{2}u}}{(1-e^{-u})^d} \frac{1+e^{-u}}{1-e^{-u}} \tag{6.2.11}$$

Plugging (6.2.11) into (6.2.10) yields a character integral expression for the unregularized $\log Z_\nu$

$$\log Z_\nu = \int_0^\infty \frac{du}{2u} \frac{1+e^{-u}}{1-e^{-u}} \Theta^{\mathrm{AdS}_{d+1}}_{\Delta_+}(u), \quad \Theta^{\mathrm{AdS}_{d+1}}_{\Delta_+}(u) = \frac{e^{-\Delta_+ u}}{(1-e^{-u})^d} \tag{6.2.12}$$

where $\Theta^{\mathrm{AdS}_{d+1}}_{\Delta_+}(u)$ is the character of scaling dimension Δ_+ representation of $\mathfrak{so}(2,d)$. Before turning to the higher spin case, let's comment on the relation between the character integral and the original heat kernel integral. The heat kernel in AdS is defined through the spectral density $\mu(\lambda)$ whose explicit construction is given in [213]. Briefly speaking, the authors of [213] found a complete set of δ-function normalizable eigenfunctions of the Laplacian operator $-\nabla^2$

in EAdS_{d+1}:

$$-\nabla^2 h^{(\lambda\sigma)}(x) = \left(\frac{d^2}{4} + \lambda^2\right) h^{(\lambda\sigma)}(x) \tag{6.2.13}$$

$$\langle h^{(\lambda\sigma)}, h^{(\lambda\sigma')}\rangle \equiv \int d^{d+1}x\sqrt{g}\, h^{(\lambda\sigma)*}(x) h^{(\lambda\sigma')}(x) = \delta_{\sigma\sigma'}\delta(\lambda-\lambda') \tag{6.2.14}$$

where σ is a discrete label for distinguishing eigenfunctions of the same λ. (Note that the inner product for $h^{(\lambda\sigma)}$ involves an integration over the whole EAdS rather than a spatial slice as in the standard Klein-Gordon inner product). The spectral density is defined via these eigenfunctions: $\mu(\lambda) \propto \sum_\sigma h^{(\lambda\sigma)*}h^{(\lambda\sigma')}(0)$. Therefore the original heat kernel method involves an integral over the whole continuous spectrum labeled by (λ, σ). On the other hand, the character can be expanded into a discrete sum

$$\Theta^{\text{AdS}_{d+1}}_{\Delta_+}(u) = \sum_{n\geq 0} \binom{d+n-1}{d-1} e^{-(\Delta_+ + n)u} \tag{6.2.15}$$

This expansion encodes a whole tower of solutions to the equation of motion $(-\nabla^2 + \Delta_+(\Delta_+ - d))\phi = 0$ in global AdS that furnish a representation of $\mathfrak{so}(2,d)$. More explicitly, $\phi_0 = \frac{e^{-i\Delta_+ t}}{(1+r^2)^{\Delta_+/2}}$ is the primary mode, i.e. ground state, in the global coordinate: $ds^2 = -(1+r^2)dt^2 + \frac{dr^2}{1+r^2} + r^2 d\Omega^2$. It solves the equation of motion, falls like $r^{-\Delta_+}$ at the boundary but its Wick rotation under $t \to -i\tau$ is not normalizable in the sense of (6.2.14). By acting the conformal algebra $\mathfrak{so}(2,d)$ on ϕ_0 repeatedly, we get a collection of modes that also solve the equation of motion and have the same boundary condition. At each frequency $\omega_n = \Delta_+ + n$, the degeneracy of these modes are exactly $\binom{d+n-1}{d-1}$. Therefore while switching from the heat kernel integral to the character integral, we effectively turn a *continuous* spectrum into a *discrete* spectrum and curiously both of them encode the information of partition function. This observation is the main point of [192]. Actually the character integral representation we found is equivalent to the "zero mode method" used in that paper. For example, using the unregularized expression (6.2.12), we obtain formally

$$\log Z_\nu = \sum_{n\geq 0} D^{d+2}_n \int_0^\infty \frac{du}{2u} e^{-(\Delta_+ + n)u} = -\frac{1}{2}\sum_{n\geq 0} D^{d+2}_n \log(\Delta_+ + n) \tag{6.2.16}$$

which recovers the result in [192] up to some holomorphic function denoted by $\mathrm{Pol}(\Delta_+)$ there. $\mathrm{Pol}(\Delta_+)$ is a polynomial in Δ_+ and depends on the UV-cutoff. In section 6.5, we'll show that it can also be recovered if we use the fully regularized character integral.

6.3 Higher spin fields in AdS_{2r+2}

In this section, we turn to the character integral representation of higher spin fields in AdS_{d+1} with $d = 2r+1$[3]. Unlike scalar fields, a spin-s field $\varphi_{\mu_1\cdots\mu_s}$ in AdS can carry either massive or massless irreducible representations [70] depending on the scaling dimension. When $\Delta = \Delta_{s,t} \equiv d+t-1$ with $t \in \{0, 1, \cdots, s-1\}$, $\varphi_{\mu_1\cdots\mu_s}$ is a partially massless field of depth t and it has a gauge symmetry $\delta\varphi_{\mu_1\cdots\mu_s} = \nabla_{(\mu_{t+1}\cdots\mu_s}\xi_{\mu_1\cdots\mu_t)} + \cdots$ [99]. In this case, we should include the contribution of the ghost field, which has spin-t and scaling dimension $\Delta_{t,s} = d+s-1$, in the one-loop partition function. When $\Delta = \frac{d}{2}+\nu$ is not in the discrete set $\{\Delta_{s,t}\}$, the field $\varphi_{\mu_1\cdots\mu_s}$ falls into the massive representations and does not have gauge symmetry. Since the character is supposed to count only the *physical* degrees of freedom, it takes very different forms for massive and massless representations [70]:

$$\begin{aligned} \text{massive } \ [\Delta, s] : \Theta^{\mathrm{AdS}_{d+1}}_{[\Delta,s]}(u) &= D^d_s \frac{e^{-\Delta u}}{(1-e^{-u})^d} \\ \text{massless } \ [\Delta_{s,t}, s] : \Theta^{\mathrm{AdS}_{d+1}}_{[\Delta_{s,t},s]}(u) &= \frac{D^d_s e^{-(d+t-1)u} - D^d_t e^{-(d+s-1)u}}{(1-e^{-u})^d} \end{aligned} \tag{6.3.1}$$

Let's start computing the partition function of a spin-s field in the massive representations

$$\log Z_{s,\nu} = \frac{(-)^{\frac{d+1}{2}}}{2}\frac{D^d_s}{d!}\int_0^\infty \frac{dt}{t}\, e^{-\frac{\epsilon^2}{4t}} \int_0^\infty d\lambda\, \mu^{(d)}_s(\lambda)\, e^{-t(\lambda^2+\nu^2)} \tag{6.3.2}$$

where the spin-s spectrum density is

$$\mu^{(d)}_s(\lambda) = \prod_{j=0}^{r-2}\left[\lambda^2 + \left(j+\frac{1}{2}\right)^2\right]\left[\lambda^2 + \left(s+r-\frac{1}{2}\right)^2\right]\lambda\tanh(\pi\lambda) \tag{6.3.3}$$

[3]We will focus on the $r \geq 1$ case because there is a discrete spectrum for each higher spin STT Laplacian in AdS_2 which corresponds to the discrete series of $\mathrm{SO}(2,1)$ and doesn't have any analogue in higher dimensions [106, 213].

Following the same steps as in the scalar case, we obtain

$$\log Z_{s,\nu} = \frac{(-)^{\frac{d+1}{2}}}{d!} D_s^d \int_0^\infty \frac{du}{2u} W_s^{(d)}(u)\, e^{-\nu\, u} \tag{6.3.4}$$

where $W_s^{(d)}(u) \equiv \int_{-\infty}^{\infty} d\lambda\, \mu_s^{(d)}(\lambda)\, e^{i\lambda u}$ is an even function in u and when $u > 0$, it is

$$W_s^{(d)}(u) = (-)^{\frac{d-1}{2}} (d-2)! \frac{e^{-(\frac{d}{2}-1)u}}{(1-e^{-u})^d} \frac{1+e^{-u}}{1-e^{-u}} \left[s(s+d-2)(1-e^{-u})^2 - d(d-1)e^{-u} \right] \tag{6.3.5}$$

Plugging (6.3.5) into (6.3.4) yields a new character integral

$$\log Z_{s,\nu} = \int_0^\infty \frac{du}{2u} \frac{1+e^{-u}}{1-e^{-u}} \left(\Theta_{[\Delta,s]}^{\mathrm{AdS}_{d+1}}(u) - D_{s-1}^{d+2} \frac{e^{-(\frac{d-2}{2}+\nu)u}}{(1-e^{-u})^{d-2}} \right) \tag{6.3.6}$$

The second term in the bracket corresponds to subtracting the partition function of D_{s-1}^{d+2} scalars on EAdS$_{d-1}$ with scaling dimension $\frac{d-2}{2}+\nu$ since it involves an so$(2,d-2)$ character of scalar representation. As in the de Sitter case, we tend to identify these scalar degrees of freedom as edge modes living on the horizon of Rindler-AdS [4] which is a Lorentzian Wick rotation of EAdS and has a EAdS$_{d-1}$ shaped horizon. More discussions about Rindler-AdS will be left to section 6.8 and appendix D.5. We can also write $\log Z_{s,\nu}$ in terms of an $\mathfrak{so}(2,d)$ character and an $\mathfrak{so}(2,d+2)$ character

$$\log Z_{s,\nu} = \int_0^\infty \frac{du}{2u} \frac{1+e^{-u}}{1-e^{-u}} \left(\Theta_{[\Delta,s]}^{\mathrm{AdS}_{d+1}}(u) - \left(e^{\frac{u}{2}} - e^{-\frac{u}{2}} \right)^4 \Theta_{[\Delta+1,s-1]}^{\mathrm{AdS}_{d+3}}(u) \right) \tag{6.3.7}$$

This form of character integral representation doesn't have the physically meaningful edge character structure but it turns out to be much more convenient than (6.3.6) computationally, in particular when we sum over all field contents in Vasiliev higher spin gravities.

Finally, let's move to our main interest: (partially) massless fields. Due to gauge symmetry, the partition function of a PM field with spin-s and depth $t \in \{0, 1, \cdots, s-1\}$ is given by

$$\log Z_{s,\nu_t} = \frac{(-)^{\frac{d+1}{2}}}{d!} \int_0^\infty \frac{du}{u} \left(D_s^d e^{-(\frac{d}{2}+t-1)u} W_s^{(d)}(u) - D_t^d e^{-(\frac{d}{2}+s-1)u} W_t^{(d)}(u) \right) \tag{6.3.8}$$

where the first term corresponds to the spin-s gauge field and the second term arises from the spin-t ghost field. By using the explicit expression of $W_s^{(d)}$ derived in eq. (6.3.5), we can rewrite (6.3.8) in the same form as (6.3.7)

$$\log Z_{s,\nu_t} = \int_0^\infty \frac{du}{2u}\frac{1+e^{-u}}{1-e^{-u}}\left(\Theta^{\mathrm{AdS}_{d+1}}_{[d+t-1,s]}(u) - \left(e^{\frac{u}{2}} - e^{-\frac{u}{2}}\right)^4 \Theta^{\mathrm{AdS}_{d+3}}_{[d+t,s-1]}(u)\right) \tag{6.3.9}$$

where the characters of PM fields are

$$\Theta^{\mathrm{AdS}_{d+1}}_{[d+t-1,s]}(u) = \frac{D^d_s e^{-(d+t-1)u} - D^d_t e^{-(d+s-1)u}}{(1-e^{-u})^d} \tag{6.3.10}$$

$$\Theta^{\mathrm{AdS}_{d+3}}_{[d+t,s-1]}(u) = \frac{D^{d+2}_{s-1} e^{-(d+t)u} - D^{d+2}_{t-1} e^{-(d+s)u}}{(1-e^{-u})^{d+2}} \tag{6.3.11}$$

Note $[d+t, s-1]$ is a massless representation of SO$(2, d+2)$ with spin-$(s-1)$ and depth-$(t-1)$.

6.4 Regularization, contour prescription and odd dimensional AdS

In the previous sections, we've derived a *formal* character integral formula for the *unregularized* one-loop partition functions of both scalar fields and higher spin fields in even dimensional AdS. However, to make sense of the character integral mathematically and apply it to actual computation of renormalized partition functions, we have to use a well-defined and efficient regularization scheme. In this section, we'll sort out this issue. Surprisingly the resolution turns out to have a very important byproduct: a character integral representation that works in odd dimensional AdS.

6.4.1 Regularization and contour prescription

We use a real scalar field to illustrate the regularization scheme. But it will be clear in the end that the same regularization also works for higher spin fields.

6.4.1.1 $d = 2r+1$

It's mentioned in section 6.2.1 that $W^{(d)}(u)$, Fourier transformation of the scalar spectral density $\mu^{(d)}(\lambda)$, is not well-defined. As a manifestation of this point, $W^{(d)}(u)$ is singular at $u = 0$. This singularity may lead to two inequivalent definitions of inverse Fourier transformation of $W^{(d)}(u)$ since the contour can either go above or below $u = 0$:

$$\tilde{\mu}_d^{(\pm)}(\lambda) \equiv \int_{\mathbb{R}\pm i\delta} \frac{du}{2\pi} W^{(d)}(u)\, e^{-i\lambda u} = (-)^{r+1} d! \int_{\mathbb{R}\pm i\delta} \frac{du}{2\pi} \frac{1+e^{-u}}{1-e^{-u}} \frac{e^{-(\frac{d}{2}+i\lambda)u}}{(1-e^{-u})^d} \tag{6.4.1}$$

where δ is a small positive number. (At this stage, the size of δ is not important as long as it's smaller than 2π. We'll later impose a more stringent constraint on it). It's clear that the two definitions of inverse Fourier transformation differ by the residue of the integrand at $u = 0$. In terms of the notation $H_{d,\nu}(u) \equiv \frac{1+e^{-u}}{1-e^{-u}} \frac{e^{-(\frac{d}{2}+\nu)u}}{(1-e^{-u})^d}$ introduced in the appendix D.4, the function $\tilde{\mu}_d^{(\pm)}(\lambda)$ can also expressed as

$$\tilde{\mu}_d^{(\pm)}(\lambda) = (-)^{r+1} d! \int_{\mathbb{R}\pm i\delta} \frac{du}{2\pi} H_{d,i\lambda}(u) \tag{6.4.2}$$

We use the function $H_{d,i\lambda}(u)$ here because its residue at $u = 0$ is given by eq. (D.4.8) and the residues at other poles $u = 2\pi i n, n \in \mathbb{Z}$ can be easily inferred by using its quasi-periodicity $H_{d,i\lambda}(u + 2\pi i n) = (-e^{2\pi\lambda})^n H_{d,i\lambda}(u)$. With the information of residues known, we close the contour at infinity and get

$$\tilde{\mu}_d^{(\pm)}(\lambda) = \prod_{j=0}^{r-1} \left(\lambda^2 + \left(j + \frac{1}{2}\right)^2\right) (\lambda \tanh(\pi\lambda) \pm \lambda) \tag{6.4.3}$$

Therefore in order to recover the spectral density $\mu^{(d)}$, cf. (6.2.9), the contour prescription should be the average of $\int_{\mathbb{R}\pm i\delta}$:

$$\mu^{(d)}(\lambda) = \frac{1}{2}(\tilde{\mu}_d^{(+)} + \tilde{\mu}_d^{(-)}) = \frac{1}{2}\left(\int_{\mathbb{R}+i\delta} + \int_{\mathbb{R}-i\delta}\right) \frac{du}{2\pi} W^{(d)}(u)\, e^{-i\lambda u} \tag{6.4.4}$$

Plugging this equation into the scalar partition function (6.2.8) , we obtain

$$\begin{aligned}\log Z_\nu &= \frac{(-)^{r+1}}{4\, d!} \left(\int_{\mathbb{R}+i\delta} + \int_{\mathbb{R}-i\delta}\right) \frac{du}{4\pi} W^{(d)}(u) \int_0^\infty \frac{dt}{t}\, e^{-\frac{\epsilon^2}{4t} - t\nu^2} \int_{-\infty}^\infty d\lambda\, e^{-t\lambda^2 - i\lambda u} \\ &= \frac{1}{4}\left(\int_{\mathbb{R}+i\delta} + \int_{\mathbb{R}-i\delta}\right) \frac{du}{2\sqrt{u^2+\epsilon^2}} \frac{1+e^{-u}}{1-e^{-u}} \frac{e^{-\frac{d}{2}u - \nu\sqrt{\epsilon^2+u^2}}}{(1-e^{-u})^d}\end{aligned} \tag{6.4.5}$$

where δ has to be smaller than ϵ, otherwise the contours would cross the branch cut of $\sqrt{u^2+\epsilon^2}$, which has two disconnected pieces with one piece going upwards from $i\epsilon$ to $i\infty$ and the other going downwards from $-i\epsilon$ to $-i\infty$. Due to the manifest $u \to -u$ symmetry of the integrand for odd d, the two contours $\mathbb{R} \pm i\delta$ are actually equivalent and it suffices to use one of them

$$\log Z_\nu = \frac{1}{2}\int_{\mathbb{R}+i\delta} \frac{du}{2\sqrt{u^2+\epsilon^2}} \frac{1+e^{-u}}{1-e^{-u}} \frac{e^{-\frac{d}{2}u-\nu\sqrt{\epsilon^2+u^2}}}{(1-e^{-u})^d} \tag{6.4.6}$$

The computation above also works for higher spin fields. It suffices to show that the higher spin generalization of (6.4.4) holds. Notice that $W_s^{(d)}(u)$ corresponding to a spin-s field is related to the scalar version $W^{(d)}(u)$ by

$$W_s^{(d)}(u) = W^{(d)}(u) + s(s+d-2)W^{(d-2)}(u) \tag{6.4.7}$$

Applying the integral (6.4.4) to this relation indeed yields

$$\frac{1}{2}\left(\int_{\mathbb{R}+i\delta} + \int_{\mathbb{R}-i\delta}\right) \frac{du}{2\pi} W_s^{(d)}(u)\, e^{-i\lambda u} = \mu_s^{(d)} \tag{6.4.8}$$

Therefore the regularized character integral for a spin-s field in massive representations is

$$\log Z_{s,\nu} = \int_{\mathbb{R}+i\delta} \frac{du}{4\sqrt{u^2+\epsilon^2}} \frac{1+e^{-u}}{1-e^{-u}} \left(\frac{D_s^d\, e^{-\frac{d}{2}u-\nu\sqrt{\epsilon^2+u^2}}}{(1-e^{-u})^d} - \frac{D_{s-1}^{d+2}\, e^{-\frac{d-2}{2}u-\nu\sqrt{\epsilon^2+u^2}}}{(1-e^{-u})^{d-2}}\right) \tag{6.4.9}$$

Starting from the eq. (6.4.9), we'll derive a complete expression for the regularized partition function $Z_{s,\nu}$ in section 6.5.

6.4.1.2 $d = 2r$

The odd d case above tells us the correct strategy to get a regularized character integral. First, we pick a function $W^{(d)}(u) \propto \frac{1+e^{-u}}{1-e^{-u}} \frac{e^{-\frac{d}{2}u}}{(1-e^{-u})^d}$ and compute its inverse Fourier transformations defined by two different contour choices. Then one proper linear combination of these choices can give the correct spectral density. Plug this integral expression of spectral density into the original partition function (6.1.10) and we finally obtain the regularized character integral. Now let's apply this strategy to the $d = 2r$ case where we choose $W^{(d)}(u)$ to be

$$W^{(d)}(u) = (-)^{r+1} d! \frac{1+e^{-u}}{1-e^{-u}} \frac{e^{-\frac{d}{2}u}}{(1-e^{-u})^d} \tag{6.4.10}$$

The two possible inverse Fourier transformations defined as in eq. (6.4.1) are

$$\tilde{\mu}_d^{(\pm)}(\lambda) = i \prod_{j=0}^{r-1} (\lambda^2 + j^2)(\coth(\pi\lambda) \pm 1) \tag{6.4.11}$$

Therefore the spectral density $\mu^{(d)}(\lambda) = \prod_{j=0}^{r-1}(\lambda^2 + j^2)$ in AdS$_{2r+1}$ is given by

$$\mu^{(d)}(\lambda) = \frac{i}{2}(\tilde{\mu}_d^{(-)} - \tilde{\mu}_d^{(+)}) = \frac{i}{2}\left(\int_{\mathbb{R}-i\delta} - \int_{\mathbb{R}+i\delta}\right) \frac{du}{2\pi} \, W^{(d)}(u) \, e^{-i\lambda u} \tag{6.4.12}$$

Substituting this equation into (6.1.10) and performing the t, λ integrals, we end up with

$$\frac{\log Z_\nu}{\log R} = \frac{i(-)^r}{2\pi d!} \left(\int_{\mathbb{R}-i\delta} - \int_{\mathbb{R}+i\delta}\right) \frac{du}{2\sqrt{u^2+\epsilon^2}} W^{(d)}(u) e^{-\nu\sqrt{u^2+\epsilon^2}} \tag{6.4.13}$$

Since there is no poles in the strip bounded by $\mathbb{R} \pm i\delta$, we can further deform the contour to be a small circle C_0 around $u = 0$. Then the integral is equivalent to evaluating the residue at $u = 0$:

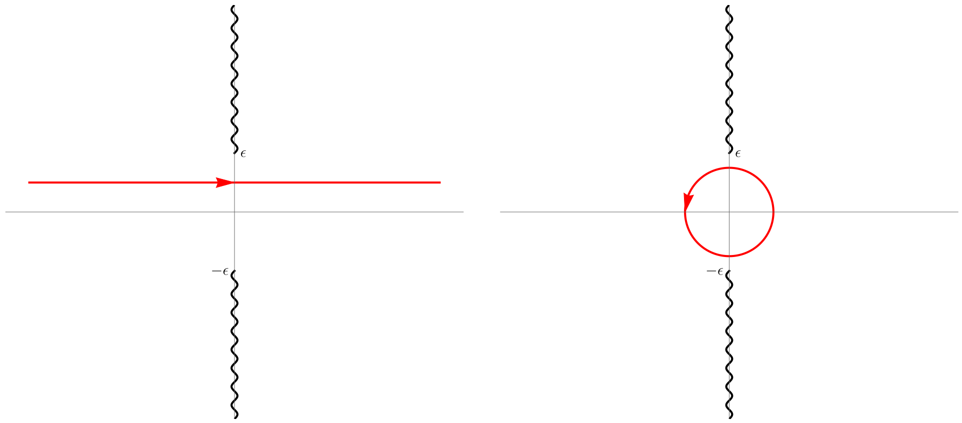
$$\frac{\log Z_\nu}{\log R} = \text{Res}_{u\to 0} \left(\frac{e^{-\nu\sqrt{u^2+\epsilon^2}}}{2\sqrt{u^2+\epsilon^2}} \frac{1+e^{-u}}{1-e^{-u}} \frac{e^{-\frac{d}{2}u}}{(1-e^{-u})^d} \right) \tag{6.4.14}$$

Similarly for higher spin fields, by using appropriate square root regularization, we can also get

$$\frac{\log Z_{s,\nu}}{\log R} = \text{Res}_{u\to 0} \left[\frac{e^{-\nu\sqrt{\epsilon^2+u^2}}}{2\sqrt{u^2+\epsilon^2}} \frac{1+e^{-u}}{1-e^{-u}} \left(\frac{D_s^d \, e^{-\frac{d}{2}u}}{(1-e^{-u})^d} - \frac{D_{s-1}^{d+2} \, e^{-\frac{d-2}{2}u}}{(1-e^{-u})^{d-2}} \right) \right] \tag{6.4.15}$$

Conclusion

Altogether, we can conclude that the regularized one-loop partition function (with the UV regularization introduced by $e^{-\frac{\epsilon^2}{4t}}$ in the original definition (6.1.5)) of a field in the $[\frac{d}{2} + \nu, s]$



(a) odd d (b) even d

Figure 6.4.1: u-contours for even dimensional AdS (left) and odd dimensional AdS (right). The black wavy lines represent branch cuts.

massive representation of SO$(2,d)$ is

$$\log Z_{s,\nu} = \frac{A_d}{2}\int_{C_d} du \frac{e^{-\nu\sqrt{\epsilon^2+u^2}}}{\sqrt{u^2+\epsilon^2}}\frac{1+e^{-u}}{1-e^{-u}}\left(\frac{D^d_s\, e^{-\frac{d}{2}u}}{(1-e^{-u})^d} - \frac{D^{d+2}_{s-1}\, e^{-\frac{d-2}{2}u}}{(1-e^{-u})^{d-2}}\right) \tag{6.4.16}$$

where the overall constant and contour choice, fig. (6.4.1), are

$$A_d = \begin{cases} \frac{\log R}{2\pi i} & d \text{ even} \\ \frac{1}{2} & d \text{ odd} \end{cases} \qquad C_d = \begin{cases} \text{a counterclockwise circle around } 0 & d \text{ even} \\ \mathbb{R} + i\delta & d \text{ odd} \end{cases} \tag{6.4.17}$$

For a field in massless representations, we need to include the corresponding ghost contribution

$$\begin{aligned} \log Z_{s,\nu_t} =& \frac{A_d}{2}\int_{C_d} \frac{du}{\sqrt{u^2+\epsilon^2}}\frac{1+e^{-u}}{1-e^{-u}}\frac{e^{-\frac{d}{2}u}}{(1-e^{-u})^d}\left(D^d_s\, e^{-\nu_t\sqrt{u^2+\epsilon^2}} - D^d_t\, e^{-\nu_s\sqrt{u^2+\epsilon^2}}\right) \\ &- \frac{A_d}{2}\int_{C_d} \frac{du}{\sqrt{u^2+\epsilon^2}}\frac{1+e^{-u}}{1-e^{-u}}\frac{e^{-\frac{d-2}{2}u}}{(1-e^{-u})^{d-2}}\left(D^{d+2}_{s-1}\, e^{-\nu_t\sqrt{u^2+\epsilon^2}} - D^{d+2}_{t-1}\, e^{-\nu_s\sqrt{u^2+\epsilon^2}}\right) \end{aligned} \tag{6.4.18}$$

where $\nu_t = \frac{d}{2} + t - 1$ and $\nu_s = \frac{d}{2} + s - 1$.

6.5 Evaluation of the regularized character integrals

In this section, we give an efficient and general recipe to compute the regularized character integral formula following the appendix C of [2]. For the simplicity of notation, we'll use scalar fields as an illustration of this recipe. But our reasoning and final result can be easily generalized to fields of arbitrary spin. In addition, the result can be used to justify that the unregularized character integral is sufficient for the application to Vasiliev gravities. First, let's briefly review the standard heat kernel method in computing one-loop partition functions

$$\log Z_\nu(\epsilon) = \int_0^\infty \frac{dt}{2t} e^{-\frac{\epsilon^2}{4t}} K_\nu(t), \quad K_\nu(t) = \frac{\text{Vol}(\text{AdS}_{d+1})}{2^d \pi^{\frac{d+1}{2}} \Gamma(\frac{d+1}{2})} \int_0^\infty d\lambda\, \mu^{(d)}(\lambda)\, e^{-t(\lambda^2+\nu^2)} \quad (6.5.1)$$

where I've plugged in the explicit form of $\text{Vol}(S^d)$ compared to eq. (6.1.5). By using the small t expansion of heat kernel, say $K_\nu(t) = \sum_{k=0}^{d+1} \alpha_k\, t^{-\frac{d+1-k}{2}} + \mathcal{O}(t)$, we can separate it into UV and IR parts without ambiguity

$$K_\nu^{\text{uv}}(t) \equiv \sum_{k=0}^{d+1} \alpha_k\, t^{-\frac{d+1-k}{2}}, \quad K_\nu^{\text{ir}}(t) = K_\nu(t) - K_\nu^{\text{uv}}(t) \quad (6.5.2)$$

The UV-part of the heat kernel expansion in AdS is fairly simple. When d is even, $K_\nu(t)$ can be evaluated exactly because it's a Gaussian integral in λ. When d is odd, using $\tanh \pi\lambda = 1 - \frac{2}{1+e^{2\pi\lambda}}$ we can split the heat kernel into two parts. The first part is a simple Gaussian integral as in the even d case. The second part is of $\mathcal{O}(t^0)$ and hence we can put $t = 0$ which yields an exactly solvable integral. In addition, by direct computation, one can show that α_k is nonvanishing only for even k. For example, in $d = 3$, we obtain the nonzero heat kernel coefficients: $\alpha_0 = \frac{1}{12}, \alpha_2 = \frac{1-4\nu^2}{48}$ and $\alpha_4 = -\frac{17}{5760} - \frac{\nu^2}{48} + \frac{\nu^4}{24}$.

Introducing an infinitesimal IR cutoff $\kappa \to 0$, we can further separate $\log Z_\nu(\epsilon)$ into two parts

$$\log Z_\nu^{\text{uv}}(\epsilon) = \int_0^\infty \frac{dt}{2t} K_\nu^{\text{uv}}(t) e^{-\frac{\epsilon^2}{4t} - \kappa^2 t}, \quad \log Z_\nu^{\text{ir}} = \int_0^\infty \frac{dt}{2t} K_\nu^{\text{ir}}(t) e^{-\kappa^2 t} \quad (6.5.3)$$

where the UV regulator has been dropped in the IR integral because it's by construction UV finite. The IR regulator $e^{-\kappa^2 t}$ is inserted in the UV integral because the integrand has a $\frac{1}{t}$ term when $\alpha_{d+1} \neq 0$, i.e. when d is odd. In the end, the $\log \kappa$ terms in $\log Z_\nu^{\text{uv}}$ and $\log Z_\nu^{\text{ir}}$ will cancel out and we're left with

$$\log Z_\nu(\epsilon) = \frac{1}{2} \sum_{k=0}^{d} \alpha_k\, \Gamma\left(\frac{d+1-k}{2}\right) \left(\frac{2}{\epsilon}\right)^{d+1-k} + \alpha_{d+1} \log\left(\frac{2}{e^\gamma \epsilon}\right) + \frac{1}{2}\zeta_\nu'(0) \quad (6.5.4)$$

where $\zeta_\nu(z) \equiv \frac{1}{\Gamma(z)}\int_0^\infty \frac{dt}{t}t^z K_\nu(t)$ is the spectral zeta function and $\alpha_{d+1} = \zeta_\nu(0)$. Next, we'll apply this UV-IR separation idea to the evaluation of partition function in the regularized character integral formalism.

6.5.1 Even dimensional AdS$_{2r+2}$

In section 6.4.1.1, we've found that the square-root regularized partition function in AdS$_{2r+2}$ is given by

$$\log Z_\nu(\epsilon) = \frac{1}{2}\int_{\mathbb{R}+i\delta} \frac{du}{2\sqrt{u^2+\epsilon^2}}\frac{1+e^{-u}}{1-e^{-u}}e^{-\frac{d}{2}u-\nu\sqrt{u^2+\epsilon^2}} \tag{6.5.5}$$

where $d = 2r+1$. Putting $\epsilon = 0$ we recover the formal UV-divergent character formula:

$$\log Z_\nu(\epsilon = 0) = \int_0^\infty \frac{du}{2u}H_{d,\nu}(u), \quad H_{d,\nu}(u) = \frac{1+e^{-u}}{1-e^{-u}}\frac{e^{-(\frac{d}{2}+\nu)u}}{(1-e^{-u})^d} \tag{6.5.6}$$

The unregularized integrand $\frac{1}{2u}H_{d,\nu}(u)$ admits a Laurent expansion around $u = 0$ with coefficients being polynomials in ν:

$$\frac{1}{2u}H_{d,\nu}(u) = \frac{1}{u}\sum_{k\geq 0} b_k(\nu)u^{-(d+1-k)}, \quad b_k(\nu) = \sum_{\ell=0}^{k} b_{k\ell}\nu^\ell \tag{6.5.7}$$

where $b_{k\ell}$ vanishes when $k+\ell$ is odd due to the symmetry $H_{d,-\nu}(-u) = H_{d,\nu}(u)$ for odd d. The terms corresponding to $0 \leq k \leq d+1$ in $H_{d,\nu}(u)$ are UV divergent in $\log Z_\nu(\epsilon = 0)$. Therefore we separate them from the remaining UV-finite terms

$$H^{\mathsf{uv}}_{d,\nu}(u) \equiv 2\sum_{k\geq d+1} b_k(\nu)\, u^{-(d+1-k)}, \quad H^{\mathsf{ir}}_{d,\nu}(u) \equiv H_{d,\nu}(u) - H^{\mathsf{uv}}_{d,\nu}(u) \tag{6.5.8}$$

Similarly using an infinitesimal IR regulator $\kappa \to 0^+$, we obtain a UV-IR separation for the regularized partition function $\log Z_\nu(\epsilon) = \log Z^{\mathsf{uv}}_\nu(\epsilon) + \log Z^{\mathsf{ir}}_\nu$:

$$\log Z^{\mathsf{uv}}_\nu(\epsilon) = \frac{1}{4}\int_{\mathbb{R}+i\delta} \frac{du}{ru}H^{\mathsf{uv}}_{d,r\nu}(u)e^{-\kappa|u|}, \quad \log Z^{\mathsf{ir}}_\nu = \int_0^\infty \frac{du}{2u}H^{\mathsf{ir}}_{d,\nu}(u)e^{-\kappa u} \tag{6.5.9}$$

where $r=\sqrt{u^2+\epsilon^2}/u$. In the IR part of the partition function, we've deformed the contour and safely put $\epsilon=0$. The $\log\kappa$ terms will drop out at the end when summing up $\log Z_\nu^{\mathsf{uv}}$ and $\log Z_\nu^{\mathsf{ir}}$.

Evaluation of UV part

Using the expansion (6.5.8), the UV part of $\log Z_\nu(\epsilon)$ can be written as

$$\log Z_\nu^{\mathsf{uv}}(\epsilon)=\frac{1}{2}\sum_{k=0}^{d}\sum_{\ell=0}^{k}b_{k\ell}\,\nu^\ell\int_{\mathbb{R}+i\delta}\frac{du}{ru}\,r^\ell u^{-(d+1-k)}+\frac{1}{2}\sum_{\ell=0}^{d+1}b_{d+1,\ell}\,\nu^\ell\int_{\mathbb{R}+i\delta}\frac{du}{ru}\,r^\ell e^{-\kappa|u|} \tag{6.5.10}$$

where the IR regulator κ is not necessary for $0\le k\le d$. Then it suffices to evaluate the following two u-integrals

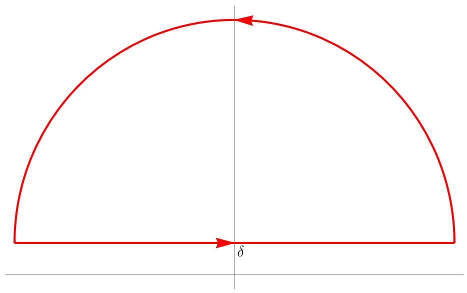
$$\begin{aligned}\mathcal{I}_\epsilon(k,\ell)&\equiv\int_{\mathbb{R}+i\delta}\frac{du}{2ru}\,r^\ell u^{-(d+1-k)}=\epsilon^{-(d+1-k)}\frac{1}{2}\int_{\mathbb{R}+i\delta}du\,\frac{(1+u^2)^{\frac{\ell-1}{2}}}{u^{d+\ell+1-k}}\\ \mathcal{J}_\epsilon(\ell)&\equiv\int_{\mathbb{R}+i\delta}\frac{du}{2ru}\,r^\ell e^{-\kappa|u|}=\frac{1}{2}\int_{\mathbb{R}+i\delta}du\,\frac{(1+u^2)^{\frac{\ell-1}{2}}}{u^\ell}e^{-\epsilon\kappa|u|}\end{aligned} \tag{6.5.11}$$

When ℓ is odd in $\mathcal{I}_\epsilon(k,\ell)$, we can close the contour in the upper half plane (fig. 6.5.1a) and the integral vanishes because $(1+u^2)^{\frac{\ell-1}{2}}$ has no pole in this case. Thus $\mathcal{I}_\epsilon(k,\ell)$ is nonvanishing only when ℓ is even. On the other hand, the coefficient of $\mathcal{I}_\epsilon(k,\ell)$, i.e. $b_{k\ell}$, vanishes for $k+\ell$ odd. Therefore, only terms with even k,ℓ survive in $\log Z_\nu^{\mathsf{uv}}(\epsilon)$. When ℓ is even in $\mathcal{I}_\epsilon(k,\ell)$, the previous "no poles" argument doesn't hold any more due to the presence of a branch cut from i to $i\infty$ in the upper half plane. However, we can deform the contour to integrate along the branch cut (fig. 6.5.1b) which yields

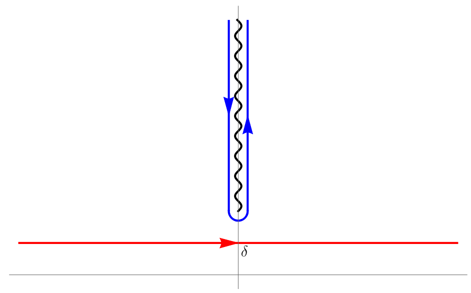
$$\mathcal{I}_\epsilon(k,\ell)=\frac{1}{2}(-)^{\frac{d+1-k}{2}}B\left(\frac{d+1-k}{2},\frac{1+\ell}{2}\right) \tag{6.5.12}$$

where the B-function is $B(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$.

For the integral $\mathcal{J}_\epsilon(\ell)$, we split it into an IR-divergent part and an IR-finite part. In the



(a) odd ℓ



(b) even ℓ

Figure 6.5.1: Contour for the integral $\mathcal{I}_\epsilon(k,\ell)$. When ℓ is odd, the contour is closed at infinity. When ℓ is even, the red contour in deformed to the blue one running along the branch cut of $\sqrt{u^2+1}$.

IR-finite part, we can drop the IR regulator $e^{-\kappa u}$.

$$\begin{aligned}\mathcal{J}_\epsilon(\ell) &= \frac{1}{2}\int_{\mathbb{R}+i\delta}\frac{du}{\sqrt{1+u^2}}\left((1+u^{-2})^{\frac{\ell}{2}}-1\right)+\frac{1}{2}\int_{\mathbb{R}+i\delta}\frac{du}{\sqrt{1+u^2}}e^{-\epsilon\kappa|u|}\\ &= \frac{1}{2}\int_{\mathbb{R}+i\delta}\frac{du}{\sqrt{1+u^2}}\left((1+u^{-2})^{\frac{\ell}{2}}-1\right)+\int_0^\infty\frac{du}{\sqrt{1+u^2}}e^{-\epsilon\kappa u}\end{aligned} \tag{6.5.13}$$

Since the coefficient of $\mathcal{J}_\epsilon(\ell)$ is $b_{d+1,\ell}$, it suffices to consider even ℓ which implies that $(1+u^{-2})^{\frac{\ell}{2}}$ can be expanded into a polynomial of u^{-2} and for each term in the polynomial we can use the same contour trick as in the $\mathcal{I}_\epsilon$ case to evaluate the integral. The IR-divergent part can be analytically evaluated in mathematica and it has a very simple small ϵ behavior. Altogether, the final result for $\mathcal{J}_\epsilon(\ell)$ is

$$\begin{aligned}\mathcal{J}_\epsilon(\ell) &= \frac{1}{2}\sum_{n=1}^{\ell/2}(-)^n\binom{\frac{\ell}{2}}{n}B\left(n,\frac{1}{2}\right)+\log\frac{2}{e^\gamma\epsilon\kappa}\\ &= -\left(H_\ell-\frac{1}{2}H_{\ell/2}+\log\frac{e^\gamma\epsilon\kappa}{2}\right)\end{aligned} \tag{6.5.14}$$

where H_ℓ is the harmonic number of order ℓ. Plugging (6.5.12) and (6.5.14) into (6.5.10) yields the regularized UV-part of the partition function

$$\begin{aligned}\log Z_\nu^{\text{uv}}(\epsilon) = \frac{1}{2}\sum_{k=0}^{r}(-)^{r+1-k}\epsilon^{-(d+1-2k)}\sum_{\ell=0}^{2k}b_{2k,\ell}B\left(\frac{d+1}{2}-k,\frac{\ell+1}{2}\right)\nu^\ell\\ -\sum_{\ell=0}^{d+1}b_{d+1,\ell}\left(H_\ell-\frac{1}{2}H_{\ell/2}\right)\nu^\ell-b_{d+1}(\nu)\log\frac{e^\gamma\epsilon\kappa}{2}\end{aligned} \tag{6.5.15}$$

For example, for $d=3$, we get

$$\log Z_\nu^{\mathsf{uv}}(\epsilon) = \frac{2}{3\,\epsilon^4} + \frac{1-4\nu^2}{24\,\epsilon^2} + \left(\frac{\nu^2}{48} - \frac{\nu^4}{18}\right) + \left(\frac{\nu^4}{24} + \frac{\nu^2}{48} - \frac{17}{5760}\right)\log\frac{2}{e^\gamma \epsilon\kappa} \tag{6.5.16}$$

which reproduces the Pol(Δ) part of $\log Z_\nu$ in [192] [4] without using the heat kernel coefficients α_k. In fact, by comparing (6.5.16) and (6.5.4), we can express the nonzero heat kernel coefficients α_k in terms of b_{kl}

$$\alpha_{2k} = (-4)^{k-\frac{d+1}{2}} \sum_{\ell=0}^{2k} \frac{\Gamma(\frac{\ell+1}{2})}{\Gamma(\frac{d+\ell}{2}+1-k)} b_{2k,\ell}\,\nu^\ell \tag{6.5.17}$$

Evaluation of IR part

The IR part can be evaluated through certain zeta function method as in [2]

$$\log Z_\nu^{\mathsf{ir}} = \frac{1}{2}\bar\zeta_\nu'(0) + b_{d+1}(\nu)\log(\kappa), \quad \bar\zeta_\nu(z) \equiv \frac{1}{\Gamma(z)}\int_0^\infty \frac{du}{u}\, u^z H_{d,\nu}(u) \tag{6.5.18}$$

Notice that the "character zeta function" $\bar\zeta_\nu(z)$ is originally defined by the integral above for z sufficiently large and then analytically continued to small z. $b_{d+1}(\nu)$ is related to the character zeta function as $b_{d+1}(\nu) = \frac{1}{2}\bar\zeta_\nu(0)$. Combing the UV part (6.5.15) and IR part (6.5.18) leads to

$$\begin{aligned}\log Z_\nu(\epsilon) = &\frac{1}{2}\sum_{k=0}^{r}(-)^{r+1-k}\epsilon^{-(d+1-2k)}\sum_{\ell=0}^{2k} b_{2k,\ell} B\left(\frac{d+1}{2}-k, \frac{\ell+1}{2}\right)\nu^\ell \\ &-\sum_{\ell=0}^{d+1} b_{d+1,\ell}\left(H_\ell - \frac{1}{2}H_{\ell/2}\right)\nu^\ell + b_{d+1}(\nu)\log\left(\frac{2}{e^\gamma\epsilon}\right) + \frac{1}{2}\bar\zeta_\nu'(0)\end{aligned} \tag{6.5.19}$$

Compared to the standard heat kernel results, we find the discrepancy between the *character zeta function* $\bar\zeta_\nu(z)$ and the *spectral zeta function* $\zeta_\nu(z) = \frac{1}{\Gamma(z)}\int_0^\infty \frac{dt}{t} t^z K_\nu(t)$:

$$\frac{1}{2}\zeta_\nu'(0) = \frac{1}{2}\bar\zeta_\nu'(0) - \sum_{\ell=0}^{d+1} b_{d+1,\ell}\left(H_\ell - \frac{1}{2}H_{\ell/2}\right)\nu^\ell \tag{6.5.20}$$

This difference is the multiplicative anomaly. Multiplicative anomaly is computed specifically for fields in AdS in [144], where it's called "secondary contribution". Though the information

[4] There is an overall $\frac{1}{2}$ factor difference because [192] computes the partition function of a *complex* scalar rather than a *real* scalar.

about multiplicative anomaly is lost, the formal factorization makes the evaluation of $\bar{\zeta}_\nu(z)$ much simpler than $\zeta_\nu(z)$. For example, we can expand $H_{d,\nu}(u)$ with respect to e^{-u} as $H_{d,\nu}(u) = \sum_{n\geq 0} D_n^{d+2} e^{-(\Delta+n)u}$ and for each fixed n, the u-integral yields $(n+\Delta)^{-z}$. Then we can immediately express $\bar{\zeta}_\nu(z)$ as a finite sum of Hurwitz zeta functions

$$\bar{\zeta}_\nu(z) = \sum_{n\geq 0} \frac{D_n^{d+2}}{(n+\Delta)^z} = D_{\delta-\Delta}^{d+2}\,\zeta(z,\Delta) \tag{6.5.21}$$

where δ is an operator defined as $\delta^n \zeta(z,\Delta) \equiv \zeta(z-n,\Delta)$. When $d=3$, we have $D_{\delta-\Delta}^{d+2} = \frac{\delta^3}{3} - \nu\delta^2 + \frac{12\nu^2-1}{12}\delta + \frac{\nu-4\nu^3}{12}$ and thus

$$\bar{\zeta}_\nu(z) = \frac{1}{3}\zeta(z-3,\Delta) - \nu\zeta(z-2,\Delta) + \left(\nu^2 - \frac{1}{12}\right)\zeta(z-1,\Delta) + \frac{\nu-4\nu^3}{12}\zeta(z,\Delta) \tag{6.5.22}$$

Altogether, the full one-loop partition function of a real scalar field with scaling dimension $\Delta = \frac{3}{2}+\nu$ in AdS$_4$ is

$$\begin{aligned}\log Z_\nu(\epsilon) = {} & \frac{2}{3\,\epsilon^4} + \frac{1-4\nu^2}{24\,\epsilon^2} + \left(\frac{\nu^2}{48} - \frac{\nu^4}{18}\right) + \left(\frac{\nu^4}{24} + \frac{\nu^2}{48} - \frac{17}{5760}\right)\log\frac{2}{e^\gamma \epsilon} \\ & + \frac{1}{6}\zeta'(-3,\Delta) - \frac{\nu}{2}\zeta'(-2,\Delta) + \frac{1}{2}\left(\nu^2 - \frac{1}{12}\right)\zeta'(-1,\Delta) + \frac{\nu-4\nu^3}{24}\zeta'(0,\Delta)\end{aligned} \tag{6.5.23}$$

consistent with [192].

6.5.2 Odd dimensional AdS$_{2r+1}$

The UV-regularized partition function in AdS$_{2r+1}$ can be written in terms of the following character integral

$$\frac{\log Z_\nu(\epsilon)}{\log R} = \frac{1}{2\pi i}\int_{C_0} \frac{du}{2\sqrt{u^2+\epsilon^2}} \frac{1+e^{-u}}{1-e^{-u}} e^{-\frac{d}{2}u - \nu\sqrt{u^2+\epsilon^2}} \tag{6.5.24}$$

where $d=2r$. As in the even dimensional AdS case, we can separate $H_{d,\nu}(u)$ into a UV-part and an IR-part. But the resulting IR partition function $\log Z_\nu^{\text{ir}} = \text{Res}_{u\to 0}\frac{1}{2u}H_{d,\nu}^{\text{ir}}(u)$ vanishes because $\frac{1}{2u}H_{d,\nu}^{\text{ir}}(u)$ doesn't have a pole at $u=0$ by our prescription for the UV-IR separation.

Therefore, it suffices to compute the UV-part of the partition function

$$\frac{\log Z_\nu(\epsilon)}{\log R} = \frac{\log Z_\nu^{\mathsf{uv}}(\epsilon)}{\log R} = \sum_{k=0}^{d+1}\sum_{\ell=0}^{k} b_{k\ell}\nu^\ell \frac{1}{2\pi i}\int_{C_0}\frac{du}{u}\frac{r^{\ell-1}}{u^{d+1-k}}$$
$$= \sum_{k=0}^{d+1}\sum_{\ell=0}^{k} b_{k\ell}\,\nu^\ell\,\epsilon^{-(d+1-k)}\mathsf{Res}_{u\to 0}\frac{(1+u^2)^{\frac{\ell-1}{2}}}{u^{d+\ell+1-k}} \tag{6.5.25}$$

where $b_{k\ell}$ is nonvanishing only when $k+\ell$ is even. By evaluating the residue explicitly, we find

$$\frac{\log Z_\nu(\epsilon)}{\log R} = \sum_{k=0}^{\frac{d}{2}}\epsilon^{-(d+1-2k)}\sum_{\ell=0}^{2k} b_{2k,\ell}\binom{\frac{\ell-1}{2}}{\frac{d-\ell+2k}{2}}\nu^\ell + b_{d+1}(\nu)$$
$$= \sum_{k=0}^{\frac{d}{2}}\epsilon^{-(d+1-2k)}\sum_{\ell=0}^{2k} b_{2k,\ell}\binom{\frac{\ell-1}{2}}{\frac{d-\ell+2k}{2}}\nu^\ell + \mathsf{Res}_{u\to 0}\left(\frac{1}{2u}H_{d,\nu}(u)\right) \tag{6.5.26}$$

where we rewrite $b_{d+1}(\nu)$ as a residue. As expected, there is no $\log\epsilon$ divergence or multiplicative anomaly and hence we can define a renormalized partition function by subtracting all the divergent terms unambiguously

$$\frac{\log Z_\nu^{\mathsf{ren}}}{\log R} = \mathsf{Res}_{u\to 0}\left(\frac{1}{2u}H_{d,\nu}(u)\right) \tag{6.5.27}$$

where the residue is evaluated in appendix D.4:

$$\frac{\log Z_\nu^{\mathsf{ren}}}{\log R} = -\frac{1}{(2r)!}\sum_{n=1}^{r}\frac{a_n(r)}{2n+1}\nu^{2n+1}, \quad \prod_{j=0}^{r-1}(x-j^2) = \sum_{n=1}^{r} a_n(r)\,x^n \tag{6.5.28}$$

For example, according to this expression, the renormalized scalar partition function in AdS_3 is $-\frac{\nu^3}{6}$, consistent with [216].

6.5.3 Summary

The computations in this section provide a well-defined and efficient rule to obtain a regularized partition function using only the unregularized character integral formula derived in section 6.2 and section 6.3. Here we summarize this rule for both even and odd dimensional AdS.

Odd dimensional AdS: $d = 2r$

Given the total character $\Theta_{\text{tot}}(u) = \Theta_b(u) - \Theta_e(u)$, where $\Theta_b(u), \Theta_e(u)$ are bulk character and edge character respectively, the renormalized partition function (with all negative powers of ϵ dropped) is given by

$$\log Z^{\text{ren}} = \text{Res}_{u\to 0}\left(\frac{1}{2u}\frac{1+e^{-u}}{1-e^{-u}}\Theta_{\text{tot}}(u)\right)\log R \tag{6.5.29}$$

For example, a massive spin-s field with $\Delta = 2+\nu$ on AdS$_5$ has bulk character $\Theta_b(u) = D_s^4\frac{e^{-(2+\nu)u}}{(1-e^{-u})^4}$ and edge character $\Theta_e(u) = D_{s-1}^6\frac{e^{-(1+\nu)u}}{(1-e^{-u})^2}$. Therefore according to (6.5.29)

$$\log Z_{s,\nu}^{\text{ren}} = \frac{(s+1)^2\nu^3}{360}\left(5(s+1)^2 - 3\nu^2\right)\log R \tag{6.5.30}$$

This result agrees with the computation of [112] based on direct spectral zeta function regularization. Another interesting example is linearized gravity in AdS$_3$. In this case, the bulk character is $\Theta_b(u) = 2\frac{e^{-2u}-e^{-3u}}{(1-e^{-u})^2}$, where the overall factor 2 is spin degeneracy for any field of nonzero spin, and the edge character is $\Theta_e(u) = 4\,e^{-u} - e^{-2u}$. Therefore, according to eq. (6.5.29) the renormalized partition function of 3D gravity is $\frac{13}{3}\log R$, consistent with [216].

Even dimensional AdS: $d = 2r+1$

For odd d, we need to consider massive and massless representations separately because in massless representations both gauge field and ghost field can contribute to the total multiplicative anomaly. Given a massive representation $\left[\frac{d}{2}+\nu, s\right]$, the unregularized partition function is

$$\log Z_{s,\nu} = \int_0^\infty \frac{du}{2u}\,\left(f_{s,\nu}^b(u) - f_{s,\nu}^e(u)\right) \tag{6.5.31}$$

where

$$f_{s,\nu}^b(u) = D_s^d\frac{1+e^{-u}}{1-e^{-u}}\frac{e^{-(\frac{d}{2}+\nu)u}}{(1-e^{-u})^d}, \quad f_{s,\nu}^e(u) = D_{s-1}^{d+2}\frac{1+e^{-u}}{1-e^{-u}}\frac{e^{-(\frac{d-2}{2}+\nu)u}}{(1-e^{-u})^{d-2}} \tag{6.5.32}$$

$f_{s,\nu}^b(u)$ induces the *bulk* character zeta function $\bar{\zeta}_{s,\nu}^b(z) \equiv \frac{1}{\Gamma(z)}\int_0^\infty \frac{du}{u}u^z f_{s,\nu}^b(u)$ and $f_{s,\nu}^e(u)$ induces the *edge* character zeta function $\bar{\zeta}_{s,\nu}^e(z) \equiv \frac{1}{\Gamma(z)}\int_0^\infty \frac{du}{u}u^z f_{s,\nu}^e(u)$. The renormalized partition function (with all negative powers of ϵ dropped) is

$$\log Z^{\text{ren}}_{s,\nu} = \frac{1}{2}\left(\bar{\zeta}^b_{s,\nu}(0) - \bar{\zeta}^e_{s,\nu}(0)\right)\log(2e^{-\gamma}/\epsilon) + \frac{1}{2}\left(\bar{\zeta}^{b\prime}_{s,\nu}(0) - \bar{\zeta}^{e\prime}_{s,\nu}(0)\right) - \sum_{\ell=0}^{d+1} b_\ell(s)\left(H_\ell - \frac{1}{2}H_{\ell/2}\right)\nu^\ell \tag{6.5.33}$$

where $\sum_{\ell=0}^{d+1} b_\ell(s)\nu^\ell = \frac{1}{2}(\bar{\zeta}^b_{s,\nu}(0)-\bar{\zeta}^e_{s,\nu}(0))$. Using the expansion $f^b_{s,\nu}(u) = \sum_{n\geq 0} P^b_{s,\nu}(n)e^{-(\Delta+n)t}$, where $\Delta = \frac{d}{2}+\nu$, the bulk character zeta function $\bar{\zeta}^b_{s,\nu}(z)$ can be written as a finite sum of Hurwitz zeta functions

$$\bar{\zeta}^b_{s,\nu}(z) = P^b_{s,\nu}(\delta - \Delta)\zeta(z,\Delta) \tag{6.5.34}$$

and similarly for $\bar{\zeta}^e_{s,\nu}(z)$. In AdS_4, for example, the bulk and edge character zeta functions are

$$\begin{aligned} \bar{\zeta}^b_{s,\nu}(z) &= (2s+1)\left(\frac{1}{3}\zeta(z-3,\Delta) - \nu\,\zeta(z-2,\Delta) + (\nu^2 - \frac{1}{12})\zeta(z-1,\Delta) + \frac{\nu(1-4\nu^2)}{12}\zeta(z,\Delta)\right) \\ \bar{\zeta}^e_{s,\nu}(z) &= \frac{1}{3}s(s+1)(2s+1)\left(\zeta(z-1,\Delta-1) - \nu\,\zeta(z,\Delta-1)\right) \end{aligned} \tag{6.5.35}$$

For a massless representation of spin-s and depth-t, the unregularized partition consists of four parts

$$\log Z_{s,\nu_t} = \int_0^\infty \frac{du}{2u}\left(f^b_{s,\nu_t}(u) - f^e_{s,\nu_t}(u) - f^b_{t,\nu_s}(u) + f^e_{s,\nu_t}(u)\right) \tag{6.5.36}$$

The renormalized partition function can be expressed as

$$\begin{aligned} \log Z^{\text{ren}}_{s,\nu_t} = &\frac{1}{2}\left(\bar{\zeta}^b_{s,\nu_t}(0) - \bar{\zeta}^e_{s,\nu_t}(0) - \bar{\zeta}^b_{t,\nu_s}(0) + \bar{\zeta}^e_{t,\nu_s}(0)\right)\log(2e^{-\gamma}/\epsilon) + \frac{1}{2}\left(\bar{\zeta}^{b\prime}_{s,\nu_t}(0) - \bar{\zeta}^{e\prime}_{s,\nu_t}(0) - \bar{\zeta}^{b\prime}_{t,\nu_s}(0) + \bar{\zeta}^{e\prime}_{t,\nu_s}(0)\right) \\ &- \left(\sum_{\ell=0}^{d+1} b_\ell(s)\left(H_\ell - \frac{1}{2}H_{\ell/2}\right)\nu_t^\ell - \sum_{\ell=0}^{d+1} b_\ell(t)\left(H_\ell - \frac{1}{2}H_{\ell/2}\right)\nu_s^\ell\right) \end{aligned} \tag{6.5.37}$$

For example, the renormalized partition function of linearized gravity in AdS_4 is

$$\log Z^{\text{ren}}_{\text{gravity}} = \frac{47\log\,A}{6} + \frac{1}{3}\zeta'(-3) + \frac{1957}{288} - \frac{\log 2}{2} - 5\log(2\pi) - \frac{571}{90}\log(2e^{-\gamma}/\epsilon) \tag{6.5.38}$$

where A is the Glaisher-Kinkelin constant. (6.5.38) reproduces the $s=2$ result in [111].

Remark 12. *Apart from the multiplicative anomaly, the remaining part of the partition function for both massive and massless representations is completely captured by the character zeta function* $\frac{1}{\Gamma(z)}\int_0^\infty \frac{du}{u^{1-z}}\frac{1+e^{-u}}{1-e^{-u}}\Theta_{tot}(u)$*, where* $\Theta_{tot}(u)$ *consists of bulk and edge characters. Thus, by inserting a UV regulator* u^z *in the unregularized partition function, we recover the correct* $\log$*-divergence and finite part up to multiplicative anomaly, which was proved to vanish when summing up the whole spectrum of Vasiliev theories [144]. We will also show in section 6.7 the vanishing of multiplicative anomaly for type-A Vasiliev gravity using the character integral formalism. Keeping this comment in mind, we are free to use the unregularized character integral formula to compute the full partition function of Vasiliev theories in section 6.7.*

6.6 Double trace deformation

A large N CFT$_d$ where a primary operator $\mathcal{O}$ has scaling dimension $\Delta_{\mathcal{O}}$ can flow to another CFT$_d$ where $\mathcal{O}$ has the shadow scaling dimension $\bar{\Delta}_{\mathcal{O}} \equiv d - \Delta_{\mathcal{O}}$ by turning on a double trace deformation $\mathcal{O}^2$ in the Lagrangian [217, 218]. On the AdS side, this RG flow is equivalent to switching the boundary conditions when we quantize the dual bulk field. Due to AdS/CFT duality, the effect of this RG flow on the partition function of the large N CFT$_d$ living on S^d can be computed from both boundary and bulk sides [138, 214, 218–221]. In particular, in [138] the authors thoroughly computed the effect of any higher spin currents. Let $\mathcal{O}_{\mu_1\cdots\mu_s}$ be a spin-s current and they found the change of free energy induced by $\mathcal{O}_{\mu_1\cdots\mu_s}\mathcal{O}^{\mu_1\cdots\mu_s}$ has log-divergence when $\Delta_{\mathcal{O}} = d+s-2$, i.e. $\mathcal{O}$ is a conserved current, and the change is of order 1 when $\Delta_{\mathcal{O}}$ takes other values. In this section, we'll reproduce the main results of [138] on bulk side by using character integral formula (6.4.9). As we've just mentioned that the double trace deformation induces the dual boundary condition, it suffices to compute $\log Z_{s,\nu} - \log Z_{s,-\nu}$. We'll focus on the odd d case (the even d case can be analyzed similarly) and see that it's extremely convenient to use the character integral representation to do this computation because flipping the boundary condition is equivalent to switching to the character of the dual representation, i.e. $\Theta^{\text{AdS}_{d+1}}_{[\Delta,s]} \to \Theta^{\text{AdS}_{d+1}}_{[\bar{\Delta},s]}$.

We'll start from considering a scalar field with complex scaling dimension $\Delta = \frac{d}{2} + i\nu$ and then Wick rotate Δ to a real number. Before performing any actual computation, we want to mention the following observation which is based on the explicit evaluation in last section, that the UV-divergent part including multiplicative anomaly of $\log Z_\nu$ is an even function in ν when d is odd (see equation (6.5.16) as an explicit example). This observation implies that $\log Z_{i\nu} - \log Z_{-i\nu}$ is UV finite. In addition, in the difference

$$\log Z_{i\nu}(\epsilon) - \log Z_{-i\nu}(\epsilon) = \frac{1}{2}\int_{\mathbb{R}+i\delta} \frac{du}{2\sqrt{u^2+\epsilon^2}} \frac{1+e^{-u}}{1-e^{-u}} \frac{e^{-\frac{d}{2}u}(e^{-i\nu\sqrt{u^2+\epsilon^2}} - e^{+i\nu\sqrt{u^2+\epsilon^2}})}{(1-e^{-u})^d} \tag{6.6.1}$$

the integrand is indeed a single-valued function because when u goes around one of the branch points of $\sqrt{u^2+\epsilon^2}$, the combination $\frac{e^{-i\nu\sqrt{u^2+\epsilon^2}} - e^{+i\nu\sqrt{u^2+\epsilon^2}}}{\sqrt{u^2+\epsilon^2}}$ keeps invariant though $\sqrt{u^2+\epsilon^2}$ itself picks an extra minus sign. Then we are free to shift the u-contour upwards such that $\delta > \epsilon$. With this new contour and using the UV-finiteness of $\log Z_{i\nu} - \log Z_{-i\nu}$, we can safely put $\epsilon \to 0$ which amounts to sending $\sqrt{u^2+\epsilon^2}$ to u:

$$\log Z_{i\nu} - \log Z_{-i\nu} = \frac{1}{2}\int_{\mathbb{R}+i\delta} \frac{du}{2\,u} \frac{1+e^{-u}}{1-e^{-u}} \frac{e^{-\frac{d}{2}u}(e^{-i\nu u} - e^{+i\nu u})}{(1-e^{-u})^d} \tag{6.6.2}$$

To proceed further, we introduce a new integral that can eliminate u in the denominator of (6.6.2) and then switch the order of integrals

$$\log Z_{i\nu} - \log Z_{-i\nu} = -\frac{i}{4}\int_{-\nu}^{\nu} d\lambda \int_{\mathbb{R}+i\delta} du \frac{1+e^{-u}}{1-e^{-u}} \frac{e^{-\frac{d}{2}u - i\lambda u}}{(1-e^{-u})^d} \tag{6.6.3}$$

After these manipulations, the u-integral is essentially the definition of $\tilde{\mu}_d^{(+)}(\lambda)$, cf. (6.4.1)

$$\log Z_{i\nu} - \log Z_{-i\nu} = \frac{i\pi(-)^{\frac{d-1}{2}}}{2\,d!}\int_{-\nu}^{\nu} d\lambda\, \tilde{\mu}_d^{(+)}(\lambda) = \frac{i(-)^r}{d!}\int_0^{\nu} d\lambda\, \mu^{(d)}(\lambda) \tag{6.6.4}$$

where we've used that the even part of $\tilde{\mu}_d^{(+)}(\lambda)$ is the spectral density $\mu^{(d)}(\lambda)$. It's straightforward to generalize this method to higher spin fields by including the edge-mode contribution and using eq. (6.4.8)

$$\log Z_{s,i\nu} - \log Z_{s,-i\nu} = \frac{i\pi(-)^r}{d!} D_s^d \int_0^{\nu} d\lambda\, \mu_s^{(d)}(\lambda) \tag{6.6.5}$$

Surprisingly, the change of free energy triggered by the higher spin double trace deformation at large N is completely encoded in the higher spin spectral density. Given the eq. (6.6.5), we make a Wick rotation $\nu \to i\nu$ to obtain result for real scaling dimension $\Delta = \frac{d}{2} + \nu$

$$\log Z_{s,\nu} - \log Z_{s,-\nu} = \frac{\pi(-)^r}{d!} D_s^d \int_0^\nu d\lambda \, \mu_s^{(d)}(i\lambda) \tag{6.6.6}$$

For example, at $d = 3$ we get

$$\log Z_{s,\nu} - \log Z_{s,-\nu} = \frac{(2s+1)\pi}{6} \int_{\frac{3}{2}}^{\Delta} dx (x - \frac{3}{2})(x + s - 1)(x - s - 2) \cot(\pi x) \tag{6.6.7}$$

where we've changed variable $\lambda = x - \frac{3}{2}$. This equation agrees with the result in [138]. When ν reaches some half integer number, say $\nu_{s-1} = \frac{d}{2} + s - 2$, which corresponds to a double trace deformation triggered by a spin-s conserved current in free $U(N)$ vector model, the integral (6.6.6) is divergent no matter what contour we use because the singularity $\nu = \nu_{s-1}$ is at the end point of the integration contour. To extract the leading divergence, we need the singular behavior of $\mu_s^{(d)}(i\lambda)$ near $\lambda = \nu_{s-1}$

$$\frac{\pi(-)^r}{d!} D_s^d \, \mu_s^{(d)}(i\lambda) = -\frac{D_{s-1,s-1}^{d+2}}{2(\lambda - \nu_{s-1})} + \mathcal{O}((\lambda - \nu_{s-1})^0) \tag{6.6.8}$$

which is a consequence of eq. (2.2.4) and $D_{p-1,s}^{d+2} = -D_{s-1,p}^{d+2}$. Since $D_{s-1,s-1}^{d+2} = n_{s-1}^{\mathsf{KT}}$ is the number of spin-$(s-1)$ Killing tensors on S^{d+1} and also the number of spin-$(s-1)$ conformal Killing tensors on S^d [138]. Therefore if we truncate the integral (6.6.6) at $\nu = \nu_s - \epsilon$, the change of $\log Z$ induced by a spin-s conserved current has a $\log$-divergence part $\frac{1}{2} n_{s-1}^{\mathsf{KT}} \log(\epsilon)$.

6.7 Application to Vasiliev theories

With the character integral method developed in the previous sections, we're finally able to compute partition function of Vasiliev theories in all even dimensional AdS (The odd dimensional AdS case can be analyzed similarly and is indeed much simpler). We'll use (non)minimal type-A$_\ell$ theory and type-B Vasiliev theory, which are reviewed below, to illustrate the application of the character integral method. Before that we want to stress again, due to the comment at the end of the section 6.5, the unregularized version of character integral formula is sufficient.

6.7.1 A brief review of Vasiliev theories and Flato-Fronsdal theorems

The simplest and best understood higher spin theory is the nonminimal type-A Vasiliev theory in AdS_{d+1}, which contains a $\Delta = d-2$ real scalar and a tower of massless higher spin gauge fields. This theory is believed to be dual to a free $U(N)$ vector model on boundary described by Lagrangian $\mathcal{L} = \frac{1}{2}\phi_i^* \,\Box\, \phi_i, 1 \le i \le N$. The $U(N)$ fundamental field ϕ_i is in the scalar singleton representation $\left[\frac{d-2}{2}, 0\right] \equiv$ Rac of $\mathfrak{so}(2,d)$. One direct result of the duality is a one-to-one correspondence between the field content in bulk and the single-trace operators in $U(N)$ vector model. In representation theory, this is confirmed by Flato-Fronsdal theorem [194]:

$$\text{Rac} \otimes \text{Rac} = \bigoplus_{s=0}^{\infty} [d+s-2, s] \tag{6.7.1}$$

which can be proved by using the following identity of characters

$$\left(\Theta_{\text{Rac}}^{\mathfrak{so}(2,d)}(u)\right)^2 = \sum_{s=0}^{\infty} \Theta_{[d+s-2,s]}^{\text{AdS}_{d+1}}(u) \equiv \Theta_{\text{A}}^{\text{AdS}_{d+1}}, \quad \Theta_{\text{Rac}}^{\mathfrak{so}(2,d)} = \frac{e^u - e^{-u}}{(e^{\frac{u}{2}} - e^{-\frac{u}{2}})^d} \tag{6.7.2}$$

There are a lot of variants of the original type-A Vasiliev theory. For example, if we relax the requirement of unitarity, we can take the boundary CFT to be $\mathcal{L} = \frac{1}{2}\phi_i^* \,\Box^\ell\, \phi_i$, where each ϕ_i is in the representation $[\frac{d-2\ell}{2}, 0] \equiv \text{Rac}_\ell$ of $\mathfrak{so}(2,d)$. For more details about the $\Box^\ell$ theory, we refer readers to [99, 222]. The bulk dual of this nonunitary CFT is called the type-A_ℓ higher spin gravity with field content given by a generalized Flato-Fronsdal theorem [222–224]

$$\text{Rac}_\ell \otimes \text{Rac}_\ell = \bigoplus_{p=1,3,\cdots}^{2\ell-1} \bigoplus_{s=0}^{\infty} [d+s-p-1, s] \tag{6.7.3}$$

where $[d+s-p-1, s]$ corresponds to a PM field of spin-s and depth-$(s-p)$ for $p \le s$. At the level of characters, this tensor product decomposition is equivalent to

$$\left(\Theta_{\text{Rac}_\ell}^{\mathfrak{so}(2,d)}(u)\right)^2 = \sum_{p=1,3,\cdots}^{2\ell-1} \sum_{s=0}^{\infty} \Theta_{[d+s-2,s]}^{\text{AdS}_{d+1}}(u) \equiv \Theta_{\text{A}_\ell}^{\text{AdS}_{d+1}}, \quad \Theta_{\text{Rac}_\ell}^{\mathfrak{so}(2,d)} = \frac{e^{\ell u} - e^{-\ell u}}{(e^{\frac{u}{2}} - e^{-\frac{u}{2}})^d} \tag{6.7.4}$$

In type-A$_\ell$ theory, we can further replace the complex scalars by real scalars that are in the fundamental representation of $O(N)$. The resulting AdS dual is called the minimal type-A$_\ell$ theory and its field content can be extracted from the symmetrized tensor product of two Rac$_\ell$:

$$\mathsf{Rac}_\ell \odot \mathsf{Rac}_\ell = \bigoplus_{p=1,3,\cdots}^{2\ell-1} \bigoplus_{s=0,2,\cdots}^{\infty} [d+s-p-1, s] \tag{6.7.5}$$

where only fields even spin exist. Summing over the characters for representations appearing in the tensor product decomposition (6.7.5) leads to

$$\Theta^{\mathsf{AdS}_{d+1}}_{\mathsf{A}^{\min}_\ell} \equiv \sum_{p=1,3,\cdots}^{2\ell-1} \sum_{s=0,2,\cdots}^{\infty} \Theta^{\mathsf{AdS}_{d+1}}_{[d+s-2,s]}(u) = \frac{1}{2}\left(\Theta^{\mathfrak{so}(2,d)}_{\mathsf{Rac}_\ell}(u)\right)^2 + \frac{1}{2}\Theta^{\mathfrak{so}(2,d)}_{\mathsf{Rac}_\ell}(2u) \tag{6.7.6}$$

Another important variant of the original nonminimal type-A theory is the so-called type-B theory. It is the AdS-dual of free $U(N)$ Dirac fermions restricted to $U(N)$ singlet sector. Each Dirac fermion carries the spinor singleton representation $[\frac{d-1}{2}, \mathbf{\frac{1}{2}}] \equiv \mathsf{Di}$ of $\mathfrak{so}(2,d)$, where $\mathbf{\frac{1}{2}}$ denotes the spin-$\frac{1}{2}$ representation of $\mathfrak{so}(d)$. The bulk field content is given by Di $\otimes$ Di, which takes the following for odd d [194]

$$\mathsf{Di} \otimes \mathsf{Di} = [d-1, 0] \bigoplus \bigoplus_{m=0}^{\frac{d-3}{2}} \bigoplus_{s=1}^{\infty} [d-2+s, (s, 1^m)] \tag{6.7.7}$$

When $d = 3$, all $(s, 1^m)$ are reduced to a spin-s representation of $\mathfrak{so}(3)$. Thus in AdS$_4$, the spectra of the type-A and type-B theory are the same except that the $m^2 = -2$ scalar is quantized with $\Delta_- = 1$ in the former and $\Delta_+ = 2$ in the latter. However, for higher d, the spectrum type-B theory is much more complicated due to the presence of fields of mixed symmetry. Let's call the collection of fields of "spin" $(s, 1^m)$ the m-sector. The $m = 0$ sector is almost the same as spectrum of type-A theory except the scaling dimension of the scalar. For the $m \geq 1$ sectors, fields with $s \geq 2$ are massless gauge fields with the corresponding ghost fields in the representation $[d-1+s, (s-1, 1^m)]$ of $\mathfrak{so}(2,d)$ while the $s = 1$ fields are massive and totally antisymmetric.

6.7.2 Type-A higher spin gravity

Nonminimal theory: Field content of the nonmonimal type-A higher spin gravity is given by eq. (6.7.1). Using (6.2.12) for the $\Delta = d-2$ scalar and (6.3.9) for the massless gauge fields, we get

$$\log Z_{\mathsf{A}}^{\mathsf{AdS}_{d+1}} = \int_0^\infty \frac{du}{2u}\frac{1+e^{-u}}{1-e^{-u}}\left(\sum_{s\geq 0}\Theta^{\mathsf{AdS}_{d+1}}_{[d+s-2,s]}(u) - \left(e^{\frac{u}{2}}-e^{-\frac{u}{2}}\right)^4\sum_{s\geq 0}\Theta^{\mathsf{AdS}_{d+3}}_{[d+2+s-2,s]}(u)\right) \tag{6.7.8}$$

where s is shifted by 1 in the edge character to match with the bulk part. Due to the Flato-Fronsdal theorem (6.7.2), both sums in (6.7.8) give the square of a Rac-character. Plugging in the explicit form of these Rac-characters, it's clear that the bulk and edge contributions exactly cancel out and hence the total one-loop free energy of type-A theory vanishes:

$$\log Z_{\mathsf{A}}^{\mathsf{AdS}_{d+1}} = \int_0^\infty \frac{du}{2u}\frac{1+e^{-u}}{1-e^{-u}}\left[\left(\Theta^{\mathfrak{so}(2,d)}_{\mathsf{Rac}}(u)\right)^2 - \left(e^{\frac{u}{2}}-e^{-\frac{u}{2}}\right)^4\left(\Theta^{\mathfrak{so}(2,d+2)}_{\mathsf{Rac}}(u)\right)^2\right] = 0 \tag{6.7.9}$$

Before moving to the minimal case, we want to check that the total multiplicative anomaly indeed vanishes. To do this, we should use the fully regularized character integral formula, with which the exact cancellation between bulk and edge contributions doesn't hold any more at the integrand level. Instead, we get

$$\log Z_{\mathsf{A}}^{\mathsf{AdS}_{d+1}} = 4^{-d}\int_{\mathbb{R}+i\delta}\frac{du}{ru}\frac{1+e^{-u}}{1-e^{-u}}\frac{(1+\cosh(ru))(\cosh(ru)-\cosh(u))}{(\sinh\frac{u}{2}\sinh\frac{ru}{2})^d} \tag{6.7.10}$$

where $r = \sqrt{u^2+\epsilon^2}/u$. The multiplicative anomaly, if exists, should appear as the coefficient of ϵ^0 in the small ϵ expansion of (6.7.10), which can be realized by a change of variable $u \to \epsilon\, u$ and expanding the integrand around small ϵ:

$$\log Z_{\mathsf{A}}^{\mathsf{AdS}_{d+1}} = 4^{-d}\int_{\mathbb{R}+i\delta}\frac{du}{\sqrt{u^2+1}}\frac{1+e^{-\epsilon u}}{1-e^{-\epsilon u}}\frac{(1+\cosh(\epsilon\sqrt{1+u^2}))(\cosh(\epsilon\sqrt{1+u^2})-\cosh(\epsilon u))}{(\sinh\frac{\epsilon u}{2}\sinh\frac{\epsilon\sqrt{1+u^2}}{2})^d} \tag{6.7.11}$$

Notice that the integrand of (6.7.11) is an odd function of ϵ and hence cannot have any ϵ^0 term in small ϵ expansion. This observation leads to the vanishing of the total multiplicative anomaly in nonminimal type-A theory.

Minimal theory: Since minimal type-A theory contains only fields of even spins, its total partition function can be written as

$$\log Z_{\mathsf{A}^{\min}}^{\mathsf{AdS}_{d+1}} = \int_0^\infty \frac{du}{2u}\frac{1+e^{-u}}{1-e^{-u}}\left(\sum_{s\ \text{even}} \Theta_{[d+s-2,s]}^{\mathsf{AdS}_{d+1}}(u) - \left(e^{\frac{u}{2}} - e^{-\frac{u}{2}}\right)^4 \sum_{s\ \text{odd}} \Theta_{[d+2+s-2,s]}^{\mathsf{AdS}_{d+3}}(u)\right) \tag{6.7.12}$$

where the spin in edge characters is shifted by 1. The sum over all bulk characters lead to $\Theta_{\mathsf{A}^{\min}}^{\mathsf{AdS}_{d+1}}$

$$\Theta_{\mathsf{A}^{\min}}^{\mathsf{AdS}_{d+1}}(u) = \frac{1}{2}\Theta_{\mathsf{A}}^{\mathsf{AdS}_{d+1}}(u) + \frac{1}{2}\Theta_{\mathsf{Rac}}^{\mathfrak{so}(2,d)}(2u) \tag{6.7.13}$$

where we've used eq. (6.7.2) and (6.7.6). The sum of edge characters, since only odd spin fields are involved, yields the difference between the nonminimal character and minimal character in AdS_{d+3}:

$$\Theta_{\mathsf{A}}^{\mathsf{AdS}_{d+3}}(u) - \Theta_{\mathsf{A}^{\min}}^{\mathsf{AdS}_{d+3}}(u) = \frac{1}{2}\Theta_{\mathsf{A}}^{\mathsf{AdS}_{d+3}}(u) - \frac{1}{2}\Theta_{\mathsf{Rac}}^{\mathfrak{so}(2,d+2)}(2u) \tag{6.7.14}$$

Plugging eq. (6.7.13) and (6.7.14) into (6.7.12), the type-A characters cancel out as in the nonminimal theory and thus the remaining term is

$$\begin{aligned}\log Z_{\mathsf{A}^{\min}}^{\mathsf{AdS}_{d+1}} &= \frac{1}{2}\int_0^\infty \frac{du}{2u}\frac{1+e^{-u}}{1-e^{-u}}\left(\Theta_{\mathsf{Rac}}^{\mathfrak{so}(2,d)}(2u) + \left(e^{\frac{u}{2}} - e^{-\frac{u}{2}}\right)^4 \Theta_{\mathsf{Rac}}^{\mathfrak{so}(2,d+2)}(2u)\right)\\ &= \int_0^\infty \frac{du}{2u}\frac{1+e^{-u}}{1-e^{-u}}\frac{e^{-\left(\frac{d-1}{2}+\frac{1}{2}\right)u} + e^{-\left(\frac{d-1}{2}-\frac{1}{2}\right)u}}{(1-e^{-u})^{d-1}}\end{aligned} \tag{6.7.15}$$

where in the second line u has been rescaled $u \to \frac{u}{2}$. Notice that $\frac{e^{-\frac{d}{2}u}+e^{-(\frac{d}{2}-1)u}}{(1-e^{-u})^{d-1}}$ is the Harish-Chandra character of the $\Delta = \frac{d-2}{2}$ representation of $\mathsf{SO}(1,d)$. Then according to the character integral representation of the sphere partition functions found in [2], the partition function of minimal type-A theory on AdS_{d+1} is the same as the partition function of a conformally coupled scalar on S^d. This result agrees with [111], where the appearance of this scalar partition function is interpreted as an $N \to N-1$ shift in the identification of Newton's constant $G_N \sim \frac{1}{N}$. Again, starting from the square root regularized character and following the same argument as in the nonminimal case, one can also show the vanishing of total multiplicative anomaly for minimal type-A theory. Let's also mention that when $d = 2r$ is even, the analogue of (6.7.15) implies the coefficient of $\log R$ of minimal type-A theory in AdS_{2r+1} matches the Weyl anomaly of a conformally couple scalar on the boundary, which is a d-dimensional sphere of radius R.

6.7.3 Type-A_ℓ higher spin gravities

Nonminimal theory: Given the spectrum of nonminimal type-A_ℓ theory (6.7.3), the total partition function can be written as

$$\log Z_{\mathsf{A}_\ell}^{\mathsf{AdS}_{d+1}} = \int_0^\infty \frac{du}{2u}\frac{1+e^{-u}}{1-e^{-u}}\left(\sum_{p=1,3}^{2\ell-1}\sum_{s\geq 0}\Theta^{\mathsf{AdS}_{d+1}}_{[d+s-p-1,s]}(u) - \left(e^{\frac{u}{2}}-e^{-\frac{u}{2}}\right)^4\sum_{p=1,3}^{2\ell-1}\sum_{s\geq 0}\Theta^{\mathsf{AdS}_{d+3}}_{[d+2+s-p-1,s]}(u)\right) \tag{6.7.16}$$

where the spin label s is shifted by 1 in the sum of edge characters. Using the generalized Flato-Fronsdal theorem (6.7.4), it's straightforward to show that the bulk and edge contributions exactly cancel out

$$\log Z_{\mathsf{A}_\ell}^{\mathsf{AdS}_{d+1}} = \int_0^\infty \frac{du}{2u}\frac{1+e^{-u}}{1-e^{-u}}\left[\left(\Theta^{\mathfrak{so}(2,d)}_{\mathsf{Rac}_\ell}(u)\right)^2 - \left(e^{\frac{u}{2}}-e^{-\frac{u}{2}}\right)^4\left(\Theta^{\mathfrak{so}(2,d+2)}_{\mathsf{Rac}_\ell}(u)\right)^2\right] = 0 \tag{6.7.17}$$

Therefore the total free energy of nonminimal type-A_ℓ also vanishes.

Minimal theory: Following the same steps as in the minimal type-A case, we can directly write down the result for minimal type-A_ℓ theory

$$\begin{aligned}\log Z_{\mathsf{A}_\ell^{\min}}^{\mathsf{AdS}_{d+1}} &= \frac{1}{2}\int_0^\infty \frac{du}{2u}\frac{1+e^{-u}}{1-e^{-u}}\left(\Theta^{\mathfrak{so}(2,d)}_{\mathsf{Rac}_\ell}(2u) + \left(e^{\frac{u}{2}}-e^{-\frac{u}{2}}\right)^4\Theta^{\mathfrak{so}(2,d+2)}_{\mathsf{Rac}_\ell}(2u)\right)\\ &= \int_0^\infty \frac{du}{2u}\frac{1+e^{-u}}{1-e^{-u}}\frac{e^{-\frac{d}{2}u}\left(e^{\ell u}-e^{-\ell u}\right)}{(1-e^{-u})^d}\end{aligned} \tag{6.7.18}$$

where u is rescaled in the second line. Naively speaking, eq. (6.7.18) doesn't look like any character integral. But using the expansion $1-e^{-2\ell u}=(1-e^{-u})\sum_{n=0}^{2\ell-1}e^{-nu}$, we can transform it into a finite sum of character integrals in the dS sense

$$\log Z_{\mathsf{A}_\ell^{\min}}^{\mathrm{AdS}_{d+1}}=\sum_{n=1}^{\ell}\int_0^\infty\frac{du}{2u}\frac{1+e^{-u}}{1-e^{-u}}\frac{e^{-(\frac{d-1}{2}+(n-\frac{1}{2}))u}+e^{-(\frac{d-1}{2}-(n-\frac{1}{2}))u}}{(1-e^{-u})^{d-1}}\tag{6.7.19}$$

where $\frac{e^{-(\frac{d-1}{2}+(n-\frac{1}{2}))u}+e^{-(\frac{d-1}{2}-(n-\frac{1}{2}))u}}{(1-e^{-u})^{d-1}}$ is the Harish-Chandra character of the $\Delta=\frac{d-1}{2}+n-\frac{1}{2}$ representation of $\mathrm{SO}(1,d)$ and the corresponding character integral represents the one-loop partition function of a scalar field of mass $m_n^2=(\frac{d}{2}-n)(\frac{d-2}{2}+n)$ on S^d [2]. Defining a collection of scalar Laplacians $\{-\nabla^2+m_n^2\}_{1\le n\le\ell}$, the partition function $\log Z_{\mathsf{A}_\ell^{\min}}^{\mathrm{AdS}_{d+1}}$ can be rewritten as

$$\log Z_{\mathsf{A}_\ell^{\min}}^{\mathrm{AdS}_{d+1}}=\sum_{n=1}^{\ell}\log\det(-\nabla^2+m_n^2)^{-\frac{1}{2}}=\log\det\left(\Box_{S^d}^\ell\right)^{-\frac{1}{2}},\quad \Box_{S^d}^\ell\equiv\prod_{n=1}^{\ell}(-\nabla^2+m_n^2)\tag{6.7.20}$$

where we are allowed to put the Laplacian into a product form because there is no multiplicative anomaly on an odd dimensional manifold. Notice that $\Box_{S^d}^\ell$, the Weyl-covariant generalization of $\Box^\ell$, is a GJMS operator on S^d [225–228] and when $\ell=1$ it is reduced to the conformal Laplacian on S^d. Therefore, the one-loop partition function of minimal type-A_ℓ theory on AdS_{d+1} is the same as the one-loop partition function of the $\Box^\ell$-theory on S^d. This is again consistent with the $N\to N-1$ interpretation.

6.7.4 Type-B higher spin gravities

AdS$_4$: Let's start considering the type-B theory in AdS$_4$. Its has the same spectrum as type-A theory except the boundary condition imposed on the scalar is flipped. Using $\log Z_{\mathsf{A}}^{\mathrm{AdS}_4}=0$, we are left with

$$\log Z_{\mathsf{B}}^{\mathrm{AdS}_4}=\int_0^\infty\frac{du}{2u}\frac{1+e^{-u}}{1-e^{-u}}\left(\Theta_{\Delta=2}^{\mathrm{AdS}_4}(u)-\Theta_{\Delta=1}^{\mathrm{AdS}_4}(u)\right)=-\int_0^\infty\frac{du}{2u}\frac{1+e^{-u}}{1-e^{-u}}\frac{e^{-u}}{(1-e^{-u})^2}\tag{6.7.21}$$

which represents the change of partition function induced by a double-trace deformation. To evaluate this integral, we can either regularize it by inserting u^z and express it in terms of

Hurwitz zeta function or directly use eq. (6.6.7)

$$\log Z_{\mathsf{B}}^{\mathsf{AdS}_4} = \frac{\pi}{6}\int_{\frac{3}{2}}^{2}(x-1)(x-\frac{3}{2})(x-2)\cot(\pi x) = \frac{\zeta(3)}{8\pi^2} \tag{6.7.22}$$

AdS$_6$ and higher: The spectrum of type-B theory in AdS$_6$ can be divided into the $m=0$ sector and the $m=1$ sector.

$$\underbrace{\left([4,0]\bigoplus\bigoplus_{s\geq 1}[3+s,(s,0)]\right)}_{m=0\text{ sector}}\bigoplus\underbrace{\left(\bigoplus_{s\geq 1}[3+s,(s,1)]\right)}_{m=1\text{ sector}} \tag{6.7.23}$$

In the $m=0$ sector we can turn to the result of type-A theory because the only difference is scaling dimension of the scalar field. Using $\log Z_{\mathsf{A}}^{\mathsf{AdS}_6}=0$, we obtain the following result without any extra effort

$$\log Z_{m=0}^{\mathsf{AdS}_6} = \int_0^\infty \frac{du}{2u}\frac{1+e^{-u}}{1-e^{-u}}\frac{e^{-4u}-e^{-3u}}{(1-e^{-u})^5} = -\int_0^\infty \frac{du}{2u}\frac{1+e^{-u}}{1-e^{-u}}\frac{e^{-3u}}{(1-e^{-u})^4} \tag{6.7.24}$$

Unlike in AdS$_4$, the $m=0$ partition function $\log Z_{m=0}^{\mathsf{AdS}_6}$ itself doesn't have the double trace-deformation interpretation because $\Delta=3$ and $\Delta=4$ are not conjugate scaling dimensions. We'll see that the double-trace deformation pattern can be restored with the $m=1$ sector taken into account. The $m=1$ sector is more involving since it consists of fields with mixed symmetry. Following the same steps as in section 6.3, we derive the character integral formula for fields in massive representation $\left[\frac{5}{2}+\nu,(s,1)\right]$

$$\log Z_{(s,1),\nu} = \int_0^\infty \frac{du}{2u}\frac{1+e^{-u}}{1-e^{-u}}\left[D^5_{s,1}\frac{e^{-(\frac{5}{2}+\nu)u}}{(1-e^{-u})^5} - D^7_{s-1}\left(\frac{3\,e^{-(\frac{3}{2}+\nu)u}}{(1-e^{-u})^3}-\frac{e^{-(\frac{1}{2}+\nu)u}}{1-e^{-u}}\right)\right] \tag{6.7.25}$$

Unlike the spin-s case, the edge part of $\log Z_{(s,1),\nu}$ should be interpreted as the 1-loop path integral of D^7_{s-1} massive spin-1 fields of scaling dimension $\frac{d}{2}+\nu$ living on EAdS$_4$, which is the horizon of the Rindler patch of AdS$_6$. Though this new observation [5] of edge modes is intriguing and may help to sharpen the understanding about edge modes, we'll not try to provide a precise

[5] More generally, we find that for a massive field with hook-like spin $(s,1^m)$ and scaling dimension $\Delta=\frac{d}{2}+\nu$ in AdS$_{d+1}$, the edge part of the 1-loop partition function corresponds to D^{d+2}_{s-1} new massive fields with totally antisymmetric spin (1^m) and scaling dimension $\Delta=\frac{d-2}{2}+\nu$ living on EAdS$_{d-1}$.

interpretation for it. Summing over all the fields in the $m = 1$ sector, including the ghosts associated with the $s \geq 2$ ones, we end up with a very simple expression

$$\log Z_{m=1}^{\mathsf{AdS}_6} = \int_0^\infty \frac{du}{2u} \frac{1+e^{-u}}{1-e^{-u}} \frac{e^{-2u}+e^{-3u}}{(1-e^{-u})^4} \tag{6.7.26}$$

Combing (6.7.24) and (6.7.26) leads to the total partition function of type-B theory in AdS_6

$$\log Z_{\mathsf{B}}^{\mathsf{AdS}_6} = \log Z_{m=0}^{\mathsf{AdS}_6} + \log Z_{m=1}^{\mathsf{AdS}_6} = \int_0^\infty \frac{du}{2u} \frac{1+e^{-u}}{1-e^{-u}} \frac{e^{-2u}-e^{-3u}}{(1-e^{-u})^5} \tag{6.7.27}$$

which apparently has the interpretation of double-trace deformation of a conformally coupled scalar field on S^5. In higher dimensions, the partition function of $m = 0$ sector is still trivial. For $m \geq 1$, the partition function restricted to the m sector is given by

$$\log Z_m^{\mathsf{AdS}_{d+1}} = (-)^{m-1} \int_0^\infty \frac{du}{2u} \frac{1+e^{-u}}{1-e^{-u}} \frac{e^{-(d-1-m)u}+e^{-(d-2-m)u}}{(1-e^{-u})^{d-1}} \tag{6.7.28}$$

which we've checked up to AdS_{16} by mathematica. Summing over all $\log Z_m^{\mathsf{AdS}_{d+1}}$, we recover the structure of double-trace deformation of a conformally coupled scalar on S^d up to a sign [113]

$$\log Z_{\mathsf{B}}^{\mathsf{AdS}_{d+1}} = \sum_{m=0}^{\frac{d-3}{2}} \log Z_m^{\mathsf{AdS}_{d+1}} = (-)^{\frac{d-1}{2}} \int_0^\infty \frac{du}{2u} \frac{1+e^{-u}}{1-e^{-u}} \frac{e^{-\frac{d-1}{2}u}-e^{-\frac{d+1}{2}u}}{(1-e^{-u})^d} \tag{6.7.29}$$

More explicitly, when $\frac{d-1}{2}$ is odd, eq. (6.7.29) means that we turn on a double-trace deformation $\mathcal{O}_{\Delta_-}^2, \Delta_- = \frac{d-1}{2}$ which induces an RG flow from the original UV fixed point to a new IR fixed point and when $\frac{d-1}{2}$ is even, it means that we turn on a double-trace deformation $\mathcal{O}_{\Delta_+}^2, \Delta_+ = \frac{d+1}{2}$ which triggers an RG flow from the original IR fixed point to a new UV fixed point. Using eq. (6.6.6) for $\nu = \frac{1}{2}$ and $s = 0$, $\log Z_{\mathsf{B}}^{\mathsf{AdS}_{d+1}}$ is alternatively expressed as

$$\log Z_{\mathsf{B}}^{\mathsf{AdS}_{d+1}} = \frac{\pi}{d!} \int_0^{\frac{1}{2}} d\lambda \prod_{j=0}^{\frac{d-3}{2}} \left[\left(j+\frac{1}{2}\right)^2 - \lambda^2\right] \lambda \tanh(\pi\lambda) \tag{6.7.30}$$

For some lower dimensions, say $d = 3, 5, 7$, eq. (6.7.30) yields

$$\log Z_{\mathsf{B}}^{\mathsf{AdS}_4} = \frac{\zeta(3)}{8\pi^2}, \quad \log Z_{\mathsf{B}}^{\mathsf{AdS}_6} = \frac{\zeta(3)}{96\pi^2} + \frac{\zeta(5)}{32\pi^4}, \quad \log Z_{\mathsf{B}}^{\mathsf{AdS}_8} = \frac{\zeta(3)}{720\pi^2} + \frac{\zeta(5)}{192\pi^4} + \frac{\zeta(7)}{128\pi^6} \tag{6.7.31}$$

consistent with [112]. For completeness, let's also give the explicit result for any even dimensional AdS

$$\log Z_{\mathsf{B}}^{\mathsf{AdS}_{2r+2}} = \sum_{n=1}^{r} \frac{(-)^{n+r}}{2(2\pi)^{2n}} \frac{(2n)!}{(2k)!} a_n(r) \zeta(2n+1) \tag{6.7.32}$$

where $\{a_n(r)\}$ are defined as $\prod_{j=0}^{r-1}(x - j^2) = \sum_{n=1}^{r} a_n(r)\, x^n$. This result contracts with the proposed boundary duality which predicts vanishing one-loop free energy and meanwhile it is too complicated to be accommodated by a shift of N.

6.8 Comments on thermal interpretations

In chapter 5, with the help of character integral representations like, it is argued that the one-loop partition function $Z_{\mathsf{PI}}^{(1)}$ of a field φ on S^{d+1} is related to the bulk quasi-canonical partition function of φ in the static patch of dS$_{d+1}$, subject to possible edge corrections localized on the dS cosmological horizon. In this section, we will explore the generalization to the path integral on EAdS$_{d+1}$.

6.8.1 AdS$_2$

In the 2D Lorentzian AdS, there exist a black hole solution [192] with coordinates, cf. (D.5.5)

$$X^0 = \rho, \quad X^1 = \sqrt{\rho^2 - 1} \cosh t_S, \quad X^2 = \sqrt{\rho^2 - 1} \sinh t_S \tag{6.8.1}$$

and metric $ds^2 = -(\rho^2 - 1) dt_S^2 + \frac{d\rho^2}{\rho^2 - 1}$, which shows a point-like horizon at $\rho = 1$ of temperature $T = \frac{1}{2\pi}$. Wick rotation $t_S \to -i\tau$ and identification $\tau \sim \tau + 2\pi$ yield the 2D Euclidean AdS. Compared to the conformal global coordinate of AdS$_2$

$$X^0 = \frac{\cos t_G}{\cos\theta}, \quad X^1 = \tan\theta, \quad X^2 = \frac{\sin t_G}{\cos\theta} \tag{6.8.2}$$

the black hole solution (6.8.1) covers the region: $\theta \in (0, \frac{\pi}{2})$ and $\sin\theta > |\sin t_G|$, cf. fig (6.8.1).

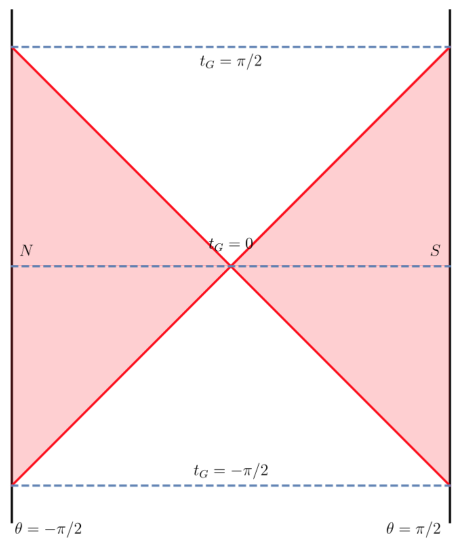


Figure 6.8.1: A portion of the periodic AdS_2 Penrose diagram near $t_G = 0$. The two vertical lines $\theta = \pm\frac{\pi}{2}$ are the boundaries of AdS_2. The black hole solution in eq. (6.8.1) corresponds to the red region denoted by "S". The region "N" is the image of "S" under the map $X^1 \to -X^1$. The red lines represent the bifurcate Killing horizon.

This scenario is very similar to its dS counter part and hence we're allowed to use the dS argument to claim that the thermal partition function of a field φ in the black hole patch of AdS_2 is given by

$$\log \mathrm{Tr}_S\, e^{-2\pi L_{21}} = \int_0^\infty \frac{du}{2\,u}\frac{1+e^{-u}}{1-e^{-u}}\,\Theta_S(u) \tag{6.8.3}$$

where the noncompact Lorentz generator $L_{21} \in \mathfrak{so}(2,1)$ generates time translation $t_S \to t_S +$ const and $\Theta_S(u)$ is the "character" defined with respect to the single-particle Hilbert space in the black hole patch. Using the Bogoliubov transformations [175], $\Theta_S(u)$ can be replaced by the $\mathrm{SO}(2,1)$ Harish-Chandra character $\Theta_\varphi^{\mathsf{HC}}(u) = \mathrm{tr}_G\, e^{-iuL_{21}}$ which is traced over the *global* single-particle Hilbert space:

$$\log \mathrm{Tr}_S\, e^{-2\pi L_{21}} = \int_0^\infty \frac{du}{2\,u} \frac{1+e^{-u}}{1-e^{-u}}\, \Theta_\varphi^{\mathsf{HC}}(u) \tag{6.8.4}$$

At this stage, we want to emphasize that the Harish-Chandra character $\Theta_\varphi^{\mathsf{HC}}(u)$, by definition, is completely different from the characters we've used in the previous sections, like $\Theta_\Delta^{\mathsf{AdS}_2}(u) = \frac{e^{-\Delta}}{1-e^{-u}}$. The latter are defined as $\mathrm{tr}_G\, e^{-uH}$ for *positive* u, where H is the global Hamiltonian generating global time translation $t_G \to t_G + \mathrm{const}$. Indeed, these characters are *not* group characters. So it seems that we cannot naively identify the thermal partition function $\mathrm{Tr}_S\, e^{-2\pi L_{21}}$ as the one-loop path integral on Euclidean AdS$_2$. However, in the appendix D.6, we explicitly compute the Harish-Chandra character $\Theta_\varphi^{\mathsf{HC}}(u)$ when φ is a scalar field of scaling dimension Δ and we find perhaps surprisingly

$$\boxed{\Theta_\varphi^{\mathsf{HC}}(u) = \Theta_\Delta^{\mathsf{AdS}_2}(|u|)} \tag{6.8.5}$$

which yields

$$\log \mathrm{Tr}_S\, e^{-2\pi L_{21}} = \int_0^\infty \frac{du}{2\,u} \frac{1+e^{-u}}{1-e^{-u}} \frac{e^{-\Delta u}}{1-e^{-u}} \tag{6.8.6}$$

in agreement with the path integral result cf. (6.2.5). Therefore the one-loop path integral on EAdS$_2$ can be interpreted as the quasi-canonical partition function $\mathrm{Tr}_S\, e^{-2\pi L_{21}}$ in the black hole patch. To further understand why the Harish-Chandra character $\Theta_\varphi^{\mathsf{HC}}(u)$ appears in the quasi-canonical partition function $\mathrm{Tr}_s\, e^{-2\pi L_{21}}$, we explore the underlying physical meanings of $\Theta_\varphi^{\mathsf{HC}}(u)$ in appendix D.7. In section D.7.1, we show that $\Theta_\varphi^{\mathsf{HC}}(u)$ encodes the quasinormal spectrum of φ in the black hole patch of AdS$_2$ and in section D.7.2, we extract a well-defined single-particle density of states in the black patch from $\Theta_\varphi^{\mathsf{HC}}(u)$ and show numerically that it can be realized as the continuous limit of the density of states in some simple model with a finite dimensional Hilbert space.

6.8.2 Higher dimensions

In higher dimensional AdS$_{d+1}$, the universe perceived by an accelerating observer is called (southern) Rindler-AdS due to the presence of a Rindler horizon [4]. The Rindler-AdS admits a dS$_{d+1}$ foliation, cf. appendix D.5:

$$ds^2 = d\eta^2 + \sinh\eta^2 \left(-(1-r^2)dt_S^2 + \frac{dr^2}{1-r^2} + r^2 d\Omega_{d-2}^2\right) \tag{6.8.7}$$

and hence has temperature $T = \frac{1}{2\pi}$. In the Rindler-AdS patch, we can use the dS type argument to show that the quasi-canonical partition function of φ with spin-s and scaling dimension Δ is

$$\log \mathrm{Tr}_S\, e^{-2\pi L_{d+1,d}} = \int_0^\infty \frac{du}{2\,u}\frac{1+e^{-u}}{1-e^{-u}}\,\Theta_\varphi^{\mathsf{HC}}(u) \tag{6.8.8}$$

where the noncompact Lorentz generator $L_{d+1,d} \in \mathfrak{so}(2,d)$ generates time translation $t_S \to t_S + \text{const}$ and $\Theta_\varphi^{\mathsf{HC}}(u)$ is the Harish-Chandra character $\mathrm{tr}_G\, e^{-iuL_{d+1,d}}$. In the appendix D.6, we argue that $\Theta_\varphi^{\mathsf{HC}}(u) = \Theta_\Delta^{\mathsf{AdS}_{d+1}}(|u|) = \frac{e^{-\Delta|u|}}{(1-e^{-|u|})^d}$ when φ is a scalar field and we believe $\Theta_\varphi^{\mathsf{HC}}(u) = \Theta_{[\Delta,s]}^{\mathsf{AdS}_{d+1}}(|u|)$ should still hold when φ is a spin-s field. Granting this relation, we are left with

$$\boxed{\log \mathrm{Tr}_S\, e^{-2\pi L_{d+1,d}} = \int_0^\infty \frac{du}{2\,u}\frac{1+e^{-u}}{1-e^{-u}}\,\Theta_{[\Delta,s]}^{\mathsf{AdS}_{d+1}}(u)} \tag{6.8.9}$$

When d is odd, (6.8.9) exhibits the the agreement between $\mathrm{Tr}_S\, e^{-2\pi L_{d+1,d}}$ and the bulk part of the one-loop path integral on Euclidean AdS. However, when d is even, (6.8.9) is different from the path integral result (6.5.29), including the volume dependence and the contour choice. We believe that the key of solving this difference is computing the properly IR regulated path integral on Euclidean AdS and understanding the mixing of UV and IR divergences. More explicitly, a functional determinant of an operator $\mathcal{D}$ can be represented as an integral transformation of the corresponding (integrated) heat kernel $K_\mathcal{D}(t) = \int_M d^D x\,\sqrt{g}\,K_\mathcal{D}(t;x,x)$, cf. (6.1.2). If the base manifold M is maximally symmetric and the operator $\mathcal{D}$ also preserves the isometry group of M, then $K_\mathcal{D}(t;x,x)$ is independent of x and hence the integral $\int_M d^D x\,\sqrt{g}$ simply yields the volume of M. When M is a compact manifold like sphere, the factorization $K_\mathcal{D}(t) = \mathrm{Vol}_M\, K_\mathcal{D}(t;x_0,x_0)$

is well-defined but when M is a noncompact manifold like flat space and Euclidean AdS, the naive factorization suffers from an IR divergence $\text{Vol}_M \to \infty$. Such an IR divergence is not a big issue while computing the free energy of ideal gas in flat space if we only care about the leading large-volume behavior, i.e. the extensive part. However, in the AdS case, as we want to extract the R^0 or $\log R$ piece [6], it's apparently more appropriate to introduce a radial cutoff R and impose certain boundary conditions on the cutoff surface. This procedure would spoil the SO$(2, d)$ symmetry and discretize the spectrum of Laplacian operators. The symmetry breaking can lead to considerable technical difficulties in computing the heat kernel $K_{\mathcal{D}}(t; x, x)$. Another approach to this difference is dimensional regularization which works for both the UV and IR divergences, along the line of [214, 229]. But the physical picture is not clear if we implement this formal regularization scheme. We will leave this to future work.

[6] It's very likely to have a R^0 piece even when d is even if we implement the IR regulator properly. Of course, the R^0 piece in this case is ambiguous because it is contaminated by the $\log R$ piece.

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Part I

Appendices

Appendix A: Appendix for chapter 3

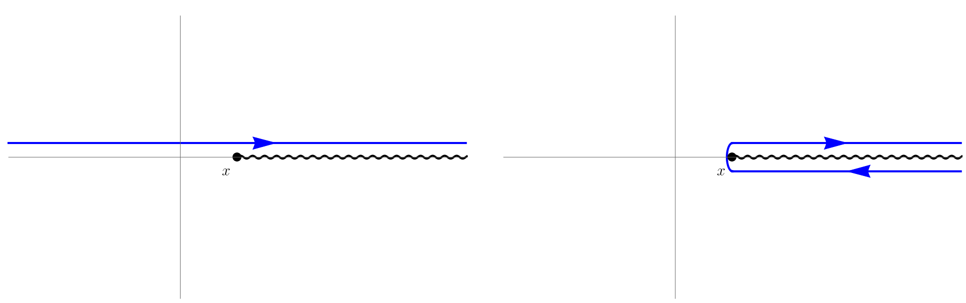
A.1 The explicit action of $\mathfrak{L}_N$ on $\psi_n^{(N)}(x)$

In the section 3.2.2.4, we have defined an operator $\mathfrak{L}_N : \mathcal{F}_N^+ \to \mathcal{F}_{1-N}^+$ such that the composition $\partial_x^{2N-1} \circ \mathfrak{L}_N$ gives the identity operator on $\mathcal{F}_N^+$. In this appendix, we aim to compute the action of $\mathfrak{L}_N$ on the basis $\psi_n^{(N)}(x)$ of $\mathcal{F}_N^+$, c.f. eq.(3.2.41)

$$\left(\mathfrak{L}_N \psi_n^{(N)}\right)(x) = \frac{1}{\Gamma(2N-1)} \int_{\mathbb{R}+i\epsilon} \frac{dy}{2\pi i} (x-y)^{2(N-1)} \log(x-y) \psi_n^{(N)}(y) \tag{A.1.1}$$

As shown in the fig. (A.1.1), we can deform the contour to go around the branch cut and then $\log(x-y)$ is replaced by its discontinuity along the branch cut

$$\left(\mathfrak{L}_N \psi_n^{(N)}\right)(x) = -\frac{1}{\Gamma(2N-1)} \int_x^{\infty} dy \, (y-x)^{2(N-1)} \frac{(1-iy)^{n-N}}{(1+iy)^{n+N}} \tag{A.1.2}$$



(a) Original contour (b) Deformed contour

Figure A.1.1: The contour deformation for the y-integral in eq. (A.1.1). The black wavy line denotes the branch cut of $\log(x-y)$.

Next, we rewrite $(1-iy)^{n-N}$ as polynomials of $1+iy$ and evaluate the y-integral for each monomial by using integration by part repeatedly

$$\begin{aligned}\left(\mathfrak{L}_N\psi_n^{(N)}\right)(x) &= \frac{(-)^{n-N+1}}{\Gamma(2N-1)}\sum_{k=0}^{n-N}(-2)^k\binom{n-N}{k}\int_x^\infty dy\,\frac{(y-x)^{2(N-1)}}{(1+iy)^{2N+k}} \\ &= \frac{i(-)^{n+1}}{1+ix}\sum_{k=0}^{n-N}\binom{n-N}{k}\frac{k!}{(2N+k-1)!}\left(\frac{-2}{1+ix}\right)^k \end{aligned} \tag{A.1.3}$$

To compute the remaining series of k, we can extend the sum to $k=1-2N$ and subtract these extra terms by hand. The sum from $k=1-2N$ to $k=n-N$ is nothing but a binomial expansion

$$\left(\mathfrak{L}_N\psi_n^{(N)}\right)(x) = \frac{(-)^N}{2^{2N-1}i}\frac{\Gamma(n-N+1)}{\Gamma(N+n)}\psi_n^{(1-N)}(x) + \mathsf{Pol}_{N,n}(x) \tag{A.1.4}$$

where $\mathsf{Pol}_{N,n}(x)$ is a polynomial in x annihilated by ∂_x^{2N-1}

$$\mathsf{Pol}_{N,n}(x) = \frac{(-)^{n+1}i}{2}\frac{\Gamma(n-N+1)}{\Gamma(N+n)}\sum_{k=0}^{2(N-1)}\binom{n+N-1}{2(N-1)-k}\left(\frac{1+ix}{-2}\right)^k \tag{A.1.5}$$

As a quick consistency check, combining eq. (3.2.37) and (A.1.4), we obtain the expected property

$$\partial_x^{2N-1}\left(\mathfrak{L}_N\psi_n^{(N)}\right)(x) = \psi_n^{(N)}(x) \tag{A.1.6}$$

A.2 Induced representation

In this appendix, we review the simple idea of induced representation and its applications in the SO$(1,d+1)$ case. Let H be a subgroup of G and let ρ be a representation of H on some vector space V. The **induced representation** $\mathsf{ind}_H^G\rho$ of G operates on the following space of equivariant maps from G to V

$$\mathsf{Map}_H(G,V) \equiv \left\{\Psi : G\to V|\Psi(gh) = \rho(h)^{-1}\Psi(g),\ g\in G, h\in H\right\} \tag{A.2.1}$$

by

$$(g \circ \Psi)(g') \equiv \Psi\left(g^{-1} g'\right) \tag{A.2.2}$$

In the case of $G = \mathsf{SO}(1, d+1)$, we take H to be the minimal parabolic subgroup $\mathsf{S} \equiv \mathsf{NAM}$ and V to be the space $\mathsf{P}_s[z^i]$ of homogeneous polynomials in a null vector z^i of degree s, which is also equivalent to the space of symmetric and traceless tensors of rank s. The representation $\rho = \rho_{\Delta,s}$ of S is chosen to be

$$\rho_{\Delta,s}\left(n e^{\lambda D} m\right) f(z) = e^{-\lambda\Delta} f\left(\rho_1(m)^{-1} z\right) \tag{A.2.3}$$

where $n \in \mathsf{N}$, $m \in \mathsf{M}$ and $\rho_1(m)$ is the fundamental representation of $\mathsf{M} = \mathsf{SO}(d)$, i.e.

$$\rho_1\left(e^{\frac{1}{2}\theta^{ij} M_{ij}}\right) : z^k \to z^k + \theta^{kj} z_j + \mathcal{O}(\Theta^2) \tag{A.2.4}$$

The induced representation $\mathsf{ind}_{\mathsf{S}}^{G} \rho_{\Delta,s}$ then acts on the space of polynomial-valued functions $\Psi(g, z)$ that satisfy

$$\Psi(gs, z) = \rho_{\Delta,s}(s)^{-1} \Psi(g, z), \quad s \in \mathsf{S} \tag{A.2.5}$$

Due to the Bruhat decomposition and the equivariant condition (A.2.5), such a function $\Psi(g, z)$ is completely fixed by its value on $\tilde{\mathsf{N}}$, which is geometrically a flat space $\mathbb{R}^d$.[1] Define $\psi(x, z) \equiv \Psi(e^{x \cdot P}, z)$ and we will show that the (infinitesimal version of) induced representation $\mathsf{ind}_{\mathsf{S}}^{G} \rho_{\Delta,s}$ agrees with the previous construction, c.f. eq.(3.3.8).

- Translations:

$$\left(e^{a \cdot P} \circ \psi\right)(x, z) = \left(e^{a \cdot P} \circ \Psi\right)(e^{x \cdot P}, z) = \Psi(e^{(x-a) \cdot P}, z) = \psi(x - a, z) \tag{A.2.6}$$

  Derivative with respect to a^i yields $\boxed{P_i \psi(x, z) = -\partial_i \psi(x, z)}$

[1] The Bruhat decomposition says that $\mathsf{SO}(1, d+1)$, up to a lower dimensional submanifold, can be written as a product $\tilde{\mathsf{N}}\mathsf{S}$. This lower dimensional manifold corresponds to the *infinity* point of $\mathbb{R}^d$ when we quotient out the S dependence.

- Dilatations:

$$\left(e^{\lambda D}\circ\psi\right)(x,z)=\Psi(e^{-\lambda D}e^{x\cdot P},z) \tag{A.2.7}$$

Using $e^{-\lambda D}P_i e^{\lambda D}=e^{-\lambda}P_i$, we obtain

$$\left(e^{\lambda D}\circ\psi\right)(x,z)=\rho_{\Delta,s}(e^{\lambda D})\psi(e^{-\lambda}x,z)=e^{-\lambda\Delta}\psi(e^{-\lambda}x,z) \tag{A.2.8}$$

which leads to $\boxed{D\psi(x,z)=-(x\cdot\partial_x+\Delta)\psi(x,z)}$.

- Rotations $m=e^{\frac{1}{2}\theta^{ij}M_{ij}}$:

$$m\circ\psi(x,z)=\rho_{\Delta,s}(m)\Psi(m^{-1}e^{x\cdot P}m,z)=\Psi(e^{x_m\cdot P},\rho_1(m)^{-1}z) \tag{A.2.9}$$

where x_m is defined via $m^{-1}e^{x\cdot P}m=e^{x_m\cdot P}$. Infinitesimally, we have

$$x_m^i=x^i-\theta^{ij}x_j+\mathcal{O}(\theta^2),\quad \rho_1(m)^{-1}z^i=z^i-\theta^{ij}z_j+\mathcal{O}(\theta^2) \tag{A.2.10}$$

and hence $\boxed{M_{ij}\psi(x,z)=(x_i\partial_j-x_j\partial_i+z_i\partial_{z^j}-z_j\partial_{z^i})\psi(x,z)}$.

- Special conformal transformations:

$$\left(e^{b\cdot K}\circ\psi\right)(x,z)=\Psi(e^{-b\cdot K}e^{x\cdot P},z) \tag{A.2.11}$$

For our purpose, we can replace $e^{-b\cdot K}$ by $1-b\cdot K$ and then use

$$-e^{-x\cdot P}\,b\cdot K\,e^{x\cdot P}=-b\cdot K-2x\cdot b\,D+2b^ix^jM_{ij}+(x^2b^i-2x\cdot b\,x^i)P_i \tag{A.2.12}$$

which yields

$$\begin{aligned}\left(e^{b\cdot K}\circ\psi\right)(x,z)&=\Psi\left(e^{((1-2x\cdot b)x+x^2b)\cdot P}e^{-b\cdot K}e^{-2x\cdot bD}e^{2b_ix_jM_{ij}},z\right)+\mathcal{O}(b^2)\\&=e^{-2\Delta x\cdot b}\psi((1-2x\cdot b)x+x^2b,2\,x\cdot z\,b-2\,b\cdot z\,x)+\mathcal{O}(b^2)\end{aligned} \tag{A.2.13}$$

Take derivative with respect to b^i and we obtain

$$\boxed{K_i\psi(x,z) = \Big(x^2\partial_i - 2x_i(x\cdot\partial_x + \Delta) + 2(x\cdot z\,\partial_{z^i} - z_i\,x\cdot\partial_z)\Big)\,\psi(x,z)} \tag{A.2.14}$$

A.3 Irreducibility of $\mathcal{F}_\Delta$

In this section we will give an elementary proof about the irreducibility of $\mathcal{F}_\Delta$ for a generic Δ. The argument heavily relies on the result of the section 3.3.1 where we have learned that the SO$(d+1)$ contents of $\mathcal{F}_\Delta$ are all single-row representations. Assuming the existence of a nontrivial SO$(1,d+1)$ subspace $\mathcal{W}_\Delta$ of $\mathcal{F}_\Delta$, it admits a decomposition into SO$(d+1)$ representations

$$\mathcal{W}_\Delta = \bigoplus_{n\in\sigma} \mathbb{Y}_n \tag{A.3.1}$$

where σ is a nontrivial subset of $\mathbb{N}$. We will show that this assumption is invalid because given an arbitrary wavefunction $\psi(x)$ in certain Y_n, by acting the dilatation operators repeatedly, it can get components in any SO$(d+1)$ content of $\mathcal{F}_\Delta$.

Pick any $\psi(x)\in\mathcal{F}_\Delta$. According to the section 3.3.1, it is a function $\hat\psi$ on S^d (in stereographic coordinate) multiplied by a Weyl factor, i.e. $\psi(x) = \left(\frac{2}{1+x^2}\right)^\Delta \hat\psi(x)$. Switching to spherical coordinates $x^i = \omega^i\cot\frac{\theta}{2}, \omega^i\in S^{d-1}$, the action of the dilatation operator on $\hat\psi(\theta,\omega)$ becomes

$$D\,\hat\psi(\theta,\omega) = (\sin\theta\partial_\theta + \Delta\cos\theta)\,\hat\psi(\theta,\omega) \tag{A.3.2}$$

The function $\hat\psi$ can be expanded in terms of spherical harmonics on S^d

$$Y_{n\ell\mathbf{m}}(\theta,\omega) = \mathcal{N}_{n\ell}\,\sin^\ell\theta\,C_{n-\ell}^{\ell+\frac{d-1}{2}}(\cos\theta)Y_{\ell\mathbf{m}}(\omega), \quad n\geq\ell \tag{A.3.3}$$

where $Y_{\ell\mathbf{m}}(\omega)$ denote the normalized spherical harmonics on S^{d-1} and $\mathcal{N}_{n\ell}$ is a constant such that $Y_{n\ell\mathbf{m}}$ is normalized

$$\mathcal{N}_{n\ell} = 2^{\ell-1+\frac{d}{2}}\,\Gamma\left(\ell+\frac{d-1}{2}\right)\sqrt{\frac{(n-\ell)!(n+\frac{d-1}{2})}{\pi\,\Gamma(n+\ell+d-1)}} \tag{A.3.4}$$

For a fixed n, all $\left(\frac{2}{1+x^2}\right)^{\Delta} Y_{n\ell\mathbf{m}}(\theta,\omega)$ span the $\mathbb{Y}_n$ part of $\mathcal{F}_\Delta$. Next, we need to figure out the action of D on all the spherical harmonics. This computation can be done by using the following recurrence relations of Gegenbauer polynomials

$$\begin{aligned}&\partial_x C_n^\lambda(x) = 2\lambda C_{n-1}^{\lambda+1}(x), \quad x\,C_n^\lambda(x) = \frac{n+1}{2(n+\lambda)}C_{n+1}^\lambda(x) + \frac{n-1+2\lambda}{2(n+\lambda)}C_{n-1}^\lambda(x)\\ &(1-x)^2 C_{n-1}^{\lambda+1}(x) = \frac{n+2\lambda}{2\lambda}xC_n^\lambda(x) - \frac{(n+1)}{2\lambda}C_{n+1}^\lambda(x)\end{aligned} \tag{A.3.5}$$

which yield

$$\begin{aligned}DY_{n\ell\mathbf{m}} &= (\Delta+n)\sqrt{\frac{(n+1-\ell)(n+\ell+d-1)}{(2n+d-1)(2n+d+1)}}Y_{n+1,\ell\mathbf{m}}\\ &\quad + (1-\bar{\Delta}-n)\sqrt{\frac{(n-\ell)(n+\ell+d-2)}{(2n+d-3)(2n+d-1)}}Y_{n-1,\ell\mathbf{m}}\end{aligned} \tag{A.3.6}$$

The coefficients of $Y_{n\pm1,\ell\mathbf{m}}$ are nonvanishing except $\Delta \in \{d, d+1, d+2, \cdots\} \cup \{0, -1, -2, \cdots\}$. Therefore, according to the argument at the beginning of this appendix, the representation $\mathcal{F}_\Delta$ is irreducible when Δ is away form these integers. For $\Delta \in \{d, d+1, d+2, \cdots\}$, the subspace with SO$(d+1)$ contents $\bigoplus_{n\geq 1-\bar{\Delta}} \mathbb{Y}_n$ is irreducible with respect to SO$(1,d+1)$ and for $\Delta \in \{0,-1,-2,\cdots\}$, the finite dimensional subspace $\bigoplus_{0\leq n\leq -\Delta} \mathbb{Y}_n$ is irreducible and it carries the spin $(-\Delta)$ representation of SO$(1,d+1)$. These finite dimensional representations are nonunitary except the $\Delta=0$ one which is the trivial representation.

Appendix B: Appendix for chapter 4

B.1 From ambient space to intrinsic coordinate: Maxwell field

In this appendix, we show the agreement between our algebraically constructed primary quasnormal modes and their intrinsic coordinate counterparts in literature for free Maxwell fields. According to [83], the quasinormal modes of Maxwell theory can be divided into the following two types:

$$\text{I}: A_t^{(I)} = 0, \quad A_r^{(I)} = R^{(I)}(r) Y_{\ell\mathbf{m}} e^{-i\omega t}, \quad A_a^{(I)} = \frac{r^{3-d}(1-r^2)}{\ell(\ell+d-2)} \partial_r (r^{d-1} R^{(I)}(r)) \partial_{\vartheta^a} Y_{\ell\mathbf{m}} e^{-i\omega t} \quad \text{(B.1.1)}$$

where ϑ^a are the spherical coordinates on S^{d-1} and $Y_{\ell\mathbf{m}}$ are scalar spherical harmonics on S^{d-1}.

$$\text{II}: A_t^{(II)} = A_r^{(II)} = 0, \quad A_a^{(II)} = R^{(II)}(r) Y_a^{\ell\mathbf{m}} e^{-i\omega t} \quad \text{(B.1.2)}$$

where $Y_a^{\ell\mathbf{m}}$ are divergence-free vector spherical harmonics on S^{d-1}. In type I solutions, the radial function $R^{(I)}(r)$ is given by

$$R^{(I)}(r) = r^{\ell-1}(1-r^2)^{\frac{i\omega}{2}} F\left(\frac{\ell+i\omega+d-2}{2}, \frac{\ell+i\omega+2}{2}, \frac{d}{2}+\ell, r^2\right) \quad \text{(B.1.3)}$$

with the quasinormal frequency ω valued in

$$i\omega_{\ell,n}^I = \ell+d-2+2n, \;\; i\tilde{\omega}_{\ell,n}^I = \ell+2+2n \quad \text{(B.1.4)}$$

In type II solutions, the radial function $R^{(II)}(r)$ is given by

$$R^{(II)}(r) = r^{\ell+1}(1-r^2)^{\frac{i\omega}{2}} F\left(\frac{\ell+i\omega+d-1}{2}, \frac{\ell+i\omega+1}{2}, \frac{d}{2}+\ell, r^2\right) \quad \text{(B.1.5)}$$

with the quasinormal frequency ω valued in

$$i\omega^{II}_{\ell,n} = \ell + d - 1 + 2n, \;\; i\tilde{\omega}^{II}_{\ell,n} = \ell + 1 + 2n \tag{B.1.6}$$

In both type I and II, $\ell \geq 1$ and $n \geq 0$. In the following, we show that the primary quasinormal modes $\alpha_i^{(1)}$ agree with the type I solutions of frequency $i\omega^I_{1,0}$ and $\gamma_{ij}^{(1)}$ agrees with the type II solutions of frequency $i\tilde{\omega}^{II}_{1,0}$.

Match the primary $\alpha_i^{(1)}$-mode

Using eq. (B.1.3), the type I quasinormal modes with $\ell = 1$ and $i\omega = i\omega^I_{1,0} = d-1$ are

$$A_t^{(I)} = 0, \quad A_r^{(I)} = \frac{e^{-(d-1)t}}{(1-r^2)^{\frac{d-1}{2}}} Y_{1\mathbf{m}}(\Omega), \quad A_a^{(I)} = \frac{r\, e^{-(d-1)t}}{(1-r^2)^{\frac{d-1}{2}}} \partial_{\vartheta^a} Y_{1\mathbf{m}}(\Omega) \tag{B.1.7}$$

On the other hand, the pull-back of $\alpha_i^{(1)} = \frac{X^+ u_i - U^+ x_i}{(X^+)^d} R^{d-2}$ yields

$$\begin{aligned}
\alpha_{i,t}^{(1)} &= -\frac{x_i \partial_t X^+}{(X^+)^d} = -\frac{r\,\Omega_i\, e^{-(d-1)t}}{(1-r^2)^{\frac{d-1}{2}}} \\
\alpha_{i,r}^{(1)} &= \frac{X^+ \partial_r x_i - x_i \partial_r X^+}{(X^+)^d} = \frac{\Omega_i\, e^{-(d-1)t}}{(1-r^2)^{\frac{d+1}{2}}} \\
\alpha_{i,a}^{(1)} &= \frac{X^+ \partial_{\vartheta^a} x_i}{(X^+)^d} = \frac{r\,\partial_{\vartheta^a}\Omega_i\, e^{-(d-1)t}}{(1-r^2)^{\frac{d-1}{2}}}
\end{aligned} \tag{B.1.8}$$

Naively, $A_\mu^{(I)}$ and $\alpha_{i,\mu}$ look different. This is because the former is solved in a modified Feynman gauge [230] while the latter follows from boundary-to-bulk propagator in de Donder gauge, which for spin-1 field is simply the Lorenz gauge. To compare the two results, we perform a gauge transformation $\alpha_{i,\mu}^{(1)} \to \tilde{\alpha}_{i,\mu}^{(1)} = \alpha_{i,\mu} + \partial_\mu \xi_i$ to set the t-component zero. The simplest choice of the gauge parameter is $\xi_i = \frac{1}{d-1}\alpha_{i,t}$. Due to this gauge choice, the new $\alpha_i^{(1)}$ modes become

$$\tilde{\alpha}_{i,t}^{(1)} = 0, \quad \tilde{\alpha}_{i,r}^{(1)} = \frac{d-2}{d-1}\frac{\Omega_i\, e^{-(d-1)t}}{(1-r^2)^{\frac{d-1}{2}}}, \quad \tilde{\alpha}_{i,a}^{(1)} = \frac{d-2}{d-1}\frac{r\,\partial_{\vartheta^a}\Omega_i\, e^{-(d-1)t}}{(1-r^2)^{\frac{d-1}{2}}} \tag{B.1.9}$$

Since $\{\Omega_i\}_i$ and $\{Y_{1\mathbf{m}}\}_{\mathbf{m}}$ are just different basis for the same vector space of spherical harmonics of eigenvalue $-(d-1)$ with respect to $\nabla^2_{S^{d-1}}$, eq.(B.1.7) and eq. (B.1.9) actually represent the same set of quasinormal modes.

Match the primary $\gamma_{ij}^{(1)}$-mode

Using eq. (B.1.5), the type II quasinormal modes with $\ell = 1$ and $i\omega = i\tilde{\omega}_{1,0}^{II} = 2$ are

$$A_t^{(II)} = A_r^{(II)} = 0, \quad A_a^{(II)} = \frac{r^2\, e^{-2t}}{1 - r^2} Y_a^{1\mathbf{m}}(\Omega) \tag{B.1.10}$$

On the other hand, the pull-back of $\gamma_{ij}^{(1)} = \frac{x_i u_j - x_j u_i}{(X^+)^2}$ yields

$$\gamma_{ij,t}^{(1)} = \gamma_{ij,r}^{(1)} = 0, \quad \gamma_{ij,a}^{(1)} = \frac{r^2\, e^{-2t}}{1 - r^2} \left(\Omega_i \partial_{\vartheta^a} \Omega_j - \Omega_j \partial_{\vartheta^a} \Omega_i\right) \tag{B.1.11}$$

One can check directly that $\Sigma_{ij,a} \equiv \Omega_i \partial_{\vartheta^a} \Omega_j - \Omega_j \partial_{\vartheta^a} \Omega_i$ are indeed divergence-free vector harmonics of $\ell = 1$. For example, let's consider the $d = 3$ case where the vector harmonics are given by $Y_a^{\ell m} = \frac{1}{\sqrt{\ell(\ell+1)}} \epsilon_{ab} \nabla^b Y_{\ell m}$ [231]:

$$Y_a^{1,0} = \frac{1}{2}\sqrt{\frac{3}{2\pi}}(0, \sin^2\theta), \quad Y_a^{1,\pm 1} = \frac{1}{4}\sqrt{\frac{3}{\pi}} e^{\pm i\varphi}(-i, \pm \sin\theta\cos\theta) \tag{B.1.12}$$

where $\vartheta^a = (\theta, \varphi)$ are the usual spherical coordinates on S^2. Meanwhile, by working out $\Sigma_{ij,a}$ explicitly, we obtain

$$\Sigma_{12,a} = (0, \sin^2\theta), \quad \Sigma_{23,a} \pm i\Sigma_{31,a} = \mp e^{\pm i\varphi}(-i, \pm \sin\theta\cos\theta) \tag{B.1.13}$$

Therefore, $\gamma_{ij,\mu}^{(1)}$ in (B.1.11) and $A_\mu^{(II)}$ in (B.1.10) represent the same quasinormal modes.

B.2 Match quasinormal spectrums

In the section 4.5, we defined a quasinormal character Θ^{QN} for a given quasinormal spectrum $\{\omega, d_\omega\}$, cf. (4.5.1). By definition, the correspondence between quasinormal characters and quasinormal spectrums is one-to-one . In this appendix, by using quasinormal characters, we show that our algebraic construction yields the same quasinormal spectrum as [83] for Maxwell fields and linearized gravity. On the algebraic side, the quasinormal character of a massless spin-s field is shown to be given by eq. (4.5.12). In particular, for $s = 1$ and $s = 2$, the quasinormal characters read

$$\Theta_1^{\mathsf{QN}}(q) = \frac{dq^{d-1} - q^d}{(1-q)^d} + \frac{dq-1}{(1-q)^d} + 1 \tag{B.2.1}$$

and

$$\Theta_2^{\mathsf{QN}}(q) = \frac{D_2^d\, q^d - D_1^d\, q^{d+1}}{(1-q)^d} + \frac{D_2^d - D_1^d\, q^{-1}}{(1-q)^d} + d(q+q^{-1}) + \frac{d^2-d+2}{2} \tag{B.2.2}$$

where $D_1^d = d, \;\; D_2^d = \frac{1}{2}(d+2)(d-1)$.

Maxwell fields

In the previous appendix, we've summarized the quasinormal modes of Maxwell fields computed in [83]. Here let's briefly recap the information about quasinormal frequencies:

$$\begin{aligned} &\text{type I}: \quad i\omega^I_{\ell,n} = \ell + d - 2 + 2n, \;\; i\tilde{\omega}^I_{\ell,n} = \ell + 2 + 2n \\ &\text{type II}: \quad i\omega^{II}_{\ell,n} = \ell + d - 1 + 2n, \;\; i\tilde{\omega}^{II}_{\ell,n} = \ell + 1 + 2n \end{aligned} \tag{B.2.3}$$

where $\ell \geq 1$ and $n \geq 0$. For fixed ℓ and n, each frequency of type I quasinormal modes has degeneracy D^d_ℓ because ℓ labels scalar spherical harmonics while each frequency of type II quasinormal modes has degeneracy $D^d_{\ell 1}$ because ℓ labels divergence-free vector spherical harmonics. So the quasinormal character associated to the spectrum (B.2.3) is

$$\begin{aligned} \Theta_1^{\mathsf{QN},intrin}(q) &\equiv \sum_{\ell=1}^\infty \sum_{n=0}^\infty D^d_\ell \left(q^{i\omega^I_{\ell,n}} + q^{i\tilde{\omega}^I_{\ell,n}}\right) + D^d_{\ell 1}\left(q^{i\omega^{II}_{\ell,n}} + q^{i\tilde{\omega}^{II}_{\ell,n}}\right) \\ &= \frac{q^2+q^{d-2}}{1-q^2} \sum_{\ell\geq 1} D^d_\ell\, q^\ell + \frac{q+q^{d-1}}{1-q^2} \sum_{\ell \geq 1} D^d_{\ell 1}\, q^\ell \end{aligned} \tag{B.2.4}$$

where the first sum over ℓ simply follows from

$$\sum_{\ell \geq 0} D^d_\ell\, q^\ell = \frac{1+q}{(1-q)^{d-1}} \tag{B.2.5}$$

The second sum over ℓ in (B.2.4) can be derived using

$$D^d_{\ell s} = D^d_\ell D^{d-2}_s - D^d_{s-1} D^{d-2}_{\ell+1} \tag{B.2.6}$$

In particular, when $s=1$, $D^d_{\ell 1}=(d-2)D^d_\ell - D^{d-2}_{\ell+1}$ and hence the second sum reduces to the (B.2.5) type:

$$\sum_{\ell\geq 1} D^d_{\ell 1}\, q^\ell = (d-2)\frac{1+q}{(1-q)^{d-1}} - \left(\frac{1+q^{-1}}{(1-q)^{d-3}} - q^{-1}\right) \tag{B.2.7}$$

Plugging (B.2.5) and (B.2.7) into the quasinormal character (B.2.4) yields

$$\Theta_1^{\mathsf{QN},intrin}(q) = \frac{dq^{d-1}-q^d}{(1-q)^d} + \frac{dq-1}{(1-q)^d} + 1 = \Theta_1^{\mathsf{QN}}(q) \tag{B.2.8}$$

which shows the agreement between our algebraic method and the traditional analytical method on the quasinormal spectrum for Maxwell theory.

Linearized gravity

Quasinormal modes of linearized gravity are divided into three categories. The three types of fluctuation can be solved simultaneously by using the so-called Ishibashi-Kodama equation [77, 83, 232]. The quasinormal frequencies are:

$$\begin{aligned}
&\text{Scalar type fluctuation}: i\omega^S_{\ell,n} = \ell+d-2+2n,\ \ i\tilde{\omega}^S_{\ell,n} = \ell+2+2n\\
&\text{Vector type fluctuation}: i\omega^V_{\ell,n} = \ell+d-1+2n,\ \ i\tilde{\omega}^V_{\ell,n} = \ell+1+2n\\
&\text{Tensor type fluctuation}: i\omega^T_{\ell,n} = \ell+d+2n,\ \ i\tilde{\omega}^T_{\ell,n} = \ell+2n
\end{aligned} \tag{B.2.9}$$

where $\ell\geq 2$ and $n\geq 0$. In these 3 types of fluctuations, ℓ labels scalar spherical harmonics, divergence-free vector spherical harmonics and divergence-free tensor spherical harmonics on S^{d-1} respectively and hence for fixed ℓ and n, each frequency has degeneracy $D^d_\ell, D^d_{\ell 1}$ and $D^d_{\ell 2}$ respectively. Altogether, the quasinormal character associated to the spectrum (B.2.9) is given by

$$\begin{aligned}
\Theta_2^{\mathsf{QN},intrin}(q) &\equiv \sum_{\ell\geq 2, n\geq 0} D^d_\ell\,(q^{i\omega^S_{\ell n}}+q^{i\tilde{\omega}^S_{\ell n}}) + D^d_{\ell 1}(q^{i\omega^V_{\ell n}}+q^{i\tilde{\omega}^V_{\ell n}}) + D^d_{\ell 2}(q^{i\omega^T_{\ell n}}+q^{i\tilde{\omega}^T_{\ell n}})\\
&= \frac{q^2+q^{d-2}}{1-q^2}\sum_{\ell\geq 2} D^d_\ell\, q^\ell + \frac{q+q^{d-1}}{1-q^2}\sum_{\ell\geq 2} D^d_{\ell 1}\, q^\ell + \frac{1+q^d}{1-q^2}\sum_{\ell\geq 2} D^d_{\ell 2}\, q^\ell
\end{aligned} \tag{B.2.10}$$

where the first two series of ℓ are essentially computed in the Maxwell field case and the last series follows from eq.(B.2.6) with $s = 2$:

$$\sum_{\ell \geq 2} D^d_{\ell 2}\, q^\ell = D^d_2 \frac{1+q}{(1-q)^{d-1}} - d\left(\frac{1+q^{-1}}{(1-q)^{d-3}} - q^{-1}\right) + \frac{d(d-1)}{2} \tag{B.2.11}$$

Combine the three series of ℓ in (B.2.10) and we obtain

$$\Theta_2^{\mathrm{QN},intrin}(q) = \frac{D^d_2\, q^d - D^d_1\, q^{d+1}}{(1-q)^d} + \frac{D^d_2 - D^d_1\, q^{-1}}{(1-q)^d} + d(q+q^{-1}) + \frac{d^2-d+2}{2} \tag{B.2.12}$$

which is exactly $\Theta_2^{\mathrm{QN}}(q)$. This computation confirms the match of quasinormal spectrum for linearized gravity. [1]

B.3 Details of $\gamma^{(s)}_{i_1\ell_1,\cdots,i_s\ell_s}$

The higher spin quasinormal mode $\gamma^{(s)}_{i_1\ell_1,\cdots,i_s\ell_s}$ defined by eq. (4.3.34) is rather schematic. In this appendix, we will write out its explicit form and then show various properties of it. Let's start from recollecting the definitions

$$\gamma^{(s)}_{i_1\ell_1,\cdots,i_s\ell_s}(X,U) = \Pi_{ss} P_{i_1}\cdots P_{i_s}(-\log(X^+)\beta^{(s)}_{\ell_1\cdots\ell_s}) - \text{trace} \tag{B.3.1}$$

$$\beta^{(s)}_{\ell_1\cdots\ell_s}(X,U) = \frac{1}{(X^+)^2}(X^+u_{\ell_1} - U^+x_{\ell_1})\cdots(X^+u_{\ell_s} - U^+x_{\ell_s}) - \text{trace} \tag{B.3.2}$$

$$P_i = -X^-\partial_{x^i} - 2\,x_i\,\partial_{X^+} - U^-\partial_{u^i} - 2\,u_i\,\partial_{U^+} \tag{B.3.3}$$

where the projection operator Π_{ss} antisymmetrizes $[i_1,\ell_1],\cdots,[i_s,\ell_s]$. Notice that $X^-\partial_{x^i}$ and $U^-\partial_{u^i}$ would introduce terms proportional $\delta_{i_j i_k}$ and $\delta_{i_j\ell_k}$. The former is killed by Π_{ss} and the latter as a pure trace term also drops out in $\gamma^{(s)}_{i_1\ell_1,\cdots,i_s\ell_s}$. Therefore only $2x_i\partial_{X^+}$ and $2u_i\partial_{U^+}$ can have nonvanishing contributions to $\gamma^{(s)}_{i_1\ell_1,\cdots,i_s\ell_s}$ and $\gamma^{(s)}_{i_1\ell_1,\cdots,i_s\ell_s}$ is independent of X^- and

[1] When $d = 3$, the tensor type fluctuations doesn't exist because there is no divergence-free tensor harmonics on S^2. However, one can still recover the quasinormal character $\Theta_2^{\mathrm{QN}}(q)$ for $d = 3$ by counting quasinormal modes in the scalar and vector type fluctuations in this case.

U^-. As a result, $\gamma^{(s)}_{i_1\ell_1,\cdots,i_s\ell_s}$ has to be of the following form

$$\gamma^{(s)}_{i_1\ell_1,\cdots,i_s\ell_s}(X,U) = \chi^{(s)}_{i_1\ell_1,\cdots,i_s\ell_s}(x,u) f(X^+,U^+) \tag{B.3.4}$$

$$\chi^{(s)}_{i_1\ell_1,\cdots,i_s\ell_s}(x,u) \equiv (x_{i_1}u_{\ell_1} - u_{i_1}x_{\ell_1})\cdots(x_{i_s}u_{\ell_s} - u_{i_s}x_{\ell_s}) - \text{trace} \tag{B.3.5}$$

where $f(X^+,U^+)$ is an unknown function to be fixed. With x_i, u_ℓ being SO(d) vectors, the tensor $\chi^{(s)}_{i_1\ell_1,\cdots,i_s\ell_s}(x,u)$ carries the $\mathbb{Y}_{ss}$ representation of SO(d). In particular, it satisfies the following set of equations which can be thought as the SO(d) analogue of uplift conditions and Pauli-Fierz conditions:

$$\begin{gathered}\partial_x^2\,\chi^{(s)}(x,u) = \partial_u^2\,\chi^{(s)}(x,u) = \partial_x\cdot\partial_u\,\chi^{(s)}(x,u) = 0\\ x\cdot\partial_u\,\chi^{(s)}(x,u) = u\cdot\partial_x\,\chi(s)(x,u) = (x\cdot\partial_x - s)\,\chi^{(s)}(x,u) = (u\cdot\partial_u - s)\,\chi^{(s)}(x,u) = 0\end{gathered} \tag{B.3.6}$$

where the subscripts of $\chi^{(s)}_{i_1\ell_1,\cdots,i_s\ell_s}$ are suppressed.

Using the definition (B.3.1), it's easy to check that $\gamma^{(s)}_{i_1\ell_1,\cdots,i_s\ell_s}$ satisfies the same conditions (4.2.6)—(4.2.10) as $\beta^{(s)}_{\ell_1\cdots\ell_s}$. In particular, the homogeneity condition and the tangentiality condition yields

$$X^+\partial_{U^+}f(X^+,U^+) = 0, \quad X^+\partial_{X^+}f(X^+,U^+) = -2f(X^+,U^+) \tag{B.3.7}$$

which have solution $f(X^+,U^+) = \frac{1}{(X^+)^2}$. Therefore, up to an unimportant normalization factor c_s, the explicit form of $\gamma^{(s)}_{i_1\ell_1,\cdots i_s\ell_s}$ is

$$\boxed{\gamma^{(s)}_{i_1\ell_1,\cdots,i_s\ell_s} = c_s\frac{\chi^{(s)}_{i_1\ell_1,\cdots,i_s\ell_s}(x,u)}{(X^+)^2} = c_s\frac{(x_{i_1}u_{\ell_1} - u_{i_1}x_{\ell_1})\cdots(x_{i_s}u_{\ell_s} - u_{i_s}x_{\ell_s}) - \text{trace}}{(X^+)^2}} \tag{B.3.8}$$

For example, for a Maxwell filed, (B.3.8) is consistent with (4.3.19). More generally, all the

γ-primaries that are generated by SO(d) action on (B.3.8) can be collectively expressed as

$$\gamma^{(s)}(X,U) = \frac{T_{ss}(x,u)}{(X^+)^2} \tag{B.3.9}$$

where $T_{ss}(x,u)$ is a polynomial in x^i, u^i carrying the $\mathbb{Y}_{ss}$ representation of SO(d) [2], i.e. it satisfies

$$T_{ss}(ax, bu) = (a\,b)^s T_{ss}(x,u), \quad u\cdot\partial_x T_{ss}(x,u) = \partial_u^2 T_{ss}(x,u) = 0 \tag{B.3.10}$$

All the linearly independent choices of T_{ss} correspond to the degeneracy of $\gamma^{(s)}$.

Near horizon $\gamma^{(s)}_{i_1\ell_1,\cdots,i_s\ell_s}$ becomes singular because $X^+ \to 0$. This singular behavior also shows the following in-going boundary condition:

$$\boxed{\gamma^{(s)}_{i_1\ell_1,\cdots,i_s\ell_s}(\rho\to\infty) \sim \frac{1}{(X^+)^2} \sim e^{-2(T-\rho)}} \tag{B.3.11}$$

where we can directly read off the quasinormal frequency $i\omega = 2$.

The next task is to show that $\gamma^{(s)}_{i_1\ell_1,\cdots,i_s\ell_s}$ is primary up to gauge transformation by using its explicit form (B.3.8) (we can also use (B.3.9) with $T_{ss}(x,u)$ subject to (B.3.10)). Acting the special conformal transformation K_m on $\gamma^{(s)}_{i_1\ell_1,\cdots,i_s\ell_s}$, we get (dropping the normalization constant c_s)

$$\begin{aligned} K_m\,\gamma^{(s)}_{i_1\ell_1,\cdots,i_s\ell_s} &= \frac{1}{X^+}\partial_{x^m}\chi^{(s)}_{i_1\ell_1,\cdots,i_s\ell_s} + \frac{U^+}{(X^+)^2}\partial_{u^m}\chi^{(s)}_{i_1\ell_1,\cdots,i_s\ell_s} \\ &= U\cdot\partial_X\left(-\frac{\partial_{u^m}\chi^{(s)}_{i_1\ell_1,\cdots,i_s\ell_s}}{X^+}\right) + \frac{1}{X^+}\left(u\cdot\partial_x\partial_{u^m} + \partial_{x^m}\right)\chi^{(s)}_{i_1\ell_1,\cdots,i_s\ell_s} \end{aligned} \tag{B.3.12}$$

where we've replaced $U\cdot\partial_X$ by $u\cdot\partial_x$ in the second term of the second line because $\chi^{(s)}_{i_1\ell_1,\cdots i_s\ell_s}$ only depends on x^i, u^i. In addition, noticing that $u\cdot\partial_x$ kills $\chi^{(s)}_{i_1\ell_1,\cdots i_s\ell_s}$, the product of operators $u\cdot\partial_x\partial_{u^m}$ can be replaced by the corresponding commutator $[u\cdot\partial_x, \partial_{u^m}] = -\partial_{x^m}$ which cancels the other derivative with respect to x^m. Altogether, $\gamma^{(s)}_{i_1\ell_1,\cdots i_s\ell_s}$ is a primary quasinormal mode

[2] Taking eq. (B.3.9) as an ansatz of quasinormal modes (which satisfy the in-going boundary condition automatically as we will see below), then it's easy to show that uplift conditions and Pauli-Fierz conditions are equivalent to $T_{ss}(x,u)$ carrying the $\mathbb{Y}_{ss}$ representation of SO(d).

in the sense that $K_m\gamma^{(s)}_{i_1\ell_1,\cdots i_s\ell_s}$ can be removed by a gauge transformation

$$\boxed{K_m\,\gamma^{(s)}_{i_1\ell_1,\cdots,i_s\ell_s}(X,U) = U\cdot\partial_X\left(-c_s\frac{\partial_{u^m}\chi^{(s)}_{i_1\ell_1,\cdots,i_s\ell_s}(x,u)}{X^+}\right)} \tag{B.3.13}$$

where we've restored the normalization constant c_s.

Appendix C: Appendix for chapter 5

C.1 Density of states and quasinormal mode resonances

The review in the section 3.4 focuses mostly on mathematical and computational aspects of the Harish-Chandra character $\Theta(t) = \mathrm{tr}\, e^{-itH}$. Here we focus on its physics interpretation, in particular the density of states $\rho(\omega)$ obtained as its Fourier transform. We define this in a general and manifestly covariant way using Pauli-Villars regularization in section 5.2. Here we will not be particularly concerned with general definitions or manifest covariance, taking a more pedestrian approach. At the end we briefly comment on an "S-matrix" interpretation and a possible generalization of the formalism including interactions.

In C.1.1, we contrast the spectral features encoded in the characters of unitary representations of the $\mathfrak{so}(1, d+1)$ isometry algebra of global dS$_{d+1}$ with the perhaps more familiar characters of unitary representations of the $\mathfrak{so}(2, d)$ isometry algebra of AdS$_{d+1}$: in a sentence, the latter encodes bound states, while the former encodes scattering states. In C.1.2 we explicitly compare $\rho(\omega)$ obtained as the Fourier transform of $\Theta(t)$ for dS$_2$ to the coarse-grained eigenvalue density obtained by numerical diagonalization of a model discretized by global angular momentum truncation, and confirm the results match at large N. In C.1.3 we identify the poles of $\rho(\omega)$ in the complex ω plane as scattering resonances/quasinormal modes, counted by the power series expansion of the character. As a corollary this implies the relation $Z_{\mathrm{PI}} = Z_{\mathrm{bulk}}$ of (5.3.4) can be viewed as a precise version of the formal quasinormal mode expansion of $\log Z_{\mathrm{PI}}$ proposed in [163].

C.1.1 Characters and the density of states: dS vs AdS

We begin by highlight some important differences in the spectrum encoded in the characters of unitary $\mathfrak{so}(1, d+1)$ representations furnished by global dS$_{d+1}$ single-particle Hilbert spaces and the characters of unitary $\mathfrak{so}(2,d)$ representations furnished by global AdS$_{d+1}$ single-particle Hilbert spaces. Although the discussion applies to arbitrary representations, for concreteness we consider the example of a scalar of mass $m^2 = (\frac{d}{2})^2 + \nu^2$ on dS$_{d+1}$. Its character as computed in (3.4.7) is

$$\Theta_{\rm dS}(t) \equiv {\rm tr}\, e^{-itH} = \frac{e^{-\Delta_+ t} + e^{-\Delta_- t}}{|1-e^{-t}|^d} \,, \qquad \Delta_\pm = \tfrac{d}{2} \pm i\nu \,, \qquad t \in \mathbb{R}. \tag{C.1.1}$$

where ${\rm tr}$ traces over the *global* single-particle Hilbert space and we recall $H = M_{0,d+1}$ is a global SO$(1,1)$ boost generator, which becomes a spatial momentum operator in the future wedge and the energy operator in the southern static patch (cf. fig. C.3.1c). This is to be contrasted with the familiar character of the unitary lowest-weight representation of a scalar of mass $m^2 = -(\frac{d}{2})^2 + \mu^2$ on global AdS$_{d+1}$ with standard boundary conditions:

$$\Theta_{\rm AdS}(t) \equiv {\rm tr}\, e^{-itH} = \frac{e^{-i\Delta_+ t}}{(1-e^{-it})^d} \,, \qquad \Delta_+ = \tfrac{d}{2} + \mu \,, \qquad {\rm Im}\, t < 0 \,. \tag{C.1.2}$$

Here the $\mathfrak{so}(2)$ generator H is the energy operator in global AdS$_{d+1}$. Besides the occurrence of both $\Delta_\pm$ in (C.1.1), another notable difference is the absence of factors of i in the exponents.

The physics content of $\Theta_{\rm AdS}$ is clear: $\Theta_{\rm AdS}(-i\beta) = {\rm tr}\, e^{-\beta H}$ is the single-particle partition function at inverse temperature β for a scalar particle trapped in the global AdS gravitational potential well. Equivalently for ${\rm Im}\, t < 0$, the expansion

$$\Theta_{\rm AdS}(t) = \sum_\lambda N_\lambda \, e^{-it\lambda} \,, \qquad \lambda = \Delta_+ + n \,, \quad n \in \mathbb{N} \,, \tag{C.1.3}$$

counts normalizable single-particle states of energy $H = \lambda$, or equivalently global normal modes of frequency λ. The corresponding density of single-particle states is

$$\rho_{\rm AdS}(\omega) = \int_{-\infty}^{\infty} \frac{dt}{2\pi} \, \Theta_{\rm AdS}(t) \, e^{i\omega t} = \sum_\lambda N_\lambda \, \delta(\omega - \lambda) \,. \tag{C.1.4}$$

For dS, we can likewise expand the character as in (C.1.3). For $t > 0$,

$$\Theta_{\rm dS}(t) = \sum_\lambda N_\lambda \, e^{-it\lambda} \,, \qquad \lambda = -i(\Delta_\pm + n) = -i(\tfrac{d}{2} + n) \pm \nu \,, \quad n \in \mathbb{N} \,. \tag{C.1.5}$$

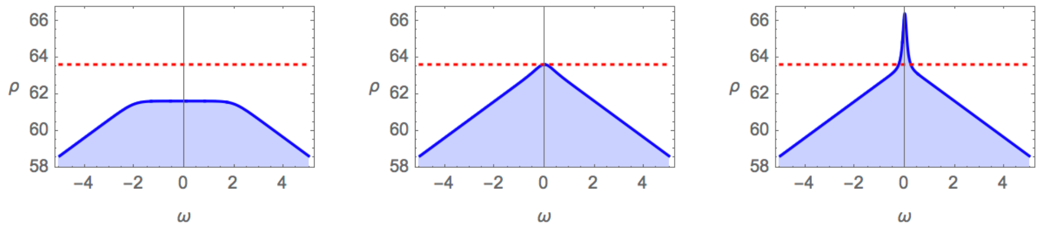


Figure C.1.1: Density of states $\rho_\Lambda(\omega)$ for dS_3 scalars with $\Delta = 1+2i$, $\Delta = \frac{1}{2}$, $\Delta = \frac{1}{10}$, and UV cutoff $\Lambda = 100$, according to (C.1.7). The red dotted line represents the term $2\Lambda/\pi$. The peak visible at $\Delta = \frac{1}{10}$ is due to a resonance approaching the real axis, as explained in section C.1.3.

However λ is now *complex*, so evidently N_λ does not count physical eigenstates of the hermitian operator H. Rather, as further discussed in section C.1.3, it counts *resonances*, or *quasinormal* modes. The density of *physical* states with $H = \omega \in \mathbb{R}$ is formally given by

$$\rho_{\mathrm{dS}}(\omega) = \int_{-\infty}^{\infty} \frac{dt}{2\pi}\, \Theta_{\mathrm{dS}}(t)\, e^{i\omega t} = \int_0^\infty \frac{dt}{2\pi}\, \Theta_{\mathrm{dS}}(t) \big(e^{i\omega t} + e^{-i\omega t}\big)\,, \tag{C.1.6}$$

where ω can be interpreted as the momentum along the T-direction of the future wedge (F in fig. C.3.1 and table C.3.5). Alternatively for $\omega > 0$ it can be interpreted as the energy in the southern static patch, as discussed in section 5.2.2. A manifestly covariant Pauli-Villars regularization of the above integral is given by (5.2.16). For our purposes here a simple $t > \Lambda^{-1}$ cutoff suffices. For example for dS_3,

$$\begin{aligned} \rho_{\mathrm{dS}_3,\Lambda}(\omega) &\equiv \int_{\Lambda^{-1}}^{\infty} \frac{dt}{2\pi}\, \frac{e^{-(1+i\nu)t} + e^{-(1-i\nu)t}}{(1-e^{-t})^2} \big(e^{i\omega t} + e^{-i\omega t}\big) \\ &= \frac{2\Lambda}{\pi} - \frac{1}{2} \sum_{\pm} (\omega \pm \nu) \coth\!\big(\pi(\omega \pm \nu)\big)\,. \end{aligned} \tag{C.1.7}$$

Some examples are illustrated in fig. C.1.1. In contrast to AdS, $\rho_{\mathrm{dS}}(\omega)$ is continuous. Indeed energy eigenkets $|\omega\sigma\rangle$ of the static patch form a continuum of scattering states, coming out of and going into the horizon, instead of the discrete set of bound states one gets in the global AdS potential well. Note that although the above $\rho_{\mathrm{dS}_3,\Lambda}(\omega)$ formally goes negative in the large-ω limit, it is positive within its regime of validity, that is to say for $\omega, \nu \ll \Lambda$.

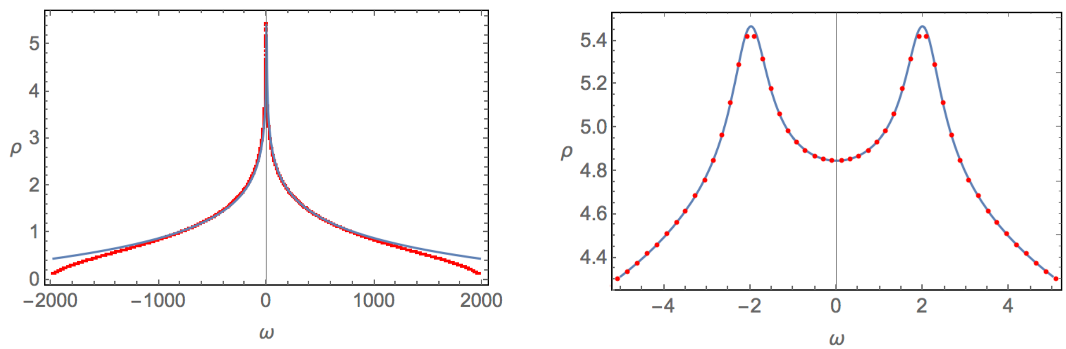


Figure C.1.2: Density of states for a $\Delta = \frac{1}{2} + i\nu$ scalar with $\nu = 2$ in dS_2. The red dots show the local eigenvalue density $\bar{\rho}_N(\omega)$, (C.1.10), of the truncated model with global angular momentum cutoff $N = 2000$, obtained by numerical diagonalization. The blue line shows $\rho(\omega)$ obtained as the Fourier transform of $\Theta(t)$, explicitly (C.1.8) with $e^{-\gamma}\Lambda \approx 4000$. The plot on the right zooms in on the IR region. The peaks are due to the proximity of quasinormal mode poles in $\rho(\omega)$, discussed in C.1.3.

C.1.2 Coarse-grained density of states in globally truncated model

For a $\Delta = \frac{1}{2} + i\nu$ scalar on dS_2, the density of states regularized by as in (C.1.7) is

$$\rho(\omega) = \frac{2}{\pi}\log(e^{-\gamma}\Lambda) - \frac{1}{2\pi}\sum_{\pm,\pm}\psi\big(\tfrac{1}{2} \pm i\nu \pm i\omega)\big)\,, \tag{C.1.8}$$

where γ is the Euler constant, $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function, and the sum is over the four different combinations of signs. To ascertain it makes physical sense to identify this as the density of states, we would like to compare this to a model with discretized spectrum of eigenvalues ω.

An efficient discretization — which does not require solving bulk equations of motion and is quite natural from the point of view of dS-CFT approaches to de Sitter quantum gravity [38, 39, 203] — is obtained by truncating the *global* dS_{d+1} angular momentum $SO(d+1)$ of the single-particle Hilbert space, considering instead of H a *finite*-dimensional matrix

$$h_{\bar{\sigma}\bar{\sigma}'} \equiv \langle\bar{\sigma}|H|\bar{\sigma}'\rangle\,, \tag{C.1.9}$$

where $\bar{\sigma}$ are $SO(d+1)$ quantum numbers. For dS_2 this is $SO(2)$ and $\bar{\sigma} = n \in \mathbb{Z}$, truncated e.g. by $|n| \le N$. The matrix h is sparse and can be computed either directly using $|n\rangle \propto \int d\varphi\, e^{in\varphi}|\varphi\rangle$ and the explicit form of H, i.e. $H = i(\sin\varphi\partial_\varphi + \Delta\cos\varphi)$, or algebraically.

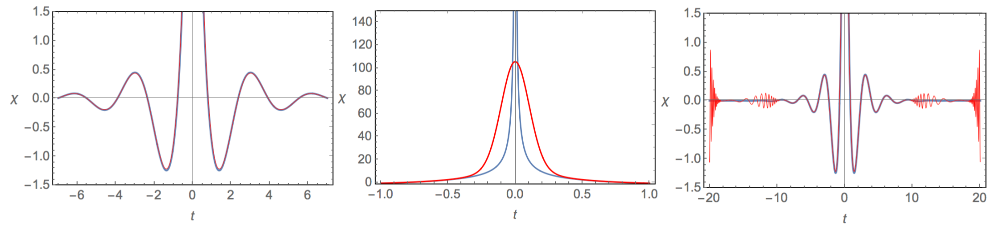


Figure C.1.3: Comparison of $d = 1$ character $\Theta(t)$ defined in (C.1.1) (blue) to the coarse-grained discretized character $\bar{\Theta}_{N,\delta}(t)$ defined in (C.1.11) (red), with $\delta = 0.1$ and other parameters as in fig. C.1.2. Plot on the right shows wider range of t. Plot in the middle smaller range of t, but larger Θ.

The algebraic way goes as follows. A normalizable basis $|n\rangle$ of the global dS$_2$ scalar single-particle Hilbert space can be constructed from the SO$(1,2)$ conformal algebra (3.2.1)[1], using a basis of generators L_0, $L_\pm$ related to H, K and P as $L_0 = \frac{1}{2}(P+K)$, $L_\pm = \frac{1}{2}(P-K) \pm iH$. Then L_0 is the global angular momentum generator $i\partial_\phi$ along the future boundary S^1 and $L_\pm$ are its raising and lowering operators. In some suitable normalization of the L_0 eigenstates $|n\rangle$, we have $L_0|n\rangle = n|n\rangle$, $L_\pm|n\rangle = (n \pm \Delta)|n \pm 1\rangle$. Cutting off the single-particle Hilbert space at $-N < n \leq N$,[2] the operator $H = \frac{i}{2}(L_- - L_+)$ acts as a sparse $2N \times 2N$ matrix on the truncated basis $|n\rangle$.

A minimally coarse-grained density of states can then be defined as the inverse spacing of its eigenvalues ω_i, $i = 1, \ldots, 2N$, obtained by numerical diagonalization:

$$\bar{\rho}_N(\omega_i) \equiv \frac{2}{\omega_{i+1} - \omega_{i-1}}. \tag{C.1.10}$$

The continuum limit corresponds to $N \to \infty$ in the discretized model, and to $\Lambda \to \infty$ in (C.1.8). To compare to (C.1.8), we adjust Λ, in the spirit of renormalization, to match the density of states at some scale ω, say $\omega = 0$. The results of this comparison for $\nu = 2$, $N = 2000$ are shown in fig. C.1.2. Clearly they match remarkably well indeed in the regime where they should, i.e. well below the UV cutoff scale.

[1] Notice that the definitions of P, K, H differ by an overall factor i from the convention in chapter 3.

[2] The asymmetric choice here allows us to use the simple coarse graining prescription (C.1.10) and keep this discussion short. A symmetric choice $|n| \leq N$ would lead to an enhanced $\mathbb{Z}_2$ and two families of eigenvalues distinguished by their $\mathbb{Z}_2$ parity, inducing persistent microstructure in the level spacing. The most efficient way to proceed then is to compute $\bar{\rho}_{N,\pm}(\omega)$ as the inverse level spacing for these two families separately and then add the contributions together as interpolated functions. For dS$_3$ with SO(3) cutoff $\ell \leq N$ one similarly gets $2N+1$ families of eigenvalues, labeled by SO(2) angular momentum m, and one can proceed analogously. Alternatively, one can compute $\bar{\rho}_N(\omega)$ directly by binning and counting, but this requires larger N.

We can make a similar comparison directly at the (UV-finite) character level. The discrete character is $\sum_i e^{-i\omega_i t}$, which is a wildly oscillating function. At first sight this seems very different from the character $\Theta(t) = \operatorname{tr} e^{-iHt}$ in (C.1.6). However to properly compare the two, we should coarse grain this at a small but finite resolution δ. We do this by convolution with a Gaussian kernel, that is to say we consider

$$\bar{\Theta}_{N,\delta}(t) \equiv \frac{1}{\sqrt{2\pi}\delta}\int_{-\infty}^{\infty} dt'\, e^{-(t-t')^2/2\delta^2} \sum_i e^{-i\omega_i t'} = \sum_i e^{-it\omega_i - \delta^2\omega_i^2/2} . \tag{C.1.11}$$

A comparison of $\bar{\Theta}_{N,\delta}$ to Θ is shown in fig. C.1.3 for $\delta = 0.1$. The match is nearly perfect for $|t|$ not too large and not too small. For small t, the $\bar{\Theta}_{N,\delta}(t)$ caps off at a finite value, the number of eigenvalues $|\omega_i| \lesssim 1/\delta$, while $\Theta(t) \sim 1/|t| \to \infty$. The approximation gets better here when δ is made smaller. For larger values of t, $\bar{\Theta}_{N,\delta}(t)$ starts showing some oscillations again. These can be eliminated by increasing δ, at the cost of accuracy at smaller t. In the $N \to \infty$ limit, the discretized approximation gets increasingly better over increasingly large intervals of t, with $\lim_{\delta\to 0}\lim_{N\to\infty} \bar{\Theta}_{N,\delta}(t) = \Theta(t)$.

Note that there is no reason to expect *any* discretization scheme will converge to $\Theta(t)$ or $\rho(\omega)$. For example it is not clear a brick wall discretization along the lines described in section C.4.3 would. On the other hand, the convergence of the above global angular momentum cutoff scheme to the continuum $\Theta(t)$ was perhaps to be expected.

C.1.3 Resonances and quasinormal mode expansion

Substituting the expansion (C.1.5) of the dS character,

$$\Theta(t) = \sum_\lambda N_\lambda\, e^{-it\lambda} \qquad (t > 0)\,, \tag{C.1.12}$$

into (C.1.6), $\rho(\omega) = \frac{1}{2\pi}\int_0^\infty dt\, \Theta(t)\,(e^{i\omega t} + e^{-i\omega t})$, we can formally express the density of states as

$$\rho(\omega) = \frac{1}{2\pi i}\sum_\lambda N_\lambda \Big(\frac{1}{\lambda - \omega} + \frac{1}{\lambda + \omega}\Big)\,, \tag{C.1.13}$$

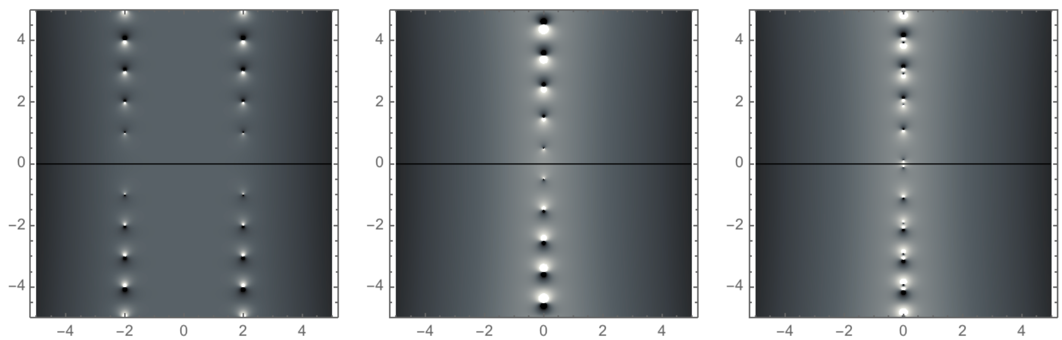


Figure C.1.4: Plot of $|\rho(\omega)|$ in complex ω-plane corresponding to the dS$_3$ examples of fig. C.1.1, that is $\Delta_\pm = \{1+2i, 1-2i\}$, $\{\frac{1}{2}, \frac{3}{2}\}$, $\{0.1, 1.9\}$, and $2\Lambda/\pi \approx 64$. Lighter is larger with plot range 58 (black) $< |\rho| < 67$ (white). Resonance poles are visible at $\omega = \mp i(\Delta_\pm + n)$, $n \in \mathbb{N}$.

From this we read off that $\rho(\omega)$ analytically continued to the complex plane has poles at $\omega = \pm\lambda$ which for massive representations means $\omega = \mp i(\Delta_\pm + n)$. This can also be checked from explicit expressions such as the dS$_3$ scalar density of states (C.1.7), illustrated in fig. C.1.4. These values of ω are precisely the frequencies of the (anti-)quasinormal field modes in the static patch, that is to say modes with purely ingoing/outgoing boundary conditions at the horizon, regular in the interior. If we think of the normal modes as scattering states, the quasinormal modes are to be thought of as scattering resonances. Indeed the poles of $\rho(\omega)$ are related to the poles/zeros of the static patch S-matrix $S(\omega)$, cf. (C.1.14) below. Thus we see the coefficients N_λ in (C.1.12) count resonances (or quasinormal modes), rather than states (or normal modes) as in AdS. This expresses at the level of characters the observations made in [59]. It holds for any SO$(1, d+1)$ representation, including massless representations, as explored in more depth in [1] (see also appendix C.6.1). Some corresponding quasinormal mode expansions of bulk thermodynamic quantities are given in (5.2.31) and (5.2.34), and related there to the quasinormal mode expansion of [163] for scalar and spinor path integrals.

“S-matrix” formulation

The appearance of resonance poles in the analytically continued density of states is well-known in quantum mechanical scattering off a fixed potential V. They are directly related to the poles/zeros in the S-matrix $S(\omega)$ at energy ω through the relation [233]

$$\rho(\omega) - \rho_0(\omega) = \frac{1}{2\pi i} \frac{d}{d\omega} \operatorname{tr} \log S(\omega) \,, \tag{C.1.14}$$

where $\rho_0(\omega)$ is the density of states at $V = 0$.

Using the explicit form of the dS_2 dimension-Δ scalar static patch mode functions (C.4.24) $\phi^{\Delta}_{\omega\ell}(r,T)$, expanding these for $r =: \tanh X \to 1$ as

$$\phi^{\Delta}_{\omega\ell}(r) \to A^{\Delta}_{\ell}(\omega)\, e^{-i\omega(T+X)} + B^{\Delta}_{\ell}(\omega)\, e^{-i\omega(T-X)} \,, \tag{C.1.15}$$

and defining $S^{\Delta}_{\ell}(\omega) \equiv B^{\Delta}_{\ell}(\omega)/A^{\Delta}_{\ell}(\omega)$, one can check that $\rho^{\Delta}(\omega)$ as obtained in (C.1.8) satisfies

$$\rho^{\Delta}(\omega) - \rho_0(\omega) = \frac{1}{2\pi i} \frac{d}{d\omega} \sum_{\ell=0,1} \log S^{\Delta}_{\ell}(\omega) \,, \tag{C.1.16}$$

where $\rho_0(\omega) = \frac{1}{\pi}(\psi(i\omega) + \psi(-i\omega)) + \text{const.}$ does not depend on Δ. This can be viewed as a rough analog of (C.1.14), although the interpretation of $\rho_0(\omega)$ in the present setting is not clear to us. Similar observations can be made in higher dimensions.

In [234], a general (flat space) S-matrix formulation of statistical mechanics for interacting QFTs was developed. In this formulation, the canonical partition function is expressed as

$$\log Z - \log Z_0 = \frac{1}{2\pi i} \int dE\, e^{-\beta E} \frac{d}{dE} \left[\operatorname{Tr} \log S(E) \right]_c \,, \tag{C.1.17}$$

where the subscript c indicates restriction to connected diagrams (where "connected" is defined with the rule that particle permutations are interpreted as interactions [234]). Combined with the above observations, this hints at a possible generalization of our free QFT results to interacting theories.

C.2 Evaluation of character integrals

The most straightforward way of UV-regularizing character integrals is to simply cut off the t-integral at some small $t = \epsilon$. However to compare to the standard heat kernel (or spectral zeta function) regularization for Gaussian Euclidean path integrals [178], it is useful to have explicit results in the latter scheme. In this appendix we give an efficient and general recipe to compute the exact heat kernel-regularized one-loop Euclidean path integral, with regulator $e^{-\epsilon^2/4\tau}$ as in (5.3.2), requiring only the unregulated character formula as input. For concreteness we consider the scalar case in the derivation, but because the scalar character $\Theta_0(t)$ provides the basic building block for all other characters $\Theta_S(t)$, the final result will be applicable in general. We spell out the derivation is some detail, and summarize the final result together with some examples in section C.2.2. Application to the massless higher-spin case is discussed in section C.2.3, where we work out the exact one-loop Euclidean path integral for Einstein gravity on S^4 as an example. In section C.2.4 we consider different regularizations, such as the simple $t > \epsilon$ cutoff.

C.2.1 Derivation

As shown in section 5.3, the scalar Euclidean path integral regularized as

$$\log Z_\epsilon = \int_0^\infty \frac{d\tau}{2\tau}\, e^{-\frac{\epsilon^2}{4\tau}}\, F_D(\tau)\,, \qquad F_D(\tau) \equiv \mathrm{Tr}\, e^{-\tau D} = \sum_n D_n^{d+2}\, e^{-\left(n+\frac{d}{2}+i\nu\right)\left(n+\frac{d}{2}-i\nu\right)}\,, \tag{C.2.1}$$

where $D = -\nabla^2 + \frac{d^2}{4} + \nu^2$, can be written in character integral form as

$$\log Z_\epsilon = \int_\epsilon^\infty \frac{dt}{2\sqrt{t^2-\epsilon^2}} \sum_n D_n^{d+2}\Big(e^{-(n+\frac{d}{2})t - i\nu\sqrt{t^2-\epsilon^2}} + e^{-(n+\frac{d}{2})t + i\nu\sqrt{t^2-\epsilon^2}}\Big) \tag{C.2.2}$$

$$= \int_\epsilon^\infty \frac{dt}{2\sqrt{t^2-\epsilon^2}}\, \frac{1+e^{-t}}{1-e^{-t}}\, \frac{e^{-\frac{d}{2}t - i\nu\sqrt{t^2-\epsilon^2}} + e^{-\frac{d}{2}t + i\nu\sqrt{t^2-\epsilon^2}}}{(1-e^{-t})^d}\,, \tag{C.2.3}$$

Putting $\epsilon = 0$ we recover the formal (UV-divergent) character formula

$$\log Z_{\epsilon=0} = \int_0^\infty \frac{dt}{2t}\, F_\nu(t)\,,$$
$$F_\nu(t) \equiv \sum_n D_n^{d+2}\Big(e^{-(n+\frac{d}{2}+i\nu)t} + e^{-(n+\frac{d}{2}-i\nu n)t}\Big) = \frac{1+e^{-t}}{1-e^{-t}}\, \frac{e^{-(\frac{d}{2}+i\nu)t} + e^{-(\frac{d}{2}-i\nu)t}}{(1-e^{-t})^d}\,. \tag{C.2.4}$$

To evaluate (C.2.3), we split the integral into UV and IR parts, each of which can be evaluated in closed form in the limit $\epsilon \to 0$.

Separation into UV and IR parts

The separation of the integral in UV and IR parts is analogous to the usual procedure in heat kernel regularization, where one similarly separates out the UV part of the τ integral by isolating the leading terms in the $\tau \to 0$ heat kernel expansion

$$F_D(\tau) := \mathrm{Tr}\, e^{-\tau D} \;\to\; \sum_{k=0}^{d+1} \alpha_k \, \tau^{-(d+1-k)/2} =: F_D^{\mathrm{uv}}(\tau)\,. \tag{C.2.5}$$

Introducing an infinitesimal IR cutoff $\mu \to 0$, we may write $\log Z_\epsilon = \log Z_\epsilon^{\mathrm{uv}} + \log Z^{\mathrm{ir}}$ where

$$\log Z_\epsilon^{\mathrm{uv}} \equiv \int_0^\infty \frac{d\tau}{2\tau}\, e^{-\frac{\epsilon^2}{4\tau}}\, F_D^{\mathrm{uv}}(\tau)\, e^{-\mu^2\tau}\,, \quad \log Z^{\mathrm{ir}} \equiv \int_0^\infty \frac{d\tau}{2\tau}\big(F_D(\tau) - F_D^{\mathrm{uv}}(\tau)\big)\, e^{-\mu^2\tau}\,. \tag{C.2.6}$$

Dropping the UV regulator in the IR integral is allowed because all UV divergences have been removed by the subtraction. The factor $e^{-\mu^2\tau}$ serves as an IR regulator needed for the separate integrals when F^{uv} has a term $\frac{\alpha_{d+1}}{2\tau} \neq 0$, that is to say when $d+1$ is even. The resulting $\log\mu$ terms cancel out of the sum at the end. Evaluating this using the specific UV regulator of (C.2.1) gives

$$\log Z_\epsilon = \frac{1}{2}\zeta_D'(0) + \alpha_{d+1}\log\big(\tfrac{2}{e^{\gamma}\epsilon}\big) + \frac{1}{2}\sum_{k=0}^{d} \alpha_k\, \Gamma\big(\tfrac{d+1-k}{2}\big)\left(\tfrac{2}{\epsilon}\right)^{d+1-k}\,, \tag{C.2.7}$$

where $\zeta_D(z) = \mathrm{Tr}\, D^{-z} = \frac{1}{\Gamma(z)}\int \frac{d\tau}{\tau}\, \tau^z\, \mathrm{Tr}\, e^{-\tau D}$ is the zeta function of D and $\alpha_{d+1} = \zeta_D(0)$.

We can apply the same idea to the square-root regulated character formula (C.2.3) for Z_ϵ. The latter is obtained from the simpler integrand of the formal character formula (C.2.4) for $Z_{\epsilon=0}$ by dividing it by $r(\epsilon,t) \equiv \sqrt{t^2-\epsilon^2}/t$ and replacing ν by $\nu r(\epsilon,t)$:

$$\log Z_{\epsilon=0} = \int_0^\infty \frac{dt}{2t}\, F_\nu(t) \qquad \Rightarrow \qquad \log Z_\epsilon = \int_\epsilon^\infty \frac{dt}{2rt}\, F_{r\nu}(t)\,, \qquad r \equiv \frac{\sqrt{t^2-\epsilon^2}}{t}\,. \tag{C.2.8}$$

Note that $0 < r < 1$ for all $t > \epsilon$, $r \sim \mathcal{O}(1)$ for $t \sim \epsilon$ and $r \to 1$ for $t \gg \epsilon$. Therefore, given the $t \to 0$ behavior of the integrand in the formal character formula for $Z_{\epsilon=0}$,

$$\frac{1}{2t}F_\nu(t) \to \frac{1}{t}\sum_{k=0}^{d+1} b_k(\nu)\, t^{-(d+1-k)} =: \frac{1}{2t}F_\nu^{\rm uv}(t)\,, \qquad b_k(\nu) = \sum_{\ell=0}^{k} b_{k\ell}\,\nu^\ell\,, \tag{C.2.9}$$

we get the $t \sim \epsilon \to 0$ behavior of the integrand for the exact Z_ϵ:

$$\frac{1}{2rt}\,F_{r\nu}(t) \to \frac{1}{2rt}\,F_{r\nu}^{\rm uv}(t) = \frac{1}{rt}\sum_{k,\ell} b_{k\ell}\,\nu^\ell\, r^\ell\, t^{-(d+1-k)}\,. \tag{C.2.10}$$

Thus we can separate $\log Z_\epsilon = \log \tilde{Z}_\epsilon^{\rm uv} + \log \tilde{Z}^{\rm ir}$, with

$$\log \tilde{Z}_\epsilon^{\rm uv} \equiv \int_\epsilon^\infty \frac{dt}{2rt}\,F_{r\nu}^{\rm uv}(t)\,e^{-\mu t}\,, \qquad \log \tilde{Z}^{\rm ir} \equiv \int_0^\infty \frac{dt}{2t}\,\big(F_\nu(t) - F_\nu^{\rm uv}(t)\big)e^{-\mu t}\,. \tag{C.2.11}$$

Again the limit $\mu \to 0$ is understood. We were allowed to put $\epsilon = 0$ in the IR part because it is UV finite.

Evaluation of UV part

Using the expansion (C.2.10), the UV part can be evaluated explicitly as

$$\log \tilde{Z}_\epsilon^{\rm uv} = \frac{1}{2}\sum_{\ell,k\le d} b_{k\ell}\,B\big(\tfrac{d+1-k}{2},\tfrac{\ell+1}{2}\big)\,\nu^\ell\,\epsilon^{-(d+1-k)} - \sum_\ell b_{d+1,\ell}\big(H_\ell - \tfrac{1}{2}H_{\ell/2} + \log(\tfrac{e^\gamma\,\epsilon\,\mu}{2})\big)\nu^\ell \tag{C.2.12}$$

where $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ is the Euler beta function and $H_x = \gamma + \frac{\Gamma'(1+x)}{\Gamma(1+x)}$ which for integer x is the x-th harmonic number $H_x = 1 + \frac{1}{2} + \cdots + \frac{1}{x}$. For example for $d = 3$, we get

$$\log \tilde{Z}_\epsilon^{\rm uv} = \tfrac{4}{3}\,\epsilon^{-4} - \tfrac{4\nu^2+1}{12}\,\epsilon^{-2} - \big(\tfrac{\nu^4}{9} + \tfrac{\nu^2}{24}\big) - \big(\tfrac{\nu^4}{12} + \tfrac{\nu^2}{24} - \tfrac{17}{2880}\big)\log\big(\tfrac{e^\gamma\epsilon\mu}{2}\big)\,. \tag{C.2.13}$$

This gives an explicit expression for the part of $\log Z$ denoted $\mathrm{Pol}(\Delta)$ in [163], without having to invoke an independent computation of the heat kernel coefficients. Indeed, turning this around, by comparing (C.2.12) to (C.2.7), we can express the heat kernel coefficients α_k explicitly in terms of the character coefficients $b_{k,\ell}$. In particular the Weyl anomaly coefficient is simply given by the coefficient $b_{d+1} = \sum_\ell b_{d+1,\ell}\nu^\ell$ of the $1/t$ term in the integrand of the formal character formula (C.2.4). More generally,

$$\alpha_k = \sum_\ell \frac{\Gamma(\frac{\ell+1}{2})}{2^{d+1-k}\Gamma(\frac{d+1-k+\ell+1}{2})}\, b_{k\ell}\,\nu^\ell\,. \tag{C.2.14}$$

For example for $d = 3$, this becomes $\alpha_0 = \frac{1}{12}b_{00}$, $\alpha_2 = \frac{1}{2}b_{20} + \frac{\nu^2}{6}b_{22}$ and $\alpha_4 = b_4$. From the small-t expansion $\frac{1}{2t}F_\nu(t) \to \sum_k b_k t^{3-k}$ in (C.2.4) we read off $b_0 = 2$, $b_2 = -\frac{1}{12} - \nu^2$ and $b_4 = -\frac{17}{2880} + \frac{1}{24}\nu^2 + \frac{1}{12}\nu^4$. Thus $\alpha_0 = \frac{1}{6}$, $\alpha_2 = -\frac{1}{24} - \frac{1}{6}\nu^2$ and $\alpha_4 = -\frac{17}{2880} + \frac{1}{24}\nu^2 + \frac{1}{12}\nu^4$.

Evaluation of IR part

As we explain momentarily, the IR part can be evaluated as

$$\log\tilde{Z}^{\rm ir} = \frac{1}{2}\zeta_\nu'(0) + b_{d+1}\log\mu\,, \qquad \zeta_\nu(z) \equiv \frac{1}{\Gamma(z)}\int_0^\infty \frac{dt}{t}\, t^z\, F_\nu(t)\,, \tag{C.2.15}$$

where like for the spectral zeta function $\zeta_D(z)$, the "character zeta function" $\zeta_\nu(z)$ is defined by the above integral for z sufficiently large and by analytic continuation for $z \to 0$. This zeta function representation of $\log Z^{\rm ir}$ follows from the following observations. If we define $\zeta_\nu^{\rm ir}(z) \equiv \frac{1}{\Gamma(z)}\int_0^\infty \frac{dt}{t}\, t^z\big(F_\nu(t) - F_\nu^{\rm uv}(t)\big)\, e^{-\mu t}$, then since the integral remains finite for $z \to 0$, while $\Gamma(z) \sim 1/z$ and $\partial_z(1/\Gamma(z)) \to 1$, we trivially have $\frac{1}{2}\partial_z\zeta_\nu^{\rm ir}(z)|_{z=0} = \log\tilde{Z}^{\rm ir}$. Moreover for z sufficiently large we have in the limit $\mu \to 0$ that $\frac{1}{2}\zeta^{\rm uv}(z) \equiv \frac{1}{\Gamma(z)}\int_0^\infty \frac{dt}{2t}\, t^z F_\nu^{\rm uv}(t)\, e^{-\mu t} = b_{d+1}\mu^{-z}$, so upon analytic continuation we have $1/2\partial_z\zeta^{\rm uv}(z)|_{z=0} = -b_{d+1}\log\mu$, and (C.2.15) follows.

In contrast to the spectral zeta function, the character zeta function can straightforwardly be evaluated in terms of Hurwitz zeta functions. Indeed, denoting $\Delta_\pm = \frac{d}{2} \pm i\nu$, we have $F_D(t) = \sum_n Q(n)\, e^{-t(n+\Delta_+)(n+\Delta_-)}$ where the spectral degeneracy $Q(n)$ is some polynomial in n, and $\zeta_D(z) = \sum_{n=0}^\infty Q(n)\,\big((n+\Delta_+)(n+\Delta_-)\big)^{-z}$, which is quite tricky to evaluate, whereas $F_\nu(t) = \sum_n Q(n)\big(e^{-t(n+\Delta_+)} + e^{-t(n+\Delta_-)}\big)$, and we can immediately express the associated character zeta function as a finite sum of Hurwitz zeta functions $\zeta(z,\Delta) = \sum_{n=0}^\infty (n+\Delta)^{-z}$:

$$\zeta_\nu(z) = \sum_\pm\sum_{n=0}^\infty Q(n)(n+\Delta_\pm)^{-z} = \sum_\pm Q(\hat{\delta} - \Delta_\pm)\,\zeta(z,\Delta_\pm)\,. \tag{C.2.16}$$

Here $\hat{\delta}$ is the unit z-shift operator acting as $\hat{\delta}^n\zeta(z,\Delta) = \zeta(z-n,\Delta)$; for example if $Q(n) = n^2$

we have $Q(\hat\delta-\Delta)\,\zeta(z,\Delta)=(\hat\delta^2-2\Delta\hat\delta+\Delta^2)\zeta(z,\Delta)=\zeta(z-2,\Delta)-2\Delta\zeta(z-1,\Delta)+\Delta^2\zeta(z,\Delta)$.

C.2.2 Result and examples

Result

Altogether we conclude that given a formal character integral formula

$$\log Z_{\rm PI}=\int_0^\infty \frac{dt}{2t}\,F_\nu(t)\,, \tag{C.2.17}$$

for a field corresponding to a dS_{d+1} irrep of dimension $\frac{d}{2}+i\nu$, with IR and UV expansions

$$F_\nu(t)=\sum_\Delta\sum_{n=0}^\infty P_\Delta(n)\,e^{-(n+\Delta)t}\,,\qquad \frac{1}{2t}F_\nu(t)=\frac{1}{t}\sum_{k=0}^{d+1}b_k(\nu)\,t^{-(d+1-k)}\;+\mathcal{O}(t^0)\,, \tag{C.2.18}$$

where $b_k(\nu)=\sum_\ell b_{k\ell}\,\nu^\ell$, we obtain the exact $Z_{\rm PI}$ with heat kernel regulator $e^{-\epsilon^2/4\tau}$ as

$$\boxed{\begin{aligned}\log Z_{{\rm PI},\epsilon}=&\frac{1}{2}\sum_\Delta P_\Delta(\hat\delta-\Delta)\,\zeta'(0,\Delta)-\sum_{\ell=0}^{d+1}b_{d+1,\ell}(H_\ell-\tfrac{1}{2}H_{\ell/2})\nu^\ell+b_{d+1}(\nu)\log(2e^{-\gamma}/\epsilon)\\&+\frac{1}{2}\sum_{k=0}^{d}\sum_{\ell=0}^{k}b_{k\ell}\,B\big(\tfrac{d+1-k}{2},\tfrac{\ell+1}{2}\big)\,\nu^\ell\,\epsilon^{-(d+1-k)}\,.\end{aligned}} \tag{C.2.19}$$

Here $B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, $H_x=\gamma+\frac{\Gamma'(1+x)}{\Gamma(1+x)}$, which for integer x is the x-th harmonic number $H_x=1+\frac{1}{2}+\cdots+\frac{1}{x}$, and $\hat\delta$ is the unit shift operator acting on the first argument of the Hurwitz zeta function $\zeta(z,\Delta)$: the polynomial $P_\Delta(\hat\delta-\Delta)$ is to be expanded in powers of $\hat\delta$, setting $\hat\delta^n\zeta'(0,\Delta)\equiv\zeta'(-n,\Delta)$. Finally the heat kernel coefficients are

$$\alpha_k=\sum_\ell\frac{\Gamma(\frac{\ell+1}{2})}{2^{d+1-k}\Gamma(\frac{d+1-k+\ell+1}{2})}\,b_{k\ell}\,\nu^\ell\,. \tag{C.2.20}$$

If we are only interested in the finite part of $\log Z$, only the first three terms in (C.2.19) matter. Note that the third and the second term $\mathcal{M}_\nu\equiv\sum_\ell b_{d+1,\ell}(H_\ell-\frac{1}{2}H_{\ell/2})$ is in general nonvanishing for even $d+1$. By comparing (C.2.19) to (C.2.7), say in the scalar case discussed earlier, we see that $\zeta_D'(0)=\zeta_\nu'(0)+2\mathcal{M}_\nu$. Thus $2\mathcal{M}_\nu$ can be thought of as correcting the formal factorization $\sum_n\log(n+\Delta_+)(n+\Delta_-)=\sum_n\log(n+\Delta_+)+\sum_n\log(n+\Delta_-)$ in zeta function regularization.

For this reason $\mathcal{M}_\nu$ is called the multiplicative "anomaly", as reviewed in [235]. The above thus generalizes the explicit formulae in [235] for $\mathcal{M}_\nu$ to fields of arbitrary representation content.

Examples

1. A scalar on S^2 $(d=1)$ with $\Delta_\pm = \frac{1}{2} \pm i\nu$ has $F_\nu(t) = \frac{1+e^{-t}}{1-e^{-t}} \frac{e^{-\Delta_+ t}+e^{-\Delta_- t}}{1-e^{-t}}$ so the IR and UV expansions are $F_\nu(t) = \sum_\pm \sum_{n=0}^\infty (2n+1) e^{-(\Delta_\pm + n)t}$ and $\frac{1}{2t} F_\nu(t) = \frac{2}{t^3} + \frac{\frac{1}{12} - \nu^2}{t} + \mathcal{O}(t^0)$. Therefore according to (C.2.19)

$$\log Z_{\mathrm{PI},\epsilon} = \sum_{\Delta = \frac{1}{2} \pm i\nu} \Big(\zeta'(-1,\Delta) - (\Delta - \tfrac{1}{2}) \zeta'(0,\Delta) \Big) + \nu^2 + \left(\tfrac{1}{12} - \nu^2\right) \log(2\, e^{-\gamma}/\epsilon) + \frac{2}{\epsilon^2} \,. \tag{C.2.21}$$

The heat kernel coefficients are obtained from (C.2.20) as $\alpha_0 = 1$ and $\alpha_2 = \frac{1}{12} - \nu^2$.

2. For a scalar on S^3, $F_\nu(t) = \sum_\pm \sum_{n=0}^\infty (n+1)^2 e^{-(\Delta_\pm + n)t}$, $\frac{1}{2t} F_\nu(t) \to \frac{2}{t^4} - \frac{\nu^2}{t^2} + \mathcal{O}(t^0)$, so

$$\log Z_{\mathrm{PI},\epsilon} = \sum_\pm \Big(\tfrac{1}{2} \zeta'(-2, 1 \pm i\nu) \mp i\nu \zeta'(-1, 1 \pm i\nu) - \tfrac{1}{2} \nu^2 \zeta'(0, 1 \pm i\nu) \Big) - \frac{\pi \nu^2}{4\epsilon} + \frac{\pi}{2\epsilon^3} \,. \tag{C.2.22}$$

The heat kernel coefficients are $\alpha_0 = \frac{\sqrt{\pi}}{4}$, $\alpha_2 = -\frac{\sqrt{\pi}}{4} \nu^2$. In particular for a conformally coupled scalar, i.e. $\Delta = \frac{1}{2}, \frac{3}{2}$ or equivalently $\nu = i/2$, we get for the finite part the familiar result $\log Z_{\mathrm{PI}} = \frac{3\zeta(3)}{16\pi^2} - \frac{\log(2)}{8}$. For $\Delta = 1$, i.e. $\nu = 0$, we get $\log Z_{\mathrm{PI}} = -\frac{\zeta(3)}{4\pi^2}$. Notice that the finite part looks quite different from (5.2.25) obtained by contour integration. Nevertheless they are in fact the same function.

3. A more interesting example is the massive spin-s field on S^4 with $\Delta_\pm = \frac{3}{2} \pm i\nu$. In this case, (5.4.6) combined with (C.5.8) or equivalently (5.4.7) gives $F_\nu = F_{\rm bulk} - F_{\rm edge}$ with

$$F_{\rm bulk}(t) = \sum_{\Delta=\frac{3}{2}\pm i\nu} \sum_{n=-1}^{\infty} D_s^3 D_n^5 \, e^{-(n+\Delta)t} = D_s^3 \, \frac{1+e^{-t}}{1-e^{-t}} \frac{e^{-(\frac{3}{2}+i\nu)t} + e^{-(\frac{3}{2}-i\nu)t}}{(1-e^{-t})^3} \,, \tag{C.2.23}$$

$$F_{\rm edge}(t) = \sum_{\Delta=\frac{1}{2}\pm i\nu} \sum_{n=-1}^{\infty} D_{s-1}^5 D_{n+1}^3 \, e^{-(n+\Delta)t} = D_{s-1}^5 \, \frac{1+e^{-t}}{1-e^{-t}} \frac{e^{-(\frac{1}{2}+i\nu)t} + e^{-(\frac{1}{2}-i\nu)t}}{(1-e^{-t})} \,, \tag{C.2.24}$$

where $D_p^3 = 2p+1$, $D_p^5 = \frac{1}{6}(2p+3)(p+2)(p+1)$. In particular note that with $g_s \equiv D_s^3 = 2s+1$, we have $D_{s-1}^5 = \frac{1}{24} g_s (g_s^2-1)$. The small-$t$ expansions are

$$\tfrac{1}{2t} F_{\rm bulk}(t) \to g_s \Big(2\, t^{-5} - (\nu^2 + \tfrac{1}{12}) t^{-3} + (\tfrac{\nu^4}{12} + \tfrac{\nu^2}{24} - \tfrac{17}{2880}) t^{-1} + \mathcal{O}(t^0) \Big) \tag{C.2.25}$$

$$\tfrac{1}{2t} F_{\rm edge}(t) \to \tfrac{1}{24} g_s (g_s^2 - 1) \Big(2\, t^{-3} + (\tfrac{1}{12} - \nu^2) t^{-1} + \mathcal{O}(t^0) \Big) \,. \tag{C.2.26}$$

Thus the exact partition function for a massive spin-s field is

$$\begin{aligned} \log Z_{\rm PI,\epsilon} = g_s & \sum_{\Delta=\frac{3}{2}\pm i\nu} \Big(\tfrac{1}{6} \zeta'(-3,\Delta) \mp \tfrac{1}{2} i\nu \zeta'(-2,\Delta) - (\tfrac{1}{2}\nu^2 + \tfrac{1}{24}) \zeta'(-1,\Delta) \pm i (\tfrac{1}{24}\nu + \tfrac{1}{6}\nu^3) \zeta'(0,\Delta) \Big) \\ & - \tfrac{1}{24} g_s (g_s^2 - 1) \sum_{\Delta=\frac{1}{2}\pm i\nu} \Big(\zeta'(-1,\Delta) \mp i\nu \zeta'(0,\Delta) \Big) - \tfrac{1}{24} g_s^3 \nu^2 - \tfrac{1}{9} g_s \nu^4 \\ & + \Big(g_s^3 (\tfrac{1}{24}\nu^2 - \tfrac{1}{288}) + g_s (\tfrac{1}{12}\nu^4 - \tfrac{7}{2880}) \Big) \log(2\, e^{-\gamma}/\epsilon) - (\tfrac{1}{12} g_s^3 + \tfrac{1}{3} g_s \nu^2) \epsilon^{-2} + \tfrac{4}{3} g_s \epsilon^{-4} \,. \end{aligned} \tag{C.2.27}$$

Finally the heat kernel coefficients are

$$\alpha_0 = \tfrac{1}{6} g_s \,, \qquad \alpha_2 = -\tfrac{1}{24} g_s^3 - \tfrac{1}{6} g_s \nu^2 \,, \qquad \alpha_4 = g_s^3 (\tfrac{1}{24}\nu^2 - \tfrac{1}{288}) + g_s (\tfrac{1}{12}\nu^4 - \tfrac{7}{2880}) \,. \tag{C.2.28}$$

Single-mode contributions

Contributions from single path integral modes and contributions of single quasinormal modes are of use in some of our derivations and applications. These are essentially special cases of the above general results, but for convenience we collect some explicit formulae here:

- **Path integral single-mode contributions**: For our choice of heat-kernel regulator $e^{-\epsilon^2/4\tau}$, the contribution to $\log Z_{\mathrm{PI},\epsilon}$ from a single bosonic eigenmode with eigenvalue λ is

$$I_\lambda = \int_0^\infty \frac{d\tau}{2\tau}\, e^{-\epsilon^2/4\tau}\, e^{-\tau\lambda} = K_0(\epsilon\sqrt{\lambda}) \to -\frac{1}{2}\log\frac{\lambda}{M^2}\,, \qquad M \equiv \frac{2e^{-\gamma}}{\epsilon}\,, \tag{C.2.29}$$

Different regulator insertions lead to a similar result in the limit $\epsilon \to 0$, with $M = c/\epsilon$ for some regulator-dependent constant c. A closely related formula is obtained for the contribution from an individual term in the sum (C.2.2) or equivalently in the IR expansion of (C.2.18), which amounts to computing (C.2.17) with $F_\nu(t) \equiv e^{-\rho t}$, $\rho = a \pm i\nu$. The small-t expansion is $\frac{1}{2t}F_\nu(t) = \frac{1}{2t} + \mathcal{O}(t^0)$, so the UV part is given by the log term in (C.2.19) with coefficient $\frac{1}{2}$, and the IR part is $\frac{1}{2}\zeta_\nu'(0) = -\frac{1}{2}\log\rho$ as in (C.2.15). Thus

$$I'_\rho = \int_0^\infty \frac{dt}{2t}\, e^{-\rho t} \to -\frac{1}{2}\log\frac{\rho}{M}\,, \qquad M = \frac{2e^{-\gamma}}{\epsilon}\,, \tag{C.2.30}$$

where the integral is understood to be regularized as in (C.2.2), $I'_\rho = \int_\epsilon^\infty \frac{dt}{2\sqrt{t^2-\epsilon^2}}\, e^{-ta-i\nu\sqrt{t^2-\epsilon^2}}$, left implicit here. The similarities between (C.2.29) and (C.2.30) are of course no accident, since in our setup, the former splits into the sum of two integrals of the latter type: writing $\lambda = a^2+\nu^2 = (a+i\nu)(a-i\nu)$, we have $I_\lambda = I'_{a+i\nu} + I'_{a-i\nu}$.

• **Quasinormal mode contributions**: Considering a character quasinormal mode expansion $\Theta(t) = \sum_r N_r\, e^{-r|t|}$ as in (5.1.14), the IR contribution from a single bosonic/fermionic QNM is

$$\int_0^\infty \frac{dt}{2t}\frac{1+e^{-t}}{1-e^{-t}}\, e^{-r\,t}\bigg|_{\mathrm{IR}} = \log\frac{\Gamma(r+1)}{\mu^r\sqrt{2\pi r}}\,, \qquad -\int_0^\infty \frac{dt}{2t}\frac{2e^{-t/2}}{1-e^{-t}}\, e^{-r\,t}\bigg|_{\mathrm{IR}} = -\log\frac{\Gamma(r+\frac{1}{2})}{\mu^r\sqrt{2\pi}} \tag{C.2.31}$$

• **Harmonic oscillator**: The character of a $d=0$ scalar of mass ν is $\Theta(t) = e^{-i\nu t} + e^{i\nu t}$, hence

$$\log Z_{\mathrm{PI},\epsilon} = \int_0^\infty \frac{dt}{2t}\,\frac{1+e^{-t}}{1-e^{-t}}\,(e^{-i\nu t}+e^{i\nu t}) = \frac{\pi}{\epsilon} - \log(e^{\pi\nu}-e^{-\pi\nu})\,. \tag{C.2.32}$$

The finite part gives the canonical bosonic harmonic oscillator thermal partition function $\mathrm{Tr}\, e^{-\beta H} = \sum_n e^{-\beta\nu(n+\frac{1}{2})} = \left(e^{\beta\nu/2}-e^{-\beta\nu/2}\right)^{-1}$ at $\beta = 2\pi$. The fermionic version is

$$\log Z_{\mathrm{PI},\epsilon} = -\int_0^\infty \frac{dt}{2t}\,\frac{2e^{-t/2}}{1-e^{-t}}\,(e^{-i\nu t}+e^{i\nu t}) = -\frac{\pi}{\epsilon} + \log(e^{\pi\nu}+e^{-\pi\nu})\,. \tag{C.2.33}$$

C.2.3 Massless case

Here we give a few more details on how to use (C.2.19) to explicitly evaluate $Z_{\rm PI}$ in the massless case, and work out the exact $Z_{\rm PI}$ for Einstein gravity on S^4 as an example.

Our final result for the massless one-loop $Z_{\rm PI} = Z_G \cdot Z_{\rm char}$ is given by (5.5.17):

$$Z_{\rm PI} = i^{-P}\,\frac{\gamma^{\rm dimG}}{{\rm vol}(G)_{\rm c}} \cdot \exp \int^{\times} \frac{dt}{2t}\,F\,, \qquad F = \frac{1+q}{1-q}\Big([\hat\Theta_{\rm bulk}]_+ - [\hat\Theta_{\rm edge}]_+ - 2\dim G\Big)\,, \tag{C.2.34}$$

where for $s=2$ gravity $\gamma = \sqrt{\frac{8\pi G_{\rm N}}{A_{d-1}}}$, $P = d+3$, $G = {\rm SO}(d+2)$ and ${\rm vol}(G)_{\rm c} = (C.3.2)$.

• **UV part:** As always, the coefficient of the log-divergent term simply equals the coefficient of the $1/t$ term in the small-t expansion of the integrand in (C.2.34). For the other UV terms in (C.2.19) (including the "multiplicative anomaly"), a problem might seem to be that we need a continuously variable dimension parameter $\Delta = \frac{d}{2} + i\nu$, whereas massless fields, and our explicit formulae for $\hat\Theta \to [\hat\Theta]_+$, require fixed integer dimensions. This problem is easily solved, as the *UV part* can actually be computed from the original *naive* character formula (C.6.7):

$$\log Z_{\rm PI}\big|_{\rm UV} = \int \frac{dt}{2t}\,\hat F\Big|_{\rm UV}\,, \qquad \hat F = \frac{1+q}{1-q}(\hat\Theta_{\rm bulk} - \hat\Theta_{\rm edge})\,, \tag{C.2.35}$$

Indeed since $\hat F \to F = \{\hat F\}_+$ in (C.6.9) affects just a finite number of terms $c_k q^k \to c_k q^{-k}$, it does not alter the small-t (UV) part of the integral. Moreover $\hat\Theta_s = \hat\Theta_{s,\nu_\phi} - \hat\Theta_{s,\nu_\xi}$, where $\hat\Theta_{s,\nu}$ is a massive spin-s character. Thus the UV part may be obtained simply by combining the results of (C.2.19) for general ν and s, substituting the values ν_ϕ, ν_ξ set by (5.5.2).

• **IR part:** The *IR part* is the ζ' part of (C.2.19), obtained from the q-expansion of $F(q)$ in (C.2.34). This can be found in general by using

$$\frac{1+q}{1-q}\,\frac{q^\Delta}{(1-q)^k} = \sum_{n=0}^{\infty} P(n)\, q^{n+\Delta}\,, \qquad P(n) = D_n^{k+2}\,, \tag{C.2.36}$$

with D_n^{k+2} the polynomial given in (2.2.3). For $k=0$, (C.2.31) is useful. In particular, using the $\int^\times$ prescription (C.6.15), the IR contribution from the last term in (C.2.34) is obtained by considering the $r \to 0$ limit of the bosonic formula in (C.2.31):

$$\int^{\times} \frac{dt}{2t}\,\frac{1+q}{1-q}(-2\dim G)\Big|_{\rm IR} = \dim G \cdot \log(2\pi)\,. \tag{C.2.37}$$

Example: Einstein gravity on S^4

As a simple application, let us compute the exact one-loop Euclidean path integral for pure gravity on S^4. In this case $G = \mathrm{SO}(5)$, $\dim G = 10$, $d = 3$ and $s = 2$. From (5.5.2) we read off $i\nu_\phi = \frac{3}{2}$, $i\nu_\xi = \frac{5}{2}$, and from (5.5.7) we get

$$\Theta_{\rm bulk} = \big[\hat\Theta_{\rm bulk}\big]_+ = \frac{10\,q^3 - 6\,q^4}{(1-q)^3}\,, \qquad \Theta_{\rm edge} = \big[\hat\Theta_{\rm edge}\big]_+ = \frac{10\,q^2 - 2\,q^3}{1-q}\,. \tag{C.2.38}$$

The small-t expansion of the integrand in (C.2.34) is $\frac{1}{2t}F = 4\,t^{-5} - \frac{47}{3}\,t^{-3} - \frac{571}{45}\,t^{-1} + O(t^0)$. The coefficient of the log-divergent part of $\log Z_{\rm PI}$ is the coefficient of t^{-1}:

$$\log Z_{\rm PI}|_{\log \rm div} = -\frac{571}{45}\,\log(2e^{-\gamma}\epsilon^{-1})\,, \tag{C.2.39}$$

in agreement with [127]. The complete heat-kernel regularized UV part of (C.2.19) can be read off directly from our earlier results for massive spin-s in $d = 3$ as

$$\begin{aligned}\log Z_{\rm PI}\big|_{\rm UV} &= \log Z_{\rm PI}(s=2, \nu=\tfrac{3}{2}i)\big|_{\rm UV} - \log Z_{\rm PI}(s=1, \nu=\tfrac{5}{2}i)\big|_{\rm UV} \\ &= \frac{8}{3}\,\epsilon^{-4} - \frac{32}{3}\,\epsilon^{-2} - \frac{571}{45}\log(2e^{-\gamma}\epsilon^{-1}) + \frac{715}{48}\,.\end{aligned} \tag{C.2.40}$$

Here $\mathcal{M} = \frac{715}{48}$ is the "multiplicative anomaly" term. The integrated heat kernel coefficients are similarly obtained from (C.2.28): $\alpha_0 = \frac{1}{3}$, $\alpha_2 = -\frac{16}{3}$, $\alpha_4 = -\frac{571}{45}$.

The IR (ζ') contributions from bulk and edge characters are obtained from the expansions

$$\frac{1+q}{1-q}(\Theta_{\rm bulk} - \Theta_{\rm edge}) = \sum_n P_{\rm b}(n)\,(10\,q^{3+n} - 6\,q^{4+n}) - \sum_n P_{\rm e}(n)\,(10\,q^{2+n} - 2\,q^{3+n})\,, \tag{C.2.41}$$

where $P_{\rm b}(n) = D^5_n = \frac{1}{6}(n+1)(n+2)(2n+3)$, $P_{\rm e}(n) = D^3_n = 2n+1$. According to (C.2.19) this gives a contribution to $\log Z_{\rm char}|_{\rm IR}$ equal to

$$5\,P_{\rm b}(\hat\delta-3)\,\zeta'(0,3)-3\,P_{\rm b}(\hat\delta-4)\,\zeta'(0,4)-5\,P_{\rm e}(\hat\delta-2)\,\zeta'(0,2)+P_{\rm e}(\hat\delta-3)\,\zeta'(0,3)\,, \quad \text{(C.2.42)}$$

where the polynomials are to be expanded in powers of $\hat\delta$, putting $\hat\delta^n\zeta'(0,\Delta)\equiv\zeta'(-n,\Delta)$. Working this out and adding the contribution (C.2.37), we find

$$\log Z_{\rm char}|_{\rm IR}=-\log 2-\frac{47}{3}\,\zeta'(-1)+\frac{2}{3}\,\zeta'(-3)\,. \quad \text{(C.2.43)}$$

Combining this with the UV part and reinstating ℓ, we get[3]

$$\begin{aligned}\log Z_{\rm char}=&\frac{8}{3}\,\frac{\ell^4}{\epsilon^4}-\frac{32}{3}\,\frac{\ell^2}{\epsilon^2}-\frac{571}{45}\log\frac{2e^{-\gamma}L}{\epsilon}\\&-\frac{571}{45}\log\frac{\ell}{L}+\frac{715}{48}-\log 2-\frac{47}{3}\zeta'(-1)+\frac{2}{3}\zeta'(-3)\,,\end{aligned} \quad \text{(C.2.44)}$$

where L is an arbitrary length scale introduced to split off a finite part:

$$\log Z^{\rm fin}_{\rm char}=-\frac{571}{45}\log(\ell/L)+\frac{715}{48}-\log 2-\frac{47}{3}\zeta'(-1)+\frac{2}{3}\zeta'(-3)\,, \quad \text{(C.2.45)}$$

To compute the group volume factor Z_G in (C.2.34), we use (C.3.2) for $G=\mathrm{SO}(5)$ to get $\mathrm{vol}(G)_{\rm c}=\frac{2}{3}(2\pi)^6$, and $\gamma=\sqrt{8\pi G_{\rm N}/4\pi\ell^2}$. Finally, $i^{-P}=i^{-(d+3)}=-1$. Thus we conclude that the one-loop Euclidean path integral for Einstein gravity on S^4 is

$$Z_{\rm PI}=-\frac{(8\pi G_{\rm N}/4\pi\ell^2)^5\,Z_{\rm char}}{\frac{2}{3}(2\pi)^6}\,, \quad \text{(C.2.46)}$$

where $Z_{\rm char}$ is given by (C.2.44).

[3] This splits as $\log Z_{\rm char}=10\,\log(2\pi)+\log Z_{\rm bulk}-\log Z_{\rm edge}$ where $\log Z_{\rm bulk}=\frac{8\ell^4}{3\epsilon^4}-\frac{8\ell^2}{3\epsilon^2}-\frac{331}{45}\log\frac{2e^{-\gamma}\ell}{\epsilon}+\frac{475}{48}-\frac{23}{3}\zeta'(-1)+\frac{2}{3}\zeta'(-3)-5\log(2\pi)$ and $\log Z_{\rm edge}=\frac{8\ell^2}{\epsilon^2}+\frac{16}{3}\log\frac{2e^{-\gamma}\ell}{\epsilon}-5+8\zeta'(-1)+\log 2+5\log(2\pi)$.

Example: Einstein gravity on S^5

For S^5 an analogous (actually simpler) computation gives $Z_{\rm PI} = i^{-7} Z_G Z_{\rm char}$ with

$$\begin{aligned} \log Z_{\rm char} &= \frac{15\,\pi}{8}\,\frac{\ell^5}{\epsilon^5} - \frac{65\,\pi}{24}\,\frac{\ell^3}{\epsilon^3} - \frac{105\,\pi}{16}\,\frac{\ell}{\epsilon} + \frac{65\,\zeta(3)}{48\,\pi^2} + \frac{5\,\zeta(5)}{16\,\pi^4} + 15\log(2\pi) \\ \log Z_G &= \frac{15}{2}\log\frac{8\pi G_{\rm N}}{2\pi^2\ell^3} - \log\frac{(2\pi)^9}{12}\,. \end{aligned} \tag{C.2.47}$$

C.2.4 Different regularization schemes

If we simply cut off the character integral at $t = \epsilon$, we get the following instead of (C.2.19):

$$\log Z_\epsilon = \frac{1}{2}\sum_\Delta P_\Delta(\hat\delta - \Delta)\,\zeta'(0,\Delta) + b_{d+1}(\nu)\log(e^{-\gamma}/\epsilon) + \sum_{k=0}^{d}\frac{b_k(\nu)}{d+1-k}\,\epsilon^{-(d+1-k)}\,, \tag{C.2.48}$$

with $b_k(\nu)$ defined as before, $\frac{1}{2t}F_\nu(t) = \sum_{k=0}^{d+1} b_k(\nu)\,t^{-(d+2-k)} + \mathcal{O}(t^0)$. Unsurprisingly, this differs from (C.2.19) only in its UV part, more specifically in the terms polynomial in ν, including the "multiplicative anomaly" term discussed below (C.2.20). The transcendental (ζ') part and the $\log\epsilon$ coefficient remain unchanged. This remains true in any other regularization.

If we stick with heat-kernel regularization but pick a different regulator $f(\tau/\epsilon^2)$ instead of $e^{-\epsilon^2/4\tau}$ (e.g. the $f = (1-e^{-\tau\Lambda^2})^k$ PV regularization of section 5.2) or use zeta function regularization, more is true: the same finite part is obtained for any choice of f provided logarithmically divergent terms (arising in even $d+1$) are expressed in terms of M defined as in (C.2.29) with $e^{-\epsilon^2/4\tau} \to f$. The relation $M(\epsilon)$ will depend on f, but nothing else.

In dimensional regularization, some polynomial terms in ν will be different, including the "multiplicative anomaly" term. Of course no physical quantity will be affected by this, as long as self-consistency is maintained. In fact any regularization scheme (even (C.2.48)) will lead to the same physically unambiguous part of the one-loop corrected dS entropy/sphere partition function of section 5.8. However to go beyond this, e.g. to extract more physically unambiguous data by comparing different saddles along the lines of (C.8.67) and (C.8.70), a portable covariant regularization scheme, like heat-kernel regularization, must be applied consistently to each saddle. A sphere-specific ad-hoc regularization as in (C.2.48) is not suitable for such purposes.

C.3 Some useful volumes and metrics

C.3.1 Volumes

The volume of the unit sphere S^n is

$$\Omega_n \equiv \mathrm{vol}(S^n) = \frac{2\,\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})} \;=\; \frac{2\pi}{n-1} \cdot \Omega_{n-2} \tag{C.3.1}$$

The volume of $\mathsf{SO}(d+2)$ with respect to the invariant group metric normalized such that minimal $\mathsf{SO}(2)$ orbits have length 2π is

$$\mathrm{vol}\big(\mathsf{SO}(d+2)\big)_{\mathrm{c}} = \prod_{k=2}^{d+2} \mathrm{vol}(S^{k-1}) = \prod_{k=2}^{d+2} \frac{2\pi^{\frac{k}{2}}}{\Gamma(\frac{k}{2})}\,. \tag{C.3.2}$$

This follows from the fact that the unit sphere $S^{n-1} = \mathsf{SO}(n)/\mathsf{SO}(n-1)$, which implies $\mathrm{vol}(\mathsf{SO}(n))_{\mathrm{c}} = \mathrm{vol}(S^{n-1})\,\mathrm{vol}(\mathsf{SO}(n-1))_{\mathrm{c}}$ in the assumed normalization.

The volume of $SU(N)$ with respect to the invariant metric derived from the matrix trace norm on the Lie algebra $\mathfrak{su}(N)$ viewed as traceless $N \times N$ matrices is (see e.g. [236])

$$\mathrm{vol}\big(SU(N)\big)_{\mathrm{Tr}_N} \;=\; \sqrt{N} \prod_{k=2}^{N} \frac{(2\pi)^k}{\Gamma(k)} \;=\; \sqrt{N}\, \frac{(2\pi)^{\frac{1}{2}(N-1)(N+2)}}{\mathtt{G}(N+1)}\,. \tag{C.3.3}$$

C.3.2 de Sitter and its Wick rotations to the sphere

Global dS_{d+1} has a convenient description as a hyperboloid embedded in $\mathbb{R}^{1,d+1}$,

$$X^I X_I \equiv \eta_{IJ} X^I X^J \equiv -X_0^2 + X_1^2 + \cdots + X_{d+1}^2 = \ell^2\,, \qquad ds^2 = \eta_{IJ} dX^I dX^J\,. \tag{C.3.4}$$

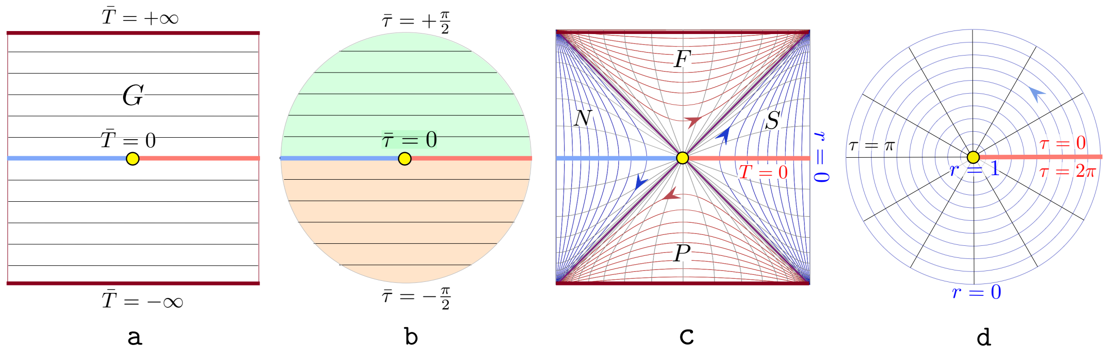


Figure C.3.1: Penrose diagrams of dS_{d+1} and S^{d+1} with coordinates C.3.5, C.3.7. Each point corresponds to an S^{d-1}, contracted to zero size at thin-line boundaries. a: Global dS_{d+1} in slices of constant $\bar{T}$. b: Wick rotation of global dS_{d+1} to S^{d+1}. c: S/N = southern/northern static patch, F/P = future/past wedge; slices of constant T (gray) and r (blue/red) = flows generated by H. Yellow dot = horizon $r=1$. d: Wick-rotation of static patch S to S^{d+1}; slices of constant τ and constant r.

Below we set $\boxed{\ell \equiv 1}$. The isometry group is $\mathrm{SO}(1,d+1)$, with generators $M_{IJ} = X_I \partial_J - X_J \partial_I$. Various coordinate patches are shown in fig. C.3.1a,c, with coordinates and metric given by

co	embedding $(X^0,\dots,X^{d+1})$	coordinate range	metric $ds^2 = \eta_{IJ} dX^I dX^J$
G	$(\sinh \bar{T}, \cosh \bar{T}\,\bar{\Omega})$	$\bar{T}\in\mathbb{R},\ \bar{\Omega}\in S^d$	$-d\bar{T}^2+\cosh^2\bar{T}\,d\bar{\Omega}^2$
S	$(\sqrt{1-r^2}\sinh T, r\Omega, \sqrt{1-r^2}\cosh T)$	$T\in\mathbb{R}, 0\le r<1,\ \Omega\in S^{d-1}$	$-(1-r^2)dT^2+\frac{dr^2}{1-r^2}+r^2 d\Omega^2$
F	$(\sqrt{r^2-1}\cosh T, r\Omega, \sqrt{r^2-1}\sinh T)$	$T\in\mathbb{R},\ r>1,\ \Omega\in S^{d-1}$	$-\frac{dr^2}{r^2-1}+(r^2-1)dT^2+r^2 d\Omega^2$

(C.3.5)

illustrated in fig. C.3.1a,c. N is obtained from S by $X^{d+1}\to -X^{d+1}$, and P from F by $X^0 \to -X^0$. The southern static patch S is the part of de Sitter causally accessible to an inertial observer at the south pole of the global spatial S^d. The metric in this patch is static, with the observer at $r=0$ and a horizon at $r=1$. The $\mathrm{SO}(1,1)$ generator $H = M_{0,d+1}$ acts by translation of the coordinate T, which is timelike in S,N and spacelike in F,P. From the direction of the flow lines in fig. C.3.1c, it can be seen that the positive energy operator is H in S, whereas it is $-H$ in N. In F/P, r is the time coordinate, and H is the operator corresponding to spatial momentum along the T-axis of the $\mathbb{R}\times S^{d-1}$ spatial slices.

A Wick rotation $X^0 \to -iX^0$ maps (C.3.4) to the round sphere S^{d+1}:

$$\delta_{IJ} X^I X^J = \ell^2\,, \qquad ds^2 = \delta_{IJ}\, dX^I dX^J\,. \tag{C.3.6}$$

The full S^{d+1} can be obtained either from global dS G by Wick rotating global time $\bar{T}\to -i\bar{\tau}$, or from a single static patch S by Wick rotating static time $T\to -i\tau$, as illustrated in fig.

C.3.1b,d. The corresponding sphere coordinates and metric are, again setting $\boxed{\ell \equiv 1}$

co	embedding $(X^0, X^1, \dots, X^{d+1})$	coordinate range	metric $ds^2 = \delta_{IJ} dX^I dX^J$
G	$(\sin\bar\tau, \cos\bar\tau, \bar\Omega)$	$-\frac{\pi}{2} \le \bar\tau \le \frac{\pi}{2}\,, \bar\Omega \in S^d$	$d\bar\tau^2 + \cos^2\bar\tau\, d\bar\Omega^2$
S	$(\sqrt{1-r^2}\sin\tau, r\Omega, \sqrt{1-r^2}\cos\tau)$	$0 \le r < 1, \tau \simeq \tau + 2\pi,\ \Omega \in S^{d-1}$	$(1-r^2)d\tau^2 + \frac{dr^2}{1-r^2} + r^2 d\Omega^2$

(C.3.7)

C.4 Euclidean vs canonical: formal & physics expectations

Given a QFT on a static spacetime $\mathbb{R} \times M$ with metric $ds^2 = -dt^2 + ds_M^2$, Wick rotating $t \to -i\tau$ yields a Euclidean QFT on a space with metric $ds^2 = d\tau^2 + ds_M^2$. The Euclidean path integral $Z_{\rm PI}(\beta) = \int \mathcal{D}\Phi\, e^{-S[\Phi]}$ on $S^1_\beta \times M$ obtained by identifying $\tau \simeq \tau + \beta$ equals the thermal partition function: $Z_{\rm PI}(\beta) = {\rm Tr}\, e^{-\beta H}$, as follows from cutting the path integral along constant-τ slices and viewing $e^{-\tau H}$ as the Euclidean time evolution operator.

At least for noninteracting theories, it is in practice much more straightforward to compute the partition function as the state sum ${\rm Tr}\, e^{-\beta H}$ of an ideal gas in a box M than as a one-loop path integral $Z_{\rm PI} = \int \mathcal{D}\Phi\, e^{-S[\Phi]}$ on $S^1_\beta \times M$, in particular for higher-spin fields. In view of this, it is reasonable to wonder if a free QFT path integral on the sphere could perhaps similarly be computed as a simple state sum, by viewing the sphere as the Wick-rotated static patch (fig. C.3.1d), with inverse temperature $\beta = 2\pi$ given by the period of the angular coordinate τ:

$$Z_{\rm PI} \overset{?}{=} {\rm Tr}_S\, e^{-2\pi H}\,. \tag{C.4.1}$$

Below we review the formal path integral slicing argument suggesting the above relation and why it fails, emphasizing the culprit is the presence of a fixed-point locus of H, the yellow dot in fig. C.3.1. At the same formal level, we show the above relation is equivalent to $Z_{\rm PI} \overset{?}{=} Z_{\rm bulk}$, with $Z_{\rm bulk}$ defined as a character integral as in section 5.2. This improves the situation, but is still incorrect for spin $s \ge 1$. In more detail, the content is as follows:

In C.4.1 we consider the $d = 0$ case: a scalar of mass ω on dS$_1$ in its Euclidean vacuum state, i.e. an entangled pair of harmonic oscillators. Though surely superfluous to most readers, we use the occasion to provide a pedagogical introduction to some standard constructions.

In C.4.2 we formally apply the same template to general d, ignoring yellow-dot issues, leading to the standard formal "thermofield double" description of the static patch of de Sitter [175], and more specifically to $Z_{\rm PI} \simeq {\rm Tr}\, e^{-2\pi H} \simeq Z_{\rm bulk}$. We review the pathological divergences that ensue when one attempts to evaluate the trace, and some of its proposed fixes such as the "brick-wall" cutoff [162] and refinements thereof. We contrast these to $Z_{\rm bulk}$ defined as a character integral.

In C.4.5, we turn to the edge corrections missed by such formal arguments, explaining from various points of view why they are to be expected.

C.4.1 S^1

Though slightly silly, it is instructive to first consider the $d = 0$ case: a free scalar field of mass ω on dS$_1$ (fig. C.4.1). Global dS$_1$ is the hyperbola $X_0^2 - X_1^2 = 1$ according to (C.3.4), which consists of two causally disconnected lines, globally parametrized according to table C.3.5 by $(\bar{T}, \bar{\Omega})$ where $\bar{\Omega} \in S^0 = \{-1, +1\} \equiv \{N, S\}$. The pictures of fig. C.3.1 still apply, except there are no interior points, resulting in fig. C.4.1. Putting a free scalar of mass ω on this space just means we consider two harmonic oscillators ϕ_S and ϕ_N, with action

$$S_L = \frac{1}{2} \int_{-\infty}^{\infty} d\bar{T} \big(\dot{\phi}_S^2 - \omega^2 \phi_S^2 + \dot{\phi}_N^2 - \omega^2 \phi_N^2 \big) \, . \tag{C.4.2}$$

The dS$_1$ isometry group is SO$(1, d+1) =$ SO$(1,1)$, generated by $H \equiv M_{01}$, which acts as forward/backward time translations on ϕ_S/ϕ_N, to be contrasted with the global *Hamiltonian* H', which acts as forward time translations on both. The southern and northern static patch are parametrized by T, and each contains one harmonic oscillator, respectively ϕ_S and ϕ_N. Introducing creation and annihilation operators $a_\omega^S, a_\omega^{S\dagger}, a_{-\omega}^N, a_{-\omega}^{N\dagger}$ satisfying $[a, a^\dagger] = 1$, we have

$$H = H_S - H_N \, , \quad H' = H_S + H_N \, , \qquad H_S = \omega \big(a_\omega^{S\dagger} a_\omega^S + \tfrac{1}{2} \big) \, , \quad H_N = \omega \big(a_{-\omega}^{N\dagger} a_{-\omega}^N + \tfrac{1}{2} \big) \, . \tag{C.4.3}$$

The subscript $\pm\omega$ refers to the H eigenvalue: $[H, a_{\pm\omega}^\dagger] = \pm\omega \, a_{\pm\omega}^\dagger$, $[H, a_{\pm\omega}] = \mp\omega \, a_{\pm\omega}$. The southern and northern Hilbert spaces $\mathcal{H}_S$, $\mathcal{H}_N$ each have a positive energy eigenbasis $|n)$ with energies $E_n = (n + \frac{1}{2}) \, \omega$. In QFT language, $|0)$ is the static patch "vacuum", and each patch has one "single-particle" state, $|1) = a^\dagger |0)$. The global Hilbert space is $\mathcal{H}_G = \mathcal{H}_S \otimes \mathcal{H}_N$, with

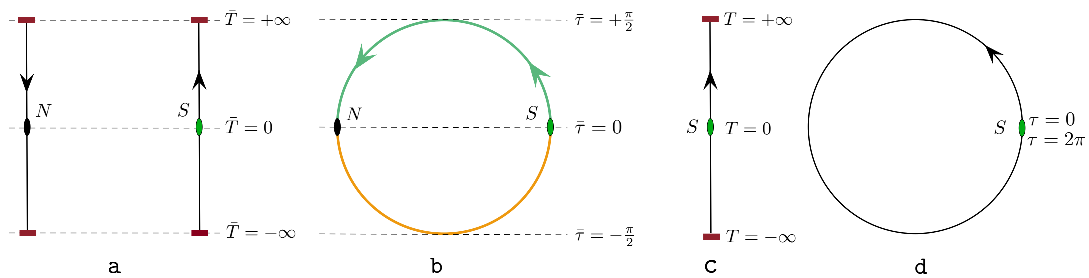


Figure C.4.1: dS_1 version of fig. C.3.1 (in c we only show S here). Wick rotation of global time $\bar{T} \to -i\bar{\tau}$ maps a $\to$ b while wick rotation of static patch time $T \to -i\tau$ maps c $\to$ d. Coordinates are as defined in tables C.3.5 and C.3.7 with $d = 0$.

basis $|n_S, n_N\rangle = |n_S) \otimes |n_N)$ satisfying $H|n_S, n_N\rangle = \omega(n_S - n_N)|n_S, n_N\rangle$.

Wick-rotating dS_1 produces an S^1 of radius $\ell = 1$. If we consider this as the Wick rotation of the static patch as in fig. C.3.1d/C.4.1d, S in table (C.3.7), the S^1 is parametrized by the periodic Euclidean time coordinate $\tau \simeq \tau + 2\pi$. The corresponding Euclidean action for the scalar is

$$S_E = \frac{1}{2}\int_0^{2\pi} d\tau\,(\dot{\phi}^2 + \omega^2\phi^2) \qquad \phi(2\pi) = \phi(0)\,. \tag{C.4.4}$$

The Euclidean path integral Z_{PI} on S^1 is most easily computed by reverting to the canonical formalism with $e^{-\tau H_S} = e^{-\tau H}$ as the Euclidean time evolution operator, which maps it to the harmonic oscillator thermal partition function at inverse temperature $\beta = 2\pi$:

$$Z_{\mathrm{PI}} = \int \mathcal{D}\phi\, e^{-S_E[\phi]} = \mathrm{Tr}_{\mathcal{H}_S}\, e^{-2\pi H} = \sum_n e^{-2\pi\omega(n+\frac{1}{2})} = \frac{e^{-2\pi\omega/2}}{1 - e^{-2\pi\omega}}\,. \tag{C.4.5}$$

We can alternatively consider the S^1 to be obtained as the Wick rotation of *global* dS_1 as in fig. C.3.1b/C.4.1b, G in (C.3.7), parametrizing the S^1 by $(\bar{\tau}, \bar{\Omega})$, $-\frac{\pi}{2} \le \bar{\tau} \le \frac{\pi}{2}$, $\bar{\Omega} \in S^0 = \{S, N\}$, identifying $(\pm\frac{\pi}{2}, S) = (\pm\frac{\pi}{2}, N)$. The global action (C.4.2) then Wick rotates to

$$S_E = \frac{1}{2}\int_{-\pi/2}^{\pi/2} d\bar{\tau}\,(\dot{\phi}_S^2 + \omega^2\phi_S^2 + \dot{\phi}_N^2 + \omega^2\phi_N^2)\,, \qquad \phi_S(\pm\tfrac{\pi}{2}) = \phi_N(\pm\tfrac{\pi}{2})\,, \tag{C.4.6}$$

which is identical to (C.4.4), just written in a slightly more awkward form. This form naturally leads to an interpretation of Z_{PI} as computing the norm squared of the Euclidean vacuum state $|O\rangle$ of the scalar on the *global* dS_1 Hilbert space $\mathcal{H}_G$, by cutting the path integral at the

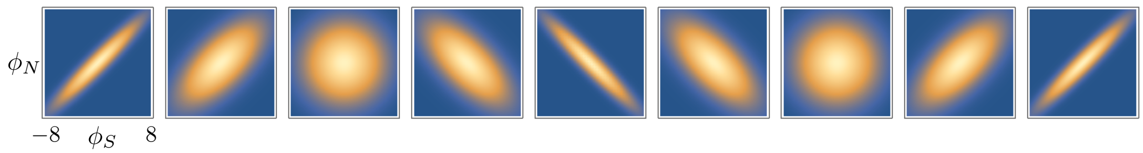


Figure C.4.2: Global time evolution of $P_{\bar{T}}(\phi_S, \phi_N) = \left|\langle \phi_S, \phi_N | e^{-iH'\bar{T}} | O\rangle\right|^2$ for free $\omega = 0.1$ scalar on dS_1, from $\bar{T} = 0$ to $\bar{T} = \pi/\omega$. $P(\phi_S) = \int d\phi_N \, P_{\bar{T}}(\phi_S, \phi_N)$ is thermal and time-independent.

$S^0 = \{N, S\}$ equator $\bar{\tau} = 0$ of the S^1 (cf. fig. C.4.1b):

$$Z_{\rm PI} = \int d^2\phi_0 \, \langle O|\phi_0\rangle\langle\phi_0|O\rangle \equiv \langle O|O\rangle \,, \qquad \langle\phi_0|O\rangle \equiv \int_{\bar{\tau}\leq 0} \mathcal{D}\phi|_{\phi_0} \, e^{-S_E[\phi]} \,, \tag{C.4.7}$$

where $\phi_0 = (\phi_{S,0}, \phi_{N,0})$. The notation $\int_{\bar{\tau}\leq 0} \mathcal{D}\phi|_{\phi_0}$ means the path integral of $\phi = (\phi_S, \phi_N)$ is performed on the lower hemicircle $\bar{\tau} \leq 0$ (orange part in fig. C.4.1b), with boundary conditions $\phi|_{\bar{\tau}=0} = \phi_0$. $\langle O|\phi_0\rangle$ is similarly defined as a path integral on the upper hemicircle (green part). It is not too difficult to explicitly compute $|O\rangle$ in the $|\phi_{S,0}, \phi_{N,0}\rangle$ basis, but it is easier to compute it in the oscillator basis $|n_S, n_N\rangle$, noticing that slicing the path integral defining $|O\rangle$ allows us to write it as $\langle n_S, n_N|O\rangle = (n_S|e^{-\pi H}|n_N) = e^{-\pi\omega(n_S+\frac{1}{2})} \, \delta_{n_S,n_N}$. Thus

$$|O\rangle = \sum_n e^{-\pi\omega(n+\frac{1}{2})}|n, n\rangle = e^{-\pi\omega/2} \exp(e^{-\pi\omega} a^{S\dagger}_{\omega} a^{N\dagger}_{-\omega})|0, 0\rangle \,. \tag{C.4.8}$$

In the Schrödinger picture, $|O\rangle$ is to be thought of as an initial state at $\bar{T} = 0$ for global dS_1: pictorially, we are gluing the bottom half of fig. C.4.1b to the top half of fig. C.4.1a. This state evolves nontrivially in global time $\bar{T}$: though invariant under $SO(1, 1)$ generated by $H = H_S - H_N$, it is *not* invariant under forward global time translations generated by the global Hamiltonian $H' = H_S + H_N$. For viewing pleasure this is illustrated in fig. C.4.2, which also visually exhibits the north-south entangled nature of $|O\rangle$.

Note that $Z_{\rm PI} = \langle O|O\rangle = \sum_n e^{-2\pi\omega(n+\frac{1}{2})}$, reproducing the dS_1 static patch thermal partition function (C.4.18). Indeed from the point of view of the static patch, the global Euclidean vacuum state looks thermal with inverse temperature $\beta = 2\pi$: the southern reduced density matrix $\hat{\varrho}_S$ obtained by tracing out the northern degree of freedom ϕ_N in the global Euclidean vacuum $|O\rangle$ is $\hat{\varrho}_S = \sum_n e^{-2\pi\omega(n+\frac{1}{2})}|n)(n| = e^{-2\pi H_S}$. In contrast to the global $|O\rangle$, the reduced density matrix is time-*independent*.

The path integral slicing arguments we used did not rely on the precise form of the action. In particular the conclusions remain valid when we add interactions:

$$|O\rangle = \sum_n e^{-\beta E_n/2}|n,n\rangle\,, \qquad \hat{\varrho}_S = e^{-\beta H_S}\,, \qquad Z_{\rm PI} = \langle O|O\rangle = {\rm Tr}_S\, e^{-\beta H} \qquad (\beta = 2\pi) \tag{C.4.9}$$

Actually in the $d = 0$ case at hand, we can generalize all of the above to arbitrary values of β. (For $d > 0$ this would create a conical singularity at $r = 1$ on S^{d+1}, but for S^1 the point $r = 1$ does not exist.) Note that since the reduced density matrix is thermal, the north-south entanglement entropy in the Euclidean vacuum $|O\rangle$ equals the thermal entropy: $S_{\rm ent} = -{\rm tr}_S\, \varrho_S \log \varrho_S = S_{\rm th} = (1 - \beta\partial_\beta)\log Z$, where $\varrho_S \equiv \hat{\varrho}_S/Z$, $Z = {\rm Tr}_S\, \hat{\varrho}_S$.

Despite appearing distinctly non-vacuous from the point of view of a local observer, and being globally time-dependent, the state $|O\rangle$ does deserve its "vacuum" epithet. As already mentioned, it is invariant under the global $SO(1,1)$ isometry group: $H|O\rangle = 0$. Moreover, for the free scalar, (C.4.8) implies $|O\rangle$ is itself annihilated by a pair of *global* annihilation operators a^G related related to a^S, $a^{S\dagger}$, a^N and $a^{N\dagger}$ by a Bogoliubov transformation:

$$a^G_{\pm\omega}|O\rangle = 0\,, \qquad a^G_\omega \equiv \frac{a^S_\omega - e^{-\pi\omega} a^{N\dagger}_{-\omega}}{\sqrt{1 - e^{-2\pi\omega}}}\,, \quad a^G_{-\omega} \equiv \frac{a^N_{-\omega} - e^{-\pi\omega} a^{S\dagger}_\omega}{\sqrt{1 - e^{-2\pi\omega}}}\,, \tag{C.4.10}$$

normalized such that $[a^G_{\pm\omega}, a^{G\dagger}_{\pm\omega}] = \delta_{\pm,\pm}$. From (C.4.3) we get $H = \omega\, a^{G\dagger}_\omega a^G_\omega - \omega\, a^{G\dagger}_{-\omega} a^G_{-\omega}$. Thus we can construct the global Hilbert space $\mathcal{H}_G$ as a Fock space built on the Fock vacuum $|O\rangle$, by acting with the global creation operators $a^{G\dagger}_{\pm\omega}$. The Hilbert space $\mathcal{H}^{(1)}_G$ of "single-particle" excitations of the global Euclidean vacuum is two-dimensional, spanned by

$$|\pm\omega\rangle \equiv a^{G\dagger}_{\pm\omega}|O\rangle, \qquad H|\pm\omega\rangle = \pm\omega\,|\pm\omega\rangle\,. \tag{C.4.11}$$

The character $\Theta(t)$ of the $\mathrm{SO}(1,1)$ representation furnished by $\mathcal{H}^{(1)}_G$ is

$$\Theta(t) \equiv {\rm tr}_G\, e^{-itH} = e^{-it\omega} + e^{it\omega}\,. \tag{C.4.12}$$

The above constructions are straightforwardly generalized to fermionic oscillators. The character of a collection of bosonic and fermionic oscillators of frequencies ω_i and ω'_j is

$$\Theta(t) = \mathrm{tr}_G \, e^{-iHt} = \Theta(t)_{\mathrm{bos}} + \Theta(t)_{\mathrm{fer}} = \sum_{i,\pm} e^{\pm i\omega_i} + \sum_{j,\pm} e^{\pm i\omega'_j} \,. \tag{C.4.13}$$

Character formula

For a single bosonic resp. fermionic oscillator of frequency ω, $\log \mathrm{Tr} \, e^{-\beta H}$ has the following integral representation:[4]

$$\begin{aligned} \log\left(e^{-\beta\omega/2}(1-e^{-\beta\omega})^{-1}\right) &= +\int_0^\infty \frac{dt}{2t} \, \frac{1+e^{-2\pi t/\beta}}{1-e^{-2\pi t/\beta}} \, (e^{-i\omega t} + e^{i\omega t}) \\ \log\left(e^{+\beta\omega/2}(1+e^{-\beta\omega})\right) &= -\int_0^\infty \frac{dt}{2t} \, \frac{2\, e^{-\pi t/\beta}}{1-e^{-2\pi t/\beta}} \, (e^{-i\omega t} + e^{i\omega t}) \,. \end{aligned} \tag{C.4.14}$$

Combining this with (C.4.13) expresses the thermal partition function of a collection of bosonic and harmonic oscillators as an integral transform of its $SO(1,1)$ character:

$$\log \mathrm{Tr} \, e^{-\beta H} = \int_0^\infty \frac{dt}{2t}\left(\frac{1+e^{-2\pi t/\beta}}{1-e^{-2\pi t/\beta}} \, \Theta(t)_{\mathrm{bos}} \, - \, \frac{2\, e^{-\pi t/\beta}}{1-e^{-2\pi t/\beta}} \, \Theta(t)_{\mathrm{fer}}\right). \tag{C.4.15}$$

The Euclidean path integral on an S^1 of radius $\ell = 1$ for a collection of free bosons and fermions (the latter with thermal, i.e. antiperiodic, boundary conditions) is then given by putting $\beta = 2\pi$ in the above:

$$\log Z_{\mathrm{PI}} = \int_0^\infty \frac{dt}{2t}\left(\frac{1+e^{-t}}{1-e^{-t}} \, \Theta(t)_{\mathrm{bos}} \, - \, \frac{2\, e^{-t/2}}{1-e^{-t}} \, \Theta(t)_{\mathrm{fer}}\right). \tag{C.4.16}$$

C.4.2 S^{d+1}

The arguments in this section will be formal, following the template of section C.4.1 while glossing over some important subtleties, the consequence of which we discuss in section C.4.5.

Wick-rotating a QFT on dS$_{d+1}$ to S^{d+1}, we get the Euclidean path integral

$$Z_{\mathrm{PI}} = \int \mathcal{D}\Phi \, e^{-S_E[\Phi]} \,, \tag{C.4.17}$$

[4] The t^{-2} pole of the integrand is resolved by the $i\epsilon$-prescription $t^{-2} \to \frac{1}{2}\left((t-i\epsilon)^{-2} + (t+i\epsilon)^{-2}\right)$, left implicit here and in the formulae below. The integral formula can be checked by observing the integrand is even in t, extending the integration contour to the real line, closing the contour, and summing residues.

where Φ collects all fields in the theory. Just like in the $d = 0$ case, the two different paths from dS$_{d+1}$ to S^{d+1}, i.e. Wick-rotating global time $\bar{T}$ or static patch time T (cf. fig. C.3.1 and table C.3.7), naturally give rise to two different dS Hilbert space interpretations: one involving the global Hilbert space $\mathcal{H}_G$ and one involving the static patch Hilbert space $\mathcal{H}_S$.

The global Wick rotation of fig. C.3.1b leads to an interpretation of $Z_{\rm PI}$ as computing $\langle O|O\rangle$, analogous to (C.4.7), by cutting the path integral on the globlal S^d equator $\bar{\tau} = 0$:

$$Z_{\rm PI} = \int_{\bar{\tau}=0} d\Phi_0 \, \langle O|\Phi_0\rangle\langle\Phi_0|O\rangle \equiv \langle O|O\rangle \, , \qquad \langle\Phi_0|O\rangle \equiv \int_{\bar{\tau}\leq 0} \mathcal{D}\Phi|_{\Phi_0} \, e^{-S_E[\Phi]} \, , \qquad \text{(C.4.18)}$$

where $\int_{\bar{\tau}\leq 0} \mathcal{D}\phi|_{\phi_0}$ means the path integral is performed on the lower hemisphere $\bar{\tau} \leq 0$ of S^{d+1} (orange region in fig. C.3.1b) with boundary conditions $\Phi|_{\bar{\tau}=0} = \Phi_0$. $\langle O|\Phi_0\rangle$ is similarly defined as a path integral on the upper hemisphere $\bar{\tau} \geq 0$ (green region). This defines the Hartle-Hawking/Euclidean vacuum state $|O\rangle$ [237] of global dS$_{d+1}$, with $Z_{\rm PI}$ computing the natural pairing of $|O\rangle$ with $\langle O|$.[5]

The static patch Wick rotation of fig. C.3.1d on the other hand leads to an interpretation of $Z_{\rm PI}$ as a thermal partition function at inverse temperature $\beta = 2\pi$, analogous to (C.4.5): slicing the path integral along constant-τ slices as in fig. C.3.1d, and viewing $e^{-\tau H}$ with $H = M_{0,d+1}$ as the Euclidean time evolution operator acting on $\mathcal{H}_S$, we formally get[6]

$$Z_{\rm PI} \simeq {\rm Tr}_{\mathcal{H}_S} \, e^{-\beta H} \qquad (\beta = 2\pi) \, . \qquad \text{(C.4.19)}$$

Like in the $d = 0$ case, this interpretation can be related to the global interpretation (C.4.18). Picking suitable bases of $\mathcal{H}_S$ and $\mathcal{H}_N$ diagonalizing H, and applying a similar slicing argument,

[5]For kind enough theories, such as a scalar field theory, this pairing can be identified with the Hilbert space inner product. However not all theories are kind enough, as is evident from the negative-mode rotation phase $i^{-(d+3)}$ in the one-loop graviton contribution to $Z_{\rm PI} = \langle O|O\rangle$ according to (5.5.17) and [131]. Indeed for gravity this pairing is not in an obvious way related to the semiclassical inner product of [238]. On the other hand, in the CS formulation of 3D gravity it appears to be framing-dependent, vanishing in particular for canonical framing (cf. (C.7.28) and discussion below it). The phase also drops out of $\langle A\rangle \equiv \langle O|A|O\rangle/\langle O|O\rangle$.

[6] The notation $\simeq$ means "equal according to these formal arguments". Besides the default deferment of dealing with divergences, we are ignoring some additional important points here, including in particular the fixed points of H: the S^{d-1} at $r = 1$ (yellow dot in fig. C.3.1), where the equal-τ slicing of (C.4.19) degenerates, and the $\mathcal{H}_G = \mathcal{H}_N \otimes \mathcal{H}_S$ factorization implicit in (C.4.20) breaks down. We return to these points in section C.4.5.

we formally get the analog of (C.4.9):

$$|O\rangle \simeq \text{``}\sum_n\text{''} e^{-\beta E_n/2}|E_n, E_n\rangle\,, \qquad \hat\varrho_S \simeq e^{-\beta H_S} \qquad (\beta = 2\pi)\,, \tag{C.4.20}$$

where we have put the sum in quotation marks because the spectrum is actually continuous, as we will describe more precisely for free QFTs below. Granting this, we conclude that an inertial observer in de Sitter space sees the global Euclidean vacuum as a thermal state at inverse temperature $\beta = 2\pi$, the Hawking temperature of the observer's horizon [17, 57, 175].

Applying (C.4.19) to a free QFT on dS$_{d+1}$, we can write the corresponding Gaussian $Z_{\rm PI}$ on S^{d+1} as the thermal partition function of an ideal gas in the southern static patch:

$$\log Z_{\rm PI} \simeq \log {\rm Tr}_S\, e^{-2\pi H} = \sum_\pm \mp \int_0^\infty d\omega\, \rho_S(\omega)_\pm \big(\log(1 \mp e^{-2\pi\omega}) + 2\pi\omega/2\big)\,, \tag{C.4.21}$$

where $\rho_S(\omega) \equiv {\rm tr}_S\, \delta(\omega - H)$ is the density of single-particle states at energy $\omega > 0$ above the vacuum energy in the static patch, split into bosonic and fermionic parts as $\rho_S = \rho_{S+} + \rho_{S-}$. Using $\Theta(t) = \Theta(-t)$, we can write the character for arbitrary SO$(1, d+1)$ representations as

$$\Theta(t) = {\rm tr}_G\, e^{-itH} = \int_0^\infty \rho_G(\omega)\,(e^{-i\omega t} + e^{i\omega t}) \tag{C.4.22}$$

where $\rho_G(\omega) \equiv {\rm tr}_G\, \delta(\omega - H)$. The Bogoliubov map (C.4.10) formally implies $\rho_G(\omega) \simeq \rho_S(\omega)$ for $\omega > 0$, hence, following the reasoning leading to (C.4.16),

$$\log Z_{\rm PI} \simeq \log Z_{\rm bulk} \equiv \int_0^\infty \frac{dt}{2t}\bigg(\frac{1+e^{-t}}{1-e^{-t}}\;\Theta(t)_{\rm bos} \,-\, \frac{2\,e^{-t/2}}{1-e^{-t}}\;\Theta(t)_{\rm fer}\bigg)\,. \tag{C.4.23}$$

C.4.3 Brick wall regularization

Here we review how attempts at evaluating the ideal gas partition function (C.4.21) directly hit a brick wall. Consider for example a scalar field of mass m^2 on dS$_{d+1}$. Denoting $\Delta_\pm = \frac{d}{2} \pm \big((\frac{d}{2})^2 - m^2\big)^{1/2}$, the positive frequency solutions on the static patch are of the form

$$\phi_{\omega\sigma}(T,\Omega,r) \propto e^{-i\omega T}\, Y_\sigma(\Omega)\; r^\ell\, \Big(1-r^2\Big)^{i\omega/2}\, {}_2F_1\big(\tfrac{\ell+\Delta_+ + i\omega}{2}, \tfrac{\ell+\Delta_- + i\omega}{2}; \tfrac{d}{2}+\ell; r^2\big), \tag{C.4.24}$$

where $\omega > 0$, and $Y_\sigma(\Omega)$ is a basis of spherical harmonics on S^{d-1} labeled by σ, which includes the total SO(d) angular momentum quantum number ℓ. A basis of energy and SO(d) angular momentum eigenkets is therefore given by $|\omega\sigma)$ satisfying $(\omega\sigma|\omega'\sigma') = \delta(\omega-\omega')\,\delta_{\sigma\sigma'}$. Naive evaluation of the density of states in this basis gives a pathologically divergent result $\rho_S(\omega) = \int d\omega' \sum_\sigma (\omega'\sigma|\delta(\omega-\omega')|\omega'\sigma) = \sum_\sigma \delta(0)$, and commensurate nonsense in (C.4.21).

Pathological divergences of this type are generic in the presence of a horizon. Physically they can be thought of as arising from the fact that the infinite horizon redshift enables the existence of field modes with arbitrary angular momentum and energy localized in the vicinity of the horizon. One way one therefore tries to deal with this is to replace the horizon by a "brick wall" at a distance δ away from the horizon [162], with some choice of boundary conditions, say $\phi(T,\Omega,1-\frac{1}{2}\delta^2) = 0$ in the example above. This discretizes the energy spectrum and lifts the infinite angular momentum degeneracy, allowing in principle to control the divergences as $\delta \to 0$. However, inserting a brick wall alters what one is actually computing, introduces ambiguities (e.g. Dirichlet/Neumann), potentially leads to new pathologies (e.g. Dirichlet boundary conditions for the graviton are not elliptic [239]), and breaks most of the symmetries in the problem.

A more refined version of the idea considers the QFT in Pauli-Villars regularization [177]. This eliminates the dependence on δ in the limit $\delta \to 0$ at fixed PV-regulator scale Λ. It was shown in [177] that for scalar fields the remaining divergences for $\Lambda \to \infty$ agree with those of the PV-regulated path integral.[7] A somewhat different approach, reviewed in [150, 240], first maps the equations of motion in the metric $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ by a (singular) Weyl transformation to formally equivalent equations of motion in the "optical" metric $d\bar{s}^2 = |g_{00}|^{-1}ds^2$. In the case at hand this would be $d\bar{s}^2 = -dT^2 + (1-r^2)^{-2}dr^2 + (1-r^2)^{-1}r^2 d\Omega^2$, corresponding to $\mathbb{R} \times$ hyperbolic d-ball. The thermal trace is then mapped to a path integral on the Euclidean optical geometry with an S^1 of constant radius β and a Weyl-transformed action. (This is not a standard covariant path integral. In the case at hand, unless the theory happens to be conformal, non-metric r-dependent terms break the SO$(1,d)$ symmetry of the hyperbolic ball to SO(d).) This path integral can be expressed in terms of a heat kernel trace $\int_x \langle x|e^{-\tau\bar{D}}|x\rangle$. The divergences encountered earlier now arise from the fact that the optical metric $d\bar{s}^2$ has infinite volume near $r=1$. This is regularized by cutting the $\int_x$ integral off at $r = 1-\delta$,

[7] This work directly inspired the use of Pauli-Villars regularization in section 5.2.

analogous to the brick wall cutoff, though computationally more convenient. For scalars and spinors, Pauli-Villars or dimensional regularization again allows trading the $\delta \to 0$ divergences for the standard UV divergences [240].

Unfortunately, certainly for general field content and in the absence of conformal invariance, none of these variants offers any simplification compared to conventional Euclidean path integral methods. In the case of interest to us, the large underlying SO$(1, d+1)$ symmetry is broken, and with it one's hope for easy access to exact results. Generalization to higher-spin fields, or even just the graviton, appears challenging at best.

C.4.4 Character regularization

The character formula (C.4.23) is formally equivalent to the ideal gas partition function (C.4.21), and indeed at first sight, naive evaluation in a global single-particle basis $|\omega\sigma\rangle = a^{G\dagger}_{\omega\sigma}|0\rangle$ diagonalizing $H = \omega \in \mathbb{R}$, obtained e.g. by quantization of the natural cylindrical mode functions of the future wedge (F in fig. C.3.1 and table C.3.5), gives a similarly pathological $\Theta(t) = \mathrm{tr}_G\, e^{-iHt} = \int_{-\infty}^{\infty} d\omega \sum_\sigma \langle\omega\sigma|e^{-i\omega t}|\omega\sigma\rangle = 2\pi\delta(t)\sum_\sigma \delta(0)$; hardly a surprise in view of the Bogoliubov relation $\rho_G(\omega) \simeq \rho_S(\omega)$ and our earlier result $\rho_S(\omega) = \sum_\sigma \delta(0)$. Thus the conclusion would appear to be that the situation is as bad, if not worse, than it was before.

However this is very much the wrong conclusion. As reviewed in the section 3.4, $\Theta(t)$, properly defined as a Harish-Chandra character, is in fact rigorously well-defined, analytic in t for $t \neq 0$, and moreover easily computed. For example for a scalar of mass m^2 on dS$_{d+1}$, we get (3.4.7:

$$\Theta(t) = \frac{e^{-t\Delta_+} + e^{-t\Delta_-}}{|1 - e^{-t}|^d} \qquad \Delta_\pm \equiv \tfrac{d}{2} \pm \sqrt{\left(\tfrac{d}{2}\right)^2 - m^2} \tag{C.4.25}$$

as explicitly computed in 3.4.2. The reason why naive computation by diagonalization of H fails so badly is explained in detail in the remark 8: it is not the trace itself that is sick, but rather the *basis* $|\omega\sigma\rangle$ used in the naive computation.

Substituting the explicit $\Theta(t)$ into the character integral (C.4.23), we still get a UV-divergent result, but this divergence is now easily regularized in a standard, manifestly covariant way, as explained in section 5.2.2. Keeping the large underlying symmetry manifest allows exact evaluation, for arbitrary particle content.

In section 5.3 we show that for scalars and spinors, the Euclidean path integral $Z_{\rm PI}$ on S^{d+1}, regularized as in (5.3.2), exactly equals $Z_{\rm bulk}$ as defined in (C.4.23), regularized as in (5.3.9):

$$Z_{{\rm PI},\epsilon} = Z_{{\rm bulk},\epsilon} \qquad \text{(scalars and spinors)}\,, \tag{C.4.26}$$

One might wonder how it is possible the switch to characters makes such a dramatic difference. After all, (C.4.21) and (C.4.23) are formally equal. Yet the former first evaluates to nonsense and then hits a brick wall, while the latter somehow ends up effortlessly producing sensible results upon standard UV regularization. The discussion in the remark 8, provides some clues: character regularization can be thought of, roughly speaking, as being akin to a regularization cutting off *global* SO$(d+1)$ angular momentum.

This goes some way towards explaining why the character formalism fits naturally with the Euclidean path integral formalism on S^{d+1}, as covariant (e.g. heat kernel) regularization of the latter effectively cuts off the SO$(d+2) \supset$ SO$(d+1)$ angular momentum.

It also goes some way towards explaining what happened above. One way of thinking about the origin of the pathological divergences encountered in section C.4.3 is that, as mentioned in footnote 6, the formal argument implicitly starts from the premise that the QFT Hilbert space can be factorized as $\mathcal{H}_G = \mathcal{H}_S \otimes \mathcal{H}_N$, like in the S^1 toy model. However this cannot be done in the continuum limit of QFT: locally factorized states, such as the formal state $|O) \otimes |O)$ in which both the southern and the northern static patch are in their minimal energy state, are violently singular objects [241]. Cutting off the global SO$(d+1)$ angular momentum does indeed smooth out the sharp north-south divide: SO$(d+1)$ is the isometry group of the global spatial slice at $\bar{T} = 0$ (fig. C.3.1b). The angular momentum cutoff means we only have a finite number of spherical harmonics available to build our field modes. This makes it impossible in particular to build field modes sharply localized in the southern or northern hemisphere: the harmonic expansion of a localized mode always has infinitely many terms. Cutting off this expansion will necessarily leave some support on the other hemisphere. Quite similar in this way again to the Euclidean path integral, this offers some intuition on why the UV-regularized character integral avoids the pathological divergences induced by sharply cutting space.

C.4.5 Edge corrections

In view of all this and (C.4.26), one might be tempted at this point to jump to the conclusion that the arguments of section C.4.2, while formal and glossing over some subtle points, are apparently good enough to give the right answer provided we use the character formulation, and that likewise $Z_{\rm PI}^{(1)}$ on the sphere for a field of arbitrary spin s, despite its off-shell baroqueness, is just the ideal gas partition function $Z_{\rm bulk}$ on the dS static patch, calculable with on-shell ease: mission accomplished. As further evidence in favor of declaring footnote 6 overly cautious, one might point to the fact that in the context of theories of *quantum gravity*, identifying $Z_{\rm PI}^{\rm grav} = {\rm Tr}_{\mathcal{H}}\, e^{-\beta H}$ elegantly reproduces the thermodynamics of horizons inferred by other means [57], and that such identifications are moreover known to be valid in a quantitatively precise way in many well-understood cases in string theory and AdS-CFT. If the formal argument is good enough for quantum gravity, then surely it is good enough for field theory, one might think.

These naive considerations are wrong: the formal relation $Z_{\rm PI} \simeq Z_{\rm bulk}$ for fields of spin $s \geq 1$ receives "edge" corrections. In sections 5.4 and 5.5, we determine these for massive resp. massless spin-s fields on S^{d+1} by direct computation. The results are eqs. (5.4.7) and (5.5.17). The corrections we find exhibit a concise and suggestive structure: again taking the form of a character formula like (C.4.23), but encoding instead a path integral on a sphere in *two lower* dimensions, i.e. on S^{d-1} rather than S^{d+1}. This S^{d-1} is naturally identified with the horizon $r = 1$, i.e. the edge of the static patch hemisphere, the yellow dot in fig. C.3.1. The results of section 5.7 then imply $S_{\rm PI} \simeq S_{\rm bulk}$ likewise receives edge corrections (besides corrections due to nonminimal coupling to curvature, which arise already for scalars).

Similar edge corrections, to the entropy $S_{\rm PI} \simeq S_{\rm bulk}$ in the conceptually analogous case of Rindler space, were anticipated long ago in [5] and explicitly computed shortly thereafter for massless spin-1 fields in [6]. The result of [6] was more recently revisited in several works including [137, 160], relating it to the local factorization problem of constrained QFT Hilbert spaces [154–159] and given an interpretation in terms of the edge modes arising in this context.

We leave the precise physical interpretation of the explicit edge corrections we obtain in this paper to future work. Below we will review why they were to be expected, and how related corrections can be interpreted in analogous, better-understood contexts in quantum gravity and QFT. We begin by explaining why the quantum gravity argument was misleading and what its correct version actually suggests, first from a boundary CFT point of view in the precise framework of AdS-CFT, then from a bulk point of view in a qualitative picture based on string theory on Rindler space. Finally we return to interpretations within QFT itself, clarifying more directly why the caution expressed in footnote 6 was warranted indeed.

C.4.5.1 AdS-CFT considerations

As mentioned above, there are reasons to believe that in theories of quantum gravity, the identification $Z_{\rm PI}^{\rm grav} = {\rm Tr}_{\mathcal{H}}\, e^{-\beta H}$ is exact as a semiclassical (small-$G_{\rm N}$) expansion.

However, the key point here is that $\mathcal{H}$ is the Hilbert space of the *fundamental* microscopic degrees of freedom, *not* the Hilbert space of the low energy effective field theory. This can be made very concrete in the context of AdS-CFT, where $\mathcal{H}$ has a precise boundary CFT definition. For example for asymptotically Euclidean AdS$_{d+1}$ geometries with $S^1_\beta \times S^{d-1}$ conformal boundary, certain analogs of the formal relations (C.4.19) and (C.4.20) then become exact in the semiclassical/large-N expansion [242, 243]:

$$Z_{\rm PI}^{\rm grav} = {\rm Tr}_{\mathcal{H}}\, e^{-\beta H} = \langle O|O\rangle \,, \qquad |O\rangle = \sum_n e^{-\beta E_n/2} |E_n)_{\mathcal{H}} \otimes |E_n)_{\mathcal{H}} \,. \tag{C.4.27}$$

Crucially, $\mathcal{H}$ here is the complete boundary CFT Hilbert space, and $|O\rangle$ is the Euclidean vacuum state of two disconnected copies of the boundary CFT, constructed exactly like in the dS$_1$ toy model of section C.4.1, but with the hemicircle $\frac{1}{2}S^1$ replaced by $\frac{1}{2}S^1 \times S^{d-1}$. From a semiclassical bulk dual point of view this can be viewed as the Euclidean vacuum of two disconnected copies of global AdS or of the eternal AdS-Schwarzchild geometry [243], depending on whether β lies above or below the Hawking-Page phase transition point β_c [244].

When $\beta > \beta_c$, where $\beta_c \sim O(1)$ assuming the low-energy gravity theory is approximately Einstein with $G_{\rm N} \ll \ell^{d-1} = 1$, $Z_{\rm PI}^{\rm grav}$ is dominated by the thermal EAdS saddle [244], with on-shell action $S_E \equiv 0$, so in the limit $G_{\rm N} \to 0$, $Z_{\rm PI}^{\rm grav} = Z_{\rm PI}^{(1)}$. Thus in this case, the relation $Z_{\rm PI}^{\rm grav} = {\rm Tr}_{\mathcal{H}}\, e^{-\beta H}$ of (C.4.27) indeed implies $Z_{\rm PI}^{(1)}$ equals a statistical mechanical partition function. There is no need to invoke quantum gravity to see this, of course: the thermal S^1 is noncontractible in the bulk geometry, so the bulk path integral slicing argument is free of subtleties, directly implying $Z_{\rm PI}^{(1)}$ equals the partition function ${\rm Tr}\, e^{-\beta H}$ of an ideal gas in global

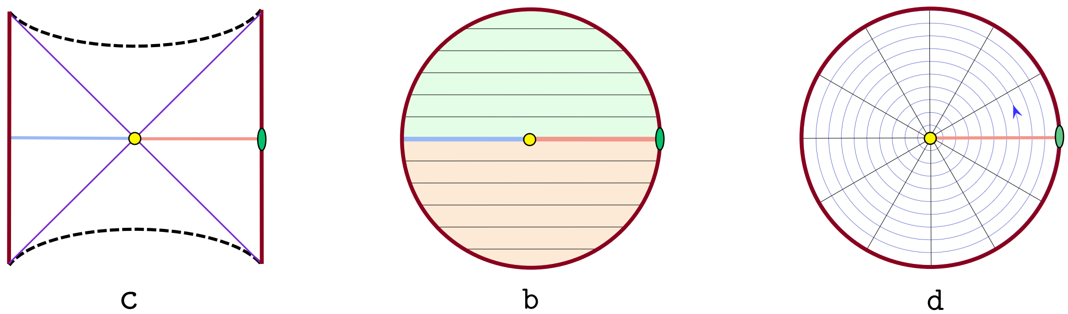


Figure C.4.3: AdS-Schwarzschild analogs of c,b,d in fig. C.3.1. Black dotted line = singularity. Thick brown line = conformal boundary.

AdS.

On the other hand if $\beta < \beta_{\rm crit}$, the dominant saddle is the Euclidean Schwarzschild geometry (fig. C.4.3), with on-shell action $\tilde{S}_E \propto -\frac{1}{G_{\rm N}}$, so in the limit $G_{\rm N} \to 0$, $Z_{\rm PI}^{\rm grav} = Z_{\rm PI}^{(0)} = e^{-\tilde{S}_E}$. In this case the identification $Z_{\rm PI}^{\rm grav} = {\rm Tr}_{\mathcal{H}}\, e^{-\beta H}$ of (C.4.27) no longer implies the one-loop correction $Z_{\rm PI}^{(1)}$ can be identified as a statistical mechanical partition function. In particular the bulk one-loop contributions $S^{(1)} = (1 - \beta\partial_\beta) \log Z_{\rm PI}^{(1)}$ to the entropy need not be positive. (More specifically its leading divergent term, which in a UV-complete description of the bulk theory would become finite but generically still dominant, need not be positive.) From the CFT point of view, these are just $\mathcal{O}(1)$ corrections in the large-N expansion of the statistical entropy. Although the total entropy must of course be positive, corrections can come with either sign. From the bulk point of view, since the Euclidean geometry is the Wick-rotated exterior of a black hole, the thermal circle is contractible, shrinking to a point analogous to the yellow dot in fig. C.3.1d, leading to the same issues as those mentioned in footnote 6.

C.4.5.2 Strings on Rindler considerations

To gain some insight from a bulk point of view, we consider the simplest example of a spacetime with a horizon: the Rindler wedge $ds^2 = -\rho^2 dt^2 + d\rho^2 + dx_\perp^2$ of Minkowski space. While not quite at the level of AdS-CFT, we do have a perturbative theory of quantum gravity in Minkowski space: string theory. In fact, that $Z_{\rm PI}^{(1)}$ on a Euclidean geometry with a contractible thermal circle cannot be interpreted as a statistical mechanical partition function in general, even if the full $Z_{\rm PI}^{\rm grav}$ has such an interpretation, was anticipated long ago in [5], in an influential attempt at developing a string theoretic understanding of the thermodynamics of the Rindler horizon.

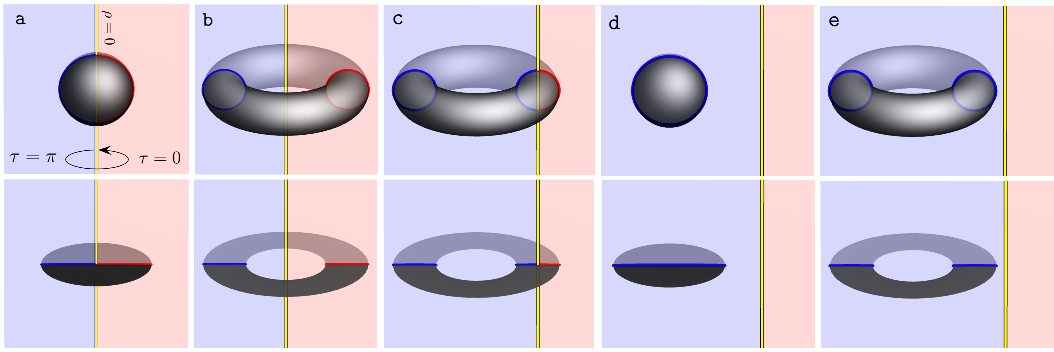


Figure C.4.4: Closed/open string contributions to the total Euclidean Rindler ($ds^2 = \rho^2 d\tau^2 + d\rho^2 + dx^2$) partition function according to the picture of [5]. $\tau =$ angle around yellow axis $\rho = 0$; blue|red plane is $\tau = \pi|0$. a,b,c contribute to the entropy. Sliced along Euclidean time τ, a and b can be viewed as free bulk resp. edge string thermal traces contributing positively to the entropy, while c can be viewed as an edge string emitting and reabsorbing a bulk string, contributing a (negative) interaction term.

Rindler space Wick rotates to

$$ds^2 = \rho^2 d\tau^2 + d\rho^2 + dx_\perp^2, \qquad \tau \simeq \tau + \beta, \qquad \beta = 2\pi - \epsilon \,. \tag{C.4.28}$$

with the conical defect $\epsilon = 0$ on-shell. The argument given in [5] is based on the point of view developed in their work that loop corrections in the semiclassical expansion of the Rindler entropy $S_{\rm PI} \equiv (1 - \beta\partial_\beta) \log Z_{\rm PI}^{\rm grav}|_{\beta=2\pi}$ are equivalent to loop corrections to the Newton constant, ensuring the entropy $S = A/4G_{\rm N}$ involves the physically measured $G_{\rm N}$ rather than than the bare $G_{\rm N}$. In $\mathcal{N} = 4$ compactifications of string theory to 4D Minkowski space (and in $\mathcal{N} = 4$ supergravity theories more generally), loop corrections to the Newton constant vanish. By the above observation, this implies loop corrections to $S_{\rm PI}$ vanish as well. Hence there must be cancelations between different particle species, and in particular the one-loop contribution to the entropy of some fields in the supergravity theory must be *negative*. Since statistical entropy is always positive, the one-loop $Z_{\rm PI}^{(1)}$ of such fields cannot be equal to a statistical mechanical partition function.

In the same work [5], a qualitative stringy picture was sketched giving some bulk intuition about the nature of such negative contributions to $S_{\rm PI}$ when the total $S_{\rm PI}$ is a statistical entropy. In this picture, all relevant microscopic fundamental degrees of freedom are presumed to be realized in the bulk quantum gravity theory as weakly coupled strings. More specifically it is presumed that $Z_{\rm PI}^{\rm grav} = {\rm Tr}_{\mathcal{H}}\, e^{-\beta H}$ where $\mathcal{H}$ is the string Hilbert space on Rindler space and H is the Rindler Hamiltonian, so $S_{\rm PI} = S$, the statistical entropy. Tree level and one-loop contributions to $\log Z_{\rm PI}$ are shown in fig. C.4.4. Diagrams `d`,`e` do not contribute to the entropy $S_{\rm PI} = (1 - \beta\partial_\beta) \log Z_{\rm PI}$ as their $\log Z_{\rm PI} \propto \beta$. Cutting `b` along constant-τ slices gives it an interpretation as a thermal trace over "bulk" string states away from $\rho = 0$ (closed strings in top row).[8] Similarly, `a` can be viewed as a thermal trace over "edge" string states stuck to $\rho = 0$ (*open* strings in top row). On the other hand `c` represents an *interaction* between bulk and edge strings, with no thermal or state counting interpretation on its own. Being statistical mechanical partition functions, `a` and `b` contribute positively to $S_{\rm PI}$, whereas `c` may contribute negatively. In fact in the $\mathcal{N} = 4$ case discussed above, `c` *must* be negative, canceling `b` to render $S_{\rm PI}^{(1)} = 0$. From an effective field theory point of view, `b` and `e` correspond to the bulk ideal gas partition function inferred from formal arguments along the lines of section C.4.2, while `c` *represents "edge" corrections missed by such arguments*.

This picture is qualitative, as the individual contributions corresponding to a sharp split of the worldsheet path integrals along these lines are likely ill-defined/divergent [245]. Moreover, even without any splitting, an actual string theory calculation of $S_{\rm PI} = (1 - \beta\partial_\beta) \log Z_{\rm PI}|_{\beta=2\pi}$ is problematic, as Euclidean Rindler with a generic conical defect $\epsilon = 2\pi - \beta$ is off-shell. Shortly after [5], [246] proposed to compute $Z_{\rm PI}$ on the orbifold $\mathbb{R}^2/\mathbb{Z}_N$ for general integer N and then analytically continue the result to $N \to 1 + \epsilon$. Unfortunately such orbifolds have closed string tachyons leading to befuddling IR-divergences [246, 247]. Recently, progress was made in resolving some of these issues: in an open string version of the idea, arranged in type II string theory by adding a sufficiently low-dimensional D-brane, it was shown in [248] that upon careful analytic continuation, the tachyon appears to disappear at $N = 1 + \epsilon$.

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[8] As a simple analog of what is meant here, consider a free scalar field on S^1 parametized by $\tau \simeq \tau + \beta$. Then $\log Z_{\rm PI} = \int_0^\infty \frac{ds}{2s}\, {\rm Tr}\, e^{-s\frac{1}{2}(-\partial_\tau^2 + m^2)} = \sum_n \int \frac{ds}{2s} \int \mathcal{D}\tau|_n \, \exp[-\frac{1}{2}\int_0^s (\dot\tau^2 + m^2)] = \sum_n \frac{1}{2|n|}\, e^{-|n|\beta m}$. Here n labels the winding number sector of the particle worldline path integral with target space S^1. Discarding the UV-divergent $\frac{1}{0}$ term, this sums to $\log Z_{\rm PI} = -\log(1 - e^{-\beta m}) - \frac{1}{2}\beta m = \log {\rm Tr}\, e^{-\beta H}$ as in (C.4.5). `b` is analogous to the $|n| = 1$ contribution $e^{-\beta m}$, `e` is analogous to $n = 0$, and higher winding versions of `b` correspond to $|n| > 1$.

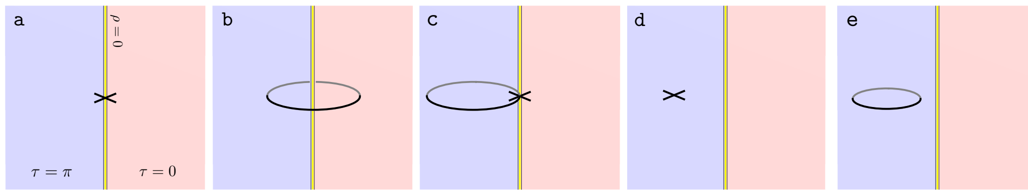


Figure C.4.5: Tree-level and one-loop contributions to $\log Z_{\rm PI}$ for massless vector field in Euclidean Rindler (C.4.28). These can be viewed as field theory limits of fig. C.4.4, with verbatim the same comments applicable to a-e. The worldline path integral c appears with a sign opposite to b in $\log Z_{\rm PI}^{(1)}$ [6].

C.4.5.3 QFT considerations

The problem of interest to us is really just a problem involving Gaussian path integrals in free quantum field theory, so there should be no need to invoke quantum gravity to gain some insight in what kind of corrections we should expect to the naive $Z_{\rm PI} \simeq Z_{\rm bulk}$. Indeed the above stringy Rindler considerations have much more straightforwardly computable low-energy counterparts in QFT.

Motivated by [5], [6] computed $Z_{\rm PI}^{(1)}$ for scalars, spinors and Maxwell fields on Rindler space. For scalars and spinors, this was found to coincide with the ideal gas partition function, whereas for Maxwell an additional contact term was found, expressible in terms of a "edge" worldline path integral with coincident start and end points at $\rho = 0$, fig. C.4.5c. This term contributes negatively to $S_{\rm PI}^{(1)} = (1 - \beta\partial_\beta) \log Z_{\rm PI}|_{\beta=2\pi}$ and thus has no thermal interpretation on its own. In fact it causes the total $S_{\rm PI}^{(1)}$ to be negative in less than 8 dimensions. The results of [6] and more generally the picture of [5] were further clarified by low-energy effective field theory analogs in [145], emphasizing in particular that whereas $S_{\rm PI}$ remains invariant under Wilsonian RG, the division between contributions with or without a low-energy statistical interpretation does not, the former gradually turning into the latter as the UV-cutoff Λ is lowered. At $\Lambda = 0$, only the tree-level contribution $S = A/4G_{\rm N}$ of fig. C.4.5a is left.

The contact/edge correction of fig. C.4.5c to $\log Z_{\rm PI}$ can be traced to the presence of a curvature coupling X linear in the Riemann tensor in $S_E = \int A(-\nabla^2 + X)A + \cdots$ [145, 146, 150]. Such terms appear for any spin $s \geq 1$ field, massless or not. Hence, as one might have anticipated from the stringy picture of fig. C.4.4, they are the norm rather than the exception.

The result of [6] was more recently revisited in [137], relating the appearance of edge

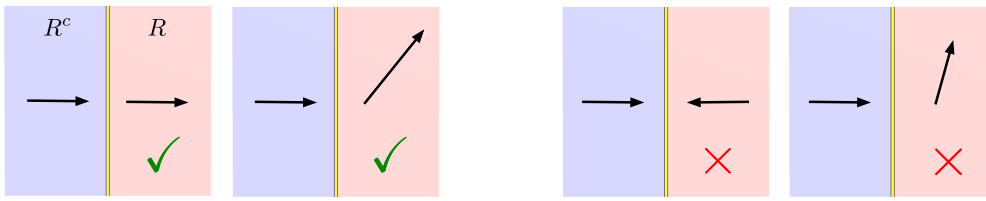


Figure C.4.6: Candidate classical initial electromagnetic field configurations (phase space points), with $A_0 = 0$, $A_i = 0$, showing electric field $E_i = \Pi_i = \dot{A}_i$. Gauss' law requires continuity $E_\perp$ across the boundary, disqualifying the two candidates on the right.

corrections to the local factorization problem of QFT Hilbert spaces with gauge constraints [154–159] like Gauss' law $\nabla \cdot E = 0$ in Maxwell theory. This problem arises more generally when contemplating the definition of entanglement entropy $S_R = -\mathrm{Tr}\,\varrho_R \log \varrho_R$ of a spatial subregion R in gauge theories. In principle ϱ_R is obtained by factoring the global Hilbert space $\mathcal{H}_G = \mathcal{H}_R \otimes \mathcal{H}_{R^c}$ and tracing out $\mathcal{H}_{R^c}$. As mentioned at the end of C.4.4, local factorization is impossible in the continuum limit of any QFT, including scalar field theories, but the issue raised there can be dealt with by a suitable regularization. However for a gauge theory such as free Maxwell theory, there is an additional obstruction to local factorization, which persists after regularization, and indeed is present already at the classical phase space level: the Gauss law constraint $\nabla \cdot E = 0$ prevents us from picking independent initial conditions in both R and R^c (fig. C.4.6), unless the boundary is a physical object that can accommodate compensating surface charges — but this is not the case here. One way to resolve this is to decompose the global phase space into sectors labeled by "center" variables located at the boundary surface [154–159], for example the normal component $E_\perp$ of the electric field. The center variables Poisson-commute with all local observables inside R and R^c. In any given sector, factorization then becomes possible.

Building on this framework it was shown in [137] that in a suitable brick wall-like regularization scheme and for some choice of measure $\mathcal{D}E_\perp$, the edge correction of [6] arises as a classical contribution $\int \mathcal{D}E_\perp \, e^{-S_E[E_\perp]}$ to the thermal statistical partition function. Here $S_E[E_\perp]$ is the on-shell action for static electromagnetic field modes in Euclidean Rindler space with prescribed $E_\perp$, localized vanishingly close to $\rho = 0$ when the brick-wall cutoff is taken to zero, and thus interpreted as edge modes. They also find a more precise form for the result of [6] for Rindler with its transverse dimensions compactified on a torus, which is identical in form to our de Sitter result (5.5.17) for $s = 1$, $G = U(1)$.

Similar results for massive vector fields were obtained in [160]. (The Stueckelberg action for a massive vector has a $U(1)$ gauge symmetry, so from that point of view it may fit into the above considerations.) An open string realization of the above ideas was proposed in [249]. It has been suggested that edge modes and "soft hair" might be related [250].

C.5 Derivations for massive higher spins

C.5.1 Massive spin-s fields

Here we derive (5.4.7) and (5.4.6). The starting point is the path integral (5.4.4). To get a result guaranteed to be consistent with QFT locality and general covariance, we should in principle start with the full off-shell system [58] involving auxiliary Stueckelberg fields of all spin $s' < s$.

Transverse-traceless part $Z_{\rm TT}$

One's initial hope might be that $Z_{\rm PI}$ ends up being equal to the path integral $Z_{\rm TT}$ restricted to the propagating degrees of freedom, the transverse traceless modes of ϕ, with kinetic operator given by the second-order equation of motion in (5.4.1). Regularized as in (5.3.2), this is

$$\log Z_{\rm TT} \equiv \int_0^\infty \frac{d\tau}{2\tau}\, e^{-\epsilon^2/4\tau}\, {\rm Tr}_{\rm TT}\, e^{-\tau\left(-\nabla^2_{s,{\rm TT}}+\overline{m}_s^2\right)}\,. \tag{C.5.1}$$

The index ${\rm TT}$ indicates the object is defined on the restricted space of transverse traceless modes. This turns out to be correct for Euclidean AdS with standard boundary conditions [138]. However, this is not quite true for the sphere, related to the presence of normalizable tensor decomposition zeromodes.

The easiest way to convince oneself that $Z_{\rm PI} \neq Z_{\rm TT}$ on the sphere is to just compute $Z_{\rm TT}$ and observe it is inconsistent with locality, in a sense made clear below. To evaluate $Z_{\rm TT}$, all we need is the spectrum of $-\nabla^2_{\rm TT}+\overline{m}_s^2$ [213, 251]. The eigenvalues are $\lambda_n = \left(n+\frac{d}{2}\right)^2+\nu^2$, $n \geq s$ with degeneracy given by the dimension $D^{d+2}_{n,s}$ of the $\mathfrak{so}(d+2)$ representation corresponding to the two-row Young diagram $\mathbb{Y}_{n,s}$

Following the same steps as for the scalar case in section 5.3, we end up with

$$\log Z_{\rm TT} = \int_0^\infty \frac{dt}{2t} \left(q^{i\nu} + q^{-i\nu}\right) f_{\rm TT}(q) \,, \qquad f_{\rm TT}(q) \equiv \sum_{n \geq s} D^{d+2}_{n,s}\, q^{\frac{d}{2}+n} \,. \tag{C.5.2}$$

Now let us evaluate this explicitly for the example of a massive vector on S^5, i.e. $d = 4$, $s = 1$. From (2.2.6) we read off $D^6_{n,1} = \frac{1}{3} n(n+2)^2(n+4)$. Performing the sum we end up with

$$f_{\rm TT}(q) = \frac{1+q}{1-q}\left(\frac{4\,q^2}{(1-q)^4} - \frac{q}{(1-q)^2}\right) + q \qquad (d = 4, s = 1)\,. \tag{C.5.3}$$

The first term inside the brackets can be recognized as the $d = 4$ massive spin-1 bulk character.

The small-t expansion of the integrand in (C.5.2) contains a term $1/t$. This term arises from the term $+q$ in the above expression, as the other parts give contributions to the integrand that are manifestly even under $t \to -t$. The presence of this $1/t$ term in the small-t expansion implies $\log Z_{\rm TT}$ has a logarithmic UV divergence $\log Z_{\rm TT}|_{\log {\rm div}} = \log M$ where M is the UV cutoff scale. More precisely in the heat-kernel regularization under consideration, the contribution of the term $+q$ to $\log Z_{\rm TT}$ is, according to (C.2.30),

$$\int_0^\infty \frac{dt}{2t}\left(q^{1+i\nu} + q^{1-i\nu}\right) = \log \frac{M}{\sqrt{1+\nu^2}} \,, \qquad M \equiv \frac{2e^{-\gamma}}{\epsilon} \,. \tag{C.5.4}$$

Note that $m = \sqrt{1+\nu^2}$ is the Proca mass (5.4.2) of the vector field. The presence of a logarithmic divergence means $Z_{\rm PI} \neq Z_{\rm TT}$, for $\log Z_{\rm PI}$ itself is defined as a manifestly covariant, local QFT path integral on S^5, which cannot have any logarithmic UV divergences, as there are no local curvature invariants of mass dimension 5.

For $s = 2$ and $d = 4$ we get similarly

$$f_{\rm TT}(q) = \frac{1+q}{1-q}\left(\frac{9\,q^2}{(1-q)^4} - \frac{6\,q}{(1-q)^2}\right) + 6\,q + 15\,q^2 \qquad (d = 4, s = 2)\,. \tag{C.5.5}$$

The terms $6\,q + 15\,q^2$ produce a nonlocal logarithmic divergence $\log Z_{\rm TT}|_{\log {\rm div}} = c \log M$, where $c = 6 + 15 = 21$, so again $Z_{\rm PI} \neq Z_{\rm TT}$. Note that $21 = \frac{7\times 6}{2} = \dim \mathfrak{so}(1,6)$, the number of conformal Killing vectors on S^5. That this is no coincidence can be ascertained by repeating the same exercise for general $d \geq 3$ and $s = 2$:

$$f_{\rm TT}^{(s=2)}(q) = \frac{1+q}{1-q}\left(D_2^d \cdot \frac{q^{\frac{d}{2}}}{(1-q)^d} - D_1^{d+2} \cdot \frac{q^{\frac{d-2}{2}}}{(1-q)^{d-2}}\right) + D_1^{d+2}\, q + D_{1,1}^{d+2}\, q^2 \,. \qquad \text{(C.5.6)}$$

The q, q^2 terms generate a log-divergence $c_2 \log M$, $c_2 = D_1^{d+2} + D_{1,1}^{d+2} = (d+2) + \frac{1}{2}(d+2)(d+1) = \frac{1}{2}(d+3)(d+2) = D_{1,1}^{d+3} = \dim \mathfrak{so}(1, d+2)$, the number of conformal Killing vectors on S^{d+1}. The identity $N_{\rm CKV} = D_{1,1}^{d+3} = D_{1,1}^{d+2} + D_{1,0}^{d+2}$ and its generalization to the spin-s case will be a crucial ingredient in establishing our claims. It has a simple group theoretic origin. As a complex Lie algebra, the conformal algebra $\mathfrak{so}(1, d+2)$ generated by the conformal Killing vectors is the same as $\mathfrak{so}(d+3)$, which is generated by antisymmetric matrices and therefore forms the irreducible representation with Young diagram $\yng(1,1)$ of $\mathfrak{so}(d+3)$. This decomposes into irreps of $\mathfrak{so}(d+2)$ by the branching rule

$$\yng(1,1) \quad \to \quad \yng(1,1) + \yng(1)\,, \qquad \text{(C.5.7)}$$

implying in particular $D_{1,1}^{d+3} = D_{1,1}^{d+2} + D_1^{d+2}$. Geometrically this reflects the fact that the conformal Killing modes split into two types: (i) transversal vector modes $\varphi_\mu^{i_1}$, $i_1 = 1, \ldots, D_{1,1}^{d+2}$, satisfying the ordinary Killing equation $\nabla_{(\mu}\varphi_{\nu)}^{i_1} = 0$, spanning the $\yng(1,1)$ eigenspace of the transversal vector Laplacian, and (ii) longitudinal modes $\varphi_\mu^{i_0} = \nabla_\mu \varphi^i$, $i_0 = 1, \ldots, D_1^{d+2}$, satisfying $\nabla_\mu \nabla_\nu \varphi^{i_0} + g_{\mu\nu}\varphi^{i_0} = 0$, with the scalar φ^{i_0} modes spanning the $\yng(1)$ eigenspace of the scalar Laplacian on S^{d+2}.

We can extend the above to general s, d by using the eq. (2.3.23), which is copied here

$$\boxed{D_{n,s}^{d+2} = D_n^{d+2} D_s^d - D_{s-1}^{d+2} D_{n+1}^d} \qquad \text{(C.5.8)}$$

This relation is crucial and together with the explicit expression (2.2.3) for D_s^d with $d \geq 3$ immediately leads to

$$f_{\rm TT}(q) = \sum_{n \geq -1} D_{n,s}^{d+2}\, q^{\frac{d}{2}+n} - \sum_{n=-1}^{s-1} D_{n,s}^{d+2}\, q^{\frac{d}{2}+n} \qquad \text{(C.5.9)}$$

$$= \frac{1+q}{1-q}\left(D_s^d \cdot \frac{q^{\frac{d}{2}}}{(1-q)^d} - D_{s-1}^{d+2} \cdot \frac{q^{\frac{d-2}{2}}}{(1-q)^{d-2}}\right) + \sum_{n=-1}^{s-2} D_{s-1,n+1}^{d+2}\, q^{\frac{d}{2}+n} \,. \qquad \text{(C.5.10)}$$

To rewrite the finite sum we used $D^{d+2}_{n,s} = -D^{d+2}_{s-1,n+1}$ and $D^{d+2}_{s-1,s} = 0$, both of which follow from (C.5.8). Substituting this into the integral (C.5.2), we get

$$\log Z_{\rm TT} = \log Z_{\rm bulk} - \log Z_{\rm edge} + \log Z_{\rm res}\,, \tag{C.5.11}$$

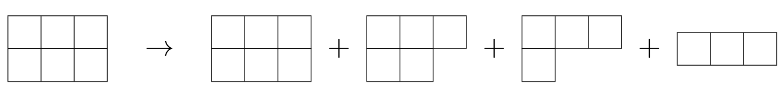
where $\log Z_{\rm bulk}$ and $\log Z_{\rm edge}$ are the character integrals defined in (5.4.7)-(5.4.8), and, evaluating the integral of the remaining finite sum as in (C.5.4),

$$\log Z_{\rm res} = \sum_{n=-1}^{s-2} D^{d+2}_{s-1,n+1} \int \frac{dt}{2t} \left(q^{\frac{d}{2}+n+i\nu} + q^{\frac{d}{2}+n-i\nu}\right) = \sum_{n=-1}^{s-2} D^{d+2}_{s-1,n+1} \log \frac{M}{\sqrt{(\frac{d}{2}+n)^2+\nu^2}} \tag{C.5.12}$$

The term $\log Z_{\rm res}$ has a logarithmic UV-divergence:

$$\log Z_{\rm res} = c_s \log M + \cdots\,, \qquad c_s = \sum_{n=-1}^{s-2} D^{d+2}_{s-1,n+1} = D^{d+3}_{s-1,s-1} = N_{\rm CKT}\,, \tag{C.5.13}$$

where $N_{\rm CKT} = D^{d+3}_{s-1,s-1}$ is the number of rank $s-1$ conformal Killing tensors on S^{d+1} [252]. This identity has a group theoretic origin as an $\mathfrak{so}(d+3) \to \mathfrak{so}(d+2)$ branching rule generalizing (C.5.7). For example for $s = 4$:



(C.5.14)

Geometrically this reflects the fact that the rank $s-1=3$ Killing tensor modes split up into 4 types: Schematically $\varphi^{i_3}_{\mu_1\mu_2\mu_3}$, $i_3 = 1,\ldots,D_{3,3}$; $\varphi^{i_2}_{\mu_1\mu_2\mu_3} \sim \nabla_{(\mu_1}\varphi^{i_2}_{\mu_2\mu_3)}$, $i_2 = 1,\ldots,D_{3,2}$; $\varphi^{i_1}_{\mu_1\mu_2\mu_3} \sim \nabla_{(\mu_1}\nabla_{\mu_2}\varphi^{i_1}_{\mu_3)}$, $i_1 = 1,\ldots,D_{3,1}$; $\varphi^{i_0}_{\mu_1\mu_2\mu_3} \sim \nabla_{(\mu_1}\nabla_{\mu_2}\nabla_{\mu_3)}\varphi^{i_0}$, $i_0 = 1,\ldots,D_3$, where the $\varphi^{i_r}_{\mu_1\cdots\mu_r}$ span the eigenspace of the TT spin-r Laplacian labeled by the above Young diagrams.

As pointed out in examples above and discussed in more detail below, the log-divergence of $\log Z_{\rm res}$ is inconsistent with locality, hence $Z_{\rm PI} \neq Z_{\rm TT}$: locality must be restored by the non-TT part of the path integral. Below we argue this part in fact *exactly cancels* the $\log Z_{\rm res}$ term, thus ending up with $\log Z_{\rm PI} = \log Z_{\rm bulk} - \log Z_{\rm edge}$, i.e. the character formula (5.4.7).

Full path integral $Z_{\rm PI}$: locality constraints

The full, manifestly covariant, local path integral takes the form (a simple example is (C.5.21)):

$$Z_{\rm PI} = Z_{\rm TT} \cdot Z_{\textsf{non-TT}} = Z_{\rm bulk} \cdot Z_{\rm edge}^{-1} \cdot \, Z_{\rm res} \cdot Z_{\textsf{non-TT}} \, . \tag{C.5.15}$$

All UV-divergences of $\log Z_{\rm PI}$ are *local*, in the sense they can be canceled by local counterterms, more specifically local curvature invariants of the background metric. In particular for odd $d+1$, this implies there cannot be any logarithmic divergences at all, as there are no curvature invariants of odd mass dimension. Recall from (C.5.13) that the term $\log Z_{\rm res}$ is logarithmically divergent. For odd $d+1$, this is clearly the only log-divergent contribution to $\log Z_{\rm TT}$, as the integrands of both $\log Z_{\rm bulk}$ and $\log Z_{\rm edge}$ are even in t in this case. More generally, for even or odd $d+1$, $\log Z_{\rm res}$ is the only *nonlocal* log-divergent contribution to $\log Z_{\rm TT}$, as follows from the result of [128, 139] mentioned below (5.4.4), combined with the observation in (C.5.13) that $c_s = N_{\rm CKT}$. Therefore the log-divergence of $\log Z_{\rm res}$ must be canceled by an equal log-divergence in $\log Z_{\textsf{non-TT}}$ of the opposite sign.

The simplest way this could come about is if $Z_{\textsf{non-TT}}$ exactly cancels $Z_{\rm res}$, that is if

$$Z_{\textsf{non-TT}} = Z_{\rm res}^{-1} = \prod_{n=-1}^{s-2} \left(M^{-1} \sqrt{(\tfrac{d}{2}+n)^2 + \nu^2} \right)^{D^{d+2}_{s-1,n+1}} \qquad \Rightarrow \qquad Z_{\rm PI} = \frac{Z_{\rm bulk}}{Z_{\rm edge}} \, . \tag{C.5.16}$$

Note furthermore that from (C.5.9), or from (C.5.12) and $D^{d+2}_{s-1,n+1} = -D^{d+2}_{n,s}$, it follows this identification is equivalent to the following simple prescription: *The full $Z_{\rm PI}$ is obtained from $Z_{\rm TT}$ by extending the TT eigenvalue sum $\sum_{n \geq s}$ in (C.5.2) down to $\sum_{n\geq -1}$*:

$$\log Z_{\rm PI} = \int_0^\infty \frac{dt}{2t} \left(q^{i\nu} + q^{-i\nu} \right) \sum_{n \geq -1} D^{d+2}_{n,s} \, q^{\frac{d}{2}+n} \, , \tag{C.5.17}$$

i.e. (5.4.6). In what follows we establish this is indeed the correct identification. We start by showing it precisely leads to the correct spin-s unitarity bound, and that it moreover exactly reproduces the critical mass ("partially massless") thresholds at which a new set of terms in the action defining the path integral $Z_{\textsf{non-TT}}$ fails to be positive definite. Assisted by those insights, it will then be rather clear how (C.5.16) arises from explicit path integral computations.

Unitarity constraints

A significant additional piece of evidence beyond consistency with locality is consistency with unitarity. It is clear that both the above integral (C.5.17) for $\log Z_{\rm PI}$ and the integral (C.5.2) for $\log Z_{\rm TT}$ are real provided ν is either real or imaginary. Real ν corresponds to the principal series $\Delta = \frac{d}{2} + i\nu$, while imaginary $\nu = i\mu$ corresponds to the complementary series $\Delta = \frac{d}{2} - \mu \in \mathbb{R}$. In the latter case there is in addition a bound on $|\mu|$ beyond which the integrals cease to make sense, due to the appearance of negative powers of $q = e^{-t}$ and the integrand blowing up at $t \to \infty$. The bound can be read off from the term with the smallest value of n in the sum. In the $Z_{\rm TT}$ integral (C.5.2) this is the $n = s$ term $\propto q^{\frac{d}{2}+s\pm\mu}$, yielding a bound $|\mu| < \frac{d}{2} + s$. In the $Z_{\rm PI}$ integral (C.5.17), assuming $s \geq 1$, this is the $n = -1$ term $\propto q^{\frac{d}{2}-1\pm\mu}$, so the bound becomes much tighter:

$$|\mu| < \frac{d}{2} - 1 \qquad (s \geq 1)\,. \tag{C.5.18}$$

This is exactly the correct unitarity bound for the spinning complementary series representations of SO$(1, d{+}1)$ found in the chapter 3 . In terms of the mass $m^2 = (\frac{d}{2}{+}s{-}2)^2{-}\mu^2$ in (5.4.2), this becomes $m^2 > (s-1)(d-3+s)$, also known as the Higuchi bound [42]. From a path integral perspective, this bound can be understood as the requirement that the full off-shell action is positive definite [58], so indeed $\log Z_{\rm PI}$ should diverge exactly when the bound is violated. Moreover, we get new divergences in the integral formula for $\log Z_{\textsf{non-TT}}$, according to the above identifications, each time $|\mu|$ crosses a critical value $\mu_{*n} = \frac{d}{2}{+}n$, where $n = -1, 0, 1, 2, \ldots, s{-}2$. These correspond to critical masses $m^2_{*n} = (\frac{d}{2}+s-2)^2 - (\frac{d}{2}+n)^2 = (s-2-n)(d+s-2+n)$, which on the path integral side precisely correspond to the points where a new set of terms in the action fails to be positive definite. [58].

This establishes the terms in the integrand of (C.5.12), or equivalently the extra terms $n = -1, \ldots, s-2$ in (C.5.17), have exactly the correct powers of q to match with $\log Z_{\textsf{non-TT}}$. It does not yet confirm the precise values of the coefficients $D^{d+2}_{n,s}$ — except for their sum (C.5.13), which was fixed earlier by the locality constraint. To complete the argument, we determine the origin of these coefficients from the path integral point of view in what follows.

Explicit path integral considerations

Complementary to but guided by the above general considerations, we now turn to more concrete path integral calculations to confirm the expression (C.5.16) for $Z_{\rm non-TT}$, focusing in particular on the origin of the coefficients $D^{d+2}_{n,s}$.

Spin 1:

We first consider the familar $s = 1$ case, a vector field of mass m, related to ν by (5.4.2) as $m = \sqrt{(\frac{d}{2}-1)^2+\nu^2}$. The local field content in the Stueckelberg description consists of a vector ϕ_μ and a scalar Θ, with action and gauge symmetry given by

$$S_0 = \int \nabla_{[\mu}\phi_{\nu]}\nabla^{[\mu}\phi^{\nu]} + \tfrac{1}{2}(\nabla_\mu\Theta - m\,\phi_\mu)(\nabla^\mu\Theta - m\,\phi^\mu)\,; \qquad \delta\Theta = m\xi\,, \quad \delta\phi_\mu = \nabla_\mu\xi\,. \tag{C.5.19}$$

Gauge fixing the path integral by putting $\Theta \equiv 0$, we get the gauge-fixed action

$$S = \int \nabla_{[\mu}\phi_{\nu]}\nabla^{[\mu}\phi^{\nu]} + \tfrac{1}{2}m^2\phi_\mu\phi^\mu + m\bar{c}c\,, \tag{C.5.20}$$

with BRST ghosts c, $\bar{c}$. Decomposing ϕ_μ into a transversal and longitudinal part, $\phi_\mu = \phi^T_\mu + \phi'_\mu$, we can decompose the path integral as $Z_{\rm PI} = Z_{\rm TT}\cdot Z_{\textsf{non-TT}}$ with

$$Z_{\rm TT} = \int \mathcal{D}\phi^T\, e^{-\frac{1}{2}\int \phi^T(-\nabla^2+\bar{m}_1^2)\phi^T}\,, \qquad Z_{\textsf{non-TT}} = \int \mathcal{D}\phi'\mathcal{D}c\mathcal{D}\bar{c}\, e^{-\int \frac{1}{2}m^2\phi'^2 + m\bar{c}c}\,, \tag{C.5.21}$$

Both the ghosts and the longitudinal vectors $\phi'_\mu = \nabla_\mu\varphi$ have an mode decomposition in terms of orthonormal real scalar spherical harmonics Y_i.[9] In our heat kernel regularization scheme, each longitudinal vector mode integral gives a factor M/m, which is exactly canceled by a factor m/M from integrating out the corresponding ghost mode.[10] However there is one ghost mode which remains unmatched: the constant mode. A constant scalar does not map to a longitudinal vector mode, because $\phi'_\mu = \nabla_\mu\varphi = 0$ for constant φ. Thus we end up with a ghost factor m/M

---

[9]Explicitly, $c = \sum_i c_i Y_i$, $\bar{c} = \sum_i \bar{c}_i Y_i$, $\phi'_\mu = \sum_{i:\lambda_i\neq 0}\phi'_i\nabla_\mu Y_i/\sqrt{\lambda_i}$ where $\nabla^2 Y_i = -\lambda_i Y_i$, $\int Y_i Y_j = \delta_{ij}$.

[10]A priori there might be a relative numerical factor κ between ghost and longitudinal factors, depending on the so far unspecified normalization of the measure $\mathcal{D}c$. But because c is local, *unconstrained*, rescaling $\mathcal{D}c = \prod_i dc_i \to \prod_i(\lambda dc_i)$ merely amounts to a trivial constant shift of the bare cc. So we are free to take $\kappa = 1$.

in excess, and

$$Z_{\textsf{non-TT}} = m/M = M^{-1}\sqrt{(\tfrac{d}{2}-1)^2+\nu^2}\,, \tag{C.5.22}$$

in agreement with (C.5.16) for $s=1$.

Spin 2:

For $s=2$, the analogous Stueckelberg action involves a symmetric tensor $\phi_{\mu\nu}$, a vector χ_μ, and a scalar χ, subject to the gauge transformations [58]

$$\delta\chi = a_{-1}\,\xi\,, \qquad \delta\chi_\mu = a_0\,\xi_\mu + \sqrt{\tfrac{d-1}{2d}}\,\nabla_\mu\xi\,, \qquad \delta\phi_{\mu\nu} = \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu + \sqrt{\tfrac{2}{d(d-1)}}\,a_0\,\xi\,, \tag{C.5.23}$$

where $a_0 \equiv m$ and $a_{-1} \equiv \sqrt{m^2-(d-1)}$. Equivalently, recalling (5.4.2), $a_n = \sqrt{(\frac{d}{2}+n)^2+\nu^2}$. Gauge fixing by putting $\chi=0$, $\chi_\mu=0$, we get a ghost action

$$S_{\rm gh} = \int a_{-1}\,\bar{c}c + a_0\,\bar{c}^\mu c_\mu\,. \tag{C.5.24}$$

We can decompose $\phi_{\mu\nu}$ into a TT part and a non-TT part orthogonal to it as $\phi_{\mu\nu} = \phi^{\rm TT}_{\mu\nu} + \phi'_{\mu\nu}$, where $\phi'_{\mu\nu}$ can be decomposed into vector and scalar modes as $\phi'_{\mu\nu} = \nabla_{(\mu}\varphi_{\nu)} + g_{\mu\nu}\varphi$. Analogous to the $s=1$ example, we should expect that integrating out ϕ' cancels against integrating out the ghosts, up to unmatched modes of the latter. The unmatched modes correspond to mixed vector-scalar modes solving $\nabla_{(\mu}\varphi_{\nu)} + g_{\mu\nu}\varphi = 0$. This is equivalent to the conformal Killing equation. Hence the unmatched modes are the conformal Killing modes. As discussed below (C.5.7), the conformal Killing modes split according to $\boxminus \to \boxminus + \square$ into $D^{d+2}_{1,1}$ vector $\boxminus$-modes and D^{d+2}_1 scalar $\square$-modes. Integrating out the $\boxminus$-modes of the vector ghost c_μ then yields an unmatched factor $(a_0/M)^{D^{d+2}_{1,1}}$, while integrating out the $\square$-modes of the scalar ghost c yields an unmatched factor $(a_{-1}/M)^{D^{d+2}_1}$. All in all, we get

$$Z_{\textsf{non-TT}} = (a_{-1}/M)^{D^{d+2}_1}(a_0/M)^{D^{d+2}_{1,1}} = \Big(M^{-1}\sqrt{(\tfrac{d}{2}-1)^2+\nu^2}\Big)^{D^{d+2}_1}\Big(M^{-1}\sqrt{(\tfrac{d}{2})^2+\nu^2}\Big)^{D^{d+2}_{1,1}} \tag{C.5.25}$$

in agreement with (C.5.16) for $s = 2$.

Spin s:

The pattern is now clear: according to [58], the Stueckelberg system for a massive spin-s field consists of an unconstrained symmetric s-index tensor $\phi^{(s)}$ and of a tower of unconstrained symmetric s'-index auxiliary Stueckelberg fields $\chi^{(s')}$ with $s' = 0, 1, \ldots, s-1$, with gauge symmetries of the form

$$\delta\chi^{(s')} = a_{s'-1}\, \xi^{(s')} + \cdots\,, \qquad \delta\phi^{(s)} = \cdots\,, \qquad a_n \equiv \sqrt{(\tfrac{d}{2} + n)^2 + \nu^2}\,, \tag{C.5.26}$$

where the dots indicate terms we won't technically need — which is to say, as transpired from $s = 1, 2$ already, we need very little indeed. The ghost action is $S = \sum_{s'=0}^{s-1} a_{s'-1} \bar{c}^{(s')} c^{(s')}$. The unmatched modes correspond to the conformal Killing tensors modes on S^{d+1}, decomposed for say $s = 4$ as in (C.5.14) into $D^{d+2}_{3,3}$ $\ydiagram{3,3}$-modes, $D^{d+2}_{3,2}$ $\ydiagram{3,2}$-modes, $D^{d+2}_{3,1}$ $\ydiagram{3,1}$-modes, and D^{d+2}_{3} $\ydiagram{3}$-modes. The corresponding unmatched modes of respectively $c^{(3)}$, $c^{(2)}$, $c^{(1)}$ and $c^{(0)}$ then integrate to unmatched factors $(a_2/M)^{D^{d+2}_{3,3}} (a_1/M)^{D^{d+2}_{3,2}} (a_0/M)^{D^{d+2}_{3,1}} (a_{-1}/M)^{D^{d+2}_{3}}$. For general s:

$$Z_{\text{non-TT}} = \prod_{s'=0}^{s-1} (a_{s'-1}/M)^{D^{d+2}_{s-1,s'}} = \prod_{n=-1}^{s-2} \left(M^{-1} \sqrt{(\tfrac{d}{2} + n)^2 + \nu^2} \right)^{D^{d+2}_{s-1,n+1}}\,, \tag{C.5.27}$$

in agreement with (C.5.16) for general s. This establishes our claims.

The above computation was somewhat schematic of course, and one could perhaps still worry about missed purely numerical factors independent of ν, perhaps leading to an additional finite constant term being added to our final formulae (5.4.7) -(5.4.6) for $\log Z_{\rm PI}$. However at fixed UV-regulator scale, the limit $\nu \to \infty$ of these final expressions manifestly approaches zero, as should be the case for particles much heavier than the UV cutoff scale. This would not be true if there was an additional constant term. Finally, we carefully checked the analogous result in the massless case (which has a more compact off-shell formulation [179]), discussed in section 5.5, by direct path integral computations in complete gory detail [253], for all s.

Also, the result is pretty.

C.5.2 General massive representations

Here we give a generalization of (5.4.6) for arbitrary massive representations of the dS_{d+1} isometry group $\mathrm{SO}(1,d+1)$.

Massive irreducible representations of $\mathrm{SO}(1,d+1)$, i.e. $\mathcal{F}_{\Delta,\mathbf{s}}$, are labeled by a dimension $\Delta=\frac{d}{2}+i\nu$ and an $\mathfrak{so}(d)$ highest weight $=(s_1,\ldots,s_r)$. The massive spin-s case considered in (5.4.6) corresponds to $\mathbf{s}=(s,0,\ldots,0)$, a totally symmetric tensor field. More general irreps correspond to more general mixed-symmetry fields. The analog of (C.5.1) in this generalized setup is

$$\log Z_{\rm PI}^{\text{“TT”}}=\pm\int_0^\infty\frac{d\tau}{2\tau}\,e^{-\epsilon^2/4\tau}\sum_{n\geq s_1}D^{d+2}_{n,\mathbf{s}}\,e^{-\tau((n+\frac{d}{2})^2+\nu^2)}\,, \tag{C.5.28}$$

where for bosons the sum runs over integer n with an overall $+$ sign and for fermions the sum runs over half-integer n with an overall $-$ sign. The dimensions of the $\mathfrak{so}(d+2)$ irreps $(n,\mathbf{s})$ are given explicitly as polynomials in $(n,s_1,\ldots,s_r)$ by the Weyl dimension formulae (2.2.1)-(2.2.2). From this it can be seen that $D^{d+2}_{n,S}$ is (anti-)symmetric under reflections about $n=-\frac{d}{2}$, more precisely

$$D^{d+2}_{n,\mathbf{s}}=(-1)^d D^{d+2}_{-d-n,\mathbf{s}}. \tag{C.5.29}$$

Moreover the exponent in (C.5.28) is symmetric under the same reflection. The most natural extension of the sum is therefore to *all* (half-)integer n, taking into account the sign in (C.5.29) for odd d, and adding an overall factor $\frac{1}{2}$ to correct for double counting, suggesting

$$\log Z_{\rm PI}=\pm\frac{1}{2}\int_0^\infty\frac{d\tau}{2\tau}\,e^{-\epsilon^2/4\tau}\sum_{n}\sigma_d(\tfrac{d}{2}+n)\,D^{d+2}_{n,\mathbf{s}}\,e^{-\tau((n+\frac{d}{2})^2+\nu^2)}\,, \tag{C.5.30}$$

where $\sigma_d(x)\equiv 1$ for even d and $\sigma_d(x)\equiv\mathrm{sign}(x)$ for odd d. Equivalently, in view of (C.5.29)

$$\boxed{\log Z_{\rm PI}=\pm\int_0^\infty\frac{d\tau}{2\tau}\,e^{-\epsilon^2/4\tau}\sum_{n}\theta(\tfrac{d}{2}+n)\,D^{d+2}_{n,\mathbf{s}}\,e^{-\tau((n+\frac{d}{2})^2+\nu^2)}} \tag{C.5.31}$$

where $n \in \mathbb{Z}$ for bosons and $n \in \frac{1}{2} + \mathbb{Z}$ for fermions, and

$$\theta(x) = 1 \text{ for } x > 0, \quad \theta(0) = \frac{1}{2}, \quad \theta(x) = 0 \text{ for } x < 0. \tag{C.5.32}$$

At first sight this seems to be different from the extension to $n \geq -1$ in (5.4.6) for the spin-s case $\mathbf{s} = (s, 0, \ldots, 0)$. However it is actually the same, as (2.2.1)-(2.2.2) imply that $D_{n,\mathbf{s}}$ vanishes for $2 - d \leq n \leq -2$ when $\mathbf{s} = (s, 0, \ldots, 0)$.

The obvious conjecture is then that (C.5.31) is true for general massive representations. Here are some consistency checks, which are satisfied precisely for the sum range in (C.5.31):

• *Locality:* For even d, the summand in (C.5.30) is analytic in n. Applying the Euler-Maclaurin formula to extract the $\tau \to 0$ asymptotic expansion of the sum gives in this case

$$\sum_n D^{d+2}_{n,\mathbf{s}} \, e^{-\tau(n+\frac{d}{2})^2} \sim \int_{-\infty}^{\infty} dn \, D^{d+2}_{n,\mathbf{s}} \, e^{-\tau(n+\frac{d}{2})^2} \, . \tag{C.5.33}$$

The symmetry (C.5.29) tells us that the integrand on the right hand side is even in $x \equiv n + \frac{d}{2}$. Since $\int dx \, x^{2k} \, e^{-\tau x^2} \propto \tau^{-k-1/2}$, this implies the absence of $1/\tau$ terms in the $\tau \to 0$ expansion of the integrand in (C.5.30), and therefore, in contrast to (C.5.28), the absence of nonlocal log-divergences, as required by locality of $Z_{\rm PI}$ in odd spacetime dimension $d+1$.

• *Bulk − edge structure:* By following the usual steps, we can rewrite (C.5.31) as

$$\boxed{\log Z_{\rm PI} = \int \frac{dt}{2t} \, F(e^{-t}) \, , \qquad F(q) = \pm(q^{i\nu} + q^{-i\nu}) \sum_n \theta(\tfrac{d}{2} + n) \, D^{d+2}_{n,\mathbf{s}} \, q^{\frac{d}{2}+n}} \tag{C.5.34}$$

Using (2.2.1)-(2.2.2), this can be seen to sum up to the form $\log Z_{\rm PI} = \log Z_{\rm bulk} - \log Z_{\rm edge}$, where $Z_{\rm bulk}$ is the physically expected bulk character formula for an ideal gas in the dS$_{d+1}$ static patch consisting of massive particles in the $\mathcal{F}_{\Delta,\mathbf{s}}$ UIR of SO$(1, d+1)$, and $Z_{\rm edge}$ can be interpreted as a Euclidean path integral of local fields living on the S^{d-1} edge/horizon.

• *Unitarity:* Note that for $\Delta = \frac{d}{2} + \mu$ with $\mu \equiv i\nu$ real, we get a bound on μ from requiring $t \to \infty$ (IR) convergence of the integral (C.5.34), generalizing (C.5.18), namely

$$|\mu| < \frac{d}{2} + n_*(\mathbf{s}) \, , \tag{C.5.35}$$

where $n_*(\mathbf{s})$ is the lowest value of n in the sum for which $D^{d+2}_{n,\mathbf{s}}$ is nonvanishing. This coincides again with the unitarity bound on μ for massive representations of SO$(1,d{+}1)$ [70, 75]. Recalling the discussion below (C.5.18), this can be viewed as a generalization of the Higuchi bound to arbitrary representations.

Combining (C.5.31) with (C.2.19), we thus arrive at an exact closed-form solution for the Euclidean path integral on the sphere for arbitrary massive field content.

C.6 Derivations for massless higher spins

In this appendix we derive (5.5.17) and provide details of various other points summarized in section 5.5.

C.6.1 Bulk partition function: $Z_{\rm bulk}$

The bulk partition function $Z_{\rm bulk}$ as defined in (5.3.3) for a massless spin-s field is given by

$$Z_{\rm bulk} = \int \frac{dt}{2t}\,\frac{1+q}{1-q}\,\Theta_{{\rm bulk},s}(q)\,, \tag{C.6.1}$$

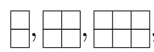
where $q = e^{-t}$, and $\Theta_{{\rm bulk},s}(q) = {\rm tr}\, q^{iH} = \Theta_{\mathcal{U}_{s,s-1}}(q)$ in the case at hand is the (restricted) q-character of the massless spin-s SO$(1,d+1)$ UIR (a.k.a., $\mathcal{U}_{s,s-1}$). These characters are quite a bit more intricate than their massive counterparts and the explicit forms are given by the eq.(3.4.20) with $t = s-1$. Here we show some low-dimensional examples

d	r	$(1-q)^d\,\Theta_{{\rm bulk},s}(q)$
3	1	$2D^3_s\,q^{s+1} - 2D^3_{s-1}\,q^{s+2}$
4	2	$D^4_{s,s}\,2\,q^2$
5	2	$D^5_{s,s}\,(q^2-q^3) + 2D^5_s q^{s+3} - 2D^5_{s-1} q^{s+4}$
6	3	$D^6_{s,s}\,(q^2+q^4) - D^6_{s,s,1}\,2\,q^3$
7	3	$D^7_{s,s}\,(q^2-q^5) - D^7_{s,s,1}\,(q^3-q^4) + 2D^7_s\,q^{s+5} - 2D^7_{s-1}\,q^{s+6}\,,$

(C.6.2)

where $D^3_s = 2s{+}1$, $D^4_{s,s} = 2s{+}1$, $D^5_s = \frac{1}{6}(s{+}1)(s{+}2)(2s{+}3)$, $D^5_{s,s} = \frac{1}{3}(2s{+}1)(s{+}1)(2s{+}3)$, $D^6_{s,s} = \frac{1}{12}(s+1)^2(s+2)^2(2s+3)$, $D^6_{s,s,1} = \frac{1}{12}s(s+1)(s+2)(s+3)(2s+3)$, etc. For $s=1$, the character can be expressed more succinctly as

$$\Theta_{\text{bulk},1}(q) = d \cdot \frac{q^{d-1}+q}{(1-q)^d} - \frac{q^d+1}{(1-q)^d} + 1\,. \qquad \text{(C.6.3)}$$

With the exception of the $d = 3$ case, the above $\mathfrak{so}(1, d+1)$ q-characters encoding the H-spectrum of massless spin-s fields in dS$_{d+1}$ are very different from the $\mathfrak{so}(2, d)$ characters encoding the energy spectrum of massless spin-s fields in AdS$_{d+1}$ with standard boundary conditions, the latter being $\Theta^{\text{AdS}_{d+1}}_{\text{bulk},s} = (D^d_s\, q^{s+d-2} - D^d_{s-1}\, q^{s+d-1})/(1-q)^d$. In particular for $d \geq 4$, the lowest power q^Δ appearing in the q-expansion of the character is $\Delta = 2$, and is associated with the $\mathfrak{so}(d)$ representation $\mathbf{s} = (s, s, 0, \cdots, 0)$, i.e. , $\ldots$ for $s = 1, 2, 3, \ldots$, whereas for the $\mathfrak{so}(2, d)$ character this is $\Delta = s + d - 2$ and $\mathbf{s} = (s, 0, \cdots, 0)$. An explanation for this was given in [70]: in dS, s should be thought of as associated with the higher-spin Weyl curvature tensor of the gauge field rather than the gauge field itself.

This fits well with the interpretation of the expansion

$$\Theta(q) = \sum_r N_r q^r\,, \qquad \text{(C.6.4)}$$

as counting the number N_r of *physical* static patch quasinormal modes decaying as e^{-rT} (cf. section 5.2 and appendix C.1.3). Indeed for $d \geq 4$, the longest-lived physical quasinormal modes of a massless spin-s field in the static patch of dS$_{d+1}$ always decay as e^{-2T} as we have shown in the section 4.3.3, which can be understood as follows. Physical quasinormal modes of the southern static patch can be thought of as sourced by insertions of gauge-invariant[11] local operators on the past conformal boundary $T = -\infty$ of the static patch, or equivalently at the south pole of the past conformal boundary (or alternatively the north pole of future boundary) of global dS$_{d+1}$ [1, 59, 60]. By construction, the dimension r of the operator maps to the decay rate r of the quasinormal mode $\propto e^{-rT}$. For $s = 1$, the gauge-invariant operator with the smallest dimension $r = \Delta$ is the magnetic field strength $F_{ij} = \partial_i A_j - \partial_j A_i$ of the boundary gauge field A_i, which has $\Delta = \dim \partial + \dim A = 1 + 1 = 2$. For $s = 2$ in $d \geq 4$, the gauge-invariant operator with smallest dimension is the Weyl tensor of the boundary metric: $\Delta = 2 + 0 = 2$. Similarly for higher-spin fields we get the spin-s Weyl tensor, with $\Delta = s+2-s = 2$. The reason

[11] More precisely, invariant under linearized gauge transformations acting on the conformal boundary.

$d = 3$ is special is that the Weyl tensor vanishes identically in this case. To get a nonvanishing gauge-invariant tensor, one has to act with at least $2s - 1$ derivatives (spin-s Cotton tensor), yielding $\Delta = (2s - 1) + (2 - s) = s + 1$. An extensive analysis is given in the section 4.4.

C.6.2 Euclidean path integral: $Z_{\rm PI} = Z_G Z_{\rm char}$

The Euclidean path integral of a collection of gauge fields ϕ on S^{d+1} is formally given by

$$Z_{\rm PI} = \frac{\int \mathcal{D}\phi \, e^{-S[\phi]}}{\rm vol(\mathcal{G})} \tag{C.6.5}$$

where $\mathcal{G}$ is the local gauge group generated by the local field ξ appearing in (5.5.1). This ill-defined formal expression is turned into something well-defined by BRST gauge fixing. A convenient gauge for higher-spin fields is the de Donder gauge. At the Gaussian level, the resulting analog of (C.5.1) is[12]

$$\log Z_{\rm TT} \equiv \sum_s \int_0^\infty \frac{d\tau}{2\tau} \, e^{-\epsilon^2/4\tau} \Big({\rm Tr}_{\rm TT} \; e^{-\tau\left(-\nabla^2_{s,{\rm TT}} + m^2_{\phi,s}\right)} - {\rm Tr}_{\rm TT} \; e^{-\tau\left(-\nabla^2_{s-1,{\rm TT}} + m^2_{\xi,s}\right)} \Big) \,, \tag{C.6.6}$$

where $\sum_s$ sums over the spin-s gauge fields in the theory (possibly with multiplicities) and $m^2_{\phi,s}$ and $m^2_{\xi,s}$ are obtained from the relations below (5.4.1) using (5.5.2). The first term arises from the path integral over the TT modes of ϕ, while the second arises from the TT part of the gauge fixing sector in de Donder gauge — a combination of integrating out the TT part of the spin-$(s - 1)$ ghost fields and the corresponding longitudinal degrees of freedom of the spin-s gauge fields. The above (C.6.6) is the difference of two expressions of the form (C.5.1). Naively applying the formula (C.5.34) or (5.4.6) for the corresponding full $Z_{\rm PI}$, we get

$$Z_{\rm PI} = \exp \sum_s \int \frac{dt}{2t} \, \hat{F}_s(e^{-t}) \qquad \text{(naive)} \tag{C.6.7}$$

---

[12] A detailed discussion of normalization conventions left implicit here is given above and below (C.6.29).

where (assuming $d \geq 3$)

$$\hat{F}_s(q) = \sum_{n \geq -1} D^{d+2}_{n,s} \big(q^{s+d-2+n} + q^{2-s+n} \big) - \sum_{n \geq -1} D^{d+2}_{n,s-1} \big(q^{s+d-1+n} + q^{1-s+n} \big) \qquad \text{(C.6.8)}$$

However this is clearly problematic. One problem is that for $s \geq 2$, the above $\hat{F}_s(q)$ contains *negative* powers of $q = e^{-t}$, making (C.6.7) exponentially divergent at $t \to \infty$. The appearance of such "wrong-sign" powers of q is directly related to the appearance of "wrong-sign" Gaussian integrals in the path integral, as can be seen for instance from the relation between (C.5.34) and the heat kernel integral (C.5.31). In the path integral framework, one deals with this problem by analytic continuation, generalizing the familiar contour rotation prescription for negative modes in the gravitational Euclidean path integral [125]. Thus one defines $\int dx \, e^{-\lambda x^2/2}$ for $\lambda < 0$ by rotating $x \to ix$, or equivalently by rotating $\tau \to -\tau$ in the heat kernel integral. Essentially this just boils down to flipping any $\lambda < 0$ to $-\lambda > 0$. Since the Laplacian eigenvalues are equal to the products of the exponents appearing in the pairs $(q^{\Delta+n}+q^{d-\Delta+n})$ in (C.6.8), the implementation of this prescription in our setup is to flip the negative powers q^k in $\hat{F}_s(q) = \sum_k c_k \, q^k$ to positive powers q^{-k}, that is to say replace

$$\hat{F}_s(q) \to F_s(q) \equiv \big\{ \hat{F}_s(q) \big\}_+ = \Big\{ \sum_k c_k \, q^k \Big\}_+ \equiv \sum_{k<0} c_k \, q^{-k} + \sum_{k \geq 0} c_k \, q^k \, . \qquad \text{(C.6.9)}$$

In addition, each negative mode path integral contour rotation produces a phase $\pm i$, resulting in a definite, finite overall phase in $Z_{\rm PI}$ [131]. The analysis of [131] translates to each corresponding flip in (C.6.9) contributing with the *same* sign,[13] hence to an overall phase i^{-P_s} with P_s the total degeneracy of negative modes in (C.6.8). Using $D^{d+2}_{n,s} = -D^{d+2}_{s-1,n+1}$:

$$Z_{\rm PI} \to i^{-P_s} Z_{\rm PI} \, , \qquad P_s = \sum_{n'=0}^{s-2} D^{d+2}_{s-1,n'} + \sum_{n'=0}^{s-2} D^{d+2}_{s-2,n'} = D^{d+3}_{s-1,s-1} - D^{d+2}_{s-1,s-1} + D^{d+3}_{s-2,s-2} \qquad \text{(C.6.10)}$$

In particular this implies $P_1 = 0$ and $P_2 = d + 3$ in agreement with [131].

After having taken care of the *negative* powers of q, the resulting amended formula $Z_{\rm PI} =$

[13] This can be seen in a more careful path integral analysis [253].

$\int \frac{dt}{2t} F_s(q)$ is still problematic, however, as $F_s(q)$ still contains terms proportional to q^0, causing the integral to diverges (logarithmically) for $t \to \infty$. These correspond to zeromodes in the original path integral. Indeed such zermodes were to be expected: they are due to the existence of normalizable rank $s-1$ traceless Killing tensors $\bar{\xi}^{(s-1)}$, which by definition satisfy $\nabla_{(\mu_1}\bar{\xi}_{\mu_2\cdots\mu_s)} = 0$, and therefore correspond to vanishing gauge transformations (5.5.1), leading in particular to ghost zeromodes. Zeromodes of this kind must be omitted from the Gaussian path integral. They are easily identified in (C.6.8) as the values of n for which a term proportional to q^0 appears. Since we are assuming $d > 2$, this is $n = s-2$ in the first sum and $n = s-1$ in the second. Thus we should refine (C.6.9) to

$$\hat{F}_s \to F_s - F_s^0 \,, \tag{C.6.11}$$

where $F_s^0 = D^{d+2}_{s-2,s}(q^{2s+d-4}+1) - D^{d+2}_{s-1,s-1}(q^{2s+d-2}+1)$. Noting that $D^{d+2}_{s-2,s} = -D^{d+2}_{s-1,s-1}$ and $D^{d+2}_{s-1,s-1}$ is the number N^{KT}_{s-1} of rank $s-1$ traceless Killing tensors on S^{d+1}, we can rewrite this as

$$F_s^0 = -N^{\mathrm{KT}}_{s-1}(q^{2s+d-4}+1+q^{2s+d-2}+1) \,, \qquad N^{\mathrm{KT}}_{s-1} = D^{d+2}_{s-1,s-1} \,, \tag{C.6.12}$$

making the relation to the existence of normalizable Killing tensors manifest. For example $N_0^{\mathrm{KT}} = 1$, corresponding to constant $U(1)$ gauge transformations; $N_1^{\mathrm{KT}} = \frac{1}{2}(d+2)(d+1) = \dim \mathsf{SO}(d+2)$, corresponding to the linearly independent Killing vectors on S^{d+1}; and $N^{\mathrm{KT}}_{s-1} \propto s^{2d-3}$ for $s \to \infty$, corresponding to large-spin generalizations thereof.

We cannot just drop the zeromodes and move on, however. The original formal path integral expression (C.6.5) is local by construction, as both numerator and denominator are defined with a local measure on local fields. In principle BRST gauge fixing is designed to maintain manifest locality, but if we remove any finite subset of modes by hand, including in particular zeromodes, locality is lost. Indeed the $-F_s^{(0)}$ subtraction results in nonlocal log-divergences in the character integral, i.e. divergences which cannot be canceled by local counterterms. From the point of view of (C.6.5), the loss of locality is due the fact that we are no longer dividing by the volume of the local gauge group $\mathcal{G}$, since we are effectively omitting the subgroup G generated by the Killing tensors. To restore locality, and to correctly implement the idea embodied in

(C.6.5), we must divide by the volume of G by hand. This volume must be computed using the same local measure defining $\mathrm{vol}(\mathcal{G})$, i.e. the invariant measure on $\mathcal{G}$ normalized such that integrating the gauge fixing insertion in the path integral over the gauge orbits results in a factor 1. Hence the appropriate measure defining the volume of G in this context is inherited from the BRST path integral measure. As such we will denote it by $\mathrm{vol}(G)_{\rm PI}$. A detailed general discussion of the importance of these specifications for consistency with locality and unitarity in the case of Maxwell theory can be found in [136]. Relating $\mathrm{vol}(G)_{\rm PI}$ to a "canonical", theory-independent definition of the group volume $\mathrm{vol}(G)_{\rm c}$ (such as for example $\mathrm{vol}(U(1))_{\rm c} \equiv 2\pi$) is not trivial, requiring considerable care in keeping track of various normalization factors and conventions. Moreover $\mathrm{vol}(G)_{\rm PI}$ depends on the nonlinear interaction structure of the theory, as this determines the Lie algebra of G. We postpone further analysis of $\mathrm{vol}(G)_{\rm PI}$ to section C.6.4.

Conclusion

To summarize, instead of the naive (C.6.7), we get the following formula for the 1-loop Euclidean path integral on S^{d+1} for a collection of massless spin-s gauge fields:

$$Z_{\rm PI} = i^{-P_s} \left(\mathrm{vol}(G)_{\rm PI}\right)^{-1} \exp \sum_s \int \frac{dt}{2t} \left(F_s - F_s^0\right) , \tag{C.6.13}$$

where $F_s = \{\hat{F}_s\}_+$ and F_s^0 were defined in (C.6.8), (C.6.9) and (C.6.12); G is the subgroup of gauge transformations generated by the Killing tensors $\bar{\xi}^{(s-1)}$, i.e. the zeromodes of (5.5.1); and i^{-P_s} is the phase (C.6.10). We can split up the integrals by introducing an IR regulator:

$$\boxed{Z_{\rm PI} = i^{-P_s}\, Z_G\, Z_{\rm char} \,, \qquad Z_G \equiv \frac{\exp(-\sum_s \int^{\times} \frac{dt}{2t} F_s^0)}{\mathrm{vol}(G)_{\rm PI}} \,, \qquad Z_{\rm char} \equiv \exp \sum_s \int^{\times} \frac{dt}{2t}\, F_s} \tag{C.6.14}$$

where the notation $\int^{\times}$ means we IR regulate by introducing a factor $e^{-\mu t}$, take $\mu \to 0$, and subtract the $\log \mu$ divergent term. For a function $f(t)$ such that $\lim_{t\to\infty} f(t) = c$, this means

$$\int^{\times} \frac{dt}{t}\, f(t) \equiv \lim_{\mu \to 0} \Big(c \log \mu + \int \frac{dt}{t}\, f(t)\, e^{-\mu t} \Big) \tag{C.6.15}$$

For example for $f(t)=\frac{t}{t+1}$ this gives $\int_0^\times \frac{dt}{t}\,\frac{t}{t+1}=\log\mu-\log(e^\gamma\mu)=-\gamma$, and for $f(t)=1$ with the integral UV-regularized as in (C.2.30) we get $\int^\times \frac{dt}{t}=\log(2e^{-\gamma}/\epsilon)$.

In section C.6.3 we recast $Z_{\rm char}$ as a character integral formula. In section C.6.4 we express Z_G in terms of the canonical group volume $\mathrm{vol}(G)_{\rm c}$ and the coupling constant of the theory.

C.6.3 Character formula: $Z_{\rm char}=Z_{\rm bulk}/Z_{\rm edge}Z_{\rm KT}$

In this section we derive a character formula for $Z_{\rm char}$ in (C.6.14). If we start from the naive $\hat F_s$ given by (C.6.8) and follow the same steps as those bringing (5.4.6) to the form (5.4.7), we get

$$\hat F_s=\frac{1+q}{1-q}\,\hat\Theta_s\,,\qquad \hat\Theta_s=\hat\Theta_{{\rm bulk},s}-\hat\Theta_{{\rm edge},s}\,, \tag{C.6.16}$$

where

$$\hat\Theta_{{\rm bulk},s}=D^d_s\,\frac{q^{s+d-2}+q^{2-s}}{(1-q)^d}-D^d_{s-1}\,\frac{q^{s+d-1}+q^{1-s}}{(1-q)^d} \tag{C.6.17}$$

$$\hat\Theta_{{\rm edge},s}=D^{d+2}_{s-1}\,\frac{q^{s+d-3}+q^{1-s}}{(1-q)^{d-2}}-D^{d+2}_{s-2}\,\frac{q^{s+d-2}+q^{-s}}{(1-q)^{d-2}}\,. \tag{C.6.18}$$

Note that $\hat\Theta_{{\rm bulk},s}$ and $\hat\Theta_{{\rm edge},s}$ take the form of "field $-$ ghost" characters obtained respectively by substituting the values of ν_ϕ and ν_ξ given by (5.5.2) into the massive spin s and spin $s-1$ characters (5.4.8). The naive bulk characters $\hat\Theta_{\rm bulk,s}$ thus obtained cannot possibly be the character of any UIR of SO$(1,d+1)$, as is obvious from the presence of negative powers of q. Indeed, it is equal to $\Theta_{\mathcal{F}_{2-s,s}}-\Theta_{\mathcal{F}_{1-s,s-1}}$ which differs from the character $\Theta_{\mathcal{U}_{s,s-1}}(q)$ by the flipping procedure (3.4.16). Now let us consider the actual $F_s=\{\hat F_s\}_+$ appearing in (C.6.14). Then we find[14]

$$F_s=\Big\{\frac{1+q}{1-q}\,\hat\Theta_s\Big\}_+=\frac{1+q}{1-q}\Big(\big[\hat\Theta_s\big]_+-2\,N^{\rm KT}_{s-1}\Big)\,, \tag{C.6.19}$$

[14] To check (C.6.19) starting from (C.6.9), observe that $\big\{\frac{1+q}{1-q}\big(q^k+q^{-k}-2\big)\big\}_+=0$ for any integer k, so $\big\{\frac{1+q}{1-q}\,q^k\big\}_+=\frac{1+q}{1-q}\big(-q^{-k}+2\big)$ for $k<0$, while of course $\big\{\frac{1+q}{1-q}\,q^k\big\}_+=\frac{1+q}{1-q}\,q^k$ for $k\geq 0$, . This accounts for the $k<0$ and $k>0$ terms in the expansion $\sum_k c_k\,q^k$ of (C.6.19). The coefficient $2N^{\rm KT}_{s-1}$ of the q^0 term is most easily checked by comparing the q^0 terms on the left and right hand sides of (C.6.19), taking into account that, by definition, $[\Theta]_+$ has no q^0 term, and that the q^0 terms of the left hand side are given by (C.6.12).

where the flipping operator $[\ ]_+$ is defined by the eq. (3.4.17) and hence the $\big[\hat{\Theta}_{\text{bulk},s}\big]_+$ part in $\big[\hat{\Theta}_s\big]_+$ is exactly the Harish-Chandra character $\Theta_{\mathcal{U}_{s,s-1}}$ of the massless spin-s representation . Thus this flipping prescription can be thought of as the character analog of contour rotations for "wrong-sign" Gaussians in the path integral[15]. Notice the slight differences in the map $\hat{\Theta} \to [\hat{\Theta}]_+$ and the related but different map $\hat{F} \to \{\hat{F}\}_+$ defined in (C.6.9).

Substituting (C.6.19) into (C.6.14), we conclude

$$\boxed{\log Z_{\text{char}} = \sum_s \int^\times \frac{dt}{2t}\,\frac{1+q}{1-q}\Big(\big[\hat{\Theta}_{\text{bulk},s}\big]_+ - \big[\hat{\Theta}_{\text{edge},s}\big]_+ - 2N^{\text{KT}}_{s-1}\Big)} \tag{C.6.20}$$

Bulk characters for (partially) massless fields

Now let us apply this to a slight generalization of the massless $\hat{\Theta}_{\text{bulk,s}}$ given in (C.6.17),

$$\hat{\Theta}^{\text{bulk}}_{ss'}(q) \equiv D^d_s\,\frac{q^{1-s'}+q^{d-1+s'}}{(1-q)^d} - D^d_{s'}\,\frac{q^{1-s}+q^{d-1+s}}{(1-q)^d}\,. \tag{C.6.21}$$

This is the naive bulk character for a *partially massless* spin-s field $\phi_{\mu_1\cdots\mu_s}$ with a spin-s' $(0 \leq s' \leq s-1)$ gauge parameter field $\xi_{\mu_1\cdots\mu_{s'}}$ [222]. The massless case (C.6.17) corresponds to $s' = s-1$. Applying the flipping procedure to it yields the Harish-Chandra character $\Theta_{\mathcal{U}_{ss'}}(q)$ of the exceptional UIR $\mathcal{U}_{ss'}$, c.f. eq. (3.4.16) and eq. (3.4.20).

Edge characters for (partially) massless fields

For the edge correction we proceed analogously. The naive PM edge character is

$$\hat{\Theta}^{\text{edge}}_{ss'}(q) = D^{d+2}_{s-1}\,\frac{q^{-s'}+q^{d-2+s'}}{(1-q)^{d-2}} - D^{d+2}_{s'-1}\,\frac{q^{-s}+q^{d-2+s}}{(1-q)^{d-2}}\,, \tag{C.6.22}$$

[15]In representation theory, we have seen that the flipping prescription arises as we overcount the kernel of $z\cdot\partial_x$, which is the boundary counterpart of the bulk gauge transformation.

reducing to the massless case (C.6.18) for $s' = s-1$. Applying (3.4.18) gives, still with $r = \lfloor \frac{d}{2} \rfloor$,

$$\boxed{\begin{aligned}(1-q)^{d-2}\big[\hat{\Theta}^{\rm edge}_{ss'}(q)\big]_+ &= \big(1+(-1)^{d+1}\big)\big(D^{d+2}_{s-1}\, q^{d-2+s'} - D^{d+2}_{s'-1}\, q^{d-2+s}\big) \\ &\quad + \sum_{m=0}^{r-2} (-1)^m\, \tilde{D}_m \big(q^{1+m} + (-1)^d q^{d-3-m}\big)\end{aligned}} \tag{C.6.23}$$

where $\tilde{D}_m \equiv D^{d+2}_{s-1}\, D^{d-2}_{s'+1,1^m} - D^{d+2}_{s'-1}\, D^{d-2}_{s+1,1^m}$.

Note that in the massless spin-1 case

$$\big[\hat{\Theta}^{\rm edge}_1(q)\big]_+ = \frac{q^{d-2}+1}{(1-q)^{d-2}} - 1\,. \tag{C.6.24}$$

It is the SO$(1,d-1)$ Harish-Chandra character of the exceptional UIR $\mathcal{V}_{s,0}$ with $s=1$ — the irreducible representation indeed of a massless scalar on dS$_{d-1}$ with its zeromode removed. The fact that $s=1$ is analogous to what happens in the 2D CFT of a massless free scalar X: the actual CFT primary operators are the spin ± 1 derivatives $\partial_\pm X(0)$.

In contrast to (3.4.20), we did not find a way of rewriting $\tilde{D}_m$ for general spin to suggest an interpretation along these lines in general. Indeed unlike (3.4.20), $\big[\hat{\Theta}^{\rm edge}_{ss'}(q)\big]_+$ in general does not appear to be proportional to the character of a *single* exceptional series irrep of SO$(1,d-1)$. This is not in conflict with the picture of edge corrections as a Euclidean path integral of some collection of local fields on S^{d-1}, since if the fields have nontrivial spins / $\mathfrak{so}(d-2)$ weights, the corresponding character integrals will have a complicated structure, involving sums of iterations of SO$(1,d-1-2k)$ characters with $k=0,1,2,\ldots$, exhibiting patterns that might be hard to discern without knowing what to look for. It should also be kept in mind we have not identified a reason the edge correction *must* have a local QFT path integral interpretation. On the other hand, the coefficients of the q-expansion of the effective edge character *do* turn out to be positive, consistent with an interpretation in terms of some collection of fields corresponding to unitary representations of dS$_{d-1}$. A more fundamental group-theoretic or physics understanding of the edge correction would evidently be desirable.

For practical purposes, the interpretation does not matter of course. The formula (C.6.23) gives a general formula for $\Theta_{\rm edge}$, which is all we need. For example for $d=3$, this gives $\big[\hat{\Theta}^{\rm edge}_s(q)\big]_+ = 2\,\frac{D^5_{s-1}\, q^s - D^5_{s-2}\, q^{s+1}}{1-q} = 2D^5_{s-1}\, q^s + 2D^4_{s-1}\frac{q^{s+1}}{1-q}$, where $D^5_{s-1} = \frac{1}{6}s(s+1)(2s+1)$

and $D^4_{s-1} = D^5_{s-1} - D^5_{s-2} = s^2$. The second form makes positivity of coefficients manifest. For $d = 4$ we get $\big[\hat\Theta^{\rm edge}_s(q)\big]_+ = D^5_{s-1}\frac{2\,q}{(1-q)^2}$.

Conclusion

We conclude that (C.6.20) can be written as

$$\log Z_{\rm char} = \log Z_{\rm bulk} - \log Z_{\rm edge} - \log Z_{\rm KT}\,, \tag{C.6.25}$$

where the bulk and edge contributions are explicitly given by (3.4.20)-(C.6.23) with $s' = s-1$,[16] and

$$\log Z_{\rm KT} = \dim G \int^\times \frac{dt}{2t}\,\frac{1+q}{1-q}\cdot 2\,, \qquad \dim G = \sum_s N^{\rm KT}_{s-1} = \sum_s D^{d+2}_{s-1,s-1}\,. \tag{C.6.26}$$

The finite (IR) part of $Z_{\rm KT}$ is given by (C.2.37): $Z_{\rm KT}|_{\rm IR} = (2\pi)^{-\dim G}$.

C.6.4 Group volume factor: Z_G

The remaining task is to compute the factor Z_G defined in (C.6.14), that is

$$Z_G = \big({\rm vol}(G)_{\rm PI}\big)^{-1} \exp \sum_s N^{\rm KT}_{s-1} \int^\times \frac{dt}{2t}\big(q^{2s+d-4} + q^{2s+d-2} + 2\big) \tag{C.6.27}$$

We imagine the spin range to be finite, or cut off in some way. (The infinite spin range case is discussed in section 5.9.) In the heat kernel regularization scheme of appendix C.2, we can then evaluate the integral using (C.2.30):

$$Z_G = \big({\rm vol}(G)_{\rm PI}\big)^{-1} \prod_s \bigg(\frac{M^4}{(2s+d-4)(2s+d-2)}\bigg)^{\frac12 N^{\rm KT}_{s-1}}\,, \qquad M \equiv \frac{2e^{-\gamma}}{\epsilon}\,, \tag{C.6.28}$$

On general grounds, the nonlocal UV-divergent factors M appearing here in Z_G should cancel against factors of M in ${\rm vol}(G)_{\rm PI}$, as we will explicitly confirm below.

[16] In the partially massless case $\log Z_{\rm KT}$ takes the same form, but with $D^{d+2}_{s-1,s-1}$ replaced by $D^{d+2}_{s-1,s'}$.

Generalities

Recall that G is the group of gauge transformations generated by the Killing tensors. Equivalently it is the subgroup of gauge transformations leaving the background invariant. $\mathrm{vol}(G)_{\mathrm{PI}}$ is the volume of G with respect to the path integral induced measure. This is different from what we shall call the "canonical" volume $\mathrm{vol}(G)_{\mathrm{c}}$, defined with respect to the invariant metric normalized such that the generators of some standard basis of the Lie algebra have unit norm. (In the case of Yang-Mills, this coincides with the metric defined by the canonically normalized Yang-Mills action, providing some justification for the (ab)use of the word canonical.) In particular, in contrast to $\mathrm{vol}(G)_{\mathrm{c}}$, $\mathrm{vol}(G)_{\mathrm{PI}}$ depends on the coupling constants and UV cutoff of the field theory.

As mentioned at the end of section C.6.2, the computation of Z_G brings in a series of new complications. One reason is that the Lie algebra structure constants defining G are not determined by the free part of the action, but by its interactions, thus requiring data going beyond the usual one-loop Gaussian level. Another reason is that due to the omission of zeromodes and the ensuing loss of locality in the path integral, a precise computation of $\mathrm{vol}(G)_{\mathrm{PI}}$ requires keeping track of an unpleasantly large number of normalization factors, such as for instance constants multiplying kinetic operators, as these can no longer be automatically discarded by adjusting local counterterms. Consequently, exact, direct path integral computationz of Z_G for general higher-spin theories requires great care and considerable persistence, although it can be done [253]. Below we obtain an exact expression for Z_G in terms of $\mathrm{vol}(G)_{\mathrm{c}}$ and the Newton constant in a comparatively painless way, by combining results and ideas from [135, 136, 138, 140–143], together with the observation that the form of (C.6.6) actually determines all the normalization factors we need. Although the expressions at intermediate stages are still a bit unpleasant, the end result takes a strikingly simple and universal form.

If G is finite-dimensional, as is the case for example for Yang-Mills, Einstein gravity and certain (topological) higher-spin theories [166–169, 182, 208–211] including the dS_3 higher-spin theory analyzed in section 5.6, we can then proceed to compute $\mathrm{vol}(G)_{\mathrm{c}}$, and we are done. If G is infinite-dimensional, as is the case in generic higher-spin theories, one faces the remaining problem of making sense of $\mathrm{vol}(G)_{\mathrm{c}}$ itself. Glossing over the already nontrivial problem of exponentiating the higher-spin algebra to an actual group [191], the obvious issue is that $\mathrm{vol}(G)_{\rm c}$ is going to be divergent. We discuss and interpret this and other infinite spin range issues in section 5.9. In what follows we will continue to assume the spin range is finite or cut off in some way as before, so G is finite-dimensional.

We begin by determining the path integral measure to be used to compute $\mathrm{vol}(G)_{\rm PI}$ in (C.6.28). Then we compute Z_G in terms of $\mathrm{vol}(G)_{\rm c}$ and the coupling constant of the theory, first for Yang-Mills, then for Einstein gravity, and finally for general higher-spin theories.

Path integral measure

To determine $\mathrm{vol}(G)_{\rm PI}$ we have to take a quick look at the path integral measure. This is fixed by locality and consistency with the regularized heat kernel definition of Gaussian path integrals we have been using throughout. For example for a scalar field as in (5.3.2), we have $\int \mathcal{D}\phi \, e^{-\frac{1}{2}\phi(-\nabla^2+m^2)\phi} \equiv \exp \int \frac{d\tau}{2\tau} \, e^{-\epsilon^2/4\tau} \, \mathrm{Tr} \, e^{-\tau(-\nabla^2+m^2)}$. An eigenmode of $-\nabla^2 + m^2$ with eigenvalue λ_i contributes a factor $M/\sqrt{\lambda_i}$ to the right hand side of this equation, with $M = \exp \int \frac{d\tau}{2\tau} \, e^{-\epsilon^2/4\tau} e^{-\tau} = 2e^{-\gamma}/\epsilon$, the same parameter as in (C.6.28) (essentially by definition). To ensure the left hand side matches this, we must use a path integral measure derived from the local metric $ds^2_\phi = \frac{M^2}{2\pi} \int (\delta\phi)^2$. To see this, expand $\phi(x) = \sum_i \varphi_i \psi_i(x)$ with $\psi_i(x)$ an orthonormal basis of eigenmodes of $-\nabla^2 + m^2$ on S^{d+1}. The metric in this basis becomes $ds^2 = \sum_i \frac{M^2}{2\pi} d\varphi_i^2$, so a mode with eigenvalue λ_i contributes a factor $\int d\varphi_i \, \frac{M}{\sqrt{2\pi}} \, e^{-\frac{1}{2}\lambda_i \varphi_i^2} = M/\sqrt{\lambda_i}$ to the left hand side, as required.

We work with canonically normalized fields. For a spin-s field ϕ this means the quadratic part of the action evaluated on its transverse-traceless part $\phi^{\rm TT}$ takes the form

$$S[\phi^{\rm TT}] = \frac{1}{2} \int \phi^{\rm TT} (-\nabla^2 + \overline{m}^2) \phi^{\rm TT}. \qquad \text{(C.6.29)}$$

Consistency with (C.5.1) or (C.6.6) then requires the measure for ϕ to be derived again from the metric $ds^2_\phi = \frac{M^2}{2\pi} \int (\delta\phi)^2$. If ϕ has a gauge symmetry, the formal division by the volume of the gauge group $\mathcal{G}$ is conveniently implemented by BRST gauge fixing. For example for a spin-1 field with gauge symmetry $\delta\phi_\mu = \partial_\mu \xi$, we can gauge fix in Lorenz gauge by adding the BRST-exact action $S_{\rm BRST} = \int iB \, \nabla^\mu \phi_\mu - \bar{c} \nabla^2 c$. This requires specifying a measure for the Lagrange multiplier field B and the ghosts $c, \bar{c}$. It is straightforward to check that a ghost measure derived from $ds^2_{\bar{c}c} = M^2 \int \delta\bar{c}\,\delta c$ (which translates to a mode measure $\prod_i \frac{1}{M^2} d\bar{c}_i\, dc_i$) combined with a B-measure derived from $ds^2_B = \frac{1}{2\pi}\int(\delta B)^2$, reproduces precisely the second term in (C.6.6) upon integrating out B, c, $\bar{c}$ and the longitudinal modes of ϕ. It is likewise straightforward to check that BRST gauge fixing is then formally equivalent to dividing by the volume of the local gauge group $\mathcal{G}$ with respect to the measure derived from the following metric on the algebra of local gauge transformations:

$$ds^2_\xi = \frac{M^4}{2\pi}\int(\delta\xi)^2\,. \tag{C.6.30}$$

Note that all of these metrics take the same form, with the powers of M fixed by dimensional analysis. An important constraint in the above was that the second term in (C.6.6) is exactly reproduced, without some extra factor multiplying the Laplacian. This matters when we omit zeromodes. For this to be the case with the above measure prescriptions, it was important that the gauge transformation took the form $\delta\phi_\mu = \alpha_1\partial_\mu\xi$ with $\alpha_1 = 1$ as opposed to some different value of α_1, as we a priori allowed in (5.5.1). For a general α_1, we would have obtained an additional factor α_1 in the ghost action, and a corresponding factor α_1^2 in the kinetic term in the second term of (C.6.6). To avoid having to keep track of this, we picked $\alpha_1 \equiv 1$. For Yang-Mills theories, everything remains the same, with internal index contractions understood, e.g. $S[\phi^{\rm TT}] = \frac{1}{2}\int\phi^{a\rm TT}(-\nabla^2+\overline{m}^2)\phi^{a\rm TT}$, $ds^2_\phi = \frac{M^2}{2\pi}\int(\delta\phi^a)^2$, $ds^2_\xi = \frac{M^4}{2\pi}\int(\delta\xi^a)^2$.

For higher-spin fields, we gauge fix in the de Donder gauge. All metrics remain unchanged, except for the obvious additional spacetime index contractions. The second term of (C.6.6) is exactly reproduced upon integrating out the $\rm TT$ sector of the BRST fields together with the corresponding longitudinal modes of ϕ, provided we pick

$$\alpha_s = \sqrt{s} \tag{C.6.31}$$

in (5.5.1), with symmetrization conventions such that $\phi_{(\mu_1\cdots\mu_s)} = \phi_{\mu_1\cdots\mu_s}$. (Technically the origin of the factor s can be traced to the fact that if $\phi_{\mu_1\cdots\mu_s} = \nabla_{(\mu_1}\xi_{\mu_2\cdots\mu_s)}$ for a TT ξ, we have $\int\phi^2 = s^{-1}\int\xi(-\nabla^2+c_s)\xi$.) Equation (C.6.31) fixes the normalization of ξ, and (C.6.30) then determines unambiguously the measure to be used to compute $\mathrm{vol}(G)_{\rm PI}$ in (C.6.28). We will see more concretely how this works in what follows, first spelling out the basic idea in detail in the familiar YM and GR examples, and then moving on to the general higher-spin gauge theory case considered in [140].

Yang-Mills

Consider a Yang-Mills theory with with a simple Lie algebra

$$[L^a, L^b] = f^{abc} L^c \,, \tag{C.6.32}$$

with the L^a some standard basis of anti-hermitian matrices and f^{abc} real and totally antisymmetric. For example for $\mathfrak{su}(2)$ Yang-Mills, $L^a = -\frac{1}{2} i\sigma^a$ and $[L^a, L^b] = \epsilon^{abc} L^c$. Consistent with our general conventions, we take the gauge fields $\phi_\mu = \phi^a_\mu L^a$ to be canonically normalized: the curvature takes the form $F^a_{\mu\nu} L^a = F_{\mu\nu} = \partial_\mu \phi_\nu - \partial_\nu \phi_\mu + g[\phi_\mu, \phi_\nu]$, and the action is

$$S = \frac{1}{4} \int F^a \cdot F^a \tag{C.6.33}$$

The quadratic part of S is invariant under the linearized gauge transformations $\delta^{(0)}_\xi \phi_\mu = \partial_\mu \xi$, where $\xi = \xi^a L^a$, taking the form (5.5.1) with $\alpha_1 = 1$ as required. The full S is invariant under local gauge transformations $\delta_\xi \phi_\mu = \partial_\mu \xi + g[\phi_\mu, \xi]$, generating the local gauge algebra

$$[\delta_\xi, \delta_{\xi'}] = \delta_{g[\xi', \xi]} \,. \tag{C.6.34}$$

The rank-0 Killing tensors $\bar\xi$ satisfy $\partial_\mu \bar\xi = 0$: they are the constant gauge transformations $\bar\xi = \bar\xi^a L^a$ on the sphere, forming the subalgebra $\mathfrak{g}$ of local gauge transformations acting trivially on the background $\phi_\mu = 0$, generating the group G whose volume we have to divide by. The bracket of $\mathfrak{g}$, denoted $[\![\cdot,\cdot]\!]$ in [140], is inherited from the local gauge algebra (C.6.34):

$$[\![\bar\xi, \bar\xi']\!] = g[\bar\xi', \bar\xi] \,. \tag{C.6.35}$$

Evidently this is isomorphic to the original YM Lie algebra. Being a simple Lie algebra, $\mathfrak{g}$ has an up to normalization unique invariant bilinear form/metric. The path integral metric $ds^2_{\rm PI}$ of

(C.6.30) corresponds to such an invariant bilinear form with a specific normalization:

$$\langle\bar{\xi}|\bar{\xi}'\rangle_{\rm PI} = \frac{M^4}{2\pi}\int \bar{\xi}^a\bar{\xi}'^a = \frac{M^4}{2\pi}\,{\rm vol}(S^{d+1})\,\bar{\xi}^a\bar{\xi}'^a\,. \tag{C.6.36}$$

We define the theory-independent "canonical" invariant bilinear form $\langle\cdot|\cdot\rangle_{\rm c}$ on $\mathfrak{g}$ as follows. First pick a "standard" basis M^a of $\mathfrak{g}$, i.e. a basis satisfying the same commutation relations as (C.6.32): $[\![M^a, M^b]\!] = f^{abc}M^c$. This fixes the normalization of the M^a. Then we fix the normalization of $\langle\cdot|\cdot\rangle_{\rm c}$ by requiring these standard generators have unit norm, i.e.

$$\langle M^a|M^b\rangle_c \equiv \delta^{ab}\,. \tag{C.6.37}$$

The explicit form of (C.6.35) implies such a basis is given by the constant functions $M^a = -L^a/g$ on the sphere. Thus we have $\langle L^a|L^b\rangle_{\rm c} = g^2\delta^{ab}$ and

$$\langle\bar{\xi}|\bar{\xi}'\rangle_{\rm c} = g^2\bar{\xi}^a\bar{\xi}'^a \tag{C.6.38}$$

Comparing (C.6.38) and (C.6.36), we see the path integral and canonical metrics on G and their corresponding volumes are related by

$$ds^2_{\rm PI} = \frac{M^4}{2\pi}\frac{{\rm vol}(S^{d+1})}{g^2}\,ds^2_c \qquad \Rightarrow \qquad \frac{{\rm vol}(G)_{\rm PI}}{{\rm vol}(G)_{\rm c}} = \left(\frac{M^4}{2\pi}\frac{{\rm vol}(S^{d+1})}{g^2}\right)^{\frac{1}{2}\dim G}. \tag{C.6.39}$$

From (C.6.28), we get $Z_G = {\rm vol}(G)^{-1}_{\rm PI}\big(\frac{M^4}{(d-2)d}\big)^{\frac{1}{2}\dim G}$, hence

$$Z_G = \frac{\gamma^{\dim G}}{{\rm vol}(G)_{\rm c}}\,, \qquad \gamma \equiv \frac{g}{\sqrt{(d-2)A_{d-1}}}\,, \qquad A_{d-1} \equiv {\rm vol}(S^{d-1}) = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}\,, \tag{C.6.40}$$

where we used ${\rm vol}(S^{d+1}) = \frac{2\pi}{d}{\rm vol}(S^{d-1})$. (Recall we have been assuming $d > 2$. The case $d = 2$ is discussed in appendix C.7.1.) The quantity γ may look familiar: the Coulomb potential energy for two unit charges at a distance r in flat space is $V(r) = \gamma^2/r^{d-2}$.

Practically speaking, the upshot is that Z_G is given by (C.6.40), with ${\rm vol}(G)_{\rm c}$ the volume of the Yang-Mills gauge group with respect to the metric defined by the Yang-Mills action (C.6.33). For example for $G = SU(2)$ with $f^{abc} = \epsilon^{abc}$ as before, ${\rm vol}(G)_{\rm c} = 16\pi^2$, because $SU(2)$ in this metric is the round S^3 with circumference 4π, hence radius 2.

The relation (C.6.35) can be viewed as defining the coupling constant g given our normalization conventions for the kinetic terms and linearized gauge transformations. Of course the final result is independent of these conventions. Conventions without explicit factors of g in the curvature and gauge transformations are obtained by rescaling $\phi \to \phi/g$, $\xi \to \xi/g$. Then there won't be a factor g in (C.6.35), but instead g is read off from the action $S = \frac{1}{4g^2}\int (F^a)^2$. We could also write this without explicit reference to a basis as $S = \frac{1}{4g^2}\int \mathrm{Tr}\, F^2$, where the trace "$\mathrm{Tr}$" is normalized such that $\mathrm{Tr}(L^a L^b) \equiv \delta^{ab}$. Then we can say the canonical bilinear/metric/volume is defined by the trace norm appearing in the YM action. We could choose a differently normalized trace $\mathrm{Tr}' = \lambda^2 \mathrm{Tr}$. The physics remains unchanged provided $g' = \lambda g$. Then $\mathrm{vol}(G)'_{\mathrm{c}} = \lambda^{\dim G} \mathrm{vol}(G)_{\mathrm{c}}$, hence, consistently, $Z'_G = Z_G$.

As a final example, for $SU(N)$ Yang-Mills with $\mathfrak{su}(N)$ viewed as anti-hermitian $N \times N$ matrices, $S = -\frac{1}{4g^2}\int \mathrm{Tr}_N F^2$ in conventions without a factor g in the gauge algebra, and Tr_N the ordinary $N \times N$ matrix trace, $\mathrm{vol}(SU(N))_{\mathrm{c}} = (C.3.3)$.

Einstein gravity

The Einstein gravity case proceeds analogously. Now we have single massless spin-2 field $\phi_{\mu\nu}$. The gauge transformations are diffeomorphisms generated by vector fields ξ_μ. The subgroup G of diffeomorphisms leaving the background S^{d+1} invariant is SO$(d+2)$, generated by Killing vectors $\bar{\xi}_\mu$. The usual standard basis $M_{IJ} = -M_{JI}$, $I = 1, \ldots, d+2$ of the $\mathfrak{so}(d+2)$ Lie algebra satisfies $[M_{12}, M_{23}] = M_{13}$ etc. We define the canonical bilinear $\langle \cdot | \cdot \rangle_c$ to be the unique invariant form normalized such that the M_{IJ} have unit norm:

$$\langle M_{12} | M_{12} \rangle_c = 1\,. \tag{C.6.41}$$

With respect to the corresponding metric ds_c^2, orbits $g(\varphi) = e^{\varphi M_{12}}$ with φ ranging from 0 to 2π have length 2π. The canonical volume is then given by (C.3.2).

To identify the standard generators M_{IJ} more precisely in our normalization conventions for $\bar{\xi}$, we need to look at the field theory realization in more detail. The $\mathfrak{so}(d+2)$ algebra generated by the Killing vectors $\bar{\xi}$ is realized in the interacting Einstein gravity theory as a subalgebra of the gauge (diffeomorphism) algebra. As in the Yang-Mills case (C.6.35), the bracket $[\![\cdot,\cdot]\!]$ of this subalgebra is inherited from the gauge algebra. Writing the Killing vectors as $\bar{\xi} = \bar{\xi}^{\mu}\partial_{\mu}$, the standard Lie bracket is $[\bar{\xi},\bar{\xi}']_{\rm L} = (\bar{\xi}^{\mu}\partial_{\mu}\bar{\xi}'^{\nu} - \bar{\xi}'^{\mu}\partial_{\mu}\bar{\xi}^{\nu})\partial_{\nu}$. If we had normalized $\phi_{\mu\nu}$ as $\phi_{\mu\nu} \equiv g_{\mu\nu} - g^0_{\mu\nu}$ with $g^0_{\mu\nu}$ the background sphere metric, and if we had normalized ξ_{μ} by putting $\alpha_2 \equiv 1$ in (5.5.1), the bracket $[\![\cdot,\cdot]\!]$ would have coincided with the Lie bracket $[\cdot,\cdot]_{\rm L}$. However, we are working in different normalization conventions, in which $\phi_{\mu\nu}$ is canonically normalized and $\alpha_2 = \sqrt{2}$ according to (C.6.31). In these conventions we have instead

$$[\![\bar{\xi},\bar{\xi}']\!] = \sqrt{16\pi G_{\rm N}}\,[\bar{\xi}',\bar{\xi}]_{\rm L}\,, \tag{C.6.42}$$

where $G_{\rm N}$ is the Newton constant. This can be checked by starting from the Einstein-Hilbert action, expanding to quadratic order (see e.g. [254] for convenient and reliable explicit expressions in dS$_{d+1}$), and making the appropriate convention rescalings. This is the Einstein gravity analog of (C.6.35). To be more concrete, let us consider the ambient space description of the sphere S^{d+1}, i.e. $X^I X_I = 1$ with $X \in \mathbb{R}^{d+2}$. Then the basis of Killing vectors $M_{IJ} \equiv -(X_I\partial_J - X_J\partial_I)/\sqrt{16\pi G_{\rm N}}$ satisfy our standard $\mathfrak{so}(d+2)$ commutation relations $[\![M_{12},M_{23}]\!] = M_{13}$ etc, hence by (C.6.41), $\langle M_{12}|M_{12}\rangle_c = 1$. The path integral metric (C.6.30) on the other hand corresponds to the invariant bilinear $\langle\bar{\xi}|\bar{\xi}'\rangle_{\rm PI} = \frac{M^4}{2\pi}\int\bar{\xi}\cdot\bar{\xi}'$, so $\langle M_{12}|M_{12}\rangle_{\rm PI} = \frac{M^4}{2\pi}\frac{1}{16\pi G_{\rm N}}\int_{S^{d+1}}(X_1^2 + X_2^2) = \frac{M^4}{2\pi}\frac{1}{16\pi G_{\rm N}}\frac{2}{d+2}{\rm vol}(S^{d+1})$. Thus we obtain the following relation between PI and canonical metrics and volumes for $G = SO(d+2)$:

$$ds^2_{\rm PI} = \frac{A_{d-1}}{4G_{\rm N}}\frac{1}{d(d+2)}\frac{M^4}{2\pi}\,ds^2_{\rm c} \qquad\Rightarrow\qquad \frac{{\rm vol}(G)_{\rm PI}}{{\rm vol}(G)_{\rm c}} = \left(\frac{A_{d-1}}{4G_{\rm N}}\frac{1}{d(d+2)}\frac{M^4}{2\pi}\right)^{\frac{1}{2}\dim G} \tag{C.6.43}$$

where $\dim G = \frac{1}{2}(d+2)(d+1)$, $A_{d-1} = {\rm vol}(S^{d-1})$ as in (C.6.40), and we again used ${\rm vol}(S^{d+1}) = \frac{2\pi}{d}{\rm vol}(S^{d-1})$. Combining this with (C.6.28), we get our desired result:

$$Z_G = \frac{\gamma^{\dim G}}{{\rm vol}(G)_{\rm c}}\,, \qquad\qquad \gamma \equiv \sqrt{\frac{8\pi G_{\rm N}}{A_{d-1}}}\,. \tag{C.6.44}$$

Higher-spin gravity

We follow the same template for the higher-spin case. In the interacting higher-spin theory, the Killing tensors generate a subalgebra of the nonlinear gauge algebra, with bracket $[\![\cdot,\cdot]\!]$ inherited from the gauge algebra, just like in the Yang-Mills and Einstein examples, except the gauge algebra is much more complicated in the higher-spin case. Fortunately it is not necessary to construct the exact gauge algebra to determine the Killing tensor algebra: it suffices to determine the lowest order deformation of the linearized gauge transformation (5.5.1) fixed by the transverse-traceless cubic couplings of the theory [140]. The Killing tensor algebra includes in particular an $\mathfrak{so}(d+2)$ subalgebra, that is to say an algebra of the same general form (C.6.42) as in Einstein gravity, with some constant appearing on the right-hand side determined by the spin-2 cubic coupling in the TT action. We *define* the "Newton constant" $G_{\rm N}$ of the higher-spin theory to be this constant, that is to say we read off $G_{\rm N}$ from the $\mathfrak{so}(d+2)$ Killing vector subalgebra by writing it as

$$[\![\bar\xi,\bar\xi']\!] = \sqrt{16\pi G_{\rm N}}\,[\bar\xi',\bar\xi]_{\rm L}\,. \tag{C.6.45}$$

The standard Killing vector basis is then again given by $M_{IJ} \equiv -(X_I\partial_J - X_J\partial_I)/\sqrt{16\pi G_{\rm N}}$, satisfying $[\![M_{12},M_{23}]\!] = M_{13}$ etc.

It was argued in [140] that for the most general set of consistent parity-preserving cubic interactions, assuming the algebra does not split as a direct sum of subalgebras, i.e. assuming the algebra is simple, there exists an up to normalization unique invariant bilinear form $\langle\cdot|\cdot\rangle_c$ on the Killing tensor algebra. We fix its normalization again by requiring the standard $\mathfrak{so}(d+2)$ Killing vectors M_{IJ} have unit norm,

$$\langle M_{12}|M_{12}\rangle_c \equiv 1\,. \tag{C.6.46}$$

Expressed in terms of the bilinears $\langle\bar\xi_{s-1}|\bar\xi_{s-1}\rangle_{\rm PI} = \frac{M^4}{2\pi}\int \bar\xi_{s-1}\cdot\bar\xi_{(s-1)}$ corresponding to (C.6.30), the invariant bilinear on the Killing tensor algebra takes the general form

$$\langle\bar\xi|\bar\xi'\rangle_c = \sum_s B_s\,\langle\bar\xi_{s-1}|\bar\xi'_{s-1}\rangle_{\rm PI}\,, \tag{C.6.47}$$

where B_s are certain constants fixed in principle by the algebra. The arguments given in [140] moreover imply that up to overall normalization, the coefficients B_s are independent of the coupling constants in the theory. More specifically, adapted (with some work, as described below) to our setting and conventions, and correcting for what we believe is a typo in [140], the coefficients are $B_s \propto (2s+d-4)(2s+d-2)$. We confirmed this by comparison to [143], where the invariant bilinear form for minimal Vasiliev gravity in AdS$_{d+1}$, dual to the free $O(N)$ model, was spelled out in detail, building on [140–142]. Analytically continuing to positive cosmological constant, implementing their ambient space X-contractions by a Gaussian integral, and reducing this integral to the sphere by switching to spherical coordinates, the expression in [143] can be brought to the form (C.6.47). This transformation almost completely cancels the factorials in the analogous coefficients b_s in [143], reducing to the simple $B_s \propto (2s+d-4)(2s+d-2)$. (The alternating signs of [143] are absent here due to the analytic continuation to positive cc.) Taking into account our normalization prescription (C.6.46) (which is different from the normalization chosen in [143]), we thus get

$$\langle \bar{\xi} | \bar{\xi} \rangle_{\mathrm{c}} = \frac{2\pi}{M^4} \cdot \frac{4G_{\mathrm{N}}}{A_{d-1}} \sum_s (2s+d-4)(2s+d-2) \, \langle \bar{\xi}^{(s-1)} | \bar{\xi}^{(s-1)} \rangle_{\mathrm{PI}} \,, \tag{C.6.48}$$

with $A_{d-1} = \mathrm{vol}(S^{d-1})$ as before. In view of the independence of the coefficients B_s of the couplings within the class of theories considered in [140], i.e. all parity-invariant massless higher-spin gravity theories consistent to cubic order, this result is universal, valid for this entire class.

As before for Einstein gravity and Yang Mills, from (C.6.48) we get the ratio

$$\frac{\mathrm{vol}(G)_{\mathrm{PI}}}{\mathrm{vol}(G)_{\mathrm{c}}} = \prod_s \left(\frac{M^4}{2\pi} \cdot \frac{A_{d-1}}{4G_{\mathrm{N}}} \cdot \frac{1}{(2s+d-4)(2s+d-2)} \right)^{\frac{1}{2} N^{\mathrm{KT}}_{s-1}} . \tag{C.6.49}$$

Combining this with (C.6.28) we see that, rather delightfully, all the unpleasant-looking factors cancel, leaving us with

$$\boxed{ Z_G = \frac{\gamma^{\dim G}}{\mathrm{vol}(G)_{\mathrm{c}}} \,, \qquad \gamma \equiv \sqrt{\frac{8\pi G_{\mathrm{N}}}{A_{d-1}}} } \tag{C.6.50}$$

This takes exactly the same form as the Einstein gravity result (C.6.44) except G is now the higher-spin symmetry group rather than the SO$(d+2)$ spin-2 symmetry group.

The cancelation of the UV divergent factors M is as expected from consistency with locality. The cancelation of the s-dependent factors on the other hand seems surprising, in view of the different origin of the numerator (spectrum of quadratic action) and the denominator (invariant bilinear form on higher spin algebra of interactions). Apparently the former somehow knows about the latter. We do not see an obvious reason why this is the case, although the simplicity and universality of the result suggests we should, and that this entire section should be replaceable by a one-line argument. Perhaps it is obvious in a frame-like formalism.

Newton constant from central charge

Recall that the Newton constant $G_{\rm N}$ appearing in (C.6.50) was defined by the $\mathfrak{so}(d+2)$ algebra (C.6.45) in our normalization conventions. An analogous definition can be given in dS$_{d+1}$ or AdS$_{d+1}$ where the algebra becomes $\mathfrak{so}(1,d+1)$ resp. $\mathfrak{so}(2,d)$. Starting from this definition, $G_{\rm N}$ can also be formally related to the Cardy central charge C of a putative[17] boundary CFT for AdS or dS, defined as the coefficient of the CFT 2-point function of the putative energy-momentum tensor. With our definition of $G_{\rm N}$, the computation of [255] remains unchanged, so we can just copy the result obtained there:

$$\boxed{C = \frac{(\pm 1)^{\frac{d-1}{2}}\,\Gamma(d+2)}{(d-1)\,\Gamma(\frac{d}{2})^2} \cdot \frac{A_{d-1}}{8\pi G_{\rm N}}} \tag{C.6.51}$$

where as before $A_{d-1} = 2\pi^{d/2}\ell^{d-1}/\Gamma(\frac{d}{2})$, and $\pm 1 = +1$ for AdS and -1 for dS. The central charge of N free real scalars equals $C = \frac{d}{2(d-1)}N$ in the conventions used here. Note that (C.6.51) reduces to the Brown-Henneaux formula $C = 3\ell/2G_{\rm N}$ for $d = 2$. In [39] it was argued that the Hartle-Hawking wave function of minimal Vasiliev gravity in dS$_4$ is perturbatively computed by a $d = 3$ CFT of N free Grassmann scalars. This CFT has central charge $C = -\frac{3}{4}N$, hence according to (C.6.51), $G_{\rm N} = 2^5/\pi N$ and $\gamma = \sqrt{2G_{\rm N}} = 8/\sqrt{\pi N}$.

The final result of this appendix, putting everything together, is stated in (5.5.17).

[17] There is no assumption whatsoever this CFT actually exists. One just imagines it exists and uses the formal holographic dictionary to infer the two-point function of this imaginary CFT's stress tensor. In dS, this "dual CFT" can be thought of as computing the Hartle-Hawking wave function of the universe [203].

C.7 One-loop and exact results for 3D theories

C.7.1 Character formula for $Z_{\rm PI}^{(1)}$

For $d = 2$, i.e. dS$_3$ / S^3, some of the generic-d formulae in sections 5.4 and 5.5 become a bit degenerate, requiring separate discussion. One reason $d = 2$ is a bit more subtle is that the spin-s irreducible representation of SO(2) actually comes in two distinct chiral versions $\pm s$, as do the corresponding SO$(1,3)$ irreducible representations $(\Delta, \pm s)$. Likewise the field modes of a spin s field in the path integral on S^3 split into chiral irreps $(n, \pm s)$ of SO(4). The dimensions $D^2_s = D^2_{-s} = 1$ and $D^4_{n,s} = D^4_{n,-s} = (1+n-s)(1+n+s)$ of the SO(2) and SO(4) irreps are correctly reproduced by the Weyl dimension formula (2.2.1), rather than (2.2.3). It should however be kept in mind that the single-particle Hilbert space of for instance a massive spin-$s \geq 1$ Pauli-Fierz field on dS$_3$ carries *both* helicity versions $(\Delta, \pm s)$ of the massive spin-s SO$(1,3)$ irrep, hence the character Θ to be used in expressions for $Z_{\rm PI}$ in this case is $\Theta = \Theta_{+s} + \Theta_{-s} = 2\Theta_{+s} = 2(q^\Delta + q^{2-\Delta})/(1-q)^2$. On the other hand for a real scalar field, we just have $\Theta = \Theta_0 = (q^\Delta + q^{2-\Delta})/(1-q)^2$.

For massless higher-spin gauge fields of spin $s \geq 2$, a similar reasoning implies we should include an overall factor of 2 in (C.6.17)-(C.6.18). For an $s = 1$ Maxwell field on the other hand, we get a factor of 2 in the first term but not in the second term (since the gauge parameter/ghost field is a scalar). The proper massless spin-s bulk and edge characters are then obtained from these by the polar term flip (3.4.17) as usual. This results in

$$\Theta_{\rm bulk,s} = 0 \quad (s \geq 2)\,, \qquad \Theta_{\rm bulk}^{(s=1)} = \frac{2q}{(1-q)^2}\,, \qquad \Theta_{\rm edge,s} = 0 \quad (\text{all } s)\,, \tag{C.7.1}$$

expressing the absence of propagating degrees of freedom (i.e. particles) for massless spin-$s \geq 2$ fields on dS$_3$.

This can also be derived more directly from the general path integral formula (C.5.34), taking into account the $\pm s$ doubling. In particular for massless $s \geq 2$, (C.6.8) gets replaced by

$$\hat{F}_s = \sum_{n\geq -1} \theta(1+n)\, 2D^4_{n,s}\big(q^{s+n} + q^{2-s+n}\big) - \sum_{n\geq -1} \theta(1+n)\, 2D^4_{n,s-1}\big(q^{s+1+n} + q^{1-s+n}\big), \tag{C.7.2}$$

which matters for the $n = -1$ term because $\theta(0) \equiv \frac{1}{2}$. For $s = 1$, we get instead

$$\hat{F}_1 = \sum_{n\geq -1} \theta(1+n)\, 2D^4_{n,1}\big(q^{1+n} + q^{1+n}\big) - \sum_{n\geq 0} D^4_{n,0}\big(q^{2+n} + q^{n}\big)\,. \tag{C.7.3}$$

For $s \geq 2$, the computation of $Z_{\rm char}$ and Z_G remains essentially unchanged. For $s = 1$ there are some minor changes. The edge character in (C.6.18) acquires an extra q^0 term in $d = 2$ because $q^{s+d-3} = q^0$, so the map $\hat{\Theta}_{\rm edge} \to [\hat{\Theta}_{\rm edge}]_+$ gets an extra -1 subtraction, as a result of which the factor -2 in (C.6.19) becomes a -3. Relatedly we get an extra q^0 term in $q^{2s+d-4} + q^{2s+d-2} + 2 = q^2 + 3$ in (C.6.27), and we end up with $Z_G = \tilde{\gamma}^{\dim G}/\mathrm{vol}(G)_{\rm c}$ with $\tilde{\gamma} = g\ell/\sqrt{A_1} = g\sqrt{\ell}/\sqrt{2\pi}$ instead of (C.6.40). Everything else remains the same.

Finally, the phase i^{-P_s} (C.6.10) is somewhat modified. For $s \geq 2$, from (C.7.2),

$$P_s = -\sum_{n=-1}^{s-3} \theta(1+n)\, 2D^4_{n,s} - \sum_{n=-1}^{s-2} \theta(1+n)\, 2D^4_{n,s-1} = \frac{1}{3}(2s-3)(2s-1)(2s+1) \tag{C.7.4}$$

Note that $P_2 = 5$, in agreement with [131]. $P_1 = 0$ as before, since there are no negative modes.

Conclusion

The final result for $Z^{(1)}_{\rm PI} = Z_G Z_{\rm char}$ in dS$_3$ replacing (5.5.17)-(5.5.18) is:

- For Einstein and HS gravity theories with $s \geq 2$,

$$Z^{(1)}_{\rm PI} = i^{-P}\, \frac{\gamma^{\dim G}}{\mathrm{vol}(G)_{\rm c}} \cdot Z_{\rm char}\,, \qquad Z_{\rm char} = e^{-2\dim G \int^{\times} \frac{dt}{2t} \frac{1+q}{1-q}} = (2\pi)^{\dim G} e^{-\dim G \cdot c\,\ell\,\epsilon^{-1}}\,, \tag{C.7.5}$$

  where as before $\gamma = \sqrt{8\pi G_{\rm N}/A_1} = \sqrt{4G_{\rm N}/\ell}$, $P = \sum_s P_s$, and $\mathrm{vol}(G)_{\rm c}$ is the volume with respect to the metric for which the standard $\mathfrak{so}(4)$ generators M_{IJ} have norm 1. We used (C.2.37) to evaluate $Z_{\rm char}$. The coefficient c of the linearly divergent term is an order 1 constant depending on the regularization scheme. (For the heat kernel regularization of appendix C.2, following section C.2.3, $c = \frac{3\pi}{4}$. For a simple cutoff at $t = \epsilon$ as in section C.2.4, $c = 2$.) The finite part is

$$\boxed{Z^{(1)}_{\rm PI,fin} = i^{-P}\,\frac{(2\pi\gamma)^{\dim G}}{{\rm vol}(G)_{\rm c}}\,, \qquad \gamma = \sqrt{\frac{8\pi G_{\rm N}}{2\pi\ell}}} \tag{C.7.6}$$

For example for Einstein gravity with $G = \mathsf{SO}(4)$, we get

$$Z^{(1)}_{\rm PI,fin} = i^{-5}\,\frac{(2\pi\gamma)^6}{(2\pi)^4} = -i\,4\pi^2\gamma^6\,. \tag{C.7.7}$$

- For Yang-Mills theories with gauge group G and coupling constant g, we get

$$Z^{(1)}_{\rm PI} = \frac{\tilde\gamma^{\dim G}}{{\rm vol}(G)_{\rm c}}\cdot e^{\dim G\int^{\times}\frac{dt}{2t}\frac{1+q}{1-q}\left(\frac{2q}{(1-q)^2}-3\right)}\,, \qquad \tilde\gamma = \frac{g\sqrt{\ell}}{\sqrt{2\pi}}\,. \tag{C.7.8}$$

Using (C.2.19), (C.2.37), the finite part evaluates to

$$Z^{(1)}_{\rm PI,fin} = \frac{(2\pi g\sqrt{\ell}\,Z_1)^{\dim G}}{{\rm vol}(G)_{\rm c}}\,, \qquad Z_1 = e^{-\frac{\zeta(3)}{4\pi^2}}\,. \tag{C.7.9}$$

As in (5.5.17), ${\rm vol}(G)_{\rm c}$ is the volume of G with respect to the metric defined by the trace appearing in the Yang-Mills action. As a check, for $G = U(1)$ we have ${\rm vol}(G)_{\rm c} = 2\pi$, so $Z = g\,e^{-\zeta(3)/4\pi^2}\sqrt{\ell}$ in agreement with [256] eq. (3.25).

- We could also consider the Chern-Simons partition function on S^3,

$$Z_k = \int \mathcal{D}A\, e^{i\,k\,S_{\rm CS}[A]}\,, \qquad S_{\rm CS}[A] \equiv \frac{1}{4\pi}\int {\rm Tr}(A\wedge dA + \frac{2}{3}A\wedge A\wedge A)\,, \tag{C.7.10}$$

with $k > 0$ suitably quantized ($k \in \mathbb{Z}$ for $G = SU(N)$ with ${\rm Tr}$ the trace in the N-dimensional representation). Because in this case the action is first order in the derivatives and not parity-invariant, it falls outside the class of theories we have focused on in this paper. It is not too hard though to generalize the analysis to this case. The main difference with Yang-Mills is that $\Theta_{\rm bulk} = 0 = \Theta_{\rm edge}$: like in the $s \geq 2$ case, the $s = 1$ Chern-Simons theory has no particles. The function $\hat{F}_1$ is no longer given by the Maxwell version (C.7.3), but rather by (C.7.2), except without the factors of 2, related to the fact that the CS action is first order in the derivatives. This immediately gives $F_1 = \hat{F}_1 = -2\,\frac{1+q}{1-q}$. The computation of the volume factor is analogous to our earlier discussions. The result (in canonical framing [257]) is

$$Z_k^{(1)} = \frac{\tilde{\gamma}^{\dim G}}{\mathrm{vol}(G)_{\mathrm{Tr}}}\, e^{-2\dim G \int^{\times} \frac{dt}{2t}\frac{1+q}{1-q}}\,, \qquad Z_{k,\mathrm{fin}}^{(1)} = \frac{(2\pi\tilde{\gamma})^{\dim G}}{\mathrm{vol}(G)_{\mathrm{Tr}}}\,, \qquad \tilde{\gamma} = \frac{1}{\sqrt{k}}\,, \quad \text{(C.7.11)}$$

where $\mathrm{vol}(G)_{\mathrm{Tr}}$ is the volume with respect to the metric defined by the trace appearing in the Chern-Simons action (C.7.10). This agrees with the standard results in the literature, nicely reviewed in section 4 of [258].

C.7.2 Chern-Simons formulation of Einstein gravity

3D Einstein gravity can be reformulated as a Chern-Simons theory [164, 259]. Although well-known, we briefly review some of the basic ingredients and conceptual points here to facilitate the discussion of the higher-spin generalization in section C.7.3. A more detailed review of certain aspects, including more explicit solutions, can be found in section 4 of [260]. Explicit computations using the Chern-Simons formulation of $\Lambda > 0$ Euclidean quantum gravity with emphasis on topologies more sophisticated than the sphere can be found in [261–263].

Lorentzian gravity

For the Lorentzian theory with positive cosmological constant, amplitudes are computed by path integrals $\int \mathcal{D}A\, e^{iS_L}$ with real Lorentzian $SL(2,\mathbb{C})$ Chern-Simons action [165]

$$S_L = (l + i\kappa) S_{\mathrm{CS}}[A_+] + (l - i\kappa) S_{\mathrm{CS}}[A_-]\,, \qquad A_+^* = A_-, \qquad \text{(C.7.12)}$$

where S_{CS} is as in (C.7.10) with $A_\pm$ an $\mathfrak{sl}(2,\mathbb{C})$-valued connection and $\mathrm{Tr} = \mathrm{Tr}_2$. The vielbein e and spin connection ω are the real and imaginary parts of the connection:

$$A_\pm = \omega \pm ie/\ell\,, \qquad ds^2 = 2\,\mathrm{Tr}_2\, e^2 = \eta_{ij} e^i e^j\,. \qquad \text{(C.7.13)}$$

For the last equality we decomposed $e = e^i L_i$ in a basis L_i of $\mathfrak{sl}(2,\mathbb{R})$, say

$$(L_1, L_2, L_3) \equiv (\tfrac{1}{2}\sigma_1, \tfrac{1}{2}i\sigma_2, \tfrac{1}{2}\sigma_3) \qquad \Rightarrow \qquad \eta_{ij} \equiv 2\,\mathrm{Tr}_2(L_i L_j) = \mathrm{diag}(1,-1,1)\,. \quad \text{(C.7.14)}$$

Note that $[L_i, L_j] = -\epsilon_{ijk} L^k$ with $L^k \equiv \eta^{kk'} L_{k'}$. When $l = 0$, the action reduces to the firs-order form of the Einstein action with Newton constant $G_{\rm N} = \ell/4\kappa$ and cosmological constant $\Lambda = 1/\ell^2$. The equations of motion stipulate $A_\pm$ must be flat connections:

$$dA_\pm + A_\pm \wedge A_\pm = 0\,, \qquad \text{(C.7.15)}$$

equivalent with the Einstein gravity torsion constraint (with $\omega^i{}_j \equiv \eta^{il}\epsilon_{ljk}\omega^k$) and the Einstein equations of motion [164]. Turning on l deforms the action by parity-odd terms of gravitational Chern-Simons type. This does not affect the equations of motion (C.7.15). We can take $l \geq 0$ without loss of generality. The part of the action multiplied by l has a discrete ambiguity forcing l to be integrally quantized, like k in (C.7.10). Summarizing,

$$0 \leq l \in \mathbb{Z}\ , \qquad 0 < \kappa = \frac{2\pi\ell}{8\pi G_{\rm N}} \in \mathbb{R}\,, \qquad \text{(C.7.16)}$$

dS$_3$ vacuum solution

A flat connection corresponding to the de Sitter metric can be obtained as follows. (We will be brief because the analog for the sphere below will be simpler and make this more clear.) Define $Q(X) \equiv 2\,(X^4 L_4 + i X^i L_i)$ with $L_4 \equiv \frac{1}{2}\mathbf{1}$ and note that $\det Q = X_4^2 + \eta_{ij} X^i X^j =: \eta_{IJ} X^I X^J$, so $\mathcal{M} \equiv \{X |\det Q(X) = 1\}$ is the dS$_3$ hyperboloid, and Q is a map from $\mathcal{M}$ into $SL(2,\mathbb{C})$. Its square root $h \equiv Q^{1/2}$ is then a map from $\mathcal{M}$ into $SL(2,\mathbb{C})/\mathbb{Z}_2 \simeq \mathrm{SO}(1,3)$, so $A_+ \equiv h^{-1}dh$ is a flat $\mathfrak{sl}(2,\mathbb{C})$-valued connection on $\mathcal{M}$. Moreover on $\mathcal{M}$ we have $Q^* = Q^{-1}$, so $h^* = h^{-1}$, $A_- = A_+^* = -(dh)h^{-1}$, and $ds^2 = -\frac{1}{2}\ell^2\,\mathrm{Tr}(A_+ - A_-)^2 = -\frac{1}{2}\ell^2\,\mathrm{Tr}\,(Q^{-1}dQ)^2 = \ell^2 \eta_{IJ} dX^I dX^J$, which is the de Sitter metric of radius ℓ on $\mathcal{M}$.

Euclidean gravity

Like the Einstein-Hilbert action — or any other action for that matter — (C.7.12) may have complex saddle points, that is to say flat connections $A_\pm$ which do not satisfy the reality constraint (C.7.12), or equivalently solutions for which some components of the vielbein and spin connection are not real. Of particular interest for our purposes is the solution corresponding to the round metric on S^3. This can be obtained from the dS$_3$ solution as usual by a Wick rotation of the time coordinate. Given our choice of $\mathrm{sl}(2,\mathbb{R})$ basis (C.7.14), this means $X^2 \to -iX^2$. At the level of the vielbein $e = e^i L_i$ such a Wick rotation is implemented as $e^2 \to -ie^2$. Similarly, recalling $\omega_{ij} = \epsilon_{ijk}\omega^k$, the spin connection $\omega = \omega^i L_i$ rotates as $\omega^1 \to i\omega^1$, $\omega^3 \to i\omega^3$. Equivalently, $A_\pm \to (\omega^i \pm e^i/\ell)S_i$ where $S_i \equiv \frac{1}{2}i\sigma_i$. Notice the S_i are the generators of $\mathfrak{su}(2)$, satisfying $[S_i, S_j] = -\epsilon_{ijk}S_k$, $-2\,\mathrm{Tr}_2(S_i S_j) = \delta_{ij}$ and $S_i^\dagger = -S_i$. Thus the Lorentzian metric η_{ij} gets replaced by the Euclidean metric δ_{ij}, the Lorentzian $\mathfrak{sl}(2,\mathbb{C}) = \mathfrak{so}(1,3)$ reality condition gets replaced by the Euclidean $\mathfrak{su}(2) \oplus \mathfrak{su}(2) = \mathfrak{so}(4)$ reality condition, and the Lorentzian path integral $\int \mathcal{D}A\, e^{iS_L}$ becomes a Euclidean path integral $\int \mathcal{D}A\, e^{-S_E}$, where $S_E \equiv -iS_L$ is the Euclidean action:

$$S_E = (\kappa - il)\, S_{\mathrm{CS}}[A_+] - (\kappa + il)\, S_{\mathrm{CS}}[A_-]\,, \qquad A_\pm^\dagger = -A_\pm\,. \tag{C.7.17}$$

This can be interpreted as the Chern-Simons formulation of Euclidean Einstein gravity with positive cosmological constant. The $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$-valued connection (A_+, A_-) encodes the Euclidean vielbein, spin connection and metric as

$$A_\pm = \omega \pm e/\ell = (\omega^i \pm e^i/\ell)S_i\,, \qquad S_i \equiv \tfrac{1}{2}i\sigma_i\,, \qquad ds^2 = -2\,\mathrm{Tr}_2\, e^2 = \delta_{ij}e^i e^j\,. \tag{C.7.18}$$

The Euclidean counterpart of the reality condition of the Lorentzian action is that S_E gets mapped to S_E^* under reversal of orientation. Reversal of orientation maps $S_{\mathrm{CS}}[A] \to -S_{\mathrm{CS}}[A]$, and in addition here it also exchanges the $\pm$ parts of the decomposition $\mathfrak{so}(4) = \mathfrak{su}(2)_+ \oplus \mathfrak{su}(2)_-$ into self-dual and anti-self-dual parts, that is to say it exchanges $A_+ \leftrightarrow A_-$. Thus orientation reversal maps $S_E \to -(\kappa - il)\, S_{\mathrm{CS}}[A_-] + (\kappa + il)\, S_{\mathrm{CS}}[A_+] = S_E^*$, as required.

Round sphere solutions

Parametrizing S^3 by $g \in SU(2) \simeq S^3$, it is easy to write down a flat $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ connection yielding the round metric of radius ℓ:

$$(A_+, A_-) = (g^{-1}dg, 0) \qquad \Rightarrow \qquad e/\ell = \tfrac{1}{2} g^{-1}dg = \omega\,, \quad ds^2 = -\tfrac{1}{2}\ell^2\, \mathrm{Tr}(g^{-1}dg)^2\,. \tag{C.7.19}$$

The radius can be checked by observing that along an orbit $g(\varphi) = e^{\varphi S_3}$, we get $g^{-1}dg = d\varphi\, S_3$ so $ds = \frac{1}{2}\ell d\varphi$ and the orbit length is $\int_0^{4\pi} ds = 2\pi\ell$. The on-shell action is $S_E = -\frac{\kappa - il}{12\pi}\int_{S^3} \mathrm{Tr}_2(g^{-1}dg)^3 = -\frac{2(\kappa-il)}{3\pi\ell^3}\int e^i e^j e^k\, \mathrm{Tr}_2(S_i S_j S_k) = -\frac{\kappa - il}{6\pi\ell^3}\int e^i e^j e^k\, \epsilon_{ijk} = -2\pi(\kappa - il)$, so

$$\exp(-S_E) = \exp(2\pi\kappa + 2\pi i l) = \exp\Big(\frac{2\pi\ell}{4G_{\mathrm{N}}}\Big)\,, \tag{C.7.20}$$

where we used (C.7.16). This reproduces the standard Gibbons-Hawking result [57] for dS_3.

More generally we can consider flat connections of the form $(A_+, A_-) = (h_+^{-1}dh_+, h_-^{-1}dh_-)$ with $h_\pm = g^{n_\pm}$, where $n_\pm \in \mathbb{Z}$ if we take the gauge group to be $G = SU(2) \times SU(2)$. These are all related to the trivial connection $(0,0)$ by a *large* gauge transformation $g \in S^3 \to (h_+, h_-) \in G$. All other flat connections on S^3 are obtained from these by gauge transformations continuously connected to the identity, which are equivalent to diffeomorphisms and vielbein rotations continuously connected to the identity in the metric description [164]. Large gauge transformations on the other hand are in general *not* equivalent to large diffeomorphisms. Indeed,

$$e^{-S_E} = e^{2\pi n\kappa + 2\pi i \tilde{n} l} = e^{2\pi n \kappa}\,, \qquad n \equiv n_+ - n_-, \quad \tilde{n} \equiv n_+ + n_-\,, \tag{C.7.21}$$

so evidently different values of n are physically inequivalent. Conversely, for a fixed value of n but different values of $\tilde{n}$, we get the same metric, so these solutions are geometrically equivalent. In particular the $n = 1$ solutions all produce the same round metric (C.7.19). For $n = 0$, the metric vanishes. For $n < 0$, we get a vielbein with negative determinant. Only vielbeins with positive determinant reproduce the Einstein-Hilbert action with the correct sign, so from the point of view of gravity we should discard the $n < 0$ solutions. Finally the cases $n > 1$ correspond to a metric describing a chain of n spheres connected by throats of zero size, presumably more appropriately thought of as n disconnected spheres.

Euclidean path integral

The object of interest to us is the Euclidean path integral $Z = \int \mathcal{D}A\, e^{-S_E[A]}$, defined perturbatively around an $n = n_+ - n_- = 1$ round sphere solution $(\bar{A}_+, \bar{A}_-) = (g^{-n_+}dg^{n_+}, g^{-n_-}dg^{n_-})$, such as the $(1,0)$ solution (C.7.19). Physically, this can be interpreted as the all-loop quantum-corrected Euclidean partition function of the dS$_3$ static patch. For simplicity we take $G = SU(2) \times SU(2)$, so $n_\pm \in \mathbb{Z}$ and we can formally factorize Z as an $SU(2)_{k_+} \times SU(2)_{k_-}$ CS partition function where $k_\pm = l \pm i\kappa$, with $l \in \mathbb{Z}^+$ and $\kappa \in \mathbb{R}^+$:

$$Z = \int_{(n_+,n_-)} \mathcal{D}A\, e^{ik_+ S_{\rm CS}[A_+] + ik_- S_{\rm CS}[A_-]} = Z_{\rm CS}\big(SU(2)_{k_+}|\bar{A}_{n_+}\big)\, Z_{\rm CS}\big(SU(2)_{k_-}|\bar{A}_{n_-}\big)\,, \tag{C.7.22}$$

Here the complex-k CS partition function $Z_{\rm CS}(SU(2)_k|\bar{A}_m) \equiv \int_m \mathcal{D}A\, e^{ikS_{\rm CS}[A]}$ is defined perturbatively around the critical point $\bar{A} = g^{-m}dg^m$. It is possible, though quite nontrivial in general, to define Chern-Simons theories at complex level k on general 3-manifolds M_3 [264, 265]. Our goal is less ambitious, since we only require a perturbative expansion of Z around a given saddle, and moreover we restrict to $M_3 = S^3$. In contrast to generic M_3, at least for integer k, the CS action on S^3 has a unique critical point modulo gauge transformations, and its associated perturbative large-k expansion is not just asymptotic, but actually converges to a simple, explicitly known function: in canonical framing [257],

$$Z_{\rm CS}\big(SU(2)_k|\bar{A}\big)_0 = \sqrt{\tfrac{2}{2+k}}\, \sin\!\big(\tfrac{\pi}{2+k}\big)\, e^{i(2+k)S_{\rm CS}[\bar{A}]} \qquad (k \in \mathbb{Z}^+)\,. \tag{C.7.23}$$

The dependence on the choice of critical point $\bar{A} = g^{-m}dg^m$ actually drops out for integer k, as $S_{\rm CS}[\bar{A}] = -2\pi m \in 2\pi\mathbb{Z}$. We have kept it in the above expression to because this is no longer the case for complex k. Analytic continuation to $k_\pm = l \pm i\kappa$ with $l \in \mathbb{Z}^+$ and $\kappa \in \mathbb{R}^+$ in (C.7.22) then gives:

$$Z_0 = \Big|\sqrt{\tfrac{2}{2+l+i\kappa}}\, \sin\!\big(\tfrac{\pi}{2+l+i\kappa}\big)\Big|^2 e^{2\pi n\kappa - 2\pi i\, \tilde{n}(2+l)} = \Big|\sqrt{\tfrac{2}{2+l+i\kappa}}\, \sin\!\big(\tfrac{\pi}{2+l+i\kappa}\big) \cdot e^{\pi\kappa}\Big|^2. \tag{C.7.24}$$

Framing dependence of phase and one-loop check

For a general choice of S^3 framing with SO(3) spin connection $\hat{\omega}$, (C.7.23) gets replaced by [257]

$$Z_{\rm CS}(SU(2)_k|\bar{A}) = \exp(\tfrac{i}{24}c(k)I(\hat{\omega}))\, Z_{\rm CS}(SU(2)_k|\bar{A})_0\,, \qquad c(k) = 3(1-\tfrac{2}{2+k})\,, \quad \text{(C.7.25)}$$

where $I(\hat{\omega}) = \frac{1}{4\pi}\int {\rm Tr}_3(\hat{\omega}\wedge d\hat{\omega} + \frac{2}{3}\hat{\omega}\wedge\hat{\omega}\wedge\hat{\omega})$ is the gravitational Chern-Simons action. The action $I(\hat{\omega})$ can be defined more precisely as explained under (2.22) of [266], by picking a 4-manifold M with boundary $\partial M = S^3$ and putting

$$I(\hat{\omega}) \equiv I_M \equiv \frac{1}{4\pi}\int_M {\rm Tr}(R\wedge R)\,, \qquad \text{(C.7.26)}$$

where R is the curvature form of M, $R^\mu{}_\nu = \frac{1}{2}R^\mu{}_{\nu\rho\sigma}dx^\rho\wedge dx^\sigma$. Taking M to be a flat 4-ball B, the curvature vanishes so $I_B = 0$, corresponding to canonical framing. Viewing B as a 4-hemisphere with round metric has ${\rm Tr}(R\wedge R) = 0$ pointwise so again $I_B = 0$. Gluing any other 4-manifold M with boundary S^3 to B, we get a closed 4-manifold $X = M - B$, with $I_M - I_B = \frac{1}{4\pi}\int_X {\rm Tr}(R\wedge R) = 2\pi p_1(X)$, where $p_1(X)$ is the Pontryagin number of X. According to the Hirzebruch signature theorem, the signature $\sigma(X) = b_2^+ - b_2^-$ of the intersection form of the middle cohomology of X equals $\frac{1}{3}p_1(X)$. Therefore, for any choice of M,

$$I_M = 6\pi r\,, \qquad r = \sigma(X) \in \mathbb{Z}\,. \qquad \text{(C.7.27)}$$

For example $r = 1$ for $X = \mathbb{CP}^2$ and $r = p - q$ for $X = p\mathbb{CP}^2\#q\overline{\mathbb{CP}}^2$. Thus for general framing, (C.7.25) becomes $Z_{\rm CS}(k|m) = Z_{\rm CS}(k|m)_0\, \exp(r\, c(k)\, \frac{i\pi}{4})$ and (C.7.24) becomes

$$Z_r = e^{ir\phi}\left|\sqrt{\tfrac{2}{2+l+i\kappa}}\,\sin(\tfrac{\pi}{2+l+i\kappa})e^{\pi\kappa}\right|^2, \qquad r \in \mathbb{Z}\,, \qquad \text{(C.7.28)}$$

where, using $c(k) = 3(1-\frac{2}{2+k})$, the phase is given by

$$r\phi = r\big(c(l+i\kappa) + c(l-i\kappa)\big)\frac{\pi}{4} = r\big(1 - \tfrac{2(2+l)}{(2+l)^2+\kappa^2}\big)\frac{3\pi}{2}\,. \qquad \text{(C.7.29)}$$

In the weak-coupling limit $\kappa \to \infty$,

$$Z_r \to (-i)^r \, \frac{2\pi^2}{\kappa^3} \cdot e^{2\pi\kappa} \,. \tag{C.7.30}$$

Using (C.7.16) and taking into account that we took $G = SU(2) \times SU(2)$ here, the absolute value agrees with our general one-loop result (C.7.6) in the metric formulation, with the phase $(-i)^r$ matching Polchinski's phase $i^{-P} = i^{-5} = -i$ in (5.5.17) for odd framing r.[18] We do not have any useful insights into why (or whether) CS framing and the phase i^{-P} might have anything to do with each other, let alone why odd but not even framing should reproduce the phase of [131]. Perhaps different contour rotation prescriptions as those assumed in [131] might reproduce the canonically framed ($r = 0$) result in the metric formulation of Euclidean gravity. We leave these questions open.

Comparison to previous results: The Chern-Simons formulation of gravity was applied to calculate Euclidean $\Lambda > 0$ partition functions in [261–263]. The focus of these works was on summing different topologies. Our one-loop (C.7.30) in canonical framing agrees with [261] up to an unspecified overall normalization constant in the latter, agrees with $Z(S^3)/Z(S^1 \times S^2)$ in [262] combining their eqs. (13),(32), and disagrees with eq. (4.39) in [263], $Z^{(1)}(S^3) = \pi^3/(2^5\kappa)$.

C.7.3 Chern-Simons formulation of higher-spin gravity

The $SL(2,\mathbb{C})$ Chern-Simons formulation of Einstein gravity (C.7.12) has a natural extension to an $SL(n,\mathbb{C})$ Chern-Simons formulation of higher-spin gravity — the positive cosmological constant analog of the theories studied e.g. in [166–169, 182, 183]. The Lorentzian action is

$$S_L = (l + i\kappa)\, S_{\rm CS}[\mathcal{A}_+] + (l - i\kappa)\, S_{\rm CS}[\mathcal{A}_-] \,, \qquad \mathcal{A}_+^* = \mathcal{A}_- \,, \tag{C.7.31}$$

where $S_{\rm CS}[\mathcal{A}] = \frac{1}{4\pi}\int {\rm Tr}_n(\mathcal{A}\wedge d\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A})$, an $\mathcal{A}$ is an ${\rm sl}(n,\mathbb{C})$-valued connection, $\kappa \in \mathbb{R}^+$ and $l \in \mathbb{Z}^+$. The corresponding Euclidean action $S_E = -iS_L$ extending (C.7.17) is given by

$$S_E = (\kappa - il)\, S_{\rm CS}[\mathcal{A}_+] - (\kappa + il)\, S_{\rm CS}[\mathcal{A}_-] \,, \qquad \mathcal{A}_\pm^\dagger = -\mathcal{A}_\pm \,, \tag{C.7.32}$$

[18] Strictly speaking for $r = 1$ mod 4, but i^P vs i^{-P} in (5.5.17) is a matter of conventions, so there is no meaningful distinction we can make here.

where $\mathcal{A}_\pm$ are now independent $\mathfrak{su}(n)$-valued connections.

C.7.3.1 Landscape of dS$_3$ vacua

The solutions A of the original ($n = 2$) Einstein gravity theory can be lifted to solutions $\mathcal{A} = R(A)$ of the extended ($n > 2$) theory by choosing an embedding R of (2) into (n). More concretely, such lifts are specified by picking an n-dimensional representation R of $\mathfrak{su}(2)$,

$$R = \oplus_a \mathbf{m}_a \,, \qquad \sum_a m_a = n \,, \qquad \mathcal{S}_i = R(S_i) = \oplus_a J_i^{(m_a)} \,. \tag{C.7.33}$$

Here $J_i^{(m)}$ are the standard anti-hermitian spin $j = \frac{m-1}{2}$ representation matrices of $\mathfrak{su}(2)$, satisfying the same commutation relations and reality properties as the spin-$\frac{1}{2}$ generators S_i in (C.7.18). Then the matrices $\mathcal{L}_i \equiv R(L_i)$ with the L_i as in (C.7.14) are real, generating the corresponding n-dimensional representation of $\mathfrak{sl}(2,\mathbb{R})$. The Casimir eigenvalue of the spin $j = \frac{m-1}{2}$ irrep is $j(j+1) = \frac{1}{4}(m^2-1)$, so

$$\mathrm{Tr}_n(\mathcal{S}_i\mathcal{S}_j) = -\tfrac{1}{2}T_R\,\delta_{ij} \,, \qquad \mathrm{Tr}_n(\mathcal{L}_i\mathcal{L}_j) = \tfrac{1}{2}T_R\,\eta_{ij} \,, \qquad T_R \equiv \frac{1}{6}\sum_a m_a(m_a^2-1) \,; \tag{C.7.34}$$

A general $SL(2,\mathbb{C})$ connection $A = A^i L_i$ has curvature $dA + A\wedge A = \big(dA^i - \frac{1}{2}\epsilon^i{}_{jk}A^jA^k\big)L_i$, and an $SL(n,\mathbb{C})$ connection of the form $\mathcal{A} = R(A) = A^i\mathcal{L}_i$ has curvature $d\mathcal{A} + \mathcal{A}\wedge\mathcal{A} = \big(dA^i - \frac{1}{2}\epsilon^i{}_{jk}A^jA^k\big)\mathcal{L}_i$, hence $\mathcal{A} = R(A)$ solves the equations of motion of the extended $SL(n,\mathbb{C})$ theory iff A solves the equations of motion of the original Einstein $SL(2,\mathbb{C})$ theory. In other words, restricting to connections $\mathcal{A} = A^i\mathcal{L}_i$ amounts to a consistent truncation, which may be interpreted as the gravitational subsector of the $n > 2$ theory. Substituting $\mathcal{A} = R(A)$ into the action (C.7.31) gives the consistently truncated action

$$S_L = (l + i\kappa)T_R\,S_{\mathrm{CS}}[A_+] + (l - i\kappa)T_R\,S_{\mathrm{CS}}[A_-] \,, \qquad A_+^* = A_- \,, \tag{C.7.35}$$

which is of the exact same form as the original Einstein CS gravity theory (C.7.12), except $l + i\kappa$ is replaced by $(l + i\kappa)T_R$. Thus we can naturally interpret the components $A_\pm^i$ again as metric/vielbein/spin connection degrees of freedom, just like in (C.7.18), i.e. $A_\pm^i = \omega^i \pm ie^i/\ell$, $ds^2 = \eta_{ij}e^ie^j$, and the lift $\mathcal{A} = R(A)$ of the original solution A corresponding to the dS$_3$ metric again as a solution corresponding to the dS_3 metric. The difference is that the original relation (C.7.16) between κ and ℓ/G_{N} gets modified to

$$\kappa\, T_R = \frac{2\pi\ell}{8\pi G_{\mathrm{N}}}\,. \tag{C.7.36}$$

Since κ is fixed, this means the dimensionless ratio ℓ/G_{N} depends on the choice of R. Thus the different solutions $\mathcal{A} = R(A)$ of the $SL(n,\mathbb{C})$ theory can be thought of as different de Sitter vacua of the theory, labeled by R, with different values of the curvature radius in Planck units ℓ/G_{N}. These are the dS analog of the AdS vacua discussed in [183]. The total number of vacua labeled by $R = \oplus_a \mathbf{m}_a$ equals the number of partitions of $n = \sum_a m_a$,

$$\mathcal{N}_{\mathrm{vac}} \sim e^{2\pi\sqrt{n/6}} \qquad (n \gg 1)\,. \tag{C.7.37}$$

For, say, $n \sim 2 \times 10^5$, this gives $\mathcal{N}_{\mathrm{vac}} \sim 10^{500}$.

Analogous considerations hold for the Euclidean version of the theory. For example the round sphere solution (C.7.19) is lifted to

$$(\mathcal{A}_+, \mathcal{A}_-) = \big(R(A_+), 0\big) = \big(R(g)^{-1} dR(g), 0\big), \qquad R(e^{\alpha^i S_i}) \equiv e^{\alpha^i \mathcal{S}_i}\,, \tag{C.7.38}$$

with the sphere radius ℓ in Planck units given again by (C.7.36). The tree-level contribution of the solution (C.7.38) to the Euclidean path integral is

$$\exp(-S_E) = \exp\big(2\pi\kappa\, T_R\big) = \exp\Big(\frac{2\pi\ell}{4\, G_{\mathrm{N}}}\Big)\,. \tag{C.7.39}$$

Note that $\mathcal{S}^{(0)} \equiv -S_E = \frac{2\pi\ell}{4G_{\mathrm{N}}}$ is the usual dS_3 Gibbons-Hawking horizon entropy [57]. Its value $\mathcal{S}^{(0)} = 2\pi\kappa\, T_R$ depends on the vacuum $R = \oplus_a \mathbf{m}_a$ through T_R as given by (C.7.34). The vacuum R maximizing e^{-S_E} corresponds to the partition of $n = \sum_a m_a$ maximizing T_R. Clearly the maximum is achieved for $R = \mathbf{n}$:

$$\max_R\, T_R = T_{\mathbf{n}} = \frac{n(n^2-1)}{6}\,. \tag{C.7.40}$$

The corresponding embedding of $\mathfrak{su}(2)$ into $\mathfrak{su}(n)$ is called the “principal embedding”. Thus the "principal vacuum" maximizes the entropy at $S_{\mathrm{GH},\mathbf{n}} = \frac{1}{6}n(n^2-1)\,2\pi\kappa$, exponentially dominating the Euclidean path integral in the semiclassical (large-κ) regime. In the remainder we focus on the Euclidean version of the theory.

C.7.3.2 Higher-spin field spectrum and algebra

Of course for $n > 2$, there are more degrees of freedom in the $2(n^2-1)$ independent components of $\mathcal{A}_\pm$ than just the 3+3 vielbein and spin connection degrees of freedom $\mathcal{A}^i_\pm \mathcal{S}_i$. The full set of fluctuations around the vacuum solution can be interpreted in a metric-like formalism as higher-spin field degrees of freedom. The precise spectrum depends on the vacuum R. For the principal vacuum $R = \mathbf{n}$, we get the higher-spin vielbein and spin connections of a set of massless spin-s fields of $s = 2, 3, \ldots, n$, as was worked out in detail for the AdS analog in [182]. Indeed $\mathfrak{su}(n)$ decomposes under the principally embedded $\mathfrak{su}(2)$ subalgebra into spin-r irreps, $r = 1, 2, \ldots, n-1$, generated by the traceless symmetric products $\mathcal{S}_{i_1\cdots i_r}$ of the generators $\mathcal{S}_i$. As reviewed in [141], this means we can identify the $\mathfrak{su}(n)_+ \oplus \mathfrak{su}(n)_-$ Lie algebra of the theory (C.7.32) with the higher-spin algebra $\mathrm{hs}_n(\mathfrak{su}(2))_+ \oplus \mathrm{hs}_n(\mathfrak{su}(2))_-$, where $\mathfrak{su}(2)_+ \oplus \mathfrak{su}(2)_- = \mathfrak{so}(4)$ is the principally embedded gravitational subalgebra. In the metric-like formalism the spin-r generators correspond to (anti-)self-dual Killing tensors of rank r. These are the Killing tensors of massless symmetric spin-s fields with $s = r+1$. As a check, recall the number of (anti-)self-dual rank r Killing tensors on S^3 equals $D^4_{r,\pm r} = 2r+1$, correctly adding up to

$$\sum_\pm \sum_{r=1}^{n-1} D^4_{r,\pm r} = 2\sum_{r=1}^{n-1}(2r+1) = 2(n^2-1)\,. \tag{C.7.41}$$

For different choices of embedding R, we get different $\mathfrak{su}(2)$ decompositions of $\mathfrak{su}(n)$. For example for $n = 12$, while the principal embedding $R = \mathbf{12}$ considered above gives the $\mathfrak{su}(2)$ decomposition $\mathbf{143}_{\mathfrak{su}(12)} = \mathbf{3}+\mathbf{5}+\mathbf{7}+\mathbf{9}+\mathbf{11}+\mathbf{13}+\mathbf{15}+\mathbf{17}+\mathbf{19}+\mathbf{21}+\mathbf{23}$, taking $R = \mathbf{6}\oplus\mathbf{4}\oplus\mathbf{2}$ gives $\mathbf{143}_{\mathfrak{su}(12)} = 2\cdot\mathbf{1} + 7\cdot\mathbf{3} + 8\cdot\mathbf{5} + 6\cdot\mathbf{7} + 3\cdot\mathbf{9} + \mathbf{11}$. Interpreting these as Killing tensors for n_s massless spin-s fields, we get for the former $n_2 = 1, n_3 = 1, \ldots, n_{12} = 1$, and for the latter $n_1 = 2, n_2 = 7, n_3 = 8, n_4 = 6, n_5 = 3, n_6 = 1$. The tree-level entropy $\mathcal{S}^{(0)} = 2\pi\ell/4G_{\mathrm{N}}$ for $R = \mathbf{12}$ is $\mathcal{S}^{(0)} = 286\cdot 2\pi\kappa$, and for $R = \mathbf{6}\oplus\mathbf{4}\oplus\mathbf{2}$ it is $\mathcal{S}^{(0)} = 46\cdot 2\pi\kappa$.

C.7.3.3 One-loop Euclidean path integral from metric-like formulation

In view of the above higher-spin interpretation of the theory, we can apply our general massless HS formula (C.7.6) with $G = SU(n) \times SU(n)$ to obtain the one-loop contribution to the Euclidean path integral (for $l = 0$). In combination with (C.7.36) this takes the form

$$Z_{\rm PI}^{(1)} = i^{-P} \, \frac{(2\pi\gamma)^{\dim G}}{\mathrm{vol}(G)_{\rm c}} \, , \qquad \gamma = \sqrt{\frac{8\pi G_{\rm N}}{2\pi\ell}} = \frac{1}{\sqrt{\kappa \, T_R}} \, . \tag{C.7.42}$$

Recall that $\mathrm{vol}(G)_{\rm c}$ is the volume of G with respect to the metric normalized such that $\langle M|M\rangle_{\rm c} = 1$, where M is one of the standard $\mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ generators, which we can for instance take to be the rotation generator $M = \mathcal{S}_3 \oplus \mathcal{S}_3$. In the context of Chern-Simons theory, it is more natural to consider the volume $\mathrm{vol}(G)_{\mathrm{Tr}_n}$ with respect to the metric defined by the trace appearing in the Chern-Simons action (C.7.32). Using the definition of T_R in (C.7.35), we see the trace norm of M is $\langle M|M\rangle_{\mathrm{Tr}_n} = -2\,\mathrm{Tr}_n(\mathcal{S}_3\mathcal{S}_3) = T_R = T_R\langle M|M\rangle_{\rm c}$, hence $\mathrm{vol}(G)_{\mathrm{Tr}_n} = (\sqrt{T_R})^{\dim G}\,\mathrm{vol}(G)_{\rm c}$. Note that upon substituting this in (C.7.42), the T_R-dependent factors cancel out. Finally, using (C.7.4), we get $P = \sum_{s=2}^{n} P_s = \frac{1}{3}(2s-3)(2s-1)(2s+1) = \frac{2}{3}n^2(n-1)(n+1) - (n^2-1)$. Because $(n-1)\cdot n \cdot (n+1)$ is divisible by 3, the first term is an integer, and moreover a multiple of 8 because either n^2 or $(n+1)(n-1)$ is a multiple of 4. Hence $i^{-P} = i^{(n^2-1)}$, which equals $-i$ for even n and $+1$ for odd n. Thus we get

$$Z_{\rm PI}^{(1)} = i^{n^2-1} \, \frac{(2\pi\tilde\gamma)^{\dim G}}{\mathrm{vol}(G)_{\mathrm{Tr}_n}} \, , \qquad \tilde\gamma \equiv \frac{1}{\sqrt{\kappa}} \, . \tag{C.7.43}$$

C.7.3.4 Euclidean path integral from CS formulation

As in the $SU(2) \times SU(2)$ Einstein gravity case, we can derive an all-loop expression for the Euclidean partition function $Z(R)$ of the $SU(n) \times SU(n)$ higher-spin gravity theory (C.7.32) expanded around a lifted round sphere solution $\bar{\mathcal{A}} = R(\bar A)$ such as (C.7.38), by naive analytic continuation of the exact $SU(n)_{k_+} \times SU(n)_{k_-}$ partition function on S^3 to $k_\pm = l \pm i\kappa$, paralleling (C.7.22) and the subsequent discussion there. The $SU(n)_k$ generalization of the canonically framed $SU(2)_k$ result (C.7.23) as spelled out e.g. in [171, 267] is

$$Z_{\rm CS}(SU(n)_k|\bar{\mathcal{A}})_0 = \frac{1}{\sqrt{n}}\frac{1}{(n+k)^{\frac{n-1}{2}}}\prod_{p=1}^{n-1}\Big(2\sin\frac{\pi p}{n+k}\Big)^{(n-p)} \cdot e^{i(n+k)S_{\rm CS}[\bar{\mathcal{A}}]}\,. \tag{C.7.44}$$

The corresponding higher-spin generalization of (C.7.24) is therefore

$$Z(R)_0 = \Big|\frac{1}{\sqrt{n}}\frac{1}{(n+l+i\kappa)^{\frac{n-1}{2}}}\prod_{p=1}^{n-1}\Big(2\sin\frac{\pi p}{n+l+i\kappa}\Big)^{(n-p)}\Big|^2 \cdot e^{2\pi\kappa T_R}\,. \tag{C.7.45}$$

Physically this can be interpreted as the all-loop quantum-corrected Euclidean partition function of the dS_3 static patch in the vacuum labeled by R. The analog of the result (C.7.25) for more general framing I_M is

$$Z_{\rm CS}(SU(n)_k|\bar{\mathcal{A}}) = \exp\big(\tfrac{i}{24}c(k)I_M\big)\, Z_{\rm CS}(SU(n)_k|\bar{\mathcal{A}})_0\,, \qquad c(k) = (n^2-1)\big(1-\tfrac{n}{n+k}\big)\,, \tag{C.7.46}$$

hence the generalization of (C.7.28) for arbitrary framing $I_M = 6\pi r$, $r \in \mathbb{Z}$, is

$$Z(R)_r = e^{ir\phi}\, Z(R)_0\,, \tag{C.7.47}$$

where $\phi = \big(c(l+i\kappa)+c(l-i\kappa)\big)\frac{\pi}{4} = \big(1-\frac{2(n+l)}{(n+l)^2+\kappa^2}\big)(n^2-1)\frac{\pi}{2}$. In the limit $\kappa \to \infty$,

$$Z(R)_r \to i^{r(n^2-1)}\,\frac{1}{n}\frac{1}{\kappa^{n-1}}\prod_{r=1}^{n-1}\Big(\frac{2\pi r}{\kappa}\Big)^{2(n-r)} = i^{r(n^2-1)}\left(\frac{2\pi}{\sqrt{\kappa}}\right)^{2(n^2-1)}\left(\frac{1}{\sqrt{n}}\prod_{s=2}^{n}\frac{\Gamma(s)}{(2\pi)^s}\right)^2. \tag{C.7.48}$$

Recognizing $n^2-1 = \dim SU(n)$ and $\sqrt{n}\prod_{s=2}^n (2\pi)^s/\Gamma(s) = {\rm vol}(SU(n))_{{\rm Tr}_n}$ (C.3.3), we see this precisely reproduces the one-loop result (C.7.43). Like in the original $n=2$ case, the phase again matches for odd framing r. (The agreement at one loop can also be seen more directly by a slight variation of the computation leading to (C.7.11).) This provides a nontrivial check of our higher-spin gravity formula (C.7.6) and more generally (5.5.17).

C.7.3.5 Large-n limit and topological string description

In generic dS$_{d+1}$ higher-spin theories, $\dim G = \infty$. To mimic this case, consider the $n \to \infty$ limit of $SU(n) \times SU(n)$ dS$_3$ higher-spin theory with $l = 0$. A basic observation is that the loop expansion is only reliable then if $n/\kappa \ll 1$. Using (C.7.36), this translates to $T_R\, n \ll \frac{\ell}{G_{\rm N}}$ For the exponentially dominant principal vacuum $R = \mathbf{n}$, this becomes $n^4 \ll \ell/G_{\rm N}$ while at the other extreme, for the nearly-trivial $R = \mathbf{2} \oplus \mathbf{1} \oplus \cdots \oplus \mathbf{1}$, this becomes $n \ll \ell/G_{\rm N}$. Either way, for fixed ℓ/G_N, the large-n limit is necessarily strongly coupled, and the one-loop formula (C.7.42), or equivalently (C.7.43) or (C.7.48), becomes unreliable. Indeed, according to this formula, $\log Z^{(1)} \sim \log\left(\frac{n}{\kappa}\right) \cdot n^2$ in this limit, whereas the exact expression (C.7.45) actually implies $\log Z^{(\rm loops)} \to 0$.

In fact, the partition function *does* have a natural weak coupling expansion in the $n \to \infty$ limit — not as a 3D higher-spin gravity theory, but rather as a topological string theory. $U(n)_k$ Chern-Simons theory on S^3 has a description [268] as an open topological string theory on the deformed conifold T^*S^3 with n topological D-branes wrapped on the S^3, and a large-n 't Hooft dual description [170] as a *closed* string theory on the *resolved* conifold. Both descriptions are reviewed in [171], whose notation we follow here. The string coupling constant is $g_s = 2\pi/(n+k)$ and the Kähler modulus of the resolved conifold is $t = \int_{S^2} J + iB = i g_s n = 2\pi i n/(n+k)$. Under this identification,

$$Z_{\rm CS}(SU(n)_k)_0 = \sqrt{\tfrac{n+k}{n}}\, Z_{\rm CS}(U(n)_k)_0 = \sqrt{\tfrac{2\pi i}{t}}\, Z_{\rm top}(g_s,t) \equiv \tilde{Z}_{\rm top}(g_s,t)\,. \tag{C.7.49}$$

Thus we can write the $SU(n)_{l+i\kappa} \times SU(n)_{l-i\kappa}$ higher-spin Euclidean gravity partition function (C.7.45) expanded around the round S^3 solution $\bar{\mathcal{A}} = R(\bar{A})$ as

$$\boxed{Z(R)_0 = \left|\tilde{Z}_{\rm top}(g_s,t)\, e^{-\pi T_R \cdot 2\pi i/g_s}\right|^2} \tag{C.7.50}$$

where T_R was defined in (C.7.34), maximized for $R = \mathbf{n}$ at $T_{\mathbf{n}} = \frac{1}{6} n(n^2-1)$, and

$$g_s = \frac{2\pi}{n+l+i\kappa}\,, \qquad t = i g_s n = \frac{2\pi i n}{n+l+i\kappa}\,. \tag{C.7.51}$$

Note that t takes values inside a half-disk of radius $\frac{1}{2}$ centered at $t = i\pi$, with ${\rm Re}\, t > 0$. The higher-spin gravity theory (or the open string theory description on the deformed conifold) is weakly coupled when $\kappa \gg n$, which implies $|t| \ll 1$. In the free field theory limit $\kappa \to \infty$, we get $g_s \sim -2\pi i/\kappa \to 0$ and $t \sim 2\pi n/\kappa \to 0$, which is singular from the closed string point of view. In the 't Hooft limit $n \to \infty$ with t kept finite, the closed string is weakly coupled and sees a smooth geometry. The earlier discussed Vasiliev-like limit $n \to \infty$ with $l = 0$ and $\ell/G_{\rm N} \sim T_{\mathbf{n}}\kappa \sim n^3\kappa$ fixed, infinitely strongly coupled from the 3D field theory point of view, maps to $g_s \sim 2\pi/n \to 0$ and $t \sim 2\pi i + 2\pi\kappa/n \to 2\pi i$, which is again singular from the closed string point of view, differing from the 3D free field theory singularity by a mere B-field monodromy, reflecting the more general $n \leftrightarrow l + i\kappa$, $t \leftrightarrow 2\pi i - t$ level-rank symmetry.

C.8 Quantum dS entropy: computations and examples

Here we provide the details for section 5.8.

C.8.1 Classical gravitational dS thermodynamics

C.8.1.1 3D Einstein gravity example

For concreteness we start with pure 3D Einstein gravity as a guiding example, but we will phrase the discussion so generalization will be clear. The Euclidean action in this case is

$$S_E[g] = \frac{1}{8\pi G}\int d^3x\sqrt{g}\Big(\Lambda - \frac{1}{2}R\Big)\,, \tag{C.8.1}$$

with $\Lambda > 0$. The tree-level contribution to the entropy (5.8.2) is

$$\mathcal{S}^{(0)} = \log \mathcal{Z}^{(0)}\,, \qquad \mathcal{Z}^{(0)} = \int_{\text{tree}} \mathcal{D}g\, e^{-S_E[g]}\,. \tag{C.8.2}$$

The dominant saddle of (C.8.2) is a round S^3 metric g_ℓ of radius $\ell = \ell_0$ minimizing $S_E(\ell) \equiv S_E[g_\ell]$:

$$\mathcal{Z}^{(0)} = \int_{\text{tree}} d\ell\, e^{-S_E(\ell)}\,, \qquad S_E(\ell) = \frac{2\pi^2}{8\pi G}(\Lambda\ell^3 - 3\ell)\,, \tag{C.8.3}$$

where $\int_{\rm tree}$ means evaluation at the saddle point, here at the on-shell radius $\ell = \ell_0$:

$$\partial_\ell S_E(\ell_0) = 0 \qquad \Rightarrow \qquad \Lambda = \frac{1}{\ell_0^2} \,, \qquad \mathcal{S}^{(0)} = -S_E(\ell_0) = \frac{2\pi\ell_0}{4G} \,, \tag{C.8.4}$$

reproducing the familiar area law $\mathcal{S}^{(0)} = A/4G$ for the horizon entropy.

We now recast the above in a way that will allow us to make contact with the formulae of section 5.7.1 and will naturally generalize beyond tree level in a diffeomorphism-invariant way. To this end we define an "off-shell" tree-level partition function at fixed (off-shell) volume V:

$$Z^{(0)}(V) \equiv \int_{\rm tree} d\sigma \int_{\rm tree} \mathcal{D}g \, e^{-S_E[g] + \sigma(\int \sqrt{g} - V)} \,. \tag{C.8.5}$$

Evaluating the integral is equivalent to a constrained extremization problem with Lagrange multiplier σ enforcing the constraint $\int \sqrt{g} = V$. The dominant saddle is the round sphere $g = g_\ell$ of radius $\ell(V)$ fixed by the volume constraint:

$$Z^{(0)}(V) = e^{-S_E(\ell)} \,, \qquad 2\pi^2 \ell^3 = V \,. \tag{C.8.6}$$

Paralleling (5.7.6) and (5.7.7), we define from this an off-shell energy density and entropy,

$$\begin{aligned} \rho^{(0)} &\equiv -\partial_V \log Z^{(0)} = -\tfrac{1}{3}\ell\partial_\ell \log Z^{(0)} \,/\, V = (\Lambda - \ell^{-2})/8\pi G \\ S^{(0)} &\equiv (1 - V\partial_V) \log Z^{(0)} = (1 - \tfrac{1}{3}\ell\partial_\ell) \log Z^{(0)} = \frac{2\pi\ell}{4G} \,. \end{aligned} \tag{C.8.7}$$

$\rho^{(0)}$ is the sum of the positive cosmological constant and negative curvature energy densities. $S^{(0)}$ is independent of Λ. It is the Legendre transform of $\log Z^{(0)}$:

$$S^{(0)} = \log Z^{(0)} + V\rho^{(0)} \,, \qquad d\log Z^{(0)} = -\rho^{(0)} dV \,, \qquad dS^{(0)} = V d\rho^{(0)} \,, \tag{C.8.8}$$

Note that evaluating $\int_{\rm tree} d\sigma$ in (C.8.5) sets $\sigma = -\partial_V \log Z^{(0)} = \rho^{(0)}(V)$. On shell,

$$\rho^{(0)}(\ell_0) = 0 \,, \qquad \mathcal{S}^{(0)} = \log Z^{(0)}(\ell_0) = S^{(0)}(\ell_0) = \frac{2\pi\ell_0}{4G} \,. \tag{C.8.9}$$

Paralleling (5.7.8), the differential relations in (C.8.8) can be viewed as the first law of tree-level de Sitter thermodynamics. We can also consider variations of coupling constants such as Λ. Then $d\log Z^{(0)} = -\rho^{(0)} dV - \frac{1}{8\pi G} V d\Lambda$, $dS^{(0)} = V d\rho^{(0)} - \frac{1}{8\pi G} V d\Lambda$. On shell, $d\mathcal{S}^{(0)} = -\frac{V_0}{8\pi G}\, d\Lambda$.

C.8.1.2 General d and higher-order curvature corrections

The above formulae readily extend to general dimensions and to gravitational actions $S_E[g]$ with general higher-order curvature corrections. Using that $R_{\mu\nu\rho\sigma} = (g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})/\ell^2$ for the round[19] S^{d+1}, $Z^{(0)}(V)$ (C.8.5) can be evaluated explicitly for any action. It takes the form

$$\log Z^{(0)}(V) = -S_E[g_\ell] = \frac{\Omega_{d+1}}{8\pi G}\big(-\Lambda \ell^{d+1} + \tfrac{d(d+1)}{2}\,\ell^{d-1} + \cdots\big)\,, \qquad \Omega_{d+1}\ell^{d+1} = V\,, \quad \text{(C.8.10)}$$

where $+\cdots$ is a sum of R^n higher-order curvature corrections $\propto \ell^{-2n}$ and $\Omega_{d+1} = (C.3.1)$. The off-shell energy density and entropy are defined as in (C.8.7)

$$\begin{aligned} \rho^{(0)} &= -\tfrac{1}{d+1}\ell\partial_\ell \log Z^{(0)} \,/\, V = \big(\Lambda - \tfrac{d(d-1)}{2}\,\ell^{-2} + \cdots\big)/8\pi G \\ S^{(0)} &= \big(1 - \tfrac{1}{d+1}\ell\partial_\ell\big)\log Z^{(0)} = \frac{A}{4G}\big(1 + \cdots\big)\,. \end{aligned} \qquad \text{(C.8.11)}$$

where $A = \Omega_{d-1}\ell^{d-1}$ and $+\cdots$ are $1/\ell^{2n}$ curvature corrections. The on-shell radius ℓ_0 solves $\rho^{(0)}(\ell_0) = 0$, most conveniently viewed as giving a parametrization $\Lambda(\ell_0)$.

As an example, consider the general action up to order R^2 written as

$$S_E = \frac{1}{8\pi G}\int\!\sqrt{g}\Big(\Lambda - \tfrac{1}{2}R - l_s^2\big(\lambda_{C^2} C^{\mu\nu\rho\sigma}C_{\mu\nu\rho\sigma} + \lambda_{R^2} R^2 + \lambda_{E^2} E^{\mu\nu}E_{\mu\nu}\big)\Big)\,, \qquad \text{(C.8.12)}$$

where $E_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{d+1} R\, g_{\mu\nu}$, $C_{\mu\nu\rho\sigma}$ is the Weyl tensor, l_s is a length scale and the λ_i are dimensionless. The Weyl tensor vanishes on the round sphere and $R_{\mu\nu} = d\, g_{\mu\nu}/\ell^2$, hence

$$\log Z^{(0)} = \frac{\Omega_{d+1}}{8\pi G}\big(-\Lambda\,\ell^{d+1} + \tfrac{1}{2}d(d+1)\,\ell^{d-1} + \lambda_{R^2}\, d^2(d+1)^2\, l_s^2\,\ell^{d-3}\big)\,, \qquad \text{(C.8.13)}$$

[19]By virtue of its $SO(d+2)$ symmetry, the round sphere metric g_ℓ with $\Omega_{d+1}\ell^{d+1} = V$ is a saddle of (C.8.5). Spheres of dimension ≥ 5 admit a plentitude of Einstein metrics that are *not* round [269–272], but as explained e.g. in [273], by Bishop's theorem [274], these saddles are subdominant in Einstein gravity. In the large-size limit, higher-order curvature corrections are small, hence the round sphere dominates in this regime.

For example for $d = 2$,

$$\log Z^{(0)} = \frac{\pi}{4G}\Big(-\Lambda\,\ell^3 + 3\,\ell + \frac{36\,l_s^2\lambda_{R^2}}{\ell}\Big)\,, \tag{C.8.14}$$

hence, using (C.8.11) and $\rho^{(0)}(\ell_0) = 0$,

$$\mathcal{S}^{(0)} = S^{(0)}(\ell_0) = \frac{2\pi\ell_0}{4G}\Big(1 + \frac{24\,l_s^2\lambda_{R^2}}{\ell_0^2}\Big)\,, \qquad \Lambda = \frac{1}{\ell_0^2}\Big(1 - \frac{12\,l_s^2\lambda_{R^2}}{\ell_0^2}\Big)\,. \tag{C.8.15}$$

C.8.1.3 Effective field theory expansion and field redefinitions

Curvature corrections such as those considered above naturally appear as terms in the derivative expansion of low-energy effective field theories of quantum gravity, with l_s the characteristic length scale of UV-completing physics and higher-order curvature corrections terms suppressed by higher powers of $l_s^2/\ell^2 \ll 1$. The action (C.8.12) is then viewed as a truncation at order l_s^2, and (C.8.15) can be solved perturbatively to obtain ℓ_0 and $\mathcal{S}^{(0)}$ as a function of Λ.

Suppose someone came up with some fundamental theory of de Sitter quantum gravity, producing both a precise microscopic computation of the entropy and a precise low-energy effective action, with the large-ℓ_0/l_s expansion reproduced as some large-N expansion. At least superficially, the higher-order curvature-corrected entropy obtained above looks like a Wald entropy [188]. In the spirit of for instance the nontrivial matching of R^2 corrections to the macroscopic BPS black hole entropy computed in [275] and the microscopic entropy computed from M-theory in [276], it might seem then that matching microscopic $1/N$-corrections and macroscopic l_s^2/ℓ_0^2-corrections to the entropy such as those in (C.8.15) could offer a nontrivial way of testing such a hypothetical theory.

However, this is not the case. Unlike the Wald entropy, there are no charges Q (such as energy, angular momentum or gauge charges) available here to give these corrections physical meaning as corrections in the large-Q expansion of a function $S(Q)$. Indeed, the detailed structure of the l_s/ℓ_0 expansion of $\mathcal{S}^{(0)} = S^{(0)}(\ell_0)$ has no intrinsic physical meaning at all, because all of it can be wiped out by a local metric field redefinition, order by order in l_s/ℓ_0, bringing the entropy to pure Einstein area law form, and leaving only the value of $\mathcal{S}^{(0)}$ itself as a physically meaningful, field-redefinition invariant, dimensionless quantity.

This is essentially a trivial consequence of the fact that in perturbation theory about the round sphere, the round sphere itself is the unique solution to the equations of motion. Let us however recall in more detail how this works at the level of local field redefinitions, and show how this is expressed at the level of $\log Z^{(0)}(\ell)$, as this will be useful later in interpreting quantum corrections. For concreteness, consider again (C.8.12) viewed as a gravitational effective field theory action expanded to order $l_s^2 R^2$. Under a local metric field redefinition

$$g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu} + O(l_s^4)\,, \qquad \delta g_{\mu\nu} \equiv l_s^2 (u_0\, \Lambda\, g_{\mu\nu} + u_1\, R\, g_{\mu\nu} + u_2\, R_{\mu\nu})\,, \tag{C.8.16}$$

where the u_i are dimensionless constants, the action transforms as

$$S_E \to S_E + \frac{1}{16\pi G} \int \sqrt{g}\, (R^{\mu\nu} - \tfrac{1}{2} R g^{\mu\nu} + \Lambda g^{\mu\nu})\, \delta g_{\mu\nu} \,+\, O(l_s^4)\,, \tag{C.8.17}$$

shifting $\lambda_{R^2}, \lambda_{E^2}$ and rescaling G, Λ in (C.8.12). A suitable choice of u_i brings S_E to the form

$$S_E = \frac{1}{8\pi G} \int \sqrt{g} \Big(\Lambda' - \tfrac{1}{2} R - l_s^2 \lambda_{C^2} C^{\mu\nu\rho\sigma} C_{\mu\nu\rho\sigma} + O(l_s^4) \Big), \qquad \Lambda' = \Lambda \big(1 - \tfrac{4(d+1)^2}{(d-1)^2}\, \lambda_{R^2} l_s^2 \Lambda\big). \tag{C.8.18}$$

Equivalently, this is obtained by using the $O(l_s^0)$ equations of motion $R_{\mu\nu} = \frac{2}{d-1} \Lambda g_{\mu\nu}$ in the $O(l_s^2)$ part of the action. Since $\lambda'_{R^2} = 0$, the entropy computed from this equivalent action takes a pure Einstein area law form $\mathcal{S}^{(0)} = \Omega_{d+1} {\ell'_0}^{d-1}/4G$, with $\ell'_0 = \sqrt{d(d-1)/2\Lambda'}$. The on-shell value $\mathcal{S}^{(0)}$ itself remains unchanged of course under this change of variables.

In the above we picked a field redefinition keeping $G' = G$. Further redefining $g_{\mu\nu} \to \alpha\, g_{\mu\nu}$ leads to another equivalent set of couplings $G'', \Lambda'', \ldots$ rescaled with powers of α according to their mass dimension. We could then pick α such that instead $\Lambda'' = \Lambda$, or such that $\ell''_0 = \ell_0$, now with $G'' \neq G$. If we keep $\ell''_0 = \ell_0$, we get

$$\mathcal{S}^{(0)} = \frac{\Omega_{d+1} \ell_0^{d-1}}{4G''}\,, \qquad \Lambda'' = \frac{d(d-1)}{2\ell_0^2}\,, \tag{C.8.19}$$

where for example in $d = 2$ starting from (C.8.15), $G'' = G\big(1 - 24\, \lambda_{R^2} l_s^2/\ell_0^2 + O(l_s^4)\big)$.

At the level of $\log Z^{(0)}(\ell)$ in (C.8.13) the metric redefinition (C.8.16) amounts to a radius redefininition $\ell \to \ell\, f(\ell)$ with $f(\ell) = 1 + l_s^2 (v_{10} \Lambda + v_{11} \ell^{-2}) + O(l_s^4)$. For suitable v_i this brings

$\log Z^{(0)}$ and therefore $\mathcal{S}^{(0)}$ to pure Einstein form. E.g. for the $d=2$ example (C.8.14),

$$\ell = \big(1 - 12\,\lambda_{R^2} l_s^2 (\Lambda + \ell'^{\,-2})\big)\ell' \qquad \Rightarrow \qquad \log Z^{(0)} = \frac{\pi}{4G}\big(-\Lambda'\,\ell'^{\,3} + 3\,\ell' + O(l_s^4)\big)\,. \tag{C.8.20}$$

The above considerations generalize to all orders in the l_s expansion. R^n corrections to $\log Z^{(0)}$ are $\propto (l_s/\ell)^{2n}$ and can be removed order by order by a local metric/radius redefinition

$$\ell \to \alpha\, f(\ell)\,\ell\,, \qquad f(\ell) = 1 + l_s^2(v_{10}\Lambda + v_{11}\ell^{-2}) + l_s^4(v_{20}\Lambda^2 + v_{21}\Lambda\ell^{-2} + v_{22}\ell^{-4}) + \cdots \tag{C.8.21}$$

bringing $\log Z^{(0)}$ and thus $\mathcal{S}^{(0)}$ to Einstein form to any order in the l_s expansion.

In $d=2$, the Weyl tensor vanishes identically. The remaining higher-order curvature invariants involve the Ricci tensor only, so can be removed by field redefinitions, reducing the action to Einstein form in general. Thus in $d=2$, $\mathcal{S}^{(0)}$ is the only tree-level invariant in the theory, i.e. the only physical coupling constant. In the Chern-Simons formulation of C.7.2, $\mathcal{S}^{(0)} = 2\pi\kappa$. In $d \geq 3$, there are infinitely many independent coupling constants, such as the Weyl-squared λ_{C^2} in (C.8.12), which are not picked up by $\mathcal{S}^{(0)}$, but are analogously probed by invariants $\mathcal{S}_M^{(0)} = \log \mathcal{Z}^{(0)}[g_M] = -S_E[g_M]$ for saddle geometries g_M different from the round sphere. We comment on those and their role in the bigger picture in section C.8.5.

The point of considering quantum corrections to the entropy $\mathcal{S}$ is that these include nonlocal contributions, not removable by local redefinitions, and thus, unlike the tree-level entropy $\mathcal{S}^{(0)}$, offering actual data quantitatively constraining candidate microscopic models.

C.8.2 Quantum gravitational thermodynamics

The *quantum* off-shell partition function $Z(V)$ generalizing the tree-level $Z^{(0)}(V)$ (C.8.5) is defined by replacing $\int_{\text{tree}} \mathcal{D}g \to \int \mathcal{D}g$ in that expression:[20]

$$Z(V) \equiv \int_{\text{tree}} d\sigma \int \mathcal{D}g\, e^{-S_E[g] + \sigma(\int \sqrt{g} - V)}\,. \tag{C.8.22}$$

[20] $Z(V)$ is reminiscent of but different from the fixed-volume partition function considered in the 2D quantum gravity literature, e.g. (2.20) in [277]. The latter would be defined as above but with $\frac{1}{2\pi i}\int_{i\mathbb{R}} d\sigma$ instead of $\int_{\text{tree}} d\sigma$, constraining the volume to V, whereas $Z(V)$ constrains the *expectation value* of the volume to V.

The quantum off-shell energy density and entropy generalizing (C.8.7) are

$$\rho(V) \equiv -\partial_V \log Z \,, \qquad S(V) \equiv (1 - V\partial_V) \log Z \,. \tag{C.8.23}$$

S is the Legendre transform of $\log Z$:

$$S = \log Z + V\rho \,, \qquad d\log Z = -\rho\, dV \,, \qquad dS = V d\rho \,. \tag{C.8.24}$$

Writing $e^{-\Gamma(V)} \equiv Z(V)$, the above definitions imply that as a function of ρ,

$$S(\rho) = \log \textstyle\int_{\text{tree}} dV\, e^{-\Gamma(V)+\rho V} = \log \int \mathcal{D}g\, e^{-S_E[g]+\rho \int \sqrt{g}} \tag{C.8.25}$$

hence $S(\rho)$ is the generating function for moments of the volume. In particular

$$V = \langle \textstyle\int \sqrt{g} \rangle_\rho = \partial_\rho S(\rho) \tag{C.8.26}$$

is the expectation value of the volume in the presence of a source ρ shifting the cosmological constant $\frac{\Lambda}{8\pi G} \to \frac{\Lambda}{8\pi G} - \rho$. $\Gamma(V)$ can be viewed as a quantum effective action for the volume, in the spirit of the QFT 1PI effective action [278–280] but taking only the volume off-shell. At tree level it reduces to $S_E[g_\ell]$ appearing in (C.8.10). At the quantum on-shell value $V = \bar{V} = \langle \int \sqrt{g} \rangle_0$,

$$\rho(\bar{V}) = 0 \,, \qquad \mathcal{S} = \log Z(\bar{V}) = S(\bar{V}) \,. \tag{C.8.27}$$

It will again be convenient to work with a linear scale variable ℓ instead of V, *defined* by

$$\Omega_{d+1} \ell^{d+1} \equiv V \,, \tag{C.8.28}$$

Since the mean volume $V = \langle \int \sqrt{g} \rangle_\rho$ is diffeomorphism invariant, (C.8.28) gives a manifestly diffeomorphism-invariant definition of the "mean radius" ℓ of the fluctuating geometry. Given

$Z(\ell) \equiv Z(V(\ell))$, the off-shell energy density and entropy are then computed as

$$\rho(\ell) = -\frac{\frac{1}{d+1}\ell\partial_\ell \log Z}{V}\,, \qquad S(\ell) = \big(1 - \tfrac{1}{d+1}\ell\partial_\ell\big)\log Z\,. \tag{C.8.29}$$

The quantum on-shell value of ℓ is denoted by $\bar{\ell}$ and satisfies $\rho(\bar{\ell}) = 0$.

The magnitude of quantum fluctuations of the volume about its mean value is given by $\delta V^2 \equiv \big\langle \big(\int\!\sqrt{g} - V\big)^2 \big\rangle_\rho = S''(\rho) = 1/\Gamma''(V) = 1/\rho'(V) = V/S'(V)$. At large V, $\delta V/V \propto 1/\sqrt{S}$.

C.8.3 One-loop corrected de Sitter entropy

The path integral (C.8.22) for $\log Z$ can be computed perturbatively about its round sphere saddle in a semiclassical expansion in powers of G. To leading order it reduces to $\log Z^{(0)}$ defined in (C.8.5). For 3D Einstein gravity,

$$\log Z(V) = \log Z^{(0)}(V) + \mathcal{O}(G^0) = \frac{\Omega_3}{8\pi G}\big(-\Lambda\,\ell^3 + 3\,\ell\big) \, + \, O(G^0)\,, \tag{C.8.30}$$

To compute the one-loop $O(G^0)$ correction, recall that evaluation of $\int_{\rm tree} d\sigma$ in (C.8.22) is equivalent to extremization with respect to σ, which sets $\sigma = -\partial_V \log Z(V) = \rho(V)$ and

$$\log Z(V) = \log \int\! \mathcal{D}g\, e^{-S_E[g] + \rho(V)\left(\int\sqrt{g} - V\right)}\,. \tag{C.8.31}$$

To one-loop order, we may replace ρ by its tree-level approximation $\rho^{(0)} = -\partial_V \log Z^{(0)}$. By construction this ensures the round sphere metric $g = g_\ell$ of radius $\ell(V)$ given by (C.8.28) is a saddle. Expanding the action to quadratic order in fluctuations about this saddle then gives a massless spin-2 Gaussian path integral of the type solved in general by (5.5.17), or more explicitly in (5.7.19)-(5.7.20). For 3D Einstein gravity, using (5.7.20),

$$\log Z = -\Big(\frac{\Lambda}{8\pi G} + c_0'\Big)\Omega_3\ell^3 + \Big(\frac{1}{8\pi G} + c_2' - \frac{3}{4\pi\epsilon}\Big)3\Omega_3\ell - 3\log\frac{2\pi\ell}{4G} + 5\log(2\pi) + \mathcal{O}(G) \tag{C.8.32}$$

Here c_0' and c_2' arise from $\mathcal{O}(G^0)$ local counterterms

$$S_{E,\text{ct}} = \int \sqrt{g}\Big(c_0' - \frac{c_2'}{2} R\Big) \,, \tag{C.8.33}$$

split off from the bare action (C.8.1) to keep the tree-level couplngs Λ and G equal to their "physical" (renormalized) values to this order. We define these physical values as the coefficients of the local terms $\propto \ell^3, \ell$ in the $V \to \infty$ asymptotic expansion of the *quantum* $\log Z(V)$. That is to say, we fix c_0' and c_2' by imposing the renormalization condition

$$\log Z(V) = \frac{\Omega_3}{8\pi G}(-\Lambda\,\ell^3 + 3\,\ell\,) + \cdots \qquad (V \to \infty)\,. \tag{C.8.34}$$

This renormalization prescription is diffeomorphism invariant, since $Z(V)$, V and ℓ were all defined in a manifestly diffeomorphism-invariant way. In (C.8.32) it fixes $c_0' = 0$, $c_2' = \frac{3}{4\pi\epsilon}$, hence $\log Z(\ell) = \log Z^{(0)} + \log Z^{(1)} + \mathcal{O}(G)$, where

$$\log Z^{(1)} = -3 \log \frac{2\pi\ell}{4G} + 5 \log(2\pi)\,. \tag{C.8.35}$$

We can express the renormalization condition (C.8.34) equivalently as

$$\boxed{\log Z^{(1)} = \log Z_{\text{PI}}^{(1)} + \log Z_{\text{ct}}\,, \qquad \lim_{\ell\to\infty} \partial_\ell \log Z^{(1)} = 0} \tag{C.8.36}$$

where $\log Z_{\text{ct}} = -S_{E,\text{ct}}[g_\ell]$ with g_ℓ the round sphere metric of volume V. On S^3 we have $\log Z_{\text{ct}} = c_0\ell^3 + c_2\ell$, and the $\ell \to \infty$ condition fixes c_0 and c_2. Recalling (5.7.6), we can physically interpret this as requiring the renormalized one-loop Euclidean energy $U^{(1)}$ of the static patch vanishes in the $\ell \to \infty$ limit.

For general d, the UV-divergent terms in $\log Z_{\text{PI}}^{(1)}$ come with non-negative powers $\propto \ell^{d+1-2n}$, canceled by counterterms consisting of n-th order curvature invariants. For example on S^5, $\log Z_{\text{ct}} = c_0\ell^5 + c_2\ell^3 + c_4\ell$. In odd $d+1$, the renormalization prescription (C.8.36) then fixes the c_{2n}. In even $d+1$, $\log Z_{\text{ct}}$ has a constant term c_{d+1}, which is not fixed by (C.8.36). As we will make explicit in examples later, it can be fixed by $\lim_{\ell\to\infty} Z^{(1)} = 0$ for massive field contributions, and for massless field contributions by minimal subtraction at scale L, $c_{d+1} = -\alpha_{d+1}\log(M_\epsilon L)$, $M_\epsilon = 2e^{-\gamma}/\epsilon$ (C.2.29), with $L\partial_L \log Z = 0$, i.e. $L\partial_L \log Z^{(0)} = \alpha_{d+1}$.

The renormalized off-shell ρ and S are obtained from $\log Z$ as in (C.8.29). For 3D Einstein,

$$\rho^{(1)} = \frac{1}{2\pi^2 \ell^3}\,, \qquad S^{(1)} = -3\log\frac{2\pi\ell}{4G} + 5\log(2\pi) + 1\,. \tag{C.8.37}$$

The on-shell quantum dS entropy $\mathcal{S} = \log Z(\bar\ell) = S(\bar\ell)$ (C.8.27) is

$$\boxed{\mathcal{S} = S(\bar\ell) = S^{(0)}(\bar\ell) + S^{(1)}(\bar\ell) \, + \, \mathcal{O}(G)} \tag{C.8.38}$$

where $\bar\ell$ is the quantum mean radius satisfying $\rho(\bar\ell) \propto \partial_\ell \log Z(\bar\ell) = 0$. For 3D Einstein,

$$\mathcal{S} = \frac{2\pi\bar\ell}{4G} - 3\log\frac{2\pi\bar\ell}{4G} + 5\log(2\pi) + 1 \, + \, \mathcal{O}(G)\,, \qquad \Lambda = \frac{1}{\bar\ell^2} - \frac{4G}{\pi\bar\ell^3} + \mathcal{O}(G^2)\,. \tag{C.8.39}$$

Alternatively, $\mathcal{S}$ can be expressed in terms of the tree-level ℓ_0, $\rho^{(0)}(\ell_0) \propto \partial_\ell \log Z^{(0)}(\ell_0) = 0$, using $\mathcal{S} = \log Z^{(0)}(\bar\ell) + \log Z^{(1)}(\bar\ell) + \mathcal{O}(G)$, $\bar\ell = \ell_0 + \mathcal{O}(G)$ and Taylor expanding in G:

$$\boxed{\mathcal{S} = \log Z(\bar\ell) = S^{(0)}(\ell_0) + \log Z^{(1)}(\ell_0) \, + \, \mathcal{O}(G)} \tag{C.8.40}$$

This form would be obtained from (5.8.2) by a more standard computation. For 3D Einstein,

$$\mathcal{S} = \frac{2\pi\ell_0}{4G} - 3\log\frac{2\pi\ell_0}{4G} + 5\log(2\pi) \, + \, \mathcal{O}(G)\,, \qquad \Lambda = \frac{1}{\ell_0^2}\,. \tag{C.8.41}$$

The equivalence of (C.8.38) and (C.8.40) can be checked directly here noting $\bar\ell = \ell_0 - \frac{2}{\pi}G + \mathcal{O}(G^2)$, so $\frac{2\pi\bar\ell}{4G} = \frac{2\pi\ell_0}{4G} - 1 + \mathcal{O}(G)$. The -1 cancels the $+1$ in (C.8.39), reproducing (C.8.41).

More generally and more physically, the relation between these two expressions can be understood as follows. At tree level, the entropy equals the geometric horizon entropy $S^{(0)}(\ell_0)$, with radius ℓ_0 such that the geometric energy density $\rho^{(0)}$ vanishes. At one loop, we get additional contributions from quantum field fluctuations. The UV contributions are absorbed into the gravitational coupling constants. The remaining IR contributions shift the entropy by $S^{(1)}$ and the energy density by $\rho^{(1)}$. The added energy backreacts on the fluctuating geometry: its mean radius changes from ℓ_0 to $\bar\ell$ such that the geometric energy density changes by $\delta\rho^{(0)} = -\rho^{(1)}$, ensuring the total energy density vanishes. This in turn changes the geometric horizon

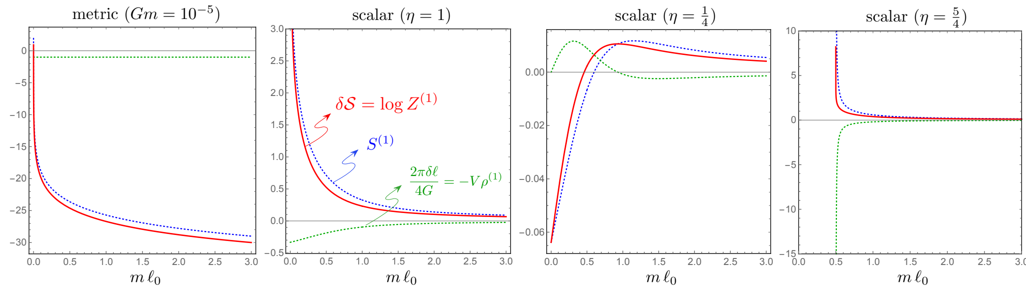


Figure C.8.1: One-loop contributions to the dS$_3$ entropy from metric and scalars with $\eta = 1, \frac{1}{4}, \frac{5}{4}$, i.e. $\xi = 0, \frac{1}{8}, -\frac{1}{24}$. Blue dotted line = renormalized entropy $S^{(1)}$. Green dotted line = horizon entropy change $\delta S^{(0)} = 2\pi\delta\ell/4G = -V\rho^{(1)}$ due to quantum backreaction $\ell_0 \to \bar{\ell} = \ell_0 + \delta\ell$, as dictated by first law. Solid red line = total $\delta\mathcal{S} = S^{(1)} - V\rho^{(1)} = \log Z^{(1)}$. The metric contribution is negative within the semiclassical regime of validity $\ell \gg G$. The renormalized scalar entropy and energy density are positive for $m\ell \gg 1$, and for all $m\ell$ if $\eta = 1$. If $\eta > 1$ and $\ell_0 \to \ell_* \equiv \frac{\sqrt{\eta-1}}{m}$, the correction $\delta\ell \sim -\frac{G}{3\pi}\frac{\ell_*}{\ell_0 - \ell_*} \to -\infty$, meaning the one-loop approximation breaks down. The scalar becomes tachyonic beyond this point. If a ϕ^4 term is included in the action, two new dominant saddles emerge with $\phi \neq 0$.

entropy by an amount dictated by the first law (C.8.8),

$$\delta S^{(0)} = V_0\,\delta\rho^{(0)} = -V_0\,\rho^{(1)}\,. \tag{C.8.42}$$

We end up with a total entropy $\mathcal{S} = S^{(0)}(\bar{\ell}) + S^{(1)} = S^{(0)}(\ell_0) - V_0\,\rho^{(1)} + S^{(1)} = S^{(0)}(\ell_0) + \log Z^{(1)}$, up to $O(G)$ corrections, relating (C.8.38) to (C.8.40). (See also fig. C.8.1.)

More succinctly, obtaining (C.8.40) from (C.8.38) is akin to obtaining the canonical description of a thermodynamic system from the microcanonical description of system + reservoir. The analog of the canonical partition function is $Z^{(1)} = e^{S^{(1)} - V_0\,\rho^{(1)}}$, with $-V_0\,\rho^{(1)}$ capturing the reservoir (horizon) entropy change due to energy transfer to the system.

C.8.4 Examples

C.8.4.1 3D scalar

An example with matter is 3D Einstein gravity + scalar ϕ as in (5.7.11). Putting $\xi \equiv \frac{1-\eta}{6}$,

$$S_E[g,\phi] = \frac{1}{8\pi G}\int\!\sqrt{g}\,(\Lambda - \tfrac{1}{2}R) \,+\, \frac{1}{2}\int\!\sqrt{g}\,\phi(-\nabla^2 + m^2 + \tfrac{1-\eta}{6}R)\phi\,, \tag{C.8.43}$$

The metric contribution to $\log Z^{(1)}$ remains $\log Z^{(1)}_{\rm metric} = -3\log\frac{2\pi\ell}{4G} + 5\log(2\pi)$ as in (C.8.35). The scalar $Z^{(1)}_{\rm PI}$ was given in (5.7.14). Its finite part is

$$\log Z^{(1)}_{\rm PI,fin,scalar} = \frac{\pi\nu^3}{6} - \sum_{k=0}^{2}\frac{\nu^k}{k!}\,\frac{{\rm Li}_{3-k}(e^{-2\pi\nu})}{(2\pi)^{2-k}}\,, \qquad \nu \equiv \sqrt{m^2\ell^2-\eta}\,. \tag{C.8.44}$$

The polynomial $\log Z_{\rm ct}(\ell) = c_0\ell^3 + c_2\ell$ corresponding to the counterterm action (C.8.33) is fixed by the renormalization condition (C.8.36), resulting in

$$\log Z^{(1)}_{\rm scalar} = \log Z^{(1)}_{\rm PI,fin,scalar} - \frac{\pi}{6}\,m^3\ell^3 + \frac{\pi\eta}{4}\,m\ell\,. \tag{C.8.45}$$

The finite polynomial cancels the local terms $\propto \ell^3, \ell$ in the large-ℓ asymptotic expansion of the finite part: $\log Z^{(1)}_{\rm scalar} = \frac{\pi\eta^2}{16}(m\ell)^{-1} + \frac{\pi\eta^3}{96}(m\ell)^{-3} + \cdots$ when $m\ell\to\infty$. The $(m\ell)^{-2n-1}$ terms have the ℓ-dependence of R^n terms in the action and can effectively be thought of as finite shifts of higher-order curvature couplings in the $m\ell \gg 1$ regime. In the opposite regime $m\ell \ll 1$, IR bulk modes of the scalar becomes thermally activated and $\log Z^{(1)}_{\rm scalar}$ ceases to have a local expansion. In particular in the minimally-coupled case $\eta = 1$,

$$\log Z^{(1)}_{\rm scalar} \simeq -\log(m\ell) \qquad (m\ell\to 0)\,. \tag{C.8.46}$$

The total energy density is $\rho = -\frac{1}{3}\ell\partial_\ell \log Z/V = \frac{1}{8\pi G}(\Lambda - \ell^{-2}) + 1 + \rho^{(1)}_{\rm scalar}$ where

$$V\rho^{(1)}_{\rm scalar} = -\frac{\pi}{6}(m\ell)^2\nu\coth(\pi\nu) + \frac{\pi}{6}(m\ell)^3 - \frac{\pi\eta}{12}\,m\ell\,. \tag{C.8.47}$$

The on-shell quantum dS entropy is given to this order by (C.8.40) or by (C.8.38) as

$$\mathcal{S} = \mathcal{S}^{(0)} + \mathcal{S}^{(1)} = S^{(0)}(\ell_0) + \log Z^{(1)} = S^{(0)}(\ell_0) - V\rho^{(1)} + S^{(1)} = S^{(0)}(\bar\ell) + S^{(1)}\,, \tag{C.8.48}$$

where $\ell_0^{-2} = \Lambda = \bar\ell^{-2} - 8\pi G\,\rho^{(1)}(\bar\ell)$ and $S^{(1)} = S^{(1)}_{\rm PI,fin} + \frac{1}{6}\pi\eta\, m\ell$, with the scalar contribution to $S^{(1)}_{\rm PI,fin}$ given by the finite part of (5.7.16). Some examples are shown in fig. C.8.1.

For a massless scalar, $m=0$, the renormalized scalar one-loop correction to $\mathcal{S}$ is a constant independent of ℓ_0 given by (C.8.44) evaluated at $\nu = \sqrt{-\eta}$, and $\rho^{(1)}_{\rm scalar} = 0$. For example for a massless conformally coupled scalar, $\eta = \frac{1}{4}$, $Z^{(1)}_{\rm scalar} = \frac{3\,\zeta(3)}{16\pi^2} - \frac{\log(2)}{8}$.

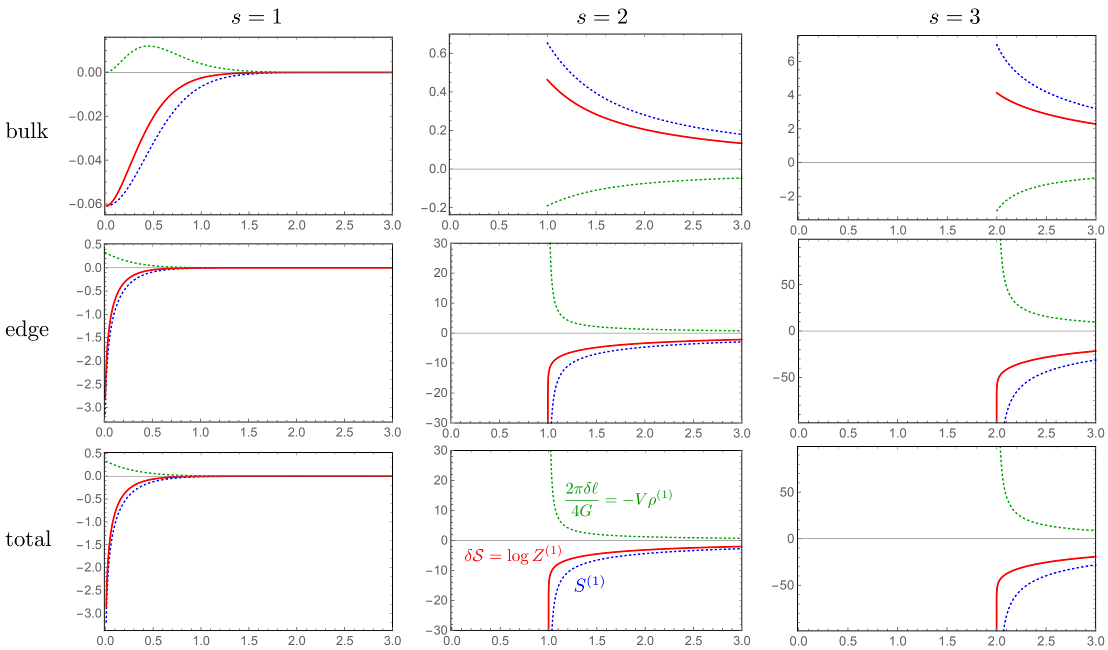


Figure C.8.2: Contributions to the dS_3 entropy from massive spin $s = 1, 2, 3$ fields, as a function of $m\ell_0$, with coloring as in fig. C.8.1. Singularities = Higuchi bound, as discussed under (5.7.16).

C.8.4.2 3D massive spin s

The renormalized one-loop correction $\mathcal{S}_s^{(1)} = \log Z_s^{(1)}$ to the dS_3 entropy from a massive spin-s field is obtained similarly from (5.4.13):

$$\mathcal{S}_s^{(1)} = \log Z_{s,\text{bulk}}^{(1)} - \log Z_{s,\text{edge}}^{(1)} , \tag{C.8.49}$$

where $\log Z_{s,\text{bulk}}^{(1)}$ equals twice the contribution of an $\eta = (s-1)^2$ scalar as given in (C.8.45), while the edge contribution is, putting $\nu \equiv \sqrt{m^2\ell^2 - (s-1)^2}$,

$$\log Z_{s,\text{edge}}^{(1)} = s^2\big(\pi(m\ell - \nu) - \log(1 - e^{-2\pi\nu})\big) . \tag{C.8.50}$$

The edge contribution to (C.8.49) is manifestly negative. It dominates the bulk part, and increasingly so as s grows. Examples are shown in fig. C.8.2.

C.8.4.3 2D scalar

As mentioned below (C.8.36), the counterterm polynomial $\log Z_{\rm ct}^{(0)}$ has a constant term in even spacetime dimensions $d+1$, which is not fixed yet by the renormalization prescription given there. Let us consider the simplest example: a $d=1$ scalar with action (5.7.11). Denoting

$$\mathtt{f}(\nu) \equiv \sum_{\pm} \zeta'(-1, \tfrac{1}{2} \pm i\nu) \mp i\nu\zeta'(0, \tfrac{1}{2} \pm i\nu)\,, \,, \tag{C.8.51}$$

and $M_\epsilon = 2e^{-\gamma}/\epsilon$ as in (C.2.29), we get from (C.2.21) with $\nu \equiv \sqrt{m^2\ell^2 - \eta}$, $\eta \equiv \frac{1}{4} - 2\xi$,

$$\log Z_{\rm PI}^{(1)} = \big(2\epsilon^{-2} - m^2 \log(M_\epsilon \ell) + m^2\big)\,\ell^2 + \big(\eta + \tfrac{1}{12}\big)\log(M_\epsilon \ell) - \eta + \mathtt{f}(\nu)\,, \tag{C.8.52}$$

In the limit $m\ell \to \infty$, using the asymptotic expansion of the Hurwitz zeta function [281],

$$\log Z_{\rm PI}^{(1)} = \big(2\epsilon^{-2} - m^2 \log(M_\epsilon/m) - \tfrac{1}{2}m^2\big)\,\ell^2 + \big(\eta + \tfrac{1}{12}\big)\log(M_\epsilon/m) \;+\; \mathcal{O}\big((m\ell)^{-2}\big)\,. \tag{C.8.53}$$

Notice the $\log \ell$ dependence apparent in (C.8.52) has canceled out. The counterterm action to this order is again of the form (C.8.33), corresponding to $\log Z_{\rm ct} = 4\pi(-c_0'\ell^2 + c_2')$. The renormalization condition (C.8.36) fixes c_0' but leaves c_2' undetermined. Its natural extension here is to pick $c_2 = 4\pi c_2'$ to cancel off the constant term as well, that is

$$c_2 = -\big(\eta + \tfrac{1}{12}\big)\log(M_\epsilon/m) \qquad \Rightarrow \qquad \lim_{\ell\to\infty} \log Z^{(1)} = 0\,, \tag{C.8.54}$$

ensuring the tree-level G equals the renormalized Newton constant to this order, as in (C.8.34). The renormalized scalar one-loop contribution to the off-shell partition function is then

$$\log Z^{(1)} = \big(\tfrac{3}{2} - \log(m\ell)\big)(m\ell)^2 + \big(\eta + \tfrac{1}{12}\big)\log(m\ell) - \eta + \mathtt{f}(\nu)\,. \tag{C.8.55}$$

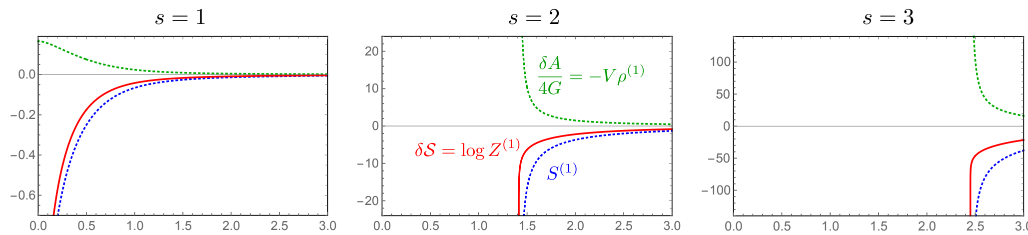


Figure C.8.3: Edge contributions to the dS_4 entropy from massive spin $s = 1, 2, 3$ fields, as a function of $m\ell_0$, with coloring as in fig. C.8.1. The Higuchi/unitarity bound in this case is $(m\ell_0)^2 - (s-\frac{1}{2})^2 > -\frac{1}{4}$.

In the large-$m\ell$ limit, $\log Z^{(1)} = \frac{240\,\eta^2+40\,\eta+7}{960}(m\ell)^{-2} + \cdots > 0$, while in the small-$m\ell$ limit

$$\log Z^{(1)} \simeq (\eta + \tfrac{1}{12}) \log(m\ell) \quad (\eta < \tfrac{1}{4}) \,, \qquad \log Z^{(1)} \simeq (\tfrac{1}{4} + \tfrac{1}{12} - 1) \log(m\ell) \quad (\eta = \tfrac{1}{4}) \,. \tag{C.8.56}$$

The extra $-\log(m\ell)$ in the minimally-coupled case $\eta = \frac{1}{4}$ is the same as in (C.8.46) and has the same thermal interpretation. The energy density is $\rho^{(1)} = -\frac{1}{2}\ell\partial_\ell \log Z^{(1)}/V$ with $V = 4\pi\ell^2$:

$$V\rho^{(1)} = -\tfrac{1}{2}(\eta + \tfrac{1}{12}) + \tfrac{1}{2}(m\ell)^2\big(2\log(m\ell) - \textstyle\sum_\pm \psi^{(0)}(\tfrac{1}{2} \pm i\nu)\big) \tag{C.8.57}$$

In the massless case $m = 0$, $\nu = \sqrt{-\eta}$ is ℓ-independent, and we cannot use the asymptotic expansion (C.8.53), nor the renormalization prescription (C.8.54). Instead we fix c_2 by minimal subtraction, picking a reference length scale L and putting (with $M_\epsilon = \frac{2e^{-\gamma}}{\epsilon}$ (C.2.29) as before)

$$c_2(L) \equiv -(\eta + \tfrac{1}{12}) \log(M_\epsilon L) \,, \tag{C.8.58}$$

The renormalized G then satisfies $\partial_L(\frac{4\pi}{8\pi G} + c_2) = 0$, i.e. $L\partial_L \frac{1}{2G} = \eta + \frac{1}{12}$, and

$$\log Z^{(1)} = (\eta + \tfrac{1}{12}) \log(\ell/L) - \eta + \mathtt{f}(\sqrt{-\eta}) \,, \qquad V\rho^{(1)} = -\tfrac{1}{2}(\eta + \tfrac{1}{12}) \,. \tag{C.8.59}$$

The total $\log Z = \frac{1}{2G}(-\Lambda\ell^2 + 1) + \log Z^{(1)}$ is of course independent of the choice of L.

C.8.4.4 4D massive spin s

4D massive spin-s fields can be treated similarly, starting from (C.2.27). In particular the edge contribution $\log Z^{(1)}_{\text{edge}}$ equals *minus* the $\log Z^{(1)}$ of $D^5_{s-1} = \frac{1}{6}s(s+1)(2s+1)$ scalars on S^2, computed earlier in (C.8.55), with the same $\nu = \sqrt{m^2\ell^2 - \eta_s}$ as the bulk spin-s field, which according to (5.4.2) means $\eta_s = (s - \frac{1}{2})^2$. The corresponding contribution to the renormalized energy density is $\rho^{(1)}_{\rm edge} = -\frac{1}{4}\ell\partial_\ell \log Z^{(1)}_{\rm edge}/V$ with $V = \Omega_4 \ell^4$, so $V\rho^{(1)}_{\rm edge}$ equals $-\frac{1}{2}D^5_{s-1}$ times the scalar result (C.8.57).

As in the $d = 2$ case, the renormalized one-loop edge contribution $\mathcal{S}^{(1)}_{\rm edge}$ to the entropy is negative and dominant. Some examples are shown in fig. C.8.3.

C.8.4.5 Graviton contribution for general d

For $d \geq 3$, UV-sensitive terms in the loop expansion renormalize higher-order curvature couplings in the gravitational action, prompting the inclusion of such terms in $S_E[g]$. Some caution is in order then if we wish to apply (5.5.17) or (5.7.19)-(5.7.20) to compute $\log Z^{(1)}$. The formula (5.5.17) for $Z^{(1)}_{\rm PI}$ depends on $\gamma = \sqrt{8\pi G_{\rm N}/A_{d-1}}$, gauge-algebraically defined by (5.5.14) and various normalization conventions. We picked these such that in pure Einstein gravity, $\gamma = \sqrt{8\pi G/A(\ell_0)}$, $\ell_0 = \sqrt{d(d-1)/2\Lambda}$, with G and Λ read off from the gravitational Lagrangian. However this expression of γ in terms of Lagrangian parameters will in general be modified in the presence of higher-order curvature terms. This is clear from the discussion in C.8.1.3, and (C.8.19) in particular. Since γ_0 is field-redefinition invariant, and since after transforming to a pure Einstein frame we have $\gamma_0 = \sqrt{2\pi/\mathcal{S}^{(0)}}$, with the right hand side also invariant, we have in general (for Einstein + perturbative higher-order curvature corrections)

$$\gamma = \sqrt{2\pi/\mathcal{S}^{(0)}}\,. \tag{C.8.60}$$

From (5.7.19) we thus get (ignoring the phase)

$$\mathcal{S} = \mathcal{S}^{(0)} - \frac{D_d}{2}\log\mathcal{S}^{(0)} + \alpha^{(2)}_{d+1}\log\frac{\ell_0}{L} + K_{d+1} + \mathcal{O}(1/\mathcal{S}^{(0)}) \tag{C.8.61}$$

where $D_d = \frac{(d+2)(d+1)}{2}$, $\alpha^{(2)}_{d+1} = 0$ for even d and given by (5.5.21) for odd d, and K_{d+1} a numerical constant obtained by evaluating (5.5.20). For odd d the constant in the counterterm $\log Z_{\rm ct}(\ell)$ is fixed by minimal subtraction at a scale L, $c_{d+1}(L) \equiv -\alpha_{d+1}\log(M_\epsilon L)$, with $M_\epsilon = 2e^{-\gamma}/\epsilon$ determined by the heat kernel regulator as in (C.2.29), and $L\partial_L\mathcal{S} = 0$, i.e.

$L\partial_L \mathcal{S}^{(0)} = \alpha^{(2)}_{d+1}$. Explicitly for $d = 2, 3, 4$, using (5.7.20), (5.1.12)

d	$\mathcal{S}$
2	$\mathcal{S}^{(0)} - 3\log\mathcal{S}^{(0)} + 5\log(2\pi)$
3	$\mathcal{S}^{(0)} - 5\log\mathcal{S}^{(0)} - \frac{571}{90}\log(\frac{\ell_0}{L}) - \log(\frac{8\pi}{3}) + \frac{715}{48} - \frac{47}{3}\zeta'(-1) + \frac{2}{3}\zeta'(-3)$
4	$\mathcal{S}^{(0)} - \frac{15}{2}\log\mathcal{S}^{(0)} + \log(12) + \frac{27}{2}\log(2\pi) + \frac{65\,\zeta(3)}{48\,\pi^2} + \frac{5\,\zeta(5)}{16\,\pi^4}$

(C.8.62)

For a $d = 3$ action (C.8.12) up to $O(l_s^2 R^2)$, with dots denoting $O(l_s^4)$ terms,

$$\mathcal{S}^{(0)} = \frac{\pi}{G}\left(\ell_0^2 + 48\,\lambda_{R^2}\,l_s^2 + \cdots\right), \qquad \Lambda = \frac{3}{\ell_0^2} + \cdots . \tag{C.8.63}$$

where $L\partial_L\lambda_{R^2} = -\frac{G}{48\,\pi\,l_s^2}\cdot\frac{571}{45}$. Putting $L = \ell_0$, and defining the scale ℓ_{R^2} by $\lambda_{R^2}(\ell_{R^2}) = 0$,

$$\mathcal{S} = \mathcal{S}^{(0)} - 5\log\mathcal{S}^{(0)} + K_4 + \mathcal{O}(1/\mathcal{S}^{(0)}), \qquad \mathcal{S}^{(0)} = \frac{\pi\ell_0^2}{G} - \frac{571}{45}\log\frac{\ell_0}{\ell_{R^2}} + \cdots . \tag{C.8.64}$$

The constant K_4 could be absorbed into λ_{R^2} at this level. Below, in (C.8.70), we will give it relative meaning however, by considering saddles different from the round S^4.

C.8.5 Classical and quantum observables

Here we address question 3 in our list below (5.8.2). To answer this, we need "observables" of the $\Lambda > 0$ Euclidean low-energy effective field theory probing independent gravitational couplings (for simplicity we restrict ourselves to purely gravitational theories here), i.e. diffeomorphism and field-redefinition invariant quantities, analogous to scattering amplitudes in asymptotically flat space. For this to be similarly useful, an infinite amount of unambiguous data should be extractable, at least in principle, from these observables.

As discussed above, $\mathcal{S}^{(0)} = \log\mathcal{Z}^{(0)} = -S_E[g_{\ell_0}]$ invariantly probes the dimensionless coupling given by $\ell_0^{d-1}/G \propto 1/G\Lambda^{(d-1)/2}$ in Einstein frame. The obvious tree-level invariants probing different couplings in the gravitational low-energy effective field theory are then the analogous $\mathcal{S}_M^{(0)} \equiv \log\mathcal{Z}_M^{(0)} = -S_E[g_M]$ evaluated on saddles g_M different from the round sphere, in the parametric $\ell_0 \gg l_s$ regime of validity of the effective field theory, with g_M asymptotically Einstein in the $\ell_0 \to \infty$ limit. These are the analogs of tree-level scattering amplitudes. The obvious quantum counterparts are the corresponding generalizations of $\mathcal{S}$, i.e. $\mathcal{S}_M \equiv \log \mathcal{Z}_M$ evaluated in large-ℓ_0 perturbation theory about the saddle g_M. These are the analogs of quantum scattering amplitudes. Below we make this a bit more concrete in examples.

3D

In $d = 2$, the Weyl tensor vanishes identically, so higher-order curvature invariants involve $R_{\mu\nu}$ only and can be removed from the action by a field redefinition in large-ℓ_0 perturbation theory, reducing it to pure Einstein form in general. As a result, $\mathcal{S}^{(0)}$ is the only independent invariant in pure 3D gravity, all g_M are Einstein, and the $\mathcal{S}_M^{(0)}$ are all proportional to $\mathcal{S}_0 \equiv \mathcal{S}_{S^3}^{(0)}$.

As discussed under (5.8.17), the quantum $\mathcal{S} = \mathcal{S}_{S^3}$ takes the form

$$\mathcal{S} = \mathcal{S}_0 = \mathcal{S}_0 - 3 \log \mathcal{S}_0 + 5 \log(2\pi) + \textstyle\sum_n c_n \, \mathcal{S}_0^{-2n} \tag{C.8.65}$$

The corrections terms in the expansion are all nonlocal (no odd powers of ℓ_0), and the coefficients provide an unambiguous, infinite data set.

odd D $\geq$ 5

In 5D gravity, there are infinitely many independent coupling constants. There are also infinitely many different $\Lambda > 0$ Einstein metrics on S^5, including a discrete infinity of Böhm metrics with SO$(3) \times$ SO(3) symmetry [270] amenable to detailed numerical analysis [271], and 68 Sasaki-Einstein families with moduli spaces up to real dimension 10 [272]. Unlike the round S^5, these are not conformally flat, and thus, unlike $\mathcal{S}^{(0)}$, the corresponding $\mathcal{S}_M^{(0)}$ will pick up couplings such as the Weyl-squared coupling λ_{C^2} in (C.8.12). It is plausible that this set of known Einstein metrics (perturbed by small higher-order corrections to the Einstein equations of motion at finite ℓ_0) more than suffices to invariantly probe all independent couplings of the gravitational action, delivering moreover infinitely many quantum observables $\mathcal{S}_M$, providing an infinity of unambiguous low-energy effective field theory data to any order in perturbation theory, without ever leaving the sphere — at least in principle.

The landscape of known $\Lambda > 0$ Einstein metrics on odd-dimensional spheres becomes increasingly vast as the dimension grows, with double-exponentially growing numbers [272]. For example there are at least 8610 families of Sasaki-Einstein manifolds on S^7, spanning all 28 diffeomorphism classes, with the standard class admitting a 82-dimensional family, and there are at least 10^{828} distinct families of Einstein metrics on S^{25}, featuring moduli spaces of dimension greater than 10^{833}.

4D

4D gravity likewise has infinitely many independent coupling constants. It is not known if S^4 has another Einstein metric besides the round sphere. In fact the list of 4D topologies known to admit $\Lambda > 0$ Einstein metrics is rather limited [282]: S^4, $S^2 \times S^2$, $\mathbb{CP}^2$, and the connected sums $\mathbb{CP}^2 \# k \overline{\mathbb{CP}^2}$, $1 \leq k \leq 8$. However for $k \geq 5$ these have a moduli space of nonzero dimension [283, 284], which might suffice to probe all couplings. (The moduli space would presumably be lifted at sufficiently high order in the l_s expansion upon turning on higher-order curvature perturbations.)

Below we illustrate in explicit detail how the Weyl-squared coupling can be extracted from suitable linear combinations of pairs of $\mathcal{S}^{(0)}_M$ with $M \in \{S^4, S^2 \times S^2, \mathbb{CP}^2\}$, and how a suitable linear combination of all three can be used to extract an unambiguous linear combination of the constant terms arising at one loop.

The Weyl-squared coupling λ_{C^2} in $S_E[g] =$ (C.8.12) $+ \cdots$ is invisible to $\mathcal{S}^{(0)}$ (C.8.63) but it is picked up by $\mathcal{S}^{(0)}_M$ by $M = S^2 \times S^2$:

$$\mathcal{S}^{(0)}_{S^2 \times S^2} = \frac{2}{3} \cdot \frac{\pi}{G} \big(\ell_0^2 + 48 \, \lambda_{R^2} \, l_s^2 + 16 \, \lambda_{C^2} \, l_s^2 \, + \, \cdots \big) \, , \tag{C.8.66}$$

with the dots denoting $\mathcal{O}(l_s^4)$ terms and $\ell_0 = \sqrt{3/\Lambda} + \cdots$ as in (C.8.63). Physically, $\mathcal{S}^{(0)}_{S^2 \times S^2}$ is the horizon entropy of the $\mathrm{dS}_2 \times S^2$ static patch, i.e. the Nariai spacetime between the cosmological and maximal Schwarzschild-de Sitter black hole horizons, both of area $A = \frac{1}{3} \cdot 4\pi \ell_0^2$. Comparing to (C.8.63), the linear combination

$$\mathcal{S}^{(0)}_{C^2} \equiv 3 \, \mathcal{S}^{(0)}_{S^2 \times S^2} - 2 \, \mathcal{S}^{(0)} = \frac{32 \pi l_s^2}{G} \big(\, \lambda_{C^2} \, + \, \cdots \big) \tag{C.8.67}$$

extracts the Weyl-squared coupling of $S_E[g]$. Analogously, for the Einstein metric on $\mathbb{CP}^2$, we get $\tilde{\mathcal{S}}^{(0)}_{C^2} \equiv 8\,\mathcal{S}^{(0)}_{\mathbb{CP}^2} - 6\,\mathcal{S}^{(0)} = \frac{48\pi l_s^2}{G}(\lambda_{C^2} + \cdots)$. Then

$$\mathcal{S}^{(0)}_{\rm cub} \equiv 2\,\tilde{\mathcal{S}}^{(0)}_{C^2} - 3\,\mathcal{S}^{(0)}_{C^2} = 16\,\mathcal{S}^{(0)}_{\mathbb{CP}^2} - 9\,\mathcal{S}^{(0)}_{S^2\times S^2} - 6\,\mathcal{S}^{(0)} \, = 0 \, + \, \cdots \, , \tag{C.8.68}$$

which extracts some curvature-cubed coupling in the effective action.

To one loop, the quantum $\mathcal{S}_M = \log \mathcal{Z}_M$ can be expressed in a form paralleling (C.8.61):

$$\mathcal{S}_M = \mathcal{S}^{(0)}_M - \frac{D_M}{2}\log \mathcal{S}^{(0)}_M + \alpha_M \log\frac{\ell_0}{L} + K_M \, + \, \cdots \, , \tag{C.8.69}$$

where D_M is the number of Killing vectors of M: $D_{S^4} = 10$, $D_{S^2\times S^2} = 6$, $D_{\mathbb{CP}^2} = 8$, and α_M can be obtained from the local expressions in [127]: $\alpha_{S^4} = -\frac{571}{45}$, $\alpha_{S^2\times S^2} = -\frac{98}{45}$, $\alpha_{\mathbb{CP}^2} = -\frac{359}{60}$. Computing the constants K_M generalizing K_{S^4} given in (C.8.62) would require more work. Moreover, computing them for one or two saddles would provide no unambiguous information because they may be absorbed into λ_{C^2} and λ_{R^2}. However, since there only two undetermined coupling constants at this order, computing them for all *three* does provide unambiguous information, extracted by the quantum counterpart of (C.8.68):

$$\mathcal{S}_{\rm cub} \, \equiv \, 16\,\mathcal{S}_{\mathbb{CP}^2} - 9\,\mathcal{S}_{S^2\times S^2} - 6\,\mathcal{S}_{S^4} = -7\,\log \mathcal{S}^{(0)} + 16\,K_{\mathbb{CP}^2} - 9\,K_{S^2\times S^2} - 6\,K_{S^4} + \cdots \tag{C.8.70}$$

The $\log(\ell_0/L)$ terms had to cancel in this linear combination because the tree-level parts at this order cancel by design and $L\partial_L \mathcal{S}_{\rm cub} = 0$.

Appendix D: Appendix for chapter 6

D.1 Partition function of Dirac spinors

As an example of applying the character integral method to fermions, let's consider a complex Dirac spinor of scaling dimension $\Delta = \frac{d}{2} + \nu$ in AdS$_{d+1}$ with $d = 2r+1$. It carries a highest weight representation $\mathbf{s} = \frac{\mathbf{1}}{\mathbf{2}}$ of SO(d) which has real dimension 2^{r+1}. The one-loop partition function of this field is given by

$$\log Z = \frac{(-2)^r}{d!}\int_0^\infty \frac{dt}{t}\, e^{-\frac{\epsilon^2}{4t}} \int_0^\infty d\lambda\, \mu_{\frac{1}{2}}(\lambda)\, e^{-t(\lambda^2+\nu^2)} \tag{D.1.1}$$

where the spinor spectral function $\mu_{\frac{1}{2}}(\lambda)$ is

$$\mu_{\frac{1}{2}}(\lambda) = \prod_{j=1}^{r}\left(\lambda^2 + j^2\right)\frac{\lambda}{\tanh(\pi\lambda)} \tag{D.1.2}$$

After using the standard Hubbard-Stratonovich trick, the partition function is completely encoded in $W_{\frac{1}{2}}(u) = \int_{-\infty}^{\infty} d\lambda\, \mu_{\frac{1}{2}}(\lambda)\, e^{i\lambda u}$:

$$\log Z = \frac{(-2)^r}{d!}\int_0^\infty \frac{du}{u}\, W_{\frac{1}{2}}(u)\, e^{-\nu u} \tag{D.1.3}$$

To perform the λ-integral in $W_{\frac{1}{2}}(u)$, we can close the contour in the upper half plane and pick up the poles at $\lambda = in, n \geq r+1$, where $\frac{\lambda}{\tanh(\pi\lambda)}$ has residue $\frac{in}{\pi}$.

$$W_{\frac{1}{2}}(u) = 2(-)^{r+1}\sum_{n\geq r+1} n\prod_{j=1}^{r}(n^2 - j^2)e^{-nu} = (-1)^{r+1}d!\frac{2\, e^{-\frac{d+1}{2}u}}{(1-e^{-u})^{d+1}} \tag{D.1.4}$$

Plugging (D.1.4) into (D.1.3) yields the unregularized partition function

$$\log Z = \int_0^\infty \frac{du}{u} \frac{-e^{-\frac{u}{2}}}{1-e^{-u}} \Theta^{\mathsf{AdS}_{d+1}}_{[\Delta,\frac{1}{2}]}(u), \quad \Theta^{\mathsf{AdS}_{d+1}}_{[\Delta,\frac{1}{2}]}(u) = \frac{2^{r+1}e^{-\Delta u}}{(1-e^{-u})^d} \tag{D.1.5}$$

We can also easily write down the regularized version following the derivation in section 6.4

$$\log Z = -\frac{1}{2}\int_{\mathbb{R}+i\delta} \frac{du}{\sqrt{u^2+\epsilon^2}} \frac{e^{-\frac{u}{2}}}{1-e^{-u}} \frac{2^{r+1}e^{-\frac{d}{2}u-\nu\sqrt{u^2+\epsilon^2}}}{(1-e^{-u})^d} \tag{D.1.6}$$

Compared to the bosonic case, the only difference is that the representation-independent factor $\frac{1+e^{-u}}{1-e^{-u}}$ gets replaced by $\frac{-2e^{-\frac{u}{2}}}{1-e^{-u}}$.

D.2 Physical interpretation of spectral density/Plancherel measure

In an ordinary quantum mechanical system, given a Hamiltonian H, the associated density of state (DOS) is defined as $\rho(E) = \operatorname{Tr}\delta(H-E)$ where we trace over the whole Hilbert space. Using the well-known distributional identity $\frac{1}{x\pm i\epsilon} = \mathsf{P}(\frac{1}{x}) \mp i\pi\delta(x)$, the DOS can also be formally expressed

$$\rho(E) = \frac{1}{2\pi i}\left(R(E+i\epsilon) - R(E-i\epsilon)\right) \tag{D.2.1}$$

where $R(E) \equiv \operatorname{Tr}\frac{1}{H-E}$ is the so-called resolvent and the limit $\epsilon \to 0^+$ is understood. In this appendix, we will show that the (scalar) Plancherel measure given by eq. (6.1.8) can be interpreted as a DOS in the sense of (D.2.1).

For a real scalar field in EAdS_{d+1}, we choose Hamiltonian H to be the Laplace-Beltrami operator $-\nabla^2$ which has a continuous spectrum $E_\lambda \equiv \frac{d^2}{4} + \lambda^2$ for all $\lambda \in \mathbb{R}_{\geq 0}$ [213]. In this case, the operator $\frac{1}{H-E}$ is nothing but a scalar Green function $G_\Delta(X,X')$ with mass $m^2 = -E \equiv \Delta(\Delta-d)$ [285–287]:

$$G_\Delta(X,X') = G_\Delta(P) = \frac{\Gamma(\Delta)(-2P)^{-\Delta}}{2\pi^{\frac{d}{2}}\Gamma(\Delta-\frac{d-2}{2})} F\left(\frac{\Delta}{2}, \frac{\Delta+1}{2}, \Delta - \frac{d-2}{2}, \frac{1}{P^2}\right), \quad P = X\cdot X' \tag{D.2.2}$$

where X, X' are points in the embedding space representation of EAdS_{d+1}. Plugging in $E = E_\lambda \pm i\epsilon$, the corresponding resolvent $R(E_\lambda \pm i\epsilon)$ is given by

$$R(E_\lambda \pm i\epsilon) = \int_{\mathsf{EAdS}_{d+1}} d^{d+1}X\, G_{\frac{d}{2}\mp i\lambda}(X,X) = \mathsf{Vol}(\mathsf{AdS}_{d+1})\, G_{\frac{d}{2}\mp i\lambda}(X,X) \tag{D.2.3}$$

Therefore, combining eq. (D.2.1) and (D.2.3), we find that the DOS per volume of $-\nabla^2$ in EAdS_{d+1} is simply

$$\frac{\rho_{d+1}(E_\lambda)}{\mathsf{Vol}(\mathsf{AdS}_{d+1})} = -\frac{1}{2\pi i}\lim_{P\to -1^-}\left(G_{\frac{d}{2}+i\lambda}(P) - G_{\frac{d}{2}-i\lambda}(P)\right) \tag{D.2.4}$$

where $P \to -1^-$ means that P approaches -1 from the left. (Technically the direction of limit is important, and physically the direction is also fixed because $X \cdot X' \leq -1$ for any two points on EAdS_{d+1}). Before showing the main result extracted from eq. (D.2.4), let's digress a bit and discuss some properties of hypergeometric functions appearing in eq. (D.2.2). In general, the hypergeometric function $F(a,b,c;x)$ with $\mathsf{Re}(c-(a+b)) < 0$ is singular around $x = 1$ [288]

$$\lim_{x\to 1-}\frac{F(a,b,c,x)}{(1-x)^{c-a-b}} = -\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} \tag{D.2.5}$$

which implies that the two Green functions in (D.2.4) have singularity as P approaches -1^-. However, amazingly all the divergences cancel out in the end when we take the difference (which of course is expected since the DOS per volume should be well-defined). We list some lower dimensional examples here

$$\begin{aligned}
\frac{\rho_2(E_\lambda)}{\mathsf{Vol}(\mathsf{AdS}_2)} &= \frac{1}{4\pi}\tanh(\pi\lambda), && d=1\\
\frac{\rho_4(E_\lambda)}{\mathsf{Vol}(\mathsf{AdS}_4)} &= \frac{1}{16\pi^2}\left(\lambda^2+\frac{1}{4}\right)\tanh(\pi\lambda), && d=3\\
\frac{\rho_6(E_\lambda)}{\mathsf{Vol}(\mathsf{AdS}_6)} &= \frac{1}{128\pi^3}\left(\lambda^2+\frac{1}{4}\right)\left(\lambda^2+\frac{9}{4}\right)\tanh(\pi\lambda), && d=5
\end{aligned} \tag{D.2.6}$$

and

$$\begin{aligned}
\frac{\rho_3(E_\lambda)}{\mathrm{Vol}(\mathrm{AdS}_3)} &= \frac{1}{4\pi^2}\lambda, && d=2 \\
\frac{\rho_5(E_\lambda)}{\mathrm{Vol}(\mathrm{AdS}_5)} &= \frac{1}{24\pi^3}\lambda(1+\lambda^2), && d=4 \\
\frac{\rho_7(E_\lambda)}{\mathrm{Vol}(\mathrm{AdS}_7)} &= \frac{1}{240\pi^4}\lambda\left(\lambda^2+1\right)\left(\lambda^2+4\right), && d=6 \qquad \text{(D.2.7)}
\end{aligned}$$

Notice that what we've obtained here is the number of states per unit "energy" E_λ rather than spectral density because the latter is the number of states per unit λ. However, they can be easily mapped to each other by a change of integral measure $dE_\lambda = 2\lambda\, d\lambda$. This observation suggests us to define the spectral density as

$$\tilde{\rho}_{d+1}(\lambda) \equiv \frac{2\lambda\, \rho_{d+1}(E_\lambda)}{\mathrm{Vol}(\mathrm{AdS}_{d+1})} = \frac{1}{2^{d-1}\Gamma(\frac{d+1}{2})^2 \mathrm{Vol}(S^d)} \frac{|\Gamma(\frac{d}{2}+i\lambda)|^2}{|\Gamma(i\lambda)|} \qquad \text{(D.2.8)}$$

where the λ-dependent factor $\frac{|\Gamma(\frac{d}{2}+i\lambda)|^2}{|\Gamma(i\lambda)|}$ is exactly what we call $\mu^{(d)}(\lambda)$ in eq. (6.2.9). As a final consistency check, let's reconstruct the scalar heat kernel associated to $(-\nabla^2+\nu^2-\frac{d^2}{4})$ from its canonical definition, i.e. "summing" over all energy eigenfunctions

$$\begin{aligned}
K_\nu(t) \equiv \mathrm{Tr}\, e^{-(-\nabla^2+\nu^2-\frac{d^2}{4})} &= \int_{\frac{d^2}{4}}^\infty dE_\lambda\, \rho_{d+1}\,(E_\lambda) e^{-(E_\lambda+\nu^2-\frac{d^2}{4})} \\
&= \mathrm{Vol}(\mathrm{AdS}_{d+1}) \int_0^\infty d\lambda\, \tilde{\rho}_{d+1}(\lambda) e^{-t(\lambda^2+\nu^2)} \\
&= \frac{\mathrm{Vol}(\mathrm{AdS}_{d+1})}{\mathrm{Vol}(S^d)} \frac{1}{2^{d-1}\Gamma(\frac{d+1}{2})^2} \int_0^\infty d\lambda\, \mu^{(d)}(\lambda) e^{-t(\lambda^2+\nu^2)} \qquad \text{(D.2.9)}
\end{aligned}$$

Altogether, the computations in this appendix help us to identify the spectral density or the Plancherel measure of $\mathrm{SO}(1,d+1)$ which has a rigorous mathematical definition in the pure group theory setup [64, 66], as the density of states associated to the Hamiltonian $H=-\nabla^2$ in a unit volume of EAdS_{d+1} up to some representation-independent normalization factors.

D.3 Comparison with dS character integral

In sections 6.2 and 6.3, we derived character integral formulae for one-loop partition functions of both scalars and spin-s fields in even dimensional AdS. These formulae are very similar with their dS counterpart derived in [[chapter ?]] , where the unregularized one-loop partition function of a *massive* spin-s field with scaling dimension $\Delta = \frac{d}{2} + i\nu$ is given by

$$\log Z_{\text{PI}} = \int_0^\infty \frac{du}{2u}(e^{-(\frac{d}{2}+i\nu)u} + e^{-(\frac{d}{2}-i\nu)u}) \sum_{n\geq -1} D^{d+2}_{n,s} e^{-nu} \tag{D.3.1}$$

Here the extension of the sum to $n = -1$ is a result of locality. Summing over n yields a (bulk+edge) type contribution as in $W_s^{(d)}$:

$$\log Z_{\text{PI}} = \int_0^\infty \frac{du}{2u}\frac{1+e^{-u}}{1-e^{-u}}\left(D^d_s \frac{e^{-(\frac{d}{2}+i\nu)u} + e^{-(\frac{d}{2}-i\nu)u}}{(1-e^{-u})^d} - D^{d+2}_{s-1}\frac{e^{-(\frac{d-2}{2}+i\nu)u} + e^{-(\frac{d-2}{2}-i\nu)u}}{(1-e^{-u})^{d-2}}\right) \tag{D.3.2}$$

In this appendix, we will show that the origin of such similarity between AdS and dS can be traced back to the eq. (6.1.9).

On the AdS side, we know that the unregularized partition function of a field carrying the *massive* representation $[\Delta = \frac{d}{2} + \nu, \mathbf{s}]$ of SO$(2, d)$ is given by (assuming $d = 2r + 1$)

$$\log Z_{\mathbf{s},\nu} = \frac{(-)^{r+1}}{d!}\int_0^\infty \frac{du}{2u}\, D^d_{\mathbf{s}}\, W^{(d)}_{\mathbf{s}}(u)\, e^{-\nu\, u} \tag{D.3.3}$$

where $W^{(d)}_{\mathbf{s}}(u) \equiv \int d\lambda\, \mu^{(d)}_{\mathbf{s}}(\lambda)e^{i\lambda u}$ can be written as a series by closing the contour at infinity:

$$W^{(d)}_{\mathbf{s}}(u) = 2(-)^{r+1}\sum_{n\geq 0}\left(n+\frac{1}{2}\right)\prod_{j=1}^{r}\left(\left(n+\frac{1}{2}\right)^2 - \ell_j^2\right) e^{-(n+\frac{1}{2})u} \tag{D.3.4}$$

Using the eq. (6.1.9), we obtain $\frac{(-)^{r+1}}{d!}D^d_{\mathbf{s}}W^{(d)}_{\mathbf{s}}(u) = \sum_{n\geq 0} D^{d+2}_{n,\mathbf{s}}e^{-(n+\frac{1}{2})u}$ which yields

$$\begin{aligned}\log Z_{\mathbf{s},\nu} &= \int_0^\infty \frac{du}{2u}\, e^{-\nu\, u}\sum_{n\geq 0} D^{d+2}_{n-r,\mathbf{s}}\, e^{-(n+\frac{d}{2})u} \\ &= \int_0^\infty \frac{du}{2u}\, e^{-\Delta\, u}\sum_{n\geq -r} D^{d+2}_{n,\mathbf{s}}\, e^{-n\, u}\end{aligned} \tag{D.3.5}$$

where in the second line we have shifted n by r. Now let's focus on the spin-s representation, i.e. $\mathbf{s} = (s, 0, \cdots, 0)$. In this case $D^{d+2}_{n,s}$ vanishes for $n \in \{-r, -(r-1), \cdots, -2\}$ (and also $n = s - 1$ but this is irrelevant to our discussion) and hence the sum in eq. (D.3.5) effectively starts from $n = -1$

$$\log Z_{s,\nu} = \int_0^\infty \frac{du}{2u}\, e^{-\Delta\, u} \sum_{n\geq -1} D^{d+2}_{n,s}\, e^{-n\, u} \tag{D.3.6}$$

Compared to (D.3.1), it's clear that the only difference is the absence of $e^{-\bar{\Delta}u}$ because in AdS only one boundary mode is dynamical and the other one is identified as a source.

D.4 Evaluation of various residues

This appendix is a collection of technical proofs and results about residues of certain functions appearing in the character integrals. The ultimate goal here is to compute the residue of

$$F_{d,\nu}(u) = \frac{1}{2u}\frac{1+e^{-u}}{1-e^{-u}}\frac{e^{-(\frac{d}{2}+\nu)u}}{(1-e^{-u})^d} \tag{D.4.1}$$

at $u = 0$ for even dimension d, which is closely related to the one-loop partition functions in odd dimensional AdS. For most of the discussions in this section, we consider a general dimension d and only restrict the result to even d in the end. An intermediate step to $\text{Res}_{u\to 0}F_{d,\nu}(u)$ is the residue of the following function

$$G_{d,\nu}(u) \equiv \frac{e^{-(\frac{d}{2}+\nu)u}}{(1-e^{-u})^{d+1}} \tag{D.4.2}$$

which itself is also very interesting because we need it to verify the contour prescription proposed in section 6.4.

First we show by induction that

$$\text{Res}_{u\to 0}G_{d,\nu}(u) = \frac{(-)^d}{d!}\frac{\Gamma(\nu+\frac{d}{2})}{\Gamma(\nu-\frac{d}{2})} \tag{D.4.3}$$

It's straightforward to check that eq. (D.4.3) holds for $d = 1$. Assuming the induction condition

(D.4.3), we show that it also works for $d+1$. Let C_0 be a small circle around $u=0$, i.e. it doesn't enclose any other poles of $G_{d,\nu}(u)$ except $u=0$. Then the residue of $G_{d+1,\nu}(u)$ at $u=0$ can be expressed as a contour integral along C_0 counterclockwisely

$$\mathrm{Res}_{u\to 0}G_{d+1,\nu}(u)=\oint_{C_0}\frac{du}{2\pi i}\frac{e^{-(\frac{d+1}{2}+\nu)u}}{(1-e^{-u})^{d+2}} \tag{D.4.4}$$

To use the induction condition, we should lower the power in the denominator which can be realized by integration by part:

$$\begin{aligned}\mathrm{Res}_{u\to 0}G_{d+1,\nu}(u)&=-\frac{1}{d+1}\oint_{C_0}\frac{du}{2\pi i}e^{-(\frac{d}{2}+\nu-\frac{1}{2})u}\frac{d}{du}\frac{1}{(1-e^{-u})^{d+1}}\\&=-\frac{\frac{d-1}{2}+\nu}{d+1}\oint_{C_0}\frac{du}{2\pi i}\frac{e^{-(\frac{d}{2}+\nu-\frac{1}{2})u}}{(1-e^{-u})^{d+1}}=-\frac{\frac{d-1}{2}+\nu}{d+1}\mathrm{Res}_{u\to 0}G_{d,\nu-\frac{1}{2}}(u)\end{aligned} \tag{D.4.5}$$

Applying the induction condition (D.4.3) to eq. (D.4.5) yields

$$\mathrm{Res}_{u\to 0}G_{d+1,\nu}(u)=\frac{(-)^{d+1}}{(d+1)!}\frac{\Gamma(\nu+\frac{d+1}{2})}{\Gamma(\nu-\frac{d+1}{2})} \tag{D.4.6}$$

This confirms that (D.4.3) holds for all d.

To bridge the gap between $\mathrm{Res}_{u\to 0}G_{d,\nu}(u)$ and $\mathrm{Res}_{u\to 0}F_{d,\nu}(u)$, we need to define another function

$$H_{d,\nu}(u)\equiv\frac{1+e^{-u}}{1-e^{-u}}\frac{e^{-(\frac{d}{2}+\nu)u}}{(1-e^{-u})^{d}}=G_{d,\nu}(u)+G_{d,\nu+1}(u) \tag{D.4.7}$$

It's direct to write down the residue of $H_{d,\nu}(u)$ at $u=0$ by using its relation with the G-functions

$$\mathrm{Res}_{u\to 0}H_{d,\nu}(u)=\frac{2(-)^{d}}{d!}\nu\frac{\Gamma(\nu+\frac{d}{2})}{\Gamma(\nu+1-\frac{d}{2})} \tag{D.4.8}$$

which is a polynomial in ν for positive integer d. In addition, the R.H.S of eq. (D.4.8) is an even function in ν when d is even and an odd function when d is odd. More explicitly, for $d = 2r$,

$$\text{Res}_{u\to 0} H_{d,\nu}(u) = \frac{2}{d!} \prod_{j=0}^{r-1} (\nu^2 - j^2) \tag{D.4.9}$$

and for $d = 2r + 1$,

$$\text{Res}_{u\to 0} H_{d,\nu}(u) = -\frac{2\nu}{d!} \prod_{j=0}^{r-1} \left(\nu^2 - \left(j + \frac{1}{2}\right)^2\right) \tag{D.4.10}$$

To achieve our original goal, the residue of $F_{d,\nu}(u)$ at $u = 0$ for even d, we need the following differential relation between $F_{d,\nu}(u)$ and $H_{d,\nu}(u)$

$$\partial_\nu F_{d,\nu}(u) = -\frac{1}{2} H_{d,\nu}(u) \tag{D.4.11}$$

which yields

$$\text{Res}_{u\to 0} F_{d,\nu}(u) = \frac{(-)^{d+1}}{d!} \int_0^\nu dx\, x \frac{\Gamma(x + \frac{d}{2})}{\Gamma(x + 1 - \frac{d}{2})} + \text{const} \tag{D.4.12}$$

The unknown constant can be easily fixed for even d without any extra effort. This claim follows from the observation that $F_{2r,0}(u)$ is an even function in u. Thus $\text{Res}_{u\to 0} F_{2r,\nu}(u)$ vanishes when $\nu = 0$ and the integration constant has to be zero:

$$\begin{aligned} \text{Res}_{u\to 0} F_{2r,\nu}(u) &= -\frac{1}{(2r)!} \int_0^\nu dx \prod_{j=0}^{r-1} (x^2 - j^2) \\ &= -\frac{1}{(2r)!} \sum_{n=1}^{r} \frac{a_n(r)}{2n+1} \nu^{2n+1} \end{aligned} \tag{D.4.13}$$

where the numerical coefficients $\{a_n(r)\}$ are defined through the following generating function

$$\prod_{j=0}^{r-1} (x - j^2) = \sum_{n=1}^{r} a_n(r)\, x^n \tag{D.4.14}$$

D.5 Various coordinate systems in Euclidean/Lorentzian AdS

We begin with the embedding space representation of Lorentzian AdS$_{d+1}$ of unit radius

$$-(X^0)^2 + (X^1)^2 + \cdots + (X^d)^2 - (X^{d+1})^2 = -1 \tag{D.5.1}$$

By Wick rotation $X^{d+1} \to -iX^{d+1}$, we obtain Euclidean AdS in embedding space

$$-(X^0)^2 + (X^1)^2 + \cdots + (X^{d+1})^2 = -1 \tag{D.5.2}$$

The global coordinate for Euclidean AdS is chosen to be

$$X^0 = \cosh\eta,\ \ X^a = \sinh\eta\,\Omega_d^a, \quad 1 \le a \le d+1 \tag{D.5.3}$$

where $\eta \ge 0$ and Ω_d denotes a point on S^d. In this coordinate, the metric is given by

$$ds^2_{\mathsf{EAdS}_{d+1}} = d\eta^2 + \sinh^2\eta\, d\Omega_d^2 \tag{D.5.4}$$

In particular when $d = 1$, choosing $\Omega_1 = (\cos\varphi, \sin\varphi)$, the metic is $ds^2_{\mathsf{EAdS}_2} = d\eta^2 + \sinh^2\eta\, d\varphi^2$. Under the Wick rotation $\varphi \to it$, we transform back to Lorentzian signature. In embedding space, it means we choose the following coordinate systems on two patches that cover different portions of Lorentzian AdS

$$\text{Southern}\ :\begin{cases} X^0 = \rho \\ X^1 = \sqrt{\rho^2-1}\cosh t_S \\ X^2 = \sqrt{\rho^2-1}\sinh t_S \end{cases}, \quad \text{Northern}\ :\begin{cases} X^0 = \rho \\ X^1 = -\sqrt{\rho^2-1}\cosh t_N \\ X^2 = \sqrt{\rho^2-1}\sinh t_N \end{cases} \tag{D.5.5}$$

where I've replaced $\cosh\eta$ by $\rho \ge 1$. The metric of southern/northern patch can be expressed as

$$ds^2_{\mathsf{AdS}_2} = -(\rho^2-1)dt^2 + \frac{d\rho^2}{\rho^2-1}, \quad t = t_S, t_N \tag{D.5.6}$$

which describes a black hole solution with a point-like horizon at $\rho = 1$, the intersection of the southern and northern patches. The temperature of this black hole is $T = \frac{1}{2\pi}$.

When $d \geq 2$, there exists a similar Wick rotation that describes a spacetime of the same temperature. Notice that the global coordinate system (D.5.3) realizes a S^d foliation of EAdS$_{d+1}$. By using the Wick rotation between de Sitter static patch and sphere, we obtain a dS foliation of AdS$_{d+1}$. More explicitly, we choose the following coordinate system for Ω_d

$$\Omega_d = (r\,\Omega_{d-2}, \sqrt{1-r^2}\,\cos\varphi, \sqrt{1-r^2}\,\sin\varphi), \quad 0 \leq r \leq 1 \tag{D.5.7}$$

where Ω_{d-2} denotes the usual spherical coordinates of S^{d-2}. Upon a Wick rotation $\varphi \to it$, Ω_d becomes a point on dS$_d$ and (D.5.4) becomes the Rindler-AdS metric [4]. As before, the Wick-rotated coordinate system describes two patches of Lorentzian AdS

$$\text{Southern}: \begin{cases} X^0 = \cosh\eta \\ X^{\bar{i}} = r\sinh\eta\,\Omega_{d-2}^{\bar{i}} \\ X^d = \sinh\eta\cosh t_S\sqrt{1-r^2} \\ X^{d+1} = \sinh\eta\sinh t_S\sqrt{1-r^2} \end{cases}, \quad \text{Northern}: \begin{cases} X^0 = \cosh\eta \\ X^{\bar{i}} = r\sinh\eta\,\Omega_{d-2}^{\bar{i}} \\ X^d = -\sinh\eta\cosh t_N\sqrt{1-r^2} \\ X^{d+1} = \sinh\eta\sinh t_N\sqrt{1-r^2} \end{cases} \tag{D.5.8}$$

in either of which the metric is

$$ds^2_{\text{AdS}_{d+1}} = d\eta^2 + \sinh^2\eta\left(-(1-r^2)dt^2 + \frac{dr^2}{1-r^2} + r^2 d\Omega_{d-2}^2\right), \quad t = t_S, t_N \tag{D.5.9}$$

The two patches intersect at the horizon $r = 1$ which has the geometry of EAdS$_{d-1}$.

Finally, let's also introduce the global coordinate of AdS

$$\text{Global}: \begin{cases} X^0 = \sqrt{R^2+1}\cos t_G \\ X^i = R\,\Omega_{d-1}^i \\ X^{d+1} = \sqrt{R^2+1}\sin t_G \end{cases} \tag{D.5.10}$$

At time $t_G = t_S = t_N = 0$, i.e. $X^{d+1} = 0$, the southern and northern patches cover the $X^d \geq 0$

and $X^d \leq 0$ parts of the global spatial slice respectively.

D.6 SO$(2,d)$ Harish-Chandra characters

SO$(2,d)$ is the isometry group of AdS$_{d+1}$. In conventions in which the generators of SO$(2,d)$ are hermitian operators $L_{MN}, 0 \leq M, N \leq d+1$, they are subject to commutation relations

$$[L_{MN}, L_{PQ}] = i(\eta_{MP}L_{NQ} + \eta_{NQ}L_{MP} - \eta_{MQ}L_{NP} - \eta_{NP}L_{MQ}) \tag{D.6.1}$$

where $\eta_{MN} = \text{diag}(-,+,\cdots,+,-)$. The physical interpretation of this algebra will be clear using the following Cartan-Weyl type basis

$$H = L_{0,d+1}, \quad L_i^{\pm} = L_{i0} \mp iL_{i,d+1}, \quad M_{ij} = L_{ij} \tag{D.6.2}$$

with commutation relations

$$[H, L_i^{\pm}] = \pm L_i^{\pm}, \quad [L_i^-, L_j^+] = 2\delta_{ij}H - 2iM_{ij}, \quad [L_{ij}, L_k^{\pm}] = i(\delta_{ik}L_j^{\pm} - \delta_{jk}L_i^{\pm}) \tag{D.6.3}$$

where the trivial commutation relations are omitted. While acting on the AdS$_{d+1}$ quantum Hilbert space, H can be identified with the Hamiltonian which generates time translation in global coordinates, and M_{ij} can be identified with angular momentum operators. The $L_i^{\pm}$ can then be viewed as raising/lowering operators for energy eigenstates. We're mainly interested in single-particle Hilbert space $\mathcal{H}_\Delta$ built from a primary state $|\Delta\rangle$ (also known as the lowest energy state), i.e. $H|\Delta\rangle = \Delta|\Delta\rangle, L_i^-|\Delta\rangle = 0$. By construction the Hilbert space $\mathcal{H}_\Delta$ furnishes a representation of $\mathfrak{so}(2,d)$.

In most of the physics literature [12, 70, 289, 290], the $\mathfrak{so}(2,d)$ character of representation $\mathcal{H}_\Delta$ is computed with respect to a compact Cartan algebra, in particular Hamiltonian and rotations. Here, for our purpose of thermal interpretations in section 6.8, we illustrate in (unitary) scalar primary representations how to compute $\mathfrak{so}(2,d)$ character associated to a noncompact generator, i.e. generator of boost in AdS$_{d+1}$.

D.6.1 SO$(2,1)$ character

As shown in eq. (D.6.2) and (D.6.3), the Lie algebra $\mathfrak{so}(2,1)$ is generated by $\{H, L^{\pm}\}$ with commutation relations $[H, L^{\pm}] = 2L^{\pm}, [L^{-}, L^{+}] = 2H$. Starting from the primary state $|\Delta\rangle$, we build a tower of descendants $|n\rangle \equiv (L^{+})^{n}|\Delta\rangle, n \geq 0$ which is a basis of the Hilbert space $\mathcal{H}_{\Delta}$. Then the character associated to a noncompact generator, say $L_{12} = \frac{i}{2}(L^{+} - L^{-})$, is defined as

$$\mathrm{Tr}_{\mathcal{H}_{\Delta}}\, e^{itL_{12}} \equiv \sum_{n\geq 0} \frac{\langle n|e^{itL_{12}}|n\rangle}{\langle n|n\rangle}, \quad t \in \mathbb{R} \tag{D.6.4}$$

Though defined through a simple and transparent way physically, it's technically very hard to figure out the character by computing this sum. Therefore, we'll use a different realization of the same representation that makes the same computation doable.

Disc realization: In [289], by using the standard coadjoint orbit method, Witten showed that the Hilbert space $\mathcal{H}_{\Delta}$ can be mapped to the space of normalizable holomorphic function $f(z)$ on disc $D = \{z \in \mathbb{C} : |z| < 1\}$ with inner product

$$||f||^2 = \int_D |f(z)|^2\,(1 - z\bar{z})^{2(\Delta-1)} d^2 z \tag{D.6.5}$$

On these holomorphic function, the generators of $\mathfrak{so}(2,1)$ act as

$$H = z\,\partial_z + \Delta, \quad L^{-} = -i\partial_z, \quad L^{+} = -i(z^2\partial_z + 2\Delta z) \tag{D.6.6}$$

The normalizable function $f(z) = z^k$ is an eigenfunction of H with eigenvalue $\Delta + k$ and hence the character associated to H is

$$\mathrm{Tr}_{\mathcal{H}_{\Delta}}\, q^{H} = \sum_{k=0}^{\infty} q^{\Delta+k} = \frac{q^{\Delta}}{1-q}, \quad 0 < |q| < 1 \tag{D.6.7}$$

Upper half-plane realization: Using a fractional linear transformation $z \to w = \frac{1-iz}{z-i}$, we can map the disc D to the upper half-plane $\mathbb{H} = \{x + iy \in \mathbb{C} : y > 0\}$ and the new Hilbert space consists of normalizable holomorphic functions $f(w)$ on $\mathbb{H}$ with inner product [66, 291]

$$||f||^2 = \int_{\mathbb{H}} \frac{dxdy}{y^2}\, y^{2\Delta}\, |f(w)|^2, \quad w = x + iy,\ y > 0 \tag{D.6.8}$$

When acting on a holomorphic function $f(w)$ on $\mathbb{H}$, the algebra $\mathfrak{so}(2,1)$ is realized as

$$H = -i\left(\frac{1+w^2}{2}\partial_w + \Delta\, w\right), \quad L_{10} = i\left(\frac{w^2-1}{2}\partial_w + \Delta\, w\right), \quad L_{12} = i(w\partial_w + \Delta) \tag{D.6.9}$$

The eigenfunctions of H are $\phi_k(w) \equiv (w-i)^k(w+i)^{-2\Delta-k}, k \in \mathbb{N}$, with $H\phi_k = (\Delta+k)\phi_k$.

$\mathbb{R}_+$ **realization [291] and evaluation of character**: Given a holomorphic function $f(w)$ on the upper half-plane, we can define a new function F on $\mathbb{R}_+$:

$$F(\xi) = \int_{\mathrm{Im}\,(w)=\text{const}} \frac{dw}{2\pi}\, f(w)^{-iw\xi}, \quad \xi \in \mathbb{R}_+ \tag{D.6.10}$$

and the inverse transformation is given by

$$f(w) \equiv \int_0^\infty d\xi\, F(\xi) e^{iw\xi}, \quad \mathrm{Im}\,(w) > 0 \tag{D.6.11}$$

Under the integral transformation (D.6.10), the inner product (D.6.8) is mapped to

$$||F||^2 = \int_0^\infty d\xi\, \xi^{1-2\Delta} |F(\xi)|^2 \tag{D.6.12}$$

and the $\mathfrak{so}(2,1)$ action (D.6.9) is mapped to

$$\begin{gathered} H = \frac{1}{2}\xi(1-\partial_\xi^2) + (\Delta-1)\partial_\xi, \quad L_{10} = \frac{1}{2}\xi(1+\partial_\xi^2) + (1-\Delta)\partial_\xi \\ L_{12} = -i(\xi\,\partial_\xi + 1 - \Delta) \end{gathered} \tag{D.6.13}$$

To find all eigenfunctions of H in (D.6.13), let's start from the primary state $\phi_0(w) = (w+i)^{-2\Delta}$ in the upper half plane realization. ϕ_0 can be expressed as a Schwinger parameterization:

$$(w+i)^{-2\Delta} = \frac{1}{i^{2\Delta}\Gamma(2\Delta)} \int_0^\infty \frac{d\xi}{\xi}\, \xi^{2\Delta} e^{-\xi(1-iw)} \tag{D.6.14}$$

Comparing (D.6.14) with (D.6.11), we immediately get that the dual function of $\phi_0(w)$ in $\mathbb{R}_+$ is $G_0(\xi) = \xi^{2\Delta-1}e^{-\xi}$ (dropping unimportant normalization constants). For the dual function of $\phi_k(w)$, we use the ansatz $G_k(\xi) = G_0(\xi)P_k(\xi)$, where $P_k(\xi)$ is a polynomial in ξ. Then the eigenequation $HG_k = (\Delta+k)G_k$ yields a second order differential equation of $P_k(\xi)$

$$\frac{\xi}{2}\,P_k(\xi)'' + (\Delta-\xi)\,P_k(\xi)' + k\,P_k(\xi) = 0 \qquad \text{(D.6.15)}$$

whose polynomial solution is the generalized Laguerre polynomial $P_k(\xi) = L_k^{(2\Delta-1)}(2\xi)$. Thus the spectrum of H is given by

$$HG_k = (\Delta+k)G_k, \quad G_k(\xi) = \xi^{2\Delta-1}\,L_k^{(2\Delta-1)}(2\xi)\,e^{-\xi} \qquad \text{(D.6.16)}$$

By using the recurrence relation of Laguerre polynomial, we can also show that $L^{\pm}$ indeed behaves like lower/raise operator

$$L^{-}G_k(\xi) = (1-2\Delta-k)G_{k-1}(\xi), \quad L^{+}G_k(\xi) = -(k+1)G_{k+1}(\xi) \qquad \text{(D.6.17)}$$

As another self-consistency check of this representation, we show the Fourier/Laplace transformation (D.6.11) of $G_k(\xi)$ is $\phi_k(w)$ up to normalization factors. To do this, we need the series expansion of a generalized Laguerre polynomial $L_n^{(\alpha)}(x) = \sum_{\ell=0}^{n}\frac{(-)^\ell}{\ell!}\binom{n+\alpha}{n-\ell}x^\ell$, which yields

$$\begin{aligned}\int_0^\infty d\xi\,G_k(\xi)e^{i\,\xi\,w} &= \sum_{\ell=0}^{k}\frac{(-2)^\ell}{\ell!}\binom{2\Delta+k-1}{k-\ell}\int_0^\infty\frac{d\xi}{\xi}\,\xi^{2\Delta+\ell}e^{-(1-iw)\xi}\\ &= \frac{\Gamma(2\Delta+k)i^{2\Delta}}{k!(w+i)^{2\Delta}}\sum_{\ell=0}^{k}\binom{k}{\ell}\left(\frac{-2}{1-iw}\right)^\ell = \frac{\Gamma(2\Delta+k)i^{2\Delta}}{k!}\phi_k(w) \qquad \text{(D.6.18)}\end{aligned}$$

Finally, we are at a stage of actually evaluating the character associated with L_{12} by using the new basis $G_k(\xi)$ of $\mathcal{H}_\Delta$:

$$\mathrm{Tr}\,e^{itL_{12}} = \sum_{k\geq 0}\frac{(G_k, e^{itL_{12}}G_k)}{(G_k,G_k)} \qquad \text{(D.6.19)}$$

where $(e^{itL_{12}}G_k)(\xi) = e^{(1-\Delta)t}G_k(e^t\xi)$ is obtained by exponentiating the action of L_{12} in (D.6.13) and $(G_k,G_k) = \frac{\Gamma(2\Delta+k)}{2^{2\Delta}\,k!}$ is a result of the orthogonality of generalized Laguerre polynomial:

$$\int_0^\infty x^\alpha L_n^{(\alpha)}\, L_m^{(\alpha)} e^{-x}\, dx = \frac{\Gamma(n+\alpha+1)}{n!}\delta_{nm} \tag{D.6.20}$$

Thus the character $\operatorname{Tr} e^{itL_{12}}$ can be expressed as

$$\operatorname{Tr} e^{itL_{12}} = e^{\Delta t} \sum_{k\geq 0} \frac{k!}{\Gamma(2\Delta+k)} \int_0^\infty d\xi \xi^{2\Delta-1} e^{-\frac{1+e^t}{2}\xi} L_k^{(2\Delta-1)}(\xi) L_k^{(2\Delta-1)}(e^t\xi) \tag{D.6.21}$$

If we switch the order of summation and integration mindlessly, the sum is not convergent. To makes sense of this procedure, we introduce a factor $(1-\delta)^k, \delta > 0$

$$\begin{aligned} &\sum_{k\geq 0} \frac{k!}{\Gamma(2\Delta+k)} L_k^{(2\Delta-1)}(\xi) L_k^{(2\Delta-1)}(e^t\xi)(1-\delta)^k \\ &\quad = \frac{(1-\delta)^{\frac{1}{2}-\Delta}}{\delta} \frac{e^{(\frac{1}{2}-\Delta)t}}{\xi^{2\Delta-1}} e^{-(\frac{1}{\delta}-1)(1+e^t)\xi} I_{2\Delta-1}\left(\frac{2e^{\frac{t}{2}}\sqrt{1-\delta}}{\delta}\xi\right) \end{aligned} \tag{D.6.22}$$

This equation is called "Hardy-Hille formula" [292]. With the summation regularized and evaluated, the remaining integral can be computed by using the result on page 91 of [288]

$$\begin{aligned} \operatorname{Tr} e^{itL_{12}} &= \frac{(1-\delta)^{\frac{1}{2}-\Delta}}{\delta} e^{\frac{t}{2}} \int_0^\infty d\xi e^{-A\xi} I_{2\Delta-1}(B\xi) \\ &= (1-\delta)^{\frac{1}{2}-\Delta} e^{\frac{t}{2}} \left(\frac{B}{A+\sqrt{A^2-B^2}}\right)^{2\Delta-1} \frac{1}{\delta\sqrt{A^2-B^2}} \end{aligned} \tag{D.6.23}$$

where

$$A = \left(\frac{2}{\delta}-1\right)\cosh(t/2) e^{\frac{t}{2}}, \quad B = \frac{2\sqrt{1-\delta}}{\delta} e^{\frac{t}{2}} \tag{D.6.24}$$

Expanding around $\delta = 0$ and keeping the leading term yield

$$\operatorname{Tr} e^{itL_{12}} = \frac{(\cosh(t/2)+\sinh(|t|/2))^{1-2\Delta}}{2\sinh(|t|/2)} \tag{D.6.25}$$

When $t > 0$, it's reduced to $\operatorname{Tr} e^{itL_{12}} = \frac{e^{-\Delta t}}{1-e^{-t}}$ and when $t < 0$, it's $\operatorname{Tr} e^{itL_{12}} = \frac{e^{\Delta t}}{1-e^{t}}$. Altogether, the character associated to the noncompact generator L_{12} can be summarized as

$$\mathrm{Tr}\, e^{itL_{12}} = \frac{e^{-\Delta|t|}}{1-e^{-|t|}} = \mathrm{Tr}\, e^{-|t|H} \tag{D.6.26}$$

This equation also holds if L_{12} is replaced by L_{10} since they are related by a conjugation of H.

D.6.2 SO$(2,d)$ character

In higher dimensional, we fix an $\mathfrak{so}(2,1)$ subalgebra, i.e. generated by $\{H, L_1^\pm\}$ and decompose the $\mathfrak{so}(2,d)$-invariant Hilbert space $\mathcal{H}_\Delta^{\mathrm{AdS}_{d+1}}$ into $\mathfrak{so}(2,1)$-invariant subspaces where the formula (D.6.26) can be used. For concreteness, we use $\mathfrak{so}(2,2)$ scalar representations to illustrate how this decomposition procedure works. Let $|\Delta\rangle$ be the scalar primary state of an $\mathfrak{so}(2,2)$ representation

$$H|\Delta\rangle = \Delta|\Delta\rangle, \quad L_1^-|\Delta\rangle = 0, \quad L_2^-|\Delta\rangle = 0, \quad M_{12}|\Delta\rangle = 0 \tag{D.6.27}$$

Decomposition of $\mathcal{H}_\Delta^{\mathrm{AdS}_3}$ into $\mathfrak{so}(2,1)$-invariant subspaces is equivalent to finding $\mathfrak{so}(2,2)$ descendants that are $\mathfrak{so}(2,1)$ primary. Suppose $|\psi_n\rangle = \sum_{k=0}^n c_k (L_1^+)^{n-k}(L_2^+)^k|\Delta\rangle$ with $c_n = 1$ is such a state. Then the $\mathfrak{so}(2,1)$ primary condition $L_1^-|\psi\rangle = 0$ imposes a nontrivial recurrence relation relation on c_k

$$(n-k)(2\Delta+n+k-1)c_k = (k+1)(k+2)c_{k+2} \tag{D.6.28}$$

which fixes the coefficients $\{c_k\}$ completely, for example

$$\begin{aligned}
&|\psi_1\rangle = L_2^+|\Delta\rangle, \quad |\psi_2\rangle = (L_2^+)^2|\Delta\rangle + \frac{1}{2\Delta+1}(L_1^+)^2|\Delta\rangle \\
&|\psi_3\rangle = (L_2^+)^3|\Delta\rangle + \frac{3}{2\Delta+3}(L_1^+)^2\, L_2^+|\Delta\rangle \\
&|\psi_4\rangle = (L_2^+)^4|\Delta\rangle + \frac{6}{2\Delta+5}(L_1^+)^2\,(L_2^+)^2|\Delta\rangle + \frac{3}{(2\Delta+3)(2\Delta+5)}(L_1^+)^4|\Delta\rangle
\end{aligned} \tag{D.6.29}$$

Each $|\psi_n\rangle$ induces an $\mathfrak{so}(2,1)$ representation $\mathcal{H}_{\Delta+n}^{\mathrm{AdS}_2}$ of scaling dimension $\Delta+n$. Thus we have

$$\mathcal{H}_{\Delta}^{\mathrm{AdS}_3} \supseteq \bigoplus_{n\geq 0} \mathcal{H}_{\Delta+n}^{\mathrm{AdS}_2} \tag{D.6.30}$$

Counting the dimension of H-eigenspace with eigenvalue $\Delta+K$ for any $K\in\mathbb{N}$ on the two sides of (D.6.30), we find it should be an isomorphism of vector spaces

$$\mathcal{H}_{\Delta}^{\mathrm{AdS}_3} \cong \bigoplus_{n\geq 0} \mathcal{H}_{\Delta+n}^{\mathrm{AdS}_2} \tag{D.6.31}$$

Applying the SO$(2,1)$ character formula (D.6.26) to the decomposition (D.6.31) yields the SO$(2,2)$ character associated with L_{10}

$$\mathrm{Tr}_{\mathcal{H}_{\Delta}^{\mathrm{AdS}_3}}\, e^{itL_{10}} = \sum_{n=0}^{\infty} \mathrm{Tr}_{\mathcal{H}_{\Delta+n}^{\mathrm{AdS}_2}}\, e^{itL_{10}} = \frac{e^{-\Delta|t|}}{(1-e^{-|t|})^2} \tag{D.6.32}$$

In higher dimensions, the decomposition formula (D.6.31) is generalized to

$$\mathcal{H}_{\Delta}^{\mathrm{AdS}_{d+1}} \cong \bigoplus_{n\geq 0} \left(\mathcal{H}_{\Delta+n}^{\mathrm{AdS}_2}\right)^{\oplus D_n}, \quad D_n = \binom{d+n-2}{n} \tag{D.6.33}$$

and thus the corresponding character becomes

$$\mathrm{Tr}_{\mathcal{H}_{\Delta}^{\mathrm{AdS}_{d+1}}}\, e^{itL_{10}} = \sum_{n=0}^{\infty} D_n\ \mathrm{Tr}_{\mathcal{H}_{\Delta+n}^{\mathrm{AdS}_2}}\, e^{itL_{10}} = \frac{e^{-\Delta|t|}}{(1-e^{-|t|})^d} \tag{D.6.34}$$

Though we've only computed character of a noncompact generator for scalar representations in this appendix, we believe

$$\mathrm{Tr}_{\mathcal{H}_{[\Delta,\mathrm{s}]}}\, e^{itL_{10}} = \mathrm{Tr}_{\mathcal{H}_{[\Delta,\mathrm{s}]}}\, q^H, \quad q = e^{-|t|} \tag{D.6.35}$$

holds for any unitary representation $[\Delta,\mathrm{s}]$, massive or massless.

D.7 Physics of SO$(2,d)$ character

We will try to build up some physical intuitions about the character $\Theta(t) \equiv \mathrm{Tr}\, e^{itL_{d,d+1}}$ that is computed by brutal force in the last appendix. In section D.7.1, we construct quansinormal modes in Rinder-AdS and show that they are counted by the character $\Theta(t)$. In section D.7.2, we compute the density of L_{21} eigenstates numerically in AdS$_2$ by imposing an upper bound on the eigenvalues of the global Hamiltonian H and compare it with the density of states defined as the Fourier transformation of $\Theta(t)$.

D.7.1 Quasinormal modes in Rindler-AdS

It is clear that the character $\mathrm{Tr}\, e^{itH}, \mathrm{Im}\, t > 0$ counts normal modes in AdS$_{d+1}$ because H is the Hamiltonian in global coordinate. However, the same interpretation does not hold for $\mathrm{Tr}\, e^{itL_{d,d+1}}$ because $L_{d,d+1}$ is not a positive definite opeator. Instead, as we will show in the following, it counts resonances/quasinormal modes in Rindler-AdS.

To construct quasinormal modes in an efficient algebraic way [1], it's convenient to define the following dS-type conformal generators:

$$D = -iL_{d,d+1}, \quad P_\mu = i(L_{\mu,d} + L_{\mu,d+1}), \quad K_\mu = i(L_{\mu,d} - L_{\mu,d+1}) \tag{D.7.1}$$

subject to commutation relations (we only show the nontrivial ones that will be used in the derivation):

$$[D, P_\mu] = P_\mu, \quad [D, K_\mu] = -K_\mu \tag{D.7.2}$$

where $0 \leq \mu \leq d-1$. In terms of embedding space coordinates, the differential operator realization of D, P_μ, K_μ is

$$D = -(X^d \partial_{X^{d+1}} + X^{d+1}\partial_{X^d}), \quad P_\mu = X_\mu(\partial_{X^d} + \partial_{X^{d+1}}) + (X^{d+1} - X^d)\partial_{X^\mu},$$
$$K_\mu = X_\mu(\partial_{X^d} - \partial_{X^{d+1}}) - (X^{d+1} + X^d)\partial_{X^\mu} \tag{D.7.3}$$

Consider a scalar field of scaling dimension Δ. Then its "primary mode" i.e. eigenfunction of

D with eigenvalue Δ which is annihilated by K_μ, is given by

$$\psi_\Delta(X) \equiv \left(X^d + X^{d+1}\right)^{-\Delta} \tag{D.7.4}$$

In the southern Rindler coordinate of AdS [1], this "primary mode" $\psi_\Delta(X)$ descends to

$$\psi_\Delta(\eta, t_S, r) = (\sinh\eta)^{-\Delta}(1-r^2)^{-\frac{\Delta}{2}} e^{-\Delta t_S} \tag{D.7.5}$$

ψ_Δ has $e^{-\Delta\eta}$-type fall-off at the future boundary and satisfies in-going boundary condition at horizon with quasinormal frequency identified as $i\omega = \Delta$. The other quasinormal modes are descendants of ψ_Δ[2] because the equation of motion and in-going boundary condition are invariant under the action of SO$(2, d)$. At level n, there are $\binom{n+d-1}{d-1}$ linearly independent quasinormal modes whose quasinormal frequency ω_n is related to their scaling dimension under D by $i\omega_n = \Delta + n$. Notice that

$$\mathrm{Tr}\, e^{-tD} = \frac{e^{-\Delta t}}{(1-e^{-t})^d} = \sum_{n\geq 0} \binom{n+d-1}{d-1} e^{-i\omega_n t}, \quad t > 0 \tag{D.7.6}$$

and thus the character $\mathrm{Tr}\, e^{-tD}$ counts quasinormal modes. The same construction can be easily generalized to higher spin fields, either massive of massless.

D.7.2 Numerical computation of density of state

For the southern Rindler-AdS Hamiltonian $L_{d+1,d}$, the associated density of (single-particle) states can be formally defined as

$$\rho(\omega) = \mathrm{tr}\, \delta(L_{d+1,d} - \omega), \quad \omega > 0 \tag{D.7.7}$$

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[1] In the AdS_2 case, we should actually use the black hole coordinate (D.5.5) and then the "primary mode" becomes $\psi_\Delta(t_S, \rho) = (\rho^2-1)^{-\Delta/2} e^{-\Delta\, t_S}$.

[2] There are two different definitions of descendants depending on the choice of Hamiltonian: either $L_{0,d+1}$ or $L_{d+1,d}$. Since the Hamiltonian in Rindler-AdS is $L_{d+1,d}$, the descendants are obtained by acting P_μ on ψ_Δ.

which is also a Fourier transformation of the character $\Theta(t) = \mathrm{Tr}\, e^{itL_{d,d+1}}$

$$\rho(\omega) = \int \frac{dt}{2\pi}\,\Theta(t)\,e^{i\omega t} = \int_0^\infty \frac{dt}{2\pi}\,\Theta(t)\left(e^{i\omega t} + e^{-i\omega t}\right) \tag{D.7.8}$$

The Fourier transformation above is UV-divergent but it can be easily regularized by using a hard cutoff $\frac{1}{\Lambda}$ for the lower bound of the t integral. For example, for a scalar field of scaling dimension Δ in AdS$_2$, this regularization yields

$$\rho_\Lambda(\omega) = \frac{1}{\pi}\log\left(e^{-\gamma_E}\Lambda\right) - \frac{1}{2\pi}\sum_{\pm}\psi(\Delta \pm i\omega) \tag{D.7.9}$$

where γ_E is the Euler constant and $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function.

On the other hand, approximating $\rho_\Lambda(\omega)$ by a model of finite dimensional Hilbert space would provide a more physical interpretation for it. Such an approximation can be easily implemented by imposing a UV cutoff on the spectrum of H. For example in the AdS$_2$ case, consider a truncated Hilbert space $\mathcal{H}_K$ generated by $G_k(\xi)$ with $0 \le k \le K$ and thus the highest energy of H is $\Delta + K$. Normalizing $G_k(\xi)$ and using the recurrence relations (D.6.17), $L^\pm$ are realized as finite dimensional matrices in $\mathcal{H}_K$

$$L^+_{k+1,k} = -\sqrt{(k+1)(k+2\Delta)}, \quad L^-_{k,k+1} = -\sqrt{(k+1)(k+2\Delta)} \tag{D.7.10}$$

Altogether, in this truncated model, the noncompact "Hamiltonian" $L_{21} = \frac{i}{2}(L^- - L^+)$ is a sparse $(K+1)\times(K+1)$ matrix, which admits an efficient numerical diagonalization. With the eigenspectrum $\{\omega_k\}_{0\le k\le K}$ (which is ordered such that $\omega_{k+1} \ge \omega_k$) obtained from diagonalization, a coarse-grained density of eigenstates can be defined as

$$\bar{\rho}_K(\omega_k) \equiv \frac{2}{\omega_{k+1} - \omega_{k-1}} \tag{D.7.11}$$

To compare the character induced density ρ_Λ and the discretized density $\bar{\rho}_K$ for a fixed K, we adjust the UV cut-off Λ such that they coincide around $\omega \approx 0$, i.e. more precisely at the lowest non-negative eigenvalue of L_{21}. Such a comparison for $\Delta = 3$ and $K = 2999$ is shown in fig. D.7.1. They agree fairly well in the IR region and hence the UV truncated model is a pretty

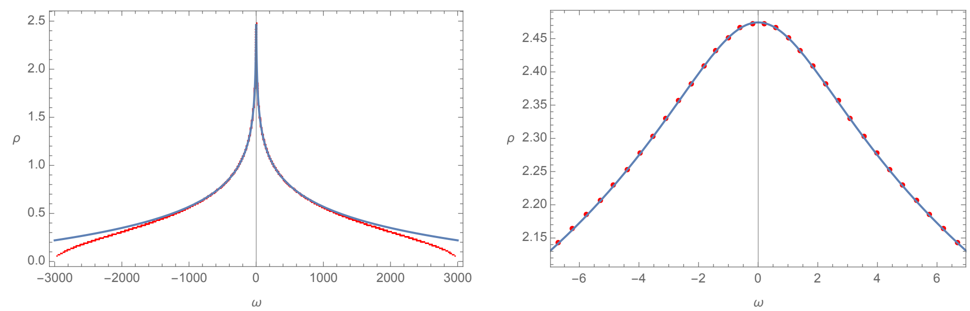

Figure D.7.1: Density of states for a $\Delta = 3$ scalar in AdS$_2$. The red dots show the coarse-grained density of states $\bar{\rho}_K(\omega)$, cf. (D.7.11), of the truncated model with global energy cut-off $K = 2999$. The blue lines show the character induced density of states $\rho_\Lambda(\omega)$, cf. (D.7.9), with the UV cut-off being $e^{-\gamma_E}\Lambda \approx 5981$. The plot on the left shows the two densities for the full spectrum of the truncated model while the plot on the right zooms in on the deep IR region.

good approximation for computing density of states.