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ORIGINAL RESEARCH

# Euclidean spacetime functionalism

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**Abstract**
We explore the significance of physical theories set in Euclidean spacetimes (i.e., theories with Riemannian rather than pseudo-Riemannian metrical structure). In particular, we explore (a) the use of these theories in contemporary physics at large, and (b) the sense in which there can be a notion of temporal evolution in these theories. Having achieved these tasks, we proceed to reflect on the lessons that one can take from such theories for Knox's 'inertial frame' version of spacetime functionalism, which seems (on the face of it) to issue incorrect verdicts in the case of theories with Euclidean metrical structure.

**Keywords** Euclidean spacetime · Physicality · Dynamical evolution · Spacetime functionalism

## Contents



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## 1 Introduction

Our purpose in this article is to reflect on the philosophical significance of physical theories set in Euclidean spacetimes. By such theories, we mean: theories with—and only with!—Riemannian rather than pseudo-Riemannian metrical structure. If one searches the literature for long enough, one can find claims that such theories are 'unphysical' (see e.g. Pelissetto & Testa, 2015, p. 340). But these claims strike us as plainly false, for (a) theories with Euclidean metric structure find widespread application in contemporary physics, and (b) (more interestingly) the elliptic partial differential equations with which these theories are associated can still be used to model dynamical evolution in time in (at least) four distinct senses (to be elaborated upon below)—so that if one means 'lacking dynamics' by 'unphysical', then the claim is again false.[1]

While the foregoing are points worth making in their own right, they are also of relevance to Knox's 'inertial frame' version of spacetime functionalism. Recall that, in general, spacetime functionalism is the view that "spacetime is as spacetime does" (Lam & Wüthrich, 2018); on Knox's particular version of the view, the 'spacetime does' in the above is to be cashed out as follows: "the spacetime role is played by whatever defines a structure of local inertial frames" (Knox, 2018, p. 9). Thus, for example, the Minkowski metric field of special relativity picks out the inertial frames of special relativistic material fields (e.g. Maxwell fields); it is for this reason that the Minkowski metric field qualifies as spatiotemporal. *Prima facie*, theories with Euclidean metric structure present problems for Knox's brand of spacetime functionalism, for although metrical structure in such theories may satisfy Knox's functional definition of spacetime, there appears to be nothing spatio*temporal* about said structure. We argue that this is not necessarily problematic for Knox: the spacetime functionalist may simply augment their functional characterisation of spacetime to include a criterion that the structure in question pick out a distinguished *temporal* direction (thereby qualifying as 'spacetime', rather than just 'space'); moreover, there are various ways in which this might be implemented. Even if such a move is made, however, we argue that one must pay careful attention to the physical scenario in question, for Euclidean metrical

[1] There are numerous distinct ways of understanding the distinction between 'kinematics' and 'dynamics', just one of which corresponds to the question of modelling evolution over time (so that a physical theory is 'dynamical' just in case it can model diachronic evolution). It is upon this notion of 'dynamics' which we focus in this article; for discussions of the kinematics/dynamics distinction more broadly, see (Curiel, 2016; Linnemann & Read, 2021).

structure may *still* satisfy this revised functional definition, in a certain restricted class of situations related to those discussed in the first half of this article.[2]

The plan for the article is this. In §2, we present some background on partial differential equations which will prove essential in the ensuing. In §3, we present and assess a recent proposal by Callender to identify temporal directions in a given theory; again, knowledge of this will be important in our later discussions. In §4, we discuss the physical significance of theories set in Euclidean spacetimes—both (a) the application of such theories in contemporary physics, and (b) the sense in which such theories can be used to model temporal evolution. In §5, we discuss the problems which—at least *prima facie*—these theories seem to pose for Knox's version of spacetime functionalism, and the various ways in which her account might be revised in order to address and overcome these problems. We close in §6.

## 2 Partial differential equations

As indicated above, before addressing the 'physicality' of theories with Euclidean metric structure, we must (i) recall some elementary background regarding partial differential equations (PDEs) upon which we will draw in the remaining sections of this article, and (ii) (in the following section) recall the basics of Callender's proposal for distinguishing temporal from spatial directions in a given world.

### 2.1 Multi-indices

In the study of PDEs, it is useful to introduce the notion of a 'multi-index'. This is simply an $n$-tuple of non-negative integers. Thus, we might define the multi-index $\alpha := \langle \alpha_1, \ldots, \alpha_n \rangle$, where all $\alpha_i \in \mathbb{N}_0$. Suppose that one is dealing with a PDE in $n$ independent variables. Then, an $n$-component multi-index can be used to specify, say, the orders of the derivatives in a particular term of that PDE. For example, in a PDE with two independent variables, $x_0$ and $x_1$, one would associate with a term of the form

$$\frac{\partial^2}{\partial x_0^2} \frac{\partial}{\partial x_1} y\,(x, t) \tag{1}$$

the multi-index $\langle 2, 1 \rangle$.

[2] It should be made clear at the outset of this article that we are agnostic with respect to the debate—promulgated by Brown (2005); Brown and Pooley (2001, 2006)—between 'dynamical' and 'geometrical' approaches to spacetime. According to the 'dynamical' view, spacetime structure (for our purposes here read: metrical structure in a given theory) is to be regarded as a codification of the symmetries of a set of antecedently-given dynamical laws; according to the 'geometrical' view, spacetime structure (again, for our purposes here read: metrical structure in a given theory) is to be regarded as independent of the symmetries of the laws, and indeed to play a constraining role over those symmetries. More pithily: on the dynamical view, dynamical symmetries explain spacetime structure; on the geometrical view, the reverse is true. When in this article we speak of theories with Euclidean metric structure, we take everything we say to be consistent with both of these positions regarding the ontological and explanatory status of that structure.

Now consider, for a given PDE, the term with the greatest number of derivative operators. We denote the combined derivative operator $D^\alpha$, so

$$D^\alpha := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}. \tag{2}$$

Here, the multi-index $\alpha$ records the orders of the derivatives associated with each of the $n$ independent variables in this term of the PDE.

## 2.2 Quasi-linear equations

A general $k$th order PDE takes the form

$$f\left(\partial^k y, \partial^{k-1} y, \ldots, \partial^1 y, y, x\right) = 0, \tag{3}$$

where $\partial^i$ is shorthand for all possible $i$th order partial derivatives. (To a practitioner of differential geometry, this notation may initially be confusing—but it is standard in the study of PDEs.) We then say that a PDE is *linear* just in case the dependent variable and all its partial derivatives occur linearly; a PDE is *quasi-linear* (a more general notion) just in case all the terms with highest order derivatives of dependent variables occur linearly (i.e., the coefficients of these terms are functions only of lower order derivatives of the dependent variables), but terms with lower order derivatives can occur in any manner.

Some of the theorems which we cite below apply to the more general case of quasi-linear PDEs, whereas others apply only to the case of linear PDEs; thus, it is important for us to have made explicit the distinction between these two notions at this stage.

## 2.3 Cauchy problems

Consider a PDE defined on some region $\Omega \subset \mathbb{R}^l$, with initial data given on some hypersurface $\Gamma \subset \Omega$. Let $\mathbf{n}(\mathbf{x}) := (n_1(\mathbf{x}), \ldots, n_l(\mathbf{x}))$ be a unit normal vector to $\Gamma$. Then, given $j \in \mathbb{N}$, the $j$th order 'normal derivative' of $y$ at $\mathbf{x} \in \Gamma$ is defined as

$$\frac{\partial^j y}{\partial \mathbf{n}^j} := \sum_{|\alpha|=j} D^\alpha y \mathbf{n}^\alpha, \tag{4}$$

where $\mathbf{n}^\alpha := n_1^{\alpha_1} \cdots n_l^{\alpha_l}$; note that this is a function on $\Gamma$.

For a $k$th order quasi-linear PDE, the 'Cauchy problem' asks for a unique solution on $\Omega$, given the following initial data—a specification of the dependent variable and its set of normal derivatives—on $\Gamma$:

$$y = g_0 \tag{5}$$

$$\frac{\partial y}{\partial \mathbf{n}} = g_1 \tag{6}$$

$$\vdots$$
$$\frac{\partial^{k-1} y}{\partial \mathbf{n}^{k-1}} = g_{k-1}. \tag{7}$$

Finally, a smooth hypersurface $\Gamma$ and (smooth) boundary data is said to be 'non-characteristic' for the PDE just in case

$$A(\mathbf{x}) := \sum_{|\alpha|=k} a_\alpha \left(\partial^{k-1} y, \ldots, y, \mathbf{x}\right) n^\alpha(\mathbf{x}) \neq 0, \quad \forall \mathbf{x} \in \Gamma. \tag{8}$$

Call this the 'non-characteristic condition' on a hypersurface $\Gamma$. This depends only on $\mathbf{x}$, for the values of $y$ are supplied by the Cauchy data (Mouhot, 2013).

One can similarly characterise a 'Dirichlet problem' and 'Neumann problem' for a given PDE, as will become relevant in later sections of this article. Specifying the boundary of $\Omega \subset \mathbb{R}^l$ as $\partial\Omega \subset \mathbb{R}^{l-1}$, the Dirichlet problem asks for a unique solution on $\Omega$ given the specification of a function $\phi(\mathbf{x})$ such that (Miersemann, 2014, p. 181)

$$\phi(\mathbf{x}) = y(\mathbf{x}), \quad \forall \mathbf{x} \in \partial\Omega. \tag{9}$$

The Neumann problem, on the other hand, asks for a unique solution on $\Omega$ given the specification of a function $\psi(\mathbf{x})$ such that

$$\psi(\mathbf{x}) = \mathbf{n} \cdot \frac{\partial y}{\partial \mathbf{n}}(\mathbf{x}), \quad \forall \mathbf{x} \in \partial\Omega, \tag{10}$$

where here $\mathbf{n}(\mathbf{x})$ is a unit normal to $\partial\Omega$ (Miersemann, 2014, p. 182).

### 2.4 The Cauchy-Kowalevski theorem

With the above in hand, we are in a position to articulate a central result in PDE theory. This is the Cauchy–Kowalevski theorem:[3]

**Theorem 1** (Cauchy–Kowalevski) *Let $\Omega \subset \mathbb{R}^l$, and let $\Gamma \subset \Omega$ be a real-analytic hypersurface. Consider a quasi-linear PDE on $\Omega$ with real-analytic Cauchy data $g_0, \ldots, g_{k-1}$. Suppose, furthermore, that the non-characteristic condition holds on $\Gamma$. Then, for any $x \in \Gamma$, there is a unique analytic solution $u$ on an open subset $U_x \subset \Omega$ containing $x$, satisfying the boundary data on $\Gamma \cap U_x$.*

The Cauchy–Kowalevski theorem tells us that, for analytic initial data, a quasi-linear PDE has a unique local solution—i.e., has a Cauchy problem.

[3] A special case of this theorem was proved by Cauchy in 1842; the full result by Kowalevski (1875).

### 2.5 Classification of partial differential equations

Since the seminal 1926 work of Hadamard (2003), it has become standard to classify PDEs as either 'hyperbolic', 'parabolic', or 'elliptic'.[4] Although often these definitions are presented for second-order equations only (and philosophers have followed suit: see e.g. (James, 2020, p. 15) and (Callender, 2017, §8.4)), we will aim for greater generality here.

For some linear partial differential operator

$$P = \sum_{|\alpha| \leq m} (i)^{|\alpha|} a^{\alpha} \partial_{\alpha}, \tag{11}$$

where $\alpha$ is a multi-index, define the 'symbol' $p(\xi)$ of $P$ as

$$p(\xi) = \sum_{|\alpha| \leq m} a^{\alpha} \xi_{\alpha}. \tag{12}$$

Roughly speaking, $P$ is classified as hyperbolic, parabolic, or elliptic depending on whether the corresponding $p(\xi)$ is a polynomial which defines as hyperbola, parabola, or ellipse (respectively).[5] It should be flagged, however, that this approach to the classification of PDEs is a delicate business and the above should be understood as heuristic only—see e.g. Birkhoff (1954) for a classic article on this matter, and AFK (2011) and references therein for more recent approaches to PDE classification. That being said, the above will suffice for our purposes in this article.

### 2.6 Well-posed Cauchy problems

A Cauchy problem is said to be *well-posed* just in case the (local) solution depends continuously on the initial data. Callender rightly points out that, in physics, we are often interested in solving PDEs which have well-posed Cauchy problems (Callender, 2017, pp. 161-163). The thought is that PDEs with well-posed Cauchy problems can be solved, with experimentally-obtained initial data, safe in the knowledge that the solutions to those equations are robust against small modifications to that initial data. This, of course, is practically of the utmost importance—as Callender writes,

> If the solution depends on the data in a discontinuous way, then that will mean that small errors in data can create large deviations in solution. Rounding off numbers, noise from perturbations, and so on, may imply very different solutions. The partial differential equation's predictive value will plummet if it is not well-posed. If we have to specify the initial data with infinite precision to get anything out of the theory, then it is worthless to us. (Callender, 2017, p. 162)

[4] Note that, in fact, this classification does not exhaust possible PDEs beyond second-order scalar equations: see Wong (2011).

[5] One can, of course, make this last condition more precise: see Wong (2011) for a rigorous presentation of the hyperbolicity condition, for example.

Here, of course, Callender is exactly correct. This, in turn, raises the following question: which PDEs actually *have* well-posed Cauchy problems? The answer, at least in the case of linear PDEs, is straightforward: only *hyperbolic* linear PDEs have well-posed Cauchy problems. The relevant theorem is this (see Hörmander (1983)):

**Theorem 2** *A linear partial differential operator of order m with smooth coefficients admit a well-posed Cauchy problem for arbitrary smooth data if and only if its symbol is a hyperbolic polynomial.*

One way in which to understand this result is as follows: for hyperbolic linear PDEs, the Cauchy-Kowalevski theorem can be upheld (in some regimes) even when the analyticity assumptions are relaxed; not so for elliptic PDEs—it is this which allows hyperbolic PDEs, but not elliptic PDEs, to have well-posed Cauchy problems.[6]

Note that, while e.g. elliptic PDEs do not have well-posed Cauchy problems, they *do* have Cauchy problems (that are not well-posed), for appropriate *analytic* initial data: this is ensured by the Cauchy-Kowalevski theorem. This observation will be important below.

### 2.7 Equations and metric fields

This paper will focus upon hyperbolic and elliptic PDEs. We close this section with some observations: (i) a quasi-linear PDE with (local) Euclidean symmetries (i.e., such a PDE set in Euclidean spacetime[7]) is elliptic; (ii) a quasi-linear PDE with (local) Poincaré symmetries is hyperbolic. (These results follow from the fact a given set of symmetries translates into a given plus/minus structure in the terms of an equation.) Interestingly, however, the reverse direction does not obtain—for example, it is not the case that all hyperbolic PDEs can be associated with Lorentzian spacetimes: see (Schuller, 2011), and (Gomes and Butterfield, 2022, §3) for philosophical discussion.

## 3 Callender's proposal

Having discussed some basics of PDE theory, we turn now to a recapitulation of Callender's discussion of the differences between time and space in a given possible world. Typically in physical theorising, one takes it that a given variable represents the temporal direction, and that one uses physical laws in order to evolve in the temporal direction given certain initial data, as basic inputs. Callender, however, proposes a means of functionally *identifying* which direction is temporal, rather than taking this as basic. His proposal is this:

> For worlds described by [a] partial differential equation [which is second-order and linear], a *temporal direction* at a point $p$ on $\langle M^d, g\rangle$ is that direction $(n, -n)$ in which our laws allow a well-posed Cauchy problem. (Callender, 2017, p. 166)

[6] Research into well-posedness for *quasi*-linear PDEs is difficult and ongoing: see (Tao, 2010).

[7] Recall from the introduction that one can make sense of this statement on both 'dynamical' and 'geometrical' approaches to spacetime, but that for the purposes of this article we are agnostic between these approaches.

Callender is motivated to give this functional definition of a temporal direction on the grounds that this accords with his vision (cf. Callender, 2017, ch. 7) of temporal directions as being those in which one can 'tell the best stories'—that is, stories which maximise simplicity and strength. In essence, he seeks to derive a temporal direction from a best systems analysis of laws. (Recall that such approaches—often referred to also as Mill-Ramsey-Lewis approaches, or 'Humean' approaches—regard laws of nature as being nothing more than the simplest and strongest codifications of the goings-on in some assumed-to-be basic 'mosaic'; recently, these approaches have been extended to other structures, e.g. spacetime structures (Esfeld & Deckert, 2018; Huggett, 2006; Vassallo & Esfeld, 2016), and Callender's approach should be viewed in this spirit—more on this below.)

As we saw in §2.6 (and as Callender himself notes), well-posed Cauchy problems are robust against small perturbations in initial data—which, plausibly, is a necessary criterion to do physics. This appears principally to be a *practical* reason to work with well-posed Cauchy problems. But Callender is motivated to propose the above functional definition of a temporal direction on the grounds that the definition picks out those directions in which one may use one's laws of physics to describe the evolution of the system under consideration—and it is not immediately obvious why the well-posedness criterion is necessary on this front.

To understand Callender's motivations to include the criterion of well-posedness in his functional definition of a temporal direction, his proposal should be viewed from the perspective of his work with Cohen on a 'Better Best Systems' account of natural laws (Cohen & Callender, 2009). In order to better accord with empiricist intuitions, the account attempts to forgo Lewis' commitment to 'perfectly natural properties' as a means of underwriting the notions of simplicity and strength in his own best systems analysis. (For further discussion of Lewis' invocation of perfectly natural properties, see (Weatherson, 2021) and references therein.) Under Cohen and Callender's account, laws are instead relativised to a chosen set of basic predicates, which in turn are relativised to the epistemic community under consideration. Two epistemic communities may therefore disagree on their sets of basic predicates, and by extension the set of laws; neither set of laws take priority over the other. Note that while the properties are relativised, they are not subjective, insofar as the predicates are objectively identifiable and are not agent-dependent.

With this in mind, then, the difference between Lewis' Humean approach to laws on the one hand, and that of Cohen and Callender on the other, is this: the former is an endeavour in pure metaphysics, assuming as it does a 'God's eye view' of the world and not taking into account considerations from scientific practice; by contrast, the latter's Humean account of laws considers the practice of agents in the world, and what is epistemically most useful to those agents. It is thus no surprise that well-posedness—a criterion that physical laws actually be practically useful—plays a greater role in Callender's Humean approach to a temporal direction than it might do for one seeking to follows Lewis' original approach more closely.

Predicated on this 'Better Best System' account of natural laws, Callender develops two different versions of his ideas regarding extending this account to offer a functional definition of the temporal dimension. The *conservative* version takes both the manifold and metrical structure of spacetime, written as $\langle M^d, g\rangle$ above, as initial assumptions—i.e., as part of the mosaic from which the analysis begins. One subsequently derives laws based on the pattern of events which occurs on that manifold. As such, the temporal direction that emerges is akin to a label, placed *on top* of the already-present metrical structure, which gives the simplest and strongest system of laws. More ambitious than this is the *radical* version of Callender's proposal, on which not only do the laws emerge from the Humean mosaic, but also the metrical structure of spacetime, *as well as* facts about what constitutes the privileged temporal direction. (From the metric, one can identify a maximally informative direction which one then takes to be temporal.) Callender further suggests that the radical version "is the most natural development of the theory" (Callender, 2017, p. 151) given its reduced initial assumptions. Callender's radical approach is adopted by Baron and Evans (2021), who further suggest a view which they call 'temporal perspectivalism'. They propose that the temporal direction according to a set of natural laws should be relativised to the epistemic community in question, just as are the basic predicates in the Better Best System account. As such, there is no prioritised notion of time, but rather simply one which suits the basic predicates of a community. (We discuss further below this temporal perspectivalism.)

There are undeniable similarities between the radical version of Callender's proposal and the 'regularity relationalism' of Huggett (2006), according to which—again—spatiotemporal structure is to be extracted from a fundamental mosaic denuded of said structure. (Cf. also the 'super-Humean' approaches of Esfeld and Deckert (2018); Vassallo and Esfeld (2016)). The central difference between these approaches are twofold: (i) Huggett presents a particular approach to extracting said spatiotemporal structure from the mosaic, based upon the symmetries of the laws (see (Huggett, 2006) for the details), and (ii) Huggett—like Lewis—assumes a 'God's eye view' of the manifold; that is, he does not embrace the practice-centred Better Best Systems approach.

We will draw upon all of the above background in the remaining sections of the article. For the time being, the point which we wish to make here is a simple one: if one accepts Callender's functional definition of a temporal direction, then a Euclidean (i.e., Riemannian) metric field in a theory featuring elliptic PDEs as dynamical equations will not pick out a temporal direction, for such theories do not have well-posed Cauchy problems, and so cannot satisfy said functional definition.[8]

## 4 Physicality

We return our attention now to theories with Euclidean spacetime structure. What is so "clearly unphysical" (Pelissetto and Testa, 2015, p. 340) about theories featuring Euclidean metric fields? While authors often do not elaborate upon such claims, it

[8] One curious thing to note is that Callender's analysis seems not to provide a distinction between past and future *along* his temporal direction, given that the Cauchy problem will be well-posed in both directions. He does note that "most agree that many or all of these temporally asymmetric phenomena do not follow directly from what we normally consider the fundamental laws of nature" (Callender 2017, p. 148), so perhaps he believes that the direction of time should be identified based on thermodynamic or statistical considerations instead.

is reasonable to infer that reservations in this regard stem from the elliptic nature of the PDEs associated with these theories (in the sense of §2.7) as—as we have seen in §2.6—such equations generally do not admit (well-posed) Cauchy problems, and thus (the thought presumably goes) cannot be used to model the evolution of a system with respect to a given parameter. In turn (again, the thought presumably goes), such equations cannot be used to model *dynamics*.

At this point, however, one must ask: why is this relevant? Examples of 'physical' elliptic PDEs abound—consider Poisson's equation in Newtonian gravitation theory, electrostatics, etc. Moreover, in spite of its name, no parameter representing temporal evolution is anywhere to be found in thermodynamics. Thus, even if we assume that elliptic PDEs cannot be used to model evolution and so are not dynamical in this sense, this would not necessarily impugn the physicality of the laws associated with those PDEs. And in any case: we will also argue in this section that claims to the effect that elliptic PDEs cannot model temporal evolution are false, for a number of reasons.

Our goal in this section is thus to explore in greater depth the 'physicality' of theories with Euclidean spacetime structure. We begin in §5.1 by noting the multifarious ways in which such theories feature in physics practice—and so, following the methodology of e.g. Belot (2018), should be taken physically seriously. Next, we note in §4.2 that what some (e.g. Brown and Read (2021); Brown and Sypel (1995)) have regarded as being a necessary condition for physicality—*viz.*, satisfaction of a relativity principle—is met by such theories (of course, this does not guarantee sufficiency for physicality, but we nevertheless regard the point as being important, and one worth making). This done, we turn in §4.3 to the question of whether such theories, and the (elliptic) PDEs associated with them, can be used to model temporal evolution, finding various senses in which this is indeed the case; in this way, we rebut the above claim that such theories are dynamically sterile.

Before we begin this section in earnest, we should note that we do not offer a *definition* of physicality. Such a task would, presumably, be difficult and nuanced—and although the results would be valuable, such an ambition transcends anything which we require here. Rather, what we wish to show in this section is simply that there are good grounds, both on the basis of physics practice and on the basis of considerations of dynamics and temporal evolution, to regard—in at least some cases—theories set in Euclidean spacetimes as being physical. We thereby offer these observations and arguments as *data*, which any author wishing to offer a definition of 'physicality' would be wise (in our view!) to incorporate.

### 4.1 Physics practice

There is much evidence from physics practice that (elliptic) PDEs embodying Euclidean spacetime structure, and the theories associated therewith, are, in fact, often regarded as being physical. Consider, for example, the widespread application of the Laplace and Poisson equations in electrostatics, gravitation, and fluid dynamics. To take just one explicit example—by now well-domesticated for philosophers of physics—recall the Newton-Poisson equation of Newtonian gravitation theory, which, in coordinate-free notation, reads

$$h^{ab}\nabla_a\nabla_b\varphi = 4\pi\rho, \tag{13}$$

where $h^{ab}$ is a degenerate metric field of signature $(0, 1, 1, 1)$, $\nabla$ is a derivative operator compatible with $h^{ab}$ (and with a degenerate metric field $t_{ab}$ of signature $(1, 0, 0, 0)$ which does not feature in the above equation, but which is nevertheless essential in the articulation of the structure of the theory), $\varphi$ is a scalar field representing the gravitational potential, and $\rho$ is a scalar field representing matter density content in the theory. This equation is elliptic—but no physicist in possession of their right minds would deny its physicality! (For further technical details on the above equation, especially in the coordinate-free presentation, see (Malament, 2012, ch. 4); for discussions of this equation relevant to the content of this section, see (Linnemann & Read, 2021), which is also discussed further below.)

There is a moral here consistent with that drawn in Doboszewski (2021): although Euclidean spacetimes might arguably be 'physically unreasonable' *qua* cosmologies,[9] they are nevertheless invaluable—and thereby arguably perfectly physical—in modelling a diverse range of phenomena. Relatedly, the morals of Belot (2018) should be stressed: if physicists take seriously theories which seem to be set in Euclidean spacetimes, then that is (at least defeasible!) reason to take seriously such spacetimes.[10]

Indeed, there are also other more speculative reasons to countenance Euclidean metric structure, deriving from quantum gravity. For example, it is argued by Bojowald and Brahma (2018); Brahma (2020) that the quantum corrections to general relativity deriving from loop quantum gravity can also be responsible for the possibility of signature change—so that some regions of the spacetime manifold would be populated with a metric field of one signature (say, Lorentzian), while others would be populated with a metric field of another signature (say, Euclidean); in a similar spirit—albeit not motivated directly by considerations from loop quantum gravity *per se*—the possibility of signature change, and the cosmological consequences thereof, is discussed by Hartle and Hawking (1983). For further discussion of these and related issues from quantum gravity, see (Huggett & Wüthrich, 2018; Oriti, 2014); although it is reasonable to inquire as to how generic the above possibilities of Euclidean signature from quantum gravity really are—for example, Le Bihan and Linnemann (2019) argue that "most approaches to quantum gravity already start with an in-built distinction between structures to which the asymmetry between space and time can be traced back", where they have in mind some 'seed' of a Lorentzian spacetime signature—we take our above point—that a Euclidean spacetime signature *might* be motivated from quantum gravity—to stand nevertheless.

[9] Actually, this is not so obvious in the case of Newtonian gravity and its defining equation (13), given that this theory was of course developed to cosmological ends. There are, however, well-known problems with Newtonian gravity understood *qua* cosmology—for which we refer the reader to (Wallace, 2017) and (Malament, 2012, ch. 4). What we really mean by the above is that nowadays no fluid dynamicist etc. would be in any way inclined to construe the (elliptic) equations of their theories cosmologically.

[10] If one is a certain kind of 'ontic structural realist'—see (Ladyman & Ross, 2007) for an influential book-length defence of such a view—one might be inclined to say that, in the domain of application of such theories, spacetime *really is* Euclidean. Although we have sympathies with such approaches, we need not endorse them for our purposes in this paper. (For more on the emergence of different spacetime structures in certain physical contexts, see (Cheng & Read, 2021; Wallace, 2020).)

In these reflections, it is important to distinguish the fundamentality of a certain spacetime structure from its physicality. In both cases such as the application of Euclidean metric structures to e.g. fluid mechanics, and the emergence of such metrical structure from quantum gravity, we are not claiming that these these structures are fundamental; nevertheless, we *are* claiming that they represent robust patterns in the physical goings-on at some level of description, and so, in this minimalist sense, can be regarded as being 'physical'.[11]

### 4.2 The relativity principle

We turn now to a separate argument which one might at least take to be a motivation for (at least in some cases) regarding theories set in Euclidean spacetime to be 'physical'; this argument proceeds by way of consideration of the relativity principle (**RP**)—which, recall, Einstein put as follows (Einstein, 1905):

**RP:** The laws by which the states of physical systems undergo change are not affected, whether these changes be referred to the one or the other of two systems of coordinates in uniform translatory motion.

For Brown (Brown & Read, 2021; Brown & Sypel, 1995), the physical content of **RP** is to be understood by way of the *boostability of subsystems*, "suitably shielded from the rest of the universe" (Brown and Sypel, 1995, p. 240). And in turn, such boostability is a necessary condition for physics *tout cout*: "it is a fundamental condition for the very possibility of kinematics such as exemplified by the Galilean transformations, that ideal clocks are boostable, meaning that as long as the forces producing the boosts are weak enough, the proper periods of such clocks in their equilibrium states are invariant under the boosts" (Brown and Read, 2021, p. 6).

With this in mind, consider the Ignatowkski transformations—those coordinate transformations derivable from the relativity principle, but without assuming the light postulate (**LP**)—which, recall, reads as follows:

**LP:** Any ray of light moves in the 'stationary' system of coordinates with the determined velocity $c$, whether the ray be emitted by a stationary or by a moving body. Hence

$$\text{Velocity} = \frac{\text{Light path}}{\text{Time interval}}.$$

(Here, 'time interval' is to be understood in the sense articulated in (Einstein, 1905, §1).)

These transformations contain Lorentz, Galilean and Euclidean transformations as special cases. Thus, one sees that Euclidean transformations embody the **RP**, so that what is (at least for Brown) a necessary condition for the physicality of said transformations is satisfied. Of course, this is not to suggest that other *sufficient* conditions for the physicality of such transformations are satisfied, but it is perhaps sufficient to shift the burden of proof to authors such as (Pelissetto and Testa, 2015, p. 340), who would claim flat-footedly that such transformations are to be excluded on physical grounds.

[11] Cf. (Wallace, 2012, p. 50).

### 4.3 Temporality

Having pointed out that it would (at least in our view) be too hasty to reject the physicality of theories set in Euclidean spacetimes, given (a) the widespread application of such theories in physics, and (b) the fact that such theories satisfy a relativity principle, we turn now to another concern which we expect underlies any rejection of such theories on physical grounds. This concern is that these theories, insofar as they are governed by elliptic PDEs, cannot be used to model *dynamics*. Over the course of this subsection, we will argue that said concern is misplaced, for (at least) four reasons. First: it is not clear why *well-posed* Cauchy problems are required for temporal evolution. Second: elliptic PDEs might be argued to have the capacity to model temporal evolution, insofar as they have well-posed Dirichlet and/or Neumann problems. Third: there are in principle other means of identifying temporal directions than via the use of well-posed Cauchy problems. Fourth: one can demonstrate that, by appropriately 'gluing' individual solutions of an elliptic PDE, one can (at least in some cases) build a model reasonably interpreted as a system (e.g. a wave) evolving or propagating in time.

#### 4.3.1 Cauchy problems

While it is true that elliptic PDEs do not have *well-posed* Cauchy problems, they may still have Cauchy problems (albeit not well-posed) for particular choices of analytic initial data—this follows from the Cauchy-Kowalevski theorem (see again §2.4). In these very particular cases, one can uniquely evolve initial data using an elliptic PDE—so in this very specialised sense, such equations can be used to model evolution/dynamics. As Callender points out (cf. §2.6 above) well-posedness may be necessary on *practical* grounds, but it is not necessary for the possibility of evolution *tout court*. To claim as much would be too fast—*a fortiori*, to claim that elliptic PDEs are unphysical because they cannot be used to model evolution would also be too fast.

Of course, whether this observation is of any practical use is a different matter—recall again Callender's reasons for insisting on well-posedness, which were discussed in §3. Nevertheless, strictly speaking, the point stands that elliptic PDEs can be used to model evolution in a given direction, in the sense that they do have Cauchy problems in a certain (admittedly highly restricted) range of circumstances.

The thought that elliptic PDEs might in principle be used to model temporal evolution is also compounded by some observations running (as it were) in the opposite direction. In particular: even well-posed Cauchy problems are not guaranteed to be informative—James suggests the counterexample of chaotic systems, which are "unstable under perturbations of initial conditions" (James, 2020, p. 20), that can arise from well-posed Cauchy problems. Due to this instability, temporal evolution of the system remains uninformative. Since even a well-posed Cauchy problem may not be informative in the way proposed by Callender, the connection between well-posed Cauchy problems, informativeness, and in turn (on Callender's account) the direction of time, is thereby weakened, potentially opening the door—if further arguments can be mustered—to the view that elliptic PDEs, which do *not* have well-posed Cauchy problems, might be reconcilable with a notion of temporal evolution after all.

#### 4.3.2 Well-posed Dirichlet and Neumann problems

A further impetus for treating elliptic PDEs that emerge from a Euclidean metric as physical is noted by James (2020), who emphasises that, while elliptic PDEs do not have well-posed *Cauchy* problems, they do in fact have well-posed Dirichlet and Neumann problems.[12,13] Nevertheless, she assumes that elliptic PDEs can describe only static systems, and thereby rejects them as a way of modelling dynamical systems:[14]

> Recall that elliptic PDEs have no real characteristics and thus no restrictions on where in the domain this boundary can be, so no particular set of directions is privileged by elliptic PDEs. We therefore have no reason to regard the informative directions of an elliptic PDE as temporal. (James 2020, p. 19)
> If we wish to be informed about dynamics, an elliptic system will not do. Given that the study of dynamics just is the study of processes evolving in time, it is to be expected that the mathematical tools designed for this purpose distinguish timelike directions, as hyperbolic PDEs do. (James, 2020, p. 27)

James' reasoning here is that, since there is no restriction on the location of acceptable boundary data in the case of Dirichlet/Neumann problems for PDEs, anything goes; thus, these PDEs cannot pick out a privileged temporal direction.[15] While we are in agreement that Dirichlet and Neumann problems can be used as 'informatively' as Cauchy problems, James' claim that, "If we wish to be informed about dynamics, an elliptic system will not do" is perhaps more controversial. To see this, suppose that, for some closed region $\Omega \subset \mathbb{R}^l$, one is in possession of the relevant (Dirichlet or Neumann) boundary data on $\partial\Omega$, but in addition one restricts one's attention to some (relatively) small region $R$ encompassing some part of $\partial\Omega$. In that case, one can still use this boundary data to evolve in a certain direction, as illustrated in figure 1.

There is also a connection here with the temporal perspectivalism of Baron and Evans (2021). Suppose that, for distinct $\Omega, \Xi \subset \mathbb{R}^l$, one community has access to boundary data on $\partial\Omega$, while the other community has access to boundary data on $\partial\Xi$. In that case, if both of these communities restrict to some salient small regions around $\partial\Omega$ or $\partial\Xi$ respectively (again as per figure 1), they may end up identifying—using

[12] James does not give a reference for this claim, but see e.g. (Jackson, 1998, p. 17) for an elegant summary of which classes of PDEs have well-posed Cauchy/Dirichlet/Neumann problems.

[13] James mentions explicitly that Dirichlet problems require less information than Cauchy problems, in the sense that one requires only the value of the dependent variable on the codimension one hypersurface, rather than the value of this variable and its normal derivatives, as in the Cauchy problem. By the same token, Neumann problems require less information than Cauchy problems in the sense that the require only the normal derivative of the dependent variable on the hypersurface.

[14] In the following quote, James understands 'characteristics' in the technical sense of the PDEs literature.

[15] Perhaps one of the reasons for which Cauchy problems seem to be seen as more informative is because they combine both conditions on the boundary variable and on its derivative. As such, they appeal to intuitions from classical dynamics, whereby problems typically are solved via specification of an initial position and velocity. However, many physical systems do not require both of these inputs to be evolved forward in time; therefore, there is no reason to consider Cauchy problems to be superior to Dirichlet or Neumann problems in such a way.

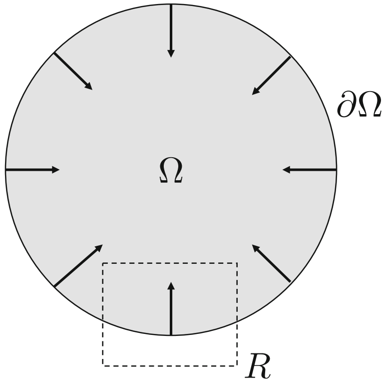


**Fig. 1** For some region $\Omega \subset \mathbb{R}^l$, restricting to a region $R$ encompassing some part of $\partial\Omega$ allows for the possibility of modelling (informative) temporal evolution using elliptic PDEs endowed with (well-posed) Dirichlet/Neumann conditions

Dirichlet or Neumann problems—*different* directions as being temporal.[16]

In sum, then: although we are sympathetic to much of James' work, we do not endorse the claim that elliptic PDEs cannot be used to model temporal evolution—in fact, for the reasons described in this subsection (as well as the other subsections of this section), this does seem to possible; moreover, doing so appears to be consistent with the temporal perspectivalism of Baron and Evans.

#### 4.3.3 Egan's proposal: temporal directions from matter

In his *Orthogonal* series of novels (the first entry of which is (Egan, 2012), Egan describes what he calls a 'Riemannian' universe (which has a Euclidean metric structure—i.e., as already clarified above, a Riemannian rather than pseudo-Riemannian metric field).[17] As a supplement to these novels, he also provides a conceptual guide to the physics in such a universe, highlighting the differences with our universe with its (assumed-to-be) Lorentzian metrical structure (Egan, 2010). Notably, he argues that although no direction is prioritised by the spacetime structure of such a universe, a temporal direction may nevertheless be defined *functionally* (in philosophers' parlance—not Egan's) just in case there is a privileged direction picked out by the worldlines of the material bodies in such a universe.

It is useful to distinguish Egan's proposal from Callender's own radical and conservative proposals, discussed in §3. Given that Callender defines a temporal direction in terms of the existence of a well-posed Cauchy problem, a Riemannian universe would simply not have a temporal direction in the absence of well-posed Cauchy problems. By contrast, Egan's own functional definition of time is weaker than Callender's, in the sense that he might still be able to identify a temporal direction (according to his own criterion), even in worlds with such metrical structure.

This is not to say that Callender's proposal excludes the metaphysical possibility of worlds with Euclidean spacetime structure—though the consequences are slightly

[16] Note that this would also qualify as a form of *spatial* perspectivalism, if the spatial directions are taken to be orthogonal to the temporal directions (as, invariably, they are). (Our thanks to an anonymous referee for pointing this out.)

[17] Callender makes passing reference to Egan at (Callender, 2017, p. 136), but his discussion is different from ours below.

different depending upon which of his two readings is deployed. On the conservative reading, the metric of the spacetime in question could be assumed to be Riemannian anyhow, though no direction would be identified as temporal according to Callender. Subsequently, one could utilise Egan's method to identify an emergent temporal direction (by Egan's definition, rather than Callender's) based upon (say) the orientation of worldlines of material bodies.

Whether Callender's radical reading could yield a world with Euclidean spacetime structure but an emergent temporal direction as per Egan (this being the kind of world envisaged in Egan's novels) is an interesting question: one initial reaction might be to argue that the radical approach precludes a Riemannian universe, if an emergent direction is nevertheless to be picked out via the worldlines of bodies *á la* Egan. Since the Humean mosaic would need to display the appropriate characteristics for a Euclidean metric to emerge, there should be a lack of any privileged direction in the distribution of material objects. In other words, if there is an emergent temporal direction as per Egan, then the Humean prescription of the radical reading would seem to imply that the structure of spacetime should *not* be Euclidean.

However, this may not be correct: symmetries of equation do not have to be shared by solutions—so certain solutions could still pick out a particular direction, which would suffice to get Egan's functional definition off the ground. For example, while Gauss' law itself is isotropic in all directions, the solution of an infinite plane sheet would immediately break this symmetry. Thus, even in a world lacking a certain symmetry, the Humean codification of that world might involve equations which *possess* that symmetry.[18,19]

In any case, our point in this subsection is again a simple one: if one deploys Egan's functional definition of a temporal direction, rather than that of Callender, then one again sees that it is possible to identify temporal directions even in worlds with Euclidean spacetime structure.

[18] Though Egan is able to derive numerous physical consequences of a Riemannian universe, there are certain concerns with regards to its implications towards Noether's theorems and energy conservation—to the latter of which Egan makes significant appeal in (Egan, 2010). Given that the principle of energy conservation is associated standardly (via the Noether theorems) with time translation invariance, the absence of a clearly defined time coordinate leaves the status of the principle unclear. It would seem that instead of momentum conservation in the three spatial directions associated with translational symmetry, there should be *four* components of momentum that are now conserved in such a universe. (Perhaps one should apply the same reasoning as above, functionally defining a particular direction as time and further labelling the component of 'momentum' in that direction as 'energy'; whether such a quantity would meet all desiderata on a notion of energy would, however, have to be worked out carefully.)

[19] There are connections here with the philosophy of statistical mechanics and thermodynamics. In particular, the point that the solutions of time-symmetric equations of motion may nevertheless be time-asymmetric (i.e., not respect the symmetries of said equations), with the behaviour of objects in a world ultimately dictated by a combination of those equations *plus the initial conditions*, has also been discussed in that context. This can be sufficient to yield a (thermodynamic) entropic gradient, which in turn has been taken by some to be a plausible candidate to which one might reduce the notion of a temporal direction (so Egan can be seen as a variant of such a proposal)—see e.g. (Wallace, 2013, §5.1). For more on these issues in the philosophy of statistical mechanics and thermodynamics, see Albert (2000); Brown (2017); Wallace (2011). (We're very grateful to an anonymous referee for inviting us to draw these connections.)

#### 4.3.4 Piecewise solutions

In (Linnemann and Read, 2021, §2.1), a novel means by which elliptic PDEs—in particular (13) (or its geometrised version: see (Malament 2012, ch. 4))—might be able to model dynamics is proposed. In particular, one takes standing wave solutions to (13), subject to different boundary conditions at different times and overall multiplied by a 'smearing' function (this is what we mean by 'piecewise solutions' in the title of this subsection), such that the net effect is a propagating wave. Although (by the authors' own admission) highly gerrymandered, this construction again suffices to demonstrate that elliptic PDEs—and, in turn, theories set in Euclidean spacetimes—can in principle be used to model evolution in time.[20]

## 5 Spacetime functionalism

Up to this point in the article, we have argued against claims that theories with Euclidean metrical structure are 'unphysical', and moreover have seen that multiple arguments militate in favour of the view that such theories may indeed admit of a notion of dynamical evolution in time. We turn now to the significance of such theories for Knox's 'inertial frame' version of spacetime functionalism.

### 5.1 Inertial frame spacetime functionalism

What is spacetime functionalism? Lam and Wüthrich put the idea pithily, when they state that "spacetime is as spacetime does" (Lam & Wüthrich, 2018). The idea is not to identify *primitively* some object in one's theories as being spatiotemporal, but rather to identify the structures in one's theories which play a certain, antecedently-specified functional *role* of spacetime.

Perhaps the best-known spacetime functionalist approach is due to Knox, whose slogan is the following: "I propose that the spacetime role is played by whatever defines a structure of local inertial frames" (Knox, 2018, p. 9). The idea is to define spacetime functionally as any structure in a given theory which itself picks out a structure of local inertial frames. In turn, Knox is required to give a functional characterisation of inertial frames—see (Knox, 2014, p. 348) for the most explicit statement in this regard. For our purposes, an inertial frame can be taken to be a coordinate system in which dynamical equations for matter fields take their simplest form, and in which unforced bodies move on uniform trajectories.

To illustrate Knox's version of spacetime functionalism, consider the case of special relativity. In this theory, symmetries of the Minkowski metric field coincide globally with those of the dynamical equations governing matter fields; in any frame in which these dynamical equations take their simplest form and in which unforced bod-

[20] Here, as Linnemann and Read (2021) stress, it is very important to distinguish between what's represented and the mathematics which is used to represent. Mathematics is a tool for representing physics—and the point of (Linnemann and Read, 2021, §2.1) is that, with sufficient ingenuity, one can engineer elliptic PDEs and their solutions to model physical systems evolving in time, even if the former might initially be regarded as being (and, in fact, might indeed well be) ill-suited to the task.

ies move on uniform trajectories, the Minkowski metric field itself takes the form diag $(-1, 1, 1, 1)$. Thus, the Minkowski metric field picks out a structure of inertial frames in this theory, and so qualifies as spatiotemporal. Essentially the same story goes through locally in general relativity, on the assumption of satisfaction of the so-called 'strong equivalence principle', which, roughly speaking, states that dynamical equations governing matter fields in general relativity take locally a Poincaré invariant form (see Brown and Read (2021); Knox (2013); Read et al. (2018) for further discussion).[21] So, for Knox, the metric field of general relativity qualifies as spatiotemporal in that theory.

### 5.2 Euclidean spacetime functionalism

Having presented Knox's version of spacetime functionalism, we are now in a position to see the problems which theories set in Euclidean spacetimes seem to raise for the position. Consider such a theory. What verdict does inertial frame spacetime functionalism issue in this case? According to Knox's programme, a very similar story to that told above in the context of special relativity goes through in this context: it is precisely the Euclidean metric field $\delta_{ab}$ which picks out a structure of local inertial frames; thus, the Euclidean metric field is spatiotemporal. At this point, however, one might ask: is it truly reasonable to regard the Euclidean metric field as space*time*, when it picks out no temporal direction (unlike e.g. the Minkowski metric field, which encodes a fundamental distinction between spatial, temporal, and null vectors, as encoded in its signature)?[22] For this reason, one might understand theories with Euclidean metric fields to constitute problem cases for Knoxian spacetime functionalism.

In our view, such a response would be too quick, though there are a number of threads here which need to be disentangled. First, the spacetime functionalist who concurs that Euclidean metric fields should not, in fact, qualify as spacetime, may simply augment Knox's functional definition of spacetime with a further criterion of the form: 'spacetime must distinguish between spatial and temporal directions'. Once this additional condition is added, Euclidean metric fields will not (in general—see below) satisfy the functional definition of spacetime.

To be more concrete here: one way of implementing this proposal would be to incorporate Callender's functional characterisation of a temporal direction as the criterion for distinguishing spatial and temporal directions: the temporal direction being that which allows for a well-posed Cauchy problem to arise. By restricting the objects in equations to those which obey this constraint, this augmented definition of spacetime

[21] It is worth flagging that the status of the strong equivalence principle in general relativity is controversial—see e.g. Weatherall (2021) for some recent discussion. Since it is not our purpose in the present paper to defend Knox's appeal to this principle (or its status in general relativity more generally), we set these concerns aside for the purposes of this article.

[22] Since it is conformal structure which encodes such a distinction, one might argue that it is this structure—strictly weaker than metrical stricture—which is the *sine qua non* of spatiotemporality. For further discussion of this idea, see (Read et al., 2018, p. 20); one reason to hesitate here, however, is that while there is clearly a difference between space and time in Newtonian theories (again, see Malament, 2012, ch. 4), the notion of conformal structure in Newtonian spacetimes is not straightforward (on this matter, see Curiel (2015); Ewen and Schmidt (1989)).

will never be satisfied by the Euclidean metric field in such theories. In this way, it would appear that amending Knox's functional definition of spacetime is sufficient to eliminate problem cases arising from a Euclidean metric.

Even if one *does* introduce this extra criterion, however, the foregoing discussion in this paper perhaps suggests that it would be too quick to always deny that Euclidean spacetime can play the functional role of spacetime—for, in the restricted class of circumstances in which elliptic PDEs are amenable to Cauchy problems (in particular, when one is dealing with analytic initial data, so that the Cauchy-Kowalevski theorem is satisfied), these equations *can* be used to describe the evolution of a system with respect to a given (temporal) parameter. Whether Euclidean metric fields can ever qualify as spatiotemporal will then depend upon whether one includes the well-posedness criterion in one's functional characterisation of the temporal direction, in turn incorporated into the revised version of Knox's functional characterisation of spacetime.

Note also that, instead of appealing to Callender's characterisation of a temporal direction, one could alternatively make use of that of Egan. In this case, if there is a privileged direction identified via (say) some commonly-shared direction in the worldlines of material bodies, then that direction qualifies as temporal, and so—on the revised version of Knox's spacetime functionalist approach—the Euclidean metrical structure can again qualify as spatiotemporal. So, on the more fine-grained level of solutions, there is an argument for a Euclidean metric field qualifying as spacetime, provided one is willing to allow for emergent properties such as the distribution of material bodies to influence their identification of spacetime.

As such, although there is nothing in the structure of a Euclidean metric field *itself* which distinguishes spatial from temporal directions—in contrast with e.g. a Minkowski metric field—and so it appears that such a structure cannot qualify as spatiotemporal on our proposed revised version of Knox's functionalist account (according to which the object so identified must draw some distinction between space and time), ultimately one must not be too hasty, for on various ways of making precise this extra criterion, objects such as Euclidean metric fields may nevertheless satisfy it. In particular: (i) on Callender's proposal for identifying temporal directions via well-posed Cauchy problems, Euclidean metric fields still cannot qualify as spatiotemporal; however, (ii) if one drops the well-posedness criterion, such structures can sometimes qualify as spatiotemporal (when the conditions of the Cauchy-Kowalevski theorem are satisfied), moreover; (iii) if one instead appeals to Egan's proposal, the Euclidean metric structure may qualify as spatiotemporal when there is an emergent direction picked out by the behaviour of material bodies.[23]

We are content to add these extra conditions in a revision of Knox's proposal, for although it seems to us correct to state that although Euclidean metric fields do not *always* qualify as spatiotemporal (as Knox's original proposal would have it), it can reasonably be said to be the case that such structures are *contingently* spatiotemporal,

[23] Of course, these are just some proposals for a functional definition of time—there may exist yet other plausible candidates, which may issue a variety of verdicts on whether Euclidean spacetimes can pick out temporal directions. Moreover, one might in principle be a pluralist about temporal directions, and maintain that there are multiple senses in which a given piece of structure is or is not temporal. (We are grateful to an anonymous referee for inviting us to say more on these issues.)

depending upon the behaviour of other fields. Thus, we recommend to Knox the revisions to her proposal articulated in this section.

## 6 Close

In this article, we have seen that it is too hasty to declare that theories set in Euclidean spacetimes are 'unphysical', for (i) their use is ubiquitous in physics practice; moreover, (ii) it does indeed appear to be possible, in various senses, to model dynamics—in the sense of temporal evolution—in such theories. Having established this, we have drawn upon our discussions in order to highlight and resolve a potential problem for Knox's 'inertial frame' version of spacetime functionalism: that she is compelled to always adjudicate the Euclidean metric structures are spatiotemporal. We have seen that, when Knox's approach is augmented with Callender or Egan's proposed functional characterisations of a temporal direction, whether Euclidean metric structures qualify as spatiotemporal, for Knox, becomes a contingent business, dependent upon the behaviour of other physical fields in the situation under consideration. This seems to us to be exactly the correct verdict—thus, we endorse the proposed modifications to Knox's account which we have articulated in this article.

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## Compliance with ethical standards

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