Constrained lagrangians in N = 2-superspace formulations for the constant magnetic field system

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ABSTRACT

Two kinds of constraints are proposed and studied in the N = 2-superspace formulation of the constant magnetic field system. The constraints differ by the number of fermionic variables left effectively independent in the superspace formulation. Both types are recognized as being associated with the so-called standard and spin orbit coupling supersymmetrization procedures respectively and deal with chiral-type constraints. Our simple change of variables connecting the two-dimensional harmonic oscillator and the constant magnetic field contexts does also work in these superspace formulations.

The N = 2-superspace [1] is characterized by the time t and a pair of anticommuting parameters ϑ and $\bar{\vartheta}$ such that :

$$[t,\vartheta] = \{\vartheta,\vartheta\} = \{\bar{\vartheta},\bar{\vartheta}\} = \{\vartheta,\bar{\vartheta}\} = 0 \qquad (1)$$

so that the usual superderivatives are defined by

$$D = \partial_{\bar{\vartheta}} - i\vartheta\partial_t \quad \text{and} \quad \bar{D} = \partial_{\vartheta} - i\bar{\vartheta}\partial_t \quad . \qquad (2)$$

The fundamental superfields take here the form :

$$Z_i(t,\vartheta,\bar{\vartheta}) = q_i(t) + i\bar{\vartheta}\psi_i(t) + i\vartheta\bar{\psi}_i(t) + \vartheta\bar{\vartheta}F_i(t) \quad . \qquad (3)$$

Then, the superlagrangian appears as

$$L = \int d\bar{\vartheta} d\vartheta \mathcal{L} = \int d\bar{\vartheta} d\vartheta \, [\tfrac{1}{2} \bar{D}Z_i DZ_i - W(Z_i^2)] \qquad (4)$$

where W refers to the superpotential associated with the central potential :

$$U(q_1^2+q_2^2) = 2(q_1^2+q_2^2)\left(\frac{\partial W}{\partial(q_1^2+q_2^2)}\right)^2 \quad . \qquad (5)$$

With the following definitions :

$$Z_\pm = Z_1 \pm iZ_2 \quad , \qquad (6)$$

the superdensity $\mathcal{L}$ reads

$$\mathcal{L}(Z_+,Z_-,DZ_+,\bar{D}Z_+,DZ_-,\bar{D}Z_-) = \frac{1}{4}\,[\bar{D}Z_+DZ_- - \bar{D}Z_-DZ_+] - W(Z_+Z_-) \quad . \qquad (7)$$

Let us constraint this density in two different ways.

a) Chiral constraints

We first impose the so-called chiral constraints [2] :

$$(a) \; : \; DZ_- = 0 \Leftrightarrow \bar{D}Z_+ = 0 \quad . \qquad (8)$$

Within the classical point of view, these equations are expressed in components by

$$F_- = -i\dot{q}_- \; ; \; \psi_- = 0 \Leftrightarrow F_+ = i\dot{q}_+ \; ; \; \bar{\psi}_+ = 0 \quad . \qquad (9)$$

They imply that only two fermionic variables are left independent. By imposing the constraints (a) in the expression (7) we obtain :

$$\mathcal{L}^{(a)} = -\frac{1}{4}\bar{D}Z_-^{(a)}DZ_+^{(a)} - W(Z_+^{(a)}Z_-^{(a)}) \quad , \qquad (10)$$

with $Z_+^{(a)} = q_+ + i\bar{\vartheta}\psi_+ + i\vartheta\bar{\vartheta}\dot{q}_+$ and $Z_-^{(a)} = q_- + i\vartheta\bar{\psi}_- - i\vartheta\bar{\vartheta}\dot{q}_-$.

The resulting lagrangian is then given by

$$L^{(a)} = \dot{q}_-\dot{q}_+ + \frac{i}{4}(\bar{\psi}_-\dot{\psi}_+ - \dot{\bar{\psi}}_-\psi_+) + iW'(q_+\dot{q}_- - q_-\dot{q}_+) - (W' + q_+q_-W'')\bar{\psi}_-\psi_+ \qquad (11)$$

where $W' = \dfrac{\partial W}{\partial(q_+q_-)}$; $W'' = \dfrac{\partial^2 W}{\partial(q_+q_-)^2}$.

Now by defining the new functions

$$eA_+ = iq_+W' \ , \ eA_- = -iq_-W' \qquad (12a)$$

with

$$B = -ie\left(\frac{\partial}{\partial q_+}A_+ - \frac{\partial}{\partial q_-}A_-\right) = 2(W' + q_+q_-W'') \ , \qquad (12b)$$

we can rewrite $L^{(a)}$ on the form :

$$L^{(a)} = \dot{q}_+\dot{q}_- + \frac{i}{4}(\bar{\psi}_-\dot{\psi}_+ - \dot{\bar{\psi}}_-\psi_+) + eA_+\dot{q}_- + eA_-\dot{q}_+ - \frac{eB}{2}\bar{\psi}_-\psi_+ \ . \qquad (13)$$

This lagrangian [2] is associated with the description of the interaction with a magnetic field $\vec{B}$ oriented along the z-axis. The corresponding equations of motion are given by

$$\begin{aligned} \ddot{q}_1 &= eB\dot{q}_2 - e(\partial_{q_1}B)\bar{\psi}_1\psi_1 \\ \ddot{q}_2 &= -eB\dot{q}_1 - e(\partial_{q_2}B)\bar{\psi}_1\psi_1 \ , \end{aligned} \qquad (14)$$

for the bosonic variables and by

$$\begin{aligned} \dot{\psi}_1 &= -ieB\psi_1 \Leftrightarrow \dot{\psi}_2 = -ieB\psi_2 \\ \dot{\bar{\psi}}_1 &= ieB\bar{\psi}_1 \Leftrightarrow \dot{\bar{\psi}}_2 = ieB\bar{\psi}_2 \end{aligned} \qquad (15)$$

for the fermionic variables.

Let us consider now the quantum point of view. In such a context the two independent fermionic variables ψ_1 and $\bar{\psi}_1$ satisfy :

$$\{\psi_1,\bar{\psi}_1\} = \mathbb{1} \quad . \tag{16}$$

Then, from the relations (9) and (16), we directly obtain the following anticommutation relations :

$$\{\psi_1,\bar{\psi}_2\} = -i\mathbb{1} \quad , \quad \{\psi_2,\bar{\psi}_1\} = i\mathbb{1} \quad , \tag{17}$$

so that we can summarize all these relations on the form :

$$\{\psi_i,\bar{\psi}_j\} = \delta_{ij} + i\varepsilon_{ij} \tag{18}$$

with $\varepsilon_{12} = -i\mathbb{1} = -\varepsilon_{21}$.
This has to be related with the spin-orbit coupling procedure of supersymmetrization [3].

b) <u>More general constraints</u>

The second way to constraint the superdensity (7) is given by considering the following relations [4] :

$$(b) \; : \; \bar{D}DZ_- = 0 \Leftrightarrow D\bar{D}Z_+ = 0 \quad . \tag{19}$$

Within the <u>classical</u> point of view, these relations are expressed in components on the form :

$$F_- = -i\dot{q}_- \; ; \; \psi_- = 0 \Leftrightarrow F_+ = i\dot{q}_+ \; ; \; \dot{\bar{\psi}}_+ = 0 \quad , \tag{20}$$

which have to be compared with Eqs. (9). Then the constraints (19) appear weaker than the chiral ones and they leave <u>four</u> fermionic variables independent :

$$\psi_1 = i\psi_2+\varphi \; , \; \bar{\psi}_1 = -i\bar{\psi}_2+\bar{\varphi} \quad . \tag{21}$$

For the associated superdensity we get :

$$\mathcal{L}^{(b)} = \frac{1}{4}\,(\bar{D}\Lambda+D\bar{\Lambda}) - \frac{1}{4}\,(\bar{D}Z_-^{(b)})DZ_+^{(b)} - W(Z_+^{(b)}Z_-^{(b)}) \tag{22}$$

with

$$Z_+^{(b)} = q_+ + i\bar{\vartheta}\psi_+ + i\vartheta\bar{\varphi} + i\vartheta\bar{\vartheta}\dot{q}_+ \quad ,$$

$$Z_-^{(b)} = q_- + i\bar{\vartheta}\varphi + i\vartheta\bar{\psi}_- - i\vartheta\bar{\vartheta}\dot{q}_-$$

and

$$\Lambda = -\tfrac{1}{2}\,\vartheta\bar{\psi}_+\psi_- \ , \ \bar{\Lambda} = -\tfrac{1}{2}\,\bar{\vartheta}\bar{\psi}_+\psi_- \quad .$$

The corresponding lagrangian reads :

$$L^{(b)} = \dot{q}_+\dot{q}_- + \frac{i}{4}\,(\bar{\psi}_-\dot{\psi}_+ - \dot{\bar{\psi}}_-\psi_+) + eA_+\dot{q}_- + eA_-\dot{q}_+ - \frac{eB}{2}\,(\bar{\psi}_-\psi_+ + \bar{\psi}_+\psi_-) +$$

$$ie[(\frac{\partial}{\partial q_+}\,A_-)\psi_+\bar{\psi}_+ - (\frac{\partial}{\partial q_-}\,A_+)\psi_-\bar{\psi}_-] \qquad (23)$$

with exactly the same definition for the functions A_+, A_- and B as before (see Eqs. (12)).

It results from all these considerations that we can also make here the connection with the interaction with a magnetic field $\vec{B}$ oriented along the z-axis.

Let us notice that the lagrangian $L^{(b)}$ is not gauge invariant. However we can restore it if and only if the magnetic field is constant. Finally, in the quantum context, we can make the connection with the standard [3],[5] procedure of supersymmetrization characterized by four independent fermionic variables which satisfy a Clifford algebra (i, j = 1, 2) :

$$\{\psi_i,\bar{\psi}_j\} = \delta_{ij} \ , \ \{\psi_i,\psi_j\} = 0 = \{\bar{\psi}_i,\bar{\psi}_j\} \quad . \qquad (24)$$

c) A change of variables between the harmonic oscillator and magnetic contexts

Let us consider now a very simple change of variables [6] between the two-dimensional harmonic oscillator and the constant magnetic field systems in the superspace formalism. It will illuminate certain points of our above considerations. The change of variables is here extended to bosonic as well as to fermionic variables. It becomes

$$q_i^0(t) = R_{ij}(t) q_j^M(t) \quad ,$$

$$\psi_i^0(t) = R_{ij}(t)\psi_j^M(t) \ , \ \bar{\psi}_i^0(t) = R_{ij}(t)\bar{\psi}_j^M(t)$$

with

$$R(t) = \begin{pmatrix} \cos \omega t & -\sin \omega t \\ \sin \omega t & \cos \omega t \end{pmatrix} \ , \ \omega = \frac{eB}{2} \quad , \tag{25}$$

where the superscripts 0 and M refer respectively to the harmonic oscillator and magnetic contexts. This change of variables puts in a one-to-one correspondence the constant magnetic field with an harmonic oscillator whose the frequency is $\frac{eB}{2}$. It implies the following relations between the corresponding superfields :

$$Z_i^0 = R_{ij}(t) Z_j^M(t,\vartheta,\bar{\vartheta}) \Leftrightarrow Z_i^M = R_{ji}(t) Z_j^0(t,\vartheta,\bar{\vartheta}) \quad . \tag{26}$$

Then, by applying the change of variables to the superdensity

$$\mathcal{L}^0 = \tfrac{1}{2}\, \bar{D}Z_i^0 D Z_i^0 - \frac{\omega}{2}\, Z_i^0 Z_i^0 \tag{27}$$

characteristic of the harmonic oscillator, we obtain

$$\mathcal{L}^M = \tfrac{1}{2}\, (\bar{D}Z_i^M)(DZ_i^M) - \frac{\omega}{2}\, (1+\omega\bar{\vartheta}\vartheta) Z_i^M Z_i^M +$$

$$\frac{i\omega}{2} \left[(\vartheta\bar{D}-\bar{\vartheta}D) Z_2^M Z_1^M + (\bar{\vartheta}D-\vartheta\bar{D}) Z_1^M Z_2^M \right] \tag{28}$$

which corresponds to the magnetic case. The classical associated equations of motion are given by

$$\begin{aligned} \ddot{q}_1^M &= eB\dot{q}_2^M \\ \ddot{q}_2^M &= -eB\dot{q}_1^M \end{aligned} \tag{29}$$

for the bosonic variables and by

$$\psi_+^M = -ieB\psi_+^M \Leftrightarrow \dot{\bar{\psi}}_-^M = ieB\bar{\psi}_-^M$$
$$\psi_-^M = 0 \quad \Leftrightarrow \quad \dot{\bar{\psi}}_+^M = 0 \tag{30}$$

for the fermionic variables. By solving the fermionic equations (30) we obtain

$$\psi_-^M = \varphi = \text{constant}$$
and
$$\bar{\psi}^M = \bar{\varphi} = \text{constant} \quad . \tag{31}$$

Such a result shows that this approach contains the two kinds of constraints we have considered before. The first one was associated to the annulation of the constants φ and $\bar{\varphi}$ (see (a)) while the second one gave a nonzero value to these same constants (see (b)).

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