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Quantum clock frames: Uncertainty relations, non-Hermitian dynamics and nonlocality in time

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Abstract. Dynamical evolution can be reconstructed within stationary, closed quantum systems by employing the Page-Wootters "timeless approach". When conditioning upon the state of a "clock" subsystem, the rest of the system regains its time dependence. This mechanism, involving entanglement between the above subsystems has gained much attention during the last few years. After a brief introduction to the topic we will elaborate on a few recent results: The derivation of new time-energy uncertainty relations, emergence of non-Hermitian dynamics when utilizing non-inertial quantum clocks and dynamical nonlocality in quantum time.

1. Introduction

The notion of time has been perplexing since the dawn of humanity. It remains a source of mystery even nowadays, with many disciplines vigorously studying various aspects of time flow, evolution in time and time perception for different observers. Modern physics has majorly dealt with time during the last century employing relativity theory and quantum mechanics. However, time has a different role in these theories which leads to a different fundamental understanding of what it actually is. In quantum mechanics time is typically seen as a background parameter, external to the system of interest and unaffected by it, while in general relativity time is dynamical, relative and affected by mass and energy.

Moreover, quantum mechanics poses the well-known problem of time: A completely isolated quantum system, e.g. the entire universe (as we suspect), must satisfy the (timeless) Wheeler-DeWitt equation [1]. Then, how can the usual notion of evolution be recovered? To answer this question, Page and Wootters presented a framework in which time was taken to be the quantity given by a physical system, a quantum clock [2]. Time, then, becomes a dynamical quantum variable rather than a mere parameter. The Page-Wootters approach has attracted much interest in recent years leading to major development, generalization and various applications [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20].

This timeless framework has been shown to lead to a unitary evolution from the clock's perspective in a large class of scenarios [6]. However, when investigating a class of time-energy uncertainty relations in this setting [17], we recently found that this is not always the case. We studied von Neumann measurements of the total energy of a system that contained an internal clock. Unexpectedly, we observed that the evolution from the internal clock's perspective is, in general, non-unitary. We now believe that although our analysis was mostly nonrelativistic,

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we can borrow a relativistic idea to interpret the result. Because mass and energy are tightly related, measurements of a system's energy, in a sense, correspond to weighing the system, which then interferes with its dynamical time flow. Given that, we started speculating whether the non-unitarity could result from the internal clock being a non-inertial frame of reference. This seems to be indeed the case according to our more recent analysis of non-inertial quantum clocks [18], which shows that when considering an accelerated or gravitating quantum particle with an internal clock degree of freedom, the evolution given by such an internal clock is non-unitary. We did so in a non-relativistic framework with a relativistic correction due to the mass-energy equivalence. With that, the internal degrees of freedom of the particle couple to its external degrees of freedom. This, together with the system's acceleration, leads to the non-unitary evolution from the perspective of the internal clock.

Below we elaborate on these two works as well as on a different result obtained lately within the Page-Wootters framework: dynamical nonlocality in time [16]. Following our earlier studies of dynamical nonlocality in space [21, 22, 23, 24, 25], an interesting notion which seems to deserve more exploration [26], we shall study equations of motion which depend on two remote points in time.

2. Quantum mechanics in the Page-Wootters framework

The Page-Wootters framework consists of a clock system, whose state is given by a vector in a Hilbert space $\mathcal{H}_A$, and the rest of the system, represented by a state in a Hilbert space $\mathcal{H}_R$, whose evolution is studied. The joint system $|\Psi\rangle\rangle \in \mathcal{H}_A \otimes \mathcal{H}_R$ is assumed to be closed and hence

$$H_T|\Psi\rangle\rangle = 0, \tag{1}$$

where H_T is the total Hamiltonian acting on systems A and R. This equation is known as the Wheeler-DeWitt equation. Observe that the imposition of $|\Psi\rangle\rangle$ being an eigenstate with a null eigenvalue by Eq. (1) is not as restrictive as it may seem [3]. In fact, Hamiltonians that differ by constant terms are physically equivalent, which is associated with quantum states being defined up to a global phase. Then, Eq. (1) implies that the total system $|\Psi\rangle\rangle$, historically being thought of as the "Universe" [2], does not evolve with respect to an external time.

The clock system should have an observable T_A associated with its time. It is still desirable that H_A generates translations in time, i.e., $|t_0+t_A\rangle = e^{-iH_A t_A/\hbar}|t_0\rangle$, where $|t_0\rangle$ and $|t_0+t_A\rangle$ are taken to be clock states, but this does not necessarily require T_A to be a self-adjoint canonical conjugate to H_A. In fact, it is possible to construct T_A as a positive operator-valued measure (POVM), which means that the clock states are not necessarily eigenstates of T_A [27, 28, 29]. Such operators obtained from these constructions are symmetric but need not be self-adjoint. Here, however, for simplicity we assume an *ideal clock*, i.e., $[T_A, H_A] = i\hbar I_A$, $T_A|t_A\rangle = t_A|t_A\rangle$, and $H_A = -i\hbar\partial/\partial t_A$. Although the Hamiltonians of these clocks are unbounded from below and, hence, unrealistic, they help us avoid technicalities associated with real clocks [30, 31], while providing approximations of them [32, 33]. Thus, corrections to the results presented here are expected when dealing with non-ideal clocks.

Moreover, let H_R denote the Hamiltonian of the system of interest and let $H_{int}(T_A)$ represent the time-dependent term of the evolution of system R set by clock A, which is an interaction between A and R. We have

$$H_T = H_A + H_R + H_{int}(T_A). \tag{2}$$

Observe that, H_{int} is being assumed to be independent of H_A. Inserting Eq. (2) into Eq.

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refeq:constraint, applying a scalar product by an eigenstate $|t_A\rangle$ of T_A on the left, and defining $|\psi(t_A)\rangle \equiv \langle t_A|\Psi\rangle\rangle$, it holds that

$$i\hbar\frac{\partial}{\partial t_A}|\psi(t_A)\rangle = [H_R + H_{int}(t_A)]\,|\psi(t_A)\rangle, \tag{3}$$

which is the time-dependent Schrödinger equation that denotes the evolution of system R with respect to the time measured by clock A. Then, the usual unitary evolution of a quantum system is recovered from the static picture introduced by Page and Wootters.

As a result, $|\Psi\rangle\rangle$ can be written as

$$|\Psi\rangle\rangle = \int dt_A\,|t_A\rangle \otimes |\psi(t_A)\rangle. \tag{4}$$

Because $|\Psi\rangle\rangle$ contains information about $|\psi(t_A)\rangle$ at every t_A, it is sometimes referred to as the *history state.*

The results in Refs. [34, 35, 11] are central to our analysis. In their approach, system R is assumed to contain a clock B, and the rest of it is simply referred to as system S. Then, while system B gives the internal time of the system $R = B + S$, system A provides time as observed by an external system. In this case, the total Hamiltonian is

$$H_T = H_A + H_B + H_S + H_{int}(T_B). \tag{5}$$

Although the time-dependent term H_{int} can be taken to be a function of both T_A and T_B in a more general scenario, we assume it does not depend on T_A for simplicity. However, nothing significant changes in our analysis if H_{int} is also a function of T_A.

With the previous H_T, the analog of the Schrödinger equation becomes

$$i\hbar\frac{\partial}{\partial t_A}|\psi(t_A)\rangle = (H_B + H_S + H_{int}(T_B))\,|\psi(t_A)\rangle. \tag{6}$$

This implies that the effective Hamiltonian acting on system R is

$$H_{eff} = H_B + H_S + H_{int}(T_B). \tag{7}$$

While the designation of which clock is internal or external to the system of interest seems to be arbitrary up until now, they will acquire a more concrete meaning below when measurements of energy are studied.

3. Time-energy uncertainty relations and the emergence of non-unitarity

While discussing time-energy uncertainty relations we shall demonstrate the emergence of non-Hermitian dynamics in two scenarios used for performing energy measurements: (1) Measurement carried out by an external system, and (2) Measurement carried out by an internal system. See also [17] for more details.

3.1. Measurement carried out by an external system

Here we study the von Neumann measurement of the total energy of system R in the Page and Wootters timeless framework. Specifically, we consider the case where the measurement is carried out by an external system. Then, in addition to the systems already introduced above, we also consider an external pointer system E, and the history state $|\Psi\rangle\rangle$ belongs to the space $\mathcal{H}_A \otimes \mathcal{H}_R \otimes \mathcal{H}_E$.

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For simplicity, the free evolution of system E will be neglected. Thus, the measurement interaction can be represented by $H_{VN} = g(T_A)H_R P_E$, where $H_R = H_B + H_S + H_{int}(T_B)$, P_E is the conjugate momentum of the pointer E, and g is a non-negative function that differs from zero exclusively during the duration of the measurement. For notation purposes, we assume the measurement starts at $t_A = 0$ and ends at $t_A = \tau$ (both according to clock A). Moreover, we let

$$\int_{-\infty}^{\infty} g(t)dt = \int_0^{\tau} g(t)dt = K, \tag{8}$$

where K is a positive real constant associated with the strength (hence, the precision) of the measurement.

Moreover, we assume that P_E only takes non-negative values. As we shall see, the duration of the measurement in clock B will linearly depend on P_E. Thus, this condition will assure that the clocks considered here are "good" clocks in the sense that they do not move backward in time and, in particular, that the duration of the measurement is non-negative.

With that, the total Hamiltonian of the composed system is $H_T = H_A + H_R + g(T_A)H_R P_E$. Then, using Eq. (1) and defining $|\psi(t_A)\rangle \equiv \langle t_A|\Psi\rangle\rangle$, we obtain the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t_A}|\psi(t_A)\rangle = [H_R + g(t_A)H_R P_E]\,|\psi(t_A)\rangle, \tag{9}$$

i.e., from the perspective of clock A, the evolution is generated by the effective Hamiltonian

$$H_1 = H_R + g(t_A)H_R P_E, \tag{10}$$

which is the usual Hamiltonian of a time-independent system during a measurement. This is expected since, in standard quantum mechanics, time is an external parameter, as it is in this case.

Using the Heisenberg equation of motion to study the evolution of T_B with respect to clock A, we conclude that

$$\frac{d}{dt_A}T_B = -\frac{i}{\hbar}[T_B, H_1] = I + g(t_A)P_E. \tag{11}$$

This shows that, when g vanishes, the flow of time in both clocks is the same. In fact, in this case, from the perspective of clock A, any uncertainty in clock B is associated with its initial uncertainty. In particular, if both clocks start localized and synchronized, they remain localized and synchronized.

Yet, clock B ticks faster than clock A whenever g is non-zero, i.e., during the measurement of energy. If g is a function whose integral over time grows smoothly, then the transition to a faster ticking rate also happens smoothly. However, this is not always the case. A dramatic example can be observed by assuming a highly idealized case where g is the delta function. In this case, from the perspective of clock A, there is a sudden "jump" of the pointer of clock B.

On the other hand, using again Eq. (1) and defining $|\phi(t_B)\rangle \equiv \langle t_B|\Psi\rangle\rangle$, we obtain

$$\begin{aligned} i\hbar[I + g(T_A)P_E]\frac{\partial}{\partial t_B}|\phi(t_B)\rangle &= H_A|\phi(t_B)\rangle \\ &+ [I + g(T_A)P_E]\,[H_S + H_{int}(t_B)]|\phi(t_B)\rangle. \end{aligned} \tag{12}$$

Moreover, recalling that P_E is assumed to only take non-negative values, which implies that $I + g(T_A)P_E$ is invertible, the effective Hamiltonian is

$$H_2 = [I + g(T_A)P_E]^{-1}\,H_A + H_S + H_{int}(t_B). \tag{13}$$

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Observe that the function g that controls the measurement continues to be a function of the operator T_A, while it was a function of the parameter t_A when considering A's perspective. In a scenario like this, a well-localized event in clock A has uncertainty in clock B which was previously discussed, for instance, in Ref. [11]. This means, in particular, that the start and end of the measurement, which are well-localized in A, have uncertainty in B.

Moreover, in a sense, Hamiltonian H_2 corresponds to a measurement of H_A. However, the measurement function g is controlled by the time in clock A, which is an observable from the perspective of B. Because of it, the measurement Hamiltonian would have been symmetrized in more standard treatments of the process, which is not the case here. In fact, H_2 is non-Hermitian and, as a consequence, the evolution of clock A from the perspective of clock B is non-unitary, a characteristic that will be further discussed later. For now, observe that the Heisenberg equation of motion for T_A with respect to t_B gives

$$\frac{d}{dt_B}T_A = -\frac{i}{\hbar}[T_A, H_2] = [I + g(T_A)P_E]^{-1}. \tag{14}$$

This result is expected in view of Eq. (11) since it gives the inverse of the passage of time on clock B with respect to clock A. However, once again, it should be noticed that here the argument of g is an operator, and no longer a parameter. Then, with respect to clock B, if clock A starts with some uncertainty, there will be some uncertainty about when the measurement starts and ends. Nevertheless, on average, the "flow of time" during the measurement of energy is expected to be smaller in clock A, from the perspective of both clock A and clock B. It could be if interest to further study the possible connections between this purely quantum effect and the well-known time-dilation in special relativity.

Although Eq. (14) shows consistency across different clock perspectives, it raises questions about the use of the Heisenberg equation here. While Schrödinger's and Heisenberg's representations are unitarily equivalent, since the evolution from clock B's perspective is, in general, non-unitary, it is not trivial to obtain the Heisenberg equation from the effective Hamiltonian in the Schrödinger equation. Nevertheless, this can be done with the introduction of an "indefinite metric" in the Hilbert space, a method proposed long ago by Dirac [36] (see also [37] for Pauli's perspective). The method consists of introducing a new metric to the space of states. A well-known class of systems that fit this approach is the class of PT-symmetric systems [38]. It seems, then, that these metrics are associated with particular spatial and temporal symmetries, among others. This raises many other research questions. For instance, How to transform these metrics when switching reference frames? Can modifications in informational and thermodynamic bounds be extracted from these metrics alone? To justify the latter, it is noteworthy that PT-symmetric systems can evolve faster than Hermitian systems [39, 40]. Then, the idea is to look for similar results for the type of non-Hermitian operators of interest to this research avenue.

3.2. Measurement carried out by an internal system

Here we consider once more the total energy measurement of system R, although now the measurement in question is carried out by an internal system. We assume that system R has access to an apparatus I, and the history state $|\Psi\rangle\rangle$ belongs to $\mathcal{H}_A \otimes \mathcal{H}_R \otimes \mathcal{H}_I$. Note that the apparatus may be assumed to be an internal degree of freedom of the system whose energy is left out of the measurement.

Similarly to what was done in the previous subsection, the free evolution of the pointer I will be neglected. As a result, the measurement can be represented by the following von Neumann interaction

$$H_{VN} = \frac{1}{2}[g(T_B)H_R + H_R g(T_B)]P_I, \tag{15}$$

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where $H_R = H_B + H_S + H_{int}(T_B)$, P_I is the conjugate momentum of apparatus I, presupposed to only take non-negative values, and g is the same function introduced in the previous section. Observe the necessary symmetrization of the product $g(T_B)H_R$ due to the lack of commutativity between T_B and H_B, as previously considered, for instance, in Ref. [41]. Also, we assume that in clock B the measurement starts at $t_B = 0$ and ends at $t_B = \tau$.

Since the total Hamiltonian of the composed system is $H_T = H_A + H_R + H_{VN}$, using Eq. (1) and defining $|\psi(t_A)\rangle \equiv \langle t_A|\Psi\rangle\rangle$, we obtain the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t_A}|\psi(t_A)\rangle = H_3|\psi(t_A)\rangle, \tag{16}$$

where

$$H_3 = H_R + \frac{1}{2}[g(T_B)H_R + H_R g(T_B)]P_I. \tag{17}$$

is the effective Hamiltonian from the perspective of clock A. With that, the Heisenberg equation of motion for T_B with respect to t_A is

$$\frac{d}{dt_A}T_B = -\frac{i}{\hbar}[T_B, H_3] = I + g(T_B)P_I. \tag{18}$$

On the other hand, using Eq. (1) and defining $|\phi(t_B)\rangle \equiv \langle t_B|\Psi\rangle\rangle$, we obtain

$$\begin{aligned} i\hbar[I + g(t_B)P_I]\frac{\partial}{\partial t_B}|\phi(t_B)\rangle &= \left[H_A - \frac{i\hbar}{2}g'(t_B)P_I\right]|\phi(t_B)\rangle \\ &+ [I + g(t_B)P_I][H_S + H_{int}(t_B)]|\phi(t_B)\rangle, \end{aligned} \tag{19}$$

where the invertibility of $I + g(t_B)P_I$ was used. This means that, from the perspective of clock B, the evolution is generated by the effective Hamiltonian

$$H_4 = [I + g(t_B)P_I]^{-1}\left[H_A - \frac{i\hbar}{2}g'(t_B)P_I\right] + H_S + H_{int}(t_B), \tag{20}$$

which, similarly to H_2, is a non-Hermitian operator.

Then, using the Heisenberg equation of motion to study the evolution of T_A with respect to t_B, it can be concluded that

$$\frac{d}{dt_B}T_A = -\frac{i}{\hbar}[T_A, H_4] = [I + g(t_B)P_I]^{-1}, \tag{21}$$

which, as expected, up to the difference that the argument of g is now a parameter, is the inverse of the relation between the passage of time in the two clocks given by Eq. (18). Here, again, the use of the Heisenberg equation is justified with the introduction of an indefinite metric in the Hilbert space.

Could, then, the non-unitarity observed here be related to gravitational effects? After all, non-unitarity is expected to play a role in quantum gravity, see e.g. [42, 43, 44, 45]. However, gravitationally interacting clocks have been considered in the literature [46, 6, 11], and it was shown that they lead to unitary dynamics.

Then, why is the result different here? The answer to this question lies in the fact that the von Neumann interaction includes products of the time operator of a clock by the individual Hamiltonian of the same or other clocks. In fact, if such a condition is matched, the dynamics of the clocks associated with the time operator in the interaction is non-unitary with respect to the clock whose individual Hamiltonian appears multiplying it. To see that, consider a system composed of n clocks $C_1, C_2, \ldots, C_n$ and let the total Hamiltonian be

$$H_T = \sum_k H_{C_k} + f(T_{C_{r_1}}, T_{C_{r_2}}, \ldots, T_{C_{r_m}})H_{C_s} + H_{int}, \tag{22}$$

where $m < n$, the indices $r_1, r_2, \ldots, r_m$, and s are different elements of $\{1, 2, \ldots, n\}$, and H_{int} is not a function of H_s. Then, from the perspective of C_s we can write

$$\begin{aligned} i\hbar[I + f(T_{C_{r_1}}, T_{C_{r_2}}, \ldots, T_{C_{r_m}})]\frac{\partial}{\partial t_{C_s}}|\psi(t_{C_s})\rangle = \\ = \left[\sum_{k\neq s} H_{C_k} + H_{int}\right]|\psi(t_{C_s})\rangle, \end{aligned} \tag{23}$$

i.e., the effective Hamiltonian, which is to be compared with H_2, is

$$H_{eff}^{C_s} = [I + f(T_{C_{r_1}}, T_{C_{r_2}}, \ldots, T_{C_{r_m}})]^{-1}\left[\sum_{k\neq s} H_{C_k} + H_{int}\right], \tag{24}$$

assuming that $I + f(T_{C_{r_1}}, T_{C_{r_2}}, \ldots, T_{C_{r_m}})$ is invertible. This Hamiltonian is manifestly non-Hermitian. More specifically, the parts associated with clocks $C_{r_1}, C_{r_2}, \ldots, C_{r_m}$ are non-Hermitian and, hence, have non-unitary evolution because of the lack of commutativity between their individual time and Hamiltonian operators. We find this general appearance of non-unitarity an interesting phenomenon worthy of further research because it could possibly account for an inherent irreversibility of time, even on purely quantum mechanical grounds.

4. Accelerating quantum clocks

In this section, we connect the emergence of non-unitarity just studied, with the existence of acceleration of the clock frame [18]. For that, we need to understand how acceleration affects the Hamiltonian of a system. This influence should come from the potential V that can be added to the Hamiltonian. Our analysis here will be restricted to potentials that functions of the position x only, but this should be generalized in future work. In this case, we know from Newtonian physics that $ma = -dV/dx$, where m is the mass of the system and a is its acceleration. This implies that $V(x) = -m\int_{x_0}^{x} a(x')dx'$, where it was assumed that $V(x_0) = 0$. Further assuming that the particle is moving in a single direction and that x_0 is a point very far away in the direction of motion of it, we can define the function $f(x) = \int_{x}^{x_0} a(x')dx'$, which is always non-negative if the particle's velocity is never lower than its initial value. In this case, $V(x) = mf(x)$. Now, with a similar argument to the one used in [47, 5, 48, 49, 6, 12], we can add a post-Newtonian correction to V by using the energy-mass equivalence, i.e., $m \mapsto m + H/c^2$, where H is the Hamiltonian of the particle (other post-Newtonian corrections could be added as well). For simplicity, we neglect the term with the static mass m and consider only H/c^2. Then, redefining f to absorb the constant $1/c^2$, we can write $V(x) = Hf(x)$.

Observe that the energy-mass equivalence implies that the potential (associated with f) also couples to internal degrees of freedom of the system [47]. Then, if the clock is an internal degree of freedom of an accelerating particle, the above discussion leads to the change

$$H_S \mapsto H_S + \frac{1}{2}\left[f(X)H_S + H_S f(X)\right], \tag{25}$$

where H_S is the full Hamiltonian of system S, which includes a particle M, before it accelerates, and a clock A, and X is its position. It can be checked that this also leads to a non-unitary evolution from the perspective of clock A. In fact, consider a system composed of two particles, M and N, each with its own internal clock, A and B, respectively. If we assume that particles A starts accelerating at some point and none of the parts interact before that, the total Hamiltonian of both systems is

$$H = H_A + H_B + H_M + H_N + \frac{1}{2}\left[f(X_M)(H_A + H_M) + (H_A + H_M)f(X_M)\right] \tag{26}$$

and the effective Hamiltonian from the perspective of clock A after the acceleration is

$$H^A_{eff} = [I + f(X_M)]^{-1}\left[H_B + H_N - \frac{1}{2}[f(X_M), H_M]\right] + H_M, \tag{27}$$

which is, generally speaking, non-Hermitian, since in most cases $[I+f(X_M)]^{-1}$ does not commute with $[f(X_M), H_M]$. Now, if the particle has a large enough momentum, it may be possible to approximate its free Hamiltonian as proportional to its momentum (this can be done in the relativistic limit or by separating the total non-relativistic momentum into a large expectation value $\langle P\rangle_0$ at an instant t_0, and a "shifted momentum operator" Π such that $P = \langle P\rangle_0 I + \Pi$ and then neglecting the quadratic term Π^2 in the free Hamiltonian). In this case, letting the proportionality constant be the unity, we first observe that $[f(X_M), H_M] = i\hbar f'(X_M)$. Moreover, the Heisenberg equation leads to $X_M/dt_A = I$. As a result, neglecting any initial uncertainty associated with X_M, we can write

$$H^A_{eff} = [I + f(t_A)]^{-1}\left[H_B + H_N - \frac{i\hbar}{2}f'(t_A)\right] + H_M. \tag{28}$$

Observe that, at this stage, we can simplify the analysis by neglecting H_M and H_N. This means that a scenario involving two clocks, A and B, with one of them, A, accelerating with an acceleration associated with $f(T_A)$, can be considered a simplification of a scenario where H_M is neglected. In this sense, the analysis of both cases where the clock is an independent particle or where it is an internal degree of freedom are equivalent.

For a concrete example, consider two inertial rocket ships, each with its own clock. Initially, both are travelling with constant speed. Assuming that there is no interaction between them, we can write

$$H = H_A + H_B, \tag{29}$$

where, without loss of generality, we simplify our discussion by not including any other system in the analysis, which is possible based on what was just discussed.

Using Eq. (1), we observe that the Hamiltonian of clock B from the perspective of clock A is simply $H^A_{eff} = H_B$. Moreover, the Heisenberg equation implies that $dT_B/dt_A = I$, i.e., the flow of time is the same in both clocks.

Now, if the rocket associated with clock A starts accelerating, as was just discussed, the effective Hamiltonian of clock B from the perspective of clock A is given by

$$H^A_{eff} = [I + f(t_A)]^{-1}\left[H_B - \frac{i\hbar}{2}f'(t_A)\right], \tag{30}$$

which implies that evolution is non-unitary. Still, from the perspective of clock B, which is still an inertial frame, clock's A evolution is unitary. In fact, the effective Hamiltonian in this case is

$$H^B_{eff} = H_A + \frac{1}{2}\left[f(T_A)H_A + H_A f(T_A)\right]. \tag{31}$$

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However, there will be no perspective from which a unitary evolution is recovered if all clocks are accelerating. In fact, if g is a function associated with B's acceleration, it holds that

$$H = H_A + H_B + \frac{1}{2}\left[f(T_A)H_A + H_A f(T_A)\right] + \frac{1}{2}\left[g(T_B)H_B + H_B g(T_B)\right], \tag{32}$$

which results in non-Hermitian effective dynamics from the perspective of either clock A or B.

In future work we would like to generalize this case to motion in curvilinear coordinates and to the case of gravitationally interacting quantum systems – first within non-relativistic quantum mechanics, and then within the Klein-Gordon and Dirac equations.

5. Gravitationally interacting quantum clocks

Next is the case of gravitationally interacting clocks. We shall only present the gist of the argument, while the detailed analysis is available in [18]. In previous studies, it was shown that this type of interaction leads to a type of time dilation, although the unitarity of the evolution persists [5, 48, 49, 6, 11]. The time dilation can be even associated with loss of interferometric visibility [50].

In those analyses, a Newtonian gravitational potential of the form $V(r) = -Gm_A m_B/r$ was assumed, together with the post-Newtonian energy-mass equivalence correction, as in the above example. Then, the gravitational interaction between two clocks A and B can be added to the Hamiltonian as a term proportional to the product of their free Hamiltonians, i.e., $\lambda H_A H_B$, where $\lambda = -G/c^4 r$.

It can be noticed that the distance between the clocks is assumed to remain constant. Thus, despite the gravitational effects, both clocks work as inertial frames. In our exploration, however, we will allow the relative position of the clocks to change as a result of the gravitational interaction.

As an example, we will start by assuming that clocks A and B are internal degrees of freedom of particles M and N, respectively. Then, the gravitational potential can be written as $V(x_N - x_M) = -Gm_M m_N/|x_N - x_M|$. Moreover, if S is much more massive than R, for some periods of time we can assume that $x_N - x_M \approx x_N - x_0$, where x_0 is the initial position of system M. Letting the latter vanish, we have $V(x_N) = -Gm_M m_N/|x_N|$.

Now, using the mass-energy equivalence, we write $V(X_N) = -GH_A H_B|X_N|^{-1}/2c^4$. For simplicity and to make sure that any change to the ticking of either clock is due to the gravitational interaction between the systems, we assume no other interaction between them. This means that the total Hamiltonian of the composed system is

$$H = [I + f(X_N, H_B + H_N)](H_A + H_M) + H_B + H_N, \tag{33}$$

where $f(X_N, H_B) = -GH_B|X_N|^{-1}/2c^4$. As a result, the dynamics from the perspective of clock A is given by

$$H^A_{\text{eff}} = [I + f(X_N, H_B)]^{-1}(H_B + H_M + H_N), \tag{34}$$

which is non-Hermitian since $f(X_N, H_B)$ does not commute with H_N. This is so in spite of system S being assumed to be approximately inertial. This might seem surprising in view of the example with the rocket ships. However, a crucial aspect in that example is that the rockets had no interaction between them whatsoever. Here, system S interacts with the non-inertial system R.

Moreover, the effective Hamiltonian from the perspective of clock B is also non-Hermitian, as expected. More precisely,

$$H^B_{\text{eff}} = [I + f(X_N, H_A)]^{-1}(H_A + H_M + H_N). \tag{35}$$

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This, however, is a simple case of Newtonian attraction. It could be interesting to study post-Newtonian and other general-relativistic corrections affecting the dynamics of gravitating quantum clocks.

It is worth highlighting that gravitationally interacting systems were also considered in a recently introduced spacetime quantum reference frame [51]. There, the weak-field limit was assumed in order to avoid the problem of ordering of operators. In this limit, the dynamics was found to be unitary. This is also consistent with our results since the non-Hermitian character of the dynamics is accentuated at higher energies. However, since the Hermiticity of the dynamics appears only as an approximation, predictions using this type of approximation might likely deviate from the ones using the non-Hermitian Hamiltonians found here in experiments that are not relatively short.

It is also noteworthy that an analysis of the dynamics of quantum systems in the presence of singularities with different clocks has revealed the manifestation of non-Hermitian dynamics and, more specifically, non-unitary evolution [52]. In fact, it was found that, in this scenario, unitarity depends on the choice of the clock — even if every clock under consideration is a counterpart of good clocks at the classical level. Moreover, it was concluded that the general covariance of general relativity turns out to be incompatible with quantum unitary dynamics.

6. Dynamical nonlocality in space and time

This section continues to employ the same timeless framework, but for a completely different purpose – introduction and preliminary analysis of dynamical nonlocality in time. However, we begin the discussion with a review of dynamical nonlocality in space. In [53, 23] we have proposed a nonlocal ontology for quantum particles based on a deterministic set of operators within the Heisenberg picture. We have shown that some operators belonging to this set may, in general, evolve in a dynamically nonlocal way. For instance, the operator $e^{iP\ell/\hbar}$, akin to the modular momentum $P_{mod} = P \bmod hI/\ell$, where ℓ has units of length, changes in time according to:

$$\frac{d}{dt}e^{iP\ell/\hbar} = -\frac{i}{\hbar}\left[e^{iP\ell/\hbar}, H\right] = -\frac{i}{\hbar}\left[V(X+\ell I) - V(X)\right]e^{iP\ell/\hbar}, \tag{36}$$

which depends on two remote spatial locations separated by a distance ℓ. This apparent nonlocality is mitigated via quantum uncertainty [23]. The operator $e^{iP\ell/\hbar}$ is important when discussing interference phenomena since its expectation value explicitly depends on the relative phase, e.g. in the case of the double slit experiment [23].

The timeless framework allows to similarly analyze the modular energy operator $e^{iH_S\tau/\hbar}$, where τ is a parameter with units of time and H_S is the Hamiltonian of the system of interest. Here, I consider the case studied in Ref. [11], where system R was assumed to be composed of the main system of interest S and a clock B, i.e.

$$H_R = H_B + H_S + H_{BS}(T_B), \tag{37}$$

where $H_{BS}(T_B)$, assumed to be such that $[H_B, H_{BS}(t)] = 0$ for a parameter t, generates the inner unitary transformation of system S controlled by the time in clock B, i.e., changes on system R when it is completely isolated.

For simplicity, the term $H_{BS}(T_B)$ is assumed to be null. This implies that the effective Hamiltonian of system R from the perspective of clock A is

$$H^A_{eff} = H_B + H_S + H_{int}(t_A). \tag{38}$$

I also restrict to cases where $[H_B, H_{int}(T_A)] = 0$, i.e., H_{int} is not a function of T_B.

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The goal is to obtain the dynamics of the systems from the perspective of clock B, although a similar conclusion can be drawn based on an analysis from the perspective of clock A. Then, writing $\langle t_B|\Psi\rangle = |\varphi(t_B)\rangle \in \mathcal{H}_A \otimes \mathcal{H}_S$, it holds that

$$i\hbar \frac{\partial}{\partial t_B}|\varphi(t_B)\rangle = [H_A + H_S + H_{int}(T_A)]|\varphi(t_B)\rangle, \tag{39}$$

i.e., from B's perspective, the effective Hamiltonian is

$$H^B_{eff} = H_A + H_S + H_{int}(T_A) \tag{40}$$

From the Heisenberg equation of motion, we obtain the t_B-evolution, i.e., the evolution from the perspective of the clock B of the modular energy $e^{-iH_A\tau/\hbar}$ of clock A, where τ is a parameter with units of time. Explicitly,

$$\begin{aligned} \frac{d}{dt_B}e^{iH_A\tau/\hbar} &= -\frac{i}{\hbar}\left[e^{iH_A\tau/\hbar}, H^B_{eff}\right] \\ &= -\frac{i}{\hbar}\left[e^{iH_A\tau/\hbar}H_{int}(T_A) - H_{int}(T_A)e^{iH_A\tau/\hbar}\right] \\ &= -\frac{i}{\hbar}\left[H_{int}(T_A + \tau I) - H_{int}(T_A)\right]e^{iH_A\tau/\hbar}, \end{aligned} \tag{41}$$

which is to be compared with the equation of motion of the modular momentum. Observe that no particular set of assumptions was made in this derivation. In fact, the result follows from the fact that $e^{iH_A\tau/\hbar}$ generates translations in the time variable of clock A and H^B_{eff} is, in general, a function of T_A. Also, although Eq. (41) seems to allow backwards-in-time signaling, this is not the case due to quantum uncertainty. Nevertheless, we find it interesting to further pursue these dynamical types of nonlocality – in space and time – and connect them with kinematic (entanglement-based) nonlocality. We have shown in [54] that kinematic nonlocality (even in theories beyond quantum mechanics) will not violate relativistic causality if generalized uncertainty relations are obeyed, which could be interesting to compare with the dynamical case. Additional analysis and specific examples are given in [16].

It may seem that the kinematic and dynamic types of nonlocality are unrelated to each other, but in some recent works [22, 55, 56, 25] connections were made between them relying, on the necessity of quantum entanglement in order to explain the latter type. We would suggest to further pursue this direction in order to unify these two types of nonlocality or alternatively, render them inherently different.

7. Discussion

Qualitative and quantitative features of time in physics remain subtle and deep but the Page-Wootters approach has shown once more to be constructive. We have utilized it above to find new time-energy uncertainty relations pertaining to the interplay between internal and external clocks during the process of energy measurement [17]. We have also employed it to introduce the phenomenon of dynamical nonlocality in quantum time, which is further exemplified in [16]. Finally, we have identified what seems to be a general feature of this approach, namely, the appearance of non-Hermiitan dynamics when non-inertial clocks are used. In particular, we showed how acceleration or (Newtonian) gravitation of massive particles lead to the emergence of non-unitarity from the perspective of quantum clocks internal to them. These results come as a consequence of the coupling between external and internal degrees of freedom of a quantum system including a clock subsystem (associated with the system's proper time).

While this latter result may seem unexpected, it is in line with the expectations of part of the community regarding the breakdown of unitarity in certain quantum gravity scenarios. In fact, using the equivalence principle, we can associate the non-unitarity to a gravitational effect, and some authors have hypothesized that deviations from unitary evolution should be an ingredient of the dynamics of quantum systems affected by gravity [42, 43, 44]. Although, in principle, it is always possible to find a locally inertial frame, in practice, every system of reference is inertial up to a certain order. Thus, further studies of these non-Hermitian systems seem to be of fundamental importance. If one wishes to approximate a clock as an inertial frame for a long period, what type of control must be applied to the clock? Analogously, if the clock is approximated as inertial, for how long can it predict the correct state of a dynamical system within some margin of error? What type of challenges does it bring to the comparison between two quantum clocks? How is the flow of information across systems affected by the non-unitary correction? These are some questions of interest for future work. There are also some possible relations to make with $\mathcal{PT}$-symmetric systems on the one hand (see some discussion in [17, 18], and gravitational approaches towards the problem of time, as well as gravitationally-related decoherence on the other hand.

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