Published for SISSA by Springer

Received: *August 5, 2022*
Accepted: *October 3, 2022*
Published: *October 17, 2022*

Shape dependence of mutual information in the OPE limit: linear responses

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Abstract: Mutual information serves as an important measure of correlation between subsystem components. In the framework of quantum field theories (QFTs) they have better regulated UV behavior than entanglement entropy, and thus provide more direct access to universal aspects of entanglement structures. In this paper, we study the linear responses under shape deformation of the mutual information in the conformal field theory (CFT) vacuum between two spheres of radius R separated by large distance $L \gg R$ or conformally equivalent configurations. Our calculations make use of the previous OPE results for mutual information [1] and the associated modular Hamiltonian [2]. In particular, we apply the entanglement first law to compute the linear responses of mutual information under shape deformation on one of the spheres. We find that the linear responses exhibit a high degree of universality for a selected class of OPE contributions. We demonstrate that there is a "little group" of symmetries associated with the set-up. Our result implies that the spherical mutual information is extremal over shape deformations of non-zero modes under the symmetry group.

Keywords: AdS-CFT Correspondence, Conformal Field Models in String Theory, Scale and Conformal Symmetries

ArXiv ePrint: 2207.05268

JHEP10(2022)101

https://doi.org/10.1007/JHEP10(2022)101

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1 Introduction

Entanglement structure has played a key role in probing deep aspects of quantum field theories (QFTs) not accessible using conventional tools and techniques. For example, it can provide the order parameter for topological phase [3, 4]; through the famous Ryu-Takayanagi formula [5] and its subsequent generalizations [6], it encodes information regarding bulk space-time geometry in holography [7–9]; it can be used to characterize universal monotonous properties of renormalization group flow [10–12]; and it is also crucial for revealing connection between information theory and energy conditions [13–15], etc. In principle, one can define various measures of the underlying entanglement structure. An important one is the entanglement entropy between subsystem A and its complement $\bar{A}$. For a global pure state $|\psi\rangle$, the entanglement entropy is defined by:

$$S_A^\psi \equiv -\mathrm{tr}_A \rho_A^\psi \ln \rho_A^\psi, \qquad \rho_A^\psi = \mathrm{tr}_{\bar{A}} |\psi\rangle\langle\psi| \tag{1.1}$$

This definition relies on the notion of reduced density matrix ρ_A^ψ, which only makes clear sense if the global Hilbert space factorizes into tensor product $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$. This is a convenient but usually problematic assumption in quantum field theories, for example due to gauge constraints imposed at the entangling boundary ∂A [16–18]. Another aspect associated with ∂A in quantum field theories is the ubiquitous ultra-violet (UV) divergences that arise when computing entanglement entropies. Roughly speaking, they come from short-range entanglements across ∂A that are only regularized by the UV cut-off of the theory. For example, in CFT the divergences take the form:

$$S = c_{d-2}\frac{R^{d-2}}{\delta^{d-2}} + c_{d-4}\frac{R^{d-4}}{\delta^{d-4}} + \ldots \begin{Bmatrix} + & c_{-1}, & d = & \text{odd} \\ + & c_0 \log\left(\frac{R}{\delta}\right), & d = & \text{even} \end{Bmatrix} \tag{1.2}$$

where R, δ is the scale of entangling region and UV cutoff respectively, $c_{d-2}, c_{d-4}, \ldots$ are coefficients that depend on the geometry of entangling regions, and c_{-1} and c_0 are universal. As a result, additional efforts are required in order to extract universal aspects of entanglement structures from the result of entanglement entropy. In order to do this, one usually construct combinations of entanglement quantities whose UV dependences cancel out. In practice, such cancellations can be arranged in a number of ways. For example, when extracting the order parameters for topological phases in 2+1 dimensional gapped systems, one computes the so-called topological entanglement entropy [3, 4] by adding and subtracting vacuum entanglement entropies associated with different regions:

$$S_{\text{topo}} \equiv S_A + S_B + S_C - S_{AB} - S_{BC} - S_{AC} + S_{ABC} \tag{1.3}$$

out of which the shape dependence of these regions cancel out completely and leaving only topological contributions. Fixing a particular entangling region A, one can also arrange the cancellation between states, for example when computing the so-called relative entropy between two global states ψ and σ:

$$S_A\left(\psi|\sigma\right) \equiv \text{tr}\rho_A^\psi \ln \rho_A^\psi - \text{tr}\rho_A^\psi \ln \rho_A^\sigma \tag{1.4}$$

The relative entropy $S_A\left(\psi|\sigma\right)$ provides a measure of distinguishability between the global states ψ and σ based on what one can access in the subsystem A, i.e. between ρ_A^ψ and ρ_A^σ. It satisfies several constraints such as positivity and monotonicity, and when applied to specific context they can sometimes reveal important physics such as the emergence of Einstein's equation in AdS/CFT [19–22] or the validity of the averaged null energy condition (ANEC) in QFTs [13], etc. Mutual information is one of the simplest combinations among these entanglement quantities. For disjoint entangling regions A and B, one can define the mutual information $I_{A,B}$ between them as:

$$I_{A,B} \equiv S_{A\cup B} - S_A - S_B \tag{1.5}$$

where we have omitted the dependence on the state ψ in these quantities, since in this paper shall focus on the vacuum state $|\psi\rangle = |\Omega\rangle$ from now on. The UV divergences associated with short-distance entanglements near $\partial\left(A \cup B\right)$ are cancelled in $I_{A,B}$, thus making it a universal quantity. Roughly speaking I_{AB} measures the correlation between the subsystems A and B. When we view $A \cup B$ together as a system in the mixed state ρ_{AB}, $I_{A,B}$ receives contributions from both classical and quantum correlations. In attempts to refine the distinction between classical and quantum correlations in mixed states, other measures have been proposed which include entanglement negativity [23–25]. Being a universal quantity, the mutual information behaves in ways that align more with intuitions. For example due to strong subadditivity (SSA) for entanglement entropies, the mutual information like relative entropies also satisfy monotonicity, i.e.:

$$I_{A,B} \leq I_{\tilde{A},B} \tag{1.6}$$

for $A \subseteq \tilde{A}$, while the entanglement entropies in general do not satisfy such constraints by themselves. On the other hand, the mutual information is less constrained by symmetries and thus encode more detailed information about the underlying theories. For example, while the single-interval vacuum entanglement entropies in 2d CFT ares constrained by symmetries to only depend on the central charges c, the mutual information between single-intervals depend on the full operator-spectrum of the CFTs.

Apart from the choice of entanglement measures, shape dependence on the entangling region can also reveal important information regarding entanglement structure. For example, in the presence of corner or cusp the entanglement entropy receives log divergent contributions whose coefficients on the other hand contain universal information [26–28]. For regions with smooth boundaries, more generally one can study shape dependence by computing perturbation theory under shape deformation [29, 30], usually about symmetric shapes i.e. spheres, half-planes etc. For spheres in CFTs, computing the first-order response indicates that its entanglement entropy is extremal under symmetry-breaking deformations [31, 32]; while it being perturbatively minimal was verified by the second-order response which also revealed universal non-local contributions known as the entanglement density [33]; the minimization was later proven beyond perturbation theory [34]. Sometimes by probing shape dependence of carefully chosen entanglement quantities, one discover surprising connections between entanglement behaviors and other aspects of QFTs, such as the irreversibility of RG flows and the validity of energy conditions, manifested through analyzing shape dependence of entanglement entropies (appropriately regularized) and relative entropies.

It is therefore natural to expect that the shape dependence of mutual information can also reflect important physics. For example, it has been shown that the shape dependence of $I_{A,B}$ satisfies constraints on the null cone of A which in the large separation limit (see (1.8) below) between A and B can give rise to the unitarity bound [35]. However, explicit results for shape dependence of mutual information using the general perturbative approach have been missing so far. One reason for this is that we do not have configurations with non-trivial mutual information that are sufficiently symmetric. In the case of a single sphere A the vacuum entanglement structure in CFT is fixed by symmetry. In particular the modular Hamiltonian which encodes the complete entanglement data is known explicitly [11]:

$$\hat{H}_A = 2\pi \int_A dx \left(\frac{R^2 - |x^2|}{2R} \right) \hat{T}_{tt}(x), \quad \rho_A = e^{-\hat{H}_A} \tag{1.7}$$

Therefore it makes sense to do perturbation theory about it as in [31–33]. On the other hand, the mutual information $I_{A,B}$ between two spheres A and B in CFT vacuum is not known

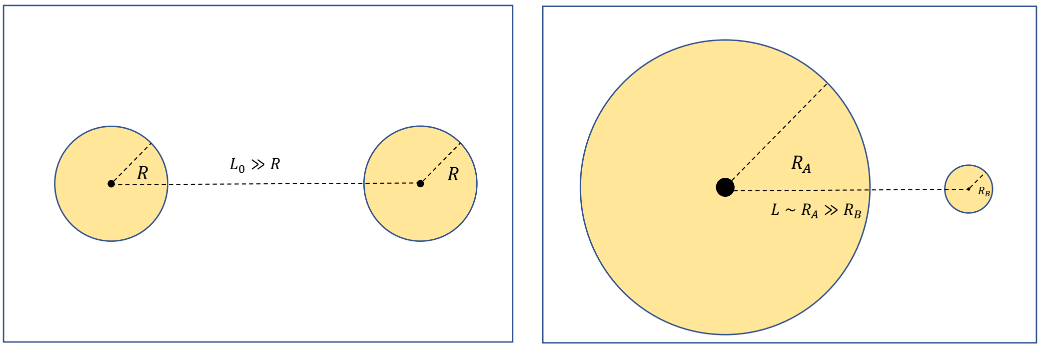

Figure 1. Two conformally equivalent configurations for the mutual information. The left: two spherical subregions of radius R separated by a distance $L_0 \gg R$. The right: two spherical subregions of radius R_A and R_B separated with distance $L \sim R_A \gg R_B$.

in general form, despite being the most symmetric configuration with non-trivial mutual information. As mentioned before it is sensitive to the details (e.g. operator spectrum) of the theory. Explicit results can be obtained by restricting to special theories, e.g. Dirac fermions in two-dimensions [36–38], in which the shape dependence on entangling region is degenerate; or by restricting to special limits in general CFTs, e.g. of large separation L between A and B (see figure 1), where I_{AB} admits an OPE-like expansion [1, 39–41]:

$$I_{A,B} = \mathcal{N}_\Delta \frac{\sqrt{\pi}\Gamma(2\Delta+1)}{4\Gamma\left(2\Delta+\frac{3}{2}\right)} \left(\frac{R_A R_B}{L^2}\right)^{2\Delta} + \dots \tag{1.8}$$

where Δ denotes the conformal dimension of an internal scalar primary operator that “carries” the correlation between A and B in some loose sense. The OPE result (1.8) is valid in any dimensional CFTs, and therefore provides a good starting point to develop perturbative expansion in shape deformations.

In this paper, we study the shape dependence of mutual information perturbatively about spheres. We will focus on the leading order contribution in the OPE (1.8). More specifically, we will compute the first-order linear response under shape deformation about one of the spheres A. The OPE expansion (1.8) is conformally invariant, i.e. depending only on the conformal-ratio:

$$\rho = \frac{2R_A 2R_B}{L^2 - R_A^2 - R_B^2} \ll 1 \tag{1.9}$$

where $R_{A,B}$ are the radius of the spheres A, B and L is the distance between them. Therefore, we have the freedom to work in a conformal frame where $R_A \sim L \gg R_B$, see figure 1, so that the fine-details in the shape-response of $I_{A,B}$ on A are optimally pronounced. A key input into our calculation is the modular Hamiltonian version of the OPE expansion (1.8) derived in [2]. This is because at the first order, the shape perturbation theory can be implemented in the context of the entanglement first law, which requires knowledge about the unperturbed modular Hamiltonian.

The paper is organized as follows. In section 2, we recall some basic ingredients regarding shape perturbation theory and its application through the entanglement first law

Figure 2. Relating shape deformation to metric deformation via a corresponding diffeomorphism.

for computing linear responses. In section 3, we summarize previous results for mutual information and the corresponding modular Hamiltonians in the OPE limit, which our calculations will be based upon. In section 4, we carry out the main computation and obtain results for linear response of mutual information under shape deformation on one of the spheres. During this we identify some subtleties associated with the integral expression of modular Hamiltonian in terms of local operators. In section 5, we explore the implications of our results from a symmetry point of view, and discuss the extremality property under shape deformations for mutual information on spheres. In section 6, we conclude the paper with some discussions and suggestions for future directions.

2 Shape perturbation theory and the entanglement first law

In this section, we quickly recall the necessary ingredients needed for our computations. More details can be found in [29–33].

2.1 Shape perturbation theory

We are only interested in the vacuum states, whose wave-functionals in QFTs we assume can be represented by Euclidean path-integrals in the lower-half-planes. As a result, the reduced density matrix elements on subsystem A can be written in terms of the Euclidean path-integrals with branch-cuts on A:

$$\langle\alpha|\rho_A|\beta\rangle = \int_{\Phi(A^-)=\beta}^{\Phi(A^+)=\alpha}[\mathcal{D}\Phi]e^{-I_E(g,\Phi)} \tag{2.1}$$

where Φ denotes the quantum fields collectively. Now we perform an infinitesimal shape deformation of the entangling region $A\to\tilde{A}$. For such a deformation, we can always find a corresponding coordinates transformation $\mathcal{F}: x^\mu \to \tilde{x}^\mu(x) = x^\mu + \zeta^\mu(x)$ on the whole Euclidean space-time such that $\mathcal{F}(\tilde{A}) = A$ (see figure 2).

Via diffeomorphism equivalence we can then trade the shape-deformation for a metric deformation $g_{\mu\nu}\to\tilde{g}_{\mu\nu}$ and write the reduced density matrix elements as:

$$\langle\alpha|\rho^g_{\tilde{A}}|\beta\rangle = \langle\tilde{\alpha}|\rho^{\tilde{g}}_A|\tilde{\beta}\rangle = \int_{\Phi(A^-)=\tilde{\beta}}^{\Phi(A^+)=\tilde{\alpha}}[\mathcal{D}\Phi]e^{-I_E(\tilde{g},\Phi)} \tag{2.2}$$

where:

$$\tilde{\alpha} = \alpha\circ\mathcal{F},\quad \tilde{\beta} = \beta\circ\mathcal{F},\quad \tilde{g}_{\mu\nu} = \nabla_\mu\tilde{x}^\alpha\nabla_\nu\tilde{x}^\beta g_{\alpha\beta} \tag{2.3}$$